The development and evaluation of a Web-based learning environment for proof-type problem solving in geometry among secondary students

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Accompanied by the CD “ANGEL”
Abstract

The major purpose of this study was to address the instructional needs of proof-type geometry problem solving. It was designed to address two research questions:

(Q1). What are the predictive indicators of successful proof-type geometry problem solving?

(Q2). Based on needs with an emphasis on formative evaluation, what is one design solution to support students solving proof-type problems in geometry?

The overall study focused on a learning need assessment in the first phase of the study (Study 1) and a development process to translate instructional needs identified into a supportive instructional environment for proof-type geometry problem solving in the second phase (Study 2).

The review of literature revealed that proof-type geometry problems have different learning requirements compared to other mathematical problems types. The solution process for proof-type geometry problems demands the adoption of a non-algorithmic approach in which students could activate problem-solving strategies that are domain specific. These strategies include heuristics such as using auxiliaries (parallel lines, bisectors and perpendiculars), alternative proving methods (indirect proof, reductio ad absurdum, method of contradiction). Equally important are the role of domain-general strategies during the solution of proof-type geometry problems such as working backward and logical inferencing. The literature review suggested that geometry content knowledge, general processes, and mathematical reasoning could be potential predictive indicators of successful proof-type geometry problem solving. However, the relative importance of these variables during the construction of geometry proofs had not been subjected to an empirical evaluation.

Study 1 takes up the above issue by determining the relative importance of these variables in proof type geometry problem solving. Data were collected from 166 Sri Lankan students on three independent variables: Geometry Content Knowledge (GCK), General Problem-Solving processes (GPS) and Mathematical Reasoning Skills (MRS); and a dependent variable Proof-Type Geometry problem-solving (PTG). The relationship among these variables was examined through a multiple linear regression analysis procedure. This analysis showed that geometry content knowledge, general problem-solving processes, and mathematical reasoning are predictive indicators of
successful proof-type geometry problem solving. Among these variables, geometry content knowledge was found to be the most influential one followed by general problem-solving processes and mathematical reasoning.

Three experts participated in a series of meetings to translate the above findings into a support framework for helping students learn to solve proof-type geometry problems in Study 2. This development process resulted in a conceptual model consisting of three major components: Remedial, Instructional and Problem Solving. The Remedial Component was suggested to address the learning needs related to geometric reasoning development, the Instructional Component focused on the development of content knowledge related to Euclidian deductive system, and the Problem-Solving Component was designed to facilitate proof-type geometry problem-solving skills among students who have the prerequisite geometric content knowledge and reasoning skills.

An iterative development process of design, development, review and revision was used to translate the Problem-Solving component into a Web-based, prototype learning environment in Part I of Study 2. This prototype, titled ANGEL (A Non-linear Geometry Environment for Learning), contained problem sets, process guidance, worked examples, diagram support and embedded content knowledge as core structural elements. Hyperlinked metacognitive supports were incorporated to facilitate the problem-solving process through guidance provided by general problem-solving processes such as analysis, representation, planning and use of knowledge retrieval by accessing embedded content. Although technology driven learning environments are mainly for student-technology interactions, ANGEL has additional advantages as it was designed for classroom use with teacher intervention to enhance social interactions: teacher-student and student-student that promote learning and construction.

The usability of ANGEL was tested in a constructivist collaborative learning environment. Six students selected from an Australian high school solved a series of proof-type geometry problems in pairs in a two-hour problem-solving session with the help of ANGEL. During their problem-solving attempts, data were collected in the form of student verbalization of the solution process, observation of problem-solving attempts, and written workings in the workbook. Having completed the problem-solving session, the students were interviewed to collect data on how they perceived ANGEL as a learning tool. The qualitative data analysis showed that the target group of students accepted ANGEL as a learning tool and that students enjoyed using ANGEL in problem
solving. These patterns of results suggest that ANGEL works as designed and assists students to construct knowledge related to proof-type geometry problem solving.
Declaration of authenticity

I certify that this thesis represents the original work of the author unless otherwise it has been acknowledged, and the material has not been previously published or submitted for a degree to any other university.

Madduma Bandara Ekanayake
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Publications

The following publications emerged from the study as a part of the thesis.


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Chapter 1: Study overview

1.0 Introduction

Problem solving is an important skill that plays an essential role in everyday activities, development, and survival. Because problem solving is important, the development of problem-solving skills has become an important social need. Education is regarded as a pivotal strategy for developing problem-solving skills (Gagne, 1980; Owen, Forman & Moscow, 1981). Hence problem solving has thus become a major concern in education (National Council of Teachers of Mathematics – NCTM, 1989; 2000; National Institute of Education – NIE, Sri Lanka, 1999).

This study focuses on the pedagogical issues associated with supporting the development of a specific type of problem solving in a particular knowledge domain within the school mathematics curriculum. It addresses the development of proof-type geometry problem solving among students at Senior Secondary Level (SSL). To do this it must identify their learning needs (needs assessment) and consider the translation of these into the instructional environment of the classroom.

The overarching research question of this study is:

How do you address the instructional needs of proof-type geometry problem solving?

This question contains two parts – needs assessment and creative translation of needs into a supportive instructional environment. The two parts are dealt with as two phases of the study (Study 1 and Study 2).

The first part of this chapter elaborates problem solving, mathematics and problem solving, the position of geometry and the unique aspects of proof-type problems. It identifies potential prerequisites for solving proof-type problems in geometry and generates the research questions. The second part of the chapter will explain the background, the significance and rationale of the study. It also provides an overview for the rest of the thesis.
1.1 Mathematics and problem solving

Problem solving requires accessing and use of knowledge (Bransford, Brown & Cocking, 2000). In most instances, the solution attempt results in changes to the problem environment. Understanding the problem environment and deciding on the changes to be made are essential to this process. Knowledge of mathematics plays a key role in the solution process. On one hand, mathematical knowledge provides the required elements for understanding the problem. On the other hand, mathematical reasoning provides powerful thinking tools that are necessary to arrive at a solution. Mathematics provides both knowledge and thinking skills that are important for problem solving.

The robustness of mathematical knowledge and associated models across domains and contexts helps integrate various domains into a coherent body of knowledge. This allows for transfer of knowledge from one domain to another during the problem-solving process. Hence, mathematical models are generic structures that fit into various situations across domains. Mathematical power is thus useful to widen the horizons in various domains (NCTM, 2000). For instance, people have used the power of mathematics to understand the universe well before they physically visited outer space.

Mathematical thinking and knowledge develop along with the use of general reasoning skills such as inductive reasoning, deductive reasoning and logical reasoning. Because of this learning mathematics also helps develop higher-order thinking skills. These skills are required to control the use of knowledge and thinking in problem solving that assist students decide what, why, how, when and where knowledge can be used in the solution process. Mathematics thus develops thinking skills required for the problem-solving process.

Another importance of mathematical problem solving is its involvement in making judgements about the validity of statements (Polya, 1973b; Rodgers, 2000; Wolf, 1998). Mathematical proof is the process that is used in making judgements, because it is regarded as a sophisticated and reliable filter that can be used to separate certainty from uncertainty (Polya, 1973 b).

People do not use mathematical proof to verify the certainty of statements in their everyday life. One might wonder why mathematics curricula emphasise reasoning and proof in standards of major curriculum documents. NCTM (2000) comments that:
Mathematical reasoning and proof offer powerful ways of developing and expressing insights about a wide range of phenomena (NCTM, 2000, p.56). 

Engagement with proof-type problems develops human insights into higher-order thinking such as formal deductive reasoning, logical reasoning and diagrammatic reasoning. Geometry has been identified as an important medium through which to introduce deductive proof (McClure, 2000).

1.2 Geometry and problem solving

Geometry knowledge is useful for the interpretation and understanding of real-life situations. Throughout civilisation, people have used geometric ideas and properties of objects even though they have not learned formal geometry. At present, angle, triangle, rectangle, parallel, and perpendicular are common terms used in everyday life, and this illustrates the growing importance of geometry in making sense of the environment.

Most products and constructions are related to geometry and geometric reasoning. Machinery, toys, furniture and clothing illustrate the closer relationships among geometric reasoning, development, culture and technology. Such widespread existence of geometric applications provides evidence of the contribution of geometry in human development.

The spatial reasoning that is associated with human intelligence is a core learning outcome of geometry (NCTM, 1989; 2000). It refers to the understanding of physical objects with spatial attributes such as shape, size, location and movement (Fujii, 1969). Among these attributes, shape and size have a close relationship to geometric concepts. Spatial reasoning is required to relate physical objects and abstract geometric concepts during problem solving. The abstraction of geometric representation brings an additional advantage to problem solving. Geometric representations can be analysed into parts and processed during problem solving (Fischbein, 1993). In contrast, the symbolic representations in algebra and other subjects do not demonstrate this attribute during the problem-solving process.

Geometry knowledge and reasoning are useful for problem solving in real-life situations. They also provide a strong basis for the development of mathematical proof.
1.3 Geometry and learning to develop mathematical proof

Mathematical proof is regarded as an effective way of expressing particular types of reasoning and justification (NCTM, 2000). It is one of the three accepted approaches to proving – the others being scientific and judicial (Polya, 1973b; Rodgers, 2000; Wolf, 1998). This significance has led curriculum reforms to include the teaching of mathematical proof for all students (NCTM, 2000, The Royal Society -TRS, 2001).

Teaching mathematical proof has been shown to be difficult. It deals with abstract mathematical objects that are defined and integrated into axiomatic deductive systems. Mathematical operations are selected from a pool of rules and theorems. Understanding the nature of both objects, the axiomatic system and the selection of appropriate rules are difficult for many students. Educationists believe that the Euclidean deductive system could reduce the complexity of learning to prove to some degree (McClure, 2000).

Why does geometry reduce the complexity of learning that is associated with the development of mathematical proof? It is because geometry knowledge can be represented in a visual form. This strategy enables students to engage in proof construction in a meaningful setting where the objects of focus such as angles, parallel lines, triangles, and circles are already familiar and visible. In other areas where students encounter proof, they often have to cope with additional difficulties such as the meaning of symbols, abstract statements and quantifiers.

Students can ‘try and feel’ the validity of a statement during proof development. For instance, there are various practical methods to verify relations like ‘angles of a triangle add up to $180^\circ$’. On the other hand, verification of statements expressed in abstract symbols such as $\Sigma r^3 = \{n(n+1)/2\}^2$ are more complex.

The development of proofs can be seen as a problem-solving activity. Solving proof-type geometry problems is regarded as a valuable opportunity to develop logical reasoning skills. Polya (1973a) once wrote:

Geometry as presented in Euclid’s Elements, is not a mere collection of facts, but a logical system. … it is the first and the greatest example of such a system, which other sciences have tried, and still are trying to imitate (Polya, 1973a, p. 217).

Euclidean deductive system provides a powerful way for students to learn reasoning formal mathematical deductive proof in a less complicated manner.
The logical methods involved in geometry tend to be less subtle than those in other introductory parts of mathematics. Students are able to use and develop their skills in logical thinking as soon as they emerge, rather than wait till a later stage in their mathematical education. This holds advantages for students’ intellectual development generally. Most deductions in school mathematics take the form of a linear sequence with each conclusion following from the previous one. Geometry involves contemplating several statements at the same time and drawing conclusions from the collection as a whole.

Solving proof-type geometry problems fosters reasoning skills. These thought processes can be enhanced by planning strategies, implementing them in order to achieve sub-goals and the global goal. It is relatively easy to solve problems in which the solution is not immediately obvious but reachable by memorized algorithms. In contrast, geometry problem solving enables students to solve problems by using their own strategies. This is an important feature of solution of proof-type mathematical problems.

Because geometry involves these skills proof-type problem solving in this domain is regarded as providing a great opportunity.

If a student failed to get acquainted with a geometric fact he did not miss so much; he may have little use for such facts in later life. But if he failed to get acquainted with geometric proof, he missed the best and simplest examples of true evidence and he missed the best opportunity to acquire the idea of strict reasoning (Polya, 1973a, pp. 216-7).

What Polya refers to is that the value of Euclidean geometry is in the development of thinking development rather than in acquiring subject knowledge. This implies that proof-type geometry problem solving is important as a means for developing advance human thinking skills.

1.4 Nature of geometry problem solving

It is acknowledged that proof-type geometry problem solving is difficult not only to learn, but also to teach (NCTM, 2000; TRS, 2001). This implies that the difficulty is located in the subject and the purpose of instruction would be to reduce subject complexity. These instructions should be tailored to suit the complexities associated with proof-type geometry problem solving.

According to an analysis of categorisations (Jonassen, 2000a; Robertson, 2001), proof-type geometry problems are well-structured, domain-specific, and non-algorithmic.
These three features can be considered as key determinants of what students need to bring to solving proof-type geometry problems as well as what pedagogical strategies teachers should develop in order to support student learning. This argument suggests domain-specific knowledge (henceforth content knowledge) and skills for solving non-algorithmic problems could be vital prerequisites for student success in solving proof-type geometry problems.

1.4.1 Role of content knowledge in proof-type geometry problem solving

Content knowledge that is related to proof-type geometry problem solving comprises geometric shapes and their relationships, definitions and axioms, representation of information such as conventions and practices, symbols and diagrams. Students should acquire those components of geometry knowledge in order to solve proof-type problems. Charalambos (1997) reports that 78% of students lack the necessary basic content knowledge for the solution of proof-type geometry problems.

The development of content knowledge is intertwined with that of related reasoning processes. Research shows that the development of geometric reasoning is different from that of other subjects. It does not follow Piagetian development phases. The van Hiele theory (Fuys, Geddes & Tischler, 1988; Lawrie, 1998; Senk, 1985) describes that the development of geometric reasoning takes place in five discrete levels namely: van Hiele Level 0, van Hiele Level 1, van Hiele Level 2, van Hiele Level 3, and van Hiele Level 4.

Different van Hiele levels have different characteristics and requirements in terms of language, reasoning and thinking (Fuys, Geddes & Tischler, 1988). The development of geometric thinking is hierarchical and takes place sequentially from vHL 0 to 4 successively. Without adequate maturity at level \(n\), the student cannot make progress to level \((n+1)\). Research on van Hiele theory suggests that acquiring geometry content knowledge can be difficult as the geometric thought process develops in discontinuous phases (Clements & Battista, 1992; Mistretta, 2000). For instance, to develop the concept of angle (content knowledge at vHL 2), students should reason out two sides and a common vertex in a closed geometric figure (maturity of reasoning at vHL1). This highlights the importance of reasoning skills in acquiring content knowledge.
It is also important that geometric reasoning develops through inductive reasoning. Table 1.1 summarises the relationship between the geometric reasoning and inductive reasoning related to the content knowledge about a triangle.

**Table 1.1 Differences between geometric reasoning across van Hiele levels about ‘triangle’**

<table>
<thead>
<tr>
<th>vHL Levels</th>
<th>Geometric reasoning</th>
<th>Inductive reasoning process</th>
<th>Nature</th>
</tr>
</thead>
<tbody>
<tr>
<td>vHL 0</td>
<td>Thinks as a physical object</td>
<td>Generalising visual perception of physical laminas and faces of solids into a shape</td>
<td>A physical shape: inside is necessary</td>
</tr>
<tr>
<td>vHL 1</td>
<td>Thinks as an object having three sides and three vertices</td>
<td>Generalising physical lamina, faces and triangular networks into a shape</td>
<td>A physical shape: inside is not necessary</td>
</tr>
<tr>
<td>vHL 2</td>
<td>Thinks as a geometric shape having three angles, three sides, and three vertices</td>
<td>Generalising physical sides into lines: vertices into points</td>
<td>A geometric shape: physical measurements for angles and sides are necessary</td>
</tr>
<tr>
<td>vHL 3</td>
<td>Thinks as an ideal geometric shape</td>
<td>No generalisation: according Euclidean deductive system</td>
<td>Triangle is perceived as a quality.</td>
</tr>
</tbody>
</table>

Information in Table 1.1 raises two issues: the influence of inductive reasoning on the development of geometric reasoning across van Hiele levels, and the importance of experiences in vHL 2 to understand the concept of triangle (ideal geometric triangle) as defined in the Euclidean system. Both themes demonstrate the importance of inductive reasoning, which is domain-general.

The reasoning level appropriate for learning proof-type geometry problem solving is van Hiele Level 3 (Senk, 1985; 1989; Shaughnessy and Burger, 1985). Students at this level are expected to possess geometry content knowledge including: concepts and properties of geometric shapes; diagrammatic representation of geometric information and interpretation; identifying required shapes or parts in a complex diagram; geometric relationships; selection of appropriate relations for a given situation and providing reasons. A student who is at a lower van Hiele level (vHL 0, vHL 1, and vHL 2) is not cognitively ready to participate in proof-type geometry problem solving activities.

Senk (1989) reports that not only do about 93% of students not possess appropriate prerequisite knowledge but also that they are located among three lower levels: 27% at
vHL0, 51% at vHL1 and 15% at vHL2. Conventional instructions in a classroom setting are not appropriate for the majority of the class unless students are provided with opportunities to make progress to vHL 3. More complex reasoning is based on advanced content knowledge and vice versa.

Traditional instructional efforts also tend to be less effective due to the gap that exists between instruction and learning. Having students at different van Hiele levels creates a paradox for the mathematics teacher within the frame of conventional instructional settings. Shaughnessy and Burger (1985) describe this situation:

Teachers and students often confront levels in a geometry class. That is, it is very likely that the teacher and students are reasoning about the same concept but at different levels. While the teacher is writing a careful definition of a rectangle on the chalkboard (Level 2), [some] students may be thinking about all the properties that the teacher has left out (Level 1) (Shaughnessy and Burger, 1985, p. 425).

This quotation indicates that the gap that exists between instructional process and the learning process is a major difficulty in learning about proof-related problems. Given that proof-type geometry problems are at found at vHL 3, and students are located at vHL 3, 2, 1 or 0, it is obvious there will be a range of gaps between instructional process and learning process.

Geometric reasoning is not the only critical issue related to content knowledge. Mathematical formal proof is a process that deals predominantly with abstract concepts that are usually not visible. In proof-type geometry problems, students work with diagrams that are visible, as well as concepts embedded in the diagrams (Fischbein, 1993). This dual nature of geometric figures creates two types of problems:

(i) Generalisation

The first difficulty is related to generalisation. Proof must reflect a general situation, whereas a diagram is usually perceived as a specific object that can be associated with that situation. Hence, most students cannot develop a generalised relationship in geometric proof (Charalambos, 1997).

(ii) Fixedness

The second difficulty is associated with fixedness to prototypical configurations. The prototypical configuration is a result of the visual property of geometric diagrams. In classroom practice, the teacher frequently draws, and draws on, typical geometric
diagrams with specific orientations to explain geometric relationships. For instance, an isosceles triangle is typically drawn with a horizontal side (most frequently the base). As a result, the perpendicular from vertex to the base is vertical. Consequently, students associate a relationship with such a prototypical configuration, and apply the relationship only when they see that configuration. As these prototypical configurations are seldom found in typical problem situations, students are less capable of retrieving the relevant relationship (Charalambos, 1997). Constructing generalisations and perceiving relevant parts from a complex diagram through concrete visual figures are also difficulties faced by students.

In summary, acquiring geometry content knowledge is a complex process as it is influenced by various factors. Some of these factors are domain-specific such as geometric reasoning while the others are domain-general such as inductive reasoning and visual perception.

1.4.2 Non-algorithmic nature of proof-type geometry problem solving

Geometry problem-solving process is complex in nature. The complexity of this process is related to the non-algorithmic nature of this class of problems. There are no formulae or other predetermined procedures to use as algorithms in the proof-type geometry problem-solving process. It requires students to repeat a chain of logical inferences. In case of failure, they need to try another inferential path.

In the process of developing a computer-based geometry proof tutor, Koedinger and Anderson (1993) calculated the number of possible inferences that has to be fed in to the computer program, and noted the following.

Of the 27 definitions, postulates and theorems that are introduced prior to such a problem in a traditional curriculum, 7 can be applied at the beginning of this problem. Some of these rules can be applied in more than one way yielding 45 possible inferences that can be made from this problem’s givens. Using the results of these inferences, essentially as new givens, we did the same thing over again and found that 563 inferences can be made at this second layer. At the third layer the options really explode as there are more than 100,000 possible inferences. The number of options continues to increase at further layers -- at minimum it takes 6 such layers of inferences to reach the problem goal (Koedinger and Anderson, 1993).
As the quotation describes, a typical proof-type geometry problem can exceed 100,000 possible inferences. From such a large number, selecting appropriate inferences is critical. Most of the other inferences could also be applied to produce a result that is not mathematically wrong, but does not produce the expected result and, therefore, is not appropriate. Extensive practice in problem solving is required to minimise inappropriate inferences.

The non-algorithmic nature of problem solving contributes to the novelty aspect of the problem. As a student as well as a secondary mathematics teacher, the investigator of the present study has experienced that students seek teacher explanation for almost all problems. Although the teacher provides the necessary information orally, all expressions cannot be recorded and students lose vital information. Their workbooks also provide incomplete information. Anderson (1995) asserts that the most important information related to the solution process such as the reason for making a decision is lost in class textbooks. Resources such as teachers, class textbooks and workbooks may not provide the required scaffolding for the majority of students. In addition heuristics such as working backward and using auxiliary objects can sometimes be helpful for the students.

Proof-type geometry problem solving is different from other mathematical problem-solving processes. The acquisition of content knowledge may not be sufficient for the success of proof-type geometry problem solving.

1.4.3. General problem-solving processes in proof-type geometry problem solving

Another feature of the non-algorithmic nature of proof-type geometry problem solving is the difficulty of finding a starting point or a method for approaching the problem (Healey and Hoyles, 1998; Riess, Kleime, and Heinze, 2001). Working backward seems to be helpful as a general problem-solving skill in proof-type geometry problem-solving process (Anderson, 1985). Cognitive processes such as planning have a role in searching for strategies and heuristics (Schoenfeld, 1985). As a control process, metacognition orchestrates the solution process. In addition to metacognitive processes, in order to deploy available content knowledge students need to activate appropriate reasoning processes.
1.4.4 Reasoning in proof development

Reasoning skills play two key roles during the solution of proof-type problems. Firstly, reasoning facilitates the construction of important links. During the course of proof development, the solver is required to show the connections between the different steps through chains of reasoning. A proof is incomplete without reasoning.

The National Council of Teachers of Mathematics considers reasoning and proof is intertwined.

Systematic reasoning is a defining feature of mathematics. Exploring, justifying, and using mathematical conjectures are common to all content areas and, with different levels of rigor, all grade levels. Through the use of reasoning, students learn that mathematics makes sense. Reasoning and proof must be a consistent part of students’ mathematical experiences in pre-kindergarten through grade 12 (NCTM, 2000-2004, Online).

Secondly, reasoning is important for the development and enrichment of mathematical content knowledge. This in turn facilitates the reasoning process and pattern generation. On the other hand, deductive proof draws on deductive reasoning skills. Both inductive and deductive reasoning are thus important in proof-type geometry problem solving.

1.5 Research questions

Study 1 – Needs Assessment

As proof-type geometry problems are domain-specific, it can be anticipated problem-solving process in this domain may be content knowledge driven. Content knowledge that is related to Euclidean geometry is a coherent body of mathematical knowledge. Understandings about the characteristics of objects such as point, angle, triangle, and shapes as well as knowledge about axiomatic reasoning are important components of this content knowledge.

The non-algorithmic nature of the proof-type geometry problem-solving process demands the use of non-algorithmic problem-solving strategies. These strategies seem to have links with domain-general processes.

As a component of mathematics, mathematical reasoning has been argued to influence proof-type geometry problem-solving process. These refer to the broad range of
reasoning skills that students activate in the context of solving a range of classroom mathematics problems. It would seem that these mathematical reasoning skills would contribute to the solution outcome of proof-type problems.

While domain-specific knowledge, domain-general processes and mathematical reasoning seem to be relevant to proof-type geometry problem solving, the relative influence of these three key knowledge-related factors has yet to be established, giving rise to the research question:

(Q1). What are the predictive indicators of successful proof-type geometry problem solving?

Study 2 – Instructional Design and Development

The answers to the above question would provide input to the following research question:

(Q2). Based on needs with an emphasis on formative evaluation, what is one design solution to support students solving proof-type problems in geometry?

1.6 Background to the study

The Sri Lankan education system has long acknowledged the importance of geometry in solving problems in personal contexts as well as in social contexts. Geometry is a component of the school mathematics curriculum, which is a compulsory subject up to senior secondary level (SSL) in the general school education system in Sri Lanka. Knowledge and skills, problem solving, communication, connections, and reasoning are major learning outcomes (NIE, 1999). All Sri Lankan students learn geometry in the mathematics curriculum throughout the span of grades 1-11 curriculum, that is up to end of SSL. About 20% of the mathematics curriculum, and 6% of the total senior secondary curriculum is devoted to learning geometry at SSL. Formal deductive proof is a key topic in the SSL geometry curriculum. It is mainly aimed at the key competency, reasoning. Some learning outcomes of formal deductive proof are also linked to other key competencies such as problem solving, communication, knowledge and skills, and connections.

At the end of SSL, students sit for the General Certificate of Education (Ordinary Level) examination conducted by the National Examinations and Testing Service (NETS). There are two papers to assess student achievement in mathematics:
Mathematics-I (on conceptual understanding and direct application), and Mathematics-II (on complex problem solving). The Chief Examiner’s Report provides information on student performance in geometry tasks in GCE examination.

- Candidates are reluctant to answer geometry questions in Mathematics II, particularly questions on deductive reasoning.
- Candidates who have answered geometry questions did not demonstrate a satisfactory level in problem solving, which require deductive reasoning.
- Answers for the Mathematics I paper suggest that candidates have not developed basic geometry concepts (NETS, 2000, p. 11).

These observations reflect that, students do not prefer proof-type geometry problem solving and demonstrate that they are less competent in tackling such problems. In these examinations, geometry problems appear in three forms: find-type; constructions; and proof-type. In addition, students have an option to drop two problems according to the provision made in the question paper. The students can make use of this advantage and drop two geometry problems as they may be less prepared for these questions. According to the same source (p.13),

- 80.23% of total candidates have answered the 5th sum, which is on construction and 47.52% of them scored expected pass mark for the problem.
- 21.55% of total candidates have answered the 7th sum, which is on deductive reasoning and 11.27% of them scored expected pass mark for the problem.
- 19.14% of total candidates have answered the 9th sum, which is also on deductive reasoning and 10.35% of them scored expected pass mark for the problem.

The above situation raises two important issues: first, students are reluctant to select proof-type problems; second, when they do decide to tackle these problems, their performance is relatively low. The above observations have been persistent during the past twenty years. The relevance of this problem has frequently been acknowledged, but it has received little attention from researchers.

Students perform differently in construction type versus proof-type geometry problems because although the underlying rules are the same for both types, the approach to solving construction problems rests on a set of procedures such as constructing triangles, bisectors, perpendiculants, circles and parallel lines. The critical task in a construction
problem seems to be the identification of the set procedure relevant to the problem. On the other hand, there are no set procedures for solving proof-type problems, and students need to find strategies throughout the process (Koedinger and Anderson, 1993). Students at SSL seem to be familiar with algorithmic problem solving, but not with non-algorithmic problem solving that requires a logical inferential approach.

1.7 Students’ achievement levels in proof-type geometry problem solving in other countries

Literature reveals that under-achievement in proof-type geometry problem solving is not specific to Sri Lanka but is common to other countries. Following are some examples to illustrate the situation.

Research reports that students in the United States of America rank formal Euclidean geometry as the least important, most disliked, and most difficult among topics of school mathematics and show that writing proof is difficult for them (Koedinger and Anderson, 1993; Senk, 1989). This is similar to the situation in Sri Lanka where students are reluctant to select proof-type geometry problems in the examination.

Harel and Sowder (1998) state that the efforts made by mathematics educators over decades to promote students’ conception of mathematical proof have not been successful. Empirical investigations show that students’ abilities in mathematical reasoning and proof are rather poor (Heinz & Kwak, 2002). In a study in the UK, Healy and Hoyles (1998) found that proof is a difficult area even to high-attaining students. In the context of Unites States, Weber (2001) reports a similar situation as ‘… research has demonstrated that students at all levels have great difficulty with the task of proof construction’ (Weber, 2001, p. 101). In addition to these examples, various similar reports can be found on the International Newsletter on proof at http://www-didactique.imag.fr/preuve/Newsletter/981112.html as it appeared in January 2004. In other words, the need for effective instructions for proof-type geometry problem-solving skills is a global concern.

Researchers in the field of mathematics as well as in other fields such as cognitive psychology have long paid attention to this problem. It is evident that research has not made significant progress in examining the issue. Hanna (2000) reports that over one hundred research papers on this topic have been published in journals only during the
1990’s. It is important to note that formal research may be little compared to the informal efforts made by teachers and other authorities at classroom, state and local levels.

In summary, instructional design for proof-type problem solving has long been a global issue. Consequently, investigations of successful instructions for learning proof-type geometry problems have become increasingly important. This analysis shows that the situation of performance in proof-type geometry problem solving in Sri Lanka exists in other countries as well. Teaching that focuses on the development of proof-type geometry problem solving has become a major concern among mathematics educators and classroom teachers both in Sri Lanka and elsewhere.

1.8 Recent research and implications for proof-type geometry problem-solving process

Students who are at lower van Hiele levels should possess the thinking level of van Hiele Level 3 to learn proof-type geometry problem solving (Senk, 1985; 1989; Shaughnessy and Burger, 1985). As they may be at different van Hiele levels, these students necessarily have different instructional needs. When individual differences are considered, two students at the same van Hiele level are also likely to have different needs. As the number of students increases the diversity of their needs also increases. The multiple needs of multilevel students demand non-linear learning environments with learner control capabilities. Although the teacher may be instrumental for teaching non-linear situations, it is not possible for a single teacher to simultaneously cope with multilevel instructional requirements.

Recent developments in information and communication technology (ICT) that permit the learner to explore a hypertext environment according to their needs may offer a potential solution that is required for non-linear learning environments. Deciding an entry point for each individual student is a key issue in designing learning environments for proof-type geometry problems, as students can be developing at different van Hiele Levels. A self-screening procedure could be employed in a computer-assisted environment. Ideas could be obtained from the diagnostic tests developed by Mayberry (1981) or Lawrie (1998) for the purpose of screening. The activities within a level could be designed in accordance with van Hiele phases of instruction: inquiry, directed orientation, explicitation, free orientation and integration with the help from the works of Clements and Battista (1992), Fuys, Geddes & Tischler (1988), Lawrie (1998), Mistretta
(2000), and Mayberry (1981). Each level would end with a post-test that would serve as a passport to the next level.

The complexity of the proof-type geometry problem-solving process can be attributed to its non-algorithmic nature. Learning to solve such problems demands guided support during the solution attempt. The worked examples of problems provide an effective model of expert problem-solving proficiency. Chinnappan (1992), and Chinnappan and Lawson (1996) provide some useful findings about incorporating general problem-solving strategies into worked examples as an instructional strategy for training in geometry problem solving. In those works they have claimed positive results in geometry problem solving.

The effectiveness of worked examples could be increased with additional strategies such as explanatory methods. Chi, Bassok, Lewis, Riemann and Glaser (1989) added self-explanations to worked examples for learning physics problem solving. Renkl (1997) implemented the worked example strategy in probability problem solving, and then the same in computer-based learning environments in 2002. Wong, Lawson and Keeves (2002) applied the self-explanation principles in the geometry problem-solving field. Sweller and others have experimented with a series of worked example strategies since 1985 to promote student problem solving in mathematics and science while minimising cognitive load. Reiss and Renkl (2002) have put forward an idea to combine Schoenfeld’s heuristic approach (1985; 1992; 1994), Sweller’s cognitive load theory (1988; 1994; with Cooper, 1985) and Boero’s proof step model (1999) to design sub-steps of proof-type geometry problems in a worked example environment. These efforts have not yet been applied in developing proof-type geometry problem-solving skills.

The nature of visual representation in geometry suggests students may value modelling of transfer of problem information into diagrammatic form, and ways to interpret information in a diagram. This could be provided through a range of strategies such as a ‘think-aloud’ that accompanies a movie demonstration, sequential unfolding of a problem with diagrammatic representation alongside text, or embedded annotations within a movie.

The use of ICT through hypertext and hypermedia environments such as web-based instructions offers some advantages over other instructional methods. The multimedia capability of web-based instructions supports the highly visual nature of geometry and
multiple information representation strategies. ICT can deal with visual objects. ICT can be used to create quality-learning environments using Web-based instructional strategies.

Recent research reveals that conventional instructional strategies are not effective for assisting students to solve proof-type geometry problems. The needs of students of different ability levels require a flexible, student-controlled, non-linear learning environment constructed by the teacher.

**1.9 Rationale for the study**

Proof development improves the growth of higher-order thinking skills such as problem solving and justification (NCTM, 2000). This has led to the inclusion of reasoning and proof in the K-12 mathematics curriculum in most countries. *Reasoning and proof* is among the ten curriculum standards of the American K-12 mathematics curriculum (NCTM, 2000). It is a key learning outcome in education systems of South Africa (Department of Education - DE, 2002), Sri Lanka (National Institute of Education, 1999) and the United Kingdom (The Royal Society - TRS, 2001).

Proof-type problem solving received less emphasis in the school curriculum during the 1970s and 1980s (Charalambos, 1997; Clements and Battista, 1992; Koedinger and Anderson, 1993). More recently this trend has been reversed.

Despite this, students persistently demonstrate poor performance in geometry proof-type problem solving. Although the research problem for this study originated in the context of Sri Lanka, it seems that the problem is persistent in other countries. Researchers in the fields of mathematics education, cognitive science, instructional technology, psychology, information technologies, artificial intelligence, computer science, and education have individually and collectively acknowledged this situation. Thus there is a need to study this type of problem solving and examine potential instructional strategies.

This study intends to help students develop their geometry problem-solving skills using Web-based strategies. Although a large number of geometry materials are placed on the Web for senior secondary students, they do not facilitate problem solving more effectively than textbooks do. A few can be seen as better instructional materials, but they do not use the full potential of Web strategies in instructional development. None has developed instructions within the context of general problem-solving processes.
This study also addresses issues related to geometry problem solving in the context of the value of general problem-solving skills. It includes developing problem-solving skills as an independent learner, which facilitates lifelong education. It also puts forward issues related to learning in an information rich, technology based learning environment, which fosters access, and use of knowledge retrievals to solve ill-structured problems that people confront in real life.

In studies of problem solving, little effort has been invested in evaluating instructional strategies for proof-type problems. Researchers have evaluated the influence of general skills in non-proof-type geometry problem solving (Chinnappan, 1992; Chinnappan & Lawson, 1996). In some areas of mathematics, the influence of problem-solving strategies has been evaluated as to their effectiveness. The present study intends to draw on findings of the aforementioned research in order to develop proof-type geometry problem-solving skills.

1.10 Significance of the study

Proof-type problem solving has long existed as a critical issue in teaching and learning mathematics at secondary level. The problem itself is significant as researchers in various fields try to understand the nature of the problem, because proof-type problem solving is a difficult area even among university undergraduates and mathematics graduate teachers (Jones, 2000). Hence, researchers in various countries try to find better instructional solutions for proof-type geometry problem solving (Anderson, 1995; Healy & Hoyles, 1998; Heinz & Kwak, 2002; Koedinger & Anderson, 1993; Reiss et al., 2001). Teaching to solve proof-type problems has thus become a key concern among professionals in the field of mathematics education.

Whilst learning as well as teaching proof-type mathematical problem solving is difficult, educators need all students to be educated in mathematical proof (NCTM, 2000; TRS, 2001). Literature reveals that research on appropriate instruction for developing skills in proof-type geometry problem solving has been growing during the last decades in various fields. It also reveals the difficulty of proof-type geometry problem solving is a common problem in many countries. This implies that solving the problem is important.

The major intended outcomes of the present study are recommendations and suggestions for appropriate instructional strategies for proof-type geometry problem solving. The
needs compiled during the study as well as recommendations and suggestions are expected to provide significant insights for researchers and professionals in mathematics education. The study considers proof-type geometry problem solving also as one form of problem solving in general and one form of non-algorithmic problem solving in particular. It attempts to address the main research problem in a general problem-solving context. Problem-solving process has been researched from cognitive perspectives. The rich body of knowledge that has been accumulated as a result of research in different fields could be incorporated to address the main research problem. In that sense, the findings of this study will be transferable to the context of teaching and learning of non-routine problem solving and non-algorithmic problem solving.

Another important intended outcome of the study is development of the prototype of a Web-based learning environment for proof-type geometry problem solving. It should enrich the literature in two ways. First the product could be a model as well as a source for producing such learning environments. This prototype should incorporate information and communication technology that is potentially available to many students. It illustrates firstly the translation of needs into a support framework. Secondly, the methodology and procedure developed in this study contribute to developmental (or applied) research (Schoenfeld, 2000; Richey & Nelson, 1996).

During the problem-solving process, the study encountered a range of significant outcomes related to methodologies. The scoring rubric designed in Chapter 3 presents procedures to analyse mathematical problem solving particularly the proof-type geometry problem-solving processes. This can be used in pedagogical activities such as evaluation, research, and diagnosis. Regression analysis in Chapter 4 contributes data to the ongoing debate on the involvement of content knowledge versus domain-general knowledge in mathematical problem solving. Chapter 6 presents a simple method to record the solution process of mathematical problem-solving process as students attempt it. The method is also transferable to other fields.
1.11 Definitions of specific terms

The following terms will be used in this thesis as follows.

*Algorithmic problems*

These are the problems in which the solution process follows pre-determined procedures.

*Non-algorithmic problems*

These are problems that could not be solved by following a particular pre-determined approach.

*Proof-type geometry problems*

These are problems that require the establishment of a logical path that demonstrates the certainty of a statement.

*Content knowledge*

Knowledge is specific to a single domain. This is also referred to as domain-specific knowledge.

*Domain-general problems*

These are problems that require knowledge, strategies, and procedures that do not belong to specific domains.

*Senior secondary students*

Senior secondary students are the students in the present Sri Lankan education system (2004) who learn Euclidean deductive geometry in their school mathematics curriculum. This is the target population for this study.

1.12 Organization of the rest of the thesis

Chapter 2 reviews literature about the contribution of other studies regarding potential factors that are relevant to the proof-type geometry problem-solving process:

- Content knowledge
• General problem-solving processes
• Mathematical reasoning

From this review a list of instructional needs emerges to support students in developing skills in solving proof-type problems in geometry. The review also indicates a grey area involving predictive indicators of geometry problem-solving skills.

Chapter 3 describes the methodology of multiple linear regression (MLR) analysis that was employed to find answers to the research question, (Q_1). What are the predictive indicators of successful proof-type geometry problem solving?

Chapter 4 presents the results of the MLR analysis and answers (Q_1).

The predictive indicators of proof-type geometry problem-solving skills are then combined with the instructional needs identified from the literature review to provide a single list of needs for teaching and learning proof-type geometry problem solving. The discussion raises the question: how do the identified issues translate into a design of a support framework for students solving proof-type problems in geometry?

Chapter 5 describes Study 2: the process of translation of the instructional needs identified into a web-base environment to support students solving proof-type problems in geometry. It will present the methodology, design and development along with a discussion of software features. Hence, Study 2 addresses the research question, (Q_2). Based on needs with an emphasis on formative evaluation, what is one design solution to support students solving proof-type problems in geometry?

Chapter 6 provides the details of methodology and results of formative evaluation of the web-based instructional model. Chapter 7 will conclude the study with a summary of findings and major implications and recommendations.
Chapter 2: Review of literature

2.0 Introduction

The purpose of the present study is to design a learning environment that helps students solve proof-type geometry problems. The quality of this learning environment will be mainly indicated by how effectively the learning environment can address the learning requirements. The purpose of this chapter is to review literature related to three key factors that influence proof-type problem-solving performance: content knowledge, general processes and mathematical reasoning, and to identify instructional needs that would lead to the design of an appropriate learning environment for senior secondary students.

2.1 Mathematical problem-solving process

Mathematics is given special emphasis in school curricula. Students in most countries are expected to learn mathematics in their primary and secondary schooling (Department of Education 2002 – South Africa; NCTM, 2000 – United States; NIE, 1999 – Sri Lanka; Owens & Perry, 2001 – Australia; TRS, 2001 – United Kingdom). The solution of problems is an integral part of learning mathematics.

Despite the importance of learning to solve proof-type problems, students constantly demonstrate low achievement levels in this area. One factor responsible for such a situation may be the inappropriateness of instructional strategies that are used in the classroom. To be effective, the instructional process has to address students’ individual learning needs that impact on their problem-solving ability. To identify these needs a thorough analysis of the mathematical problem-solving process is required.

2.1.1. Mathematical problems and the solution of those problems

The nature of problems is an important element in the understanding of the problem-solving process. Although problems range from simple to complex, it would be useful to understand features that are common to all problems.

Research on problem solving reveals that a problem arises due to the gap that exists between the solver’s prior knowledge and that required to reach the goal. The problem
represents a novel situation that is outside the current experience of the problem solver (Robertson, 2001). Schoenfeld (1985) characterises mathematical problems as tasks that provide an intellectual impasse to the problem solver. According to this analysis, the solver (hereafter the student) has to generate knowledge to fill the gap in order to overcome the impasse. If the problem still proves to be difficult, then the student requires external help to solve the problem.

What students bring to solving problems has been researched in various fields. Literature related to this stream of research reveals that problem-solving processes are driven by prior knowledge (Greeno, 1973; 1978). Schoenfeld (1985) refers to one aspect of this knowledge as resources. Bransford, Brown & Cocking (2000) emphasise the thinking that underlies problem solving. The student has to engage thinking to extend the horizons of existing knowledge to overcome difficulty in the problem (Gagne, 1980; Owen, Forman & Moscow, 1981).

There is a common element to the solution of many types of problems. During the problem-solving process, the student generates new information (Chinnappan, 1992). The problem solution process starts with initial information provided in the problem statement and continues until the goal is achieved. This flow of information during the solution process is illustrated in Figure 2.1.

![Mathematical problem-solving process](image)

**Figure 2.1 - Mathematical problem-solving process**

During the non-linear process the student applies knowledge and thinking to generate new information on the basis of available information in order to accomplish the task. If the generated new information is erroneous, then the process has to be reviewed and changed. If the goal is not reached, then generated information could provide direction for alternative solution paths. The process could continue until the goal is reached.
The nature of knowledge and thinking related to mathematical problem solving is central to understanding the solution. When the task is to solve a mathematical problem, it requires knowledge of mathematical concepts and principles.

However, having mathematical content knowledge is not sufficient to solve mathematical problems. The student needs to be able to access, select, retrieve, and make use of that knowledge (Lawson & Chinnappan, 2000). In other words, there should be a process of selection, decision-making, and judgement of appropriate knowledge resources. Reasoning supports this process (Manktelow, 1999).

To generate new information from problem information during the solution attempt, the student needs to activate control processes. Schoenfeld (1985) refers to these as metacognitive skills that are argued to be domain-independent.

In summary, mathematical problem solving requires mathematical content knowledge to provide resources, mathematical reasoning to select resources, and metacognition to control cognitive functions. Facilitating mathematical problem solving requires support of mathematical content knowledge, appropriate reasoning skills and metacognitive skills.

2.1.2. Cognitive processes during problem solving

Researchers have studied general problem-solving processes and identified cognitive processes that constitute the problem solving process. Table 2.1 presents the perspectives of several authors.
Table 2.1 - Cognitive processes in problem solving

<table>
<thead>
<tr>
<th>Author/ researcher</th>
<th>Identified cognitive processes in the problem-solving process</th>
</tr>
</thead>
<tbody>
<tr>
<td>Polya (1973 a)</td>
<td>Understanding the problem, planning, carrying out, and looking back</td>
</tr>
<tr>
<td>Greeno (1973)</td>
<td>Interpretation of the problem, retrieval of information, planning, and carrying out</td>
</tr>
<tr>
<td>Scandura (1977)</td>
<td>Breaking problem into parts, formulating sub-goals, searching for strategies and assessing them, and achieving by sub-goals and the goal</td>
</tr>
<tr>
<td>Schoenfeld (1985)</td>
<td>Analysis, design, exploration, implementation, and verification</td>
</tr>
<tr>
<td>Hayes (1989)</td>
<td>Analysing the problem, representation, planning, carrying out, and evaluation, consolidating gains of the problem</td>
</tr>
</tbody>
</table>

Table 2.1 shows that different authors/researchers have identified common cognitive processes underlying the problem-solving process. In order to integrate them into a single profile that describes the proof-type geometry problem-solving process, the cognitive processes identified by Polya’s (1973a) theoretical analysis and Schoenfeld’s (1985) empirical analysis are compared in Figure 2.2 as they relate most to the non-algorithmic nature of these problems.

![Figure 2.2 – Comparison of the work of Polya (1973a) and Schoenfeld (1985)](image)

The models illustrated in Figure 2.2 represent the cognitive processes of general problem solving. Understanding the problem in Polya’s model (1973a) and Analysis in Schoenfeld’s model (1985) are about understanding the problem. Table 2.1 also suggests that the process of analysis is a common and important cognitive process. This cognitive process involves identification of the given information and relations.
The planning process is also a common feature in these analyses. Schoenfeld sees planning as an iterative process that follows analysis, design, and exploration in a cyclic manner. It can be expected that reasoning plays a significant role leading to decision making on how and what resources to invest in the problem-solving process.

The models mention implementation of planning, in which knowledge retrieval plays a vital role. The process can be seen as use of knowledge retrieval. It can also be inferred that new information is generated in this process until the goal is found.

The format of problem information may need to be converted into another format so that problem solving could proceed. For instance, proof-type geometry problems cannot be solved unless verbal problem information is translated into diagrammatic information. Some theorists (Hayes, 1989; Robertson, 2001) suggest a separate cognitive process called representation. The model of Schoenfeld (1985) or Polya (1973a) does not highlight representation as a process.

Greeno (1973) proposes that these cognitive processes are not likely to be carried out in a strict sequential manner. This could be a reasonable argument as the same cognitive process could reappear several times during the problem-solving process. For instance, as the decision making process, planning can exist during use of knowledge retrieval.

For the purposes of the present study, the investigator has developed a schematic representation (see Figure 2.3) of the non-algorithmic proof-type geometry problem-solving process that integrates four key processes (analysis, representation, planning and use of knowledge retrieval) with the information earlier presented in Figure 2.1. Figure 2.3 highlights the role of general processes in the solution of proof-type problems.
In summary, analysis, representation, planning, and use of knowledge retrieval are seen as cognitive processes in the problem-solving process. Once activated, it is an iterative process until the goal is found. The following section provides details of each of these cognitive processes.

2.1.2.1 Analysis

The role of analysis is to understand the problem. Students must completely understand the given information and the goal or what the question is really asking. Schoenfeld (1985) argues that if a student does not understand the question that can almost result in failure.

Mathematical problems are typically seen in text form. Students are provided with all information required to solve the problem. During analysis, the student reads the problem and analyses it to identify Problem Information (Figure 2.3). In a proof-type geometry problem, students have to understand the problem: parts, components, and the situation. In the proof-type geometry problem-solving process, the student has to identify: key terms, phrases and sentences, and the goal.

2.1.2.2 Representation

The format of information provided in the problem statement may not be easy to process. It may need to be represented in a different format so that it is possible to process. Representation could be seen internally (in working memory) as well as
externally (in the physical environment). These two types of representations are interrelated (Schnotz & Bannert, 2003; Zhang, 1997).

Internal information representation takes place in working memory. Newell & Simon (1972) acknowledge geometric diagram as an external memory. Geometric diagram has been accepted as a useful external representation (Polya 1973a; Schoenfeld, 1985). In this study, representation generally refers to external representation. The typical representation in geometry appears in diagrams. This representation is sensible, analysable, and process oriented.

Representation does not just mean the initial problem representation as it happens throughout the process. For instance, generated new information also has to be represented. Representation could be in the forms of symbols, objects and non-conventional signs. They represent geometric concepts, rules, constraints, or relations (Charalamboss, 1997, Fischbein, 1993). As the problem-solving process progresses, diagrammatic representation grows rich with information. The disadvantage is that the diagram does not demonstrate the sequence of how information was grown, hence proof is not represented by diagram and it has to be stated in semantic form.

2.1.2.3 Planning

Planning process in non-algorithmic problem solving is different to that of algorithmic problem solving. During this process the student has to search for strategies. As shown in Figure 2.2 (b), Schoenfeld (1985) has described two components: exploration, and design as an iterative process. This means that at most times students will need to use the ‘try and accept’ strategy.

In non-algorithmic problem solving, the planning for a solution path and selection of strategies are critical (Schoenfeld, 1985). This step may not be critical in the algorithmic approach, because the strategies are embedded in the algorithm. Since proof-type geometry problems are not algorithmic, reasoning may also have a role for the selection of strategies such as heuristics and inference-based on working backward.

Simplifying the situation and reformatting the problem are regarded as useful strategies in instructional design for well-structured non-algorithmic problems (Robertson, 2001). Structuring the argument and hierarchical decomposition from global to local are included in the design process. The exploration seems to be highly effective when the
student can retrieve memories about equivalent problems, slightly modified problems and broadly modified problems (Shoenfeld, 1985).

2.1.2.4 Use of knowledge retrieval

This cognitive process involves the retrieval and appropriate use of activated knowledge. As a result of this process, new information is generated. Polya (1973a) labels this process ‘carry out’ whereas Schoenfeld (1985) calls it ‘implementation’. When strategies, procedure or algorithm are available, the student can execute them appropriately. Usually, they can act on given problem information or related information retrieved from memory.

The outcome of the previous three cognitive processes is mainly concerned with use of knowledge retrieval until the goal is found. The importance of this process is related to generating new information. If new information does not lead to the goal, further new information is generated until the goal is reached. While the outcome of analysis, representation and planning involve use of activated knowledge, for the purposes of the present study use of knowledge retrieval is limited to instances where a student generates new information.

In summary, problem solving involves the use of the following general cognitive processes: analysis, representation, planning, and use of knowledge retrieval. The activation of these processes could be influenced by the type of problem that is being attempted.

2.1.3. Proof-type mathematical problems

As proof-type problems are the focus of this study, it is important to examine their structure vis-à-vis other forms of problems. According to Polya, (1966; 1973a), mathematical problems are of two types: find-type and proof-type. Table 2.2 summarises the main differences between them.
Table 2.2 - The difference between find type and proof-type problems

<table>
<thead>
<tr>
<th>Feature</th>
<th>Find-type</th>
<th>Proof-type</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution</td>
<td>Specific value or a product</td>
<td>Existence, or a process</td>
</tr>
<tr>
<td>Appearance of the goal</td>
<td>A simplified value of statement which relates to a specific situation</td>
<td>A logical chain to convince the certainty of a mathematical statement</td>
</tr>
<tr>
<td>Problem-solving strategy</td>
<td>Algorithmic or a predetermined process</td>
<td>Non-algorithmic, insightful strategies</td>
</tr>
<tr>
<td>Knowledge resources</td>
<td>Procedures, ready-made algorithms such as formulae or rules</td>
<td>Definitions, axioms and theorems as rules</td>
</tr>
<tr>
<td>Reasoning</td>
<td>Convergent and top-down non-formal deduction</td>
<td>Formal and top-top formal deduction</td>
</tr>
</tbody>
</table>

As shown in Table 2.2, the problem-solving approach and required reasoning are two factors that differentiate find-type problems from proof-type problems. In find-type problems, the goal is to find a product. The solving strategy for these problems is to use a pre-determined path or process to reach an unknown target. The student is expected to use algorithms to deduce the nature of a specific case (general to specific) (Polya, 1973b). The process involves quantitative deduction and a positivist perspective. Reasoning process related to find-type problems flows down from the top. It is convergent in the sense that it satisfies the conditions of the problem.

In contrast, the goal of a proof-type problem is to establish the existence of a given object. The student has to devise a legal path that links the given and the goal with a step-by-step logical chain. In the solving process, the student is supposed to use a dialectic strategy (Hersh, 1993) and basic rules to make the required path (Jonassen, 2000a; Polya, 1973b). The reasoning process is insightful and divergent, as the student has to consider all possibilities that are based on logical inference and check them one at a time. Although the rules are used for deduction, the process does not converge as a specific situation, because mathematical proof always represents a valid existence for all cases. Particularly in the case of geometry, information processing is based on qualitative geometric relationships.

Find-type problems are commonly encountered in most classroom learning activities. They involve numeric calculations and algorithmic problem solving procedures. Asking
students to find the value of an angle of a triangle when other angles are given is an example of a find-type problem. Most research works on mathematical problem solving are about find-type problems.

The nature of the non-algorithmic mathematical problem-solving process has been researched and showed very different results from those about algorithmic problem solving. Non-algorithmic problems are difficult even for students who have high content knowledge (Healy & Hoyles, 1998; Reiss, Klieme & Heinze, 2001) and easy for experienced problem solvers with high content knowledge (DeFranco & Hilton 1999; Schoenfeld, 1985). According to Table 2.2, knowledge types and the nature of the reasoning process seem to be key factors that differentiate proof-type problems from find-type problems.

In summary, proof-type problems have features that are over and above those found in other mathematical problems including the solution procedure. Therefore, a) research findings relevant to find-type problems do not represent the entire set of needs related to proof-type problems and b) instructional strategies appropriate to find-type problems may not be appropriate to proof-type problems.

As proof-type geometry problems are non-algorithmic in nature, the use of particular approaches or algorithms by students may not be productive. Students may have to draw on processes that would aid them in making inferences. These inferences seem to be supported by four key problem-solving processes: analysis, representation, planning, and use of knowledge retrieval.

2.2 Requirements of mathematical problem solving

What students bring to the problem drives their understanding of the problem-solution process. It has been discussed that mathematical content knowledge, mathematical reasoning, and metacognition are requisites in the mathematical problem-solving process. In an empirical analysis Schoenfeld (1985) lists four categories of knowledge and behaviour necessary for an adequate characterisation of mathematical problem solving. His analysis indicates the four main categories: resources, heuristics, control, and belief systems as requisites in mathematical problem solving. This section considers the issues related to these and other mathematical problem solving requisites.
2.2.1 Mathematical content knowledge

Content knowledge plays a key role in mathematical problem solving. It refers to facts, algorithmic procedures, routine non-algorithmic procedures, and understandings about conventions and rules (Schoenfeld, 1985). In other words, mathematical content knowledge provides rich resources for the problem-solving process. Chinnappan (1992) researched the difference in content knowledge organization between high achievers and low achievers. High achievers were able to produce a greater number of correct solutions and a greater number of generative processing events than low achievers. They also showed that they have more content knowledge and made better use of that knowledge than low achievers. The researcher concluded that high achievers’ greater content knowledge was an important factor in the solution outcomes. Figure 2.1 shows that problem solving is an information generative process. It can be inferred that content knowledge plays a key role in this process.

Anderson (1985) states that two types of knowledge are involved in the problem-solving process: declarative knowledge and procedural knowledge. In addition, some authors (McInerney & McInerney, 2001) mention a third type called conditional knowledge. The following sections will discuss these three types of content knowledge.

2.2.1.1 Declarative knowledge

Declarative knowledge includes factual knowledge, episodic knowledge, and abstract knowledge (Anderson, 1985; Robertson, 2001; McInerney & McInerney, 2001; Bruning, Schraw & Ronning, 1999). It is generally known as the ‘what’ aspect of knowledge that can be used to explain a thing, an incident, a concept, a process, or a principle. For instance, one can declare that the ‘angle sum of a triangle is 180°’. Declarative knowledge develops through generalizations in the form of propositions, propositional networks, images, and linear orderings that are organized and integrated into knowledge structures (Gagne, Yekovich & Yekovich, 1993). A proposition is the smallest bit of declarative information (Greeno, 1973), whereas propositional networks refer to stored arrangements of semantic information (Kintsch, 1974; Kintsch & van Dijk, 1978). The term linear orderings refers to a basic unit of declarative knowledge that encodes the order of the elements. This implies that declarative knowledge can be simple to complex. It can be organised as records, which can be retrieved on request.
(Anderson, 1985). The records signify a level of organization of declarative information.

In a study on proof-type geometry problem solving Reiss, et al. (2001) identified declarative knowledge as a prerequisite for proof-type geometry problem solving. This is the only one of three content knowledge types identified by Riess et al. (2001) as influencing proof-type geometry problem solving. This suggests that declarative knowledge plays a key role in solving proof-type geometry problems. In the analysis process, the student has to understand and interpret the parts and sentences in the problem statement. Concepts, properties, and rules are required for this part of the solution attempt. Almost all geometric aspects such as concepts, axioms, conventions, and theorems constitute declarative knowledge.

Declarative knowledge is also essential in the representation process in proof-type geometry problem solving too. For instance, the student has to convert each and every piece of information into diagrammatic form. The meaning of each part is related to declarative knowledge.

In the planning process, students need to infer, decide on or select the appropriate rule from a pool of rules (Jonassen, 2000a). In proof-type geometry problem solving, students need to use rules and relationships. Declarative knowledge is required in generating new information, i.e. the use of knowledge retrieval process. That means that the role of declarative knowledge is important throughout the proof-type geometry problem-solving process.

2.2.1.2 Procedural knowledge

Another type of knowledge useful in the problem-solving process is procedural knowledge (Anderson, 1985). It is viewed as ‘knowing how’, and includes memories about instructions, procedures and rules (McInerney & McInerney, 2001). It consists of knowledge related to procedure (Anderson, 1985). A procedure refers to a set of instructions that has to be furnished to perform a task. Computer programs, food recipes, and mathematical algorithms are examples of procedural knowledge. Procedural knowledge is rooted in declarative knowledge (Anderson, 1985) and relates to specific situations.

Procedural knowledge is required to transform information during the problem-solving process (Anderson, 1985). For example, the declarative knowledge: ‘angle sum of a
triangle is 180° provides instructions to calculate the magnitude of an unknown angle in a triangle. The process of using this declarative knowledge becomes a procedure: ‘get the sum of two angles and deduct the result from 180°’. Knowledge about this process is procedural knowledge.

The process of converting declarative knowledge into procedural knowledge is called *proceduralisation* and at the end of the complete transition, the action becomes automatic (Anderson, 1985). Once declarative knowledge is converted into procedural knowledge, then it can be used and demonstrated as a skill.

Procedural knowledge and declarative knowledge seem to be interrelated. Generally, procedural knowledge is situational whereas declarative knowledge is global. This property is important in proceduralisation of the same declarative knowledge for different situations. For example, the declarative knowledge the ‘angle sum of a triangle is 180°’ can be used to calculate an unknown angle. It can also be used as procedural knowledge to establish other declarative knowledge elements: the ‘sum of complementary angles in a right-angled triangle is 90°’ or the ‘angle sum of a polygon with $n$ sides is $2n - 4$ right angles’. One can interpret procedural knowledge as the knowledge of how declarative knowledge is used in problem solving.

Declarative knowledge contains knowledge elements the flexibility of which allows it to combine with other knowledge elements to solve problems or to make procedural knowledge. Procedural knowledge provides tools for solving mathematical problems. This can be used even without conceptual understanding.

Usually procedural knowledge contains a set of instructions that generates a set of steps in the solution. For instance, the procedure for ‘construction of the perpendicular bisector of a line segment’ contains a set of instructions. However, the contribution of procedural knowledge to proof-type geometry problems seems to be slight, because proof-type geometry problem solving is not rule driven (Jonassen, 2000a) where rules belong to declarative knowledge. For the development of a proof, all relationships are taken from theorems, which are declarative knowledge. This suggests that procedural knowledge seems to be less effective in proof-type geometry problem solving.

Both declarative knowledge and procedural knowledge have limitations. They cannot be used everywhere. Declarative knowledge is about *what to use*, and procedural
knowledge is about *how to use*. It has to be decided when to use knowledge and where to use knowledge. Knowledge about limitations is known as *conditional knowledge*.

### 2.2.1.3 Conditional knowledge

As procedural knowledge is situational, it always binds with a certain condition or conditions. The procedure can be used only when these conditions are satisfied. For instance, the declarative knowledge the ‘angle sum of a triangle is 180°’ can be used to calculate an unknown angle only when (i) the magnitude of the other two angles is known or (ii) the magnitude of one angle and the fact that the triangle is isosceles with other information is known or (iii) the fact that the triangle is equilateral is known. The same element of declarative knowledge is used differently in three cases, as conditions are different. This ‘why’ and ‘when’ aspects of knowledge are known as conditional knowledge (McInerney & McInerney, 2001). Conditional knowledge is important in the selection of information for sequential processing (Reynolds, 1992).

Conditional knowledge is invoked in making decisions about when and where to use other knowledge. Decision making about when to use knowledge and where to use knowledge, and why it is used can be viewed as reasoning. For this reason, some authors do not mention conditional knowledge as a separate type of knowledge. Conditional knowledge is important in proof-type geometry problem solving. The student has to select appropriate rules, axioms, and strategies. This selection requires conditional knowledge.

### 2.2.1.4 Content knowledge organisation - Problem schema

Figure 2.1 illustrates that problem solving enhances the development of knowledge and reasoning. How knowledge and reasoning that is developed from one problem-solving attempt is reflected in a new situation is explained by the notion of *problem schema*. A problem schema is a generic structure that can be applied to various similar novel situations. For instance, once the rate problem schema is formed, students can apply it to various similar situations such as time and distance, work and time, work and wages, or currency exchange. Hinsley, Hayes, and Simon (1977) observed that students who have experience in algebra word problem schema categorise related problems very quickly.

Hinsley, Hayes, and Simon (1977) generalise their findings as follows.
(1) People can categorize problems into types.

(2) People can categorize problems without completely formulating them for solution. If the category is to be used to cue a schema for formulating a problem, the schema must be retrieved before formulation is complete.

(3) People have a body of information about each problem type which is potentially useful in formulating problems of that type for solutions... directing attention to important problem elements, making relevant judgments, retrieving information concerning relevant equations, etc.

(4) People use category identifications to formulate problems in the course of actually solving them. (Hinsley et al., 1977, p. 92).

Hinsley et al. also say simply reading a glossary term of a text-based problem becomes a clue to retrieve additional information related to the given problem. Different students appear to solve the same problem differently depending on the extent to which they can relate the problem schema. For instance, the following two examples illustrate this difference.

**Problem 1.**

A candle factory has two workers, Jones and Smith. Jones makes candles at the rate of 60 candles per hour and Smith, at the rate of 75 candles per hour. Jones spends one hour more each day making candles than Smith. If Jones makes the same number of candles each day as Smith, how many hours a day does Smith work?

**Problem 2.**

In a ‘Fathers and sons’ tennis match, the Greens are playing doubles against the Browns. Mr. Green is four times as old as his son, and Mr. Brown is five times older than his son. Mr. Green’s son is one year older than Mr. Brown’s son. If Mr. Green and Mr. Brown are the same age, how old is Mr. Brown’s son? (Hinsley et al., 1977, pp. 98-99).

Problem 1 is related to rate-problem schema: the number of units made = rate * time. Most students obtained the equation, \(60(x + 1) = 75x\) using a single step due to activation of rate problem schema. On the other hand, Problem 2 does not come under that category although both problems follow the same procedure. Students found it difficult to relate the son’s age to the father’s age and obtain a similar equation \(4(x+1) = 5x\). Instead, they tended to go through \(Gf = 4Gs, Bf = 5 Bs, Gs = Bs+1, 4Gs = 5 Bs, 4(Bs +1) = 5Bs\). Hinsley et al. (1977) state that ‘The formula \(60(x+1) = 75x\) appears to be put together in a single step … while the corresponding formula in the tennis
problem \(4(Bs +1) = 5Bs\) takes at least 3 steps to assemble’ (Hinsley et al., 1977, p. 99) using ages of the four people and writing down relationships among them. Despite the impossible age for fathers (both are 20 years old), even for competent problem solvers, the features of isomorphic problems can activate problem schemata.

In another study to investigate the relationship between problem-schema activation and the quality of knowledge base, Chinnappan (1998) observed that the quality of geometry knowledge base is a powerful predictor of activating problem schemata. In this study, 18 problem schemas were tested with 30 students in grade 10. These 30 students comprised 15 high achievers (having high quality knowledge base) and 15 low achievers. Out of 270 possible schema activations, the high achievers and low achievers accounted for 105 and 24 schemas respectively. The high achievers’ knowledge base seems to be better connected than that of the low achievers. The higher levels of schema activation were argued to be due to the better connected content knowledge of trigonometry of the high achieving students.

In sum, problem schemas are knowledge structures that develop as a result of generalisation or proceduralisation of experience related to similar-type problem solving. Schemas can also contain content knowledge, conceptual knowledge, strategies, thinking and reasoning related to a class of problems as a single chunk. They can generate feasible solution procedures to new situations. On the basis of problem similarities, problem schema can be transferred to situations even though they are contextually different. For instance, a student might have to learn (statistical) mean to solve a mathematical problem. When a situation arises in another subject, the ‘statistical-mean problem schema’ can come into play.

**Summary**

Content knowledge provides resources for the solution process. Although declarative knowledge plays a greater role in *analysis* and *representation* under certain situations it can contribute to all cognitive processes of problem solving. Procedural knowledge is mainly used to reduce the *planning* process and increase the efficiency of *use of knowledge retrieval* process. Although conditional knowledge does not provide resources it is directly related to the content knowledge. A problem schema is a structure that links various kinds of content knowledge such as procedures, strategies and concepts in a single unit.
With regard to proof-type geometry problem solving, both declarative and conditional knowledge play important roles. Procedural knowledge provides knowledge about how to use declarative knowledge in the problem-solving process. However, not every problem necessarily requires the use of such knowledge. Students find instances where procedural knowledge is not available and they have to find their ‘own’ ways or strategies.

2.2.2 Problem-solving strategies

Once the problem is presented, the student searches the knowledge base for an appropriate strategy. A strategy is basically a problem-solving method involving use of an algorithm or heuristic (Robertson, 2001; Bruning et al., 1999). Algorithms are prescribed methods that lead to solutions. When an appropriate algorithm cannot be found in the knowledge base, the student has to select a feasible alternative strategy based on prior experience. This type of strategy is not only idiosyncratic, but also situational and instance-based leading to multiple solutions. This class of approach is generally classified as heuristics.

2.2.2.1 Algorithms as problem-solving strategies

Algorithms are pre-determined rules (Bruning et al., 1999), or set procedures. A basic characteristic of an algorithm is that it guarantees the solution. On the other hand, not using an algorithm may be expensive and unproductive. Algorithms are mostly associated with domain-specific problems. Therefore, the problem and the algorithm are activated in the same context. The algorithm can be viewed as a set of instructions when the student has only to think about applying variables in the algorithm. Algorithms minimise divergent thinking and lead to a quick outcome with little or no ambiguity.

The algorithmic strategy is very common in mathematical problem solving. Some algorithms are seen as formulae. For instance, \( \int x^2 \, dx \) is an algorithmic problem, as the solution is guaranteed by the use of formula \( \int x^n \, dx = x^{n+1}/(n+1) \). On the other hand \( 2x + 3 = 9 \) is not solved with the help of a formula, but there is a definite procedure. Thus, in algorithmic-type problem solving, the solver’s planning process is limited to retrieving the predetermined procedure.
2.2.2.2 Heuristics as problem-solving strategies

Most problems are not associated with set procedures such as algorithms. The student has to think of and decide on a strategy to apply. This decision impacts on the effectiveness of the problem-solving process and the quality of the solution depends on selection of the procedure. Sometimes the selected procedure does not match the problem and consequently the result does not satisfy the goal. Although it has generally been accepted that mathematical problem solving is algorithmic, the solution of problems like $\int x \sin x \, dx$ have to start with a decision about what method of integration has to be applied (Schoenfeld, 1985). As there are different methods of integration, the student has to select the method. To narrow the range of possible options, the student can reason out a plausible method (Robertson, 2001).

People often use heuristics for searching problem spaces. In problem-solving research, a heuristic is a rule of thumb that will generally get one to the correct solution, but does not guarantee the correct solution. Some heuristics are shown to be more efficient than algorithms. However there is no guarantee that this heuristic will be productive and sometimes use of a particular heuristic may be an incorrect decision on the part of the solver (Tversky & Kahneman, 1974).

The success of the heuristic depends on the problem-solving experience of the student in the domain concerned. Because of this, for novices, heuristics become a risky venture. Heuristics such as working backward could minimise generating inferences that are not relevant in proof-type geometry problem solving. Dennet (1996) argues that the risk is not in the heuristic, but in the search.

Polya (1973a) presents a range of heuristics useful in mathematics problem solving. His four-step approach to problem solving can also be considered as a heuristic itself. Some of the other heuristics he suggests are: draw a diagram; work backward; think of a related problem; re-state the goal; and use sub-goals. Out of these, drawing a diagram is not regarded as a heuristic in proof-type geometry problem solving, as it is essential in the problem-solving procedure. Working backward is useful in organising a solution path, particularly for beginners to proof-type geometry problem solving (Anderson, 1985).

There are heuristics that are specific to proof-type geometry problem solving. For example introducing auxiliary elements such as parallel lines, perpendicular lines, angle
bisectors which have to be used in proof-type geometry problem solving. In deductive proof, other methods of proof such as indirect proof, exhaustion, arguing by contradiction, *reductio ad absurdum* have to be mixed with formal deductive proof. Working forward from data, decomposing and recombining, drawing figures, working backward are also useful heuristic strategies. These are powerful strategies, but there is no an explicit rule to guide the student as to when and where these heuristics can be used.

Mathematical problems become very complex when the student cannot find an appropriate algorithm. This necessarily happens in non-algorithmic problem solving. In such instances, heuristics are very important in solving non-algorithmic or unfamiliar problems, as they are not associated with algorithmic instructions. Schoenfeld (1985) also noted the importance of heuristics.

> Heuristic strategies are rules of thumb for successful problem solving, general suggestions that help an individual to understand a problem better or to make progress toward its solution. Such strategies include exploring analogies, introducing auxiliary elements in a problem or working auxiliary problems, arguing by contradiction, working forward from data, decomposing and recombining, exploiting related to problem, drawing figures, … using *reductio ad absurdum* and indirect proof, … working backward … (Schoenfeld, 1985, p. 23).

Since these heuristics are non-standard, they are not taught specifically. Instead, they can be familiarised to students with experience gained from various situations, worked examples, and other exercises.

Focusing on the instructional role of teaching for problem solving, Schoenfeld (1985) suggests introspection or systematic observation of expert problem solving to promote the selection of effective heuristics. Schoenfeld (1985) recommends that one should provide direct instruction in these strategies, thereby saving students the trouble of having to discover the strategies on their own. The meaning of familiarising strategies is to make use of introspection or systematic observation of expert problem solving in students problem-solving tasks.

### 2.2.2.3 Modelling heuristic strategy

In general, the selection of a strategy in non-algorithmic problem solving is difficult. This situation frequently appears in proof-type geometry problem solving. For instance:
Many students argued in these interviews that they were not able find something similar to a starting point in a proof or to identify correct arguments with respect to the specific context of proof (Reiss and Renkl, p.30, 2002).

Many students face similar situations, as they cannot find suitable heuristics. There are at least two difficulties in teaching heuristics. First, the logical underpinning of selecting an algorithm is always obvious and explainable, but this cannot be considered as a heuristic. Second, a heuristic that is successful in one situation may not be successful in the next. These two reasons suggest that heuristics cannot be taught but need to be modelled.

Modelling involves demonstrating and describing component parts (Bruning et al., 1999). There are three main characteristics of modelling. First, novices see how the heuristic works. That means, modelling should have strong visual effects. Second, modelling has to contain descriptive features. That means semantic expression is important. Third, within the modelling process, the skills should be broken down into lower level skills. The teacher in a conventional class can perform these three effectively to a certain extent. The demonstration of modelling needs to be followed by students practising the heuristics.

Modelling of heuristics can be seen as providing information about solution processes. By modelling heuristics, the problem-solving schema becomes enriched. One strategy for the development of such schema is the use of worked examples.

2.2.3 Use of worked example strategy as an instructional model for proof-type geometry problem solving

Worked examples demonstrate effective problem-solving strategies and heuristics used by experts and their use has drawn the attention of educationists for the last two decades. The success of worked-out examples has been demonstrated in different domains (Anderson, Farrell & Sauers, 1984; Pirolli & Anderson, 1985; Pirolli & Recker, 1994). Recent research has shown that learning from worked examples is of major importance for the initial acquisition of schemas and associated processes. However, only learners who actively process the presented examples benefit from this learning mode.

Worked examples can model expert problem-solving strategies. The purpose of modelling is to provide opportunities to practise expert strategies and problem-solving
processes. Although, experts use larger chunks, these can be decomposed into manageable steps in the practice of the worked-example method. The learner can be provided with vital instructions to learn and become familiar with strategies. According to the principles of modelling, students practise, obtain feedback, improve, and correct where necessary.

Worked examples help students manage the limited cognitive resources available during the solution process. A great deal of research has been undertaken into cognitive load that is involved during the processing of worked examples (Sweller & Cooper, 1985; Ward & Sweller, 1990; Pass & van Merrienbore, 1994; Kalyuga, Chandler, Tuovinen & Sweller, 2001). The theory emphasises the advantage of a schema induction thereby reducing cognitive load during the analysis of worked examples.

Self-explanation is an effective strategy that can be incorporated into worked examples. This involves students’ self-reflection while learning from worked examples. In the domain of physics, Chi, Bassok, Lewis, Reimann & Glaser (1989) observed that performance was increased when learners explain the material to themselves.

In an experiment with Grade 9 students in Australia, the effect of self-explanation training in geometry problem solving was tested by Wong, Lawson & Keeves (2002). The variables concerned were knowledge access and knowledge generation. Students showed improvements in making novel connections with respect to knowledge generation. In another experiment with a computer-based cognitive tutor, Aleven & Koedinger (2002) showed that self-explanation of solution steps promoted the problem transfer process. They claimed improvement in acquisition of visual and verbal declarative knowledge.

Thus worked examples are useful for a number of reasons: to model heuristics for problem familiarisation, to demonstrate expert behaviour, and to reduce cognitive load. Worked examples can be coupled with self-explanation and structured problems to increase their effectiveness. The effect of modelling heuristics can also be explained with problem schemas that are found in a worked example. Worked examples encapsulate everything required for solving problems such as declarative knowledge, procedural knowledge, reasoning, strategies, and generated new information. They may also model general problem-solving processes such as *analysis, representation*, etc.
During the solution process of proof-type geometry problems, information generation takes place on the basis of geometric content knowledge and relationships. Among the three types of geometric knowledge, declarative knowledge provides information such as definitions, axioms, theorems, postulates, conventions, and concepts of geometry. Conditional knowledge also seems to play a role in decision making and appropriate reasoning. Students' solutions are also complemented by non-algorithmic strategies such as heuristics and metacognition. Some heuristics such as using auxiliaries, and using other proof methods belong to content knowledge, whereas other heuristics such as working backward and metacognition are domain-general in nature.

2.2.4 Mathematical reasoning

Mathematical reasoning and content knowledge enrich each other. Reasoning is important during the problem-solving process, particularly with non-algorithmic problems.

Being able to reason is essential to understanding mathematics. … Building on the considerable reasoning skills that children bring to school, teachers can help students learn what mathematical reasoning entails. By the end of secondary school, students should be able to understand and produce mathematical proofs—arguments consisting of logically rigorous deductions of conclusions from hypotheses—and should appreciate the value of such arguments (NCTM, 2000; p. 56).

Development and application of mathematical content knowledge may be associated with several reasoning skills. Reasoning exists in two forms: inductive and deductive.

2.2.4.1 Inductive reasoning

In inductive reasoning the cognitive process involves generalisation of observations into abstract concepts. As Figure 2.4 illustrates the first step in inductive reasoning is identification of patterns. A pattern developed on few observations may not be adequate to produce certainty. Therefore the certainty of this type of generalisation can be challenged. For instance, the student develops the concept of a triangle as the result of generalisation of the spatial configuration of an adequate number of physical objects having triangular shapes or faces. In inductive reasoning, the student experiences a set of triangular shapes with different attributes such as colour, size, material type and so on. In the process the student removes unnecessary differences and filters a concrete
triangular shape, which is still object-based. Adequate examples are required to develop the concept more strongly.

As Figure 2.4 illustrates, development of inductive reasoning is a long process that generates abstractness. Usually, this abstractness may exist as concepts, patterns, relationships, hypotheses, or even theories. Scientific enquiry uses inductive reasoning (Inhelder and Piaget, 1958), but is not designed to produce mathematical certainty (Polya, 1966). Since it starts with specific forms and ends with more general conclusions, its formation is called ‘bottom-up process’.

Inductive reasoning is a knowledge-generating process. The process of utilization of the generated knowledge involves deductive reasoning.

2.2.4.2 Deductive reasoning

The ability to utilise generalised abstractness in real-life situations can be seen in deductive reasoning. It starts with (top) generalised level, and ends with specific value. It is a ‘top-down’ approach (Figure 2.4). Deductive reasoning could be viewed as an information-generation process. Because the generalised rules are valid, deductive reasoning does not generate a doubtful answer. For instance, when the sum of digits of a whole number is a multiple of 3, the resulting number is divisible by three is a generalised rule. This can be used to generate information about divisibility by 3. When the deduction process is applied to abstract objects, the result is also valid.

Inductive reasoning and deductive reasoning have important roles in mathematical problem solving. Further, deductive reasoning has an important role in proof-type geometry problem solving.
2.2.4.3 Inductive reasoning and deductive reasoning in mathematics

Mathematical activities require both inductive reasoning and deductive reasoning. Concept formation and knowledge building take place according to inductive reasoning, and the application of conceptual understanding and theories takes place according to deductive reasoning. For instance, students perform several hands-on activities to develop the abstract mathematical relationship: ‘The angle sum of a triangle is $180^\circ$’; this is an inductive approach. When they use this relationship to calculate an unknown angle, they start with the general rule of angle sum and end with a specific value by using deductive reasoning.

Polya (1973b) classified mathematical reasoning into two types: plausible and demonstrative. Inductive reasoning is plausible, whereas deductive reasoning is demonstrative.

We secure our mathematical knowledge by demonstrative reasoning, but we support our conjectures by plausible reasoning. A mathematical proof is demonstrative reasoning, but the inductive evidence of the physicist, the circumstantial evidence of the lawyer, the documentary evidence of the historian, and the statistical of the economist belong to plausible reasoning (Polya, 1973b, p. v).

Deduction starts with a generalised statement, which is used to interpret particular cases. There are three types of deductive reasoning methods: direct reasoning, indirect reasoning, and transitive reasoning, which are used in mathematical problem solving (see Table 2.3).
Table 2.3 Illustrated basic features of three types of deductive reasoning.

<table>
<thead>
<tr>
<th>Method</th>
<th>Procedure</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Direct</td>
<td>Generalised form</td>
<td>( p \rightarrow q )</td>
</tr>
<tr>
<td>Known</td>
<td></td>
<td>( p )</td>
</tr>
<tr>
<td>Deduction</td>
<td></td>
<td>( q )</td>
</tr>
<tr>
<td>Indirect</td>
<td>Generalised form</td>
<td>( p \rightarrow q )</td>
</tr>
<tr>
<td>Known</td>
<td></td>
<td>(~ p )</td>
</tr>
<tr>
<td>Deduction</td>
<td></td>
<td>(~ q )</td>
</tr>
<tr>
<td>Transitive</td>
<td>Generalised form</td>
<td>( p \rightarrow q \text{ and } q \rightarrow r )</td>
</tr>
<tr>
<td></td>
<td>Deduction</td>
<td>( p \rightarrow r )</td>
</tr>
</tbody>
</table>

Direct reasoning and indirect reasoning help students shift from the generalisation to the specific, whereas transition reasoning involves activation of general rules in order to generate new general rules. Direct reasoning is important in solving find-type mathematical problems, whereas transitive reasoning plays a key role in solving proof-type mathematical problems.

Deductive reasoning, which is a domain-independent process, influences the success of geometry proof-type problem solving.

In summary, both inductive and deductive reasoning are domain-general. Inductive reasoning plays a key role in knowledge generation whereas deductive reasoning is important in applying knowledge of patterns in situations for the purpose of generating new information. Proof-type geometry problem-solving process is mainly based on formal deductive reasoning. Among three forms of deductive reasoning methods, transition reasoning appears to be dominant in proof-type geometry problem solving.

While the role of both types of reasoning appears to be critical in the solution process, the influence of these reasoning skills in solving proof-type geometry problems has yet to be investigated. This issue is taken up in the present study.

Although knowledge and reasoning can be activated, problem-solving processes may not follow a productive path unless appropriate control processes regulate these cognitive functions, an important component of metacognitive processes.
2.2.5 Metacognitive control of problem-solving process

Problem solving requires conscious decision-making on the part of the problem solver. Researchers (Brown, Hedberg & Harper, 1994; Derry, 1992; Flavell, 1979; Schoenfeld, 1985; 1992) agree that there should be appropriate processes to regulate and control cognitive actions during the solution process. The activation of appropriate resources is required for a productive solution attempt (Chinnappan, 2000). This activation needs to be mediated by control processes broadly referred to as metacognition (Brown et al., 1994).

Metacognition deals with the way that individuals marshal the use of information in problem solving. Metacognition can be seen in two forms: either as the activation power of cognitive activities; or as a monitoring device of solving actions. This conscious cognitive control process activates relevant links to access and retrieve knowledge that is subsequently used in the search for the solution. Selection of information, maintaining the right path, and right process take place under metacognition. Metacognition can activate domain-specific knowledge components as well as domain-general processes that were identified in section 2.1.2.

Metacognition refers to one’s awareness about one’s own cognitive process (Flavell, 1979). It includes metacognitive knowledge and metacognitive regulation. Schoenfeld (1987) describes the nature of metacognition as follows.

1. Your knowledge about your own reasoning process. How accurate are you in describing your own thinking?
2. Control or self-regulation. How well do you keep track of what you are doing when (for example) you are solving problems, and how well (if at all) do you use the input from those observations to guide your problem-solving actions? (Schoenfeld, 1987, p. 190).

This indicates that metacognition is involved in cognitive functions and their management. Derry and Hawkes, (1993) support the above view on self-regulation. Metacognitive experiences influence cognitive processes that demand conscious thinking about processes involved in solving problems (Brown et al., 1994).

The influence of metacognition has been studied in expert–novice comparison studies. In an analogical problem-solving session, Hatano and Inagaki (1986) observed that adaptive experts (who are able to approach new situations flexibly) monitor self-understanding, identify additional information required and decide the consistency of
new information. Research has shown that metacognitive skills are teachable. As metacognition often occurs as internal conversation, metacognitive skills can be trained using two strategies: reflection and self-explanation.

Reflection is also a metacognitive skill as it facilitates the student becoming aware of their own cognitive process (Puntambekar & du Boulay, 1999). It also fosters problem-solving processes, especially in selecting and using heuristics. The ability to reflect upon one’s own performance and continue the process of learning develops students’ reasoning power to acquire a valid, rich knowledge base, which is useful to cope with a range of situations (Hartmann, 2001; Koschmann, T., Kelson, A. C., Feltovich, P. J. & Barrows, H. S. 1996; Schraw, 2001). In other words, reflection fosters meaningful learning.

Reflection provides not only a better understanding of what the student knows, but also a way of improving metacognitive strategies, because the student can examine how a specific learning task was performed. Reflection enhances the learning benefit of an exercise because it enables the person to make a better decision through reviewing past experiences (Goodman et al., 1998). In complex domains the student needs to plan and organise their own solution paths in the problem-solving process. Hence, it is important that students can study and explore their own problem-solving efforts such as: analysing own performance, contrasting their actions to those of others, abstracting the actions they used in similar situations, and comparing their actions with those of novices and experts (Goodman, Soller & Gaimari, 1998).

As mentioned before, metacognition is instrumental in controlling the action of four general processes: analysis, representation, planning, and use of knowledge retrieval. Proof-type geometry problem-solving process begins with analysing the problem in text. The student reads the problem and understands the situation. It requires conscious effort to figure out the geometric situation. The student interprets the problem in the light of earlier experiences of the same type of problems and existing resources. Metacognition is the process that provides the sense of what is known and what is unknown in the problem.

Metacognitive knowledge also guides goal-oriented reasoning and procedure towards the solution. (Davidson & Sternberg, 2001; Schoenfeld, 1987). After constructing a mental model of a problem the student has to search for strategies and plan the solution.
process. As geometry problem solving at this level is not supported by algorithms, the student has to decide on an heuristic to use in a particular context. Metacognitive knowledge thus has an important role in facilitating planning as it encourages students to invoke inference. It also involves, through conscious choices, decision making, which is important in developing and solving sub-problems (Davidson & Sternberg, 2001; Schoenfeld, 1987).

Having developed the sub-problems, the student applies *use of knowledge retrieval* to the planned solution path. This requires the use of a range of metacognitive skills. Metacognitive knowledge and belief systems allow students to select and use heuristics for the problem-solving processes and monitor the consequences of their actions (Flavell, 1979; Schoenfeld, 1985; 1987; Brown, 1987). Metacognitive knowledge also facilitates searching the domain for specific information in order to retrieve appropriate procedural and declarative knowledge. Metacognitive actions are required to make sure that the reached solution is correct. This process updates metacognitive knowledge. Through control and use of declarative knowledge the student becomes aware of how the given information relates to the solution process.

Students need to learn strategies to apply their existing knowledge in problem-solving situations and control ongoing actions. This promotes the development of the solution as well as metacognitive skills. While constructing a solution to a problem, expert mathematicians spend most of their time analyzing, exploring and monitoring their own cognitive processes (Schoenfeld, 1987). Experts question and update their own knowledge to make the knowledge more sophisticated. Students, as novices, should learn to make effective use of their existing procedural knowledge and proceduralise declarative knowledge in new problem-solving situations.

In summary, metacognition can be seen as the control process of four key processes that are involved in the geometry proof-type problem solving: *analysis, representation, planning,* and *use of knowledge retrieval*. It has a role in solving proof-type geometry problems, as they are non-algorithmic. Metacognitive action helps students select, access, retrieve, and use knowledge during problem-solving process. The above analysis suggests that in geometry both content knowledge and general processes interact during the solution process, and the investigation of solution of proof-type geometry problems needs to consider this important interaction.
2.2.6 Content knowledge versus general processes in mathematical problem solving

The researchers who argue that content knowledge is sufficient for mathematical problem solving use the findings on expert versus novice performance. It is evident that expert mathematicians possess exceptional skills and wider knowledge base. The research on expert problem solvers reveals that experts have better memory skills, content knowledge and categorising abilities. Experts also prefer to work forward and do not use means-ends analysis (Owen & Sweller, 1989; Sweller, 1999). According to this view, the success of experts over novices could be attributed to their use of heuristic strategies.

In the domain of geometry, Chinnappan (1992) designed and conducted a study to investigate the influence of general strategies on problem-solving success. A student group was trained in general strategies to perform geometry problem-solving tasks. Results of this study showed that students who were trained in the use of general problem-solving strategies performed significantly better than those who did not receive the training.

While some researchers argue that mathematical content knowledge contains all the skills that are required to solve all mathematical problems, other research provides counter evidence. In Germany, Riess, Klieme and Heinze (2001) carried out a study with 600 high-achieving students about the requirements of proof-type problem solving. Students showed that they were comfortable in judging the correctness of proof, but not in constructing proof. A follow-up interview suggested that most of these high achievers (those who have higher content knowledge) could not find a starting point. This implies that adequate content knowledge is not sufficient for finding appropriate strategies. In this study, researchers found that metacognitive skills and methodological knowledge are also important in addition to content knowledge.

In a comparison study with four doctoral students and four undergraduate students, Weber (2001) observed that some undergraduates could not convert syntactic knowledge about the proof procedure into establishing the proof. Further, they could not decide what facts have to be applied for a particular proof. The researcher argues that the deficiency is due to strategic knowledge as opposed to content knowledge.
In a related research, DeFranco and Hilton (1999) concluded that content knowledge is not sufficient requirement for problem solving. They assert that experience is an important factor in the success of mathematical problem solving. They carried out the study with two groups of eight professional mathematicians, each with the same qualifications, but with different experience. It was observed that the experienced group outperformed the inexperienced by a highly significant margin. The members in both groups were equal in content knowledge standards, but were different in heuristics, metacognition and belief systems as the result of experience.

Experience in solving a particular array of problems can make problems familiar. It enhances the possibilities of transfer effect (Lawson, 1991; Lawson & Chinnappan, 1994). The degree to which the extent of the transfer effect exists is a function of the degree to which the cognitive elements are shared rather than the degree to which the content elements are shared. For instance, rules of geometry are the same in construction-type problems, find-type problems, and proof-type problems. But during the transfer process students perform differently.

In a survey of 2459 high-achieving students (all the students were from top mathematics stream) in 94 classes in 90 schools to examine the impact of the national curriculum on high achieving students in the UK, Healy and Hoyles (1998) sought to find out how students attempt proof-type problem solving. They found that these students showed a consistent pattern of poor performance in constructing proof. They also found that more students were able to select a correct proof than to write it. These students had the required content knowledge in proof argumentation but no skills to perform it.

There is a lack of agreement among the research community about the relative role of content knowledge during mathematical problem solving. On the one hand, expert-novice research suggests that content knowledge plays a dominant role in the solution process. On the other hand, a second stream of evidence suggests that content knowledge alone is not sufficient for success in mathematical problem solving. According to this research, we need to consider the influence of domain-general processes in problem solving. The lack of agreement about the relative role of content knowledge and general processes reflects a grey area in research related to requirements for success in problem solving, an issue that is addressed in this study. More specifically, this issue is addressed in the context of solving proof-type geometry problems.
2.3 Mathematical proof

As Polya (1966) states, among different types of proving processes, only mathematical proof is demonstrative. Mathematical proof is a linearly-ordered sequence of sentences where each sentence comes from one of the following three categories:

- Sentences that are assumed to be true.
- Sentences that are already known to be true.
- Sentences that are derived from a previous line.

The last sentence of a proof is the goal that has to be proved (Rodgers, 2000). A mathematical proof involves a reasoning pattern that consists of a sequence of deductive arguments. Other forms of arguments like empirical-inductive generalization (scientific approach), reference to a higher authority (citation) or perceptual evidence (judiciary) are not considered as mathematically acceptable proof.

The transitive nature of deductive reasoning is evident in the development of mathematical proof. According to this, to prove the statement \( p \Rightarrow q \), a logical deductive chain is build up as a deductive formal proof. That is, if \( p \Rightarrow q \) and \( q \Rightarrow r \) it follows that \( p \Rightarrow r \) and \( p \) is assumed to be true. It is used to deduce an implied statement \( p \Rightarrow p_i \). From \( p_i \), \( p_i \Rightarrow p_{i+1} \) is deduced and so on until \( p_k \Rightarrow q \) is obtained. Using this transitive reasoning, the validity of the theorem: \( p \Rightarrow r \) is deduced.

Euclidean geometry also follows the above principles of formal deductive reasoning. It is a complete axiomatic deductive system. This system is based on 23 definitions and 10 axioms (5 common notions and 5 postulates exclusively geometric) (Heath, 1956). These definitions and axioms constitute the content knowledge related to proof-type geometry problem solving.

2.3.1 Learning to develop formal mathematical proof

Some researchers are surprised by the effort taken by students to prove empirically such obvious geometrical relationships (it is said that the great mathematician Sir Isaac Newton also was one of them). According to De Villiers (2000), Students at vHL1 or 2 may not doubt the validity of their empirical observations, formal proof is meaningless to them—they see it as justifying the obvious.
Despite persistent learning difficulties educationists value teaching proof-type problem solving because of its importance. Educational value of proof-type problem solving has been acknowledged in the American Curriculum standard (NCTM, 2000) and it has been accepted as a formal way of expressing reasoning and justification. As a principle, educationists believe ‘reasoning and proof should be a consistent part of a student’s mathematical experience’ (NCTM, 2000, p. 56). This means that reasoning and proof development are useful mathematical practices for future generations of students.

There is a closer connection between reasoning and proof. Proof is basically rooted in reasoning. As mathematical reasoning is objective, mathematical proof is readily verified.

The Royal Society (UK) (2001) describes three importance forms of proof. First, proof is a logical argument, which demonstrates the truth of some claim. Second, proof is a conclusion, accepted as a true analysis of scientific observation. Third, proof is an acceptance of a statement as plausible. Mathematical proof contains no doubts, and one counter example is quite sufficient to reject the claim as not true. That implies truth can be demonstrated but proof is about how that demonstration has been justified.

### 2.3.2 Pedagogical values of proof-type problem solving

The pedagogical values stated in the curriculum provide some important indicators to understand the role of domain-general processes related to proof-type geometry problem solving. According to curricular materials, students are expected to develop skills related to proof-type problem solving from early classes (NCTM, 2000).

> Reasoning and proof cannot simply be taught in a single unit on logic, for example, or by "doing proofs" in geometry. … Reasoning and proof should be a consistent part of students' mathematical experience in pre-kindergarten through grade 12 (NCTM, 2000, p. 56).

This indicates that mathematical proof is regarded as important even though such proof has no direct connection with everyday activities.

The NCTM (2000) suggests a procedure for the development of proof in the American Mathematics curriculum in the following phases;

- Recognize reasoning and proof as fundamental aspects of mathematics.
- Make and investigate mathematical conjectures.
- Develop and evaluate mathematical arguments and proofs.
• Select and use various types of reasoning and methods of proof.

Accordingly, mathematical proof develops along with general processes such as reasoning, conjecturing and deducing.

Boero (1999) presents a model for developing elements of the expert proving process. The model consists of six phases, but it is not meant to be a linear model. Conjecturing, exploring, testing results, and writing a formal proof are activities that are most likely to iterate during the process. The following is Boero's model.

(1) The production of a conjecture. This includes the exploration of the problem situation as well as the identification of arguments to support the evidence.

(2) The formulation of the statement according to shared textual conventions is the second phase. This phase aims at providing a precisely formulated conjecture, which will then be the basis for all further activities. It may be revised in the process but this revision would have consequences for most activities performed by the mathematician.

(3) The exploration of the conjecture and the identification of appropriate arguments for its validation. Only the last three phases are subject to public communication. They include:

(4) The selection and combination of coherent arguments in a deductive chain.

(5) The organization of these arguments according to mathematical standards.

(6) The proposal of a formal proof.

This expert model of proof illustrates that proving is a complex cognitive activity. It is characterized by logical argumentation, inferencing and exchange between explorative, inductive, and deductive reasoning processes. In addition, steps (1) to (3) in Boero’s model draw attention to the role of planning during the solution process, a point that was highlighted by Schoenfeld (1985).

This section has discussed the nature of mathematical proof and its development during solution attempts. The development of mathematical proofs involves logical deductive reasoning in order to verify statements, which constitute the problem goal. General problem-solving processes such as analysis, representation, planning and use of knowledge retrieval could be invoked during proof development. The reasoning processes appear to be domain-general in character. The construction of proofs also
enriches students' learning experiences in ways that permit understandings to be transferred to other areas of school mathematics. The above review also shows that there may be overlaps between content-knowledge and domain general-knowledge components when students are required to solve proof-type geometry problems.

2.4 Proof-type geometry problem solving

As was discussed in section 2.1.2, proof-type geometry problem solving involves a range of cognitive processes. Learning proof-type geometry problem solving in schools has a long history. However, less is known about the instructional requirements related to solving proof-type geometry problems.

2.4.1 Role of geometry in proof-type mathematical problem solving

It has been established that mathematical proof construction is a valuable learning activity. Clements and Battista (1992, p. 88) once wrote: 'No one would deny that establishing the validity of ideas is critical to mathematics, both for professional mathematicians and for students'. However, learning proof-type mathematical problem solving is very difficult (Koedinger & Anderson, 1993; Reise et al., 2001, Senk, 1989). Euclidean deductive geometry has long been used to teach mathematical proof because educationists in the field of mathematics believe that the Euclidean deductive system is effective in minimizing learning difficulties. This section examines the reasons for this approach.

The Royal Society of UK (2001) provides the following ten reasons for why geometry is used as an approach to teach mathematical proof.

1. Geometry enables pupils to engage in proof as it contains familiar objects to students such as angles, parallel lines, and triangles.

2. The existence of the situation and related mathematical relationships are meaningful to students as they are in visual form although they are abstract.

3. The statements are readily accessible for verification. For example, students can ‘try and feel’ the validity of a statement such as ‘the angles of a triangle add up to 180°’. This is more appropriate than dealing with objects which are represented in abstract symbols such as $\sum r^3 = \{n (n+1)/2\}^2$. To verify the latter requires a complex process.
4. The logical methods involved in geometry at school level tend to be less subtle: they involve fewer quantifiers.

5. Proof in geometry is seen as an early start, because students develop their logical skills as soon as they emerge.

6. Proof in geometry is synthetic deduction with various options which involves contemplating several statements in a non-linear access.

7. Route-finding in geometry is easy, as the all situations of the solution process are in a visual scenario. This allows students to check the progress, and to decide the next step and how to get to it.

8. Geometry provides the taste of higher mathematics without serious approach of axiomatic approaches that other mathematics do.

9. Proof with geometry develops practices in students not having to take things on trust, as all geometric relationships are generally proved.

10. It has surprising effect, as there is a situation where a small number of plausible assured points leads to a large number of surprising and appealing results.

Within geometry, the Euclidean system provides rules and guidelines for reasoning that underlie proof development. Solving problems using this system is regarded as a valuable learning opportunity. Polya wrote:

Geometry as presented in Euclid’s Elements, is not a mere collection of facts, but a logical system. … it is the first and the greatest example of such a system, which other sciences have tried, and are still trying to imitate (Polya, 1973a, p. 217).

The Euclidean logical deductive system thus provides a strong base for students to learn and develop formal mathematical deductive proof.

Geometry is thus suitable for introducing the complex process of mathematical proof. It is also useful for activating and developing reasoning skills to recognise intermediate steps in proof related problems. This reasoning process can be extended to planning strategies, implementing them and achieving a set of sub-goals that lead to the final goal. It is relatively easy to solve problems in which the solution is not immediately obvious but reachable by memorized algorithms. In contrast, geometry problem solving enables students to solve problems by finding their own strategies. This is a key requirement in the solution of proof-type mathematical problems.
Because geometry offers these advantages, proof-type problem solving is regarded as providing a deep and meaningful context. Polya wrote:

If a student failed to get acquainted with [a] geometric fact he did not miss so much; he may have little use for such facts in later life. But if he failed to get acquainted with geometry proof, he missed the best and simplest examples of true evidence and he missed the best opportunity to acquire the idea of strict reasoning (Polya, 1973a, pp. 216-7).

Polya thus highlights the development of thinking strategies rather than mathematical subject-knowledge development in learning proof-type geometry problem solving. Proof-type geometry problem solving develops advanced thinking skills in addition to providing a powerful base for developing proof-type mathematical problem solving skills.

2.4.2 Requirements of proof-type geometry problem-solving process

As has been discussed, mathematical problem solving requires the activation of knowledge that is both domain-specific and domain-general. Proof-type geometry problems require knowledge about the conventions and methods of formal mathematical proof. Although it was introduced by Euclid (Polya, 1973a; Hersh, 1993), principles of axiomatic reasoning are now shared by all axiomatic deductive systems. In that sense, objects and principles of axiomatic systems are not in the domain of proof-type geometry problems.

Components of domain-general knowledge related to proof-type geometry problem solving include heuristic strategies, formal deductive reasoning and inference-based reasoning other than metacognition. The content knowledge for proof type geometry problems encompasses concepts of Euclidean geometry, geometric reasoning and diagrammatic representation.

2.4.3 Development of geometric reasoning - van Hiele Theory

The van Hiele Theory describes how geometric reasoning process develops in children. This theory has been evaluated during the past thirty years and is being used as the framework for geometry curriculum development in many countries. For instance, it was evaluated and used in the United States (NCTM, 2000), United Kingdom (TRS, 2001), and South Africa (DE, 2002).
According to van Hiele theory, the development of geometric reasoning takes place in a hierarchical manner in that a learner cannot operate with understanding on one level without having been through and attained concepts from the previous levels. This has been confirmed in research by Fuys, Geddes & Tischler (1988) and Shaughnessy & Burger (1985). There are five discrete reasoning levels in this system, from holistic thinking to analytical thinking to rigorous mathematical deduction. The levels: visual, analysis, non-formal deductive, deductive and rigor will be referred to as van Hiele Levels 0 – 4 respectively. This development is strictly sequential, hierarchical, and independent of biological maturation. Students at any age level should be able to make progress through levels 0 to 4. The development can be viewed as two-dimensional: horizontal development and vertical development (Figure 2.5).

1. Vertical development – this development shifts the student’s reasoning level from one level to the next.
2. Horizontal development – this development takes the student along the same level through instructions

Completion of horizontal development within the level is the prerequisite for moving to the next van Hiele Level.

The theory can be summarised as shown in Figure 2.5.

Figure 2.5 – Student geometric reasoning development

2.4.3.1. Vertical development of geometric reasoning

The vertical development of geometric reasoning referred to making progress from one van Hiele level to the next level and van Hiele Theory suggests that this development is the most difficult one. The reason for the difficulty is that two levels belong to two
different levels of reasoning in terms of concept formation, the attributes of concepts and language used. As a result, instructional support could face two difficulties.

1. When the student is not at the same level as the instructions, the student either understands differently or cannot understand.

2. The classroom includes multiple levels and instructional process and learning process fail.

The following section describes the nature of each van Hiele Level and competencies that can be expected from a student at each level.

**Level 0 -Visual (vHL 0)**

At this level, students recognise certain shapes as they see. They are able to visualise the holistic images without paying attention to their component parts. At this level, some relevant attributes of a shape such as straightness of sides might be ignored and irrelevant attributes, such as the orientation of figures on the page might be stressed.

The reasoning capacity at this level is important in recognising instructional needs. At this level, the student develops geometric reasoning related to shapes according to visual appearance (Clements & Battista, 1992; Fuys, Geddes & Tischler, 1988; Lawrie, 1996; Mayberry, 1983). This emphasises the importance of visual experience. As the starting point of geometric reasoning development, van Hiele Level 0 is important. According to van Hiele, it starts with non-verbal reasoning.

Nonverbal thinking is of special importance; all rational thinking has its roots in nonverbal thinking, and many decisions are made with only that kind of thought (van Hiele, 1999).

van Hiele (1999) recommends playful activities such as mosaic puzzles to develop decision making and nonverbal reasoning. Students relate names to shapes through physical objects such as *door* and *box*. In this phase students seem to generalise the images of physical objects into geometric shapes, and remember their names accordingly.

The content knowledge related to vHL 1 develops with general processes such as inductive reasoning, visual reasoning, and spatial reasoning.
**Level 1 - Analysis (vHL 1)**

At this level, the child focuses analytically on the component parts of a figure, such as its sides and specific angles, especially right angles. Component parts and their attributes are used to describe and characterize figures. Relevant attributes are understood and are differentiated from irrelevant attributes. For example a child who is reasoning analytically would say that a square has four "equal" sides and four "square" corners. The child also knows that turning a square on the page does not affect its squareness. A child reasoning analytically might not believe that a figure can belong to several general classes and have several names. For example, a square is also a rectangle since a rectangle has 4 sides and 4 square corners, but a child reasoning analytically may object, reasoning that square and rectangle are entirely separate types even though they share many attributes.

**Level 2 – Formal Deduction (vHL 2)**

There are two general types of reasoning at this level. First, a child understands abstract relationships among figures. For example, a rhombus is a four-sided figure with equal sides and a rectangle is a four-sided figure with square corners. A child who is reasoning at level 2 realizes that a square is both a rhombus and a rectangle since a square has 4 equal sides and 4 square corners. Second, at level 2 a child can use deduction to justify observations made at level 1. This includes identifying geometric relationships, applying them to find values of angles and sides, and verifying those geometric relationships using specific values (NCTM, 2000). These activities are associated with two types of mathematical reasoning skills. First, they have gained experience related to inductive reasoning. They use empirical methods to see patterns and develop conjectures through experimental methods such as measuring angles and paper folding or cutting activities.

**Level 3 – Formal Deduction (vHL 3)**

Students bring non-formal deductive abilities to vHL3. Students at this level have some geometric concepts about basic plane figures, geometric relationships, and use them to apply into specific situations. Reasoning at this level includes the study of geometry as a formal mathematical system. A child who reasons at level 3 understands the notions of mathematical postulates and theorems and can write formal proofs of theorems. At this level the significance of deduction as a way of establishing geometric theory within
an axiomatic system is understood. The student at this level can understand interrelationships and roles of undefined terms, axioms, definitions, theorems and formal proofs.

**Level 4 Rigor (vHL 4)**

The understanding of geometry at level 4 is highly abstract and does not require concrete or pictorial models. At this level the student can deal with absolute abstract concepts, postulates or axioms rigorously.

In vertical development the student changes geometric reasoning skills. Although the basis for these changes is inductive reasoning, the result is geometric-specific. This reflects development of geometric content knowledge that influences the success of proof-type geometry problem solving.

The geometry curriculum being implemented at SSL corresponds to vHL 3. Senk (1989) analysed the reasoning level of 241 students in American School. According to this study, 27% of the students were at vHL 0, 51% were at vHL 1, 15% were at vHL 2 and only 7% were at vHL 3. This pattern also reflects the trend in SSL classes where only 7% of students are ready to learn proof-type geometry problem solving. Students at SSL are mostly a heterogeneous group. The van Hiele theory suggests that students who have not attained van Hiele Level 3 should make necessary shifts through the van Hiele levels. Instructional strategies have to be found to cater for the heterogeneity of SSL students here.

**2.4.3.2 Horizontal development of geometric reasoning**

Horizontal development refers to maturation of reasoning within a level. This maturation is required to move from one level to the next. The Van Hiele Theory suggests appropriate instructional phases to facilitate the required horizontal development. The following details characteristics of each phase.

**Phase 1: Inquiry/Information**

At this initial stage the student strengthens reasoning skills at the same level. There should be a continuous dialogue that emphasises level-specific vocabulary between teacher and the student on observations as the student engages with hands-on activities. The student relates the visual geometric shape or its appearance with the name.
Phase 2: Directed Orientation

The aim of this phase of instructions is to widen the student’s experience so that the student is allowed to generalise the shape. The students themselves explore and investigate materials provided under the guidance of teacher. These activities should gradually be revealed so that students get more acquainted with the material.

Phase 3: Explication

Individual instructions are provided to enhance social negotiation and individual experience. Students express and exchange views about the materials to extend previous experiences and to promote self-reliance. Students are encouraged to use precise and appropriate vocabulary. The teacher's role is more remedial.

Phase 4: Free Orientation

More individual opportunities are assigned to promote cognitive organization. Students engage with complex tasks and identify explicit relations among the objects. Tasks can be completed in more than one way with different approaches. They are inventive, multi-step, goal-free, novel and problem oriented tasks.

Phase 5: Integration

Improvement of declarative knowledge as well as proceduralisation of declarative knowledge seems to be the main target of this phase. Students are encouraged to summarise the experience they acquired during the learning event. This strengthens relationships that have been built. At this phase students have completed the horizontal development and are ready to leave the present cognitive paradigm in order to start the same learning cycle at the next van Hiele level.

During horizontal development students strengthen their reasoning skills belonging to the same van Hiele Level through: personal experience → social negotiation → higher-order thinking skills. The instructional flow suggests the relevance of domain-general processes in developing content knowledge.

The development of geometric reasoning at lower van Hiele levels takes place on the basis of relevant inductive and non-formal deductive reasoning. These are domain-general processes and that allow students to make progress from one van Hiele level to the next. Although these domain general processes influence the early developmental
levels, development of geometric reasoning is dependent on the activation of appropriate content knowledge.

### 2.4.4 Role of geometric diagram in proof-type geometry problem-solving process

As a part of geometry content knowledge, diagram plays a vital role in the development of proof-type geometry problem solving. Diagram construction and use has a powerful effect on the solving process (Figure 2.6).

![Figure 2.6 – Components of information processing system related to geometry problem solving](image)

Diagrams play different roles during the problem-solving process. The diagram, given information about the diagram, and the integration of information generated within the diagram constitute the task environment. The diagram also acts as an external memory (Newell & Simon, 1972) that keeps information and helps retrieve information. Newell & Simon (1972) also state that the geometric diagram is a powerful auxiliary problem space. During geometry problem solving, there is a regular information flow that takes place between the diagram and reasoning. Thus, diagram in geometry problem solving plays a vital role. The diagram influences problem-solving success in geometry more than in any other subject.

The geometric diagram serves two functions: *figural expression*, and *conceptual existence* (Fischbein, 1993). Figural expression is useful to represent a geometric situation as all geometric objects and relationships can be represented on a single figure. Therefore, decisions and conjectures can be easily made. On the other hand, geometric diagrams hold concepts, geometric objects and relationships. According to this conceptual property geometric figures should not exist in real life (Charalambos, 1997). In addition these concepts, objects and relationships are perfectly semantic. However,
the figural expression and the conceptual existence match to a great extent, and there is a blend between the reality and abstractness that emerges in the geometric diagram. More precisely, the geometric diagram is not a symbolic or analogical representation of a geometric concept, but visualisation of the concept in reality.

A geometric diagram has a high degree of generalisability (Charalambos, 1997). For instance, a quadrilateral figure represents all quadrilaterals. Generally, spatial reasoning refers to five features related to a situation: shape, size, location, orientation, and movement (Fujii, 1969). The quadrilateral can be drawn to any size, therefore it is independent of size. Similarly, it is also independent of location and orientation. Movement does not exist in a geometrical diagram. Because of this independence geometric diagrams can be drawn anywhere, at any location in any size.

The geometric diagram can also be seen as a schema (Fischbein, 1993; Koedinger & Anderson, 1993). It contains various types of related information, and gaining information from it depends on the level of expertise. Figures may contain geometric objects such as a square which may be just a figure to some students but dozens of geometric relationships and concepts to others. As a schema, a geometric diagram can be broken down into information entities, and combined to form different geometric objects which also contain information entities. As a result, different geometric diagrams and objects can be obtained from a single geometric diagram. Thus, student schema development has to be an instructional design consideration.

Because of these properties, Fischbein (1993) names abstract concepts associated with visual geometric diagrams as figural concepts. Fischbein (1993) asserts that these figural concepts demonstrate essential properties of the domain of concepts: ideality, abstractness, absolute perfection, and universality. Even though other concepts demonstrate these essential properties, their visual presentations are not similar to visual forms of geometric concepts. For instance, the numeral (visual form) does not bind the reality with the concept, only symbolises it.

Students develop misconceptions such as prototype configurations because of diagrammatic complexity. These are mainly due to instructional defects. For instance, teachers draw a right triangle’s two sides in vertical - horizontal directions and the hypotenuse to slant. This results in an inability to recognise other right triangles. Hershkowitz, Bruckheimer and Vinner (1987) report that 70% of grade 5 students, 77%
of grade 8 students and 23% of teachers could not identify a right triangle with hypotenuse horizontal as a right triangle. Two common misconceptions about rectangles are: breadth is larger than height; and they are lying in a horizontal orientation. Because of these misconceptions, students add irrelevant features into the diagram. Hershkowitz et al. (1987) reports that teachers could mislead students in their understanding of the following concepts: the orientation of isosceles triangles; angle independence of the lengths of its arms; exterior diagonal of concave quadrilateral; and exterior altitude of obtuse triangle.

Proof-type geometry problem-solving process takes place within an environment that includes a diagram. It represents concepts, geometric relationships, problem information, and generated new information. In addition, at most times, the goal is also represented in the geometric diagram. As the solution progresses, the number of information entities increases. Eventually, the diagram represents a set of related information. The diagram development by itself does not represent the sequence of its development, or the underpinning reasoning of it. Therefore, diagram has no capability to represent the proof or convincing steps of the proof without semantic representation. In addition, it cannot distinguish between given and generated information. Because of this, proof has to be presented in two parts: the logical chain presented in words, and the convincing evidence presented in the diagram. The interplay between information during diagram construction and inferencing demonstrate that diagram influences and is influenced by the use of content knowledge and general processes during the solution attempt.

The analysis of proof-type geometry problem solving suggests that students need to adopt a non-algorithmic approach when addressing this class of problems. However, the above review of reasoning associated with van Hiele's levels and students' workings with geometric diagrams suggests that the outcome of a productive solution process is driven by a rich body of content knowledge that is specific to geometry. The elucidation of the nature of this content knowledge and how it is deployed in the search for the solution of proof-type geometry problems is an important task for researchers and is a major concern of the present study.
2. 5 Summary

Although there exists a general consensus among some researchers that content knowledge is adequate for mathematical problem solving, several researchers have recently reported that content knowledge is necessary but not sufficient for the solution of proof-type problems. The literature suggests that other factors may also influence proof-type geometry problem-solving process.

The literature review also reveals that the nature of proof-type geometry problems and their solution is different from other types of problems that can be solved by adopting an algorithmic approach. The demands of non-algorithmic approach to proof-type geometry problem solving emphasise the role of four key processes: analysis, representation, planning and use of knowledge retrieval.

A major difference between proof-type geometry problem solving and algorithmic mathematical problem solving is the dominance of declarative knowledge elements: conventional and unconventional representations; concepts; and rules like axioms, theorems, and relationships. A lack of procedural knowledge in the content knowledge related to the proof-type geometry problem-solving process may be a source of difficulty in finding an appropriate problem solving strategy. This situation probably demands logical inference to select elements from a pool of declarative knowledge. The literature further suggests that the lack of procedural knowledge and absence of algorithms requires students to access appropriate domain-specific heuristics, domain-general heuristics as well as metacognitive processes during the solution attempt.

How students go about processing information in proof-type geometry problem solving is an emerging issue. Although declarative knowledge may exist in semantic form, information related to the problem situation is processed in an external representation, i.e. in the geometric diagram. Students find difficulties related to this visual form of the problem. The interplay between conceptual application and visual expression is a key feature of problem-solving attempts that involve proof development.

Despite the emphasis on the difference between proof-type geometry problem solving and other mathematical problem-solving processes, and the evidence of role of content knowledge, domain-general processes and mathematical reasoning, the relative importance within the context of proof-type geometry problem-solving process has not been adequately acknowledged or researched. Instead instructional strategies for proof-
type geometry problem-solving process continue to be based on research findings about mathematical problems that are predominantly algorithmic in nature.

While the review helped identify some key requirements for geometry proof-type problem solving, it did not provide evidence about the relative roles of these knowledge components and reasoning skills, giving rise to the first of two key research questions in this study:

(Q1). What are the predictive indicators of successful proof-type geometry problem solving?

This question will be addressed in Chapters 3 and 4.

The application of these findings for instructional support is addressed in the second research question:

(Q2). Based on needs with an emphasis on formative evaluation, what is one design solution to support students solving proof-type problems in geometry?

This is addressed in Chapters 5 and 6.
Chapter 3: Methodology

3.0 Introduction

The literature review suggested that proof-type geometry problem solving is complex and non-algorithmic. It could be seen that the solution of this type of problem involves the interplay of domain-specific knowledge and domain-general strategies. The analysis raised a key issue regarding the relative contributions of content knowledge, general processes and mathematical reasoning to the solution outcome of proof-type geometry problems. This provided the basis for investigating the following research question:

(Q1). What are the predictive indicators of successful proof-type geometry problem solving?

To address the above research question, an empirical investigation (Study 1) was carried out to identify predictive indicators of successful proof-type geometry problem solving among students. This chapter presents the design of the investigation. The next chapter will present results of the investigation in order to address the research question: what are the predictive indicators of successful geometry problem solving? It also prepares the ground to address the other research question, which is addressed in Chapter 5:

(Q2). Based on needs with an emphasis on formative evaluation, what is one design solution to support students solving proof-type problems in geometry?

This chapter starts with an overview of issues that emerged in the literature review. It then introduces the potential variables that can influence the outcome of proof-type geometry problem-solving attempts. The chapter then provides details of some key issues related to the design of this study. It includes comprehensive details of the instruments, scoring rubric and an exemplar scoring for each.

3.1 Prerequisites for proof-type geometry problem solving

This section reiterates some important issues related to proof-type geometry problem solving that emerged from the literature review, particularly the role of content knowledge, and general problem-solving processes.
Proof-type geometry problem solving is difficult for most students (Koedinger & Anderson, 1993; Healy & Hoyles, 1998; Riess et al., 2001; Senk, 1985; 1989). These problems are domain-specific and require the use of non-algorithmic procedures. As they are domain-specific, proof-type geometry problems demand the use of content knowledge during the solution process. As they are non-algorithmic, students do not benefit from procedural support such as algorithms, or formulae.

### 3.1.1 Geometry content knowledge

Proof-type geometry problems are domain-specific in that content knowledge is an important requirement for the solution of these problems (Greeno, 1980). Greeno describes three types of geometrical knowledge that is related to proof-type geometry problems: theorems and rules; visual patterns - like the image of corresponding angles; strategic principles related to the construction of proofs. Declarative knowledge about geometric relationships and concepts are the basic resources used in proof-type geometry problem solving. Research reveals that reasoning skills equivalent to van Hiele Level 3 (Senk, 1989) is required to learn solving proof-type geometry problems. Euclidean deductive system that includes elements of logical reasoning is also a part of the content knowledge.

Diagrammatic reasoning is an important component of content knowledge for solving of proof-type geometry problems. The diagram is an essential part of proof-type geometry problem solving (Charalambos, 1997; Duval, 2001; Fischbein, 1993; Jones, 1998). It embodies the relations and various geometric concepts. In addition, solving process takes place in the diagram with assistance from the diagram. When students fail to deal with diagrams, then the relevant problem solving process also might fail.

### 3.1.2 Questions about sufficiency of content knowledge

Some researchers (De Franco & Hilton, 1999; Healey & Hoyles, 1998; Schoenfeld, 1985) provide evidence to show that mere content knowledge is not a sufficient condition for some mathematical processes, particularly for the solution of non-algorithmic problems. Solution of these types of problems seems to be process-oriented (Schoenfeld, 1985) rather than content-oriented.

Proof-type geometry problems deviate from algorithmic mathematical problems. Healy and Hoyles (1998) observed that even high-attaining students could not construct
proofs, although they demonstrate their ability to judge a proof. Kuchemann & Hoyles (2002) confirmed this in a subsequent study which demonstrated that students with substantial content knowledge also were unsuccessful in deductive proof. DeFranco and Hilton (1999) proved that proof problems need something more than content knowledge. This situation raises the issue: if content knowledge is necessary, but not sufficient, what else is required?

3.1.3 Arguments about the influence of domain-general processes

The non-algorithmic nature of proof-type geometry problems constrains students to decide and select strategies during the solution process. As they have to infer appropriate rules (Koedinger and Anderson, 1993), they are frequently required to access relevant knowledge during problem solution attempts (Chinnappan, 1992; 1998; Chinnappan & Lawson, 1996) and to check the progress via self-management and self-regulation strategies (Renkl, 2002; Wong, Lawson and Keeves, 2002), which are domain-general.

Researchers (Koedinger & Anderson, 1993; Reiss & Renkl, 2002) have observed that influences of general processes are also involved in proof-type geometry problem solving. Koedinger & Anderson (1993) assert that mathematicians do not construct proofs by retrieving definitions and theorems from their memory and putting these together to form logical deductions. Instead, they consider the line of argumentation in broad terms and recognise important properties and connections. Koedinger (1998) states that, for geometrical competence, general skills play an important role in addition to declarative knowledge and argues that a learning environment for proof should also provide specific help and support with heuristic respect to the proving process.

In sum, it would seem that success in proof-type geometry processes is not solely governed by content knowledge. It appears that general problem-solving processes are also important in the solution process.

3.2 Potential predictors of proof-type geometry problem solving

Proof-type geometry problems are domain-specific and content knowledge is a necessary component that provides resources for the solving process. Content knowledge related to proof-type geometry problem solving consists of declarative knowledge such as axioms, theorems, and concepts.
The proof-type geometry problem-solving process also requires mathematical reasoning to understand the properties of axioms and axiomatic approaches, and to use them to generate new relationships from available relationships. Although proof-type geometry problems are different from other mathematical problems, they do involve the use of mathematical operations such as addition, subtraction, and division. These problems also demonstrate quantitative attributes such as comparisons with equality, and inequality. All these are developed within broader mathematical principles such as conservation of length (of a line segment), angle, triangle or geometric figure. This means that mathematical reasoning skills may be another potential variable that influences proof-type geometry problem solving.

Researchers such as DeFranco and Hilton (1999), and Schoenfeld (1985) argue that some mathematical processes are process-dependent rather than content-dependent. There is empirical evidence that factors other than content knowledge might influence proof-type geometry problem solving. Students have to access and use mathematical reasoning skills, domain-general strategies and content-knowledge components to the task.

The lack of consensus about the importance of the three factors is compounded by the fact that previous research has been mainly confined to the study of traditional and non-proof type geometry problems. Thus it was necessary here to undertake research in order to establish whether and to what extent these factors were relevant in the solution of proof-type geometry problems. This was the rationale for the first research question of the study:

(Q1). What are the predictive indicators of successful proof-type geometry problem solving?

The next section of this chapter is devoted to the description of the design of the investigation to address this research question.

3.3 Variables

There are three key potential factors that can influence the progress that students could make with proof-type geometry problem solving: geometry content knowledge, mathematical reasoning skills, and general problem-solving skills. These constitute the independent variables of the study and will be referred to as predictive indicators. This
section discusses the present study’s strategies to collect data related to these predictive indicators.

### 3.3.1 Indicators of content knowledge

Information about student academic achievement has been an important concern in current educational practices. Written tests are the most commonly used method for gathering such information. However, it is difficult to collect information about content knowledge related to proof-type problems from these tests, as achievement related to proof-type geometry is not measured directly. Assessment scores at school level or at public examinations represent information on mathematical reasoning rather than information related to content knowledge of proof-type geometry problem solving. Thus there is a need to develop specific tests to assess content knowledge.

The content knowledge related to proof-type problem solving has been assessed in various research using written tests. For instance, Heinze and Kwak (2002) conducted a written test containing 10 test items to understand informal prerequisites for informal proof-type problems. The written answer was quantified according to the conventional methods such as 0 for wrong responses and other scores for acceptable responses according to predetermined criteria.

For these tests, content knowledge is an important requirement. *Geometry Content Knowledge* is well documented in most curricular materials. According to such documents, students should bring competencies associated with the following content knowledge to proof-type geometry problem solving tasks:

**Declarative knowledge**

- Axioms
- Basic concepts such as point, line, angle
- Right angle, angle types and angle measures and straight (flat) angle
- Parallel lines
- Concepts of geometric figures such as triangle and other polygons, circle
- Concept of congruency and similarity to comparison figures
- Identification of properties such as angles, sides
• Classification of geometric figures according to properties
• Relationships among parts as properties
• Theorems and other geometric relationships

*Diagrammatic representations, interpretation and analysis*

• Conventions of diagrammatic representation: naming points, line segments
• Representation and interpretation of geometric figures
• Identification of parts in diagrammatic forms as sides, angles, triangles

*Deduction*

• Using axioms to deduce new information
• Using postulates to deduce new relationships
• Using theorems to deduce new information

Measurement of content knowledge could be obtained from written answers. The critical requirement is how well the question performs its role in obtaining the response concerned. In this study, the indicators were identified from a set of competencies. A detailed list of competencies related to the content knowledge is found in Table 3.8 of this chapter.

The following indicators of competencies were used to construct a written test to collect data related to the content knowledge:

• Understand/ express with: geometric terminology, concepts, and representations
• Recognise/ use geometric conventions and standard notations
• Obtain information from/ represent information with geometric diagrams
• Obtain implicit information from diagrams
• Select/ use rules to generate new information

Competencies related to the above categories are widely used to prepare items to examine content knowledge required to solve problems in normal mathematics classroom as well as those that appear at public examinations.
3.3.2 Indicators of general problem-solving process

The literature revealed that the content knowledge alone is not adequate for solving problems. It also revealed that there are general processes involved in the problem-solving process.

Researchers use various methods to collect data related to the problem-solving process. Think-aloud (Newell and Simon, 1972; Schoenfeld, 1985; Chinnappan, 1992) and written tests (Hinsley et al., 1977; Senk, 1985) are commonly used methods. There are positives and negatives in both approaches. For instance, although the think-aloud method provides rich data, it consumes time and only a few subjects can participate. On the other hand, a written test can be administered among a large number of participants. Some statistical analyses such as multiple regression analysis require large samples depending on the number of variables.

In Chapter 2, it was proposed that the problem-solving process involves four general processes: analysis, representation, planning and use of knowledge retrieval. The following sections will discuss how information on written answers was scored (or coded) to identify the successful activation of each of the above four general processes. Students’ responses to written tests provide the source of data for evidence of activation of the four general processes. These indicators will be discussed in the following sections.

3.3.2.1. Analysis

This is a general process that involves understanding the problem. The student has to read and identify key information in the problem. It requires decomposition of the problem into parts within the given context. A text-based structure of such non-algorithmic problem is given in the first column of Table 3.1.
Table 3.1 – Possible evidence in the written answer to a well-structured general problem for observing indicators of analysis

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>A pharmacist has a three-litre container and a five-litre container. A mixture needs four litres of water. Without any other container, but with an unlimited supply of water how does he get four litres in either measure?</td>
<td>The student analyses the problem into following parts for understanding the information and the task.</td>
</tr>
</tbody>
</table>

**Recognition of key terms**

- Two containers
  - One contains five litres, the other three,
- No other containers
- Unlimited supply of water

**Recognition of phrases and sentences**

- 4 litres of water in one container

**Recognition of the task**

- Devise a strategy to end up with 4 litres of water in one container.
- A process of filling water (from source), transferring water (from one measure to the other), and removing water to outside

The table shows the indicators as recognition of key terms, recognition of phrases and sentences and recognition of the task.

Similar evidence could also be provided for the analysis process for text-based problems. This section will describe an example from geometry problem solving and another example from general problem solving. Table 3.2 presents ways in which we can identify indicators of analysis through a written answer to Problem 5 of the proof-type geometry problem-solving test (PTG).

Table 3.2 provides information on how to identify indicators of the analysis process in an answer to a typical proof-type geometry problem. It shows that identification of indicators is similar in both proof-type geometry problems and other text-based well-structured problems.
Table 3.2 – Possible evidence in the written answer to a proof-type geometry problem for observing indicators of analysis

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of analysis</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a parallelogram. P and Q are points on AB and CD respectively such that AP = CQ. Prove that the perpendicular distances from P and Q to the diagonal BD are equal.</td>
<td>The student analyses the problem into the following key terms in understanding the information and the task.</td>
</tr>
<tr>
<td><strong>Recognition of key terms</strong></td>
<td></td>
</tr>
<tr>
<td>(A concept) parallelogram</td>
<td></td>
</tr>
<tr>
<td>(A convention) ABCD.</td>
<td></td>
</tr>
<tr>
<td>P, (as a point) on AB</td>
<td></td>
</tr>
<tr>
<td>Q, (as a point) on CD</td>
<td></td>
</tr>
<tr>
<td>AP, CQ, AP = CQ</td>
<td></td>
</tr>
<tr>
<td>Perpendicular distance</td>
<td></td>
</tr>
<tr>
<td>P, BD and Q, BD</td>
<td></td>
</tr>
<tr>
<td>Perpendicular distance from P to BD, Q to BD</td>
<td></td>
</tr>
<tr>
<td><strong>Recognition of phrases and sentences</strong></td>
<td></td>
</tr>
<tr>
<td>ABCD is a parallelogram</td>
<td></td>
</tr>
<tr>
<td>P is a point on AB (and Q is a point on CD)</td>
<td></td>
</tr>
<tr>
<td>Perpendicular distances from P to BD (and Q to BD)</td>
<td></td>
</tr>
<tr>
<td><strong>Recognition of the task and the type</strong></td>
<td></td>
</tr>
<tr>
<td>To prove that two sides in two triangles are equal</td>
<td></td>
</tr>
</tbody>
</table>

The student analyses the problem in order to understand the different parts of the problem.

### 3.3.2.2 Representation

As was discussed, *representation* in this study refers to external representation of the information while the student is attempting to solve the problem. The diagrammatic representation or symbolic representation can be analysed in written test: evidence for three types of *representations* could be found in a written answer. They are:

- *representation* of problem information or given
- *representation* of implicit information generated
- *representation* of goal
The information in Table 3.3 concerns diagrammatic representation as the problem is solved. This example illustrates that *representation* need not necessarily be in diagrammatic form. Sometimes it is possible that representation may be in symbolic form.

### Table 3.3 – Possible evidence in the written answer to a well-structured general problem for observing indicators of *representation*

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of <em>representation</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>A pharmacist has a three-litre container and a five-litre container. A mixture needs four litre of water. Without any other container, but with an unlimited supply of water how does he get four litres in either measure?</td>
<td>The student converts the analysed information into more convenient form to process. <strong>What is known?</strong> (Converting following information into a figure) Two containers: One contains five litres, the other three, <strong>New information</strong> Moves and current situations (either in pictorial form or symbolic form)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Move</th>
<th>Status of Volume</th>
<th>Nature of Change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>2</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>5</td>
<td>5</td>
<td>2</td>
</tr>
<tr>
<td>6</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>

The importance of representation is not in the recording, but in making the solving process more convenient. In that sense, representation will be more effective when it is presented in symbolic form. For instance, problems related to algebra word problems may be more conveniently representable in algebraic symbolic form.

The Pharmacist’s problem can be represented in either symbolic (numeric) form or in diagrammatic form. Table 3.3 represents information in numeric symbols, whereas in Appendix 3 it is represented in the diagrammatic form.

Among these, implicit information is more common in mathematical problem solving than elsewhere. For instance, the term *perpendicular* implies a mathematical indication of an angle is 90°. Table 3.4 provides information about diagrammatic representation.
seen in a solution to the proof-type geometry problem that was taken up in Section 3.3.2.1.

Table 3.4 – Possible evidence in the written answer to a proof-type geometry problem for observing indicators of representation

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of representation</th>
</tr>
</thead>
</table>
| ABCD is a parallelogram. P and Q are points on AB and CD respectively such that AP = CQ. Prove that the perpendicular distances from P and Q to the diagonal BD are equal. | The student converts the analysed information into a more convenient form to process  
**What is given and what is explicit?**  
(Converting following information into a geometrical figure)  
Draws a parallelogram and names it as ABCD  
Indicates the point P on AB and the point Q on CD  
Draws a line from P to BD and another from Q to BD  
Indicates AP = CQ  
Indicates points of intersection of perpendiculars and diagonal (Let L and M)  
**What is implied?**  
(implicit information)  
Marks parallelism  
Marks equal sides of the parallelogram  
Marks perpendiculars  
**What is goal?**  
Marks the goal (sometimes) |

### 3.3.2.3 Planning

Evidence for planning can be traced with the identification of the solution path such as identification of intermediate steps or strategy in the written answer as the student thinks and devises a plan or finds a starting point to begin changes. In solving well-structured problems, intermediate steps are prescriptive and identifiable in the answer.

The evidence related to identification of a planning process is seen in how the student has identified the intermediate steps and appropriate operators in the solution process. Thus indicators for the planning process will be deciding of intermediate steps and deciding appropriate operators/strategies. Information about identification of the indicators of the planning process is shown in Table 3.5.
Table 3.5 – Possible evidence in the written answer to a well-structured general problem for observing indicators of planning

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of planning</th>
</tr>
</thead>
<tbody>
<tr>
<td>A pharmacist has a three-litre container and a five-litre container. A mixture needs four litres of water. Without any other container, but with an unlimited supply of water how does he get four litres in either measure?</td>
<td>The student explores the diagram and searches for a solution strategy and representation. A process with: Filling water (from source) to a container Transferring water (from one container to the other) Removing water from a container (to outside) Decides a mean for representation</td>
</tr>
</tbody>
</table>

It is common that students try to find answers for these questions starting from the goal (working backwards) which is given in proof-type geometry problems. Table 3.6 represents evidence of planning in proof-type geometry problems that was used in Section 3.3.2.1.

Table 3.6 – Possible evidence in the written answer to a proof-type geometry problem for observing indicators of planning

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of planning</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a parallelogram. P and Q are points on AB and CD respectively such that AP = CQ. Prove that the perpendicular distances from P and Q to the diagonal BD are equal.</td>
<td>The student explores the diagram and searches for a strategy. The student sets the following path Inference to prove a pair triangles are congruent Infers the (appropriate) conditions for congruency</td>
</tr>
</tbody>
</table>

3.3.2.4 Use of knowledge retrieval

In a successful solution, the solver generates new information. This new information is the result of activation use of knowledge retrieval. New information in the solution process can be used as indicators for use of knowledge retrieval. The written answers are a rich resource of such evidence.

The process of use of knowledge retrieval comprises two actions: retrieve knowledge and use it to generate new information. Retrieved knowledge is not seen as new information. For instance, knowledge is retrieved to analyse the problem (to recognise
meanings), to represent information (properties, convention and conversions), or to plan (to retrieve previously learned strategies). Thus, use of knowledge retrieval is observed at all generative attempts, from identification to processing information.

The information related to identification of indicators of use of knowledge retrieval in a domain-general problem is shown in Table 3.7

Table 3.7 – Possible evidence in the written answer to a well-structured domain-general problem for observing indicators of use of knowledge retrieval

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of use of knowledge retrieval</th>
</tr>
</thead>
<tbody>
<tr>
<td>A pharmacist has a three-litre container and a five-litre container. A mixture needs four litres of water. Without any other container, but with an unlimited supply of water, how does he get four-litres in either measure?</td>
<td>The student</td>
</tr>
<tr>
<td></td>
<td>retrieves required knowledge</td>
</tr>
<tr>
<td></td>
<td>uses related knowledge to generate information, processes information or transforms information</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Retrieves mathematical meaning of</th>
<th>Uses retrievals</th>
</tr>
</thead>
<tbody>
<tr>
<td>filling water</td>
<td>Adds quantities</td>
</tr>
<tr>
<td>removing water</td>
<td>Deducts</td>
</tr>
<tr>
<td>transferring water</td>
<td>Sum of water constant</td>
</tr>
</tbody>
</table>

Writing the reason for decisions is one requirement in presenting the written answer in solving proof-type geometry problems. Because of this, retrieving knowledge is easily determined in the written answer of a proof-type geometry problem. This particular advantage is not available to the same extent in other well-structured problem solving.

Table 3.8 illustrates identification of indicators of the use of knowledge retrieval process in a proof-type geometry problem.
Table 3.8 – Possible evidence in the written answer to a proof-type geometry problem for observing indicators of *use of knowledge retrieval*

<table>
<thead>
<tr>
<th>Problem</th>
<th>Indicators of <em>use of knowledge retrieval</em></th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a parallelogram. P and Q are points on AB and CD respectively such that AP = CQ. Prove that the perpendicular distances from P and Q to the diagonal BD are equal.</td>
<td>The student 1. retrieves required knowledge, 2. uses related knowledge to generate new information, process information or transform information.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Retrieves</th>
<th>Uses retrievals</th>
</tr>
</thead>
<tbody>
<tr>
<td>Opposite sides of a parallelogram are equal. When equals are subtracted from equals, the result is also equal. The necessary and sufficient conditions to prove congruency. Correspondent pairs are equal in congruent triangles.</td>
<td>AB = CD AB – AP = DC – CQ Prove that triangles BPL and DQM are congruent Prove that PL = QM</td>
</tr>
</tbody>
</table>

There is a difference between algorithmic and non-algorithmic problem solving and the use of knowledge retrieval. Algorithmic moves are either correct, or incorrect, because incorrect ones mathematically involve illegal moves. In contrast, non-algorithmic problem solving involves appropriate moves, or inappropriate moves (inappropriate moves that are not illegal, but may not be goal oriented) and wrong moves. For instance, most of the properties of a parallelogram are not relevant in problem-solving process illustrated in Table 3.8, but if the student mentions a relationship of opposite sides in order to make a change, the change is not mathematically wrong, but not relevant in this problem. Senk (1985) has used this difference to code students' answer scripts.

3.3.3 Indicators of *Mathematical Reasoning Skills (MRS)*

Reasoning is the tool of thinking (NCTM, 2000) that supports the process of selection of appropriate knowledge resources (Manktelow, 1999) in the problem-solving process. It is an important aspect of proof-type geometry problem solving. Faced with a problem, the student has to think about and decide on the solution process. In this
regard, mathematical reasoning has an important role. For instance, axioms are used in proof-type geometry problem solving. They provide support for proof development. At most times the student has to select appropriate axioms to combine two or more geometric relationships.

Reasoning is essential in mathematical problem solving (NCTM, 2000) and tests have been developed to examine this skill. *Ability to Do Quantitative Thinking (ATDQT)* is a test for mathematical reasoning that was developed by the Iowa Tests of Educational Development. The use of this test among high school students has shown that there is a high correlation between mathematical reasoning and mathematical achievement (Schoen, Hirsch & Ziebarth, 1998; Schoen, Fey, Hirsch & Coxford, 1999). This implies mathematical reasoning constitutes a component of mathematical achievement.

*School Based Assessments (SBA)* are used to assess student achievement in Sri Lanka for all subjects. SBA comprises a series of continuous tests that are used to calculate a final score at the end of each year (Appendix 4 – Circular No. 1998/04 of Sri Lanka Ministry of Education).

The assessment process is conducted in schools under the supervision of the Ministry of Education. Assessment standards for Grade 10 an 11 are set by the National Evaluation of Testing Service (NETS), the mandatory agency for educational evaluation of Sri Lanka. The NETS is guided by the National Institute of Education, the mandatory agency for curriculum and evaluation standards of Sri Lanka for general education including SSL (Appendix 5 – Circular No. 98/42 of Sri Lanka Ministry of Education).

The collaboration of the three authorities provides a high level of credibility and reliability about certification of student achievement through SBA. SBA scores are used for certification of achievement at national levels. These scores are used to represent mathematical reasoning of students who participated in the present study.

### 3.4 Multiple Linear Regression (MLR) analysis

The literature review conducted for the present study generated three key variables: *Geometry Content Knowledge* (GCK), *General Problem-Solving processes* (GPS), and *Mathematical Reasoning Skills* (MRS) that can influence *Proof-Type Geometry problem-solving skills* (PTG). MLR procedure was used to examine the hypothesis that GCK, GPS, and MRS influence PTG.
A number of methods could have been adopted to investigate the hypothesis. The literature related to research methods however suggests that a linear regression analysis is the best method to investigate the simultaneous effects of several independent variables on one dependent variable.

When researchers are interested in understanding the relationship between more than two variables, they often use a technique called multiple regression analysis which measures the relationship between one interval level dependent variable and several independent variables (Polit and Hungler, 1995, p. 358).

Linear Multiple Regression (MLR) is not just a data analysis technique but a research design strategy as well (Borg & Gall, 1983; Norwood, 2000; Punch, 1998). The main advantage of the MLR is its straightforwardness in addressing the research question.

… MLR is useful, because it addresses directly questions of key substantive significance. … First it is flexible, in being able to accommodate different conceptual arrangements among the independent variables including their joint effects on a dependent variable. … Second, it is not difficult to understand, conceptually or operationally. (Punch, 1998, p. 83).

MLR can be used to identify, compare and estimate the contribution of the independent variables that affect the dependent variable. This study was designed to identify the influence of a set of three predictive variables.

This study aims to compare the relative strengths of three independent variables in influencing the dependent variable: \textit{Proof-Type Geometry problem-solving skills} (PTG). The independent variables are: \textit{Mathematical Reasoning Skills} (MRS), \textit{General Problem-Solving processes} (GPS) and \textit{Geometry Content Knowledge} (GCK). The hypothesised linear relationship among the independent variables and the dependent variable can be expressed as:

\[ PTG = \beta_0 + \beta_1 \text{(MRS)} + \beta_2 \text{(GPS)} + \beta_3 \text{(GCK)} \]

\textbf{3.4.1 Design consideration for MLR analysis}

Although MLR analysis involves a straightforward procedure the design should satisfy various conditions to ensure the validity and reliability of results. In other words, the validity of the result exists if and only if the data is appropriate for analysis.
3.4.1.1 Variables

The independent variables of interest are Geometry Content Knowledge, General Problem-Solving processes and Mathematical Reasoning Skills. The dependent variable is Proof-Type Geometry problem-solving skills.

3.4.1.2 Data type

According to the above mathematical relationship, observations for all MRS, GPS and GCK and PTG should be quantified. They can be either in ratio scale or in rank scale.

3.4.1.3 Sample size

The sample size is another consideration in the design for MLR. Different authors present different formulae to decide the sample size. Tabachnick and Fidel (2001, p. 117) explain that sample size depends on a number of factors including desired power, alpha level, number of predictors and expected effect sizes. The simplest rules of thumb are \(N \geq 50 + 8m\) (where \(m\) is the number of independent variables) for testing the multiple correlation and \(N \geq 104 + m\) for individual predictors. Since this study is interested in three independent variables \((m = 3)\) and the appropriate minimum sample size is 74 for multiple correlation and 107 for individual predictors. Thus samples larger than 107 satisfy both of these formulae for three independent variables.

Another formula for MLR is found in Norwood (2000, p. 370). According to this, 30 subjects (40 in the case of stepwise method) for each independent variable are required for the proper representation. Since the number of independent variables in this study is 3, 90 (or 120 for stepwise method) subjects are required. A minimum sample size of 120 for three independent variables satisfies the requirements of the above method.

3.5 Method

3.5.1 Participants

Participants in this study were Grade 11 (Age 16 – 17 years) students in Sri Lanka. All senior secondary students in Sri Lanka are required to complete a common course of mathematics. The course, senior secondary mathematics, is taught in three types of schools: boys, girls and mixed.
As discussed earlier, the sampling procedure is very important in regression analysis to ensure the reliability of results. The sample size depends on the number of variables that are involved in the regression analysis. According to the discussion in 3.4.1.3, 120 subjects in the sample would be appropriate for three independent variables and purposes of this study.

The schools were located in two major urban areas (Colombo and Kandy) and rural areas of Sri Lanka. On the advice provided by the Ministry of Education and schools, the schools were chosen to represent a range of student mathematical ability.

166 students from four schools were involved in all tests in the present study. They were from a boys’ school, a girls’ school and two mixed schools. All schools had parallel classes - that is they were not streamed. At each school a class was chosen at random.

3.5.2 Design of the study

Tests were developed in order to collect data on each of the three variables PTG, GPS, and GCK. Students’ Grade 10 final scores for SBA were collected to represent MRS scores. Figure 3.1 provides an overview of the tests and associated scores for the variables.

![Figure 3.1 - Design of Study 1](image)

For each test a scoring rubric was developed. During the development of each of the rubrics, three experienced teachers of mathematics were invited to review the different components of the rubric, and comment on potential areas of ambiguity. Any disagreements were resolved through discussion. Students’ answer scripts were scored
on the basis of the final rubrics. The scores were subsequently fed into the SPSS computer-based statistical program and multiple regression analysis was conducted.

3.6 Materials

This section provides a detailed account of materials, the purpose and the procedures that were adopted during the administration of the tests. As discussed earlier, three tests were developed in order to generate scores for three independent variables:

1. A written test for PTG
2. A written test for GPS
3. A written test for GCK

Tests were prepared by a panel of experienced mathematics educators and experienced teachers. The items for the test were selected from a pool of resources such as textbooks, examination papers and research papers. Some items were modified to fit the purpose of the study. The items were reviewed by two colleagues and edited by another colleague. Each test was piloted with a group of 15 students. The piloting provided a better indicator of the duration for each test and potential areas of difficulties for students. The duration for each of the tests were as follows:

- PTG 80 minutes
- GPS 80 minutes
- GCK 60 minutes

The next section will present information about the tests and the construction process. For each test, the scoring rubric and an exemplar of the scoring system will be presented.

3.6.1 Proof-Type Geometry problem-solving test (PTG) and scoring rubric

In this section, information about the following two materials is presented.

1. Proof-Type Geometry problem-solving test
2. Scoring rubric of Proof-Type Geometry problem-solving test
3.6.1.1 Proof-Type Geometry problem-solving test

Five geometry problems of different difficulty levels were selected and modified for this test. Mathematics textbooks and previous examination papers were the first source of these problems that were subsequently modified with guidance from mathematics teachers. All problems were well-structured proof-type problems. One of them was not a typical proof, but a proof followed by a find-type task.

The next section presents details of each problem and the knowledge required for solving the problem.

**Problem 1**

*The line AB has been extended to either side so that AX = BY. Prove that AY = BX.*

This problem is a simple modification of a problem that was discussed in Anderson (1983, p. 219). Although the diagram is given, it has to be understood. In the original problem, all lines of the solution were given, and the student was asked to give the reason for each line. In this test, the student was expected to construct the entire solution. The problem is a non-routine proof problem. The student requires knowledge about axioms, and the norm of conservation of a line segment.

**Problem 2**

*If $\angle AOB = \angle COD$ then prove that $\angle AOC = \angle BOD$.*

This problem illustrates the use of production rules (Anderson, 1995, p. 331). The original problem was a two-step problem on proving congruency. As in Problem 1, this problem requires knowledge and the norm of conservation of angles. Students do not
need to draw the diagram; instead they have to understand the given diagram. They have to use non-standard conventions to mark the relationships between angles.

Problem 3

WXYZ is a parallelogram and WZQ and PXY are equilateral triangles. Prove that \( WP = QY \).

The diagram provided was incomplete. The students were expected to modify the diagram in a manner that is appropriate to the goal. In addition, they had to use non-standard conventions to mark relationships between angles and between sides. They had to search for a strategy, and use the strategy for proving congruency of triangles WPX and QYZ by planning and executing a path. They had to use knowledge retrieval of theorems concerning properties of a parallelogram, and equilateral triangles, side-angle-side postulate, and properties of congruency.

Problem 4

CDEF is a quadrilateral in which \( \angle CDE = \angle EFC \) and \( \angle DEF = \angle FCD \). Draw a diagram to represent the information. If \( CD = 11 \) cm, what is the length of EF? Give reasons for your answer.

This question is not typical of a proof-type problem. Students have to draw the diagram. After proving, they are required to apply the relationship established on a find-type problem. Students have to read, analyse and understand the question to draw a diagram. Students have to mark information in the diagram and explore it for necessary retrievals. They have to find the necessary and sufficient conditions to show that \( CDEF \) is a quadrilateral.

Problem 5

ABCD is a parallelogram. P and Q are points on AB and CD respectively such that \( AP = CQ \). Prove that the perpendicular distances from P and Q to the diagonal BD are equal.
In this problem students have to activate all cognitive actions that are involved in the solution of a proof-type geometry problem. Students are expected to read, analyse and understand important parts of the question. They need to represent problem information in the diagram and infer implicit information. They also need to understand the goal and convert it into diagrammatic information. They have to search for sub-goals and the necessary knowledge. Once they have established the sub-goal as congruency, they could prove it thereby reaching the goal.

3.6.1.2 Scoring rubric of the Proof-Type Geometry problem-solving test

Problem solving can be assessed by two different methods: process-based or product-based. The selection of method of assessment depends on the purpose of assessment. When the assessment is strictly about the product, it is reasonable to select product-based assessment. This study concerns general cognitive processes involved in the problem-solving process, and interests of process-based assessment.

The following two sections represent two scoring procedures. Senk (1985) directly scores information generated during the problem-solving process. Chinnappan on the other hand, regroups generated information into cognitive processes and then scores. As both were in the domain of geometry, the scoring was deemed to be relevant for the present study.

Two-dimensional matrix based coding procedure (Chinnappan, 1992)

A basic approach of this method was to score problem-solving process according to the general process. The advantage of this type of scoring rubric is that it focuses on the process coding rather than product coding. Chinnappan (1992) developed a set of similar criteria to analyse student think-aloud sessions. The criteria reflect how a student processes information related to geometry textbook-type problems. In this system 73 criteria related to information processing were identified under 14 main categories. These main categories are: reading; identification of given information; identification of implicit information; identification of goal; association of problem; initial problem transformation; problem simplification; planning; drawing diagram; strategy; solution; review; and self assessment. For practical convenience, the criteria were regrouped into the following five cognitive processes and scored according to a two dimensional matrix. Table 3.9 shows the group coding design.
Table 3.9 – The Group Coding Design (Chinnappan, 1992)

<table>
<thead>
<tr>
<th>Code</th>
<th>Incorrect - 0</th>
<th>Partially correct - 1</th>
<th>Correct - 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identification</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Management</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Generation</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Self-assessment</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Error detection</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Identification includes: reading; identification of given; labelling of diagram; and written expression. Problem management includes: rereading; planning; goal identification; checking; error correction. Generation includes: identification of new information; generation of new information; extension of diagram; problem categorisation; retrieval; organization; exploration; reasoning; strategy; decomposition; and computation. Self-assessment is evident with think-aloud statements and expression. This matrix was used to code answer scripts, and video transcripts of think-aloud sessions of 10 students.

One-dimensional linear scoring rubric (Senk, 1985)

Senk (1985) administered a written test comprising of three non-overlapping test items to 2699 Grade 10 students to measure student achievement in proof-type geometry problem-solving skills. The answer scripts were analysed according to five criteria. Table 3.10 shows the details.
Table 3.10 – Scoring Design (Senk, 1985)

<table>
<thead>
<tr>
<th>Score</th>
<th>Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Student writes nothing. Writes only the given or writes only invalid or useless deduction.</td>
</tr>
<tr>
<td>1</td>
<td>Student writes at least one valid deduction and gives reason.</td>
</tr>
<tr>
<td>2</td>
<td>Student shows evidence of using a chain of reasoning, either by deducing about half the proof or stopping or by writing a sequence of statements that is invalid because it is based on faulty reasoning early in the steps.</td>
</tr>
<tr>
<td>3</td>
<td>Student writes a proof in which all steps follow logically but in which errors occur in notation, vocabulary or names of theorems.</td>
</tr>
<tr>
<td>4</td>
<td>Student writes a valid proof with at most one error in notation.</td>
</tr>
</tbody>
</table>

Senk analysed student answer scripts in a linear manner, as the scope of the study was focused on the development of proof only. These criteria do not provide information about the problem solving process underlying the construction of proof.

In a sense, Senk’s design (1985) was focused on the final outcome of the solution process. The criterion-based method of evaluation of the product provides the starting point for the five-point scale used in the present study.

**Process-based scoring rubric for Proof-Type Geometry problem-solving test**

The scoring rubric of *Study I* was developed by drawing on the matrix coding strategy of Chinnappan (1992) and the one-dimensional linear model of Senk (1985). The criteria for scoring were developed from a proof-type problem-solving perspective. The scoring rubric includes features of a two-dimensional matrix. The general processes: *analysis, representation, planning* and *use of knowledge retrieval* were used as coding criteria. This design is shown in Table 3.11.
Table 3.11 – The design of student answer script analysis

<table>
<thead>
<tr>
<th>Level</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3 2 1 0</td>
</tr>
<tr>
<td>analysis</td>
<td>Identifies given, the goal.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Identifies the type of problem.</td>
<td></td>
</tr>
<tr>
<td>representation</td>
<td>Represents problem information in the diagram.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Represents the goal information in the diagram.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Represents generated new information in the diagram.</td>
<td></td>
</tr>
<tr>
<td>planning</td>
<td>Selection of problem steps.</td>
<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Transformation of information.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Final outcome.</td>
<td></td>
</tr>
</tbody>
</table>

Key: 0 – not attempted  1 – attempt incorrect  2 – attempt contains faults  3 – attempt correct: excluded one fault

The same coding sheet can be applied for both product-based scoring and process-based scoring. The shaded cell in the bottom row of Table 3.11 represents the successful final outcome. A score of 3 in that cell shows a successful outcome. In product-based scoring, students receive a score of 3 for correct outcome and 0 for incorrect outcome.

Process-based scoring is comprehensive. It is anticipated that each general process will consist of multiple steps. Each process (analysis, representation, planning, and use of knowledge retrieval) is scored on a four-point scale (0-3) and scores for all four processes are totalled. The following process is used for all questions.

0 – All step scores are 0, which means the student has not attempted any step in the general process.

1 – No correct step performed in the process (all steps at either level 0 or 1)

2 – Some correct steps appear in the process (none of steps is at level 0 or 1)

3 – All steps including reasons are correct. Only one error was neglected

Components in each could receive a maximum of 3 points. As there are four rows, the maximum score for a problem was 12. A test consisting of 5 questions carries a maximum score of 60.

The present scoring rubric has specific features. First, it extends the 5-point scoring rubric of Senk (1985) into a 12-point scheme for scoring proof-type geometry problem
solving tasks. Second, this rubric can be used for process-based scoring, and to deduce product-based scores from that.

There are different views about the solution of proof-type problem solving. Although problem solving is considered as a process, some researchers see proof as either acceptable or not. They oppose using process-based scoring procedure. For instance, Rota (1997) argues:

The expression “correct proof” is redundant. Mathematical proof does not admit degrees (Rota, 1997, p. 183).

This argument confines the scoring rubric to points 0 and 1, which represents whether the proof is right or wrong, with no intermediate degrees. This type of scoring has less value for research about instructional purposes. Too many divisions can also cause practical difficulties. For instance,

An important consideration in the development of [a] coding scheme [is] its capacity to code wide rating behaviours. One disadvantage of such an exhaustive and specific scheme [is] that it [does] not permit discussion of categories of behaviour (Chinnappan, 1992, p. 70).

Senk awards the same score to two students: one writes nothing and the other writes something that is not effective. Although they are the same from the solution’s point of view, they are different from an instructional point of view. In terms of solution attempt, doing something has a greater impact than doing nothing. However an attempt indicates some attitudinal effect or disposition (Schoenfeld, 1985). For these reasons, not attempting to answer is different from attempting to solve a problem.

In the scoring of proof-type problems, Senk (1985) does not take geometric diagram into account. As discussed in the literature, geometric diagram has a key role in proof-type problem solving. The present scoring rubric considers diagrammatic representation as an essential collection of competencies in the proof-type geometry problem-solving process that fosters interpretation, reasoning and processing of geometric information with the help of a diagram.

**Product-based scoring on dichotomous nature of mathematical proof**

A significant property of formal deductive mathematical proof is its demonstrative nature (Polya, 1973b). According to this view, only one counter example is sufficient to disprove the (non) mathematical statement concerned. Because of this, a proof is either
accepted or rejected. As it was mentioned earlier, the acceptability of proof is dichotomous and from the point of view of proof, the solution can be accepted as either “correct” or “wrong”. This leads to consider the product (the proof) as a whole.

Accordingly the answer scripts for proof-type geometry problem solving in Study 1 were scored as 0 and 3 for incorrect and correct proofs respectively. 0 and 3 were selected to match with the scoring code for the process which will be described later in this section. Thus, as was discussed earlier in this section, the scoring rubric developed for the present study is capable of addressing needs of both process-based scoring and product-based scoring.

The complete solution of Problem 1 of the Geometry Problem-Solving test is shown in Table 3.12.

**Table 3.12 – Presentation of the answer for Problem 1**

<table>
<thead>
<tr>
<th><strong>Problem 1</strong></th>
<th>The line AB has been produced to either side so that AX = BY. Prove that AY = BX</th>
</tr>
</thead>
</table>

**Given**  
The line AB has been produced to either sides so that AX = BY

**Goal**  
To prove that AY = BX

<table>
<thead>
<tr>
<th><strong>Statement</strong></th>
<th><strong>Reason</strong></th>
</tr>
</thead>
<tbody>
<tr>
<td>AX = BY</td>
<td>Given</td>
</tr>
<tr>
<td>AX + AB = BY + AB [\Rightarrow (1)]</td>
<td>AB is added to equals</td>
</tr>
<tr>
<td>AX + AB = BX</td>
<td>BX is divided into two parts by line segments AX and AB</td>
</tr>
<tr>
<td>BY + AB = AY</td>
<td>AY is divided into two parts by line segments BY and AB</td>
</tr>
<tr>
<td>Therefore, BX = AY</td>
<td>From (1) Or,</td>
</tr>
<tr>
<td>XY − AX = XY − BY [\Rightarrow (2)]</td>
<td>Equals are subtracted from XY</td>
</tr>
<tr>
<td>XY − AX = AY</td>
<td>From (2)</td>
</tr>
<tr>
<td>XY − AX = BX</td>
<td></td>
</tr>
<tr>
<td>Therefore, BX = AY</td>
<td></td>
</tr>
</tbody>
</table>

The answer in Table 3.12 is the theoretically expected form and it provides the minimum possible information. However in practice, students will provide much information such as evidence of errors, various attempts, rough works and strategies. This information provides
evidence for understanding difficulties students face during the process. The general processes related to these difficulties are also important from the instructional point of view.

In the design of conventional scoring rubrics, only the processing steps, outcome, and reasoning expressions are considered as the proof. Generating a proof is a complex cognitive process rather than a product. The written proof is the data source for the scoring process. In that sense, an answer script provides a rich information source for capturing data for the processes activated by the solver. The present scoring rubric emphasises the process of generating proof rather than producing a solution outcome. For this reason, proof-type geometry problem-solving process is considered as a subset of general problem-solving processes including analysis, representation, planning and use of knowledge retrieval. For instance, the content in the scoring sheet related to Question 1 of the PTG test is shown in Table 3.13.

### Table 3.13 – Procedure for scoring Question 1 of PTG with indicators

<table>
<thead>
<tr>
<th>Level</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>analysis</strong></td>
<td>The student writes given, goal.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The student identifies the problem type as conservation of length in line segments.</td>
<td></td>
</tr>
<tr>
<td><strong>representation</strong></td>
<td>The student marks the relationship (AX = BY) in the diagram</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The student identifies the need for linking the line segments to form AY and BX</td>
<td></td>
</tr>
<tr>
<td><strong>planning</strong></td>
<td>The student selects the relevant axiom as the strategy.</td>
<td></td>
</tr>
<tr>
<td><strong>use of knowledge retrieval</strong></td>
<td>Uses the required axiom (addition/ substraction the same to/ or from equals).</td>
<td></td>
</tr>
</tbody>
</table>

The scoring rubric presented in Table 3.13 considers the activation of each of the four cognitive processes. Each step in each process is scored as 0,1,2, or 3.

### 3.6.2 General Problem-Solving test (GPS) and scoring rubric

This section describes the following:

1. General Problem-Solving test
2. Scoring rubric of the General Problem-solving test
3.6.2.1 General Problem-Solving test (GPS)

The test contained five non-familiar items. The written test was constructed so that students could present evidence related to all four afore-mentioned general processes of problem solving: (a) analysis (b) representation (c) planning and (d) use of knowledge retrieval. As was discussed in Chapter 2, these four general processes are involved in all kinds of problem solving. This section presents solutions for the selected problems and the anticipated use of the general processes.

Problem 1

You are to organise a tea party for the class at the end of the year. How would you find out the food-item preferences of your classmates?

This is an ill-structured problem, and the solving process is not algorithmic. More information is required for solving the problem than is given in the problem statement. Planning will be a critical event in this problem. The problem, knowledge required, strategies, and information representation are domain-general.

The student reads and understands the problem. Searching for more information implies that representation event has taken place. With the given information and the domain-general knowledge at hand, the student plans to solve the problem. The generation of further information and the way of accomplishing the transformation current problem state has to be planned. Prior knowledge about tea parties would be helpful throughout the process, particularly in planning and generating the solution.

Problem 2

A classroom is 7 m x 7 m and it has a 1m wide door at a corner. This classroom has to be arranged as an examination hall. After leaving a space of 3 m x 3 m, each candidate is accommodated in a 1m x 1m space. Draw a lay out of the seating arrangement and label locations for candidates as C1, C2, C3, ...

This is a well-structured and non-algorithmic problem. Mathematical reasoning would be desirable to understand and represent the given information. Spatial reasoning and knowledge related to standard notation of space (for instance, 7 m x 7 m) and scale drawings are required knowledge components. Students usually acquire this knowledge in their mathematics classes. The problem is mainly focused on analysis and
representation. The information provided has to be clearly analysed, as there is a relationship between the free space in the classroom and the location of the door.

Problem 3

* A committee is to have at least 3 women. The number of women should be less than men and the number on the committee must be between 7 and 9. What are the possible compositions?

This problem is also well structured and non-algorithmic, but could be solved by using inequalities. However, solving it with a non-algorithmic approach requires logical inference. Inferences are suggestions that needed to be tried and accepted or rejected in order to satisfy the given restrictions. Evidence for representation is anticipated.

Problem 4

* Ruwan said to Piyal, “If you give me one marble, then we will have an equal number of marbles.” Piyal replied with delight, “If you give me one marble, then I will have double the number you have!” What was the total number of marbles they had?

This problem is well structured, algorithmic and familiar. Although, there are non-algorithmic approaches, they are harder than an algorithmic strategy for the students at this level. Analysis is pivotal here, as identification of parts invokes algorithms related to the algebraic simultaneous equations that are appropriate for solving the problem.

Problem 5

* A pharmacist has a three-litre container and a five-litre container. A mixture needs four litres of water. Without any other container, but with an unlimited supply of water, how does he get four litres in either measure?

This solution of Problem 5 demands the construction of an appropriate representation. Analysis is also critical, as the conditions given in the problem statement have to be clearly understood. Not using diagrams could increase the cognitive load, and loss of information that is processed. The problem is well structured and unfamiliar to most students.
3.6.2.2 Scoring rubric for GPS

Answer scripts of GPS were scored according to the scoring rubric suggested in this study: That is, 3 for correct, 2 for partially correct, 1 for not correct, 0 for absence of any attempt, for each of the general processes: (a) analysis, (b) representation, (c) planning, and (d) use of knowledge retrieval.

The solution presentation is parallel to the PTG test, as these problems are also associated with representations that are common to geometry. Problems in both PTG and GPS tests were presented in texts. Therefore one could expect the activation and use of cognitive events such as analysis, representation, planning and use of knowledge retrieval. Thus it was decided that the same scoring rubric could be applied to both tests.

Subjects were provided with an orientation session to practise answers to non-routine problems. Then the tests were administered on 3 days over 3 weeks, in the order of PTG, GCK and then GPS to minimize the influence of two possible effects: the training effect of the orientation session on GPS; the same effect of GCK on PTG. Table 3.14 illustrates an example coding sheet.

Table 3.14 – Scoring rubric for Question 2 of the GPS test.

<table>
<thead>
<tr>
<th>Door</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C5</td>
<td>C6</td>
<td>C7</td>
<td>C8</td>
</tr>
<tr>
<td></td>
<td>C9</td>
<td>C10</td>
<td>C11</td>
<td>C12</td>
</tr>
<tr>
<td></td>
<td>C13</td>
<td>C14</td>
<td>C15</td>
<td>C16</td>
</tr>
<tr>
<td></td>
<td>C17</td>
<td>C18</td>
<td>C19</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C20</td>
<td>C21</td>
<td>C22</td>
<td>C23</td>
</tr>
<tr>
<td></td>
<td>C24</td>
<td>C25</td>
<td>C26</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C27</td>
<td>C28</td>
<td>C29</td>
<td>C30</td>
</tr>
<tr>
<td></td>
<td>C31</td>
<td>C32</td>
<td>C33</td>
<td></td>
</tr>
<tr>
<td></td>
<td>C34</td>
<td>C35</td>
<td>C36</td>
<td>C37</td>
</tr>
<tr>
<td></td>
<td>C38</td>
<td>C39</td>
<td>C40</td>
<td></td>
</tr>
</tbody>
</table>

A classroom is 7 m x 7 m and it has a 1m wide door at a corner. This classroom has to be arranged as an examination hall. After leaving a space of 3 m x 3 m, each candidate is accommodated in a 1m x 1m space. Draw a lay out of the seating arrangement and label locations for candidates as C1, C2, C3, ...
During the *analysis* process, students have to recognise the following features: room and its shape; the location of the door; available free space and space requirements for individual students in the classroom. The implications and constraints placed by the above requirements have to be inferred by the solver. For instance, students have to come to a decision that the space for a candidate includes the total space for a candidate. Students should also recognise that information is represented in diagrammatic form. They are required to understand the use of notation such as C1, C2, C3, C4.

The solution involves the location of the free space adjacent to the door. Otherwise, the entrance will be blocked by candidates and will not be accessible. This practical necessity has to be considered. Table 3.15 shows the scoring rubric for Question 2.

**Table 3.15 - Scoring rubric for Question 2**

<table>
<thead>
<tr>
<th>Level</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3</td>
</tr>
<tr>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td></td>
<td></td>
<td>0</td>
</tr>
<tr>
<td><em>analysis</em></td>
<td>Space of the room,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>student space,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>location of the door,</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and the free space.</td>
<td></td>
</tr>
<tr>
<td><em>representation</em></td>
<td>Location, measurements and labels</td>
<td></td>
</tr>
<tr>
<td><em>planning</em></td>
<td>Where the free space is located.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Where the door is located.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>How to draw diagrams for spaces.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>How to label candidates.</td>
<td></td>
</tr>
<tr>
<td><em>use of knowledge retrieval</em></td>
<td>Drawing the room.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Drawing the door.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Drawing the free space.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Drawing the seat arrangement.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Labelling the seats as C1, C2 …</td>
<td></td>
</tr>
</tbody>
</table>

### 3.6.3 Geometry Content Knowledge test (GCK) and scoring rubric

In this section information on the following two materials is presented.
1. *Geometry Content Knowledge* test (GCK)

2. Scoring rubric of the *Geometry Content Knowledge* test (GCK)

### 3.6.3.1 *Geometry Content Knowledge* test (GCK)

Proof-type geometry problem solving requires the activation of *Geometry Content Knowledge*. The topics for the content area are clearly and comprehensively documented in curriculum resource materials. The *Geometry Content Knowledge* (GCK) test was designed to measure various rules and declarative knowledge components that are associated with content knowledge. This knowledge was broadly classified into 15 components (see Table 3.16) because these were considered to be the content requirements for the solution of five proof-type geometry problems that were the focus of the present study.

**Table 3.16 – Components of the Geometry Content Knowledge test (GCK)**

<table>
<thead>
<tr>
<th>Competency</th>
<th>Nature of knowledge</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. The student writes the notation of the angle given with labelled arms.</td>
<td>Knowledge about angles Conventions about notation</td>
</tr>
<tr>
<td>2. The student identifies</td>
<td>Diagrammatic reasoning</td>
</tr>
<tr>
<td>a) parts of the given diagram,</td>
<td>Logical inference</td>
</tr>
<tr>
<td>b) builds the relationship between the parts and whole</td>
<td>Using axioms</td>
</tr>
<tr>
<td>3. The student accesses the information about simple angle</td>
<td>Concept of straight angle</td>
</tr>
<tr>
<td></td>
<td>Understanding the conventions</td>
</tr>
<tr>
<td>4. The student identifies</td>
<td>Diagrammatic reasoning: obtaining information from</td>
</tr>
<tr>
<td>a) standard conventions</td>
<td>intelligible diagrams</td>
</tr>
<tr>
<td>b) non standard conventions</td>
<td></td>
</tr>
<tr>
<td>5. The student</td>
<td>Diagrammatic information processing</td>
</tr>
<tr>
<td>a) obtains information on diagrammatic interpretation,</td>
<td>Logical inference</td>
</tr>
<tr>
<td>b) analyses diagrams,</td>
<td>Axiom-based deduction</td>
</tr>
<tr>
<td>c) identifies conservation,</td>
<td></td>
</tr>
<tr>
<td>d) uses axioms.</td>
<td></td>
</tr>
<tr>
<td>6. The student generates information using relevant geometric relationships</td>
<td>Access and use declarative knowledge</td>
</tr>
<tr>
<td>between angles associated parallel lines.</td>
<td>Theorem-based informal deduction</td>
</tr>
<tr>
<td>Competency</td>
<td>Nature of knowledge</td>
</tr>
<tr>
<td>---------------------------------------------------------------------------</td>
<td>----------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>7. The student generates information using relevant geometric relationships between angles on parallel lines.</td>
<td>Transferring knowledge, Property-based formal deduction</td>
</tr>
<tr>
<td>8. The student generates information using properties of parallelogram.</td>
<td>Properties of shapes, Generating implicit information, Activation of diagram schema</td>
</tr>
<tr>
<td>9. The student converts semantic information into diagrammatic representation.</td>
<td>Diagrammatic representation, Standard convention, Non-standard convention</td>
</tr>
<tr>
<td>10. The student recognises standard notation and derives implicit information.</td>
<td>Interpretation: generating information</td>
</tr>
<tr>
<td>11. The student applies geometric relationships.</td>
<td>Theorem based non-formal deduction, Access and use of theorems</td>
</tr>
<tr>
<td>12. The student converts geometric relationships to construct an algebraic equation</td>
<td>Find -type problem solving, Identification of required knowledge, accessing and use</td>
</tr>
<tr>
<td>13. The student demonstrates the comprehension of geometric concepts.</td>
<td>Conceptual understanding</td>
</tr>
<tr>
<td>14. The student demonstrates the ability to disassemble a geometric figure into parts</td>
<td>Reasoning complex diagrams</td>
</tr>
<tr>
<td>15. The student demonstrates the ability to transform conceptual understanding and declarative knowledge into new information.</td>
<td>The level of abstraction, Conceptual understanding, Formal deductive reasoning, Necessary and sufficient conditions</td>
</tr>
</tbody>
</table>

3.6.3.2 Scoring rubric of the Geometry Content Knowledge test (GCK)

The following schema was used in the scoring of items in the Geometry Content Knowledge test:

I - correct response

0 - incorrect response

The expected answers are shown in bold text in Table 3.17
<table>
<thead>
<tr>
<th></th>
<th>Table 3.17 – The scoring rubric of Geometry Content Knowledge test</th>
</tr>
</thead>
</table>
| 1. | OA and OB are straight lines.  
Name the angle between OA and OB  
Naming the angle as $\angle AOB$ |
|   | ![Diagram](image1) |
| 2. | OA, OB, OC are straight lines. State three angles formed by these three lines. 
(a) Naming angles $\angle AOB$, $\angle BOC$, $\angle COA$  
(b) Stating the relationship among these three angles as  
$\angle AOB + \angle BOC = \angle AOC$ |
|   | ![Diagram](image2) |
| 3. | AB is a straight line. O is on AB. Write the magnitude of the angle (in degrees) made of sides OA and OB.  
$\angle AOB = 180^\circ$ |
|   | ![Diagram](image3) |
| 4. | Provide two pieces of information conveyed by the diagram.  
(a) ABC is a triangle  
(b) $AB = AC$ |
|   | ![Diagram](image4) |
| 5. | The diagram on the right-hand side describes that,  
(a) ABC is a TRIANGLE. X is on BC so that  
$\angle CAX = \angle BAX$  
(b) How many triangles are in the diagram? 3  
(c) Name them. ACX, ABX AND ACB.  
(d) State the relationship among the areas of those triangles.  
$ACX + ABX = ACB$ |
|   | ![Diagram](image5) |
| 6. | A transversal cuts the parallel lines. The magnitude of an angle has been marked as “x”. Show other angles with magnitude of “x”.  
Marking opposite angles,  
Marking corresponding angles  
Marking alternate angles. |
|   | ![Diagram](image6) |
7. ABC and PQR are congruent triangles. \( \angle A = \angle R \), \( AB = PR \).
   State other pairs of angles and sides that are congruent.
   \( \angle C = \angle Q \), \( \angle B = \angle P \) (two pairs of angles),
   \( AB = PR \), \( AB = PR \) (two pairs of sides)

8. ABCD is a parallelogram. Show all possible relationships between angles as well as between sides.
   **Marking parallelism,**
   **Marking equal pairs of sides,**
   **Marking equal pairs of angles.**

9. Draw a diagram to describe the following.
   ABCD is a quadrilateral. \( \angle BAC = \angle DCA \). The midpoint of AC is O.
   **Drawing the diagram according to the givens,**
   **Marking givens.**

10. Identify as many as information that is provided in the diagram.
    **Identifying O, A, B, C, D,**
    **Identifying bisectors,**
    **Identifying the point P,**
    **Identifying perpendiculars.**

11. Find the values of angles other than given on the diagram.
    **Applying the concept straight angle,**
    **Applying the concept opposite angle,**
    **Applying the concept parallelism,**

12. Calculate the value denoted by \( x \).
    **Applying the relationship,**
    **Obtaining the answer.**
13 The diagram shows an angle denoted by BAC. What do you understand by the term “angle”?  
The difference between the two states initial and existing,  
According to rotation

14 How many triangles are in the picture?  
Name them  
Stating the number of triangles,  
Naming them.

15. What do you understand by the term “congruency”?  
Being identical of two objects  
What is the meaning of “congruence of two triangles”  
Being identical of two triangles or, being equal of corresponding parts of two triangles  
If you know that two triangles are congruent, what relationships can you deduce?  
Relationships between corresponding pairs  
What are tests for congruency of two triangles?  
SSS  
SAS  
ASA  
Sp. Case (RHS)  
How can you use the idea of congruency of two triangles in problem solving?  
To relate equality of two angles  
To relate equality of two sides in proof or calculation

3.7 Test administration and scoring

Tests were administered in February 2002. In the first instance, students were made aware of the purpose of the study and were informed that their performance in the tests would not be affect their grades in the normal school mathematics examination. The implementation plan is shown in Appendix 6. Details of each visit to the school were provided in advance.

The following steps were followed during data collection.

- A 2-hour session was conducted to practise answering the GPS test. The problems used in the practice session were not similar to the test items in the
GPS test, because the aim was to practise written presentation of general problem solving.

- In order to avoid possible practice effects of the training session on performance in GPS, the practice session was followed by the PTG test instead of GPS.

- In order to avoid possible practice effects of the PTG on GCK, PTG was followed by the GPS test instead of GCK.

- In order to avoid possible practice effects of the GCK on PTG, GCK was administered last.

From the participating four schools, a total of 166 students completed all three tests. SBS scores of these students were collected in the fifth week.

Answer scripts were scored and the detailed scores were entered to Excel worksheets so that the process score for each item of the PTG and GPS tests for each student could be obtained. Figure 3.2 shows a part of the worksheet related to the PTG test.

Figure 3.2 – A part of the Excel worksheet containing detailed scores

Figure 3.2 shows that the worksheet provides detailed scores for the problems used in the test. For each problem it provides the score for the process elements (1 – analysis, 2 – representation…). The final outcome for product-based scoring is also obtained from the fourth column of each problem. For instance, Code No. 103 is awarded 3 for Item No. 1, 3 for Item No. 2, 0 for each of the other items for the final outcome.

3.8 Summary

Study I was aimed at investigating predictive indicators of successful proof-type geometry problem solving. The three independent variables of interest to the study were
geometry content knowledge (GCK), general problem-solving skills (GPS), and mathematical reasoning (MRS). The dependent variable was proof-type geometry problem solving (PTG) skills.

This chapter presented information about the development, administration and scoring procedures of tests designed to represent the above four variables. The results of the data analysis and discussion will be presented in the next chapter.
Chapter 4: Results of Study 1

4.0 Introduction

Chapter 3 presented the design of the linear multiple regression (MLR) analysis that addressed the research question:

(Q1). What are the predictive indicators of successful proof-type geometry problem solving?

The regression analysis was carried out in two stages. In Stage 1, the score for PTG represents the final outcome of the students’ solution attempts. Stage 1 provides a preliminary analysis of the influence of three independent variables.

During Stage 2 analysis, it was decided to provide a more complete picture as to the steps and final outcome of the solution process. Following this line of reasoning, PTG was scored using the scheme that was discussed in section 3.6.1.2.

This chapter presents the results of the regression analysis. The multiple linear regression procedure commenced with screening data for the purpose of verification of suitability (Francis, 2001; Hair, Anderson, Tatham & Black, 1995; Tabachnick and Fidel, 2001). The first section of this chapter presents results of the data analysis.

The second part is concerned with validity, and examines the manner in which the underlying assumptions of the regression analysis were addressed. The third section of the chapter interprets the results using evidence from students’ answer scripts for the proof-type geometry problem-solving test.

The data analysis was computed with the SPSS program and a statistics consultant provided advice on the appropriateness of statistical procedures.

In this chapter, abbreviations are used to denote variables. The dependent variable, *Proof-Type Geometry problem-solving skills* is denoted by PTG; the independent variables: *Geometry Content Knowledge, General Problem-Solving processes* and *Mathematical Reasoning Skills* are shown as GCK, GPS, and MRS respectively.
4.1 Stage 1 - Results of the preliminary analysis

Preliminary data analysis was carried out with all variables: PTG, GCK, GPS, and MRS. As was discussed in Chapter 3, PTG was scored on final outcome for the preliminary data analysis.

4.1.1 Modeling through multiple linear regression analysis

There are several methods for conducting a regression analysis. The selection of method depends on the nature of the question to be addressed. Table 4.1 provides a summary of purposes for each method.

<table>
<thead>
<tr>
<th>Method</th>
<th>Purpose</th>
</tr>
</thead>
<tbody>
<tr>
<td>Standard (Enter) method</td>
<td>To understand how the predictors combine to influence the dependent variable (Francis, 2001).</td>
</tr>
<tr>
<td>Forward selection method</td>
<td>To understand the relative importance and the order of predictors (Francis, 2001).</td>
</tr>
<tr>
<td>Backward elimination method</td>
<td>To understand the relative importance and the order of predictors (Francis, 2001).</td>
</tr>
</tbody>
</table>

As Table 4.1 shows, the focus of the Standard Method (or Enter Method) in SPSS is to examine “how the predictors combine to influence the dependent variable” (Francis, 2001, p.109). In this method, all selected variables are entered into the model at once. The method filters significantly influential factors and removes non-influential factors. It also yields the collective degree of the regression in the form of statistics.

In this instance the analysis starts with the forward selection regression analysis, entering the variable that is most strongly correlated with the dependent variable, then the next variable entered is the predictor making the biggest change to $R^2$ (the proportion of the variation in the dependent variable that can be explained by the predictor). “The process continues until none of the remaining variables would make a significant change to $R^{2*}$ (Francis, 2001, p. 117). The backward procedure starts with all variables entered and drops variables one by one in the order of least contribution to $R^2$. It stops when elimination makes a significant change to $R^2$ (Hair et al., 1995).

There were some necessary adjustments in the execution of the SPSS program. PTG was specified as the dependent variable and GCK, GPS and MRS were specified as
independent variables. In the statistics option dialog box of the SPSS program, descriptive and histogram were selected in a 95% confidence interval for $\alpha = 0.05$. Residuals were saved to allow examination of normality.

### 4.1.2 Standard regression analysis

All variables were entered into the SPSS program for analysing through standard regression analysis. Results are shown in Table 4.2

#### Table 4.2 – Results of standard regression analysis

<table>
<thead>
<tr>
<th>Variables Entered</th>
<th>Variables Removed</th>
<th>$R$</th>
<th>$R^2$</th>
<th>Adjusted $R^2$</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRS</td>
<td>-</td>
<td>.704</td>
<td>.496</td>
<td>.487</td>
<td>2.2724</td>
</tr>
<tr>
<td>GPS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GCK</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a. All requested variables entered.  
b. Dependent Variable: PTG.  
c. Predictors: (Constant), MRS, GPS, GCK.

Results in Table 4.2 reveal that all independent variables: GCK, GPS, and MRS to be predictive indicators. These results are significant at $\alpha < .000$ with $F = 53.196$

As these results are highly significant and all variables are shown to be predictive indicators, data analysis continued in detail. In the detailed analysis, scores for proof-type geometry problem solving were used according to the discussion in section 3.6.1.1 in Chapter 3. The next section provides the results of the detailed analysis.

### 4.2 Stage 2 – Results of the detailed analysis

#### 4.2.1 Descriptive statistics

Descriptive statistics for each variable were computed to gain a broader picture of the nature of variables across the sample. The means and standard deviations are the typical statistics that provide information about the distribution of the independent variable. The information is also useful to compare the distributions of different variables.

The means and standard deviations for the sample consisting of 166 students are shown in Table 4.3. Which also contains the possible range of scores for each variable.
Table 4.3 - Descriptive statistics

<table>
<thead>
<tr>
<th></th>
<th>Possible Range</th>
<th>Observed Range</th>
<th>Mean</th>
<th>Std. Deviation</th>
</tr>
</thead>
<tbody>
<tr>
<td>PTG</td>
<td>0 - 60</td>
<td>8 - 55</td>
<td>30.09</td>
<td>10.12</td>
</tr>
<tr>
<td>GPS</td>
<td>0 - 60</td>
<td>18 - 53</td>
<td>35.83</td>
<td>8.68</td>
</tr>
<tr>
<td>GCK</td>
<td>0 - 36</td>
<td>6 - 33</td>
<td>20.37</td>
<td>6.21</td>
</tr>
<tr>
<td>MRS</td>
<td>0 - 100</td>
<td>28 - 77</td>
<td>50.06</td>
<td>10.04</td>
</tr>
</tbody>
</table>

Table 4.3 shows the possible range, observed range, mean and standard deviation for the dependent variable: PTG, and independent variables: GPS, GCK, and MRS. PTG, as appears in the first row, varies in a range of 47, from 8 to 55 within the possible interval 0 – 60. The mean 30.09 lies in the mid-region of this range with the standard deviation of 10.12. This suggests that there is a good spread of scores over the possible range.

The second row represents GPS, which was also scored with a parallel scheme to PTG. While the possible range is 0 – 60, the observations were between 18 and 53. The sample mean is 35.83 and the standard deviation is 8.86. Although the distributions of scores of GPS appear to have a shift from PTG scores, its mean lies mid-region of the distribution. This distribution can also be considered as reasonable for the regression analysis.

Descriptive statistics related to GCK are shown in the third line. The total mark allotted for the paper was 36. The sample shows a mean of 20.37 with 6.21 standard deviation for the variable in the range of 6 to 33. This also shows a fair spread.

Table 4.3 shows that MRS lies between 28 and 77 with a mean score of 50.06 and a standard deviation of 10.04. The possible range for MRS is 0 – 100.

4.2.2 Pearson correlation coefficient (r)

The Pearson correlation coefficient indicates the degree of the linear relationship between two numeric variables. Pearson correlation coefficient is usually a decimal figure between -1 and 1. Higher values of correlation coefficients, regardless of their sign, indicate more strong relationships, whereas values closer to 0 from either side
indicate weaker relationships. When the two variables are not related, their correlation coefficient is zero.

As multiple linear regression analysis deals with several variables, correlation coefficients are presented as a matrix. This matrix allows a comparison of correlation coefficients of different variables with each other. It also provides information as to which independent variables have significant correlation coefficients. The correlation matrix is shown as Table 4.4.

Table 4.4. - Pearson correlation coefficient matrix

<table>
<thead>
<tr>
<th>Correlation Coefficients</th>
<th>GCK</th>
<th>GPS</th>
<th>MRS</th>
</tr>
</thead>
<tbody>
<tr>
<td>PTG</td>
<td>0.820*</td>
<td>0.703*</td>
<td>0.536*</td>
</tr>
<tr>
<td>GCK</td>
<td></td>
<td>0.667*</td>
<td>0.529*</td>
</tr>
<tr>
<td>GPS</td>
<td></td>
<td></td>
<td>0.410*</td>
</tr>
</tbody>
</table>

* Correlation is significant at the $\alpha = 0.01$ level (2-tailed).

The first row of Table 4.4 represents the Pearson correlation coefficients between the dependent variable PTG and each of the other independent variables. It reveals that the dependent variable is positively correlated with each of the independent variables with correlation coefficients 0.82, 0.70 and 0.54. These correlation coefficients are significant at $\alpha = 0.01$ level.

The next two rows of Table 4.4 show the correlation coefficients among independent variables. As it appears there, they are correlated with each other with correlation coefficients 0.67, 0.53 and 0.41. All of these are significant at $\alpha = 0.01$ level.

According to these correlations, GCK is the independent variable most strongly related to proof-type geometry problem-solving performance. The other variables: GPS, (General Problem-Solving processes) and MRS (Mathematical Reasoning Skills) are also significantly correlated to PTG. This suggests that while Geometry Content Knowledge (GCK) is the major predictor of proof-type geometry problem-solving performance, the other variables GPS and MRS also seem to be powerful predictors.
4.2.3 Partial correlation coefficients and part correlation coefficients

The Pearson correlation coefficient is a statistical indicator that describes the relationship between two variables. A correlation coefficient between two variables is not affected by the variations of other variables. Because of this, the Pearson correlation coefficient is not sufficient to compare the individual relative influences of each independent variable on the dependent variable.

There are correlation coefficients among independent variables too. These correlation coefficients suggest the existence of other relationships. It is an indicator of the influence of one independent variable on the other. Such influences also eventually affect the variability of the dependent variable. Because of these interrelationships among independent variables, the magnitude of the Pearson correlation coefficient is not sufficient to compare and contrast the individual influence of each independent variable (Hair et al., 1995). In this regard, two specific correlation coefficients: the *partial correlation coefficient* and the *part correlation coefficient* (some authors title part correlation as semi partial correlation) are important.

The partial correlation coefficient refers to the strength of the relationship between the dependent variable and a single predictor, when the effects of other independent variables are held constant. The use of this coefficient is to identify the independence with the greatest incremental predictive power beyond the predictor variables already in the model (Hair et al., 1995). It can be viewed as the influence of the independent variable assuming that other variables do not change their influences.

Second, the part correlation coefficient refers to the strength of the relationship between the dependent variable and a single independent variable when the effect of the other independent variables in the regression models is removed. This is used to describe the unit predictive effect on a single dependent variable among a set of independent variables (Hair et al., 1995). This can be viewed as the influence of the independent variable assuming that the other variables do not have influence.

Table 4.5 shows the Pearson correlation coefficient, partial correlation coefficient and part correlation coefficient of each variable with PTG.
Table 4.5 – Partial and part correlation coefficients with PTG

<table>
<thead>
<tr>
<th></th>
<th>Pearson coefficient (zero)</th>
<th>Partial coefficient</th>
<th>Part coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>GCK</td>
<td>0.820</td>
<td>0.606</td>
<td>0.398</td>
</tr>
<tr>
<td>GPS</td>
<td>0.703</td>
<td>0.357</td>
<td>0.200</td>
</tr>
<tr>
<td>MRS</td>
<td>0.536</td>
<td>0.192</td>
<td>0.102</td>
</tr>
</tbody>
</table>

Table 4.5 shows the order of the influence demonstrated by each independent variable GCK, GPS and MRS at three levels (zero, partial and part). It is similar to the order that is suggested by the correlation coefficient matrix. When different coefficients are compared, the relative differences among three variables in the Pearson correlation coefficients column are different to the relative differences among three variables in the column of partial correlation coefficients. Both partial correlation coefficients and the part correlation coefficients magnify the relative strength of the *Geometry Content Knowledge*.

The differences in the figures between the first and the second columns imply how great is the influence of GCK on PTG compared to the other two variables. For instance, while the figure of GCK decreases from 0.820 to 0.606 (by 26%), GPS decreases from 0.703 to 0.357 (by 49%) and MRS decreases from 0.536 to 0.192 (by 64%). This indicates that, when the process holds the influence of other variables at a constant level, the difference is comparatively small, indicating the importance GCK. On the other hand, as it appears in the third column of Table 4.5 at part correlation coefficient level, the magnitude of the correlation coefficient drops, from 0.82 to 0.39, by 49%. This suggests that the influence of the other two variables is also significant in the success of GPT.

In summary, the correlation coefficient matrix provides information to determine the relative contributions of the independent variables to the success of PTG. It suggests the order of contribution of independent variables GCK, GPS, MRS according to the relative magnitude of their influence on PTG. However, the matrix does not provide information about the proportional amounts of contributions. Determining the proportional contribution is complex when independent variables are interrelated. This issue will be addressed in the next section.
4.2.4 Analysis with standard regression method

To examine how the predictors combine to influence the dependent variable, standard regression analysis was executed on the SPSS program. In this method, all selected variables are entered into the model at once.

The Multiple Regression coefficient (R) is an important statistic in the regression analysis. It is the square root of the coefficient of determination or the correlation squared (R²), which is the total proportion of variation of the dependent variable explained by dependent variables. The results of the analysis are shown in Table 4.6.

Table 4.6 – The regression figures

<table>
<thead>
<tr>
<th>Entered</th>
<th>Removed</th>
<th>R</th>
<th>R²</th>
<th>Adjusted R²</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>MRS</td>
<td></td>
<td>.852</td>
<td>.726</td>
<td>.721</td>
<td>5.34</td>
</tr>
<tr>
<td>GPS</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>GCK</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- All requested variables entered.
- Dependent Variable: PTG.
- Predictors: (Constant), MRS, GPS, GCK.

R² indicates the collective variation of all three independent variables on the dependent variable. According to this, 72.6% (almost 73%) of PTG is explained by these three variables: MRS, GPS, and GCK. The adjusted R² is an estimated value to use as the population estimator, as small samples tends to overfit. The difference between R² and adjusted R² is not large. This is also a good indicator of the strength of the prediction. Standard error is an estimate of the standard deviation that represents the variation of the actual value of dependent variables around the regression line. In other words, it is a measure of the absolute size of the prediction error. If the standard error is too large, R will not be significant.

4.2.5 ANOVA

The significance of R is determined by the F value, which is generated in the analysis of variance (ANOVA). The ANOVA, thus, is useful to test the null hypothesis:

H₀: There is no significant linear relationship in the population between the dependent variable and the independent variables.
The ANOVA is shown in Table 4.7

Table 4.7 – The ANOVA

<table>
<thead>
<tr>
<th></th>
<th>Sum of Squares</th>
<th>df</th>
<th>Mean Square</th>
<th>F</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td>Regression</td>
<td>12283.391</td>
<td>3</td>
<td>4094.464</td>
<td>143.440</td>
<td>.000</td>
</tr>
<tr>
<td>Residual</td>
<td>4624.254</td>
<td>162</td>
<td>28.545</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>16907.645</td>
<td>165</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a Predictors: (Constant), MRS, GPS, GCK
b Dependent Variable: PTG

Since $R^2$ is significant according to the ANOVA ($F_{3, 162} = 143.440$, $p = .000$), the prediction is also significant. This is a statistical requirement for explaining the variation of the dependent variable in terms of the independent variables.

4.2.6 Coefficients related to the regression analysis

$R^2$ in Table 4.6 indicates that all three variables can describe 72.6% of the variation of the score for PTG. Since this value represents a collective effect, it cannot be used to explain the variation in terms of the individual contribution of each independent variable. Regression coefficients for each of the independent variables are required in that regard. The information related to regression coefficients is shown in Table 4.8.

Table 4.8 – Coefficients related to the regression analysis

<table>
<thead>
<tr>
<th></th>
<th>Unstandardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>Std. Error</td>
<td>Beta</td>
<td></td>
</tr>
<tr>
<td>(Constant)</td>
<td>-6.399</td>
<td>2.337</td>
<td>-2.738</td>
<td>.007</td>
</tr>
<tr>
<td>GCK</td>
<td>.939</td>
<td>.097</td>
<td>.576</td>
<td>9.684</td>
</tr>
<tr>
<td>GPS</td>
<td>.314</td>
<td>.065</td>
<td>.269</td>
<td>4.865</td>
</tr>
<tr>
<td>MRS</td>
<td>.122</td>
<td>.049</td>
<td>.121</td>
<td>2.492</td>
</tr>
</tbody>
</table>

a Dependent Variable: PTG

The coefficients that can be used to build up the regression model are shown in the unstandardised coefficients (B) column. The figures in the significance column show that all coefficients are significant at $\alpha = 0.05$ level. If the standard error is large, the
SPSS program indicates the respective B as not significant. *Beta (β) coefficients* are the coefficients of the independent variables in the linear regression equation that represent the standard scores. It is important when scores are presented in different scales, because standard scores are in the same scale with mean = 0 and standard deviation = 0.

Accordingly, the regression equation for predicting PTG can be expressed as a linear combination of independent variables:

\[
PTG = -6.399 + .939 \text{GCK} + .314 \text{GPS} + .122 \text{MRS}
\]

Unstandardised coefficients or B coefficients are partial regression coefficients. Each of these coefficients further explains the effect on the variation of the dependent variable for the increment of 1 unit of the respective independent variable, while other variables are not changed. Thus, theoretically, when the other two independent variables are statistically controlled:

- increasing GCK by 1 mark can increase the predicted performance of PTG by 0.939 marks;
- increasing GPS by 1 mark can increase the predicted performance of PTG by 0.314 marks and;
- increasing MRS by 1 mark can increase the predicted performance of PTG by 0.122 marks.

The other advantage of this coefficient table is in the sign of the *partial coefficient*. A positive sign indicates an increment whereas a negative sign indicates a decrement in per-unit variation.

As discussed under Table 4.4, the correlation matrix shows that there are inter-correlations among independent variables too. This suggests, the variation of the dependent variable cannot be decided in a simple manner. $R^2$ in Table 4.6 also indicates that 62.7% of the total variation of the score for PTG is described by these three independent variables. This percentage is a combined figure which does not provide information about individual contributions.

**4.2.7 Analysis with *Forward Selection* Method**

The strength and extent of the contribution of each independent variable to the regression can be estimated by building hierarchical models (Francis, 2001). This can
be done by hierarchical regression analysis methods. One approach involves the *Forward Selection* Method, which adds independent variables starting from the most influential independent variable. This process ends with the least significantly influential variable. Another method is the *Backward Elimination* Method. In the same manner, it enters all influential independent variables and removes them one by one until the change yields no significance. Table 4.9 provides the result generated by SPSS in the regression analysis by *Forward Selection* method and its model summary.

**Table 4.9 – Hierarchical regression analysis - *Forward Selection* Method**

<table>
<thead>
<tr>
<th>Model</th>
<th>R</th>
<th>R²</th>
<th>Adjusted R Square</th>
<th>Std. Error of the Estimate</th>
</tr>
</thead>
<tbody>
<tr>
<td>1a</td>
<td>.820</td>
<td>.672</td>
<td>.670</td>
<td>5.82</td>
</tr>
<tr>
<td>2b</td>
<td>.846</td>
<td>.716</td>
<td>.713</td>
<td>5.43</td>
</tr>
<tr>
<td>3c</td>
<td>.852</td>
<td>.726</td>
<td>.721</td>
<td>5.34</td>
</tr>
</tbody>
</table>

a Predictors: (Constant), GCK.
b Predictors: (Constant), GCK, GPS. c Predictors: (Constant), GCK, GPS, MRS.

Table 4.9 presents three models generated in the regression analysis. Model 1 on the first row of the table consists only of GCK (the most influential independent variable) and the constant. The second row of Table 4.9 represents another model. This contains GPS in addition to the first model. As a result, the values of R, R² and adjusted R² are increased. This, the successive selection of each variable has been continued up to the last variable indicating that all independent variables contribute to explain the variation of PTG. Summary of the analysis shows an increment in R² in each attempt, and this increment is significant for the criterion: Probability-of-F-to-enter ≤ .050. When GPS is added to the regression model, R² increases significantly from 0.672 to 0.716 (4.4%) and similarly when MRS is added, it increases by 1% (also significant). Hence, all three independent variables contribute in explaining the proof-type geometry problem-solving skills.
4.2.8 Selection of the best fit model through hierarchical model

The value of the coefficients, standard error and their significance according to the successive selection of each variable during the *Forward Selection* method is presented in Table 4.10.

**Table 4.10 – Changes in Regression Coefficients in Hierarchical Models**

<table>
<thead>
<tr>
<th>Model</th>
<th>Un-standardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>Std. Error</td>
<td>Beta</td>
<td></td>
</tr>
<tr>
<td>1 (Constant)</td>
<td>2.883</td>
<td>1.552</td>
<td>1.858</td>
<td>.065</td>
</tr>
<tr>
<td></td>
<td>GCK</td>
<td>.073</td>
<td>.820</td>
<td>18.326</td>
</tr>
<tr>
<td>2 (Constant)</td>
<td>-2.658</td>
<td>.181</td>
<td>-1.461</td>
<td>.146</td>
</tr>
<tr>
<td></td>
<td>GCK</td>
<td>.103</td>
<td>.632</td>
<td>11.281</td>
</tr>
<tr>
<td></td>
<td>GPS</td>
<td>.329</td>
<td>.282</td>
<td>5.032</td>
</tr>
<tr>
<td>3 (Constant)</td>
<td>-6.399</td>
<td>.237</td>
<td>.976</td>
<td>4.865</td>
</tr>
<tr>
<td></td>
<td>GCK</td>
<td>.939</td>
<td>.576</td>
<td>9.684</td>
</tr>
<tr>
<td></td>
<td>GPS</td>
<td>.314</td>
<td>.269</td>
<td>4.65</td>
</tr>
<tr>
<td></td>
<td>MRS</td>
<td>.122</td>
<td>.121</td>
<td>2.492</td>
</tr>
</tbody>
</table>

Dependent Variable: PTG

Table 4.10 shows the three models that can be constructed according to their relative contributions to the PTG. The previous discussions raised the point that GCK would be the main predictor. Model 1, taking only the GCK as the predictive indicator and assuming that the full strength predicting power of 72% is contributed by it, shows that the constant is not significant at $\alpha < 0.05$ level. It seems that Model 1 is not appropriate for the prediction. Model 2 has also to be rejected as the constant is again not significant at $\alpha < 0.05$ level. The figures in Model 3 prove the appropriateness of this model in predicting PTG proving that all coefficients are significant at $\alpha < 0.05$. The significance level of regression coefficients $\alpha < 0.05$ for each independent variable shows that each coefficient is significant and that the null hypothesis $H_0$: regression coefficient $= 0$ is rejected.
4.2.9 Analysis with *Backward Elimination* Method

The appropriateness of Model 3 in Table 4.10 was retested with the *Backward Elimination* method as shown in Table 4.11.

**Table 4.11 Coefficients of the Regression analysis – *Backward Elimination* Method**

<table>
<thead>
<tr>
<th>Model</th>
<th>Unstandardized Coefficients</th>
<th>Standardized Coefficients</th>
<th>t</th>
<th>Sig.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>Std. Error</td>
<td>Beta</td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>(Constant) -6.399</td>
<td>2.337</td>
<td>-2.738</td>
<td>.007</td>
</tr>
<tr>
<td>1</td>
<td>GCK .939</td>
<td>.097</td>
<td>.576</td>
<td>9.684</td>
</tr>
<tr>
<td>1</td>
<td>GPS .314</td>
<td>.065</td>
<td>.269</td>
<td>4.865</td>
</tr>
<tr>
<td>1</td>
<td>MRS .122</td>
<td>.049</td>
<td>.121</td>
<td>2.492</td>
</tr>
</tbody>
</table>

Dependent Variable: PTG

These values can be compared with the values in Model 3 of Table 4.10, which is the appropriate model selected. No change appeared between *Backward Elimination* method and other methods. This reveals that the results are consistent across regression methods: standard; *Forward Selection*; and *Backward Elimination*.

The regression analysis suggests that all three independent variables, GCK, GPS and MRS can collectively explain the variability of PTG. Further, the main predictor is GCK and the other independent variables, GPS and MRS contribute to the increase in predictive power. That is, GPS and MRS can contribute to the development of proof-type geometry problem-solving skills. Based on these facts, the regression model can be constructed.

4.2.10 Detailed regression model

Table 4.12 provides complete details of the linear regression model for predicting geometry problem-solving skills. In addition to the partial regression coefficient (B), there is another partial regression coefficient denoted as $\beta$. This is the standard partial coefficient. It shows the relative importance of each predictor, as all variables are in the same scale when they are stated in standard forms. For instance, the respective standard values for GCK, GPS and MRS are: 0.576, 0.269, 0.121 compared to (un-standardised) B values 0.939, 0.314, 0.122.
Table 4.12 – Detailed regression model

<table>
<thead>
<tr>
<th></th>
<th>Un-standardised coefficients</th>
<th>Standardised coefficients</th>
<th>t</th>
<th>Sig.</th>
<th>95% Confidence interval for B</th>
<th>Correlation coefficients</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>B</td>
<td>Std Err.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Const</td>
<td>-6.399</td>
<td>2.337</td>
<td>-2.738</td>
<td>0.007</td>
<td>-11.014 - 1.784</td>
<td></td>
</tr>
<tr>
<td>GCK</td>
<td>0.939</td>
<td>0.097</td>
<td>9.684</td>
<td>0.000</td>
<td>0.747 - 1.130</td>
<td>0.820 - 0.606 - 0.398</td>
</tr>
<tr>
<td>GPS</td>
<td>0.314</td>
<td>0.065</td>
<td>4.865</td>
<td>0.000</td>
<td>0.187 - 0.442</td>
<td>0.703 - 0.357 - 0.200</td>
</tr>
<tr>
<td>MRS</td>
<td>0.122</td>
<td>0.049</td>
<td>2.492</td>
<td>0.014</td>
<td>0.025 - 0.219</td>
<td>0.536 - 0.192 - 0.102</td>
</tr>
</tbody>
</table>

Table 4.12 shows the complete regression model. According to this, the independent variables, GCK, GPS and MRS are linearly combined with a constant to predict PTG so that:

\[ PTG = -6.399 + 0.939 \times GCK + 0.314 \times GPS + 0.122 \times MRS \]

As independent variables were scored differently, standard values (\( \beta \) values) can be used to compare the coefficients in the same scale with mean = 0, and standard deviation = 1. In such situations, it is appropriate to use the values in the Standardised Coefficients (\( \beta \) values) column. The significance of \( \beta \) values at \( \alpha < 0.05 \) level indicate that they can be accepted as appropriate according to the corresponding t values. The confidence level column provides the anticipated statistical variation of each \( \beta \) coefficient within the 95% confidence limit of estimate for the population. The correlation coefficients are required to understand the relationships between the dependent variable and each independent variable in three different conditions: with the presence of influence of the other variables, when the influence of other variables is controlled as constant, and when the influence of other variables have been removed.

### 4.3 Validity of the results related to regression analysis

Although the regression analysis is a straightforward and powerful method, its validity rests on how well the investigation was designed and how appropriate were the data collected for analysis. The theory of regression is an abstract mathematical model constructed on a set of assumptions. As was discussed in Chapter 3, violation of
assumptions brings a threat to the validity of the prediction. Before the analysis, the researcher has to verify that variables satisfy the underlying assumptions of Linear Multiple Regression (LMR) method (Allison, 1999; Tabachnick and Fidel, 2001). Tabachnick and Fidel (2001) list the assumptions as involving the ratio of cases to individual variables; absence of outliers; absence of multicollinearity and singularity; normality, linearity, and homoscedasticity of residuals; independence of errors; and absence of outliers in the solution.

4.3.1 Ratio of cases to independent variables
This issue was taken up in Chapter 3. The sample size of the present study is 166, and it exceeds the margin of 107 set by Tabachnick and Fidel (2001), and the margin of 120 set by Norwood (2001). Thus, the ratio of cases to independent variables of Study 1 is appropriate for the use of LMR.

4.3.2 Absence of outliers among independent variable and dependent variables
Outliers are considered as unrepresentative observations and their existence is inappropriate unless the investigator is interested in other influences (Hair et al., 1995). Presence of outliers can reduce this precision and lead to faulty decisions and conclusions. The presence of outliers can shift the actual regression coefficient to an unreasonable value so that it affects generalisability (Tabachnick & Fidel, 2001). It brings a degree of risk to the precision of regression analysis. As the objective of detecting outliers is to ensure the representation of population, most often outliers are eliminated or discounted from the sample. The process for this verification is via data screening.

The usual practice of determining the presence of outliers is by using box plots. The presence of outliers is indicated outside the box plots. The SPSS program was employed to screen data. The program was set to 3 standard deviations in the 95% confidence limit to detect outliers. The Figure 4.1 shows the box plots obtained in the SPSS analysis of the data in Study 1 (in order – PTG, GCK, GPS, MRS).
4.3.3 Absence of multicollinearity

Existence of correlations among independent variables is not unusual. If one independent variable is correlated with another variable more than with the dependant variable, then the independence becomes a question. When such strong correlations exist, the regression analysis likely to be unstable (Francis, 2001). This issue is statistically addressed under *multicollinearity*. However, multicollinearity is not assessed with a correlation matrix.

The presence of multicollinearity is signalled by very large standard errors for regression coefficients or very high squared multiple correlations (SMC) among independent variables. The statistic called *tolerance* is employed to test the multicollinearity.

\[
\text{Tolerance} = 1 - \text{squared multiple correlation (SMC)}
\]

The facility to obtain the *tolerance* is available on SPSS. When tolerance is exceptionally small, the software itself warns (Francis, 2001). Apart from the low magnitude of the *tolerance*, SPSS also computes another statistic called *Variance Inflation Factor* (VIF).

\[
\text{VIF} = \frac{1}{\text{Tolerance}}
\]

Tolerance is the amount of variability of the selected independent variable not explained by other variables. … very small tolerance value (and large VIF values) denotes high
collinearity. A common value of threshold is 0.10, which corresponds to VIF values above 10 (Hair et al., 1995, p. 127)

Francis (2001) suggests that the appropriate threshold for the tolerance is 0.3. The corresponding value to threshold of VIF is 3.3. Hence, for valid variables VIF should not exceed than 3.33.

The SPSS output for the variables of Study 1 is shown in Table 4.13.

**Table 4.13 – The tolerance and the VIF**

<table>
<thead>
<tr>
<th>Independent variable</th>
<th>Tolerance</th>
<th>VIF</th>
</tr>
</thead>
<tbody>
<tr>
<td>GCK</td>
<td>0.477</td>
<td>2.096</td>
</tr>
<tr>
<td>GPS</td>
<td>0.551</td>
<td>1.815</td>
</tr>
<tr>
<td>MRS</td>
<td>0.715</td>
<td>1.400</td>
</tr>
</tbody>
</table>

As Table 4.13, shows, none of the independent variables show a value less than 0.3 for tolerance or a value greater than 3.33 for VIF. It indicates that all independent variables related to Study 1 do not violate the condition of multicollinearity.

**4.3.4 Normality, linearity, and homoscedasticity**

It is assumed that all predictive indicators in the linear regression have a linear relationship with the dependent variable. Variables that are not linearly correlated to the dependent variable may weaken the regression. Therefore, the linearity between each pair of variables has to be determined. Although some useful information related to the linearity could appear in the correlation matrix, as even nonlinear relationships can produce a number for correlation, the correlation matrix is not appropriate for screening linearity.

The most common way to examine linearity is by scatter plot matrix (Allison, 1999; Francis, 2001; Tabachnick & Fidel, 2001). The scatter plot matrix related to the variables in Study 1 is shown in Figure 4.2.
The scatter plot matrix in Figure 4.2 suggests that each independent variable has a linear relationship with PTG. The diagonal of the matrix denotes the variable labels. The relationship between the PTG and the GCK is greater than each of the relationships between PTG and the other variables. It suggests that the major determinant of PTG is GCK.

The correlation matrix provides information about the linearity of pair-wise relationships. It does not provide much information about the regression, which is the overall relationship across several variables. The linearity represents the degree to which the change in the dependent variable associated with the predictor variable should be constant across the range of values for the independent variable (Hair et al., 1995). That property is measured using residual scatter plot. The scatterplot of the residuals is the graphical representation of the relationship between standardised residuals and predictive values. The existence of any relationship between the residuals and predictive value automatically adds another predictor. This means the prediction has to be independent of the residuals. If the scatterplot shows any pattern, then collinearity does not exist.

The scatter plot related to the present study is shown in Figure 4.3.
Figure 4.3 – Scatterplot of the residuals

The scatterplot shown in Figure 4.3 does not represent a pattern between predicted value and residuals. It suggests that the validity of Study 1 is confirmed in terms of non-linearity.

The homoscedasticity refers to the variance of the residuals. If homoscedasticity exists, then the variance becomes constant. In such instances, the scatterplot of the residuals does not show any pattern. As Figure 4.3 shows, the dependent variable of Study 1 satisfies the assumption of homoscedasticity.

For the validity of analysis, residuals should be normally distributed. This is a fundamental assumption, as the normality relates to the tests for significance. Hair et al. (1995) state that this is the assumption that could be violated very frequently. The diagnosis for this condition is done with the standard normal curve related to the residual histogram.

Figure 4.4 shows the distribution of residuals related to Study 1. It implies that residuals of PTG approximates to the standard normal distribution with 0 mean and .99 standard deviation. It confirms that the independent variable meets the conditions of normality.
In summary, the data set ensures the validity of the regression analysis in that the dependent variable (PTG) and independent variables (GCK, GPS and MRS) satisfy all assumptions: appropriate ratio of cases to individual variables; absence of outliers; absence of multicollinearity and singularity; normality, linearity, homoscedasticity of residuals; independence of errors; and no outliers in the solution.

### 4.4 Summary of results

The standard multiple linear regression analysis showed that the independent variables of the study: Geometry Content Knowledge, General Problem-Solving processes, Mathematical Reasoning Skills have significant effect on the dependent variable, Proof-Type Geometry problem solving skills. Furthermore, the results indicate that all three variables can explain almost 73% of students’ scores on geometry proof-type problem-solving skills. The hierarchical regression analysis procedures confirmed this result and generated details of individual contributions. Accordingly, 67.2% of the contribution comes solely from Geometry Content Knowledge. The contribution of Geometry Content Knowledge could be increased by 4.4%, when effective General Problem-Solving processes are added into the analysis. The gain can further be extended by 1% by improving Mathematical Reasoning Skills.

The F test performed according to the analysis of variance showed that the related statistic $R^2$ correspondent to this value is significant enough to represent the variation of proof-type geometry problem-solving skills in a linear relationship of the three independent variables.

The model generated by the analysis

$$PTG = - 6.399 + 0.939GCK + 0.314GPS + 0.122MRS$$
The model describes the amount of variation in PTG that could be affected by 1 unit of independent variables. According to this, increasing GCK by 1 unit can make 0.939 unit increment in PTG. Similarly increasing GPS by 1 unit can make 0.314 unit increment in PTG whereas increasing MRS by 1 unit can make 0.939 unit increment in PTG.

This model yields the following conclusions:

1. *Geometry Content Knowledge* (GCK), *General Problem-Solving processes* (GPS) and *Mathematical Reasoning Skills* (MRS) are predictive indicators of the success of students’ proof-type geometry problem solving skills.

2. *Geometry Content Knowledge* (GCK) is the major determinant of the success of proof-type geometry problem solving.

3. Both *General Problem-Solving processes* (GPS) and *Mathematical Reasoning Skills* (MRS) can promote the contribution of *Geometry Content Knowledge* in the success of proof-type geometry problem solving.

4. Emphasis on mathematical reasoning in the instructional process for proof-type geometry problem solving yields less but significant improvement.

### 4.5 Detailed description of solution attempts

The linear multiple regression analysis generated three predictive indicators of success in proof-type geometry problem solving: *Geometry Content Knowledge*, *General Problem-Solving processes* and *Mathematical Reasoning Skills*. The results further showed the relative contributions of these three variables are not equal. Instead, a greater contribution comes from content knowledge. The contribution of *General Problem-Solving processes* in proof-type geometry problem solving is also significant.

These results of *Study 1* support the findings of previous research by Charalambos (1997), Chinnappan (1992), Harel and Sowder (1998), Riess, Kleime and Heinze (2001), and Senk (1985). The following section provides descriptions of individual student problem solving attempts where the interactions between content knowledge, general processes, and mathematical reasoning are identified and analysed.

#### 4.5.1 The influence of content knowledge in geometry proof-type problem solving

The greater contribution of content knowledge to success in proof-type geometry problem solving among students is an important result of the analysis. The linear
multiple regression analysis showed not only that *Geometry Content Knowledge* is the major determinant of the success of proof-type geometry problem solving. The analysis also indicates that about 67% of that success is attributable to *Geometry Content Knowledge*.

The correlation coefficient between content knowledge and proof-type geometry problem solving was found in this study to be 0.82. In a comparison study carried out in Australia, Chinnappan (1992) obtained virtually the same correlation coefficient of 0.83. Both studies point to the fact that success in proof-type geometry problem solving has a significant positive relationship with *Geometry Content Knowledge*. This suggests that improving *Geometry Content Knowledge* will result in the success in students’ geometry problem solving performance in general.

This result is consistent with that reported by Senk (1985). In that study, which involved the participation of 2567 students from the United States, Senk obtained a value of 0.67 for Pearson correlation coefficient between proof-type geometry problem solving and *Geometry Content Knowledge*. This further demonstrates the importance of *Geometry Content Knowledge* in successful proof-type geometry problem solving. Senk (1985) also suggested the need to prepare students with *Geometry Content Knowledge* that is based on van Hiele theory.

Figure 4.5 illustrates a typical successful solution attempt by a student who participated in the present study. The answer shown in Figure 4.5 provides further support to the claim made by Reiss, Klieme and Heinz (2001) that claims methodological knowledge to be a prerequisite for proof-type geometry problem solving. The student has demonstrated all forms of methodological knowledge: knowledge about proof scheme (Harel and Sowder, 1998), proof structure and correct logical chain. In other words, the answer contains the correct mathematical proof procedure. The answer also demonstrates that the student has knowledge about geometric concepts, relationships and diagrams. Now it is important to see how these features could exist in the answer.

The answer shown in Figure 4.5 demonstrates that the solving process requires content knowledge and mathematical reasoning skills. For instance, the student has combined two known relationships: AP = CQ (given) and AB = DC (opposite sides of the parallelogram) to deduce a new relationship: BP = DQ. This demonstrates formal deductive reasoning. The deduction was made on the basis of mathematical reasoning in
order to decide that subtraction is appropriate. Then the student has established a logical chain: if $AB = DC$ and $AP = CQ$ then $AB - AP = DC - CQ$ leading to $BP = DQ$. To do this, the student needs to have content knowledge about the properties of parallelograms and axioms. In addition, the student has represented information on the diagram showing knowledge related to the geometric diagram.

Figure 4.5 A successful solution for Question No. 5

Proof-type geometry problems have their own solving standards and conventions. The content knowledge related to geometry problem solving includes methodological knowledge that contains knowledge about geometric concepts, knowledge about geometric relationships and knowledge about geometric diagrams. Knowledge related to this methodology has been acknowledged by Reiss et al. (2001) as a part of content knowledge associated with proof-type geometry problem solving. Without adequate applications of formal deductive reasoning, the student cannot demonstrate evidence in the construction of a proof. For instance, the answer in Figure 4.6 demonstrates a lack of methodological knowledge by one of the students.
The answer in Figure 4.6 shows three difficulties resulting from lack of content knowledge that is required for proof-type geometry problem solving. First, any proof-type problem in geometry could not be solved without a diagram. Second, the student does not know that the proof represents a generalized certainty, and the difference between mathematical proof and verification. Third, the student does not know the conventions of presentation. In summary, the student does not possess the appropriate content knowledge related to methodological knowledge.

4.5.2 The influence of general processes in proof-type geometry problem solving

General processes are: analysis, representation, planning and use of knowledge retrieval in generating new information.

This section provides evidence to demonstrate how successful students have used these processes during the solving process or how students have faced difficulties if they did not possess each of these skills. The section also discusses why each of these skills is important in the problem-solving process through evidence drawn from students’ solution attempts. The following examples of student efforts in proof-type problem solving highlight the importance of the above processes of general problem solving.

During proof-type geometry problem-solving process, the student has to convert text-based information into diagrammatic form. For this conversion, the student has to understand the problem. Figure 4.5 provides evidence that the student has correctly recognised the problem information such as parallelogram, its name (ABCD), locations of P and Q, perpendicular distance (to BD). Without skills in analysis process, the student cannot understand the problem. This emphasises that analysis process is essential in making a start and further progress.

Secondly, the student has represented the problem information as a diagram. The representation process seems to be content knowledge-dependent. The student has
converted geometric information in the problem situation from one form to the other. Skills in diagrammatic representation are not confined to converting text information into diagrammatic form; they are also required to generate goal-directed new information. This particular student has marked newly found equal segments and alternate angles.

In the *planning* process, the student has identified all structural steps that were not related to any particular algorithm. In order to prove the triangles PBX and DQY to be congruent, the student has planned to prove DQ and PB to be equal as a sub-goal. One of the effective ways to identify and achieve this sub goal is to work backward. This process shows the influence of planning in proof-type geometry problem-solving processes.

The solving process related to the answer presented in Figure 4.5 exemplifies the *use of knowledge retrieval*. Retrieving appropriate knowledge, and accurate use of those retrievals are influenced by metacognitive skills. In the answer shown in Figure 4.5, the student has proven the ability to access and retrieve required theorems and required geometric concepts, and to use them in generating new information in a goal-directed manner.

### 4.5.2.1 Role of analysis process

The results show that *analysis* is a critical process. Proof-type geometry problems are presented in text-based form. The student has to recognise the meaning of key words and phrases where content knowledge is required. As the *analysis process* is mostly content knowledge driven, the student cannot make progress in the solving process (Hayes, 1989) without recognizing important information.

When the *analysis* is erroneous, a successful solution cannot be expected. Figure 4.7 illustrates how students fail in the problem-solving process when they cannot analyse the problem properly.
Figure 4.7 A wrong answer caused by erroneous analysis

The answer illustrated in Figure 4.7 shows that the student could not analyse the problem appropriately. The student has erroneously recognised P and Q as mid points. In the diagram it is evident that the student was also not able to recognise the goal. The goal, ‘prove that the perpendicular distance to BD from P and Q are equal’ needs to be understood through analysing the semantic organization of the sentence. Absence of the use of analysis has led the student to generate an incorrect answer.

4.5.2.2 Role of representation process

Problem solving and representation have a continuous relationship. It is clear that after the analysis process, the configuration of problem information needs to be changed in order to plan for further actions. This change almost always involves some kind of simplification and transformation. Representation in proof-type geometry problems can have two major effects. First, it helps students to gather and integrate all bits of relevant important information into a smaller space so that solver can view it as a coherent whole. Second, it provides insight into useful relationships.

Representation mostly takes place as the result of comprehension of the relationship between the text form and the diagrammatic form. In a think-aloud protocol analysis, Chinnappan (1992) observed that representation was an important process that distinguishes high achievers from low achievers. It highlights the contribution of
content knowledge in diagrammatic representation. When the diagram is provided with the problem statement the problem becomes simpler to some extent. Activation of the reasoning process may be expected during the completion of the diagram. As a strong means to non-verbal reasoning (van Hiele, 1999), diagrams promote the generation of new information. Content knowledge does not ensure success in proof-type geometry problem solving without skills in diagrammatic representation.

Lack of content knowledge directly influences the *representation process* in several ways. Although the student possesses the content knowledge to analyse the problem, deficiencies in this knowledge might cause difficulties in *representation*. The attempt shown in Figure 4.8 is an example of this situation.

![Wrong representation of perpendicular distance from P to BD ...](image)

**Figure 4.8** An example of how insufficient content knowledge affects the *representation*

Figure 4.8 shows that this student has identified the parts of the problem to a great extent, but one defect in the content knowledge has inhibited the solution process. This may be due to lack of understanding of the concept of ‘the perpendicular distance from (P) a point to (BD) a line’.

The results of proof-type geometry problem-solving tests provide evidence where the written proof violates the notion of generalisability when the student tends to use specific cases of figures (Charalambos, 1997). Figure 4.8 illustrates such a case. Although this proof is not valid, the student could have constructed such a proof. Taking P and Q as mid points which meets the given information that AP = CQ could also lead to a solution. Instead of the general solution however, it results only in specific relationships such as AP = PB. This solution distracts the solver and the solution process deviates from the goal.

Another deficiency that students frequently demonstrate involves prototypical effect. In geometry, students specifically rely on superficial visual features, such as the fact that
angles look the same in the diagram (Aleven & Koedinger, 2002; Charalambos, 1997). These visual features sometimes reflect an understanding of unexpected relationships such as parallelism, perpendicularity, equal sides or equal angles that do not necessarily exist. Students provided evidence for this effect at various instances. The common prototypical figures reduce the generalisability of the proof. For instance, Problem 5 of the *Geometry Problem Solving test* is a problem involving parallelograms. Drawing a rectangle instead of a parallelogram is too specific, and not accepted as a proof. Figure 4.9 shows such an instance.

![Figure 4.9 Some examples for prototypical effect of diagrams](image)

Drawing of parallelograms with incorrect horizontal orientation is another common prototypical configuration. This student has drawn such a parallelogram with an angle of approximately 60°. Consequently, visual properties generate a straight line-like appearance to QWX and ZYP, which is not necessarily expected.

![Figure 4.10 Some examples for prototypical effect of diagrams](image)

The diagram shown in Figure 4.10 was used to generate an answer. Taking XW extended to Q, and ZY up to P, the student can prove that QW and PY are equal and parallel, which results in QWYPY being a specific parallelogram.

The diagram should also be intelligible. It should not display wrong information. Figure 4.11 provides an instance of such a situation.
This diagram does not display the relationship that \( AP = CQ \). Instead it looks like \( AP = DQ \), which is incorrect information. As a result, the student has decided that \( PQ \) is parallel to \( AD \) and \( BC \). The student could not achieve the goal with this form of representation.

Sometimes a geometric diagram becomes very complex as it contains too many lines, angles, and geometric shapes. Students with poor diagrammatic reasoning and representation find it difficult to identify and select the appropriate parts from the diagram. Figure 4.12 shows how a student has made the diagram more complex by adding irrelevant lines.

This student was not able to label the required points and related right angles. As a result, even though all other relationships were recognised, the student was not able to achieve the goal.

In summary, lack of content knowledge tends to produce inappropriate representations, which can generate a solution that is not generalisable, or one that leads to an incorrect solution outcome. In addition, inappropriate diagrammatic representations inhibit generation of new information and generating insights about critical relationships relevant to the solution.
4.5.2.3 Role of the planning process

During the planning process, the student searches for geometric relationships that are useful in the solving process. Planning facilitates the inferential process and aids students in the exploration of a pool of relevant rules. These rules are axioms, definitions, theorems and other geometric and mathematical relationships. These are directly related to content knowledge in the proof-type geometry problem-solving process.

Planning is a deliberate process and requires more reasoning than analysis and representation. Because of this students find it difficult to activate this strategy. The answer shown in Figure 4.13 constitutes evidence of this.

![Figure 4.13 Examples for how insufficient content knowledge impedes the planning process](image)

As shown in Figure 4.13, the student has analysed and represented the given information appropriately. However, the student’s reasoning process could not make further progress towards proving the congruency between triangles, PBY and DQX.

A common heuristic used in such instances in solving proof-type geometry problem solving is that of adding auxiliary objects such as extending a line, drawing a perpendicular. Such a heuristic is necessary to solve some problems. It works for some problems but not always. Sometimes an auxiliary line has no role at all. A student has suggested a useful auxiliary for Problem 2. Figure 4.14 illustrates such a solving attempt.
The given information for this problem was $\angle AOC = \angle BOD$. To prove that $\angle AOB = \angle COD$, the student has added an auxiliary line OP as the bisector of the $\angle BOC$. This heuristic does not simplify the problem as it extends the single step to more steps as shown in the Table 4.14.

Table 4.14 Example showing a less effective heuristic

<table>
<thead>
<tr>
<th>Without auxiliary</th>
<th>With auxiliary</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\angle AOC = \angle BOD$ (Given)</td>
<td>Construction: Draw OP the bisector of BOC</td>
</tr>
<tr>
<td>$\angle AOC - \angle BOC = \angle BOD - \angle BOC$</td>
<td>$\angle BOP = \angle POC$ (OP is the bisector)</td>
</tr>
<tr>
<td>(Axiom: deducted the same)</td>
<td>$\angle AOC = \angle BOD$ (Given)</td>
</tr>
<tr>
<td>$\angle AOB = \angle COD$</td>
<td>$\angle AOC - \angle BOP = \angle BOD - \angle POC$</td>
</tr>
<tr>
<td></td>
<td>(Axiom: deducted equals)</td>
</tr>
<tr>
<td></td>
<td>$\angle AOB = \angle COD$</td>
</tr>
</tbody>
</table>

As illustrated in the Table 4.14, the auxiliary was cleverly suggested, but this student could not make use of it in solving the problem. This example provides rich information about heuristics was activated during the planning process. First, it supports the earlier claim that heuristics cannot be taught as a rule. In the planning process, this student has used a ‘rule of thumb’ as the student could not think of a strategy such as subtraction. Second, the heuristics are not global strategies, and usefulness may be limited. Third, this shows the value of planning process when the student has to search the problem space for a strategy and sequence a series of moves involving further information generation.
4.5.2.4 Role of use of knowledge retrieval process

Use of knowledge retrieval is also based on content knowledge. It is process that involves the construction of new information on the basis of given problem information. For the problem-solving process, the student has to retrieve previously acquired knowledge. This knowledge is used to transform problem information into the goal. Retrievals could include: theorems, properties of geometric figures, definitions, axioms, and mathematical operations such as addition. Most of these are related to Geometry Content Knowledge. Students can effectively activate use of knowledge retrieval process at any stage of their solution attempt.

For instance, use of knowledge retrieval could accompany planning. Although they are successful in previous processes, planning process could be difficult for some students. Their answer scripts reveal reasons for failure in the use of knowledge retrieval process. A major reason is the problem of generalisation. Students use specific triangles such as equilateral triangles, right triangles and isosceles triangles in proof-type problems that are about any triangle.

Planning is required during analysis and representation. During the analysis process, it is required for identification and recognition. In the representation process, it is necessary to convert text-mediated information into diagrammatic information. In contrast to other processes, planning is used here for the purpose of transformation. Figure 4.15 shows an answer to Problem 5 by a student who seems to have a reasonable content knowledge.

This student knows, that the triangles QYZ and WPX have to be proved as congruent in order to achieve the goal. The student could not retrieve knowledge about some properties of a parallelogram to establish the angle requirement as $\angle QZY = \angle WXP$. 
4.6 Proof-type geometry problem solving

The discussion in the previous section highlighted how students failed to perform successfully when their content knowledge was weak or used inappropriately. It also highlighted some instances where students failed to solve problems although they appeared to have sufficient content knowledge. This was also evident in the research reported by Lawson and Chinnappan (1994) which showed that a group of less successful high school problem solvers failed to access available knowledge independently, but were able to do so when prompted.

These shortcomings related to the proof-type geometry problem-solving process can be categorised according to where students experienced difficulties in

(a) accessing available knowledge that is required to identify important parts of the problem and recognise implicit meanings;

(b) representing identified information appropriately;
(c) inferring a goal-directed path, and its transit points when algorithms are not available for exploring the diagram;

(d) generating new information from rules to transform available information into new information.

Research studies have identified two forms of metacognitive processes that are involved in the solution process (Kwang, 2003; Schoenfeld, 1985; 1992; Schraw, 2001; Sternberg, 2001): knowledge of cognition, and regulation of cognition. Knowledge of cognition involves selecting the right cognitive resources. In the problem-solving process, these resources are used in the processing of information. Regulation of cognition refers to planning (pre-active phase), monitoring (interactive phase), and evaluation (post-active phase) in processing.

The nature of the proof-type geometry problem-solving process demands more metacognitive involvement because of two requirements. First, unlike other types of mathematical problems, proof-type geometric problem solving always involve the active use of reasoning. As discussed in relation to algorithmic problem solving, some students can retrieve schemata that contain rich information. Because these schemata are information-rich, the student does not need to retrieve information frequently. In contrast, during proof-type geometry problem solving the student has to retrieve required knowledge components at each point the process. The knowledge retrieval process requires metacognitive involvement (Riess et al., 2001).

Second, an algorithm contains rich instructions and therefore aids the solution process. In such instances, the student does not need to pay more attention to regulation because of the rich instructions. The outcomes of non-algorithmic problems such as proof-type problems have to be constantly monitored. This process also requires metacognitive involvement. Especially, in the planning process, the student needs frequent feedback on the appropriateness of inferences.

Problem solving requires knowledge and reasoning. Knowledge refers to domain-specific knowledge (content knowledge) as well as domain-general knowledge (Heuristic). Conditional knowledge links domain-specific knowledge and domain-general knowledge as it provides information about conditions (when and where) applied.
The reasoning process is important in selecting what type of knowledge needs to be used on the basis of conditional knowledge. Metacognitive awareness enables this process. Another requirement appears to be important in relation to the selection of appropriate knowledge. This involves metacognitive control that governs the other cognitive processes that are activated during problem solving.

4.7 Summary

The primary aim of Study 1 was to address the research question: what are the predictive indicators of successful proof-type geometry problem solving? In order to address this question, a multiple linear regression analysis was carried out using three independent variables. The dependent variable was Geometry Proof-Type problem-solving skills (PTG). The independent variables were General Problem-Solving processes (GPS), Geometry Content Knowledge (GCK), and Mathematical Reasoning Skills (MRS).

The results show:

1. That Geometry Content Knowledge, General Problem-Solving processes and Mathematical Reasoning Skills are predictive indicators of successful proof-type geometry problem solving.

2. That among those predictive indicators, Geometry Content Knowledge is the principal predictive indicator, whereas General Problem-Solving processes and Mathematical Reasoning Skills are the other predictive indicators in the order of significance.

3. That these three predictive indicators can collectively describe 72% of the success of proof-type geometry problem solving.

4. The linear relations among the variables can be represented by the regression model:

   \[ \text{PTG} = -6.399 + .939 \text{GCK} + .314 \text{GPS} + .122 \text{MRS} \]

The above results provide strong support for including content knowledge and general processes such as analysis, representation, planning and use of knowledge retrieval in teaching programs that foster learning proof-type geometry problem solving among high school students. Issues about how to incorporate them into a learning-support environment will be addressed in following Chapters 5 and 6.
Chapter 5: Development of a learning environment

5.0 Introduction

The extensive literature review in Chapter 2 identified instructional needs such as content knowledge including diagrammatic reasoning, geometric reasoning (particularly at VHL3), general problem-solving processes and metacognitive skills. It also highlighted student-based factors that could influence the success of proof-type geometry problem-solving process. Study 1 provided empirical support via multiple linear regression analysis for the existence of three predictive indicators for success in proof-type geometry problem-solving: content knowledge, general problem-solving processes, and mathematical reasoning.

In keeping with the overarching purpose of this study to design a learning environment that helps students solve proof-type geometry problems, the instructional needs and predictive indicators provide a starting point for practical translation into the design of a supportive classroom environment that takes advantage of current advances in information and communication technology (ICT).

A non-linear approach has been identified as one of the appropriate strategies to support students (Chen, 2002), and it could be argued that the classroom teacher has long performed this role. However, due to the wide variation in students’ content knowledge required for proof-type geometry problem solving across van Hiele levels (Senk, 1985; Shaughnessy & Burger, 1985), it is simply not possible for one classroom teacher to concurrently meet the unique needs of approximately thirty students. However, with additional resource support, it is likely that the teacher’s ability to meet the needs of their students can be enhanced.

This chapter presents Part I of Study 2: the process of translation of the instructional needs identified into a Web based environment to support students solving proof-type problems in geometry. Study 2 addresses the research question:

(Q2). Based on needs with an emphasis on formative evaluation, what is one design solution to support students solving proof-type problems in geometry?

The first section of this chapter will present a review of literature related to recent developments in ICT focusing on hypertext, multimedia, and constructivist learning
environments. It also revisits the literature review to provide a brief note about worked examples and related strategies. The second section presents the data collection and analysis methodology for Part I of Study 2. The last section presents the results of Web based prototype design and development along with a discussion of prototype features. These features are formatively evaluated in Part II of Study 2, presented in the following chapter (Chapter 6).

Information and communication technologies potentially offer alternate instructional strategies by extending the resources available to both students and teachers. Some broad findings of recent developments in ICT are now explored to justify the selection of a Web based resource development environment.

5.1 Recent developments in ICT

Over the last two decades, core literature in ICT has shifted from terms like tutor (McAleese, 1986), learner control (Banderson & Inouye, 1987; Gentry & Csete, 1991) and navigation (Hedberg, Harper & Brown, 1993) through hypertext (Jonassen, 1986) and multimedia (Ellington, Addinal & Caudill, 1986)) to the World Wide Web (otherwise referred to as the “Web”). There has been a parallel shift in language associated with learning theory – a shift from terms like stimulus and feedback in programmed instruction (Shroock, 1991) to terms such as learner engagement, social negotiation and higher-order thinking (Mayer, 2003). Concurrent with the realisation that students should be active in knowledge construction in a constructivist learning environment (Jonassen, 1997) was the appreciation that the role of the teacher has shifted from information transmitter to classroom facilitator or co-learner.

Three terms from this collection are explored:

- **Hypertext** - Given the high volume and vital role of content knowledge for students solving proof-type problems in geometry, hypertext structures potentially offer a mechanism to access information on a need basis.
- **Multimedia** - Given the visual complexity of diagrams, multimedia offers a way to present and concurrently explain visual information.
- **Constructivist learning environments** - Given the non-algorithmic nature of proof-type problems, a collaborative and supportive constructivist environment offers a way for students to discuss initial strategies and construct the solution. These three terms are further explored.
5.1.1 Hypertext

Early forms of programmed teaching were branched, but the materials contained linear and sequential instructions. It was very difficult to develop non-linear learning environments with old technologies:

Traditional computer-based learning (CBL) systems are linear in their format and hence do not support the associative nature of the human mind. They offer very little referential branching that is slow and inconvenient to follow. The inadequacy of linear text for representing referential and associative links is one of the main arguments for the development of hypertext-based learning systems (Khalifa and Quock, 1999, p. 196).

Hypertext-based learning environments were useful to increase the range of navigational options to make the learning environments non-linear:

Hypertext is a natural medium for information access. …Using hypertext as access structures, users may also browse through related documents to acquire any information on any subject at any time. Access structures provide visual structural cues that can signal the structure of the text and facilitate access to it (Jonassen, Dyer, Peters, Robinson, Harvy, King & Loughner, 1997, p.119).

A number of researchers have identified positive aspects of the hypertext non-linear structure. Jacobson and Spiro (1995) studied active learning (learner engagement) in hypertext learning environments and found strong positive effects on certain transfer tasks. They argue that the advantage of hyper-based material is that it allows learners to be active and self-directed in the way they access knowledge. Rouet (1994) reports that students improved their domain knowledge and concludes that even inexperienced students benefited with hypertext environments. Jacobson, Maouri, Mishra & Kolar (1996) demonstrated the potential for hypertext learning environments to improve learning outcomes.

Despite positive findings, some consider that the effects of hypermedia are limited to tasks in which learners search for and manipulate information, and the distribution of these effects may differ across learners according to their ability levels (Dillon and Gabbard, 1998).

From the instructional perspective, the purpose of hypertext is to organise information in a coherent manner so that students can access it on a needs basis. From the learner perspective, hypertext provides increased access to information. Hypertext is not a form of instruction, but it can indicate the presence of related information. Selection of a
hyperlink is not considered as interaction either, because what is important is cognitive interaction with the material (Schnotz & Lowe, 2003).

Research has shown that hypertext is not effective unless the student is cognitively active in the process of knowledge construction. However, material quality is still important. Zumbach and Reimann (2002) emphasise that hypertext is effective only when linked to quality resources.

5.1.2 Multimedia

In its most simplistic form, multimedia refers to the use of multiple media formats to present information—such as combinations of text, graphic, sound, animation and video (Hackbart, 1993).

Others interpret the term multimedia according to the context in which it is used. Mayer & Moreno (2002) and Schnotz & Lowe (2003) report three types of such contexts: technical, semiotic, and sensory. In the technical context, multimedia refers to physical devices such as computers that can carry multimedia signs. The semiotic context of multimedia refers to external representational formats of information such as text, graphic, and sound. The third context refers to sensory modality of internal sign representation.

Multimedia can be defined in terms of sensory modalities (eg. visual vs. auditory), representational mode (eg. pictorial vs. verbal), or delivery media (screens vs. speakers) (Mayer & Moreno, 2002, p. 88).

Schnotz and Lowe (2003) assert that research in learning and instruction should focus on the second and third contexts. The present study is in the second context as it deals with the complexity of concepts and diagrams in geometry.

Multiple-representation is a major consideration in multimedia. The effectiveness of multiple representation has long been researched. Presenting text with illustrations was found to be more effective than presenting the text alone (Larkin & Simon, 1987). Schontz and Bannert (2003) argue that the success of multiple-representation depends on various factors such as relevance of the partial role of each presentation format within the message, and the context of the subject matter.

Multimedia can convey messages more effectively and powerfully than single formats. In a review of three studies, Moreno and Mayer (1999) found that learning is not
guaranteed with multimedia unless research-based design principles are applied. In a subsequent theoretical analysis, they reconfirm the same:

Computer-based multimedia learning environments consisting of pictures (such as animations) and words (such as narration) offer a potentially powerful venue for improving student understanding. However all multimedia messages are not equally effective, so our focus is how to design multimedia messages that promote meaningful learning (Mayer & Moreno, 2002, p. 107).

The aforementioned design principles are found in a cognitive theory put forward by Mayer (2001). It predicts that students learn more deeply:

(i) with pictures and words rather than words only;
(ii) when extraneous materials is excluded rather than included;
(iii) when printed text is placed close to rather than far from the picture;
(iv) when words are presented in a conversational rather than formal style.

However, learning is active processing of information and construction of knowledge built upon prior knowledge. Carefully prepared multimedia material full of multiple representations remains a collection of information unless the information in the multimedia material is processed into knowledge through active cognitive engagement (Khalifa and Quock, 1999). This is discussed further in the next section.

5.1.3 Constructivist approach to design of ICT learning environments

Constructivist theory emphasises that knowledge is constructed by learners through active cognitive engagement and interaction with the environment. According to Cunningham, Duffy & Knuth (1993), constructivist learning environments:

• provide students with experience with the knowledge construction process;
• provide experience in and appreciation for multiple perspectives;
• maintain the authentic context of the learning task;
• allow for a student-centred learning process whereby students play an important role in setting the goals for learning;
• provide for collaboration;
• use multiple modes of representation; and
encourage metacognitive and reflexive activities.

Because learning involves the active construction of knowledge, the teacher has to facilitate the process. This process involves personal factors such as active cognitive engagement, knowledge construction, and existing knowledge. Because of these personal factors, the teacher cannot expect a homogeneous classroom with equal abilities. In a normal classroom, students are at different ability levels and they have different processing paces. This situation raises two complementary and paradoxical needs that must be met for reducing disparities among students. One is self-paced self-regulation of learning which features individualisation. The second is negotiation for and sharing of knowledge among students and groups which feature socialisation (Resnick, 1989). The instructional process in the normal classroom has to cope with this paradox and implement appropriate tactics.

In a constructivist learning environment, the teacher has no role in transmission of knowledge, as students construct it. In that sense, autonomy and the ownership of learning reside with the student. As the teacher facilitates the process, the teacher becomes a learning partner in the learning process which targets higher-order thinking such as problem solving and transfer of problem solving knowledge (Mayer and Moreno, 2003). Learning to solve problems, solving problems and transferring problem solving are also the tasks of students. In this context the teacher becomes an expert partner to the student.

Learners interact with new information, interpret and build new personal knowledge representation by relating the new information to their prior knowledge. Information itself is not knowledge, but rather the stimuli. Knowledge is constructed through the cognitive processing of information (Khalifa and Quock, 1999, p. 196).

Web based instructional strategies may be useful to engage students through active participation in learning (Jacobson & Spiro, 1995; Jonassen & Wang, 1993), by multiple means of communication. Navigational options such as hypertext and hypermedia can be used to select and learn.

Mayer (2003) states that the role of multimedia in technology-based environments is to serve as a tool that enhances learner-centred learning environments for students to construct learning.
Interactive learning environments, if well designed, can support learner construction of knowledge through problem solving experience or through more creative expressions (Hedberg and Harper, 2002, p.89).

Multiple information representation strategies can support instructional design (Mayer, 2003; Moreno and Mayer, 1999). Visual representations such as texts, graphics, movies, and animations are powerful strategies. Information-highlighting strategies such as colours, sizes, locations, thickness, 3D effects, and dynamic features promote the knowledge-construction process.

Web based instructional strategies also promote the use of metacognitive skills (Zumbach & Reimann, 2002). As was discussed earlier, the proof-type geometry problem-solving process requires non-linear instructions. To learn in a non-linear learning environment, the student should have and activate skills in metacognition.

Learners are required to be active participants in their own knowledge construction in constructivist environments. This demands metacognitive skills to self-monitor and self-regulate, but it also allows learners to be more self-regulated. In a classroom of thirty students, all at differing levels of understanding with geometry, the ability to provide resources to meet these differing needs could potentially be met with an appropriately designed ICT environment. Thus design was pursued in principle.

The specific nature of geometry content knowledge as it relates to proof-type problem solving provides the scope or focus for ICT-based resource design in the present study.

5.2 Worked Examples as a strategy for problem familiarisation

Worked example method is an acknowledged instructional strategy for solving well-structured problems in well-structured complex domains such as mathematics, computer programming, and physics (Chi, Bassok, Lewis, Reinmann & Glaser, 1989; Renkl, 1997a, 1997b; Sutherland, 2002; Wong, Lawson & Keeves, 2002). Worked examples reveal solutions generated by an expert in the domain therefore it can be assumed that worked examples contain information about expert strategies, thinking and procedures. Jonassen (1999; 2003) states that worked examples that model expert problem-solving strategies and desired performance can promote constructivist learning approaches. This strategy has been shown to have positive effects (Chinnappan and Lawson, 1996; Lawson, 1991) on executive processing of problem solving in the domain of geometry.
Faded examples

Faded examples strategy refers to gradually reducing the amount of help in a series of worked examples. Faded examples provide more guidance than a regular problem, but give less guidance than a completely worked example, as the student may still require guidance for the rest of the worked example.

Renkl, Atkinson and Maier (2000) compared two student groups; one worked with faded worked examples, whereas the other with traditional worked examples. The results of this study confirmed that the group of subjects exposed to faded examples learned more effectively than the other group, even though both groups had similar pre-test performances. A more thorough, but similar, experiment produced comparable results.

The design of an ICT-based resource to support students solving proof-type geometry problems required attention to the non-algorithmic but well-structured nature of the problems. Fading of worked examples was one possible structure to consider.

5.3 Design methodology – Part I of Study 2

Ill-structured problems are complex, requiring multiple approaches, strategies, paths and solutions. Learners are often required to make judgements and express personal opinions and beliefs about the problem (Meacham & Emont, 1989). Ill-structured problems are also ill-defined in the sense that one or more of the problem elements are not known (Wood, 1983). Cognitive Flexibility Theory (Spiro, Feltovich, Jacobsen and Coulson, 1992) emphasises the inadequate nature of the initial information found in ill-structured problems. Unlike well-structured problems, ill-structured problems do not possess unique solutions, and instead the solution can be expected from a range of possible answers. The solution process demands continuous iterative improvement, so at different stages in the solution process, the problem must be reviewed and re-interpreted in the light of new understanding.

The design and development of a prototype environment to support students solving proof-type geometry problems typifies the ill-structured problem-solving process. Figure 5.1 illustrates an appropriate production cycle for such a prototype development.
The successive iterations of the production cycle increase the quality of the solution. Another way of coping with the lack of problem information is by incorporating different views, experience and expertise. Increasing the number of participants is appropriate in this regard as each member analyses and represents the problem differently and contributes different expertise, resources and strategies.

Group prototype development is presented here with the benefit of hindsight as a rather systematic process. It reflects the specific nature of the group involved in the design, the available resources at different stages in the design process, and the unique and varied nature of the development environment. The aim is to develop a meaningful rather than an accurate solution to the problem. Multiple perspectives on the complex design process emerge from three key people: the investigator, Supervisor_Tech and Supervisor_Math.

5.3.1 Participants

The investigator was a participant in all meetings and was involved in generating data. His experience as a secondary mathematics teacher, teacher educator, examiner and developer of distance resources provided raw materials for instructional design. The investigator also had some knowledge and experience in web page development and use of software for learning.

Supervisor_Tech possessed expertise in ICT in education. She provided the leadership, guidance and critical supervision throughout the software development process. She had a proven record in the field of development of ICT-supported learning environments ranging from introducing pre-service teachers to the benefits of ICT for learning, to providing leadership in large-scale software production.
Supervisor_Math was an expert in mathematics, mathematics education and researcher in geometry problem solving. He had a substantial role in ensuring the quality of the learning environment. He had lengthy experience in teaching high school mathematics as well as mathematics education and developing learning environments at university level. The research that the Supervisor_Math engaged in was mostly related to high school geometry education with some leadership in projects on use of technology for mathematics education.

5.3.2 Data collection

Richey and Nelson (1996) assert that the development process in a developmental study is itself a form of study. The reporting style of the development phase of most studies is focused on describing the developmental process. Although documentation provides a source of data for subsequent analysis of the development process, its prime purpose is to provide developmental information of use to anyone who wishes to pursue a further design. For this reason such documents can be considered reliable and authentic data sources that require no pre-determined data collection instruments.

Participants met to contribute expertise to the development process through their experience and readings. Meeting notes and relevant literature were stored as documentation. Once web based prototype development began, each version of the web site was dated and archived. Subsequent development began with a copy of the previous version. Consequently, the following data was available for analysis:

(i) Documentation of all meetings by each participant as meeting notes.

(ii) Multiple versions of the evolving web site, tagged by date.

5.3.3 Analysis procedure

At the end of the development process the documentation generated by the investigator, Supervisor_Tech and Supervisor_Math from all meetings was collated and browsed. Three broad filters were selected for first pass analysis of these meeting notes - scope of tool, activity sequence, and metacognitive support. Scope of tool might provide an indication of the proportion and pattern of meetings focused on the prototype versus the broader issues of establishing a learning environment. The development of the activity sequence might highlight key design decisions. Metacognitive skills (Reiss et al, 2001)
were important for proof type geometry problem solving, so it was important to trace designer attention to metacognitive support.

These identified categories were used to summarise meeting notes in a spreadsheet. Table 5.1 (next page) presents three rows of this data at different stages of the development process.

Three kinds of meetings were identified: Concept Meetings, Focus Group Meetings and Development Meetings. Meeting notes of concept meetings and development meetings were then separately studied to identify further patterns that were more relevant to the meeting type.

The investigator and Supervisor_Tech reviewed all web site designs. Some substantial pages were identified and screen captures were collected into a separate folder. As the decisions of meetings were translated into web pages, the same table was used to summarise their details. Chronological order was used as the sequencing criterion. The column headings of this table were: Date and focus of meeting, core focus, design ideas and outcome. With this, a third table was formed (Appendix 7). This table was used in the present analysis.
Table 5.1: Sample entries from meeting summary spreadsheet

<table>
<thead>
<tr>
<th>Date, focus of meeting</th>
<th>Scope of tool</th>
<th>Activity Sequence</th>
<th>Metacognitive Support</th>
</tr>
</thead>
</table>
| April 23, 2002 Initial discussions on the problem environment | Ill-defined at this stage, but want:  
- Ease of use (teacher likely to adopt)  
- Appropriate look and feel (students enjoy the environment)  
- Technical proficiency (needs to work)  
- Curriculum fit (so not wasting time)  
| Activity ideas: Individual student interaction with software  
Some group tasks due to possible hardware restrictions  
Self paced assessment over time. | Can be provided by teacher and fellow students. |
| | Designed to accompany classroom activities and teacher support (process and product balance)  
Aim was to minimise the problem for students below the required vHL and maximise incidental learning in the group activity. | | |
| August 7, 2002 Initial meeting to design the learning environment. Data analysis complete. | Top-level analysis – students require geometry content knowledge and general problem solving strategies (identified from Stage 1) to solve proof type geometry problems. The required scope of support was:  
- Background material  
- A reflective space  
- Clues  
- Worked examples  
| Sequence within a problem  
Provide a challenge  
Phase 1 – student has a go; correct → next problem  
Phase 2 – if incorrect → clues  
Phase 3 - Worked examples  
Phase 4 - reflections page if still having problems – ask questions or verbalise their understanding.  
Sequence of problems: Problem → 3 similar problems → 2 far transfer problems | Reflection, extended interaction enhances to social level |
| August 30, 2002 (V2) The design of the learning environment. | Strategies to overall learning environment to address top level needs particularly content knowledge | Strategies for directing to multiple levels, the idea of screening (learning). It should be optional so student is free to decide the learning point.  
Although WBLE suggests direction, student can freely select path.  
Main options: learning, practicing, problem solving, sharing, and reflection represent screening, content development, problem solving, social interaction, and self-evaluation respectively. | |
5.4 Prototype development

As Study 2 is a development process that generates a product for formative evaluation, from the study’s point of view, both process and product are important. Therefore, results of the first part of Study 2 are presented in a developmental sequence.

The first section presents the development process. It begins with an analysis of the team meetings and describes the evolution of the prototype. The second section presents the key features of the prototype.

5.4.1 Pattern of meetings

Prototype development took place from April 23, 2002 to February 15, 2003 through a series of meetings. It included 32 meetings each of 4 – 6 hours duration. Meetings were of three kinds: initial concept development meetings, ongoing development meetings, and periodic focus meetings. The distribution of these meetings is shown in Figure 5.2.

![Distribution of meetings in the prototype development process](image)

The boxes denoted by ‘C’ in the first row of Figure 5.2 represent initial concept development meetings. Similarly boxes ‘F’, and ‘D’ in the next two rows of the figure represent periodic focus group meetings and ongoing development meetings respectively. The first focus group meeting was held at the end of the initial concept development meetings. It was focused on planning of the learning environment.

As illustrated in Figure 5.2, there were four initial concept development meetings, two focus group meetings, and 26 ongoing developmental meetings. Seventeen meetings (shaded dark) were devoted to web page review and revision. The rest of the meetings were discussions focused on pedagogical issues related to the development of a learning
environment. The progress of the web page development process was evaluated in the Second focus group meeting, i.e. halfway through the development process.

5.4.2 Initial concept development meetings

The purpose of the initial concept development meetings was to develop a common language, common goal and common orientation to a learning tool that could support students learning proof-type problem solving. This was important to find common team ground, as two participants were from different areas of expertise. These meetings adopted a brainstorming approach to enable sharing of experience. Triggers or ground information for initiating discussions were literature, software demonstrations and past classroom experiences. The aims were to understand and identify the scope of the instructional problem, search for analogies from similar problems, and gather relevant information as preparation for the design of a solution.

Four concept development meetings were held. Each had a key focus: finding common ground, instructional approaches, generation of design ideas, and scope of the study prototype.

5.4.2.1 Finding common ground

In the first meeting developing software to facilitate students in the problem-solving process of proof-type geometry was identified as an ill-structured problem. The investigator raised some issues arising from his experience in teaching mathematics, evaluation of answer scripts at public examinations and the outcomes of recent readings. Supervisor_Tech raised issues like individualised learning opportunities, avenues for sharing ideas, access to requisite knowledge as well as group dynamics.

The research findings on geometric reasoning of children (van Hiele Theory) were given special emphasis while literature also revealed the heterogenous nature of the secondary geometry class (Senk, 1985). Minimising this was a top priority, due to the disadvantage to students at lower levels. Multiple levels of need demand multiple facilities from the learning environment leading to individualised learning opportunities. On the other hand, a moderate heterogeneity might have advantages in terms of incidental learning within groups. A decision was taken that the learning environment needed mixed features like individual and interactive learning, sharing of ideas and group tasks. At the
same time it was decided that this type of a learning environment should be implemented in the classroom.

Supervisor_Tech stressed the following design concerns:

- Ease of use (teacher likely to adopt)
- Appropriate look and feel (students enjoy the environment)
- Technical proficiency (need to work)
- Curriculum fit (so not wasting time)

The considerations and outcomes of the brainstorming session for this first meeting are shown in Table 5.2.

**Table 5.2 Outcomes of the first concept development meeting**

<table>
<thead>
<tr>
<th>Issue</th>
<th>Suggestion</th>
</tr>
</thead>
<tbody>
<tr>
<td>Place of implementation</td>
<td>Classroom use</td>
</tr>
<tr>
<td>Multiple level of geometric reasoning among students</td>
<td>Individual needs have to be addressed</td>
</tr>
<tr>
<td>Learning</td>
<td>Collaborative, incidental (possible)</td>
</tr>
<tr>
<td>Interaction patterns</td>
<td>Interaction: student with peers, student with teacher, student with technology</td>
</tr>
<tr>
<td>Teacher’s role</td>
<td>Learning (expert) partner</td>
</tr>
<tr>
<td>Role of technology</td>
<td>Interactive</td>
</tr>
</tbody>
</table>

5.4.2.2 *Instructional approaches*

While the previous meeting focused on classroom use, this one emphasised finding a strategy for presentation of instructions. The non-algorithmic nature of proof-type geometry problems was discussed. Even in proof-type geometry problem solving, there are some essential tasks that involve general procedures such as understanding the problem, converting the problem into a geometric diagram, applying geometric relationships to the existing situation, selecting geometric objects to apply these relationships, and presentation of the proof in semantic form. In the discussion, Supervisor_Tech suggested to integrate these solution steps and Polya’s (1973 a) four-step approach. The decision was made to consider strategies presented in other resources such as those by Mayer (1992), Schoenfeld (1985) and Robertson (2001).
5.4.2.3 Generation of design ideas

Problem similarities (Robertson, 2001) enhance problem information for the problem to be solved through the study of the solution of a similar problem. To reduce the unfamiliar and complex nature of the present ill-structured problem, Supervisor_Tech demonstrated a learning environment on radiology to analyse instructional supports that can be provided during the problem-solving process. The similarities between the demonstrated and the proposed learning environment were their multi-level learning requirements and metacognitive support.

The demonstration highlighted how the learner can make progress from level to level from no background knowledge to problem solving level. Throughout the process, metacognitive indicators were provided so that the learners would know where they are in each section, how far they are through the process and what aspects they still have to deal with. This was seen as metacognitive awareness of learning. In summary, multilevel learner needs, metacognitive support, problem solving and worked examples were identified as similarities between the demonstrated and proposed learning environments.

5.4.2.4 Scope of the study and prototype

As the investigator had sufficient information to approach the problem, a schematic diagram of a learning environment was sketched and submitted. The diagram was central to a brainstorming session that was focused on software features related to knowledge components. The outcomes of the literature review and experience of the previous sessions were also revisited. The Worked example method was identified as an appropriate central strategy.

A problem was raised about the entire design, time constraints and the scope of the study. It was concluded that the key element of the design was the worked example strategy for developing proof-type geometry problem solving. It was generally agreed that concepts and instruction, explained application, and exercises for improvement might be useful as other components (Figure 5.3).
Figure 5.3 – Components of the prototype

Figure 5.3 illustrates that the complete loop for a given topic in geometry problem solving might include:

- A pre-test to screen students for activities at their current vHL;
- Appropriate measures to develop geometry reasoning up to vHL3;
- A post-test for students to exit and move to the next level;
- Appropriate activities to provide content knowledge at vHL3; and
- Sufficient opportunities to learn proof-type geometry problem solving.

In review, Table 5.3 outlines when five design concepts were mentioned in the four initial concept development meetings.

Table 5.3 Emergence of concepts during brainstorming in concept development meetings

<table>
<thead>
<tr>
<th>Meeting Number</th>
<th>Focus</th>
<th>Problem solving</th>
<th>Worked example</th>
<th>Multilevel learner needs</th>
<th>Meta-cognitive support</th>
<th>Constructivist approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Finding a common ground</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Instructional approach</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>Generation of design ideas</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>Scope of study and prototype</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
5.4.3 Periodic focus group meetings

All the members participated in two focus group meetings to plan and monitor the quality of the learning environment. It was vital to relate back to conceptual needs to remain ‘on track’ with prototype development. Members assessed and suggested improvements and looked into contingencies. In a complex ill-structured problem-solving process, this was necessary to minimise the possibility of long-term negative consequences of a pragmatic view of short-term achievements. Various possibilities were assessed to establish the most appropriate ones. It was also necessary to search for missing required elements and to minimise them. Supervisor_Math contributed expertise in mathematics education towards improving the quality of instruction related to domain-specific aspects at the focus group meetings.

5.4.3.1 First focus group meeting

Study 1 identified that general problem-solving processes can influence the success of proof-type geometry problem solving. This influenced the prototype development process to find strategies to improve the activation of analysis, representation, planning and use of knowledge retrieval.

The sketch shown in Figure 5.4 demonstrates some key features of the decisions.

![Figure 5.4– Some key features planned for the prototype](image)

The prototype has to be a teaching tool to develop proof-type geometry (denoted by GPT) problem solving skills. The development process progressed along two dimensions. One was design of the learning environment, and the other was development of the prototype software.
The role of general problem solving in proof-type geometry problem solving became a central issue. The meeting concluded with important decisions. The focus of the learning environment is to help students in the general school curriculum become competent problem solvers.

- Content knowledge and general problem-solving processes must be at the core of the learning environment.
- The learning environment must be a supportive learning tool.
- Knowledge elements of the learning tool may include: background material (knowledge), a reflective space, clues, demonstrated solutions (worked examples) and non-demonstrated solutions (formal answer only).
- Media-rich Web based instructions may be appropriate for the prototype.
- The classroom teacher is an assumed participant in this learning environment, affording the advantages of knowledge of students and ability to customise personal support.
- The learning tool should assist both teacher and students in its goal to develop students to the stage of being competent proof-type geometry problem solvers equivalent to VHL 4.

These then became design considerations in the development of the learning environment for proof-type geometry problem solving. This focus group meeting was the turning point for meetings from initial concept to ongoing development.

### 5.4.4 Ongoing development meetings

Most of the pedagogical strategies related to teaching and learning proof type geometry problem solving were translated into software development at the development meetings. Supervisor_Tech and the investigator participated in these meetings to develop and review the software. Their purpose was to:

- Decide on the nature of the learning environment;
- Identify the focus for development, given the extensive scope of the problem;
- Review core ideas for software;
• Discuss issues from the literature, literature review and Study 1 (These were frequently the trigger to view new and emerging perspectives);

• View the status of the prototype;

• Identify changes to design;

• Edit Web pages and simplify design.

Frequent review was helpful to guide the path of this ill-structured problem-solving exercise in which micro level components were added, evaluated and removed or further developed. The investigator implemented the decisions of meetings in the development of the prototype and submitted this at the next meeting, engaged for at least 6 hours a day irrespective of weekends throughout the period mentioned. Seventeen of the ongoing development meetings were focused on web page development, and the other nine were focused on pedagogical development.

5.4.4.1 Pedagogy development

The multi-level nature of the geometry classroom environment was a major concern in pedagogy meetings. Research revealed that students should possess prerequisites such as geometric thinking at vHL3 and relevant content knowledge of Euclidean geometry to solve proof-type geometry problems (Senk, 1985; Senk, 1989; Shaughnessy and Burger, 1985). Senk (1989) reports that, in general, only 7% of students are at vHL3, and others are scattered at different levels. The development meetings took up this issue of multi level student distribution in the secondary class and discussions focused on underachievement of proof-type geometry problem solving due to insufficient maturity in geometric reasoning across van Hiele levels. Comments from earlier meeting notes were re-visited:

Whole experience in class is not Web site focused. Minimising the effect in terms of disadvantage to those outside the current level is important (Meeting notes, 23. 04. 2002).

The focus of the learning environment is to help students become competent problem solvers in proof-type geometry problem solving (Meeting notes, 07. 08. 2002).

These decisions echoed in the development meeting on 30. 08. 2002. It was agreed that an appropriate learning environment for proof-type geometry problem solving that caters for senior secondary students at all van Hiele levels should consider multi-level
student needs, the benefits of heterogeneity for incidental learning and should make the assumption that inadequate prerequisite knowledge is to be expected in a proof-type geometry problem solving class.

A conceptual model containing three key components was proposed to meet multi level needs:

1. A remedial component
2. An instructional component
3. A problem-solving component

A remedial component would require the following functionality:

- Identify student/s’ van Hiele level of geometric thinking.
- Direct the student/s to the relevant van Hiele level.
- Provide learning activities appropriate to the student/s’ current van Hiele level.
- Help the student/s to make progress up to van Hiele level 3.

The suggested remedial learning environment is represented in Figure 5.5. It is anticipated that this type of learning environment would promote students' geometric reasoning skills up to vHL3.

![Figure 5.5 – Schematic structure of remedial components.](image)

An instructional component would provide declarative knowledge required for proof-type geometry problem solving. A problem-solving component would support students in solving proof-type geometry problems. All components would relate as illustrated in Figure 5.6.
The scope of this overall conceptual model was vast. Given the pragmatic considerations of this study, and the selected focus on students solving proof-type problems, the problem-solving component was chosen for prototype development. If such a vast system were to be developed, the critical question related to the effectiveness of instructional strategies to support problem solving. Subsequent research could investigate development of the other components. The separation of screening from the remedial component in Figure 5.6 could only occur once the complete system was developed.

5.4.4.2 Web-page development – the emergence of ANGEL

At the end of the concept development phase the investigator had sufficient information to initiate the web development process. The following single problem was selected to begin the prototype development:

ABCD is a square as shown in the figure. X is the mid point of AD. Prove that XB = XC

To facilitate the proof-type geometry problem-solving process with instruction, the investigator was bound to adhere to several design constraints: concepts, needs and conditions decided in previous meetings. The investigator first thought about using the software in a classroom setting, engaging students in active participation, a constructivist approach (especially, student’s autonomy in control of learning) and learning through worked examples.
Different presentations of worked examples emerge

As seen in textbooks, the traditional worked example approach presents the problem and the worked example together but this may not meet the condition of engaging in active learning and construction. When the student finds it difficult or even thinks it difficult, instead of construction of knowledge the student can go directly to the worked example to read and understand. This support for problem solving may not be appropriate for proof-type geometry problem solving where insightful thinking is required.

To minimise moving straight from a problem to its solution, the investigator wanted to present the worked example rather late. For this, a range of possible student needs was projected in order to provide options for each of the following:

1. The student might need to solve it alone and check the answer.
2. The student might need to check progress during the process such as in the diagram drawn with explicit information and the diagram with implicit information transferred.
3. The student might need to see the answer and to pick an information entity such as a strategy to support the solution process.
4. The student might require the worked example explained further to widen problem information.
5. The student might be able to solve the problem if they are provided with the structured problem.

The investigator devised options (Figure 5.7) to cater for all these needs so that each comes between the problem and the worked example. This was helpful not only to keep the problem and worked example apart, but also to enhance opportunities for engagement in active learning. This attempt resulted in the following page and its related pages.
ABCD is a square as shown in the figure. X is the mid point of AD. Prove that XB = XC.

Solve the problem

I need to check the explicit information I marked on the diagram
I need to check the implicit information I marked on the diagram
I need to check my proof writing
I need to see the worked example
I need to answer the structured question
I need to see the explained worked example

As Figure 5.7 shows the worked example was not presented along with the problem. Instead, avenues were provided to engage in problem solving. The second option from the bottom provides the structured version of the problem. In addition, other options contain clues to urge students to engage. For instance, the first option signals the need for drawing a diagram and marking givens.

Then the investigator linked options to the worked example or a part of it with appropriate hyperlinks. The investigator thus kept the worked example away from the students by one layer. In the case of a structured question, the second layer was the structured question, and the third layer was the worked example.

After preparing Web pages with relevant parts of the worked example, the investigator met Supervisor_Tech seeking opinions for improvement. She did not say that the design was not what she expected, but implied her unhappiness with her note: “Wish list for a commercial designer” on the design. However, she was happy with the way that the worked examples were presented and complemented with another option: “I need more instructions” for those who need the problem explained for engagement. It was re-emphasised that the focus of the study is (i) proof-type geometry problem solving, (ii) for secondary students. It was also decided to reduce much of the congested information and to limit the prototype to proof-type geometry problem solving in congruency.

The development meeting identified that the two navigation options to check the explicit or implicit information marked on the diagram could serve as diagram supports. The option to see the explained worked example would also function to unpack content
knowledge as required. This support for content knowledge would be trivial since only a limited number of students might select this option.

A revision was required to reduce the congestion of information. In this regard, the investigator identified two types of students: students who are able to attempt, and students who need additional help such as through a structured problem or problem explained. The first group of students require information to check their progress, which can also be accomplished with the worked example.

The investigator separated the problem from the list of options. Then, two pages were designed for options that are required by two student groups. Figure 5.8 shows these.

The shaded pages represent different forms of the problems, whereas others show the options. These pages were not only less congested, but also contained less extraneous information.

This design was discussed in the next development meeting and it was decided to avoid terms such as explicit information. It was also observed that the first four instructions in Page 3 are phases of the presentation of a worked example. A suggestion was made to present an evolving worked example with three steps:

1. Representation of explicit information of the worked example;
2. Representation of implicit information of the worked example; and

3. The complete worked example.

Integrating these in the same worked example would cut down the first four options on Page 3 into one. This configuration is seen in later revisions. The case of the explained worked example is an issue raised in the literature, as traditional worked examples typically contain only the answer. In contrast, the explained worked example (see Figure 5.9) includes additional information such as mathematical reasoning, process indicators, and problem-solving strategies, which are useful in the problem-solving process and identified as necessary by results of Study 1.

Process Guidance emerges

The explained worked example in Figure 5.9 was discussed in the next development meeting in terms of language, amount of information, and appropriateness. To improve information presentation some changes were suggested:

- The first column should be removed.
- Some terminology in the second column such as conjecture and retrieval should be removed.
• The instructional part (e.g., Draw the diagram) should be separated from the answer part.

• Explanations should be given in simple language.

Implementation of these suggestions can be seen in the Design 09/06 on the data CD accompanying this thesis. The problem provides a hyperlink to access the structure that has been labelled as *Guidance*. The new structure is shown in Figure 5.10:

![Guidance structure](image)

*Figure 5.10 – Formation of process guidance under the label of “Guidance”*

The structure *Guidance* shown in Figure 5.10 has several functions. First it represents the structured form of the problem. Second, it is general to any proof-type geometry problem. Third, it provides the steps in the problem-solving process. Fourth, it serves as the user interface to the worked example. The ‘Show me’ link introduced in this version separates the instruction from the navigation. It allows access to a stepwise breakdown of the worked example.

For the purpose of problem familiarisation, a single problem is not sufficient for the worked example strategy for non-algorithmic problems. Initially a prototype with 10 problems was planned and a menu was developed for problem selection. *Supervisor_Tech* had long suggested that problems have to be in increasing order of difficulty, so to increase flexible use of different proof-type geometry problems with the same data, the *investigator* replaced the problem with:

ABCD is a quadrilateral so that AB = DC and angle BAC = angle BDC. AC and BD cross at F. Prove that AFB and DFC are congruent triangles.
This structure of Figure 5.10 incorporating the new problem was revised with two improvements in the subsequent development meeting. First, the number of steps was increased from 6 to 9. Second, another instructional column was added. As is illustrated in Figure 5.11, the new column contains instructions specific to the problem.

![Figure 5.11 – The view of process guidance in Design 09/13](image)

The guidance represents various options from previous versions - structured problem, problem explained and the explanation part of the problem explained. It also links diagram support and part or complete versions of the worked example. The first column is process-oriented and provides steps in the proof-type geometry problem-solving process. The second column is content-oriented and is specific to the problem. The ‘Show me’ links in the third column interface each process/content step with the relevant development of the worked example.

Although students can use this process guidance as a special support, they are initially required to solve the problem without it. It was decided to provide a second problem attempt for those who solved the problem with the help of process guidance. While ‘Show me’ links retrieve related parts of the worked example, another hyperlink ‘Try the problem again’ was provided.

At this point, the prototype for a single problem, its workout example, and process guidance were relatively complete. To extend the complexities within a single problem
to a problem sequence, a second problem was added to the prototype as shown in the box below:

<table>
<thead>
<tr>
<th>Problem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a quadrilateral so that AB = DC and angle BAC = angle BDC. AC and BD cross at F. Prove that AFB and DFC are congruent triangles.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a quadrilateral so that AB = DC and angle BAC = angle BDC. AC and BD cross at F. Prove that BF = FC.</td>
</tr>
</tbody>
</table>

The relationship between these two problems is that Problem 2 extends Problem 1 by one step. This selection was carefully done for the purpose of problem transfer; knowledge learned to solve Problem 1 can be transferred to solve Problem 2. Process guidance for both problems is the same for the common parts of both problems. Therefore it was decided to use process guidance to provide reduced help. Consequently, process guidance for Problem 2 was provided only with process support.

**Problem Set emerges**

In the second focus group meeting, the option to ‘try the problem again’ was seen as a negative approach. A suggestion was made to replace the same problem with a similar (most times structurally identical) problem so that the student could make further progress. The labels of the problems were changed and the problem was converted into a similar problem.

<table>
<thead>
<tr>
<th>Problem 1</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a quadrilateral so that AB = DC and angle BAC = angle BDC. AC and BD cross at F. Prove that AFB and DFC are congruent triangles.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Similar Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>PQRS is a quadrilateral so that PQ = SR and angle QPR = angle QSR. PR and QS cross at X. Prove that PXQ and RXS are congruent triangles.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Problem 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABCD is a quadrilateral so that AB = DC and angle BAC = angle BDC. AC and BD cross at F. Prove that BF = FC.</td>
</tr>
</tbody>
</table>

This new development was termed a problem set. According to this notion, a problem set consists of base problem (Problem 1), a similar problem and an advanced problem (Problem 2). The problem set works as follows:
1. The base problem contains the full process guidance. It introduces a problem-solving strategy within a problem set. When the strategy is new, the student can practise it with process guidance.

2. The similar problem is provided to improve the use of the strategy. As the problem solving process is now familiar, process guidance is not provided. However, the worked example is provided so students can check the answer.

3. The purpose of the advanced problem is to apply the knowledge learned in a complex situation. Much of the process guidance of the base problem is valid for this and the rest is trivial. Only the process part of process guidance is provided as a support.

Five problem sets were considered sufficient for prototype development. All problem sets were similar in terms of instructional help.

*Embedded Content Knowledge emerges*

Although problems and strategies differ, the knowledge required for different problem situations comes from a common pool of declarative knowledge. For instance, the relationship between vertically opposite angles is common for many proof-type geometry problems. This means that providing content knowledge with each problem encourages the activation of new knowledge. At a development meeting it was decided to hold content knowledge common and link from various access points within worked examples or certain critical instructional points such as ‘What is missing?’ in process guidance.

In summary, worked examples and process guidance, diagram support, problem sets and embedded content emerged through many development meetings. At times developments were concurrent and at other times there was an emphasis on one structural component. The pattern of focus is illustrated in Table 5.4.
Table 5.4 – Development meetings with a focus on Web page design

<table>
<thead>
<tr>
<th>Meeting Number</th>
<th>Date</th>
<th>Core Structures</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td><strong>Process Guidance</strong></td>
</tr>
<tr>
<td>8</td>
<td>23.08.2002</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>30.08.2002</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>06.09.2002</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>13.09.2002</td>
<td></td>
</tr>
<tr>
<td>16</td>
<td>20.09.2002</td>
<td></td>
</tr>
<tr>
<td>17</td>
<td>27.09.2002</td>
<td></td>
</tr>
<tr>
<td>18</td>
<td>27.09.2002 (pm)</td>
<td></td>
</tr>
<tr>
<td>20</td>
<td>17.10.2002</td>
<td></td>
</tr>
<tr>
<td>21</td>
<td>24.10.2002</td>
<td></td>
</tr>
<tr>
<td>22</td>
<td>31.10.2002</td>
<td></td>
</tr>
<tr>
<td>24</td>
<td>01.11.2002</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>07.11.2002</td>
<td></td>
</tr>
<tr>
<td>26</td>
<td>15.11.2002</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>15.01.2003</td>
<td></td>
</tr>
<tr>
<td>30</td>
<td>30.01.2003</td>
<td></td>
</tr>
<tr>
<td>31</td>
<td>31.01.2003</td>
<td></td>
</tr>
<tr>
<td>32</td>
<td>15.02.2003</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.4 shows that process guidance received more attention during the early phase of the development process. It was almost complete when attention turned to problem set development. Diagram support was taken up at the beginning and end. The last two diagram support meetings focused on animation to unpack the sequence and highlight key information in diagrams. Embedded content developed within the appropriate context of the problem sets that highlighted the issue of potential content duplication.
5.5 Core structural elements of Web based prototype

*ANGEL* (A Non-linear Geometry Environment for Learning) emerged from this development process. While coded in html, this “Web based” prototype was tested locally on machines, which did not capitalise on the communication features of the WWW.

Five key elements emerged from the design process:

- **Problem sets** – A problem set contains a base problem, a similar problem and an advanced problem.

- **Process guidance** - The structured form of a base problem or an advanced problem broken down into a step-by-step procedure.

- **Worked examples** – Each of the problems and the process guidance has a worked example.

- **Diagram support (DS)** – This uses visual effects to highlight parts of the diagram to reduce diagrammatic complexity. DS is embedded in each worked example.

- **Embedded content base** – Knowledge that is required for solving problems. *ANGEL* has an embedded content base required for worked examples.

Since process guidance is also a form of a problem, there are five *forms* of problems in a problem set - base problem (BP), process guidance of base problem (PG-BP), similar problem (SP), advanced problem (AP) and process guidance of advanced problem (PG-AP).

Each *form* of problems has a separate form of worked example. The terms and abbreviations for these are – worked example of base problem (WE-BP), worked example of process guidance of base problem (WE-PG-BP), worked example of similar problem (WE-SP), worked example of advanced problem (WE-AP) and worked example of process guidance of advanced problem (WE-PG-AP).

Embedded content is a common set of information that can be accessed from relevant sites within *ANGEL*.

Five problem sets or units are available in *ANGEL*, each with the same configuration and features: three problems, two forms of process guidance, and five worked examples.
with visual support in each, and embedded content. The problem-set structure represents the core of the instructional strategies embedded in ANGEL. Figure 5.12 represents the problem-set structure diagrammatically and reveals potential user pathways.

The shaded boxes identify the problem and process guidance, whereas plain boxes show the respective worked example. Regular arrows indicate the navigation options and directions while the dashed arrows represent points to access embedded content. From the base problem, users can access either the worked example (WE-BP) or seek process guidance (PG-BP). If they choose the WE-BP, they can then either advance to the advanced problem (AP) or seek out the process guidance of the base problem (PG-BP). If they choose the PG-BP, then their pathway to the advanced problem (AP) will step them through the worked example process guidance of base problem (WE-PG-BP), a similar problem (SP), and the worked example of the similar problem (WE-SP) before they get to the advanced problem (AP). From the AP the user can view the worked example (WE-AP) or seek process guidance (PG-AP) that will provide access to the worked example (WE-PG-AP). Each worked example of the advanced problem leads the student to the next problem set.
Each of the core structures is now explored to unpack the content and design features of the prototype.

### 5.5.1 Problems across and within problem sets

Five problem sets (see Table 5.5) were included in ANGEL. Across each row a strategy is presented, practised and applied. Down a column different strategies are introduced.

**Table 5.5 – Problems incorporated in ANGEL**

<table>
<thead>
<tr>
<th>Base Problem</th>
<th>Similar Problem</th>
<th>Advanced Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td>AB and CD are two equal and parallel line segments. AD meets BC at X. Show that ABX and CDX are congruent triangles.</td>
<td>PQ and RS are two equal and parallel line segments. PS meets QR at M. Show that PQM and RSM are congruent triangles.</td>
<td>AB and CD are two equal and parallel line segments. AD intersects BC at X. (i). Show that ABX and CDX are congruent triangles (ii). State the relationship between CX and BX. Give reason. (iii). State any other similar relationship.</td>
</tr>
<tr>
<td>XYZ is an isosceles triangle in which XY = XZ. M is the mid point of YZ. W and P lie on XY and XZ respectively so that angle WMY = angle PMZ. Prove that YMW and ZMP are congruent.</td>
<td>ABC is an isosceles triangle in which AB = AC. L is the mid point of BC. J and K lie on AB and AC respectively so that angle JLB = angle KLC. Prove that BJ = CK.</td>
<td>XYZ is an isosceles triangle in which XY = XZ. M is the mid point of YZ. W and T lie on XY and XZ respectively so that angle WMY = angle TMZ. Prove that YT = ZW.</td>
</tr>
<tr>
<td>ABCD is a quadrilateral. AC and BD cross at F. AB = DC and angle BAF = angle CDF. Prove that AFB and CFD are congruent triangles.</td>
<td>PQRS is a quadrilateral so that PQ = SR and angle QPR = angle QSR. AC and BD cross at O. Prove that triangles QOP and SOR are congruent.</td>
<td>ABCD is a quadrilateral. AC and BD cross at F. AB = DC and angle BAF = angle CDF. Prove that angle FAD = angle FDA.</td>
</tr>
<tr>
<td>ABC is an isosceles triangle. AB = AC. The bisector of the angle BAC cuts BC at D. Prove that triangles DBA and DCA are congruent.</td>
<td>Prove that angles opposite to equal sides of an isosceles triangle are equal in measure. (Hint: Draw the bisector of the angle BAC from A to BC).</td>
<td>ABC is an isosceles triangle. AB = AC. Prove that the perpendicular distance from B to AC and C to AB are equal.</td>
</tr>
<tr>
<td>AB and CD mutually bisect each other at M. Prove that ACM and BDM are congruent triangles.</td>
<td>Prove that angles opposite to equal sides of an isosceles triangle are equal in measure. (Hint: Draw the bisector of the angle at the vertex to meet the base).</td>
<td>AB and CD mutually bisect each other at M. EF passes through M so that E is on AC and F is on BD. Prove that M is the mid point of EF.</td>
</tr>
</tbody>
</table>

Each problem set increases in difficulty and students are expected to follow a particular sequence. Within a problem set the base and similar problems are as the latter name suggests of similar difficulty (see Table 5.6).
Table 5.6 – Comparison of the first problem set by stem, goal and solution

<table>
<thead>
<tr>
<th>Base Problem</th>
<th>Similar Problem</th>
<th>Advanced Problem</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stem</strong></td>
<td>AB and CD are two equal and parallel line segments. AD intersects BC at X.</td>
<td>PQ and RS are two equal and parallel line segments. PS meets QR at M. AD intersects BC at X.</td>
</tr>
<tr>
<td><strong>Goal</strong></td>
<td>Show that ABX and CDX are congruent triangles</td>
<td>Show that PQM and RSM are congruent triangles</td>
</tr>
<tr>
<td><strong>Solution</strong></td>
<td>In ( \triangle ABX ) and ( \triangle CDX ):</td>
<td>In ( \triangle PQM ) and ( \triangle RSM ):</td>
</tr>
<tr>
<td>(Given)</td>
<td>( AB = DC )</td>
<td>( PQ = RS )</td>
</tr>
<tr>
<td>(Alternate)</td>
<td>( \angle BAX = \angle CDX )</td>
<td>( \angle QPM = \angle RSM )</td>
</tr>
<tr>
<td>(Alternate)</td>
<td>( \triangle ABX \equiv \triangle DCX )</td>
<td>( \triangle PQM \equiv \triangle RSM )</td>
</tr>
<tr>
<td>(ASA)</td>
<td>( \angle BAX = \angle CDX )</td>
<td>( \angle QPM = \angle RSM )</td>
</tr>
<tr>
<td>(Properties of congruency)</td>
<td>( \angle BAX = \angle CDX )</td>
<td>( \angle QPM = \angle RSM )</td>
</tr>
</tbody>
</table>

Close examination of Table 5.6 shows:

(i) The stem of all problems is the same
(ii) The base problem and similar problem are of similar problem structure
(iii) The advanced problem contains an additional line in the solution
(iv) The advanced problem is the most difficult

In the worked example strategy, the arrangement of problems according to the complexity is important (Wong et al., 2002). In the present study, the problem sets provide a base problem, a similar problem to provide a second attempt and an advanced problem for learning transfer. Students are expected to complete the problem off-screen in a workbook then check the answer or access process guidance. Both of these options provide worked examples with extended information. Students could then move on to the next problem set.

Comparison of consecutive base problems (see Table 5.7) illustrates the difference in problem difficulty. Relative to problem 1, problem 2 has a more complex diagram, uses more rules, generates the same number of information entities and has an additional step.
Table 5.7 – Contrast between two consecutive base problems

Problem 1

AB and CD are two equal and parallel line segments. AD intersects BC at X. Show that ABX and CDX are congruent triangles.

In \( \triangle ABX \) and \( \triangle CDX \):
- \( AB = DC \) (given)
- \( \angle BAX = \angle CDX \) (alternate)
- \( \angle ABX = \angle DCX \) (alternate)

\( \therefore \triangle ABX \cong \triangle CDX \) (ASA)

Problem 2

XYZ is an isosceles triangle in which XY = XZ. M is the mid point of YZ. W and P lie on XY and XZ respectively so that and angle WMY = angle PMZ. Prove that YMW and ZMP are congruent.

\( \angle WYM = \angle PZM \) (Base angles of isosceles tri)

Consider triangles WYM and PMZ
- YM = ZM (M is mid point)
- \( \angle WMY = \angle PMZ \) (given)
- \( \angle WYM \equiv \angle PZM \) (proved)

Triangles AMC and BMD are congruent (ASA)

Although Problem 2 is more complex than Problem 1, it is less complex than the advanced problem (1B) in the first set. The pattern of difficulty level across problem sets in ANGEL is illustrated in Figure 5.13.
As Figure 5.13 illustrates, there is no difference in complexity between a base and similar problem. Similar problems are placed after process guidance. The strategy used in early versions of the prototype was to provide a "Try Again" link to go to the same problem. A similar problem was suggested in the second focus group meeting as a more positive strategy.

Another feature of the problem set is the difference between the base and advanced problems. The advanced problem is at least one step ahead of the base problem – the solution to the base problem is a part of the solution to the advanced problem. This has several effects. First, the first part of the advanced problem is a third attempt at the base problem. Second, it is an opportunity for transfer of training (Lawson, 1991). Third, the problem schema (Chi et al., 1989) related to the base problem is easily transferred to the advanced problem.

The notion of a problem set meets the design configuration set by Lawson (1991). According to this the worked example can have one or all of three functions: acquisition; maintenance; and transfer. Lawson further says that 'acquisition and maintenance are similar in design' (p. 213). Trafton and Reisor (1993) assert that the most efficient way to present the material to acquire skill is to present an example then a similar problem to solve immediately after. The configuration of similar problem strongly aligns with the ideas of Lawson (1991) and Trafton and Reiser (1993). The major intention of the advanced problem is problem transfer, as has been emphasised by Lawson (1991).

In summary, the pedagogical aim of the problem sets is to present an effective sequence of problem complexity that can provide opportunities for familiarisation of problem solving. It also aims to ensure skill acquisition, maintenance or improvement and problem transfer.

5.5.2 Process guidance

Information in the problem page mentions that process guidance is meant for students who require help to solve the problem. Students can access process guidance when they need assistance to initiate the solution process, to continue the solution process or after realising the solution is not complete.
Process Guidance is associated with following features:

1. Process guidance provides a set of structured problems related to the problem.
2. The similar problem does not contain process guidance.
3. Process guidance of the base problem is different to that of the advanced problem.
4. Process guidance is linked to a separate worked example.
5. Three problems of the problem set and two forms of process guidance generate five worked examples up to the level of the advanced problem.

The screen capture of process guidance of a typical base problem is shown in Figure 5.14.

![Figure 5.14 – Process guidance structure of a base problem](image)

The general instruction on the process guidance page provides the following description of the information available in process guidance:

- Some or all of the following steps may be useful to you.
  - The instructions in the first column are useful for solving most proof-type geometry problems.
  - The second column provides additional information useful to solve the problem you are now trying.
  - The links in the third column help you to check your answer step-by-step.
It is important to analyse the composition of process guidance. The common portion for all problems (base or advanced) appears in the first column (see Table 5.8):

**Table 5.8 – The content available in the first column of process guidance**

<table>
<thead>
<tr>
<th>Draw your diagram</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Highlight the goal</td>
<td></td>
</tr>
<tr>
<td>Think about the key concept of the problem</td>
<td></td>
</tr>
<tr>
<td>Think about: What is missing?</td>
<td></td>
</tr>
<tr>
<td>Deduce new information</td>
<td></td>
</tr>
<tr>
<td>Derive the solution</td>
<td></td>
</tr>
<tr>
<td>Present the solution</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.8 shows that process guidance comprises the steps generally found in the solution process of proof-type geometry problems. First, the diagram has to be drawn (*analysis* and *representation*). Then the student has to relate the goal to the diagram (*representation*). For planning (think about what is missing), the student needs a clue (Think about the key concept of the problem). Strategies are required to deduce new information (*use of knowledge retrieval*) using information at hand, i.e. as given and as knowledge. Since the solution is first derived on the diagram, then it has to be presented in semantic form.

The strategies used for the presentation of instruction vary. For instance, responding to the third instruction of the first column in Table 5.8: ‘Think about the key concept of the problem’ may initially be difficult. It was clarified by providing a list box.

The middle or second column of process guidance (see Table 5.9) for the first instruction provides additional information ‘useful to solving the problem you are now trying’.
Table 5.9 – The content available in the first and second columns of process guidance

<table>
<thead>
<tr>
<th>Draw your diagram</th>
<th>Draw the diagram and label it fully</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mark the information you are given on your diagram</td>
</tr>
<tr>
<td></td>
<td>Write given and the goal.</td>
</tr>
</tbody>
</table>

Table 5.9 shows that the instruction in the first column has been detailed out in the second column. When a student cannot comprehend the broader instruction in the first column, more focused instructions are available in the second column. One purpose of the instructions is to clarify broader instructions. The other purpose of the instructions in the second column is to provide context-oriented information. For instance, the clarification for the fifth instruction in Figure 5.14 is: ‘Search the diagram to find something new. Remember, we still haven't used that AB and CD are parallel’. This instruction provides a strategy applicable for the problem that is currently being solved. In that sense, the instruction is context-oriented. On the other hand it provides a strategy.

The third column of process guidance (see Table 5.10) provides ‘Show me’ hyperlinks each of which corresponds to a broad instruction in the first column. Table 5.10 illustrates the ‘Show me’ link of the first instruction:

Table 5.10 – The content available in all three columns of process guidance

<table>
<thead>
<tr>
<th>Draw your diagram</th>
<th>Draw the diagram and label it fully</th>
<th>Show me</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mark the information you are given on your diagram</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Write given and the goal.</td>
<td></td>
</tr>
</tbody>
</table>

Table 5.10 illustrates that the ‘Show me’ link is common to all instructions in the second column. The role of the ‘Show me’ link is to provide a retrieval path to the answer, which is the relevant part of the worked example. In that sense, it is the interface to the worked example.

Non-algorithmic problem solving is both process oriented and content oriented (DeFranco & Hilton, 1999, Schoenfeld, 1985). Process guidance helps develop problem-solving skills and has been arranged according to the general problem-solving processes: analysis, representation, planning and use of knowledge retrieval. Among these, analysis and representation are collectively included in the first two steps in
process guidance. The set of structured steps in the first column can be used as a procedure to reduce the complexity of the non-algorithmic nature of the problem-solving process. Absence of procedural knowledge such as algorithms and formulae is a limitation in the proof-type problem-solving process. The generic structure provided in the first column of process guidance is useful for a range of proof-type geometry problems. The possibility of developing a template to solve proof-type problem was an early consideration in the first focus group meeting. It is argued that the process guidance in the prototype is an effective heuristic as well as a strong procedure for solving geometry proof problems.

A major difficulty in proof-type geometry problem solving is finding a starting point (Reiss et al., 2001). Problem simplification provides a manageable challenge to students, when they are dealing with complex problems, because process guidance simplifies the problem by breaking down a single problem into various sub-goals. Process guidance is a set of instructions that suggests a path to students working forward towards the goal. Forward direction in problem solving is regarded as an expert problem solving behaviour (Chi et al., 1989). Schoenfeld (1985) recommends sub-goaling as one of five problem solving heuristics. Nunokawa (2001) found that the effect of sub-goaling could be negative or positive. Process guidance here uses sub-goaling as an instructional strategy, rather than a solving strategy.

‘Think about the key concept of the problem’ is supposed to provide a metacognitive cue in the information retrieval process that helps students activate related knowledge components (Chinnappan, 1992; Lawson & Chinnappan, 2000). As the instruction itself does not recall the key word related to the problem, a list of key words has been provided in the second column. It is hoped that this list will promote metacognitive awareness.

‘Think about what is missing’ and its instruction in the second column, ‘Search the diagram to detect what else you need to know’ provide the metacognitive support for planning an approach to the problem. This process sometimes uses backward inference as a strategy to detect what is missing. Backward inference is also regarded as an effective way of finding the solution path particularly in proof-type geometry problem solving (Anderson, 1985). Process guidance across a range of different problems helps students to generalise problem planning, which differs from problem to problem.
The third column of process guidance provides point and click type ‘Show me’ links to illustrate the step. The illustration is the feedback for the respective step. Essential background knowledge is embedded in this illustration for “just in time” access.

The similar problem and base problem were designed with the same problem configuration. Since they are structurally identical problems separate process guidance is not required. Students are provided with process guidance at the phase of solving a base problem for familiarisation. The purpose of the similar problem is to provide an opportunity for improvement of the skill gained from solving the base problem.

Process guidance for the advanced problem has less information than that for the base problem. It has only two columns. The general instruction related to process guidance for an advanced problem states:

---

In Process Guidance for Advanced Problem, you do not get additional instructions. The following table will provide you necessary guidance. Perform the task in the first column. At the end of each task, check your answer and come back to this page.

---

Figure 5.15 shows the typical configuration of process guidance for advanced problems:

![Figure 5.15 - Process guidance for an advanced problem](image-url)
Two advantages were targeted in this configuration. First, it aligns with the principles introduced in faded instructions of the worked example strategy (Renkl, 2002). Second, it directs the student to follow the generic structure of the first column, which is based on the general problem-solving process. The three-column table structure of process guidance for base problems reinforces problem-solving steps for beginners. The first column provides instructions to identify general events in geometry problems (“Highlight the goal”). The second column provides information specific to the particular problem (“Highlight triangles AFB and CFD”). The third column provides a link (“Show me”) to illustrate the step. At this point students gain access to a number of screens of essential background knowledge for “just in time” support.

The geometric diagram provides both relevant and irrelevant information. Knowing the critical area of the diagram helps to reduce the amount of unnecessary information and reduce concentration on irrelevant geometric relationships.

The next instruction to “Think about what is missing” aims to facilitate planning the problem-solving process. At this phase, instructional strategies are rather complex, as students need support in developing thinking skills. Students are linked to extensive background knowledge as well as problem-solving strategy selection.

The last three steps of process guidance relate to use of knowledge retrieval. This process is supported by success at previous phases of problem analysis, representation and planning.

The intention of the support provided in process guidance can be summarised as problem simplification, linking to steps of the general problem solving process, provision of metacognitive support and problem familiarisation. A more detailed summary is presented in Table 5.11:
Table 5.11 – Summary of the support embedded in process guidance

<table>
<thead>
<tr>
<th>Learner engagement</th>
<th>Process guidance is a collection of assigned works placed between the problem and the worked example to engage the students with a range of direct instructions.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>- It requests students to work out problems in a workbook.</td>
</tr>
<tr>
<td></td>
<td>- Problem is structured and presented.</td>
</tr>
<tr>
<td></td>
<td>- Structured problem is further structured and presented.</td>
</tr>
<tr>
<td></td>
<td>- Problem planning is further guided.</td>
</tr>
<tr>
<td>General instructions</td>
<td>The first column of process guidance provides general instructions such as ‘Draw the diagram’, and ‘Highlight the goal’.</td>
</tr>
<tr>
<td>Detailed instructions</td>
<td>Instructions in the second column provide detailed instructions such as Draw the diagram, label, mark information.</td>
</tr>
<tr>
<td>Specific instruction</td>
<td>Some instructions in the second column are specific to the situation such as outline or highlight the triangles ABX and CDX.</td>
</tr>
<tr>
<td>Worked example</td>
<td>Each instruction of the first column is linked with Show me links to the respective solution step of the evolving worked example of the problem. The complete worked example is found only on the last Show me link.</td>
</tr>
<tr>
<td>Metacognitive support to activate knowledge</td>
<td>Some instructions provide metacognitive support to select, retrieve, or activate knowledge such as how many relationships at hand, how many are required.</td>
</tr>
<tr>
<td>Planning strategies</td>
<td>The Show me link related to the instruction: “Think about what is missing” retrieves a page where student can engage thinking and reasoning to plan the solution path.</td>
</tr>
</tbody>
</table>

5.5.3 Diagram support

Although content knowledge is semantic, the verbal form alone cannot be used in proof-type problem solving. New information is generated based on diagrammatic information, and it is also represented in the diagram. Students need support in this process.

ANGEL provides step-by-step diagram development through worked examples. For the problem situation: \(ABCD\) is a quadrilateral. \(AC\) and \(BD\) cross at \(F\). \(AB = DC\) and \(\text{angle } BAF = \text{angle } CDF\), in problem 3, the worked example provides diagram support as illustrated in Figure 5.16.
Figure 5.16 – A step-by-step development of diagram

The upper part of the diagram represents the situation given in the first sentence in the problem statement. The bottom part completes the situation by adding information given in the second statement of the problem.

ANGEL uses visual effects to highlight critical parts of a geometric diagram. Figure 5.17 shows examples of such effects.

Figure 5.17 – Using visual effects to highlight critical parts

As shown in Figure 5.17, visual effects are useful to highlight parts of a diagram and help maintain focus. In addition to colour effects and fill effect, thickness of lines, text colour, and styles of lines are also useful in this regard.

ANGEL uses dynamic visual strategies to hide and lighten unimportant parts to leave critical parts highlighted. For example, Figure 5.18 shows two related frames of the animation related to problem 3B. Lines AD and BC shown in the left-hand side of the diagram disappear in the right-hand side.
Solutions to complex problems in geometry are not obvious without diagrams. In the solving process, the problem information is first transferred into diagrammatic form. Then all workings are done with and within the diagram. After reaching the goal with the diagram, the student attempts to write the solution as a text. The importance of the text is that it presents the solution as a step-by-step progress. A diagram cannot present information as a logical chain. On the other hand, the text-based solution without a diagram is meaningless. The unbreakable connection between the diagrammatic and text-based information representation distinguishes geometric problem solving from others.

Several researchers (Anderson & Koedinger, 1993; Charalambos, 1997; Fischbein, 1993; Hegarty and Kozhevnikov, 1999) have investigated the influence of diagrams in the problem-solving process. One of the major obstacles in solving proof-type geometry problems is identification of the key parts of the problem from a complex geometric diagram. Figure 5.19 shows such a situation.

The light (blue) colour of lines WZ and YT takes them to the background and allows the students to see triangles WYM and ZTM more clearly than less relevant parts.
Table 5.12 lists some diagram supports available in ANGEL. The purpose of the instruction to outline or highlight the goal in process guidance is to identify the key component of the problem in the diagram. In particular, students need to avoid the interference of unnecessary relations raised in less relevant parts of the diagram.

Table 5.12 - Diagram support required for proof-type geometry problem solving

<table>
<thead>
<tr>
<th>Types of information</th>
<th>Diagram Support provided in ANGEL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Problem situation</td>
<td>Step-by-step development process</td>
</tr>
<tr>
<td>Equal pairs of sides</td>
<td>Non-conventional signs</td>
</tr>
<tr>
<td>Equal pairs of angles</td>
<td>Non-conventional signs</td>
</tr>
<tr>
<td>Highlighting sides</td>
<td>Colours, thickness, dynamic features</td>
</tr>
<tr>
<td>Highlighting angles</td>
<td>Colours, dynamic features</td>
</tr>
<tr>
<td>Highlighting areas</td>
<td>Shading, colours, dynamic features</td>
</tr>
</tbody>
</table>

The goal of problem 2 in the prototype is to prove that the triangles AFB and CFD are congruent (see Figure 5.20). The information rich parts as well as parts related to the goal are triangles AFB and CFD. Information-rich areas are very useful in generating new information. Students need to focus their attention on these areas.

![Figure 5.20– Discriminating key parts from unnecessary parts of diagrams](image)

Instead of supporting, the remaining areas such as triangles CFB and AFD disturb the process. In particular, the relationship between vertically opposite angles AFD and BFC is completely redundant to the solving process, and considering that relationship takes the student away from the required solution path. Separating this less relevant information from necessary information is critical to a learner. The meaning of the instruction: “highlight the goal” supports students in this process.
5.5.4 Embedded content knowledge

During the problem-solving process, students need to access and retrieve appropriate content knowledge. ANGEL holds embedded content for students to access from suggested points.

Content knowledge related to the 15 problems in ANGEL can be divided into two categories: theorems and postulates, and concepts. Table 5.13 shows this content knowledge divided into categories. The content focus of ANGEL is congruency and related content knowledge for solving these problems is embedded in the software.

Table 5.13 – Content knowledge embedded

<table>
<thead>
<tr>
<th>Concept</th>
<th>Theorem/ postulate/ idea for relationships</th>
</tr>
</thead>
<tbody>
<tr>
<td>Congruency</td>
<td>SSS, SAS, ASA, HS</td>
</tr>
<tr>
<td></td>
<td>Properties of congruency</td>
</tr>
<tr>
<td>Parallelism</td>
<td>Alternate angles, corresponding angles</td>
</tr>
<tr>
<td>Angles on two intersected lines</td>
<td>Vertically opposite angles</td>
</tr>
<tr>
<td>Isosceles triangle</td>
<td>Base angles, opposite sides to base angles</td>
</tr>
<tr>
<td>Concept of common side</td>
<td>Can be taken as equal sides in two different triangles</td>
</tr>
<tr>
<td>Concept of mid point</td>
<td>Bissection</td>
</tr>
<tr>
<td>Concept of perpendicular</td>
<td>Right angles</td>
</tr>
<tr>
<td>Concept of angle bisector</td>
<td>Equal angles in two different triangles</td>
</tr>
</tbody>
</table>

The planning process is critical in the proof-type geometry problem-solving process. The organization of the retrieval process is shown in Figure 5.21. The suggested links to access content knowledge are called up based on the need of the student. According to the numbers: 1, 2, 3, and 4 labelled in Figure 5.21 the organization of the retrieval process is described as follows.
Screen 1 shown in Figure 5.21 is process guidance. The ‘What is missing?’ instruction on that screen assigns the student the following planning activity.

How many relationships are required to prove that two triangles are congruent?
How many relationships are already available?
Check whether they are sufficient to prove the congruency.
What else you need?

The corresponding ‘Show me’ link directs the student to screen 2 where the planning is discussed. Screen 2 provides information to devise a strategy. To facilitate that, the student is directed to screen 3 ("Critical Knowledge Base"), the required part of content knowledge. However, information on screen 3 includes the answer and the student has to select it. This process is fostered with various retrievals such as screen 4 that represents the Congruency case: SSS (Side - Side - Side). Another piece of embedded content knowledge that can be retrieved through another hyperlink on screen 3 is shown as Figure 5.22. It is Congruence case SAS (Side – Angle - Side).

Figure 5.22 – How content knowledge is provided
Figure 5.22 illustrates the way content knowledge is presented. Information between screen 1 and screen 2 is associated with reasoning related to the selection, what is lacking in traditional worked examples.

Sometimes students might need to access reasons, or additional information. ANGEL provides such information, for example, embedded within a worked example as shown in Figure 5.23:

![Figure 5.23 – Embedded content accessed in a worked example](image)

Several researchers (Chinnappan, 1992; Schoenfeld, 1985) emphasise the importance of content knowledge in problem solving, and these findings gained support in Study 1. Chinnappan further highlights instances where students may possess content knowledge, but are unable to access or retrieve required elements of content knowledge.

The purpose of embedded content knowledge in ANGEL is to provide students with opportunities to access required content knowledge elements. Embedded content knowledge is composed of declarative knowledge (Reiss et al., 2002) such as geometric relationships in forms, both text and diagrams.

In the broader perspective, a learning environment should provide knowledge components related to content knowledge and activate the relevant cognitive functions for problem-solving skills. For the former, the learning environment or the software features should provide content knowledge-related help such as required declarative knowledge related to concepts, relationships, conventions, representation, and
diagrammatic reasoning related to problem-solving strategies. In this regard students need help to activate knowledge, retrieve it, plan a solution path, and regulate cognitive information processing. Students then need to improve the knowledge acquired and apply it in more complex situations.

During the proof-type problem-solving process, students need to access and retrieve related content knowledge and they may need help to select problem-solving strategies to scaffold the non-familiar problem-solving process. Results of Study I indicate the need for support for general problem-solving processes that are critical at the planning stage of the problem-solving process, while literature reveals the importance of metacognitive skills during that process. Self-awareness of knowledge promotes the retrieval process. Awareness is also important for regulating that process. The learning environment should provide metacognitive support through access to the relevant knowledge base, clues, explanatory information, problem simplification in worked examples and capacity for reflection.

5.6 How ANGEL could be used in the classroom

A major design consideration for the prototype was the need to use it in the classroom setting in a constructivist-learning environment as a learning tool. Although ANGEL can be used as an individual learning support, it was expected that students could construct knowledge in groups. Students could solve problems in a workbook while involved in small-group discussion. A major purpose of ANGEL is to enhance the teacher's capacity to address individual differences.

Learning needs vary from student to student. For instance, Table 5.13 shows content knowledge related to five problem sets in a single area of congruency. Considering the volume of a large number of students in a class and the range of their learning needs across a variety of non-algorithmic geometry problems, students' diversity is likely to be broad. On the other hand students need quick attention and assistance to overcome difficulties. It may be hard for a single teacher to cope with the situation. The role of ANGEL is to make available avenues to access information that is not available within the group. As the teacher is expert in proof-type geometry problem solving, the teacher can facilitate the process of knowledge construction as an expert learning partner.
In general, one student in the group can read the problem given in the prototype on the screen. Writing the problem in individual workbooks needs to be encouraged. Then students can share ideas and discuss the solution process in the group. During the discussion, they draw the diagram in a workbook as a collaborative effort, and then in their individual workbooks as an enhanced engagement. They then attempt the problem as a group. The process continues either up to completion of the solution process, or until they are blocked by a difficulty. Completion directs students to compare the answer with the worked example whereas a blockage lets them go to process guidance. This is the place where students obtain information for regulation and construction of learning.

Students can solve problems individually or collaboratively as a group. According to the performance in problem-solving attempts, the student (or the group) belongs to one of the following three categories:

A - Solve successfully
B - Complete with a solution not acceptable
C - Need helps to begin or continue

ANGEL instructs the student (or the group) to solve the problem in workbook. It engages all types as follows (See Figure 5.24 along with the description on the right hand side).

The students A, B, and C access Base Problem, and read it.

1. The students A and B, who attempt, solve the problem in the workbook.

2. The student C needs instructional assistance, therefore moves to Process Guidance of Base Problem (PG-BP), gets instructions, and solves the problem in the workbook.

3. The students A and B develop solutions and access the worked example of Base

Figure 5.24 Different paths through a Problem Set
Problem WE-BP to check the answer.

4. A realises that the solution is successful and then goes to Advanced Problem (AP).

5. B realises that the answer is not acceptable and then goes to PG-BP for more instructions.


7. After completion BP, B and C go to Similar Problem.

8. B, and C complete the Similar Problem and go to WE-SP to check, correct if any, and also go to Advanced Problem (AP).

9. C needs assistance and goes to PG-AP.

10. A and B work on AP (Advanced Problem) and go to WE-AP to check the answer.

11. B realises that the answer is not appropriate, and then moves to PG-AP. Successful A goes to the next problem set.

12. B and C in PG-AP complete work and access WE-PG-AP to check the answer, and also go to Advanced Problem.

The three paths from base problem to advanced problem can be described as follows.

(i) base problem $\rightarrow$ worked example of base problem $\rightarrow$ advanced problem

(ii) base problem $\rightarrow$ process guidance $\rightarrow$ worked example of process guidance $\rightarrow$ similar problem $\rightarrow$ worked example of similar problem $\rightarrow$ advanced problem

(iii) base problem $\rightarrow$ worked example of base problem $\rightarrow$ process guidance $\rightarrow$ worked example of process guidance $\rightarrow$ similar problem $\rightarrow$ worked example of similar problem $\rightarrow$ advanced problem

In addition to these main moves, students can access the embedded content knowledge base through the worked examples. Figure 5.25 shows how students can explore the learning environment for information.
Figure 5.25 shows the key elements of the problem set. Diagram support (DS) is included in the related worked example whereas embedded content is accessible from all types of worked examples. Each horizontal line segment in Figure 5.25 represents a step in process guidance. Each node at the end of a line indicates a “show me” link that connects process guidance to the worked example. Nodes in the embedded content area denote various pieces of declarative knowledge. ANGEL contains five problem sets on congruence. Each dot represents a node in an organised knowledge structure.

The entire map shown in Figure 5.25 illustrates that ANGEL contains possibilities for construction of knowledge.

5.7 Summary

Based on literature review and empirical evidence from Study 1, ANGEL (A Non-linear Geometry Environment for Learning) prototype has been designed as one possible resource to support teachers and students solving proof-type problems in geometry within a secondary school classroom environment.
The target group of *ANGEL* is students who need to learn to solve proof-type problems in geometry. Research has shown that progress to vHL3 is an essential prerequisite for learning proof-type geometry problem solving. The basic assumption in the development of *ANGEL* was that its target group would possess the aforementioned prerequisite.

Literature revealed other learning requirements such as:

- Access and retrieval of declarative knowledge related to proof-type geometry problem solving;
- Representation of information as a geometric figure and obtaining information from it;
- Metacognitive support for information retrieval and process monitoring. It requires quick access to information;
- Opportunities to be familiar with proof-type geometry problem-solving strategies.

Learning such a problem-solving process is non-linear and hypertext strategies can offer support when linked to high quality multimedia resources. However, learners must actively engage to construct their understanding.

The assumption was made in the development of *ANGEL* (the problem solving component of Figure 5.6) that the target group would be at vHL3 and the above learning requirements should be met.

The potential of any hypermedia environment is only realised when it is put to use. Still in early developmental phase, *ANGEL* requires formative evaluation to guide further development and identify appropriate strategies for teachers. The next chapter presents this formative evaluation as Part II of *Study 2*.
Chapter 6: Formative evaluation of ANGEL

6.0 Introduction

Chapter 5 presented the development of a Web-based prototype environment (ANGEL) for students learning to solve proof-type geometry problems. The present chapter presents Part II of Study 2: a formative evaluation to identify how a small number of the target group of learners use ANGEL, what weaknesses exist in the current interface and site structure, and what critical points require improvement.

The first section of this chapter will present a review of literature related to learning in classroom settings, obtaining information from learners about proof-type geometry problem-solving processes and formative evaluation. The second section presents the data collection and analysis methodology for Part II of Study 2. The last section presents and discusses the results of formative evaluation of ANGEL.

6.1 Learning in classroom settings

The main research question focuses on the secondary classroom setting. Explaining the learning process is very difficult using a single theory as both individual and social change takes place.

Cognitive information processing and learning in groups catalyse each other (Hoek, Van den Eaden and Terwel, 1999). In a concept paper, Anderson, Greeno, Reder & Simon (2000) suggest that cognitive and social perspectives on learning are both fundamentally important and complement each other. They argue that the cognitive approach should not be read as denying the value of learning in groups.

The effect of combining the social and cognitive strategies in training problem solving was investigated by Hoek, Van den Eaden and Terwel (1999). In order to examine the nature of mathematical problem solving, the researchers designed a comparison study between an experimental and control group. The experimental group was trained in cognitive strategies such as modeling reality, making different representations, planning, monitoring, and checking in a dynamic group-learning setting. They report that the students in the experimental group gained more than the students in the control
group confirming that cognitive strategies applied in a social interactive atmosphere are effective.

Thus the nature of interaction in the group may be an effective factor in the learning setting. The group may consist of two or more members, and interaction may be one way or both ways.

When two individuals collaborate, they often have to justify themselves to each other, to explain what they are doing and why they are doing it. Intuitively, these efforts should be learning that is frequently observed during collaboration. … When an individual explains to a second individual, learning might take place by both individuals (Ploetzner, Dillenbourg, Preier & Troum, 1999, p. 103).

In a similar vein, Webb (1989) asserts that the explanation provided by the individual in an explanatory session is rooted in cognitive activities. Cognitive research has shown that learning is most effective when there are avenues for active engagement in learning, discourse and participation in groups (Roschelle, Pea, Hoadley, Gording & Means, 2000) in the learning process.

6.1.1 Active engagement in learning problem solving

The contribution of active engagement to learning has been investigated. Learning is a complex cognitive process that involves constructing new knowledge with the help of existing knowledge (Bransford, Brown & Cocking, 2000). In this process, the student has to actively engage in constructing knowledge – active participation plays a critical role in this conscious cognitive process. Active participation is viewed as engagement beyond physical activity where deliberate involvement in cognitive information processing takes place.

… educational reforms appear to agree with the theoreticians and experts that to enhance learning, more attention should be given to actively engaging children in learning process. Curricular frameworks now expect to take active roles in solving problems, communicating effectively, analyzing information, and designing solutions … (Roschelle, Pea, Hoadley, Gording & Means, 2000, p.79).

Learning problem solving is also a deliberate process that requires active engagement in cognitive processes such as analysis, representation, planning, and use of knowledge retrieval. Learning problem solving requires active cognitive engagement not only in construction of knowledge on available knowledge, but also in generating new knowledge on the basis of available knowledge.
6.1.2 Participating in groups in learning of problem solving

The purpose of learning problem solving is to develop skills that transfer knowledge to new and unfamiliar situations (Jonassen, 2000a). Assuming that the gap between the situation and personal knowledge also differs from person to person, the possibility of solving the same problem differs from person to person. In a group the member with the smallest gap is most likely to contribute to problem solving. This emphasises the importance of learning problem solving in groups.

People differ in their potential to solve a particular problem; therefore knowledge of several people should be more effective than knowledge held by any single person. This emphasises the importance of social influence in problem solving and learning problem solving.

6.1.3 Social intervention in learning

The influence of social intervention in learning has been highlighted in social constructivism. Social intervention seems to be useful in confronting a novel situation such as problem solving and learning problem solving. In this regard, intervention of experienced partners such as teachers, parents and peers is useful in learning problem solving (Galbraith and Goos, 2003).

An important aspect of social constructivism is the Zone of Proximal Development (ZPD) that is defined as the difference between what the student can achieve alone and what the student can achieve with the help of social intervention (Galbraith and Goos, 2003) such as that of teacher, peers, parents, or worked examples (Roschelle et al., 2000). Figure 6.1 illustrates the ZPD in relation to a problem-solving task and implies a couple of points. First, it accepts that social intervention can promote individual skills in problem solving. When the student alone cannot solve the problem the difficulty can be overcome with social intervention from a more experienced partner. In some cases, social intervention could be essential. Second, the depth of the ZPD seems to be a personal matter, and it is wider when individual capacity is lower. As learning takes place and individual capacity increases, the ZPD decreases due to increased practice. The main purpose of instruction is to scaffold and bridge the ZPD. Structuring the problem into sub-goals is an effective scaffolding strategy in problem solving (Hausfather, 1996; Nunokawa, 2001).
Individuals participating in peer collaboration or guided teacher instruction must share the same focus in order to access the ZPD. Furthermore, it is essential that the partners should be on different developmental levels. If one partner dominates, the interaction is less successful (Driscoll, 1994; Hausfather, 1996). Group work thus promotes bridging the ZPD, and advanced peers and teachers become learning partners in the learning environment.

There are various ways to scaffold the proximal development in proof-type geometry problem solving. Problem simplification (Robertson, 2001) including structuring the problem into sub-problems is one way. Since structured questions simplify complex problems, the ZPD in each question is smaller than the ZPD of the main problem. Providing some clues such as supportive content knowledge to activate relative knowledge also has a significant positive effect (Lawson & Chinnappan, 2000). Diagrammatic support and worked examples may be effective aids in learning proof-type geometry problem solving. The teacher and expert students can use these aids to scaffold others in the ZPD (Figure 6.2).

Grouping of students with mixed abilities may be desirable, providing a better opportunity for all students to exchange information as each student has a peer with a close ZPD. According to Figure 6.2, all students in the group gain from the small ZPD.
of the expert student (ZPD$_1$) and the relative smaller ZPD with other students makes a desirable group climate for learning.

### 6.1.4 Learning with explanatory strategies of worked examples

Various researchers (Chi et al., 1989; Chinnappan, 1992; Chinnappan and Lawson, 1996; Renkl, 1997a; 1997b; Reiss & Renkl, 2002; Schoenfeld, 1985; Wong, Lawson & Keeves, 2002) have emphasized the importance of the use of worked examples to support learning and transfer. Learning from worked examples is common in mathematics. Worked examples are problems solved by experts, and therefore, it can be assumed that they access expert strategies and procedures. Jonassen (2003) explains that worked examples are an effective method of modeling that promotes constructive learning approaches. As discussed earlier, the worked example method is relevant in proof-type geometry problem solving in terms of familiarisation with various strategies.

Introducing self-generated explanations, Chi et al. (1989) state that “good” students can generate self-explanations by learning from worked examples. Self-explanation involves the process of generating explanations and justifications to oneself when studying an example. Similar to self-explanation, generated explanation can be presented to other people. Depending on the recipient, there are several explanatory methods such as:

1. Self-explanation – explanations are aimed at self.
2. One to another – explanations are aimed at a second person.
3. One to others - explanations are aimed at a group of people.
4. Discussion in a group – explanations are exchanged.

All of these methods can be considered learning environments that allow the learner to construct knowledge. Methods 2, 3, and 4 can be used in groups. Ploetzner, Dillenbourg, Preier & Troum (1999) argue that when two individuals collaborate, they often have to justify themselves to each other, to explain what they are doing and why. Intuitively, these efforts should enhance learning that is frequently observed during collaboration.
6.2 Obtaining information about proof-type geometry problem-solving process

Although problem solving is viewed as a cognitive process, the solver continuously interacts with the environment during the process of externalization (Chi et al., 1989). The externalized process can be recorded by one or more means:

1. Self-explanatory verbalisation
2. Concurrent explanation to one or more
3. Explanation to others about how the process took place
4. Discussion in groups
5. Written answers
6. Observation of the solving process by someone else

An observer can use these methods to gather rich information about proof-type geometry problem solving during the problem-solving process. Think-aloud methods provide rich information about the thinking process of the solver and their use of knowledge. Various researchers have used and still accept this method as appropriate to investigate the problem-solving process (Chinnappan, 1992; Newell and Simon, 1972; Schoenfeld, 1985). However, there are practical requirements such as the need for the solver to be trained thoroughly on how to verbalise generated information during the problem-solving process, the difficulty of maintaining two concurrent tasks at different paces: verbalisation and problem solving, and the high demands of advances in technology such as videoing that cost time, energy and money.

Explaining to one or more may be more familiar to students in a classroom setting. In this setting the solver (may be a teacher or some other) explains the step-by-step progress of the solution (Ploetzner, Dillenbourg, Preier & Troum, 1999). This method doesn’t require training and explanations in order to represent the cognitive process of the solver. Recorded explanations bring natural information to the analysis of the phenomenon of interest. The weakness is that only the solver is active, while others may be passive listeners.

Interactive explanation (Ploetzner et al., 1999) is a mutual and collaborative effort. All subjects in the group can contribute to the solving process, pose questions, clear doubts,
and discuss possibilities. This method not only yields verbal information for the purpose of the researcher, but also provides interactive and fully engaged meaningful learning situations.

Two individuals might mutually explain to each other without any imposed constraints. In this case, explanation is no longer something that is exclusively directed from one individual to a second, but rather corresponds to a process in which two individuals attempt to negotiate and, at least partially, share their understanding of the domain under consideration (Ploetzner, Dillenbourg, Preier & Traum, 1999, p. 104).

The above quotation implies that learning problem solving in pairs forces both individuals to be actively engaged in the learning process (Ploerzner et al., 1999).

Written answers provide concrete information about the understanding that underpins the solution process – related to sub-goals, and solution steps generated by the solver. In particular, mistakes and errors can provide valuable information about common difficulties experienced by students in the learning process. Students' workbooks provide vital information about what students can and cannot do.

When solvers externalise the problem-solving process, an external observer can watch and record important events in that process. This externalization may be either in oral form or in written form or both. Usually, students write the solution to a mathematical problem in their workbooks, as they proceed step by step. During the process the observer can take notes on what students verbalise and do.

In order to capture information about the thinking process and reasoning, working in pairs seems to be effective. Explaining the solution process to each other not only verbalises the solving process, but also creates an effective collaborative learning setting. Working in pairs seems to provide natural conversation rather than trained verbalization that can be seen in the think-aloud method.

In summary, information on problem solving can be externalised as verbalization, explanation or in written form. These externalizations can be recorded by several means such as videotapes, audiotapes, and the student's workbook. Obtaining information on the same activity from multiple sources provides higher validity. For instance, recording observations of student behaviour, their verbalization, and collecting their bookwork during the same problem solving-session is an example of a multiple data-collection approach.
6.3 Mathematics teacher’s role in a Web-based learning environment

Web-based learning environments contain various types of information such as facts, explanations, artefacts, additional information, definitions, and reasoning. They are developed to cater for multiple levels of students, to provide ‘everything’ that is required by the students to construct knowledge. Weaker students might face two problems in handling the information-rich environment. First, students might see everything as important, which takes time. Second, they might need clarification to understand the given material. The teacher has a role in helping students to work and manage information in an ICT-driven learning environment so they become competent and independent users of the environment.

The information-rich nature of Web instruction forces the student to activate a high degree of self-regulation skills. The student’s role in learning in a Web-based environment is to construct meaning through interacting in a hypertext environment by self-directed inquiry, guided activity, and discovery (Brown, 2000). Students try to discover principles and knowledge for themselves so that the ownership of learning remains with the student. The student has to determine which link or step to take next. This is possible because of the connective and communicative nature of the Web. The teacher has to support the student’s active engagement in knowledge construction (O’shea & Scanlon, 1997). In that sense, the teacher must be an experienced learning partner for the student.

Mathematical problem solving is a deliberate activity during which each student has to engage fully in the search for the solution. Mathematics curricular frameworks now expect students to take an active role in solving problems, communicating effectively, analyzing information, and designing solutions – skills that go far beyond the mere recitation of a correct response (Roschelle et al., 2000, p. 79).

According to Roschelle et al. (2000), a primary task of a mathematics teacher is to foster the active participation of the student in knowledge construction. In learning problem solving and related strategies, the learner becomes a cognitive apprentice (Jarvela, 1995). Hence, the teacher has a role of experienced learning partner to the student, so that shared cognition in the learning interaction between teacher and students is assured.
At the beginning, students may need frequent help using technology, but this need decreases with practice. The teacher has to expect that students seek persistent help in crossing the ZPD in learning to solve proof-type geometry problems, as there are non-routine problems. Being an experienced learning partner is significant in scaffolding the ZPD.

In summary, the role of the mathematics teacher in a technology-based learning setting is to create a social constructivist learning environment, whilst ANGEL plays the cognitive constructivist role.

6.4 Formative evaluation of a prototype

Evaluation is an important component in a development process. The point of the evaluation, the strategies used and the type of information obtained can vary according to the purpose of evaluation. The purpose of formative evaluation is to find information on the workability of a process with the intention to improve that process through revision (Reeves and Hedberg, 2003; Tessmer, 1993). Reeves and Hedberg (2003) see this as analogous to “life blood” in the instructional development process.

According to the explanations of these authors, formative evaluation can be seen as a continuous and simultaneous process. Although issues related to pedagogy and design of ANGEL were continuously addressed through reviewing and revising in a series of design and development meetings, issues related to implementation remained to be addressed.

It is important to identify potential participants in such an evaluation. Tessmer (1993) listed four commonly used methods of formative evaluation:

1. expert review - experts review the instructions;
2. one-to-one - one learner at a time reviews the instructions;
3. small-group - the evaluator tries out the instruction with a group of learners and records their performances and comments;
4. field test - the evaluator observes the instruction being tried out in a realistic situation with a group of learners.

Reeves and Hedberg (2003) also describe four methods as follows:

- expert review – review made by content experts, instructional experts, graphic designers, and teaching and training experts;
• *user review* – based on the analysis of user behaviour during the use of the product;
• *usability testing* – based on user oriented characteristics such as ease of learning, high speed of user tasks performance, user retention over time, low user error rate, and user satisfaction;
• alpha, beta and field tests of the prototype program.

Recommendations of Reeves and Hedberg (2003), and Tessmer (1993) emphasise the need for formative evaluation to be done by users at various levels external to the production group. Reeves and Hedberg (2003) emphasise the importance of evaluating the following factors:

1. Functionality – Does the prototype work as designed?
2. Usability – Can the intended users actually use the prototype?
3. Appeal – Do they like it?
4. Effectiveness – Did they learn anything?

Skov, Lee, Walker, and Berger (2003) launched a formative evaluation of three Visible Human Browsers: Edgewarp 3D (EW) program developed by Fred Bookstein & William Green; the Pittsburgh Supercomputing Center Volume Browser (PVB) developed by Art Wetzel & Stuart Pomerantz; and worldwide web interface to an anatomy content database. The following procedure was adopted with small groups comprising three four-student groups.

1. A member of the research team demonstrated features and controls of one software program.
2. One student in each group volunteered to operate the software controls while the other three students coached the operator.
3. Students were encouraged to converse and talk-aloud as they explored and used the software.
4. This session was video recorded and replayed to participants for a discussion on how they used software and why they did so.
5. Researchers and participants reviewed software features in a common discussion.
6.5 Methodology

The factors identified by Reeves and Hedberg (2003): functionality, usability, appeal and effectiveness were adopted as the basis for formative evaluation of ANGEL, and they generated the following questions:

(q₁) Do students accept ANGEL?
(q₂) Can the target group of students actually use ANGEL?
(q₃) Does ANGEL work as designed?
(q₄) Does ANGEL help students to construct knowledge necessary for proof-type geometry problem solving?

6.5.1 Design of Part II of Study 2

Part II of Study 2 was a formative test of a learning tool. The central idea of the design was a problem-solving session. ANGEL was designed for use in a collaborative learning setting. There were four sources of data during the problem-solving session that was conducted in a natural learning setting. They were:

1. Student workbook, in which the solution was recorded.
2. Verbal report, in which evidence of internal view of engagement was recorded.
3. Observation record, in which evidence of external view of engagement was recorded.
4. Interview protocol, in which students' reflection on ANGEL was recorded.

Figure 6.3 illustrates the student/researcher interactions and data sources for Part II of Study 2:

![Figure 6.3 – Student/researcher interactions and data sources for Part II of Study 2](image-url)
In addition, acknowledging the importance of user preparation in formative evaluation (Reeves and Hedberg, 2003), and the limited time students were available, an introductory session was designed to prepare students to interact with ANGEL and generate data. Each group was introduced via demonstration, discussion and hands-on activity to the features and facilities in the learning environment and how to use it. This is shown in Figure 6.3 as “Introducing ANGEL” and “Preparing for talk aloud”. Unlike the think aloud method, talk aloud in groups is natural, however, since high school students rarely problem solve in groups in a conventional classroom setting, a short practice session was organised. User review should be done in realistic conditions and without interruption from the observer (Reeves and Hedberg, 2003). Consequently the scope of the introductory session included: learning proof-type geometry problems; solving as a group; talking to each other; discussing the solution process; exchanging information about the solution process of the problem; and contributing individual ideas to the discussion so that the solution process took place as a group effort. Students were informed that their verbalization was to be recorded.

In the group activity, one student was asked to operate the mouse and read the text aloud. The other was encouraged to engage in silent reading while listening, and to write and draw the solution process. The discussion may range across strategies, suggestions, action to be taken, arguments, disagreements or exploration.

6.5.2 Nature of the target user group

Since ANGEL was designed for implementation with students who had a specific level of geometric thinking maturity – vHL3, student selection was a critical issue. Students at this level are found in Higher School Certificate (HSC) classes in New South Wales (NSW), Australia. Students were selected from a high school in the Illawarra district of Wollongong, Australia. The mathematics teacher selected 6 students from the Year 12 class who met the following criteria:

1. Students possess geometric thinking at van Hiele Level 3.
2. Students follow Euclidean deductive geometry.
3. Students possess declarative knowledge related to congruency of two triangles.
6.5.2.1 Procedure of engaging students in problem-solving session in pairs

According to the design of ANGEL, students can bypass both process guidance and the similar problem. If all students do this, data related to these components will not be generated. To observe student response to process guidance, the link to the worked example of the base problem of Problem 1 was disabled. All students had to access process guidance as well as the similar problem at least for Problem 1.

The six students were engaged in learning proof-type geometry problem solving in pairs. The session was held on three consecutive Thursdays on the basis of one pair a day. Table 6.1 shows how the six students participated in a problem-solving session.

Table 6.1 – Student groups in Part II of Study 2

<table>
<thead>
<tr>
<th>Day</th>
<th>Group</th>
<th>Participants</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thursday 1</td>
<td>Group A</td>
<td>Amjad, Alicia</td>
</tr>
<tr>
<td>Thursday 2</td>
<td>Group B</td>
<td>Adam, Adrian</td>
</tr>
<tr>
<td>Thursday 3</td>
<td>Group C</td>
<td>Alex, Nick</td>
</tr>
</tbody>
</table>

The school authority agreed to allocate three hours for each group. This duration was divided into three time slots. The first half-hour slot was devoted to introducing ANGEL to the students and preparing them to verbalize the solution process. The transcription of this introductory session for Group A is presented in Appendix 8.

During the next time slot of two hours, the problem-solving session, students engaged in learning problem solving with ANGEL. In the final half-hour time slot, students participated in an interview. Table 6.2 summarises the activity distribution of each group during the data collection.

Table 6.2 – Activity allocation in Problem-Solving Session for each group

<table>
<thead>
<tr>
<th>Time slot</th>
<th>Activity</th>
</tr>
</thead>
<tbody>
<tr>
<td>08.00 – 08.30</td>
<td>Introduction to using ANGEL and verbalizing the problem-solving process</td>
</tr>
<tr>
<td>08.30 – 10.30</td>
<td>Learning with ANGEL</td>
</tr>
<tr>
<td>10.30 – 11.00</td>
<td>Interview</td>
</tr>
</tbody>
</table>
6.5.3 Data sources

Students' problem solving with ANGEL was the basis for three sources of data: student workbooks, verbal reports and observation records, details of which follow.

6.5.3.1 Recording of student workings: student workbook

Students solved problems in a workbook. They read the problem on the screen and drew the diagram in their workbook. If they needed more instructions, or needed to check the current situation, the related information was available in ANGEL. According to these instructions or their own understanding, or both, they were expected to complete the solving process for every problem. They were instructed to record the solution steps in the workbook. One member in each pair recorded the solutions of the group.

6.5.3.2 Recording of student verbalization: verbal report

In the introductory session, students were asked to discuss and produce the solution in pairs. They were informed to contact the investigator if they had any difficulty (See Reeves and Hedberg, 2003). At the beginning of the problem-solving session, the position counter (henceforth counter) of the audio recorder was initialized. Students' verbalizations were recorded while they were attempting the solution process.

6.5.3.3 Recording of observations: observation record

The investigator made observation notes about the students' interactions with ANGEL. The observation record contained three types of information: counter, event in ANGEL, and observed student behaviour. Event in ANGEL referred to the instructional event of ANGEL such as Problem 1, process guidance or worked example with which students were engaged at the moment of observation. Counter referred to the position on the tape at which the verbalization of the particular event took place. Observed student behaviour referred to the student behaviour in terms of reading, drawing, or writing at the moment concerned. All observations contained one or both of “event in ANGEL” and “observed student behaviour” together with counter. Notes were added to the structure shown in Table 6.3 as the observation record.
Table 6.3– Observation record

<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

6.5.3.4 Student interview

After the problem-solving session each pair of students was interviewed. The focus questions of the semi-structured interview are shown in the Appendix 9. Once a question was posed, one or both could respond. If the question was about ANGEL, the corresponding Web page was displayed as a focus of the question. Students' responses were recorded on audiotape and then transcribed.

6.5.4 Table of Merged Observational and Verbal (MOV) data

Observational and verbal data was merged to provide a more complete picture of how students dealt with ANGEL. The counter column linked the two data sets as illustrated in the following steps:

Step 1: The observation record (see example in Table 6.4) was generated electronically.

Table 6.4 – Profile of observational data record

<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>Problem 1</td>
<td>Alicia reads the problem on screen for the group. Amjad also reads, but not aloud</td>
</tr>
</tbody>
</table>

Step 2: Two columns (second and third columns shown shaded) were added into the observation report as shown in Table 6.5.
Table 6.5 – Extension of observational data record

<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Student</th>
<th>Conversation</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>Problem 1</td>
<td></td>
<td></td>
<td>Alicia reads the text on screen for the group. Amjad also reads, but not aloud.</td>
</tr>
</tbody>
</table>

Step 3: Counter indicator was initialized then the tape was played and student conversation was transcribed into the first cell of the fourth column until 0031 appeared on the counter on the tape player (see Table 6.6).

Table 6.6 – Inclusion of verbal transcript in fourth column

<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Student</th>
<th>Conversation</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>Problem 1</td>
<td></td>
<td>Problem one. AB and CD are two equal and parallel line segments ... AD intersects BC at X ... Show that ABX and CDX are congruent triangles Now we need Process Guidance. Click here.</td>
<td>Alicia reads the text on screen for the group. Amjad also reads, but not aloud.</td>
</tr>
</tbody>
</table>

Step 4: The third column was filled with the help of the change of the voice heard, content of the dialogue, and details in the other columns. Then rows were split appropriately as is shown in Table 6.7.
Table 6.7 – Student identification and separation in third column

<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Student</th>
<th>Conversation</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>Problem 1</td>
<td>Al</td>
<td>Problem one. AB and CD are two equal and parallel line segments ... AD intersects BC at X ... Show that ABX and CDX are congruent triangles</td>
<td>Alicia reads the text on screen for the group. Amjad also reads, but not aloud.</td>
</tr>
<tr>
<td>0031</td>
<td>Process Guidance</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The resulting Table of MOV Data (Appendix 11) provided a source of further tables. For example columns one and two were selected to produce the event analysis (Appendix 12). This provides information about how each group accessed the different events in ANGEL. It contains the following events students can perform when they are working with ANGEL.

1. Problem event refers to an instance, at which the student obtains information about a problem. Problem events are seen as Problem 1, 1A, 1B, 2, …

2. Worked example event refers to an instance when the student checks the answer against a worked example. These events are denoted as WE in the event analysis. In the case of Base Problem, up to 10 events can be found in the solution process, up to eight events for an Advanced Problem, and one for a Similar Problem, resulting in 18 possible WE events in a problem set.

3. Process guidance event refers to an instance when the student obtains support information to continue the solution process. Under each problem, two process guidance events are possible: PGBP for the process guidance of base problem, and PGAP for the process guidance of advanced problem. Further, there are seven events possible in relation to the process guidance in each problem. The maximum number of events for process guidance is 14 in a problem set.

The event analysis provides information about 15 problem events 90 worked example events, and 70 process guidance events.
6.5.5 Data analysis procedure

Data analysis aims to answer the following questions in relation to student engagement with ANGEL.

(q₁) Do students accept ANGEL?
(q₂) Can the target group of students actually use ANGEL?
(q₃) Does ANGEL work as designed?
(q₄) Does ANGEL help students to construct knowledge related to proof-type geometry problem solving?

Three data sets were used for data analysis: interview data, table of MOV data (Table of observation and verbal report), and student workbook. They were used separately or merged as shown in Table 6.8 to answer the questions framing this formative evaluation.

<table>
<thead>
<tr>
<th>Data source</th>
<th>Result</th>
<th>q₁</th>
<th>q₂</th>
<th>q₃</th>
<th>q₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interview</td>
<td>Reaction analysis</td>
<td>√</td>
<td>√</td>
<td></td>
<td></td>
</tr>
<tr>
<td>MOV data</td>
<td>Event analysis</td>
<td></td>
<td></td>
<td>√</td>
<td>√</td>
</tr>
<tr>
<td>Workbook + MOV data</td>
<td>Problem-solving analysis</td>
<td></td>
<td></td>
<td></td>
<td>√</td>
</tr>
</tbody>
</table>

6.5.5.1 Reaction analysis

Student interview data was analysed to prepare the reaction analysis. This analysis holds some indicators to understand the students' reaction to ANGEL in terms of acceptance or trust. Reaction analysis mainly answers the research question: Do students accept ANGEL?

6.5.5.2 Event analysis

Event analysis was derived from MOV data. It holds information about how students access various events of the problem-solving process in ANGEL. It addresses the questions: can the target group of students actually use ANGEL, and does ANGEL work as designed?
6.5.5.3 Problem-solving analysis

The combination of student workbook and MOV data was used for problem-solving analysis to provide information about how learning took place during the problem-solving process. Problem solving analysis addresses the question: does ANGEL help students to construct knowledge related to proof-type geometry problem solving?

6.6 Results

The reaction, events and problem solving analyses are presented in this order to progressively unfold richer answers to the questions framing this formative evaluation.

6.6.1 Reaction analysis

6.6.1.1 How they remember the problem-solving session

When students were asked, “Did you enjoy the problem solving session”, as a whole they stated that it was interesting, useful and they enjoyed the session. In Group A, Alicia said they enjoyed the session and Amjad admitted that they liked to learn with ANGEL because they were successful in problem solving through its help. Adam (Group B) said that the problem solving session was interesting and useful. Adrian added that they learnt more things in the session and they found all help they required in ANGEL. Nick in Group C said he liked to learn geometry with ANGEL because it was easy to understand, he could obtain assistance and the diagrams were clear.

Students were asked whether they were tired of the session. ‘No … no … no …we like it’ Alicia replied. She appeared unhappy to stop the session. Although Group A had completed 10 of the expected 11 problems they had problem 5B left when the investigator asked them to stop. Asked whether they would like to continue the problem-solving session, both answered positively. Group C explained that they were concentrated on problem solving tasks and therefore they did not feel tired.

Comment

Students seemed to be satisfied with the problem-solving session. They said it was useful, interesting, they were not bored and they wanted to continue. This implied that they learned and obtained assistance from ANGEL.
6.6.1.2 What they thought about working in pairs

Working in pairs was a part of the learning environment associated with ANGEL. Students said that although this was new to them, working in pairs was effective and promoted learning.

Al: We really enjoyed it. It was warm and quick. We helped each other.

In Group C it was observed that Alex tried to use his own method and this made Nick unhappy. Alex admitted that he was used to working alone.

Q: Do you like work in pairs?

Alex: I don’t know … it looks good. I am used to work alone

When the question was posed to Nick his response was, “… amazing, I learned a new method from Alex”

In Group B Adrian noted, "It was interesting. Adam is a very good partner."

Comment

Students appreciated the work in pairs as a technique or a strategy. However, the strategy may not be effective unless partners match.

6.6.1.3 What they felt about ‘process guidance’ and’ check your answer’

Students were asked the difference between the purposes of options process guidance and check your answer.

Group A: Yes, Process Guidance was useful to us several times when we had doubts.

Group B: Check your solution is after solving … and Process Guidance is during …

Group C: Check your solution is to check and Process Guidance is to proceed.

The students admitted that worked examples would be helpful in their work. According to Alicia, "It was really useful to us. My impression … I like it." To Nick, "It’s fine. We used it couple of times".

They were happy with the instructions in the first column except Adrian, "I prefer some steps, … they are useful … but … not all …". The instruction in the first column was elaborated in the second column of process guidance. Both students in Group A appreciated process guidance and were aware of the value and purpose of it, but could not see the need for the detailed and specific instructions available in the second
column, although they had used it to clarify the instruction ‘Highlight the goal’ in the first column. Their opinion was that the second column contained too much information, and splitting instructions to lower levels was not meaningful.

The purpose of instructions in the third column of process guidance i.e. ‘Show me’ links was clear to them. They knew how to access the worked example to ‘Check the answer’ using ‘Show me’ links. Their understanding was that ‘Show me’ links brought up useful information with diagrams. They appreciated the clear presentation of the animation, which they enjoyed.

The worked example was presented in three steps to provide step-by-step regulation. Students had no clear idea about the purpose of the three-step presentation (of worked example of the Base Problem) called up by ‘Check your answer’. Amjad started to think about this at the interview and commented that there may be advantages to the three-step presentation.

**Comment**

Interview data showed that students were happy with the instructions and information in process guidance and found it helpful in their work. They believed that process guidance was useful to overcome difficulties and worked examples were good to check the correctness of the process. They also appreciated the opportunity to check the answer with worked examples that are accessed via ‘Check your answer’ or ‘Show me’ links.

**6.6.2 Results drawn from Event Analysis**

Event analysis considers the events of the problem-solving process and can trace information about how students incorporate various supports in ANGEL during their problem-solving process.

The total number of problems in ANGEL is 15 although students can skip solving the similar problems. They are compelled to attempt to solve 10 problems: five base problems, and five advanced problems. In the problem-solving session, the configuration was changed so that students had to solve the first similar problem (Problem 1A) as well. For this reason, students were provided with an opportunity to solve 11 to 15 problems. In other words, 11 compulsory problems were included in the version used in the problem-solving session.
6.6.2.1 Problems students attempted

Table 6.9 provides a summary of the problems attempted by each group.

Table 6.9 – Problems attempted by each group

<table>
<thead>
<tr>
<th>Group</th>
<th>Problem</th>
<th>1</th>
<th>1A</th>
<th>1B</th>
<th>2</th>
<th>2A</th>
<th>2B</th>
<th>3</th>
<th>3A</th>
<th>3B</th>
<th>4</th>
<th>4A</th>
<th>4B</th>
<th>5</th>
<th>5A</th>
<th>5B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>B</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
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<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
<tr>
<td>C</td>
<td></td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
<td>✔</td>
</tr>
</tbody>
</table>

Key - ✔ Problem completed

The symbol ‘✔' represents problem attempt and completion. A blank cell indicates the problem was not attempted. The problems in shaded columns are similar problems that were not compulsory in the problem-solving session.

Table 6.9 indicates that: Group A completed 10 problems, Group B completed 11 problems, and Group C completed 12 problems. Among 5 Problem Sets, Group C was able to complete all five sets of problems. Other groups were able to complete four problem sets and a part of the fifth set.

In terms of 11 compulsory problems, Groups A and B completed 10 problems, and Group C completed the lot. In addition, Groups B and C completed the non-compulsory problem (2A).

Comment

Results presented in Table 6.9 show that that all groups engaged in the problem solving session to a great extent. They successfully completed every problem they attempted. This implies that students can access the problem information that ANGEL provides.

6.6.2.2 Student access to process information in ANGEL

Table 6.9 does not indicate whether students accessed information to facilitate the solution process – such as process guidance and worked examples.

Table 6.10 presents this additional information.
Table 6.10 – Student access to information to facilitate the problem-solving process

<table>
<thead>
<tr>
<th>Group</th>
<th>Problem</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 1A 1B 2 2A 2B 3 3A 3B 4 4A 4B 5 5A 5B</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓ ✓</td>
<td></td>
</tr>
</tbody>
</table>

Shaded cells in Table 6.10 represent the problems that were attempted by the groups. The symbol ✓ represents the events in which students accessed either worked examples or process guidance as they solved the problem. Table 6.10 shows that students accessed support for all the problems they attempted.

Comment
All groups demonstrated that they could obtain solution information from ANGEL for all problems attempted. This implies that students can access the solution information that ANGEL provides.

6.6.2.3 How students sought process guidance support
Table 6.10 provides information to show that students have obtained either guidance or solution information during their solution attempts but it does not provide sufficient information to understand how ANGEL contributed to the learning process.

Students can access ANGEL and obtain information for two different purposes – firstly to check their solution and decide if correct to proceed; secondly to obtain guidance information and overcome a difficulty. The summary event analysis related to process guidance events is presented in Table 6.11.

Table 6.11 – Student access to process guidance in ANGEL

<table>
<thead>
<tr>
<th>Group</th>
<th>Problem</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 1A 1B 2 2A 2B 3 3A 3B 4 4A 4B 5 5A 5B</td>
<td></td>
</tr>
<tr>
<td>A</td>
<td>4 2 2 1 1</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>4 2 1 1</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>2 3 1</td>
<td></td>
</tr>
</tbody>
</table>


The shaded cells in Table 6.11 represent problems for which students obtained guidance information. The number in the cell represents how many times the group accessed ANGEL seeking guidance. All three groups obtained guidance for problems 1, 2 and 2B. Two groups obtained guidance for problem 3B and one group for problem 4. According to information in Table 6.11, the numbers along a row decrease. Students accessed guidance information from ANGEL 24 times. Collectively they took guidance 10 times for Problem 1, 7 times for Problem 2, 4 times for Problem 2B, 2 times for 3B, and once for Problem 4. The pattern shows that the need for process guidance decreased as students made progress. No group sought assistance for their last problems, although they are more difficult problems.

Comment

The above analysis indicates that the students had no trouble obtaining process guidance from ANGEL when they had difficulties – they knew how to access ANGEL to overcome the difficulty. The decreasing rate of access to process guidance implies that the degree of difficulty faced by students decreased as they worked through problems. In summary, all groups have demonstrated that they can access ANGEL to obtain guidance instructions to overcome difficulties, and ANGEL can provide information required to guide students.

6.6.2.4 How students used ANGEL for regulation of problem solving

ANGEL provides the facility to check the answer for all problem events through presentation of the relevant part of the worked example. The worked example for a base problem provides three such points of reference, process guidance provides seven points to check the answer and a similar problem provides the complete worked example in one step. In that sense, students can check their answers when they are in process guidance or solving mode.

Table 6.12 shows how students used ANGEL to obtain guidance (check solutions) in the problem-solving process.
Table 6.12 – Student access to check the answer in ANGEL

<table>
<thead>
<tr>
<th>Group</th>
<th>Problem 1</th>
<th>1A</th>
<th>1B</th>
<th>2</th>
<th>2A</th>
<th>2B</th>
<th>3</th>
<th>3A</th>
<th>3B</th>
<th>4</th>
<th>4A</th>
<th>4B</th>
<th>5</th>
<th>5A</th>
<th>5B</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>5</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>B</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>C</td>
<td>2</td>
<td>1</td>
<td>1</td>
<td>3</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

Shaded cells in the Table 6.12 represent problems in which students obtained solution information. The numbers show how many times they accessed ANGEL to obtain information during the process of solving the respective problem.

Table 6.12 shows that the students have checked each answer at least once. Sometimes they checked their answers more than once and it is likely that they have checked their answer in a stepwise fashion. The number of checks declines from left to right along a row of the table.

Comment

Students can use ANGEL to access the worked example and check their answer, as indicated by the initially high access numbers. There is an apparent decrease in the need for multiple checks as they become more familiar with the problem-solving process. In summary, all groups have demonstrated that they can access ANGEL to obtain information for regulation of the process, and ANGEL can provide information required for regulation.

6.6.3 Results of Problem-solving analysis

Problem solving analysis shows in richer detail how successfully students attempted a problem-solving task. It provides indicators to understand the effectiveness of the instructions presented in the learning environment. The following analysis provides information required to address the question \((q_e)\) as evidence.

Students require information from the learning environment for two purposes during the problem solving process:

1. To overcome a difficulty; and
2. To regulate the process.
The success of the learning environment depends on the extent to which ANGEL can provide information to facilitate students learning problem-solving tasks.

6.6.3.1 Report 1: Group A in the first problem set

The following is a complete report on how Group A students obtained information from ANGEL to solve the first problem set.

Problem 1

Alicia read the problem with a clear voice. Amjad also followed problem information as Alicia read. On Amjad’s request, Alicia turned to process guidance for further instructions. She read the first instruction in process guidance to complete the diagram and mark information. While drawing the diagram Amjad obtained information from the problem box in the process guidance page to complete the diagram.

During the drawing process Alicia was watchful, however Amjad did not want help from her. At this point Amjad asked Alicia to retrieve the answer. She used the ‘Show me’ link to retrieve it and compare their answer. Realising that their answer was correct Alicia expressed their success with an extended voice "Great!", and asked to click “back to process guidance”. After that they found two vertically opposite angles. This was not sufficient to prove the congruency.

However, they could not find the way out until they again used the ‘Show me’ link of the instruction ‘what is missing’ to find a clue. Students realized that they had not used the fact that AB and CD are parallel. This clue was very important to them.

AI: You need to use … OK … AB is parallel to CD! … We didn’t use this. …

Alicia quickly found two other relationships of alternate angles on parallel lines and the final answer in a shorter time. Amjad cut down the vertically opposite angle relationship and started writing the proof.

The entire process took approximately 20 minutes.
Problem 1A

After reading the problem, the students started drawing. Suddenly Alicia shouted:

"Same as the previous diagram!"

by that time Amjad also had identified the connection between two problems. They completed solving, recording and checking in approximately five minutes. Students directly gathered information for angle – angle – side postulate.

Problem 1B

Without error they identified the correct path to Problem 1B. They spent little time reading. They were able to prove the congruency part in four minutes. The group was happy at the end of the first part. In Part II, they stated an answer that is not wrong, but not acceptable. In checking the answer they found the expected answer, which can be derived from the previous part. Students transferred this knowledge into Part III. They did not use process guidance support for this part.

Comment

Students learned a strategy (ASA) to prove congruency. They demonstrated their willingness to actively participate and learn collaboratively. They also demonstrated that they could access the required information in ANGEL. They experienced success in Problem 1 after accessing the hint in ‘What is missing’. The decreased time taken for the similar task shows that they have increased awareness of the problem-solving process.
6.6.3.2 Report 2: Group A in the second problem set

The following report highlights evidence of the need for feedback and collaboration among students. It was also based on the work of Group A as they attempted the second problem set.

Problem 2

This time Group A simultaneously read the problem and drew the diagram. Amjad joined X to M, which was not relevant to the problem.

Based on this, Amjad marked another unnecessary relationship. Alicia opposed this, and argued that it was out of the required pair of triangles.

They were stuck here for a while. They did not use the properties that the triangle is isosceles and M is the mid point of the base.

They got information from process guidance.

“Yep, we did not realize it!” says Amjad.

At this point they laugh at their own mistake.

Once they got the clue, they reached an answer in approximately six minutes.

Problem 2B

The diagram for Problem 2B is considerably more complex. Amjad watched while Alicia was reading the problem. Alicia helped to draw. They wrote the goal this time. Alicia suggested they mark base angles as equal. Amjad made a conjecture that this problem may be associated with circle geometry. A long pause and some deep thought, but no result led Alicia to suggest comparing the diagram.
Alicia did not agree with this but she had no counter argument or alternative. She suggested checking the answer, "Shall we see the answer?" …

As the diagram appeared, Amjad admitted that he was wrong. Alicia suggested that they draw the diagram again and another struggle started to find a starting point.

However, they again added more information that was not given. Meanwhile, Alicia studied the process diagram and obtained clues and instruction. She realized the relationship between the present problem and the previous problem. Based on proving congruency first, Alicia explained the solution to Amjad. "No Amjad … first take this triangle and this triangle. These are congruent [we proved it earlier], then you get WY equals to TZ. Then take this triangle and this triangle. This angle equals this … they are base angles… This side is shared … This side equals this side, just as we proved. Then they are congruent. Now write".

Amjad carefully listened to the arguments and statements made by Alicia and asked Alicia to complete the writing. Alicia took the pencil from Amjad and explained again like a teacher. She completed writing the problem-solving process of Problem 2B without getting process guidance support again.
Comment

At the first attempt, the diagram was not correct for Problem 2. Alicia was doubtful of their solution process. In problem 2B, the diagram was wrong because of the faulty conjecture made that the problem was about circle geometry. Both situations highlight the pair’s collaborative learning process.

6.7 Discussion

The reaction, event and problem solving analyses have presented rich and meaningful data to answer the questions framing this formative evaluation.

6.7.1 Do students accept ANGEL?

Reaction analysis showed that students accept ANGEL as a useful learning environment. They stated their acceptance in three ways. First they stated that the problem solving session presented in ANGEL was useful to them and they learned more. Second, they appreciated working in pairs as a quick, warm and effective learning method for learning proof-type problem solving. Third, they admitted that information and instruction presented in ANGEL was useful.

Students preparing for public examinations are reluctant to waste time. At the interview, students stated that the problem solving session was successful and interesting. They were willing to devote their time to continuing the session. This may be because they had gained something worthwhile for their time.

Tiredness can be an indicator of a boring lesson. During the problem solving session, the investigator observed that students were actively engaged – a point that they verified at interview. They had spent approximately three forty-minute periods excluding the 30 minutes hands on session solving proof type geometry problems. This is also a good indicator of their acceptance of ANGEL in learning geometry.

ANGEL was developed for classroom use. Active learning engagement is an essential requirement of learning (Roschelle et al., 2000). A strong feature of an effective learning environment is collaborative construction of knowledge (Jonassen, 1994). The problem solving session was designed so that students were working in pairs. This problem-solving session constantly demonstrated collaborative construction of
knowledge. The investigator observed that students enjoyed working in pairs and the learning environment was appropriate to conduct collaborative group learning activities.

Nick appreciated working in pairs as he was able to pick an effective technique to cope with complex geometric figures. In his words,

“… amazing, I learned a new method from Alex”

Other groups also appreciated this method. In their words it was warm, quick, and interesting. Students saw that they could help each other and mutually enrich understanding.

Reaction analysis showed that students had a good impression of software features such as navigational options, information, and instructions. They found process guidance and feedback useful and admitted that ‘Show me’ links, dynamic and static visualization were useful to them.

6.7.2 Can the target group of students actually use ANGEL?

Event analysis showed that students used ANGEL to access information about 12 problems that span all problem sets. In the process they accessed information in various forms of worked examples and process guidance. They were able to access and retrieve problem information (Table 6.9), support information (Table 6.10), specific process guidance (Table 6.11) and specific feedback information (Table 6.12).

In the reaction analysis they described the meaning of navigational options and they demonstrated their awareness of accessing such sources intentionally and purposefully. In addition, the investigator observed that students demonstrated precise use of software – the introductory session (30 minutes) was more than adequate.

6.7.3 Does ANGEL work as designed?

ANGEL works as designed. The three groups were able to solve ten problems and two groups solved 11 problems. Tables 6.9 to 6.12 showed that students accessed information from different access points and there was no access failure reported.

According to the design, the problem set is the basic unit of configuration. That means the same configuration is applied to all problem sets. The first problem set was completed by three different groups and during that attempt the students accessed different types of information in different ways. There was no failure reported for the
first problem set that worked as designed. This was confirmed in the second problem set that was completely solved by two groups. In subsequent sets, students indicated they were freely able to bypass the similar problem when it was not required.

6.7.4 Does ANGEL help students to construct knowledge related to proof-type geometry problem solving?

Table 6.13 shows that students can access and retrieve information to overcome difficulties. Students found more difficulties at the beginning, but gradually reduced access to guidance information. From Problem 4 onwards students did not have difficulties. This indicates that ANGEL helped students learning to solve proof type problems. At the interview, students themselves admitted that the problem solving session was successful and working in pairs was useful.

There was ample evidence of activation and use of geometric content knowledge and general processing skills. Report 1 shows that the students in Group A learned the Angle-Side-Angle postulate in Problem 1 as a strategy for proving congruency. They improved in Problem 1A and transferred this knowledge in Problem 1B. Students recognized the structural similarity between Problem 1 and Problem 1A. This connection was useful for solving Problem 1A quickly and without difficulty. They also used this structural knowledge in Problem 1B. For instance, they spent five minutes to prove congruency in Problem 1A whereas it took four minutes to prove the congruency part in Problem 1B.

Report 1 also highlights the importance of support for planning through ‘What is missing’. Students accessed the missing clue for information in Problem 1 (parallel lines) and overcame the difficulty. In Problem 1B, their answer was not relevant and they were able to correct it with information in ANGEL. Students learned this and then applied it later successfully.

Report 2 shows the importance of analysis via diagrammatic support promoted by ANGEL. In Problem 2, their diagram was confusing due to an irrelevant line. As discussed, these irrelevant attempts are not wrong, but they take the solver away from the answer. As one member of the group did not agree with this, they did not deviate much from the correct answer. In Problem 2B, their diagram was wrong because they did not analyse the problem properly. However, students were able to correct this with
the support of the readily available feedback in ANGEL. Students thus used ANGEL to overcome difficulties as well as to regulate the learning process.

In both reports, it was highlighted that the students worked collaboratively to solve problems. Although both students were competent, they could enrich each other because one was quicker than the other at accessing the required knowledge element. On the other hand, when one student is more competent, they can scaffold the other.

6.8 Summary

This chapter presented the design, results and the discussion of Part II of Study 2 – a formative evaluation of ANGEL as a learning environment. Results presented through reaction, event and problem-solving analysis showed that the Year 12 students involved in this study liked to work with ANGEL and found it easy to use. It worked as designed and helped these six students to construct knowledge of geometry proof-type problem solving.
Chapter 7: Summary, conclusions and implications

7.0 Introduction

The primary purpose of the present study was to develop a learning environment that addresses the needs of proof-type geometry problem solving. This required identification of learning and instructional needs, and designing a learning environment including the development of a prototype followed by an evaluation based on the needs identified.

The literature review in Chapter 2 revealed that proof-type geometry problem solving could be difficult even though the student has built a substantial body of knowledge about geometry. There also appears to be a significant difference in the knowledge and processes that are involved in the solution of proof-type problems in comparison to other mathematical problems. The review suggested that three variables, namely, content knowledge, general problem-solving processes and mathematical reasoning are likely to influence the success of the proof-type geometry problem-solving process. The relative significance of these three potential variables on the construction of proof in geometry was investigated in Study 1.

The design, instrumentation, and procedures of Study 1 were detailed in Chapter 3 and the results of Study 1 were reported in Chapter 4. A multiple linear regression (MLR) analysis showed that geometry content knowledge, general problem-solving processes, and mathematical reasoning are predictive indicators of successful proof-type geometry problem solving. The results showed that nearly 73% of the score in proof-type geometry problem solving is explained by these three variables, while geometry content knowledge alone contributes to 67.2%. This contribution increased by the inclusion of general problem-solving processes and mathematical reasoning.

The identified needs from the literature and results of Study 1 were used as inputs for Study 2, which was the design and development of a learning environment to support the proof-type problem-solving process. This development process generated two outcomes: an overall conceptual model; and a prototype of a learning environment for learning to solve proof type geometry problems. This development process was presented in Chapter 5 as Part I of Study 2.
The overall conceptual model that resulted from Part I of Study 2 was a design of a learning environment that addresses the learning needs of all students at senior secondary level. It contained three components:

1. **Remedial** – to provide opportunities to make progress up to van Hiele Level 3
2. **Instructional** – to develop content knowledge at van Hiele Level 3
3. **Problem solving** – to familiarise problem-solving strategies and transfer

Although the overall conceptual model can address most of the instructional needs that were identified, it was too large to develop as a prototype within the time constraints of this study. On this ground, the problem-solving component was given priority in translating into a prototype. This component also was deemed to be important as it is given priority in reforms (NCTM, 2000; TRS, 2001) about students’ understandings and current classroom practice. The prototype developed was named *ANGEL* (A Non-linear Geometry Environment for Learning).

*ANGEL* addresses the needs based on a non-algorithmic approach, general problem-solving processes, and non-linear learning strategies. These needs are not addressed in the conventional classroom. In addition, *ANGEL* was designed for use with teacher support and with enhanced ICT-based metacognitive features.

Students should possess a maturity level in geometric reasoning equivalent to van Hiele Level 3 as well as prerequisite declarative knowledge to learn effectively with *ANGEL*. The usability of *ANGEL* was tested through a formative evaluation with end users at such maturity level. The results of the formative evaluation showed that *ANGEL* is appropriate for learning proof-type geometry problem solving. Details of this evaluation were reported in Chapter 6 as Part II of Study 2.

### 7.1 Findings

A summary of the research questions and the findings are presented in Table 7.1 on the following page.
### Table 7.1: Summary of research questions and answers

<table>
<thead>
<tr>
<th>(Q1). What are the predictive indicators of successful proof-type geometry problem solving?</th>
<th>(Q2). What is one design solution to support students solving proof-type problems in geometry?</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Study 1</strong></td>
<td><strong>Study 2</strong></td>
</tr>
<tr>
<td>Multiple regression Analysis</td>
<td>Iterative development process followed by formative evaluation</td>
</tr>
<tr>
<td>1. Predictive indicators of successful proof-type geometry problem-solving (PTG) are:</td>
<td>1. One overall conceptual model of a learning environment that must have three components:</td>
</tr>
<tr>
<td>Geometry Content Knowledge (GCK),</td>
<td>Remedial Component – to develop geometric reasoning up to van Hiele Level 3;</td>
</tr>
<tr>
<td>General Problem-Solving processes (GPS), and</td>
<td>Instructional Component – to develop content knowledge at van Hiele Level 3; and</td>
</tr>
<tr>
<td>Mathematical Reasoning Skills (MRS)</td>
<td>Problem-solving Component – to familiarise problem-solving strategies and transfer.</td>
</tr>
<tr>
<td>2. Geometry Content Knowledge is the principal predictive indicator, whereas General Problem-Solving processes and Mathematical Reasoning Skills are the other predictive indicators in order of significance.</td>
<td>2. ANGEL, the Web-based, non-linear learning environment that incorporates general problem-solving processes, and metacognitive control support both through worked examples that supports students at van Hiele Level 3 in learning proof-type geometry problem solving.</td>
</tr>
<tr>
<td>3. ANGEL can facilitate students solving proof-type geometry problems.</td>
<td>3. ANGEL can facilitate students solving proof-type geometry problems.</td>
</tr>
<tr>
<td>4. Poof-type geometry problem solving is well supported in a constructivist-learning environment with collaborative participation.</td>
<td>4. Poof-type geometry problem solving is well supported in a constructivist-learning environment with collaborative participation.</td>
</tr>
</tbody>
</table>

### 7.1.1 Major findings of Study 1

Multiple linear regression analysis was employed to determine predictive indicators and their relative influence. The scores of 166 students related to the independent variables (Geometry Content Knowledge, General Problem-Solving processes and Mathematical Reasoning Skills) and the scores related to the dependent variable (Proof-Type Geometry problem solving skills) were analysed on SPSS. The analysis resulted in the following findings.

- *Geometry Content Knowledge* (GCK), *General Problem-Solving processes* (GPS), and *Mathematical Reasoning Skills* (MRS) are predictive indicators of high school students’ success in *Proof-Type Geometry problem solving* (PTG).
• Geometry Content Knowledge is the major determinant of the success of proof-type geometry problem solving.

• Both General Problem-Solving processes and Mathematical Reasoning Skills can promote the use of geometry content knowledge in successful proof-type geometry problem solving attempts of high school students.

7.1.2 Major findings of Study 2

The inputs for the Study 2 were the instructional needs identified in Study 1, recent relevant research findings as revealed by literature search, and the expertise of the design group in geometry education. The design process was iterative and each cycle included design, development, review and revision.

The development process resulted in an overall conceptual design of a learning environment, and the development of a prototype website (ANGEL) along with the documentation of the development process. ANGEL was tested at the end user level through a formative evaluation as an essential part of the development process.

The major findings of Study 2:

• One overall conceptual model of a learning environment that addresses instructional needs of learning proof-type geometry problem solving for high school students. It was decided that the conceptual model must have three components: Remedial Component – to develop geometric reasoning up to van Hiele Level 3; Instructional Component – to develop content knowledge at van Hiele Level 3; and Problem-solving Component – to familiarise problem-solving strategies and transfer.

• ANGEL, the Web-based, non-linear learning environment that incorporates general problem-solving processes, and metacognitive control support both through worked examples, is one of the possible solutions appropriate to support students at van Hiele Level 3 in learning proof-type geometry problem solving.

• Proof-type geometry problem solving is well supported in a constructivist learning environment with collaborative participation.
7.2 Issues concerning learning

7.2.1 Importance of geometry content knowledge in proof-type problem solving

The greater contribution of content knowledge to the success of proof-type geometry problem solving is an important result of the analysis of Study 1. The linear multiple regression analysis showed that geometry content knowledge is a major determinant of the success of proof-type geometry problem solving.

The correlation coefficient between the content knowledge and the proof-type geometry problem solving was found in this study to be 0.82. In a study carried out in Australia, Chinnappan (1992) obtained a correlation coefficient of 0.83 for the same variables. This result of correlation is also consistent with those reported by Senk (1985). In that study, which involved 2567 students from the United States, Senk obtained a Pearson correlation coefficient of 0.67 for proof-type geometry problem solving and geometry content knowledge. This further demonstrates the role of geometry content knowledge in successful proof-type geometry problem solving.

Content knowledge related to proof-type geometry problem solving could exist in various forms. Among them, declarative knowledge, diagrammatic reasoning, axiomatic reasoning, and formal deductive reasoning are important.

7.2.1.1 Importance of declarative knowledge

Declarative knowledge related to proof-type geometry problem solving could be divided into two basic types: concepts and relationships. Concepts refer to basic geometric elements such as points, angles, triangles, various types of quadrilaterals and other polygons. Relationships refer to qualitative comparisons between parts in a same geometric figure or different figures such as theorems. Some times concepts hold relationships. For instance, the relationships: opposite sides of a parallelogram are parallel, and they are equal in length are some relationships within the concept of ‘parallelogram’. These relationships can be viewed as properties of a parallelogram.

Proof-type geometry problems are presented with various geometric terms, which are related to declarative knowledge of geometry. As it was discussed in Chapter 3, these key terms are important for students to understand the problems.
Representation of problem information in diagrammatic form can be viewed as the spatial representation of the given problem. In this process, declarative knowledge plays a key role. For instance, in diagrammatic representation of ‘ABC is a triangle’, the student must know the properties of a triangle such as three sides and three angles.

During planning a move, the student searches for geometric relationships that also constitute declarative knowledge components. This search seems to be accompanied by reasoning process across the declarative knowledge base that consists of various relationships and theorems.

It is difficult to classify declarative knowledge according to types of proof-type geometry problems. Declarative knowledge components required to solve a proof-type geometry problem can form any component of the content. For instance, the first problem of the exercise may be related to isosceles triangles while the next problem could be associated with parallelograms. Knowledge components used in one problem are not necessarily the required ones in subsequent problems in the same exercise. Hence, the required knowledge components are problem-dependent. This may explain the non-algorithmic nature of proof-type geometry problems. As a result, the student must possess sound declarative knowledge in geometry to access. It can be expected that lack of or poorly connected geometry content knowledge would result in low achievement in proof-type geometry problem solving.

Declarative knowledge thus plays an important role in the problem-solving process. However, declarative knowledge in geometry alone cannot transform a state into the next state in the problem space. The student needs to use problem information and construct the solution through a transformation process. In addition to the content knowledge required, operators and strategies are required to transform one state to the next. Operators could be supported by axiomatic reasoning whereas strategies involve formal deductive reasoning.

As was discussed, the requisite declarative knowledge is problem dependent. All pieces of declarative knowledge, the selection of required declarative knowledge appropriate for the solution (Koedinger & Anderson, 1993; Healy & Hoyles, 1998) is difficult. Through a theoretical analysis of possibilities, Koedinger and Anderson (1993) suggest that the student has to select a single entity from a large pool of rules and they argue that students need to infer reasonable rules to try and apply.
7.2.1.2 Importance of geometric diagram

Declarative knowledge related to proof-type geometry problems cannot remain in semantic form alone in the solution process. It has to be integrated with a geometric figure in a meaningful manner.

During problem solving, the diagram provides support for the students to hold declarative information in a visual and conceptual form. Although the notions of visual and conceptual forms may not always coexist, recent studies raise arguments that geometry diagrams are not analogies, but one kind of conceptual representation similar to symbolic representations (Charalambos, 1997; Fischbein, 1993). Without geometric diagram, geometric proof development could be difficult to achieve.

Although the solution is presented in sequential semantic array, the actual information processing takes place in the geometric diagram or based on the geometric diagram. The geometric diagram can represent concepts, geometric relationships, given information and new information that has been generated. In most cases, the goal is also represented in the geometric diagram. As the solution progresses, the number of information entities increases. Eventually, the diagram represents a large amount of information. Diagram is involved not only in generating new information, but also in making inferences or conjectures and selecting rules and strategies. Geometry diagram is thus an important component in the solution process.

Although the solution process is developed on the basis of the diagram, the information in the diagram with arbitrary marks made by the student may not represent the logical sequence of the transformation from given to the goal unless it is consistent with the logical chain of semantic symbolic form. The solution process of a proof-type geometry problem is thus a result of interplay between the diagrammatic representation and semantic symbols, both can be considered as declarative knowledge.

7.2.1.3 Role of mathematical reasoning

Proof-type geometry problem-solving process requires three types of knowledge elements: geometric content knowledge such as geometric concepts, properties and relationships; knowledge elements related to axioms; and appropriate deductive forms. Selection of these elements can be viewed as involving three types of reasoning, geometric reasoning, axiomatic reasoning, and deductive reasoning.
The need for the development of geometric reasoning has been acknowledged in the literature review in this study. It develops through a process of inductive reasoning. Research reveals that students cannot solve proof-type geometry problems unless their geometric reasoning is equivalent to van Hiele Level 3. According to Senk (1985; 1989), the majority of students (93%) in high school had not achieved this level of reasoning. This suggests the mathematics teacher needs to help senior secondary students by preparing a "ground work" necessary to get students prepared before starting work at van Hiele Level 3.

This study provides a framework for how to develop students’ geometric reasoning. As was revealed in literature, development of geometric reasoning is hierarchical, and each student needs to make progress through van Hiele Level 0, 1, and 2 in lower grades. The teacher has to identify students who have not reached the prerequisite geometric reasoning level. This requires a diagnostic test to identify the van Hiele Level of the student.

As students are at different van Hiele Levels, each student should be provided with remedial activities appropriate to his or her current van Hiele Level so that the student can gradually make progress up to van Hiele Level 3. How to achieve this outcome has been discussed in 5.4.4.1 of Chapter 5. It requires a series of learning activities that enable students’ inductive reasoning in order to develop appropriate abstract concepts of geometric shapes, their properties, and relationships between parts of geometric shapes.

Axiomatic reasoning provides operators to transform relationships into new relationships. The student has to select the appropriate one from 10 possible axioms. No rules or algorithms exist for this selection. For instance, the axiom: when the same is added to equals, the results also equal, is used to transform the relationship $AX = BY$ into $BX = AY$ in Problem 1 of the Geometry Problem Solving Test used in Study 1 (Figure 7.1).
Question:
The line AB has been produced to either sides so that AX = BY. Prove that AY = BX

Answer:

\[ AX = BY \text{ (given) } \]
\[ AX + AB = BY + AB \text{ (AB is added to equals) } \]
But \( AX + AB = BX \text{ and } BY + AB = AY \)
Therefore, \( BX = AY \)

**Figure 7.1 - Using the theorem: the same is added to equals, the results also equal**

On the other hand, Problem 2 requires the axiom: *when the same is subtracted from equals, the result also equal* in its solution process. The first axiom mentioned above corresponds to the normal mathematical addition, and the second axiom corresponds to the mathematical deduction. Selection of the appropriate axiom is thus critical and it is determined by the appropriateness of the solution path. This may be a reason for the influence of mathematical reasoning of proof-type geometry problem solving.

As it was revealed in Section 2.2.4.3 of Chapter 2, deductive reasoning is also an essential requirement for proof-type geometry problem solving. In proof type-geometry problem solving, a new statement is deduced from previous statements. For instance, the following example illustrates a chain of such deductions. According to the illustration, the statement: \( AX + AB = BY + AB \) was deduced from the statement: \( AX = BY \), whereas the answer was deduced from the statement: \( AX + AB = BY + AB \). This kind of goal directed abstract deduction may be difficult for students who are beginning to learn to solve proof-type geometry problems.

Geometric reasoning is unique in proof-type geometry problem solving whereas axiomatic reasoning and deductive reasoning is common to all mathematical proofs.

### 7.2.2 Inert knowledge in proof-type geometry problem solving

This study highlighted some instances where students failed to solve problems even though they appeared to have the necessary content knowledge. That is, the student does possess the appropriate knowledge, but cannot apply it in the situation where it can be used. Bereiter and Scardamalia (1985) referred to this phenomenon as *inert knowledge*. Various studies have shown that traditional approaches to instruction on declarative and
procedural knowledge often result in students developing inert knowledge (Bereiter & Scardamalia, 1985; Bransford, Franks, Vye & Sherwood, 1989). However, strategies such as modelling expert behaviour and coaching students to imitate expert skills until they are competent in their performance can address the problem of inert knowledge (Collins, Brown & Newman, 1989).

Study 1 provides evidence of inert knowledge (see Figure 4.13) where students failed to solve problems although they appeared to have sufficient content knowledge. This pattern of results is consistent with those reported by Lawson and Chinnappan (1994) which showed that a group of less successful high school problem solvers failed to access available knowledge independently, but were able to do so when prompted.

The solution of various example problems with the same declarative knowledge could be expected to reduce the inert knowledge problem. Inert knowledge may hinder reasoning or use of conditional knowledge. Knowledge related to proof-type geometry problems is mainly declarative; it can hold generic features and aid transfer. Therefore, declarative knowledge may be more useful for a range of situations than procedural knowledge. Opportunities to use the same declarative knowledge for different situations would help students to hold information of conditional knowledge (of when it could be used) or reasoning (why it is used) about the declarative knowledge used.

Providing clues through metacognitive support to stimulate the activation of existing knowledge may be a useful strategy to address the inert knowledge problem. Metacognitive awareness might help students access required knowledge components to be applied in problem solving which requires students to extend their knowledge to novel or complex situations and make links among knowledge components.

ANGEL attempts to address the inert knowledge problem through the worked example approach. It provides metacognitive support to access clues to identify required declarative knowledge in different situations. It also provides different situations that use the same declarative knowledge.

7.2.3 Relevance of general problem-solving processes in proof-type geometry problem solving

There is an on-going debate about the relative roles of content knowledge and general processes in problem solving (DeFranco & Hilton, 1999; Lawson, 1989; Sweller; 1989).
This issue received considerable attention in the present study in the context of high school students solving proof-type problems. The literature review indicated the existence of divergence of views on this issue.

Study 1 demonstrates how successful students used these skills during the solution process as well as the difficulties experienced by students who did not activate and use some of the skills. Lawson and Chinnappan (1994) and Lawson (1991) have discussed the importance of general processes in activating available knowledge components during problem solving processes. The algorithmic nature of proof-type geometry problem solving places heavy demands on the use of general processes such heuristics. Heuristics have a role in proof-type geometry problem-solving process as algorithms or predetermined procedures are not available to the student. The heuristics that are found in proof-type geometry problem solving can be categorised as domain specific and domain general.

During the solution process, when working forward from data is difficult, decomposing problem into sub-goals and recombining, working backward are useful domain general heuristics in the proof-type geometry problem solving. There is no explicit rule to guide the student when and where these heuristics are to be used. Analysis, planning and representation could aid the students in the selection of appropriate heuristics.

Among domain-specific heuristics, the drawing of auxiliary elements such as parallel lines, perpendicular lines, and angle bisectors during proof-type geometry problem solving may be powerful strategies for particular situations. There may be instances where strategies such as indirect proof, exhaustion, and reductio ad absurdum also have to be mixed with deductive proof. These cases also exemplify the role of domain specific heuristics. Reasoning could also be a factor in selection of appropriate heuristics.

Proof-type problem solving requires a novel approach in comparison to solving algorithmic type mathematical problems. The instructional process is more difficult for a problem-solving process when the process does not demonstrate significant patterns among problems. To solve proof-type problems students must be familiarised with different problem types as well as different situations. Modeling problem-solving strategies with a range of worked examples may be useful in this regard, an issue that was addressed in the present study.
Cognitive processes such as identification of problem information and goal, selection of problem-solving strategies, accessing and use knowledge components in generating new information, making decisions about the various points in the solving process seem to have important roles in the solution process. These actions could be supported by four general processes: analysis, representation, planning and use of knowledge retrieval. These were evident in all solution processes related to proof-type problems completed by the participating students in the present study.

7.3 Implications for instruction

The present study attempts to foster deductive reasoning skills. It was argued that students are developing at multiple levels in geometric thinking. Thus they need support that is consistent with their level of achievement in order help them acquire the prerequisite level of geometric content knowledge and reasoning that is necessary for proof-type geometry problem solving.

7.3.1 Implications for instruction: general perspectives

A major outcome of Study 1 is the influence of content knowledge in proof-type geometry problem solving. Literature review in Chapter 2 revealed that proof-type geometry problem solving is difficult for a majority of the students in many countries. Research literature also revealed that the main reason for such a situation is their poorly developed geometric reasoning skills.

In the instructional process, it is very important to consider the student's readiness for learning proof-type geometry problem-solving process in terms of van Hiele levels. Student cannot solve proof-type geometry problems if the geometric reasoning level of the student is below vHL3. In other words, if the student does not possess the required level of thinking the instructional effort may not be effective. Therefore, developing an appropriate level of geometric reasoning is an essential consideration of instruction.

The overall conceptual model of learning environment designed through expert contribution in Chapter 5 recommends the following instructional process for learning proof-type geometry problem solving at secondary level:
1. Identification of the level of geometric reasoning of each student.

2. Providing with remedy for developing appropriate geometric reasoning among students who have not achieved vHL3 so that they can make progress to that level.

3. Providing the students with appropriate content knowledge for those who are at vHL 3.

4. Providing opportunities for problem solving sessions.

Research review in Chapter 2 revealed that a proof-type geometry solving class may include students who are achieving below the required van Hiele level. This problem seems to be prevalent in mathematics classes throughout the world. Hence, the identification of student's current van Hiele level is important. However, research has shown that the nature of the diagnostic tests used for the purpose of describing students’ van Hiele level may vary from country to country (Lawrie, 1998).

Once the current level of the student is established as being not below van Hiele Level 3, theoretically the student can be provided with the remedy. The practical issues related to such a process are critical. Firstly, there is the problem of how can a single teacher cater for all students with various levels of geometric thinking. The second problem is how do teachers find time to provide for the range of learning experiences, as this remedial support is not accounted for in the time frame of the school curriculum. Third problem concerns the nature of the remedial programme that can be provided that is effective.

Providing content knowledge for those at vHL3 may not be difficult as some of content knowledge related to proof-type geometry problem solving such as geometric figures, concepts, notations, and basic relationships, is familiar to students at vHL2.

The instructional process suggested in Study 2 for proof-type problem solving has been developed as a prototype ANGEL and tested in a target group of six students. As indicated, the purpose of the present study is to address the instructional needs of proof-type geometry problem solving. The reason for using Web instructions in ANGEL was to provide assistance to the class teacher in providing non-linear instructions. The features incorporated in ANGEL are valid for the proof-type geometry problem-solving process and their implications are relevant to such instructions.
The problem-solving component requires strategies for:

- Problem familiarisation
- Transfer of knowledge
- Access content knowledge
- Visual support

### 7.3.1.1 Problem solving in small groups

Part II of Study 2 showed that the Problem-Solving Session is effective in pairs. It can be inferred that the same results may be possible in small groups. However, the teacher has to decide how to group and which students should be in each group. The teacher also has to cope with two contradictory situations: if the number of students in a group increases, some students tend to hide from participation; if the number of students in a group decreases the number of groups increases and it might become hard to manage groups.

During students’ work in groups, the teacher needs to provide scaffold and act as a learning partner or motivate expert students in scaffolding others.

### 7.3.1.2 Worked examples

After presenting a theorem or related theorems, the normal practice is for the teacher to demonstrate one or two examples. While presenting the working on the board, the teacher clearly explains the reasons and how decisions were made. Eventually, only the answer is written, which is almost the same as that in the textbook. The traditional worked example method is also an expert presentation similar to teacher presentation that presents an answer without solution process and necessary explanations.

In contrast, worked example used in ANGEL is more effective in two ways. First, it follows a problem-solving attempt whether the problem is unstructured or structured. Second, the worked examples used here are information rich and intelligible.

### 7.3.1.3 Problem set

Problem-solving skills improve according to the depth and length of training. 'Drill and practice' is not appropriate for non-algorithmic problem situations like proof-type geometry problems. One appropriate instructional strategy for non-algorithmic
problems is problem familiarization (Robertson, 2001). Although problem similarity is an important factor in this regard, it is very difficult to find different examples with the same problem structure in non-algorithmic problem domains. A problem set was introduced to overcome this difficulty in the present study. The formative test of ANGEL showed that problem sets configuration was effective for proof-type geometry problem solving as they: obtained less help from Process Guidance for Similar problems and Advanced Problems (Table 6.11 of Chapter 6); and, wanted fewer checks from the worked example (Table 6.12 of Chapter 6).

The principle behind the problem set was to familiarise students with a problem solving strategy, provide an opportunity to improve the strategy, and provide another opportunity to transfer knowledge from previous experience of problem solving into novel situations.

There are three problems in a set: Base Problem, Similar Problem, and Advanced Problem. Problems in a problem set have a relationship in relative difficulty as shown in Figure 5.24.

Step 1 – Introduce a strategy in a problem situation (the relevance of Base Problem).

Step 2 – Provide a problem situation structurally similar to previous problem to improve the strategy (the relevance of Similar Problem).

Step 3 – Provide a situation one step ahead from the other two (the relevance of Advanced Problem).

As the notion of problem set is unique, and was introduced in this thesis, notes on construction of the problem set are important for instruction and future work in the area. Construction of Base Problem is not difficult; but it must be a harder problem than the Base Problem in the previous problem set. The Similar Problem is the same problem with different labelling. The Advanced Problem is just one step ahead of the Base Problem.

7.3.1.4 Process guidance

Chapter 2 revealed that the solution process of proof-type geometric problems is process oriented, and Chapter 3 confirmed that general problem solving is a predictive indicator of successful proof-type geometry problem solving. The process guidance was
introduced in Part I of Study 2 and evaluated in Part II of Study 2. The evaluation showed that the Process Guidance is a useful strategy to overcome the absence of the algorithmic approach.

Although geometry problem solving is not algorithmic in terms of problem structure, the solution processes involved share four major cognitive processes: analysis; representation; planning; and use of knowledge retrieval. These provide the following framework for examining proof-type geometry problem solving.

**analysis:** Reading and understanding the text-based problem that includes given and the goal.

**representation:** Drawing the geometric diagram and marking information in the diagram.

**planning:** Exploring through the diagram for strategies that might be helpful to generate information towards the goal.

**use of knowledge retrieval:** Using knowledge to generate new information with the strategies conjectured.

Students need not necessarily know the names of the cognitive processes, but they need to know what they are doing in each case as indicated. The instructions about the processes can be displayed in the classroom while the class is engaged in proof-type geometry problem solving.

### 7.3.1.5 Providing content knowledge during the solution process

Knowledge is a basic requirement in the problem-solving process. In solving of proof-type geometry problems, students need to know particular geometric relationships. When they ask about a certain theorem, the teacher may provide such knowledge. However, students cannot ask if they do not know what they need. On the other hand, if the student's knowledge remains inert, it is difficult to identify the student's knowledge requirements. How to fill this gap is instructionally strategic. The approach of ‘What is missing?’ introduced in Process Guidance is one option. Once the student's need is known, then the teacher can provide for it. Figure 5.19 illustrates how ANGEL
accomplishes this and shows how the process can be difficult with conventional instructional approaches.

The second option is providing the answer. Some knowledgeable students can recognize the knowledge components that have been used in the solution. When they are working in a group, such knowledgeable students can help others in knowledge construction.

7.3.1.6 Diagrammatic representation and visual support

The facility to obtain and generate information from the diagram is important in proof-type geometry problem solving. Part II of Study 2 provides examples of students' use of various methods.

Generally, the geometry diagram can provide relevant information as well as irrelevant information to the solution process of proof-type geometry problems. At the beginning of proof-type geometry problem solving, the students might lose the way to the answer because of irrelevant information. For instance, Group B and Group C here tried to separate required parts from diagrams to avoid irrelevant information.

The development process of Chapter 5 suggested some strategies such as colours thickness and animation in visual support. Students need to be guided to use such strategies for the success of proof-type geometry problem solving.

7.3.2 Confluence of two methodologies

This study draws on and attempts to integrate cognitivist and constructivist principles in the formative evaluation of ANGEL. The need to adopt this approach has been supported by Anderson, Greeno, Reder, and Simon (2000), Brown (1996), Hoek et al., (1999). The advantages are that each approach can complement the advantages of the other.

It is sometimes asserted or suggested that the situative perspective accords too little importance to individuals because it emphasises participation in social practice; and it is sometimes asserted or suggested that the cognitive perspective neglects processes of social interaction because it emphasises individual development in the acquisition of intellectual skills (Anderson et al., 2000, p. 11).

According to this, students need to learn problem solving as a socio-cognitive activity. For instance, Anderson et al, (2000) recommend:
• Individual and social perspectives on activity are both fundamental in education;
• Situative and cognitive approaches can cast light on different aspects of the educational process, and both should be pursued vigorously;
• Learning can be general, and abstraction can be efficacious, but they sometimes are not;
• Educational innovations should be informed by the available scientific knowledge base and should be evaluated and analysed with rigorous research methods (Anderson et al., p.11).

Part II of Study 2 showed that the individual cognitive aspects can be carried out in social interactive situations by allowing students to work in pairs, thus blending the two approaches.

7.3.3 Implications for policy makers and curriculum developers

Proof-type geometry problem solving has been given little emphasis in comparing educational systems. The following reasons may be taken into consideration in examining the above situation.

1. In Chapter 1, it was discussed that students can avoid learning deductive geometry within current evaluation systems. When students can take options, it is a common trend that students leave out such questions. Even deductive geometry problems that are compulsory would not affect this unless the effect of leaving problems reduces the student's mark. This requires solutions at policy level and curriculum design level.

2. Development of geometric reasoning has to be given a special emphasis. Research reveals that student ability can be spread at multiple levels. The reason for this is related to lack of geometric reasoning development in lower grades. It is appropriate to provide facilities for developing geometric reasoning according to van Hiele Theory.

3. The need to identify students' level of geometric reasoning has become an important issue. This has to be acted upon at national level and should be remedied at the earliest opportunity. Developing separate standard tests for geometric reasoning would be appropriate rather than using teacher-made tests at classroom level.
7.3.4 Implications for teacher educators

Teachers have to be trained with the relevant skills at in-service and pre-service programmes. *Study 2* emphasises the following roles for teachers in teacher education courses and session.

1. Preparation of appropriate diagnostic tests.
2. Facilitating students' group activities as expert partners.
3. Preparing suitable problem sets, Process Guidance, and appropriate worked examples with fading features.
4. Encouraging active cognitive engagement during the group problem solving attempts.
5. Providing opportunities to improve self-regulation in individual attempts.
6. Facilitating explanations to others as a learning method.

7.3.5 Implications for classroom mathematics teachers

Teachers have to identify students' levels of geometric reasoning and take appropriate remedial action for preparing them for learning at van Hiele Level 3. This remedial process can take time, energy and resources. However, in teaching students without remedy students would not perform well. If the teacher does have access to a diagnostic test, then one must be developed and provided.

In Chapter 2 it was discussed that because proof-type geometry problem solving is non-algorithmic, problem familiarisation and metacognitive support are important for students. In Chapter 4, it was found that general problem-solving skills could foster proof-type geometry problem solving skills. These features and guidance are embedded in *ANGEL*.

However, software would not be effective unless the teacher uses it strategically. *ANGEL* was designed for use in a constructivist-learning environment in a classroom setting. Using this with students working in pairs would be most appropriate, but depending on physical resources this may be extended to small groups.


7.4 Contribution to the research methodology

The methodology employed in the present study necessitated the development of specific instruments and techniques. These are considered as significant in that they add to the current body of literature.

- The rubric developed (presented in Chapter 3) to analyse proof-type geometry problem-solving process presents a procedure that can also be used to analyse mathematical problem-solving processes. The rubric can also be used as a tool for classroom assessments.
- The Table of merged observational and verbal (MOV) data developed (presented in Chapter 6) constitutes a simple and less expensive technique that can be used to combine discourse and observations in a naturalistic situation.

7.5 Limitations of the study and implications for researchers

In addressing a long existing problem, the present study suggests one possible solution for supporting senior secondary students in learning proof-type problem solving. The design process translated empirical findings from the literature as well as from Study 1. In addition, this thesis presents a single solution among various possible solutions. There are various limitations associated with the study. The following limitations are noted for possible future investigations.

1. In Study 1, Mathematical Reasoning Skills were represented by achievement in mathematics.
2. Only a part of the suggested conceptual design was translated into a prototype.
3. Translating instructional features of ANGEL into other instructional strategies.
4. Part II of Study 2 was carried out in a ‘laboratory’ setting.

The next section will provide some suggestions for further research.

7.5.1 Further investigating the influence of mathematical reasoning

In the design of Study 1, Mathematical Reasoning Skills was measured by achievement in mathematics tests. The premise for this was based in research findings such as those of Schoen, Hirsch & Ziebarth, (1998) which state that there is a high correlation between mathematical reasoning and mathematical achievement.
The present study was limited to collecting data about mathematical reasoning from school-based assessments and limited to students' learning time to six forty-minute lessons on answering test papers and one period for preparation. It can be estimated that this process was helpful to cut down what could have taken two periods on mathematical reasoning. For instance, the mathematical reasoning part of the SAT I test conducted by the College Board takes 75 minutes (Sample Paper Available: http://www.collegeboard.com/prod_down loads/sat/satguide/SAT_Full.pdf).

It is suggested that mathematical reasoning tested with SAT 1 or Ability To Do Quantitative Thinking (ATDQT), which is a test for mathematical reasoning made by Iowa Tests of Educational Development, may be more appropriate for future investigations.

7.5.2 Use of suggested conceptual design for proof-type problem solving

The overall conceptual model was designed to address the complete set of needs that were identified. Due to time constraints, only the problem-solving component of the overall conceptual model designed in Study 2 was translated into ANGEL. Thus, only students who are at van Hiele Level 3 of geometric reasoning will benefit from ANGEL.

The two other instructional components in the conceptual model have yet to be physically developed and evaluated:

- Remedial component;
- Instructional component.

The overall conceptual model for geometry proof-type problem solving needs to be related to geometric reasoning development, content knowledge development, problem solving, and multilevel student needs.

It is suggested that one future study develop an overall conceptual model of a learning environment for proof-type problem solving.

7.5.3 Translating instructional features in ANGEL to other strategies

While ANGEL was developed within a potentially hypertext environment (scripted in html), a number of the core structures and strategies to emerge from this prototype can be translated to non-web based environments. The basic features of the model are valid
for many kinds of instructions and for multi-level students from conventional classroom level to ICT supported environments. It can be used as the source for a series of studies. Following are some possibilities.

1. The notion of problem sets
2. Presenting worked examples in an evolving form
3. Process guidance
4. Embedded content knowledge
5. Presentation of geometric diagrams

7.5.4 Evaluating ANGEL in a normal classroom setting

In the formative evaluation, ANGEL was tested while students were working in pairs. They were called one pair at a time because the evaluation aimed to initially determine detailed use of the software in pairs rather than more widespread general use in a naturalistic classroom setting. However, ANGEL was designed for classroom use with teacher support, and given the results of Chapter 6, its use within such a complex situation should now be evaluated.
References


Mayer, R.E. (2003). The promise of multimedia learning: Using the same instructional design methods across different media. Learning and Instruction, 13(2), 125-139.


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Appendix 1 – Approval of the Review Committee for the study

M B Ekanayake  
Faculty Education  
University of Wollongong  
WOLLONGONG NSW 2522

Dear Ekanayake

Re: Proposal Review of your Ded

Thesis Title: Information Technology in Education: using Web Based instructions to Develop Geometry Problem Solving Skills Among Senior Secondary Students.

The Review Panel has agreed that it would be appropriate for you to proceed with your thesis and has made the following recommendations:

1. In writing research design, you need to be clearer and more explicit about ways in which schools are selected in Phases 1 & 3.

2. Recognition that drawing on theories/models of learning/learning through geometry, eg information processing theory as the larger/overarching theory and cognitive load as a sub theory.

Please accept my congratulations on achieving success at this stage in your candidature.

Yours sincerely

[Redacted]

Associate Professor Jan Wright  
Graduate School Research Coordinator

cc Dr Christine Brown  
Dr Mohan Chinnappan

Faculty of Education  
University of Wollongong NSW 2522 Australia  
www.uow.edu.au
Appendix 2 – Approval of the Ethics Committee

In reply please quote: DC:KM HE01/235
Further information: Karen McRae PH 42214457

4 February 2002

Mr B. Ekanayake
Locked Bag 9812
South Coast Mail Centre NSW 2521

Dear Mr Ekanayake,

Thank you for your response to the Ethics Committee’s requirements for your Human Research Ethics application HE01/235 “Information Technology in Education: Using Web Based Instructions To Develop Geometry Problem Solving Skills Among Senior Secondary Students”.

Your response and amendments meet with the requirements of the Committee and your application was formally approved on 31/01/2002. You can now proceed with your research.

Yours sincerely,

Karen McRae
Secretary to the
Human Research Ethics Committee
Appendix 3 – Indicators for Analysis of Answer scripts

1. Analysis of Geometry Problem-solving Test

Question 1

Question:
The line AB has been produced to either sides so that AX = BY. Prove that AY = BX

Answer:

\[ AX = BY \text{ (given)} \]
\[ AX + AB = BY + AB \text{ (AB is added to equals)} \]
But \( AX + AB = BX \) and \( BY + AB = AY \)
Therefore, \( BX = AY \)
Or,
\[ XY – AX = XY – BY \text{ (equals are subtracted from XY)} \]
Therefore, \( BX = AY \)

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<tr>
<td>terms</td>
<td>Meaning of producing</td>
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<tr>
<td>Phrases</td>
<td>AB has been produced</td>
<td></td>
</tr>
<tr>
<td>and</td>
<td>AX = BY</td>
<td></td>
</tr>
<tr>
<td>sentences</td>
<td>AB has been produced to either sides so that AX = BY</td>
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</tr>
<tr>
<td>Task</td>
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<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Applies axioms on data</td>
<td></td>
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<tr>
<td>Final outcome</td>
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</table>
Question 2

Question:
If $\angle AOB = \angle COD$ then prove that $\angle AOC = \angle BOD$

Answer:

AOB = COD (Given)

AOB − COB = COD − COB (subtracted same)

AOC = BOD

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<td>use of knowledge retrieval</td>
<td>Applies axioms on data Final outcome</td>
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Question 3

Question:
WXYZ is a parallelogram and WZQ and PXY are equilateral triangles. Prove that WP = QY

Answer:

\[ \begin{align*}
\text{WP} &= \text{QY} \\
\text{XY} &= \text{WZ} \quad \text{(opposite sides of parallelogram WXYZ)} \\
\text{XY} &= \text{XP}, \text{WZ} = \text{ZQ} \quad \text{(sides of equilateral triangles)} \\
\text{XP} &= \text{ZQ} \\
\text{QZW} &= \text{YXP} = 60^\circ \quad \text{(angles of equilateral triangles)} \\
\text{WXY} &= \text{WZY} \\
\text{WXY} + \text{QZW} &= \text{WZY} + \text{YXP} \\
\text{QZY} &= \text{WXP} \quad \text{(opposite angles of parallelogram WXYZ)} \\
\text{In the triangles QZY and WXP,} \\
\text{ZQ} &= \text{PX} \\
\text{ZY} &= \text{WX} \\
\text{QZY} &= \text{WXP} \\
\text{Triangles are congruent (SAS); Hence, QY = WP} 
\end{align*} \]

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<td>planning</td>
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<td></td>
</tr>
<tr>
<td></td>
<td>Selects congruency as the strategy</td>
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<td>Decides SAS as the case of congruency</td>
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<td>Derives equal sides, equal angles</td>
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<tr>
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<td>Apply axiom on relations</td>
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<td>Final outcome</td>
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</table>
Question 4

Question:
CDEF is a quadrilateral in which, CDE = EFC and DEF = FCD. Draw a diagram to represent the information. If CD = 11 cm, what is the length of EF?

Answer:

Since, CDE = EFC and DEF = FCD, pairs of opposite angles of the quadrilateral are equal.
Therefore, CDEF must be a parallelogram.
That implies, CD = EF (opposite angles of the parallelogram)
But it has been given that, CD = 11 cm.
Therefore, EF is also 11 cm.

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<td>Represents the relationships: ∠ CDE = ∠ EFC, ∠ DEF = ∠ FCD</td>
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<td>Retrieves properties of a parallelogram Derives the final outcome</td>
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Question 5

Question:
ABCD is a parallelogram. P and Q are points on AB and CD respectively such that AP = CQ. Prove that the perpendicular distances from P and Q to the diagonal BD are equal

Answer:
Let PL and QM be the perpendiculars to BD
AB = CD (opposite sides of parallelogram)
AP = CQ (given)
\[ AB - AP = CD - CQ \]
i.e., BP = DQ
In triangles BPL and DQM, BP = DQ (proved)
\[ PLB = QMD = 90^\circ \] (PL and QM are perpendiculars)
PBL = QMD
Triangles are congruent (Sp. Case):
Hence, PL = QM (perpendiculars)

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<tr>
<td></td>
<td>perpendicular distances from Q and P to the diagonal BD</td>
<td></td>
</tr>
<tr>
<td></td>
<td>perpendicular distances from Q to the diagonal BD</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Task: To prove that two sides in two triangles are equal</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Type: Congruency</td>
<td></td>
</tr>
<tr>
<td>representation</td>
<td>Draws a parallelogram and names it as ABCD</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Indicates the point P on AB and the point Q on CD</td>
<td>2</td>
</tr>
<tr>
<td></td>
<td>Draws a line from P to BD and another from Q to BD</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Indicates AP = CQ</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>Indicates points of intersection of perpendiculars</td>
<td></td>
</tr>
<tr>
<td>planning</td>
<td>Plans to prove BP and DQ are equal</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Decides to establish the congruency between triangles PBL and QDM</td>
<td>2</td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Retrieves that: Opposite sides of a parallelogram are equal.</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>When equals are subtracted from equals, the result is also equal.</td>
<td>0</td>
</tr>
<tr>
<td></td>
<td>The necessary and sufficient conditions to prove congruency.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Correspondent pairs are equal in congruent triangles.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Derives: BP and DQ are equal; PBL and QDM are congruent.</td>
<td></td>
</tr>
</tbody>
</table>
2. General Problem-solving Test

Question 1

Question:
You are to organise a tea party for the class at the end of the year. How would you find out the food-item preferences of your classmates?

Answer: In solving this problem, different student can generate different answers. However, each has to suggest:

(a) a method to prepare a list of students

(b) a practicable method to collect each student’s food item preference

(c) a method to summarise preferences such as tabulating, using tally marks

<table>
<thead>
<tr>
<th>Process</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>3 2 1 0</td>
</tr>
<tr>
<td>analysis</td>
<td>Key Organisation, a tea party, food preferences</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Phrases A tea party for the class</td>
<td></td>
</tr>
<tr>
<td></td>
<td>and at the end of the year</td>
<td></td>
</tr>
<tr>
<td></td>
<td>sentences: of your classmates</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Task To find out the food-item preferences</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Type: Ill structured; Data collection and tabulating</td>
<td></td>
</tr>
<tr>
<td>representation</td>
<td>Constructs tables to enter data</td>
<td></td>
</tr>
<tr>
<td>planning</td>
<td>Strategy to find out food preferences</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Strategies to summarise data</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Strategies to present the result</td>
<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Retrieves past experience or persons who organized the last year tea party; usual customs; Data collecting procedures; Summarizing procedures; presentation procedures</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Informs classmates to fill the form; collects forms and records information</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Presents the outcome</td>
<td></td>
</tr>
</tbody>
</table>
Question 2

Question:
The floor space of a classroom is 7m. x 7 m. It has door of one meter wide at one of its corners. This classroom has to be arranged as an examination hall. After leaving a space of 3 m x 3 m. space, each candidate is allocated a 1m. x 1m. space in the remaining part. Draw a lay out of the seating arrangement and label locations for candidates as C1, C2, C3, ……..

Answer:
In the solution, the free space should be associated with the location of the door. Otherwise, the entrance will be occupied by the space of a candidate. The answer looks like.

<table>
<thead>
<tr>
<th>Door</th>
<th>C1</th>
<th>C2</th>
<th>C3</th>
<th>C4</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>C5</td>
<td>C6</td>
<td>C7</td>
<td>C8</td>
</tr>
<tr>
<td></td>
<td>C9</td>
<td>C10</td>
<td>C11</td>
<td>C12</td>
</tr>
<tr>
<td>C13</td>
<td>C14</td>
<td>C15</td>
<td>C16</td>
<td>C17</td>
</tr>
<tr>
<td>C18</td>
<td>C19</td>
<td>C20</td>
<td>C21</td>
<td>C22</td>
</tr>
<tr>
<td>C23</td>
<td>C24</td>
<td>C25</td>
<td>C26</td>
<td>C27</td>
</tr>
<tr>
<td>C28</td>
<td>C29</td>
<td>C30</td>
<td>C31</td>
<td>C32</td>
</tr>
<tr>
<td>C33</td>
<td>C34</td>
<td>C35</td>
<td>C36</td>
<td>C37</td>
</tr>
<tr>
<td>C38</td>
<td>C39</td>
<td>C40</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Process</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td>analysis</td>
<td>Key terms: examination hall; floor space, location of the door; leaving a space; space for each candidate; remaining part; a lay out; seating arrangement The notation:C1, C2, C3, …….. Phrases and sentences: The floor space of a classroom is 7m. x 7 m. It has door of one meter wide It is located at one of its corners. The classroom has to be arranged as an examination hall. After leaving a space of 3 m x 3 m. space each candidate is allocated a 1m. x 1m. space in the remaining part. Task To draw a lay out of the seating arrangement and label locations for candidates as C1, C2, C3, …….. Type: Well structured; Non-routine; diagrammatic</td>
<td>3 2 1 0</td>
</tr>
<tr>
<td>representation</td>
<td>Represents the classroom Represents the door Represents the common area Represents candidate spaces Labels</td>
<td></td>
</tr>
<tr>
<td>planning</td>
<td>Plans lay out strategy</td>
<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Retrieves needs of an examination hall Finds location of the door Determines the free space Divides the remaining floor space</td>
<td></td>
</tr>
</tbody>
</table>
**Question 3**

**Question:**
A committee is to have at least 3 women. The number of women should be less than men and the number on the committee must be between 7 and 9. What are the possible compositions?

**Answer:**

<table>
<thead>
<tr>
<th>Women</th>
<th>Men (more than women)</th>
<th>Number on committee</th>
<th>Possible?</th>
</tr>
</thead>
<tbody>
<tr>
<td>3</td>
<td>4</td>
<td>7</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>5</td>
<td>8</td>
<td>✓</td>
</tr>
<tr>
<td>3</td>
<td>6</td>
<td>9</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>5</td>
<td>9</td>
<td>✓</td>
</tr>
<tr>
<td>4</td>
<td>6</td>
<td>10</td>
<td>(no combination beyond this)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Process</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td>analysis</td>
<td>Key terms: A committee is to have at least 3 women.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Phrases and sentences: The number of women should be less than Men.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>The number on the committee must be between 7 and 9.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Task: To find the possible compositions?</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Type: Possibility</td>
<td></td>
</tr>
<tr>
<td>representation</td>
<td>Representation of possibilities</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Representation of acceptable possibilities</td>
<td></td>
</tr>
<tr>
<td>planning</td>
<td>Infer possibilities</td>
<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Retrieves knowledge of conditional possibilities</td>
<td></td>
</tr>
</tbody>
</table>
Question 4

Question:
Ruwan said to Piyal, “If you give me one marble, then we will have an equal number of marbles.”
Piyal replied with delight, “If you give me one marble, then I will have double the number you have!” What was the total number of marbles they had?

Answer:
Students will apply the knowledge of algebra. Let R and P represent the number of marbles possessed respectively by Ruwan and Piyal.

Then, \[ P - 1 = R + 1, \quad \text{i.e.,} \quad P = R + 2 \quad (1) \]
\[ 2(R - 1) = P + 1, \quad \text{i.e.} \quad 2R - 3 = P \quad (2) \]

From (1), \[ 2R - 3 = R + 2 \]
\[ R = 5, \quad P = 7 \]

<table>
<thead>
<tr>
<th>Process</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td>analysis</td>
<td>Key terms: If you give me one marble, then we will have an equal number of marbles and sentences: number of marbles If you give me one marble, then I will have double the number you have</td>
<td>3</td>
</tr>
<tr>
<td></td>
<td>Task: To find the total number of marbles they had Type: Equations</td>
<td>2</td>
</tr>
<tr>
<td>representation</td>
<td>Represents each one’s numbers algebraically Represents problems in equations</td>
<td>1</td>
</tr>
<tr>
<td>planning</td>
<td>Plans to solve using simultaneous equations</td>
<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Retrieves knowledge about equations Build appropriate equations Solve equations Presents the final outcome</td>
<td></td>
</tr>
</tbody>
</table>
Question 5

Question:
A pharmacist has a three-litre container and a five-litre container. A mixture needs four litre of water. Without any other container, but with an unlimited supply of water how does he get four litters in either measure?

<table>
<thead>
<tr>
<th>Process</th>
<th>Indicator</th>
<th>Rating*</th>
</tr>
</thead>
<tbody>
<tr>
<td>analysis</td>
<td>Key terms: Two containers; One contains five litres, the other- three, No other containers Unlimited supply of water</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Phrases and sentences: a three-litre container and a five- litre container. 4 litres of water in one container Without any other container A mixture needs four litre of water</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Task To devise a strategy to end up with 4 litres of water in one container. Type: Adding and removing</td>
<td></td>
</tr>
<tr>
<td>representation</td>
<td>Representation of containers</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Representation of adding</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Representation of removing</td>
<td></td>
</tr>
<tr>
<td></td>
<td>Represents the volume in each container</td>
<td></td>
</tr>
<tr>
<td>planning</td>
<td>Find a strategy to retain a particular amount of water in a container</td>
<td></td>
</tr>
<tr>
<td>use of knowledge retrieval</td>
<td>Retrieves similar problem and applies Repeats until the goal is reached. Presents the final outcome</td>
<td></td>
</tr>
</tbody>
</table>
Appendix 4 – Circular No. 1998/04 of Ministry of Education of Sri Lanka on School-Based Assessments

1998 ඉදිරිපත්කේණ වශයෙන් මෙම පදක්කම් සමාන්‍ය මේයින් පවුල්ලුන් පෙන්වා මෙම විදේශීය මූලිකයන්ගේ පුළුල් ක්‍රමයක් පිළිතුරු කරන්න, යමන්තර සිහි පදක්කම් 20 පිටතුරු සිහි පදක්කම් පෙන්වා විශේශීය මූලිකයන්ගේ පුරාම පිළිතුරු විශේශීය මූලිකයන්ගේ පුරාම පිළිතුරු කරන්න.
06. ඉන්දියා දිස්ත්‍රීකකිමිති ප්‍රදේශී පැහැදිලි නැමුත් ප්‍රශ්නයක් සමබන්ධ ස්කීටරෙක් පැවතියේ විස්රාඵය ලබා පැළක්වාමුට පැහැදිළි නැත. ප්‍රදේශ දිස්ත්‍රීකකිමිති සැලසුම පැහැදිළි දිස්ත්‍රීකකිමිති බාධාව, පරිදලම සහ ප්‍රශ්නයක් සමබන්ධ ප්‍රශ්න පැවතියේ විස්රාඵය ලබා පැළක්වාමුට පැහැදිළි දිස්ත්‍රීකකිමිති සැලසුමක් පැහැදිළි නැත.

උණුසාම්:
1. ප්‍රදේශ දිස්ත්‍රීකකිමිති පැහැදිළි
2. ප්‍රශ්නයක් පැහැදිළි පැවතියේ විස්රාඵය
Appendix 5 – Circular No. 1998/42 of Ministry of Education of Sri Lanka on School-Based Assessments

<table>
<thead>
<tr>
<th>Year</th>
<th>6th Grade</th>
<th>7th Grade</th>
<th>8th Grade</th>
<th>9th Grade</th>
<th>10th Grade</th>
<th>11th Grade</th>
<th>12th Grade</th>
<th>13th Grade</th>
</tr>
</thead>
<tbody>
<tr>
<td>1998</td>
<td>Trial</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1999</td>
<td>Impl</td>
<td>Trial</td>
<td>Trial</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>Impl</td>
<td>Trial</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2001</td>
<td>Impl</td>
<td>Impl</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2002</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2003</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>Impl</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>1 පළලින්මය</td>
<td>2 පළලින්මය</td>
<td>3 පළලින්මය</td>
<td>4 පළලින්මය</td>
<td>5 පළලින්මය</td>
<td></td>
<td></td>
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<tr>
<td>----</td>
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<td>--------------</td>
<td>--------------</td>
<td>--------------</td>
<td>--------------</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1999</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2000</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2001</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2002</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2003</td>
<td>Impl</td>
<td>Trial</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2004</td>
<td>Impl</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

(3) සුභාජා ආගමික කොටස

03. දක්වා පුළුලින් ආරම්භක හෝ අනතුරු උපාධි හෝ නිදසුන් සැකසීමට එකම් යනු ලෙසවේ?

04. පාමණ කරනු කොත්තියක්

(5) පාමණ සඳහා ආරම්භක හෝ අතර උපාධි සඳහා පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස හා පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන්

05. 9 පළලින්මය අන්තර්ගතව අපේක්ෂක කොත්තියක් ලෙස

6 පළලින්මයේදී 9 පළලින්මයේදී මාය විකල්පයන් අන්තර්ගතව කොත්තියක් ලෙස පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන් පාමණ කරනු කොත්තියක් හෝ නිදසුන් සැකසීමට එකම් යනු ලෙස වශයෙන්

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06. 9, 11, 13 අතර සිසුන් ලියවිසා දැන්වම්ෂාවීමේ කාරスター කිහිපයක් අතින් සමාන කරන්නේ සිසුන් සිසුන් පැහැදිලිවී ලැබූ සියළු අතින් සමාන කරීමේ සිසුන් පැහැදිලිවීමේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය.

9 පෙන්තිනිය - 2000 ආර්ථිකව මොණ
11 පෙන්තිනිය සියළුමතක - 2002 ආර්ථිකව මොණ
13 පෙන්තිනිය සියළුමතක - 2005 ආර්ථිකව මොණ

(කිසිදු විශේෂීය ප්‍රකාශිකව සිසුන් සිසුන් පැහැදිලිවී ලැබූ සියළු අතින් සමාන කරන්නේ සිසුන් සිසුන් පැහැදිලිවීමේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය)

07. ගොඩාරු ක්‍රමාංකයේ අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න

(කිසිදු විශේෂීය ප්‍රකාශිකව සිසුන් සිසුන් පැහැදිලිවී ලැබූ සියළු අතින් සමාන කරන්නේ සිසුන් සිසුන් පැහැදිලිවීමේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය)

1. අධිකාර කමුම්භ කමුම්භ (School Boards of Assessment)
2. කාර්යාංක කමුම්භ කමුම්භ (Zonal Boards of Assessment)
3. අධිකාර කමුම්භ කමුම්භ (Provincial Boards of Assessment)

(ගොඩාරු ක්‍රමාංකයේ අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න, අදායම්වත් ප්‍රශ්න)

08. ජනාධිපති සංඛ්‍ෂීය කමුම්භ

(කිසිදු විශේෂීය ප්‍රකාශිකව සිසුන් සිසුන් පැහැදිලිවී ලැබූ සියළු අතින් සමාන කරන්නේ සිසුන් සිසුන් පැහැදිලිවීමේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය)

1. අධිකාර කමුම්භ - අධිකාර කමුම්භ කමුම්භ
2. කාර්යාංක කමුම්භ - කාර්යාංක කමුම්භ කමුම්භ
3. අධිකාර කමුම්භ - අධිකාර කමුම්භ කමුම්භ
4. අධිකාර කමුම්භ කමුම්භ

(විශේෂීය ප්‍රකාශිකව සිසුන් සිසුන් පැහැදිලිවී ලැබූ සියළු අතින් සමාන කරන්නේ සියළු අතින් සමානකර්ණය සිදු කරන්නේ සියළු අතින් සමානකර්ණය)
Appendix 6 – Implementation Plan

<table>
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<tr>
<th>School</th>
<th>A</th>
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<th>C</th>
<th>D</th>
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<td>Test</td>
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<td>GeoCK</td>
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## Appendix 7 – Analysis of Meeting Notes

<table>
<thead>
<tr>
<th>Date</th>
<th>Type</th>
<th>Focus</th>
<th>Requirements</th>
<th>Decision</th>
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</thead>
<tbody>
<tr>
<td>23.04.2002</td>
<td>C</td>
<td>Students difficulties</td>
<td>Nature of learning environment</td>
<td>Classroom based Teacher as a learning partner Collaborative interaction for individual engagement Flexibilities within curriculum</td>
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<tr>
<td>16.05.2002</td>
<td>C</td>
<td>Instructional approach</td>
<td>Problem solving strategy</td>
<td>Polya’s four steps</td>
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<tr>
<td>04.06.2002</td>
<td>C</td>
<td>Similar learning environments for instructional strategies</td>
<td>Applicable software features</td>
<td>Worked example method Metacognitive support Learner regulation</td>
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<tr>
<td>21.06.2002</td>
<td>C</td>
<td>Overall design</td>
<td>Development sequence</td>
<td>Group students on instructional levels Provide activities and problems accordingly Provide opportunities for transition from one level to the next</td>
</tr>
<tr>
<td>07.08.2002</td>
<td>F</td>
<td>Requirements of the learning related to proof type geometry problem solving process</td>
<td>Incorporating content knowledge and general problem solving</td>
<td>Devise strategies worked example focusing problem transfer Prepare the software as a teaching tool Provide content knowledge in background</td>
</tr>
<tr>
<td>07.08.2002</td>
<td>D</td>
<td>Content knowledge, problem solving</td>
<td>Learner options</td>
<td>Provide multiple access points to access information. Pages for problem, clues, worked example, reflection</td>
</tr>
<tr>
<td>16.08.2002</td>
<td>D</td>
<td>Content knowledge and problem solving</td>
<td>Learner engagement</td>
<td>Provide interactive and multiple representation</td>
</tr>
<tr>
<td>23.08.2002</td>
<td>D</td>
<td>Initial Web design Process guidance Content knowledge Diagram representation</td>
<td>Problem solving with worked examples. Helps for control process</td>
<td>Structure the problem (worked example) Anchor related knowledge to Process Guidance (clues), and problem. (Explained worked example) Provide a metacognitive support (Check diagram, Check generated information, check proof development) Use stepwise representation to show the development of the diagram (Stepwise diagram)</td>
</tr>
<tr>
<td>30.08.2002</td>
<td>D</td>
<td>Development across van Hieles</td>
<td>Requirements of a learning environment</td>
<td>Identify van Hiele levels; provide appropriate activities, problem solving</td>
</tr>
<tr>
<td>Date</td>
<td>Type</td>
<td>Focus</td>
<td>Requirements</td>
<td>Decision</td>
</tr>
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<tr>
<td>30.08.2002</td>
<td>D</td>
<td>Content knowledge</td>
<td>Context of prototype</td>
<td>Limit Prototype is to congruency</td>
</tr>
<tr>
<td>02.09.2002</td>
<td>D</td>
<td>Content knowledge</td>
<td>Presentation sequence</td>
<td>Provide students with relevant complete content knowledge (be) for solving problems.</td>
</tr>
<tr>
<td>06.09.2002</td>
<td>D</td>
<td>Process guidance</td>
<td>Indicating helps Incorporating visual effects to highlight parts</td>
<td>Provide metacognitive awareness (<a href="#">Show me link</a>) Use Colour effects to represent diagrammatic information (<a href="#">Colours to highlight sides and angles</a>)</td>
</tr>
<tr>
<td>13.09.2002</td>
<td>D</td>
<td>Content knowledge</td>
<td>Presentation content knowledge of congruency</td>
<td>Provide instructions and activities for knowledge construction.</td>
</tr>
<tr>
<td>13.09.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Structural reorganisation</td>
<td>Expand Process Guidance. 2nd column to provide specific guidance (<a href="#">3 column representation</a>) Anchor to planning process in the Process Guidance support (<a href="#">what is missing</a>)</td>
</tr>
<tr>
<td>20.09.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Learner diversities</td>
<td>Categorise students on need. If successful check; if difficult, guidance; if problematic: reflection. Provide multiple options (<a href="#">enhanced non-linear</a>)</td>
</tr>
<tr>
<td>20.09.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Learner independence</td>
<td>Use student friendly language (<a href="#">more conversational language</a>)</td>
</tr>
<tr>
<td>27.09.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Problem familiarisation, because it has been argued that geometry problems are different to each other</td>
<td>Provide opportunities to transfer knowledge to extended situations (<a href="#">Advance problem</a>)</td>
</tr>
<tr>
<td>27.09.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Structural change</td>
<td>Fade helps and give a second attempt to the same question (<a href="#">Try again</a>)</td>
</tr>
<tr>
<td>11.10.2002</td>
<td>F</td>
<td>Focus group</td>
<td>For the feedback on work done</td>
<td>Directing to the same question may be psychologically negative: Change labels and give as a new question to use the same strategy.</td>
</tr>
<tr>
<td>17.10.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Practice problem Transfer problem solving Control process</td>
<td>Use Similar problem Use Advance problem Provide faded metacognitive support</td>
</tr>
<tr>
<td>24.10.2002</td>
<td>D</td>
<td>Problem set</td>
<td>Introducing new problem solving strategies</td>
<td>Provide different examples that require different strategies (<a href="#">Problem Sets</a>)</td>
</tr>
<tr>
<td>31.10.2002</td>
<td>D</td>
<td>Process Guidance</td>
<td>Learner independence</td>
<td>Provide an introduction to process guidance</td>
</tr>
<tr>
<td>Date</td>
<td>Type</td>
<td>Focus</td>
<td>Requirements</td>
<td>Decision</td>
</tr>
<tr>
<td>------------</td>
<td>------</td>
<td>----------------------------</td>
<td>----------------------------------</td>
<td>--------------------------------------------------------------------------</td>
</tr>
<tr>
<td>23 01.11.2002 D</td>
<td>Problem solving learning execution</td>
<td>Reducing cognitive load</td>
<td>Display the problem near worked example</td>
<td></td>
</tr>
<tr>
<td>24 01.11.2002 D</td>
<td>Content knowledge</td>
<td>A retrieval process</td>
<td>Prepare the complete content knowledge separately (Knowledge base)</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Achor required pieces of knowledge in the problem solving process</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>Provide metacognitive awareness on that</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(knowledge base is hyperlinks on worked example pages)</td>
</tr>
<tr>
<td>25 07.11.2002 D</td>
<td>Problem set</td>
<td>To introduce a new set of strategies</td>
<td>Add another set of problems with</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(i) Base Problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(ii) Similar Problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td>(iii) Advance Problem</td>
</tr>
<tr>
<td>26 15.11.2002 D</td>
<td>Problem set</td>
<td>Transition from one problem set to the other</td>
<td>Provide opportunity after the advance problem (Second problem set comes with all features)</td>
<td></td>
</tr>
<tr>
<td>27 15.01.2003 D</td>
<td>Problem sets</td>
<td>Enhance knowledge about strategies</td>
<td>Provide more opportunities for familiarisation</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td></td>
<td></td>
<td><strong>Five problem sets complete</strong></td>
</tr>
<tr>
<td>28 20.01.2002 D</td>
<td>Diagrammatic reasoning</td>
<td>Preparation for the upcoming meeting with Web designer</td>
<td>Possible visual effects Enhancing metacognitive support</td>
<td></td>
</tr>
<tr>
<td>29 22.01.2003 D</td>
<td>Meeting with Web designer</td>
<td>To discuss technical concerns</td>
<td>Improve it technically Prepare a progress bar Provide dynamic effects</td>
<td></td>
</tr>
<tr>
<td>30 30.01.2003 D</td>
<td>Diagrammatic reasoning</td>
<td>Identification of parts in a diagram</td>
<td>Highlight the parts dynamically.</td>
<td></td>
</tr>
<tr>
<td>31 31.01.2003 D</td>
<td>Issues related to Metacognition</td>
<td>Mode of distribution of software Reflection</td>
<td>Distribute on CDs Let teacher organise</td>
<td></td>
</tr>
<tr>
<td>32 15.02.2003 D</td>
<td>Integration of animation</td>
<td>Feedback</td>
<td>Prototype completes</td>
<td></td>
</tr>
</tbody>
</table>
Appendix 8 – Initial instructions

Example: Group A

In – Investigator; Am – Amjad; Al – Alicia

In  You may know that you are going to solve proof type geometry problems.
Am  Yep

In  You will work together. This CD will provide problems as well as instructions to solve them. Do you know how to open this?
Am  Yep

In  Alicia?
Al  Yes we know. We use the CDs for learning.
In  OK, then open this CD ROM … OK … thank you Amjad … Now … open the folder.
Am  OK …
In  Thank you Amjad … open this file too.
      Here we go … Now … it is your turn Alicia. You read the text on the screen for Amjad, OK?
Al  A Non linear Geometry Environment for Learning …
In  Thank you Alicia … Amjad … take the pointer to here … (Amjad directly clicked the thumbnail link and turn the next page). Its great, you know what to do. … OK … Yes Alicia
Al  This learning environment … (continues reading)
In  Thank you Alicia. Information in this page is about the purpose of our meeting today. Read it silently.
Am  Shall I click ENTER?
Al  Yes

In  Yes please … Thank you Amjad … Now … this is the first problem. You will solve it in this workbook … OK? Alicia, hereafter you will control the mouse. … hear is the mouse, you will read the problem for Amad. Amjad … you also can read the problem for your comprehension. But silently, as Alicia reads loud OK?
Am  Yep.

In  When you solve problems, always pass your ideas to each other. I mean … you tell every thing you think to the other. You can agree, or argue on that. Discuss everything when you solve problems. You can suggest the next step, or you can tell difficulties as they appear… I record your conversation throughout the session. Thank you Let’s start . Yes … Alicia
AB and CD are two equal and parallel line segments. AD intersects BC at X. Show that ABX and CDX are congruent triangles.

If you solved the problem go this way to check your answer. If you need additional help to solve the problem, try this way. Now yes your turn. If you need my help only call me OK?

AB and CD are two equal and parallel line segments. AD intersects BC at X. Show that ABX and CDX are congruent triangles.

For the first problem, go through Process Guidance. That will help you to understand how to use Process Guidance. Alicia, please go to Process Guidance.

Thank you Alicia. This is the Process Guidance page. You can see the problem on top. You can read it whenever you need. Then you will see some information. Read it loud Alicia.

Process Guidance … Some or all … (continues reading)

Yes … Have a look into this column.… you can use steps in this column for solving any proof type geometry problem.

This column is specific for the problem you are currently solving. See these hyperlinks in this column … you can check your answer against the answer provides here.
Appendix 9 – Focus questions for student interview

Did you enjoy this problem solving session?

You spent more than two hours on the screen. Are you tired?

Are you ready to continue the session now?

You used these both paths: Check your solution and Process Guidance. Did you notice the difference between Check your solution and Process Guidance?

When do you use Check your solution?

This is a Check your solution page. What is your impression on this type of stepwise Check? Was it useful to you?

When do you use Process Guidance?

This is a Process Guidance page. What is your impression on this type of stepwise Process Guidance? Was it useful to you?

How do you use it?

These are general problem solving steps. Were they useful to you?

This step (the first step) has been split into four steps in this column. What is your impression on that?

What is the purpose of these Show me links?

Do you think this instruction (highlight the goal) would be useful?

For this instruction you get this list? Will it be useful to you?

Is this instruction (what is missing) useful to you?

The Show me link of this instruction takes you to this page. You used this. How useful was that?

You played this animation. Did you enjoy it?

Do you like to continue work with ANGEL? Why?
Appendix 10 – Student interview protocol

Example: Group C

In – Investigator; Alex; Nick

Q: You used these Check your solution and Process Guidance. Were they both useful?
Alex: Yes
Q: Are they different?
Alex: Yes
Q: What is the difference?
Alex: Check your solution is to check and Process Guidance is to proceed.
Q: This is a Check your solution page. What is your impression on this type of stepwise presentation?
Alex: It is useful. But, we did not realize it.
Q: Explain
Alex: First you complete the diagram. Then check. If you find some thing wrong you can correct it.
Q: This is a Process Guidance page. Nick, what is your impression on this type of stepwise Process Guidance?
Nick: It’s fine. We used it couple of times.
Q: These problem-solving steps are generally used to solve proof type geometry problems. Were they useful to you?
Nick: Really, we used same steps earlier
Q: All steps?
Nick: No … This one (Draw your diagram) … yes … this (Highlight the goal)…no … this (Think about the key idea of the problem) … no … Think about: What is missing? … no … yes, now I realise … Process Guidance could be useful.
Q: This step (the first step) has been broken into four steps in this column. What is your impression on that?
Nick: Some students might need it. But I do not.
Q: When do you use these Show me links?
Nick: Oh, they are useful. You can check the answer
Q: Do you think this instruction (highlight the goal) was useful Alex?
Alex: I am not so certain. I feel yes, because some times I need two diagrams.
Q: For this instruction you get this list? Will it be useful to you?
Alex: Not certain … still I did not feel it.
Q: Is this instruction (what is missing) useful to you?
Alex:
Q: The Show me link of this instruction takes you to … this page. You used this. How useful was that?
Alex: Information in this page looks useful. We did not come to this page. We were not aware about this.
Q: You played this animation. Did you enjoy it?
Alex: Yes, we did. It was marvelous.
Q: Do you like work in pairs?
Alex: I don’t know … it looks good. I am used to work along
Q: Nick, what is your Impression of working in groups?
Nick: … amazing, I learned a new method from Alex
Q: Do you think you can solve geometry problems?
Alex: Yes, I do.
Q: Nick, if you are provided with this CD, would you happy to learn geometry?
Nick: Yes, certainly.
Q: Why?
Nick: You can get assistance in learning … it contains clear diagrams „, presentation is also very easy
to understand.

Thank you Nick, Thank you Alex.
Appendix 11 – Table of Merged Observational and Verbal Data

Example: Group B

<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Student</th>
<th>Conversation</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td>0000</td>
<td>Problem 1</td>
<td>Adr</td>
<td>AB and CD are two equal and parallel line segments. AD intersects BC at X. show that ABX and CDX are congruent triangle</td>
<td>Reads the problem</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>AB and CD are equal ... and parallel. AD intersects BC at X</td>
<td>Draws the diagram</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Mark that AB equals to CD and AB and that they are parallel.</td>
<td>Adam marks</td>
</tr>
<tr>
<td>0065</td>
<td>Process Guidance</td>
<td>Adr</td>
<td>Draw the diagram ... we have done it ... Label ... we’ve done it too. Write the goal</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>All right, check ... click this link</td>
<td>Points the “show me” link. Adrian turns the page to Process Guidance.</td>
</tr>
<tr>
<td>0080</td>
<td>Worked example</td>
<td>Adr</td>
<td>The diagram is all right we’ve drawn an excess line</td>
<td>Adrian shows an extra line drawn on their diagram.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Forget it ... come back to the Process Guidance</td>
<td></td>
</tr>
<tr>
<td>0090</td>
<td>Process Guidance</td>
<td>Adr</td>
<td>Highlight the goal</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Triangle ABX and triangle CDX are congruent.</td>
<td>Adam writes it. Adrian clicks the “show me” hyperlink.</td>
</tr>
<tr>
<td>0104</td>
<td>Worked example</td>
<td>Adr</td>
<td>These areas are dotted.</td>
<td>Shows triangles.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>You do not need it, because you get only two triangles here.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Yes you do not. … Excuse me, do we need to shade these triangles … there are only two triangles here</td>
<td></td>
</tr>
<tr>
<td>0123</td>
<td>Process Guidance</td>
<td>In</td>
<td>Not essential. … Come back to Process Guidance …Thank you Adrian…. It says instructions in this column are useful for solving most proof type problems. You can skip any instruction or more. Instructions are just for your convenience.</td>
<td>Adrian takes the mouse</td>
</tr>
<tr>
<td>Counter</td>
<td>Event in ANGEL</td>
<td>Student</td>
<td>Conversation</td>
<td>Observed student behaviour</td>
</tr>
<tr>
<td>---------</td>
<td>---------------</td>
<td>---------</td>
<td>-------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>0133</td>
<td>Process Guidance.</td>
<td>Adr</td>
<td>Think about key idea of the problem.</td>
<td>Adam points the second column. Adrian tries Congruency from the list. They were happy</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Not clear, what does it mean.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Its here … you need to select the matching item from this list box … OK … may be congruency … excellent!</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>For this how many relationships</td>
<td>Adrian points</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Three …</td>
<td>Identifies vertical opposite angles.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>We have already got only one. This base equals to this. These angles are equal … they are vertically opposite</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>But … AB is parallel to CD</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>It’s not a relationship required for congruency! …</td>
<td></td>
</tr>
<tr>
<td>0187</td>
<td>3rd layer</td>
<td>Adm</td>
<td>This opinion says to use that AB and CD are parallel</td>
<td>Points angle ABX and DCX</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Parallel? … Yes … you can derive relationships. Mark these as equal.</td>
<td>Identifies that angle BAX equals to angle CDX</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>OK … here we go … this pair is also equal.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Now we have four relationships write the proof</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>AB equals to CD … given … angle CXD equals to angle AXB … vertically opposite angles … angle XCD equals to angle XBA … corresponding …</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Those are not corresponding … corresponding angles are like this… They are called alternate angles!</td>
<td>Adrian draws exemplar set of corresponding angles and clarifies the difference.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>All right, I admit … because I confused … alternate angles on parallel lines … angle XDC equals to angle XAB … also alternate congruent triangles … ASA</td>
<td>Adam writes, and puts S, A, A, A and take only two pair of angles Adrian selects the last show me link</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>It hasn’t take vertical angles. No worries…. Answer is correct we will move to the next … A similar problem….</td>
<td>Adam finds the link</td>
</tr>
<tr>
<td>0207</td>
<td>Problem 1A</td>
<td>Adr</td>
<td>You draw this I will assist</td>
<td>Adam starts drawing. No one verbalizes the problem, but both involve</td>
</tr>
<tr>
<td>Counter</td>
<td>Event in ANGEL</td>
<td>Student</td>
<td>Conversation</td>
<td>Observed student behaviour</td>
</tr>
<tr>
<td>---------</td>
<td>---------------</td>
<td>---------</td>
<td>--------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Everything is same except letters. All right …</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>This time take only three relationships</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>PQ equals to RS….given angle MRS equals angle MQP alternate … angle RMS equals angle PMQ … vertically opposite … all right … Side – Angle - Angle … congruent Angle-Side-angle</td>
<td>Adam writes, Adrian watches</td>
</tr>
<tr>
<td>0251</td>
<td>Worked example</td>
<td>Adr</td>
<td>Check the answer</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Sides all right, here two pair alternate angles, ours is one alternate and the other vertically opposite</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>That doesn’t matter, we have right relationships for Angle-Side-Angle. Ok ……next.</td>
<td>They decide that their answer is correct although the presentations are different.</td>
</tr>
<tr>
<td>0289</td>
<td>Problem 1B</td>
<td></td>
<td>Both read the problem without verbalization.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>In</td>
<td>Please keep talk</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Shall we talk for workings only?</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>In</td>
<td>Yes. That’s fine.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>AB and CD … equal … and parallel … intersection of AD and BC is X</td>
<td>Adrian watches</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Same problem … All right … prove that ABX and DCX are congruent?</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>All right … we will write the proof.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>First mark relationships.</td>
<td>Adams marks relationship and starts writing.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>AB equals CD … given … angle XCD equals to angle XBA alternative angles … angle XDC equals to angle XAB … congruent Angle-Side-Angle</td>
<td>Denotes ASA</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>State the relationship between CX and BX</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Hang on … CX … BX … all right … they are equal CX equals to BX</td>
<td>Writes without checking the congruency</td>
</tr>
<tr>
<td>0330</td>
<td></td>
<td>Adr</td>
<td>Give reasons</td>
<td></td>
</tr>
<tr>
<td>Counter</td>
<td>Event in ANGEL</td>
<td>Student</td>
<td>Conversation</td>
<td>Observed student behaviour</td>
</tr>
<tr>
<td>---------</td>
<td>---------------</td>
<td>---------</td>
<td>--------------</td>
<td>---------------------------</td>
</tr>
<tr>
<td>Adm</td>
<td>Because … they are congruent.</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adr</td>
<td>State any other similar relationship</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>adm</td>
<td>AX equals to DX</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Adr</td>
<td>Check now</td>
<td>Checks</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0354</td>
<td>Worked example</td>
<td>Adm</td>
<td>Wonderful! … Correct, go to the next</td>
<td></td>
</tr>
<tr>
<td>0363</td>
<td>Problem 2</td>
<td>Adr</td>
<td>XYZ …</td>
<td>Reads</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>XYZ in isosceles, XY equals to XZ</td>
<td>Draws</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>M in the mid point of YZ. Then this part equals to this part mark it.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Yes. … YM equals to MZ</td>
<td>Marks the relationship</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Now angle WMY equals to angle PMZ</td>
<td>Adam marks that angles are equal</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Still we have two pairs</td>
<td>A long pause</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>This is a common side … shall we prove the congruency?</td>
<td>Students take a decision to prove XMW and XMP are congruent.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>No take YZ in common</td>
<td>Students stuck</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Shall we go to Process Guidance</td>
<td></td>
</tr>
<tr>
<td>0404</td>
<td>Process Guidance</td>
<td>Adm</td>
<td>That’s fine.</td>
<td>They go to Process Guidance compare the diagrams</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Check the diagram</td>
<td>Compare the diagrams</td>
</tr>
<tr>
<td>0417</td>
<td>Worked example</td>
<td>Adm</td>
<td>No any difference.</td>
<td>Compare full information.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Now we will go to this and check. Aha … we missed this relationship …</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Yes, we haven’t used the idea of isosceles triangle. Now we have YM equals to MZ … angle WMY equals to angle PMZ … given … angle AZM equals to angle WYM base angles. Therefore they are congruent angle side angle … Done.</td>
<td>Marks base angles to be equal. Puts SAA</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Now the similar problem</td>
<td>Turns to problem 2A</td>
</tr>
<tr>
<td>0449</td>
<td>Problem 2A</td>
<td>Adm</td>
<td>AB equals to AB. Base angles are equal … angle at B equal to angle at C. L is mid point therefore BL equals to CL given that JLB equals to angle XLC</td>
<td>Adrian watches</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>BL equals to LC given, angle LCK equals to angle LBJ base angles. Angle KLC equals to angle LJB given S … AA. Therefore congruent … ASA.</td>
<td>Adam writes.</td>
</tr>
<tr>
<td>Counter</td>
<td>Event in ANGEL</td>
<td>Student</td>
<td>Conversation</td>
<td>Observed student behaviour</td>
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<td></td>
<td>Adm</td>
<td>Therefore … KC equals to BD Do you need to check?</td>
<td></td>
</tr>
<tr>
<td>0480</td>
<td>Problem 2B</td>
<td>Adm</td>
<td>No we got the idea from the answer. Better proceed.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>XYZ is isosceles training to XY equals to XZ … OK … W and T lie on XY and XZ. M is mid point. Prove that YT equals to ZW</td>
<td>Adam draws in diagram marks Relationships</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>We will take T on XY and W on XZ</td>
<td>Adam interchanges W and T</td>
</tr>
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<td></td>
<td></td>
<td>Adr</td>
<td>I don’t think you can change that way, but we will try it first</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Ok, because M is mid point … YM equals to MZ</td>
<td></td>
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<tr>
<td></td>
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<td>Adr</td>
<td>It’s given that these two angles are equal. Congruency is proved. Now we will write.</td>
<td></td>
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<td></td>
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<td>Adm</td>
<td>YM equals to MZ given angle YMT equals to angle ZMW, given MYT and ZW are equal angles base angles. Congruency proved. Angle side angle. Then YT equals to ZW. Now we will check.</td>
<td>Adrian turns</td>
</tr>
<tr>
<td>0502</td>
<td>Worked example</td>
<td>Adr</td>
<td>Our answer is wrong. This is too long.</td>
<td>They realise that their answer is wrong</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Yes…you did not get angle WMY here and WMY either</td>
<td>They compare details</td>
</tr>
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<td></td>
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<td>Adr</td>
<td>We will work it out from the beginning.</td>
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<td></td>
<td>Adm</td>
<td>Ok, now you take the turn.</td>
<td>Decided to work out from the beginning</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Read it first and then we will draw</td>
<td>Adrian and Adam change roles</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>XYZ is isosceles training to XY equals to XZ … OK … W and T lie on XY and XZ. M is mid point. Prove that YT equals to ZW</td>
<td>Adam reads the problem Adrian reads the screen</td>
</tr>
<tr>
<td>0552</td>
<td>Worked example</td>
<td>Adr</td>
<td>All right. First we will prove that the triangle WYM and the triangle TZM are congruent. Angle YMW equals to angle TMZ given</td>
<td>Both seam the drawn diagram. Adrian writes the proof</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>YM equals to MZ and angle TZM equals to angle WYZ because isosceles</td>
<td>Adam notes A-S-A in the workbook. Adrian writes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Angle – Side – Angle congruency proved. … Because they are congruent, WY equals to TZ.</td>
<td>Adam seems to be agreed.</td>
</tr>
</tbody>
</table>
Adr Now we have another pair of triangles: YZW and YZT.
YZ is common
YW equals to TZ proved
Angle TZY equals to angle WYZ
Side-Side-Angle

Adr Turn the next question.  Adam turns

0616 Problem 03 Adm ABCD is a quadrilateral
Adm Hang on … AB equals to DC. You did not draw that way. Draw it again
Adr ABCD is a quadrilateral. Angle BAF equals to angle CDF … All

Adm Prove that AFB and CFD are congruent
Adr Simple – these are vertically opposite, these are equal because given and its given that these sides

Adm DC equals to AB given angle FDC equals to angle FAB given.
Angle AFB equals to angle DFC vertically opposite.
Adm These four lines are all right
Adr Triangle AFB is congruent to triangle DFC Side-Angle-Side. Turn to check

0650 Worked example Adm These four lines are all right
Adr OK go to the next

0660 Problem 3B Adam ABCD … F … AB equals to DC angle BAF equals to angle CDF … All right, you need to prove that angle FAD is equals to angle FDA.
Adr All right, you need to prove that this angle and this angle are equal.
Adm These two angles could be 90° degrees

Adm These four lines are all right
Adr Triangle AFB is congruent to triangle DFC Side-Angle-Side. Turn to check

Adm These four lines are all right
Adr OK go to the next

0660 Problem 3B Adam ABCD … F … AB equals to DC angle BAF equals to angle CDF … All right, you need to prove that angle FAD is equals to angle FDA.
Adr All right, you need to prove that this angle and this angle are equal.
Adm These two angles could be 90° degrees
<table>
<thead>
<tr>
<th>Counter</th>
<th>Event in ANGEL</th>
<th>Student</th>
<th>Conversation</th>
<th>Observed student behaviour</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Not necessary. Could we prove that these triangles are congruent</td>
<td></td>
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<td></td>
<td></td>
<td>Adm</td>
<td>Useless. Both of these angles are in the same triangle</td>
<td></td>
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<td></td>
<td></td>
<td>Adr</td>
<td>Turn guidance</td>
<td></td>
</tr>
<tr>
<td>0717</td>
<td>Process Guidance</td>
<td>Adm</td>
<td>Check the diagram</td>
<td>Turns the answer</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Diagram is all right</td>
<td></td>
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<td></td>
<td></td>
<td>Adm</td>
<td>Then the goal</td>
<td>Turn to the answer</td>
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<td></td>
<td></td>
<td>Adr</td>
<td>It is also all right</td>
<td></td>
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<td></td>
<td>Adm</td>
<td>Key idea?</td>
<td></td>
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<td></td>
<td>Adr</td>
<td>Obvious congruency better check</td>
<td>Clicks Congruency</td>
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<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Correct</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>What is missing</td>
<td>They go into information</td>
</tr>
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<td></td>
<td></td>
<td>Adm</td>
<td>Wonderful. You need to prove that this is isosceles.</td>
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<td>Adr</td>
<td>Yes when these triangles are congruent then this will become isosceles.</td>
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<td>Adm</td>
<td>We start from there.</td>
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<td></td>
<td>Adr and Adm</td>
<td>Try this animation.</td>
<td>Follow the animation carefully and curiously</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Excellent</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Really</td>
<td></td>
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<td></td>
<td>Adm</td>
<td>Now write form the beginning</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Triangle DFC and triangle AFB are congruent through Side – Angle – Angle …</td>
<td>Adam marks S-A-A with a red pen</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Oh … no … no … you have to prove it first.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>All right … because it has been proved … AB equals to DC … given … angle FDC and angle FAB … given … angle CFD equals to angle BFA … vertically opposite … Triangles are congruent.</td>
<td>Completes the proof.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Therefore DF equals to AF and then AFD is isosceles.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>Hang on … DF equals to AF … therefore angle FAD equals to angle FDA as required.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Now check.</td>
<td>Closes the animation window</td>
</tr>
<tr>
<td>0820</td>
<td>Worked example</td>
<td>Adr</td>
<td>Excellent …</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Next question. Problem 4 … ABC is an isosceles triangle … AB equals to AC … The bisector of the angle BAC cuts</td>
<td>Turns to the Problem 4</td>
</tr>
<tr>
<td>Counter</td>
<td>Event in ANGEL</td>
<td>Student</td>
<td>Conversation</td>
<td>Observed student behaviour</td>
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<td></td>
<td>BC at D ... Prove that triangles DBA and DCA are congruent</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0833</td>
<td>Problem 4</td>
<td>Adr</td>
<td>All right … ABC is an isosceles triangle. AB equals to AC. The bisector of the angle BAC cuts BC at D.</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Now, … in these triangles, AD is common.</td>
<td>Adrian writes</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>… and … AC equals to AD … given …. Angle BAD equals to angle DAC … given …</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>OK … then they are congruent … done! … Check</td>
<td>A quick finish; goes to Worked example</td>
</tr>
<tr>
<td>0922</td>
<td>Worked example</td>
<td>Adr</td>
<td>It’s fine … All lines are correct. Checks: both are happy</td>
<td></td>
</tr>
<tr>
<td>0924</td>
<td>Problem 4B</td>
<td>Adm</td>
<td>Next problem … OK? … ABC is an isosceles triangle. AB equals to AC. Prove that the perpendicular distance from B to AC and C to AB are equal. Draws ABC and marks AB equals to AC</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>ABC is an isosceles triangle … AB equals to AC … all right</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Now mark base angles. … Great … draw a perpendicular line from C to AB and C are equal</td>
<td>Adrian denotes base angles at B</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>OK … let the line be line is BX … OK</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Yes … looks like … then another perpendicular from B to AC.</td>
<td></td>
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<td></td>
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<td>Adr</td>
<td>All right …That is CY.</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Now you need to show CX equals to BY.</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>If you take this triangle: ACX and this triangle: BCY, BC is common Angle BCY equals to angle CBX … base angles … Angle BXC equals to angle BYC … given perpendicular … Side-Angle-Angle</td>
<td>Writes: Notes that S,A, A on respective lines.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Wonderful … therefore, triangles ABY and Triangle CXB are congruent … therefore BY equals to CX. Now check … continue … continue … check</td>
<td></td>
</tr>
<tr>
<td>1133</td>
<td>Worked example</td>
<td>Adr</td>
<td>Perfect … next please</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Look … this progress bar says that we have already completed …</td>
<td></td>
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<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>I agree … may be two more problems …let’s read</td>
<td></td>
</tr>
<tr>
<td>Counter</td>
<td>Event in ANGEL</td>
<td>Student</td>
<td>Conversation</td>
<td>Observed student behaviour</td>
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<tr>
<td>1154</td>
<td>Problem 5</td>
<td>Adm</td>
<td>AB and CD mutually bisect each other at M ... Prove that … ACM and BDM … are congruent triangles</td>
<td>joins A to C and B to D</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adr</td>
<td>AB … BC … Mutually bisect at M. now triangles ACM and BDM …</td>
<td></td>
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<td></td>
<td>Adm</td>
<td>Mary that AM equals to BM and CM equals to DM</td>
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<td>Adr</td>
<td>All right … Angle DMB equals to angle AMC … vertically opposite angles … DM equals to MC … bisects at centre … AM equals to MB …also bisects at centre Angle- Side-Side … are congruent</td>
<td>adam writes ASS to denote Angle- Side-Side with red.</td>
</tr>
<tr>
<td></td>
<td></td>
<td>Adm</td>
<td>Nicely done …</td>
<td></td>
</tr>
<tr>
<td>1200</td>
<td>End of session</td>
<td>In</td>
<td>Thank you. You have done a great work … only one problem remains, … but we don’t have time.</td>
<td>stop at the end of Problem 5</td>
</tr>
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</table>
Appendix 12 - Event Analysis

Problem Set 1

<table>
<thead>
<tr>
<th>Problem</th>
<th>Event</th>
<th>Group A</th>
<th>Group B</th>
<th>Group C</th>
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<td>WE</td>
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<td>√</td>
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<tr>
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<td>PGBP</td>
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<td>PGAP</td>
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## Problem Set 2

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