2001

On circulant best matrices and their applications

S. Georgiou  
*National Technical University of Athens, Greece*

C. Koukouvinos  
*National Technical University of Athens, Greece*

Jennifer Seberry  
*University of Wollongong, jennie@uow.edu.au*

Publication Details
This article was originally published as Georgiou, S, Koukouvinos, C and Seberry, J, On circulant best matrices and their applications, Linear and Multilinear Algebra, 48, 2001, 263-274. Copyright Taylor & Francis. Original journal available [here](#).
On circulant best matrices and their applications

Abstract
Call four type 1(1,-1) matrices, \(x_1, x_2, x_3, x_4\); of the same group of order \(m\) (odd) with the properties (i) \((X_i - I)^T = -(X_i - I)\), \(i = 1, 2, 3\), (ii) \(X_4^T = X_4\) and the diagonal elements are positive, (iii) \(X_i X_j = X_j X_i\) and (iv) \(X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4mI_m\), best matrices. We use a computer to give, for the first time, all inequivalent best matrices of odd order \(m \leq 31\). Inequivalent best matrices of order \(m\), \(m\) odd, can be used to find inequivalent skew-Hadamard matrices of order \(4m\). We use best matrices of order \(\frac{1}{4}(s^2 + 3)\) to construct new orthogonal designs, including new OD(\(2s^2 + 6; 1, 1, 2, 2, s^2, s^2\)).

Keywords
Circulant matrices, supplementary difference sets, orthogonal designs, Hadamard matrices, AMS Subject Classification: Primary 05B20, Secondary 05B30

Disciplines
Physical Sciences and Mathematics

Publication Details
This article was originally published as Georgiou, S, Koukouvinos, C and Seberry, J, On circulant best matrices and their applications, Linear and Multilinear Algebra, 48, 2001, 263-274. Copyright Taylor & Francis. Original journal available here.
On circulant best matrices and their applications

S. Georgiou, C. Koukouvinos and Jennifer Seberry

Abstract

Call four type 1 $(1, -1)$ matrices, $X_1, X_2, X_3, X_4$, of the same group of order $m$ (odd) with the properties

(i) $(X_i - I)^T = -(X_i - I)$, $i = 1, 2, 3$,
(ii) $X_i^T = X_4$ and the diagonal elements are positive,
(iii) $X_iX_j = X_jX_i$, and
(iv) $X_1X_1^T + X_2X_2^T + X_3X_3^T + X_4X_4^T = 4mI_m$, best matrices. We use a computer to give, for the first time, all inequivalent best matrices of odd order $m \leq 31$. Inequivalent best matrices of order $m$, $m$ odd, can be used to find inequivalent skew-Hadamard matrices of order $4m$. We use best matrices of order $\frac{1}{3}(s^2 + 3)$ to construct new orthogonal designs, including new $OD(2s^2 + 6; 1, 1, 2, 2, s^2, s^2)$.

AMS Subject Classification: Primary 05B20, Secondary 05B30

Key words and phrases: Circulant matrices, supplementary difference sets, orthogonal designs, Hadamard matrices.

1 Introduction and basic definitions

A $(1, -1)$ matrix of order $n$ is called a Hadamard matrix if $HH^T = H^T H = nI_n$, where $H^T$ is the transpose of $H$ and $I_n$ is the identity matrix of order $n$. A $(1, -1)$ matrix $A$ of order $n$ is said to be of skew type if $A - I_n$ is skew-symmetric. If $A$ is a skew Hadamard matrix then $A$ is said to be a skew-Hadamard matrix. Two $(1, -1)$ matrices $A, B$ of order $n$ are said to be amicable if $AB^T = BA^T$.

Let $G$ be an additive abelian group of order $n$ with elements $g_1, g_2, \ldots, g_n$ and $X$ a subset of $G$. Define the type 1 $(1, -1)$ incidence matrix $M = (m_{ij})$ of order $n$ of $X$ to be

$$m_{ij} = \begin{cases} +1 & \text{if } g_j - g_i \in X \\ -1 & \text{otherwise} \end{cases}$$

and the type 2 $(1, -1)$ incidence matrix $N = (n_{ij})$ of order $n$ of $X$ to be

$$n_{ij} = \begin{cases} +1 & \text{if } g_j + g_i \in X \\ -1 & \text{otherwise} \end{cases}$$

In particular, if $G$ is cyclic the matrices $M$ and $N$ are called circulant and back circulant respectively. In this case $m_{ij} = m_{1,j-i+1}$ and $n_{ij} = n_{1,i+j-1}$ respectively (indices should be reduced modulo $n$).

Definition 1 Let $X_1, X_2, X_3, X_4$ be four type 1 $(1, -1)$ matrices on the same group of order $m$ (odd) with the properties

(i) $(X_i - I)^T = -(X_i - I)$, $i = 1, 2, 3$
(ii) $X_4^T = X_4$ and the diagonal elements are positive

---

*Department of Mathematics, National Technical University of Athens, Zografou 15773, Athens, Greece.
†School of IT and Computer Science, University of Wollongong, Wollongong, NSW, 2522, Australia.
(iii) $X_i X_j = X_j X_i$

(iv) $X_1 X_1^T + X_2 X_2^T + X_3 X_3^T + X_4 X_4^T = 4m I_m$

Call such matrices best matrices of order $m$.

Pre and post multiplying equation (iv) by $e$ and $e^T$, respectively, where $e$ is the $1 \times m$ matrix of all ones gives that $4m - 3 = s^2$ where $s$ is odd integer.

In this paper we consider circulant best matrices, so condition (iii) is trivially satisfied. Hence, multiplying on the left by $e^T$ (the $1 \times m$ vector of one’s) and on the right by $e$ both sides of (iv) we conclude that circulant (or type 1) best matrices can only exist for orders $m$ of which $4m = 1^2 + 1^2 + 1^2 + a^2$, where $a$ is the sum of the elements of the first row of the symmetric matrix $X_4$ and $a$ is an odd integer.

An orthogonal design of order $n$ and type $(s_1, s_2, \ldots, s_u)$ $(s_i > 0)$, denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$, is an $n \times n$ matrix $A$ with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n$$

Alternatively, the rows of $A$ are formally orthogonal and each row has precisely $s_i$ entries of the type $\pm x_i$. In [1], where this was first defined, it was mentioned that

$$A^T A = \left(\sum_{i=1}^{u} s_i x_i^2\right) I_n$$

and so our alternative description of $A$ applies equally well to the columns of $A$. It was also shown in [1] that $u \leq \rho(n)$, where $\rho(n)$ (Radon’s function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^b$, $b$ odd, $a = 4c + d$, $0 \leq d < 4$. For more details and constructions of orthogonal designs the reader can consult the book of Geramita and Seberry [2].

In section 2 we describe briefly the method of construction, in section 3 we give all inequivalent circulant best matrices of odd order $m \leq 31$, and in section 4 we use best matrices to construct some new orthogonal designs and families of Hadamard matrices.

## 2 Method of construction

In order to describe our construction for best matrices, we need a few more definitions. Let $n$ be a positive integer.

**Definition 2** Four subsets $S_0, S_1, S_2, S_3$ of $\{1, 2, \ldots, n - 1\}$ are called $4-(n; n_0, n_1, n_2, n_3; \lambda)$ supplementary difference sets (sds) modulo $n$ if $|S_k| = n_k$ for $k = 0, 1, 2, 3$ and for each $m \in \{1, 2, \ldots, n - 1\}$ we have $\lambda_0(m) + \ldots + \lambda_3(m) = \lambda$, where $\lambda_k(m)$ is the number of solutions $(i, j)$ of the congruence $i - j \equiv m (\text{mod} \ n)$ with $i, j \in S_k$.

Suppose that $S_k$ are $4-(n; n_0, n_1, n_2, n_3; \lambda)$ sds modulo $n$ having the following additional properties:

$$n + \lambda = n_0 + n_1 + n_2 + n_3 \quad (1)$$

$$i \in S_k \iff n - i \not\in S_k, \quad k = 0, 1, 2 \quad (2)$$

$$i \in S_t \iff n - i \in S_t, \quad t = 3 \quad (3)$$
where in (2) and (3) it is assumed that \( i \in \{1, 2, \ldots, n-1\} \).

Let \( a_k = (a_{k_0}, a_{k_1}, \ldots, a_{k_{n-1}}) \), \( k = 0, 1, 2, 3 \), be the row vector defined by

\[
a_{k_i} = \begin{cases} 
-1 & \text{if } i \in S_k \\
1 & \text{otherwise}
\end{cases}
\]

Furthermore let \( A_k \), \( k = 0, 1, 2, 3 \) be the circulant matrices with first row \( a_k \). Then it can be easily verified that \( A_0, A_1, A_2, A_3 \) are four matrices of order \( n \) as described in definition 1.

Let \( r \) be an integer relatively prime to \( n \), and set

\[
S'_k = \{ r_i \pmod{n} : i \in S_k \} \subset \{1, 2, \ldots, n-1\}
\]

for \( k = 0, 1, 2, 3 \). These sets are also \( 4 - (\bar{n}; n_0, n_1, n_2, n_3; \lambda) \) sds modulo \( n \) satisfying the conditions (1), (2), (3). We shall say that such quadruples \( S_0, S_1, S_2, S_3 \) and \( S'_0, S'_1, S'_2, S'_3 \) are equivalent.

We now give a brief description of the method of computation used to find the necessary sds’s. The numbers \( n_i \) are easy to determine (see [6]). We first generate a number of subsets of size \( n_i \) of \( \{1, 2, \ldots, n\} \) having the required symmetry properties (2) or (3), and at the same time compute the corresponding set of differences. We store the multiplicities of these differences in a file, say \( f_i \), saving only sets of differences with different multiplicities. After creating these files for each of the sizes \( n_0, \ldots, n_3 \), we try to match the items in the four files to produce an sds. This is done by examining items in two files only, say \( f_0 \) and \( f_1 \) and creating a new file in which we record the pairs which produce different total multiplicities of the differences. The procedure is repeated with the remaining two files \( f_2 \) and \( f_3 \). Finally the resulting two files are examined in order to find a perfect match. The results that we found applying this algorithm are presented in the next section.

### 3 The inequivalent supplementary difference sets

In this section we give for the first time all inequivalent supplementary difference sets which satisfy the condition (1), (2), (3) for all odd \( m \leq 31 \).

<table>
<thead>
<tr>
<th>( m = 3 ): 4 - (3; 1, 1, 0; 0)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 = {1} ), ( S_1 = {1} ), ( S_2 = {1} ), ( S_3 = \emptyset )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m = 7 ): 4 - (7; 3, 3, 6; 8)</th>
</tr>
</thead>
<tbody>
<tr>
<td>( S_0 = {1, 3, 5} ), ( S_1 = {1, 2, 3} ), ( S_2 = {1, 4, 5} ), ( S_3 = {1, 2, 3, 4, 5, 6} )</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>( m = 13 ): 4 - (13; 6, 6, 10; 15)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. ( S_0 = {1, 3, 5, 6, 9, 11} ), ( S_1 = {2, 6, 8, 9, 10, 12} ), ( S_2 = {1, 4, 5, 6, 10, 11} ), ( S_3 = {2, 3, 4, 5, 6, 7, 8, 9, 10, 11} )</td>
</tr>
</tbody>
</table>
2. \[ S_0 = \{2, 6, 89, 10, 12\}, \quad S_1 = \{1, 4, 5, 6, 10, 11\}, \]
\[ S_2 = \{1, 2, 3, 4, 6, 8\}, \quad S_3 = \{1, 2, 3, 4, 6, 7, 9, 10, 11, 12\} \]

Table 1. (continued)

\[ m = 21; \quad 4 - (21; 10, 10, 10, 6; 15) \]
1. \[ S_0 = \{1, 3, 6, 10, 12, 13, 14, 16, 17, 19\}, \quad S_1 = \{1, 2, 3, 4, 7, 8, 10, 12, 15, 16\}, \]
\[ S_2 = \{1, 2, 5, 11, 12, 13, 14, 15, 17, 18\}, \quad S_3 = \{4, 6, 10, 11, 15, 17\}, \]
2. \[ S_0 = \{1, 3, 4, 5, 6, 8, 10, 12, 14, 19\}, \quad S_1 = \{1, 2, 3, 4, 7, 9, 10, 13, 15, 16\}, \]
\[ S_2 = \{3, 4, 6, 7, 8, 9, 11, 16, 19, 20\}, \quad S_3 = \{2, 8, 9, 12, 13, 19\}, \]
3. \[ S_0 = \{1, 3, 4, 6, 10, 12, 13, 14, 16, 19\}, \quad S_1 = \{2, 3, 4, 7, 8, 9, 10, 15, 16, 20\}, \]
\[ S_2 = \{1, 2, 3, 4, 5, 7, 8, 11, 12, 15\}, \quad S_3 = \{1, 3, 8, 13, 18, 20\}, \]
4. \[ S_0 = \{1, 2, 4, 6, 7, 9, 10, 12, 16, 17, 19\}, \quad S_1 = \{4, 5, 10, 12, 13, 14, 15, 18, 19, 20\}, \]
\[ S_2 = \{1, 2, 3, 4, 5, 6, 9, 11, 13, 14\}, \quad S_3 = \{1, 5, 9, 12, 16, 20\}, \]
5. \[ S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}, \quad S_1 = \{1, 2, 3, 4, 8, 11, 12, 14, 15, 16\}, \]
\[ S_2 = \{1, 6, 11, 12, 13, 14, 16, 17, 18, 19\}, \quad S_3 = \{2, 6, 8, 13, 15, 19\}, \]
6. \[ S_0 = \{1, 3, 7, 10, 12, 13, 15, 16, 17, 19\}, \quad S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}, \]
\[ S_2 = \{1, 3, 4, 5, 6, 11, 12, 13, 14, 19\}, \quad S_3 = \{1, 5, 9, 12, 16, 20\}, \]
7. \[ S_0 = \{1, 2, 4, 6, 7, 9, 10, 13, 16, 18\}, \quad S_1 = \{1, 2, 3, 4, 6, 7, 8, 11, 12, 16\}, \]
\[ S_2 = \{1, 7, 8, 9, 11, 15, 16, 17, 18, 19\}, \quad S_3 = \{2, 4, 9, 12, 17, 19\}, \]

\[ m = 31; \quad 4 - (31; 15, 15, 15, 10; 24) \]
1. \[ S_0 = \{1, 5, 6, 8, 10, 13, 14, 16, 19, 20, 22, 24, 27, 28, 29\}, \]
\[ S_1 = \{2, 3, 4, 5, 6, 7, 10, 12, 13, 14, 16, 20, 22, 23, 30\}, \]
\[ S_2 = \{1, 3, 4, 5, 12, 16, 17, 18, 20, 21, 22, 23, 24, 25, 29\}, \]
\[ S_3 = \{3, 6, 8, 12, 13, 18, 19, 23, 25, 28\}, \]
2. \[ S_0 = \{1, 4, 7, 10, 12, 14, 15, 18, 20, 22, 23, 25, 26, 28, 29\}, \]
\[ S_1 = \{1, 2, 6, 12, 13, 14, 16, 20, 21, 22, 23, 24, 26, 27, 28\}, \]
\[ S_2 = \{5, 6, 7, 9, 10, 11, 12, 14, 15, 18, 23, 27, 28, 29, 30\}, \]
\[ S_3 = \{2, 3, 10, 12, 14, 17, 19, 21, 28, 29\}, \]

4 Constructions using best matrices

Theorem 1 Suppose there exist best matrices of order \( t = \frac{1}{4}(s^2 + 3) \), then there exists an O\( D(8t; 1, 1, 2, 2, s^2, s^2) \).

Proof. Suppose \( I + X, I + X_2, I + X_3 \) and \( X_4 \) are the circulant best matrices of order \( t \). Let \( a, b, c, d, e, f \) be commuting variables. Define

\[ X = \frac{1}{2}(X_2 + X_3); \quad Y = \frac{1}{2}(X_2 - X_3) \]

So

\[ X^T = -X, \quad Y^T = -Y, \quad XX^T + YY^T = \frac{1}{2}(X_2X_2^T + X_3X_3^T) \]

Now define
\[ A_1 = aI + bX_1 \]
\[ A_2 = dX_4 \]
\[ A_3 = cI - dX_1 \]
\[ A_4 = bX_4 \]
\[ A_5 = cI + bX + dY \]
\[ A_6 = fI - dX + bY \]
\[ A_7 = cI - bX - dY \]
\[ A_8 = fI + dX - bY \]

It is straightforward to check, using the properties of best matrices, that \( A_1, A_2, \ldots, A_8 \) satisfy the additive property and

\[ \sum_{i=1}^{8} A_i A_i^T = (a^2 + c^2 + 2c^2 + 2f^2 + (4t - 3)b^2 + (4t - 3)d^2)I_t. \]

We now check if the matrices form an amicable set. First we see

\[ A_1 A_2^T = adX_4 + bdX_1X_4 \]
\[ A_2 A_1^T = adX_4 - bdX_1X_4 \]
\[ A_3 A_4^T = cbX_4 - bdX_1X_4 \]
\[ A_4 A_3^T = cbX_4 + bdX_1X_4 \]

So

\[ A_1 A_2^T - A_2 A_1^T + A_3 A_4^T - A_4 A_3^T = 0. \]

Then we have

\[ A_5 A_6^T = efI + bfX + dfY + edX + bdX^2 + a^2XY - ebY - b^2XY - dbY^2 \]
\[ A_6 A_5^T = efI - bfX - dfY - edX + bdX^2 + a^2XY + ebY - b^2XY + dbY^2 \]
\[ A_7 A_8^T = efI - bfX - dfY - edX + bdX^2 + a^2XY + ebY - b^2XY + dbY^2 \]
\[ A_8 A_7^T = efI + bfX + dfY + edX + bdX^2 + a^2XY - ebY - b^2XY - dbY^2 \]

So

\[ A_5 A_6^T - A_6 A_5^T + A_7 A_8^T - A_8 A_7^T = 0. \]

Hence \( A_1 \ldots A_8 \) are amicable set of circulant matrices satisfying the additive property. Hence we may use them in Kharaghani array [3] to form \( OD(8t; 1, 1, 2, 2, 4t - 3, 4t - 3) \).

**Remark 1** We note there is no construction known which gives \( OD(8t; 1, 1, 2, 2, 4t - 3, 4t - 3) \).

Hence we have \( OD(56; 1, 1, 2, 2, 25, 25) \), \( OD(104; 1, 1, 2, 2, 49, 49) \), \( OD(168; 1, 1, 2, 2, 81, 81) \) and \( OD(248; 1, 1, 2, 2, 121, 121) \) for the first time.

**Theorem 2** Suppose there are best matrices of order \( m \) then there exists an \( OD(4m; 1, 1, 1, 4m-3) \).
Proof. Let $x_1$, $x_2$, $x_3$ and $x_4$ be four commuting variables. Write $I + B_1$, $I + B_2$, $I + B_3$ and $B_4$ for the best matrices of order $m$. Further write $A_1 = x_1 I + x_4 B_1$, $A_2 = x_2 I + x_4 B_2$, $A_3 = x_3 I + x_4 B_3$ and $A_4 = x_4 B_4$ for the four circulant (or type 1) matrices of order $m$ satisfying

$$A_1 A_1^T + A_2 A_2^T + A_3 A_3^T + A_4 A_4^T = (x_1^2 + x_2^2 + x_3^2 + (4m - 3)x_4^2)I_m.$$ 

Let $R = r_{ij}$, where $r_{ij} = 1$ for $i + j = m + 1$ and 0 otherwise. Then using the Goethals-Seidel array

$$\begin{bmatrix} A_1 & A_2 R & A_3 R & A_4 R \\ -A_2 R & A_1 & A_4^T R & -A_3^T R \\ -A_3 R & -A_4^T R & A_1 & A_2 R \\ -A_4 R & A_3^T R & -A_2^T R & A_1 \end{bmatrix},$$

is the required $OD(4m; 1, 1, 1, 4m - 3)$. \hfill \Box

Corollary 2 Let $m$ be the order of best matrices. Then an $OD(4m; 1, 1, 1, 4m - 3)$ exists.

Corollary 3 Let $m \in \{3, 7, 13, 21, 31\}$. Then an $OD(4m; 1, 1, 1, 4m - 3)$ exists.

Corollary 4 Let $m$ be the order of best matrices. Then there exist up to 8 inequivalent skew-Hadamard, and Hadamard, matrices of order $4m$.

Proof. Let $X_1, X_2, X_3, X_4$ be best matrices of order $m$. Then choosing $A_1 = X_1$, $A_2 = I \pm (X_2 - I)$, $A_3 = I \pm (X_3 - I)$ and $A_4 = \pm X_4$, in the Goethals-Seidel array gives the required result, (Note choosing $A_2 = \pm I + (X_2 - I)$, and $A_3 = \pm I + (X_3 - I)$ is an alternative choice.) \hfill \Box

We have constructed the Hadamard matrices of order 28 made, using as $A_1$, $A_2$, $A_3$ and $A_4$, the first rows given below in the Goethals-Seidel array

$$\begin{array}{cccccccc} 1 & 1 & 1-1 & 1-1-1 & ; & 1 & 1 & 1-1 & 1-1-1 & ; & 1 & 1 & 1-1 & 1-1-1 & ; & 1-1-1 & 1-1-1-1 & ; \\
1 & 1 & 1-1 & 1-1-1 ; & 1 & 1 & 1-1 & 1-1-1 ; & -1 & 1 & 1-1 & 1-1-1 ; & 1-1-1 & 1-1-1-1 & ; \\
1 & 1 & 1-1 & 1-1-1 ; & -1 & 1 & 1-1 & 1-1-1 ; & 1 & 1 & 1-1 & 1-1-1 ; & 1-1-1 & 1-1-1-1 & ; \\
1 & 1 & 1-1 & 1-1-1 ; & -1 & 1 & 1-1 & 1-1-1 ; & -1 & 1 & 1-1 & 1-1-1 ; & 1-1-1 & 1-1-1-1 & ; \\
\end{array}$$

We believe that the four Hadamard matrices thus produced are H-inequivalent and inequivalent skew-Hadamard matrices.

Corollary 5 Suppose there are best matrices of order $m$ and an Hadamard matrix, $H$, of order $4m/3$, then there is an Hadamard matrix of order $4m(4m - 3)/3$.

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$. Write $J$ for the $4m/3 - 1 \times 4m/3 - 1$ matrix of all ones. Normalize the Hadamard matrix, $H$, of order $4m/3$ so that its first row and column is all ones, then discard the first row and column to obtain the core of the Hadamard matrix, $B$, of order $4m/3 - 1$, which satisfies $BJ = -J$ and $BB^T = 4m/3I_{4m/3-1} - J_{4m/3-1}$. Then replacing the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by $J$, $J$, $J$ and $B$, which satisfy

$$3J^T J + (4m - 3)BB^T = (4m - 3)J + 4m(4m - 3)/3I - (4m - 3)J = 4m(4m - 3)/3I,$$

gives the required matrix. \hfill \Box
Example 1 We have found best matrices of orders $m = 3$ and 21. These give Hadamard matrices of orders 36 and 2268. These orders are not new, but, since Kimura [4, 5] has found some 487 inequivalent Hadamard matrices of order 28 which can be used in the corollary for $m = 21$ we may have constructed new, inequivalent, Hadamard matrices of order 2268. Since the variables can also be replaced by $J$, $±J$, $±J$ and $±B$ there is further potential for inequivalent Hadamard matrices.

Corollary 6 Suppose there are best matrices of order $m$ and a symmetric Hadamard matrix of order $h$

1. $h = 4(m + 1)/3$;
2. $h = 4(m + 2)/3$;
3. $h = 4(m + 3)/3$,

then there is an Hadamard matrix of order $4m(h - 1)$.

Proof. Use the best matrices to make an $OD(4m; 1, 1, 1, 4m - 3)$.

Normalize the symmetric Hadamard matrix of order $h$ so that its first row and column is all ones, then discard the first row and column to obtain the symmetric core of the symmetric Hadamard matrix, $B$, which satisfies $BJ = -J$ and $BB^T = hI - J_I$. Write $K = J - 2I$. Then

$$KJ^T = JK^T; \quad KB^T = BK^T; \quad JB^T = BJ^T.$$ 

Then replacing the variables of the $OD(4m; 1, 1, 1, 4m - 3)$ by

1. $J$, $J$, $K$ and $B$;
2. $J$, $K$, $K$ and $B$;

which satisfy

$$2JJ^T + 2K^T(4m - 3)BB^T = 2(h - 1)J + (h - 5)J + 4I + h(4m - 3)I - (4m - 3)J = 4m(h - 1)I;$$

$$JJ^T + 2K^T(4m - 3)BB^T = (h - 1)J + 2(h - 5)J + 8I + h(4m - 3)I - (4m - 3)J = 4m(h - 1)I;$$

$$3K^T + (4m - 3)BB^T = 3(h - 5)J + 12I + h(4m - 3)I - (4m - 3)J = 4m(h - 1)I,$$

respectively giving the required matrices. □

Example 2 From above we have four sequences of lengths $m = 3$, 7, 13, 21 and 31 which are the first rows for best matrices. Then using Corollary 5 and the best matrices of orders 3 and 21 we obtain Hadamard matrices of order 36 and 2268. Using Corollary 6 we obtain Hadamard matrices of orders $84 = 4 \cdot 21$, $308 = 4 \cdot 77$, $988 = 4 \cdot 247 = 4 \cdot 13 \cdot 19$, $2604 = 4 \cdot 851 = 4 \cdot 21 \cdot 31$ and $5332 = 4 \cdot 31 \cdot 43$. None of these orders are new but there are possibly inequivalent Hadamard matrices. □

Corollary 7 Suppose there are best matrices of order $m$, a back-circulant $SBIBD(v, k, \lambda)$ and an Hadamard matrix with circulant core, $B$, of order

$$2268$$
1. \( v = 4(k - \lambda) + 4m/3 - 1; \)
2. \( v = (8k - 8\lambda + 4m)/3 - 1; \)
3. \( v = 4(k - \lambda + m)/3 - 1; \)

then there is an Hadamard matrix of order \( 4mv. \)

**Proof.** Form the \( OD(4m; 1, 1, 1, 4m - 3) \) as before.

As before \( B \) satisfies \( BJ = -J \) and \( BB^T = (v + 1)I_v - J_v. \) Let \( A \) be the \( \pm 1 \) incidence matrix of the \( SBIBD(v, k, \lambda) \) then \( AJ = (2k - v)J \) and \( AA^T = 4(k - \lambda)I + (v - 4(k - \lambda))J. \)

We note \( AB^T = BA^T \) as \( A \) is back-circulant and \( B \) is circulant. We now replace the variables of the \( OD(4m; 1, 1, 1, 4m - 3) \) by (1) \( A, A \) and \( B, \) (2) \( A, A, J \) and \( B, \) and (3) \( A, J, J \) and \( B, \) respectively, which satisfy

\[
3AA^T + (4m-3)BB^T = 12(k-\lambda)I + 3(v-4(k-\lambda))J + (4m-3)(v+1)I - (4m-3)J = 4mvI,
\]

\[
2AA^T + JJ^T + (4m-3)BB^T = 8(k-\lambda)I + 2(v-4(k-\lambda))J + vJ + (4m-3)(v+1)I - (4m-3)J = 4mvI,
\]

\[
AA^T + 2JJ^T + (4m-3)BB^T = 4(k-\lambda)I + (v-4(k-\lambda))J + 2vJ + (4m-3)(v+1)I - (4m-3)J = 4mvI,
\]

gives the required matrices.

\[ \square \]

**References**


