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Orthogonal Designs with Quaternion Elements

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Abstract

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Key words and phrases: Orthogonal designs, quaternions
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1 Introduction

The introduction of Space-Time Codes to harness the benefits of combined space and time diversity was a major step in moving the capacity of wireless communication systems towards the theoretical limits. The technique has been adopted in the 3G standard in the form of an Alamouti code [1] and in the newly proposed standard for wireless LANs IEEE 802.11n [2]. Application of other forms of diversity together with STCs can improve this even further. The two obvious techniques to be considered together with STCs are frequency diversity and polarisation diversity.

Polarisation diversity has been widely studied in the past, e.g. [3] with an assessment of the diversity gain under Rayleigh fading presented in [4]. This form of diversity is usually considered separately
from the others and there is no well-known mechanism of utilising it jointly with the other forms rather than through a simple concatenation. In [5], Isaeva and Sarytchev showed that polarisation state can be nicely modeled by means of quaternion representation. Hence, an orthogonal design with the quaternion elements can become a basis of an Orthogonal Space-Time-Polarization code where polarisation diversity can be considered jointly with space and time diversities.

An Hadamard matrix $H$ of order $n$ is a square $(1, -1)$ matrix having inner product of distinct rows zero. Hence $HH^T = nI_n$. We note that $n = 1, 2$ or $n \equiv 0 \pmod{4}$.

Traditionally an orthogonal design of order $n$ and type $(s_1, s_2, \ldots, s_u)$ ($s_i > 0$), denoted $OD(n; s_1, s_2, \ldots, s_u)$, on the commuting variables $x_1, x_2, \ldots, x_u$ is an $n \times n$ matrix $A$ with entries from $\{0, \pm x_1, \pm x_2, \ldots, \pm x_u\}$ such that

$$AA^T = \left(\sum_{i=1}^{u} s_i x_i^2\right)I_n.$$ 

Alternatively, the rows of $A$ are formally orthogonal and each row has precisely $s_i$ entries of the type $\pm x_i$. In [6], where this was first defined, it was mentioned that

$$A^T A = \left(\sum_{i=1}^{u} s_i x_i^2\right)I_n$$

and so our alternative description of $A$ applies equally well to the columns of $A$. It was also shown in [6] that $u \leq \rho(n)$, where $\rho(n)$ (Radon’s function) is defined by $\rho(n) = 8c + 2^d$, when $n = 2^s b$, $b$ odd, $a = 4c + d$, $0 \leq d < 4$.

Orthogonal designs with complex elements are discussed in [7]. We now consider the quaternion elements $i, j, \text{ and } k$, where $i^2 = j^2 = k^2 = -1$, and $ij = k, jk = i, ki = j$, with 1 the unit.

We will say a number $a$ is a quaternion number if

$$a = a_1 + a_2 i + a_3 j + a_4 k$$

$$= (a_1 + a_2 i) + (a_3 + a_4 i)j,$$

where $a_i, i = 1, \ldots, 4$ are real numbers. We say a variable $a$ is a quaternion variable if $a = a_1 + a_2 i + a_3 j + a_4 k$, where $a_i, i = 1, \ldots, 4$ are real variables.
We define the quaternion transform \( q^Q \) of a quaternion \( q \) by analogy with complex conjugation and hermitian transforms. \( q^Q \) is the quaternion such that \( q^Q q = q q^Q = 1 \). For example, \( i^Q = -i \). When \( q \) is real, \( q^Q = q \).

Let \( q, r \) be quaternions. We define the quaternion transform of their product as follows: \( (qr)^Q = r^Q q^Q \).

Let \( a \) be a quaternion number (or variable). Then its quaternion transform \( a^Q \) is

\[
a^Q = (a_1 + a_2 i + a_3 j + a_4 k)^Q
= a_1^Q + (a_2 i)^Q + (a_3 j)^Q + (a_4 k)^Q
= a_1 + i^Q a_2 + j^Q a_3 + k^Q a_4
= a_1 - ia_2 - ja_3 - ka_4
= a_1 - a_2 i - a_3 j - a_4 k
\]

Further we define the inner product of two quaternion variables \( \mathbf{a} = a_1 + a_2 i + a_3 j + a_4 k \), and \( \mathbf{b} = b_1 + b_2 i + b_3 j + b_4 k \), as

\[
\mathbf{a} \cdot \mathbf{b} = ab^Q
= (a_1 + a_2 i + a_3 j + a_4 k)(b_1 - b_2 i - b_3 j - b_4 k)
= (a_1 b_1 + a_2 b_2 + a_3 b_3 + a_4 b_4)
+ (-a_1 b_2 + a_2 b_1 - a_3 b_4 + a_4 b_3) i
+ (-a_1 b_3 + a_2 b_4 + a_3 b_1 - a_4 b_2) j
+ (-a_1 b_4 - a_2 b_3 + a_3 b_2 + a_4 b_1) k.
\]

We define the quaternion transform of a matrix \( A = [a_{ij}] \) as

\[A^Q = [a_{ij}^Q].\]

2 Preliminary results

**Lemma 1** Let \( a \) be a quaternion variable (or number) then \( aa^Q = \sum_{i=1}^{4} a_i^2 \), which is real.

**Lemma 2** Let \( a \) be a quaternion variable (or number). Then \( a + a^Q \) is real.
Lemma 3 Let \( a \) and \( b \) be quaternion variables (or numbers) then 
\[
ab^Q = ba^Q \text{ only if } \\
- a_1 b_2 + a_2 b_1 - a_3 b_4 + a_4 b_3 = \\
- a_1 b_3 + a_2 b_4 + a_3 b_1 - a_4 b_2 = \\
- a_1 b_4 - a_2 b_3 + a_3 b_2 + a_4 b_1 = 0.
\]

Proof. We expand \( ab^Q \) and \( ba^Q \) and equate the terms in \( i, j \) and \( k \) to get the result. \( \square \)

We now define a quaternion orthogonal design of order \( n \) and type \( (s_1, s_2, \ldots, s_u) \) \( (s_i > 0) \), denoted \( QOD(n; s_1, s_2, \ldots, s_u) \), on the quaternion commuting variables \( x_1, x_2, \ldots, x_u \) as an \( n \times n \) matrix \( A \) with entries from \( \{0, q_1, q_2, \ldots, q_u x_u \} \), where each \( q_i \) is a linear combination of \( \{\pm 1, \pm i, \pm j, \pm k\} \) such that
\[
AA^Q = \left( \sum_{i=1}^{u} s_i x_i^2 \right) I_n.
\]

Example 1 Suppose \( a \) and \( b \) are quaternion variables such that
\[
a_1 b_3 - a_2 b_4 - a_3 b_1 + a_4 b_2 = 0.
\]
Then \( D = \begin{bmatrix} a & jb \\ ib & -ka \end{bmatrix} \) is a \( QOD(2; 1, 1) \). This follows as
\[
DD^Q = \begin{bmatrix} a & jb \\ ib & -ka \end{bmatrix} \begin{bmatrix} a^Q & -b^Q i \\ -b^Q j & a^Q k \end{bmatrix} = \begin{bmatrix} aa^Q + bb^Q & -iab^Q + iab^Q \\ iba^Q - iab^Q & aa^Q + bb^Q \end{bmatrix} = (aa^Q + bb^Q) I_2.
\]

Example 2 Suppose \( a, b, x, y \) are quaternion variables such that
\[
ax^Q = xa^Q, \\
by^Q = yb^Q, \\
a_1 y_2 - a_2 y_1 - a_3 y_4 + a_4 y_3 = 0, \text{ and } \\
b_1 x_2 - b_2 x_1 - b_3 x_4 + b_4 x_3 = 0.
\]
Then the matrices

\[
A = \begin{bmatrix} a & bj \\ bi & -ak \end{bmatrix}, \quad B = \begin{bmatrix} x & -yj \\ yi & xk \end{bmatrix}
\]

have the property that \( AB^Q = BA^Q \). Such matrices, by analogy with the real case, will be called quaternion amicable matrices. Thus the matrices \( A, B \) are quaternion amicable orthogonal designs QAOD(2; 1, 1; 1, 1).

**Proof.** Let \( aiy^Q = \alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k \). Then \( (aiy^Q)^Q = \alpha_1 - \alpha_2i - \alpha_3j - \alpha_4k \). Now, \( yia^Q = -(aiy^Q)^Q \). Hence \( yia^Q = -\alpha_1 + \alpha_2i + \alpha_3j + \alpha_4k \). But \( \alpha_1 = a_1y_2 - a_2y_1 - a_3y_4 + a_4y_3 \). By equation (1), \( \alpha_1 = 0 \). So \( aiy^Q = yia^Q \).

Likewise, it can be shown that \( bix^Q = xib^Q \) by equation (2).

\[
AB^Q = \begin{bmatrix} a & bj \\ bi & -ak \end{bmatrix} \begin{bmatrix} x^Q & -iy^Q \\ jy^Q & -kx^Q \end{bmatrix} = \begin{bmatrix} ax^Q - by^Q & -aiy^Q - bix^Q \\ bix^Q + aiy^Q & by^Q - ax^Q \end{bmatrix}
\]

\[
= \begin{bmatrix} xa^Q - yb^Q & -yia^Q - xib^Q \\ xib^Q + yia^Q & yb^Q - xa^Q \end{bmatrix} = BA^Q
\]

\[\square\]

3 Conclusion

We have established the existence of quaternion orthogonal designs and quaternion amicable orthogonal designs. Their use in signal processing will be explained in future work.

References

[1] [http://www.3gpp.org/specs/specs.htm](http://www.3gpp.org/specs/specs.htm)


