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Abstract

Cochran-Mantel-Haenszel (CMH) statistics are reviewed and continuous analogues of the CMH general association and mean score statistics are given. CMH generalised correlation statistics are defined and analogues given.

Key Words: CMH general association statistic; CMH mean score statistic; generalised correlation.

1. Introduction

Subsequently Cochran-Mantel-Haenszel (CMH) statistics are reviewed. They are defined for tables of counts, but it is helpful to define continuous analogues. These analogues will be indicative of the performance of CMH tests without requiring the assessment of a broad range of categorisations. CMH generalised correlation statistics are also defined, as are their continuous analogues. Davis (2002, section 8.7) recommends the use of continuous analogues of the CMH mean score and correlation statistics and gives examples of their use.

2. Review of the Cochran-Mantel-Haenszel Statistics

The Cochran-Mantel-Haenszel statistics apply to counts $N_{ijh}$ in which $i = 1, \ldots, t$, $h = 1, \ldots, c$ and $j = 1, \ldots, b$. The layer index $j$ reflects the block in which the treatment occurs; the row index $i$ reflects the treatment of interest, and the column index $h$ reflects the value of the response variable.

The marginal totals $\{n_{bh}\}$ and $\{n_{ih}\}$ for each of the $b$ blocks are taken to be known. For each block, the vector of counts $N_j = (n_{11j}, \ldots, n_{1cj}, \ldots, n_{t1j}, \ldots, n_{tcj})^T$ has probability function

$$
\frac{n_{ij}!}{i_1! \cdots i_t!} \frac{n_{bh}!}{h_1! \cdots h_c!} / \frac{n_{ij}!}{i_1! \cdots i_t!} \frac{n_{bh}!}{h_1! \cdots h_c!} .
$$
Initially no assumption is made about ordering of the row and column variables: both are taken to be nominal.

The null hypothesis of interest, that there is no association between row and column variables in any of the \( b \) tables, is tested against its negation. In the following expectations and variances are taken under the null hypothesis.

### 2.1 The CMH General Association Statistic

Consider
- counts in \( b \) independent \( t \times c \) nominal-nominal tables and
- the vectors of counts \( N_j = (N_{ij}) \) for each of the \( j \) blocks, \( j = 1, \ldots, b \), each being of dimension \((t - 1)(c - 1) \times 1\) after removal of the redundant counts.

Take
- \( E[N_j] \) to be the expected value under the null hypothesis of no association,
- \( G_j = N_j - E[N_j] \), observed minus expected for the counts for the \( j \)th block,
- \( G = G_1 + \ldots + G_b \), to be the aggregation over all blocks of differences between observation and expectation, and
- the covariance matrix of \( G \) under the null hypothesis to be \( V_G \) say.

Then as the total sample size \( n_{..} = n_{..1} + \ldots + n_{..b} \) approaches infinity, the CMH general association statistic \( Q_G = G^T V_G^{-1} G \) has asymptotic distribution \( \chi^2_{(t-1)(c-1)} \).

### 2.2 CMH Mean Score Statistic

Assume now that the row variable is nominal, while the column variable is ordinal or interval, and that every observation in the \( h \)th column of the \( j \)th block is scored as \( b_{hj} \), \( j = 1, \ldots, c \). The null hypothesis of no association between row and column variables in any of the \( b \) tables, is now tested against the alternative that the \( t \) row mean scores differ, on average, across strata.

Take
- \( M_j \) to be the vector containing the first \( t-1 \) row means for the \( j \)th block
- \( M = \sum_j (M_j - E[M_j]) \), where the \( E[M_j] \) are expectations under the null hypothesis,
- \( V_M \) to be the covariance matrix of \( M \) under the null hypothesis.

Then as the total sample size \( n_{..} = n_{..1} + \ldots + n_{..b} \) approaches infinity \( Q_M = M^T V_M^{-1} M \), the CMH mean score statistic, has asymptotic distribution \( \chi^2_{(t-1)} \).

### 2.3 CMH Correlation Statistic

Assume now that the row and column variables are both ordinal or interval, and that the \( i \)th treatment row scores are \( a_{hi} \), \( i = 1, \ldots, t \), and the \( j \)th block column scores are \( b_{hj} \), \( j = 1, \ldots, b \).

The null hypothesis of no association between row and column variables in any of the \( b \) tables, is tested against the alternative that across blocks there is a consistent association, positive or negative, between the row scores and column scores.

Take
- \( C_j = \sum_i a_{hi} b_{hj} \{N_{ihj} - E[N_{ihj}]\} \)
- \( C = \sum_j C_j \)
• $V_C$ to be the covariance matrix of $C$ under the null hypothesis

Then as the total sample size $n = n_1 + \ldots + n_s$ approaches infinity $Q_C = C^T V_C^{-1} C$, the CMH correlation statistic, has asymptotic distribution $\chi^2_1$.

3. Continuous Analogues of the CMH Statistics

In constructing continuous analogues the CMH general association statistic is of no interest.

3.1 Continuous Analogue of the CMH Mean Score Statistic

The CMH mean score statistic $Q_M$ is based on the treatment means $\bar{Y}_i$, obtained by averaging over blocks. In the following the marginal totals are assumed to be unknown and hence there are no redundant responses. If the marginal totals were known then the means would no longer be mutually independent.

As in Best et al. (2012, section 2) we assume the model

$$Y_{ij} = \mu + \tau_i + \beta_j + \epsilon_{ij}$$

where $\tau_i$ are treatment effects with $\sum_i \tau_i = 0$, $\beta_j$ are block effects with $\sum_j \beta_j = 0$, $\mu$ is an overall mean and the $\epsilon_{ij}$ are independent random variables with mean 0 and variance $\sigma^2$. For this model the treatment means over blocks, $\bar{Y}_i$, are, by the Central Limit Theorem, approximately $N(\mu + \tau_i, \sigma^2/b)$ for the number of blocks, $b$, sufficiently large. Now we may apply the result that if $X_1, \ldots, X_n$ are $N(\mu, \sigma^2)$ then

$$\sum_{i=1}^n (X_i - \bar{X})^2 / \sigma^2 \equiv \frac{\chi^2_n}{n}$$

is independently distributed. It follows that under the null hypothesis that $\tau = (\tau_i) = 0$, $b \sum_{i=1}^t \sum_{j=1}^b (\bar{Y}_i - \bar{Y})^2 / \sigma^2 = M'$ say, is $\chi^2_{n-1}$ distributed. Further routine analysis reveals that when testing $\tau = 0$ against $\tau \neq 0$ the null hypothesis is rejected for large values of $M'$.

In practice is $\sigma^2$ unknown. On the $j$th block, because the sample variance is an unbiased estimator of the population variance, $E[\sum_{i=1}^t (\bar{Y}_{ij} - \bar{Y})^2 / (t-1)] = \sigma^2$. Writing $V = \sum_{i=1}^t \sum_{j=1}^b Y_{ij}^2 / b \sum_{j=1}^b \bar{Y}_{ij}^2 / (t-1)$ as in Best et al. (2012, section 1) and summing over blocks gives $E[V] = b \sigma^2$. In $M'$ replacing $\sigma^2$ by its unbiased estimate $V/b$ gives $M = b^2 \sum_{i=1}^t (\bar{Y}_i - \bar{Y})^2 / V$ as a test statistic for testing $\tau = 0$ against $\tau \neq 0$. Its approximate distribution is $\chi^2_{n-1} / n$.

3.2 Continuous Analogue of the CMH Correlation Statistic

First a contrast that can be the basis of an analogue of $Q_C$ is constructed. If $X_i$ has mean $\mu_i$ and standard deviation $\sigma_i$ then a contrast in these random variables is a function $a_i X_i$ such that $E[\sum_i a_i X_i] = 0$ and $a_i^2 = 1$. Thus if $\lambda_1, \ldots, \lambda_t$ are such that $\lambda_1 + \ldots + \lambda_t = 0$ then because $E[\sum_i a_i Y_{ij}] = \mu a_i = 0$, $C' = \sum_i Y_{ij} / \sqrt{\sum_i a_i^2}$ is a contrast.
From 3.1 immediately above we know that the \( \bar{Y}_i \) are, under the null hypothesis \( \tau = 0 \), approximately distributed as \( N(\mu, \sigma^2/b) \). Hence \( C' \) is approximately distributed as \( N(0, \sigma^2/b) \) and \( b(C')^2/\sigma^2 \) is approximately distributed as \( \chi^2_i \). A further approximation is introduced if, again as in 3.1, \( \sigma^2 \) by its unbiased estimate \( \hat{\sigma}^2 \). Then \( b^2(C')^2/\hat{\sigma}^2 = b^2(\bar{Y}_i - \mu)^2/\hat{\sigma}^2 = C \) say is approximately distributed as \( \chi^2_i \). So \( C \) is approximately the square of a contrast in the response means with approximate distribution \( \chi^2_i \).

The CMH correlation statistic, \( Q_C \), is based on the correlation between the treatment scores and the responses aggregated over blocks. First we construct the sample correlation between the treatment scores \( \{ \lambda_i \} \) and the response means \( \{ \bar{Y}_j \} \). The treatments scores are assumed to sum to zero; their sample variance is \( \sum_{i=1}^n \frac{\lambda_i^2}{N} \). The \( i \)th treatment has population variance \( \sigma^2/b \), which is estimated by \( V^2/b^2 \). It is this quantity that is used instead of the sample variance of the \( \{ \bar{Y}_j \} \). The sample correlation between the \( \{ \lambda_i \} \) and the \( \{ \bar{Y}_j \} \) is, subject to this adjustment,

\[
\frac{\bar{Y}_j}{\sqrt{\sum_{i=1}^n (\lambda_i^2)V^2/b^2}}.
\]

The square of the random variable corresponding to this approximate correlation is \( C \), a continuous analogue of \( Q_C \).

4. CMH Generalised Correlation Statistics and their Analogues

The CMH correlation statistic \( C \) is defined in section 2.3 as

\[
\sum_{a,b} a_i b_j (N_{ij} - E[N_{ij}]).
\]

Suppose that \( a_{ii} = a_{ii}(i) \) for all \( h \) with \( \sum_i a_{ii}(i)N_i / N = \delta_i \), and that \( b_{ii} = b_{ii}(j) \) for all \( h \) with \( \sum_j b_{ij}(j)N_j / N = \delta_j \). See the discussion in Rayner and Beh (2009) about generalised correlations. Call the modified statistic \( C_{uv} \), say, given by

\[
C_{uv} = \sum_{a,b} a_{ii}(i) b_{jj}(j) (N_{ij} - E[N_{ij}]).
\]

Suppose \( V_{Cuv} \) is the covariance matrix of \( C_{uv} \) under the null hypothesis Then as the total sample size \( N = n_1 + \ldots + n_s \) approaches infinity \( Q_{uvC} = C_{uv}^2 V_{Cuv}^{-1} C_{uv} \) the CMH generalised correlation statistic, has asymptotic distribution \( \chi^2_1 \).

From Rayner and Best (2001, section 8.2) \( C_{uv} \) standardised, that is, \( Q_{uvC} \), will detect departures of the data from the model of independence in the \( uv \)th bivariate moment.

From a data analytic perspective analogues of the generalised correlations that are of most interest are of order \( (u,1) \) between the treatment scores and the responses. It may be of interest to know how quadratic and cubic treatment scores correlate with responses, but the correlation between quadratic and cubic treatment scores and quadratic responses is of far less interest. Hence an analogue of interest is

\[
\sum a_{ii}(i) \bar{Y}_i (bV)
\]

that reflects polynomial effects of degree \( u \) in treatments. The square of this random variable is a continuous analogue of \( Q_{uvC} \), and is approximately distributed as \( \chi^2_1 \).
5. Conclusion

Suppose now that $Y_{ij}$ is a (continuous) measurement on the $i$th of $t$ treatments in the $j$th of $b$ blocks. Suppose that the ordered values of the observed $Y_{ij}$ are $y_1 < y_2 < \ldots < y_{b_t}$. If $a_0, a_1, \ldots, a_{b_t}$ are such that $a_0 < y_1 < a_1 < y_2 < \ldots < y_{b_t} < a_{b_t}$ then if a response falls in $(a_{h-1}, a_h)$ it is given the score $y_h, h = 1, \ldots, c$. Thus the observations $y_1, \ldots, y_{b_t}$ generate counts $N_{ihj}$ that are 1 or 0 as the response for the $i$th treatment on the $j$th block falls in the $h$th category $(a_{h-1}, a_h)$ or not.

If the data provide the categories in the sense just described, the for all data sets we have tested the analogue $M$ gives the same value as $Q_M$ and the analogue $C$ gives the same value as $Q_C$.

References