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Abstract

It is generally believed that in order to solve initial value problems using Lie symmetry methods, the initial condition needs to be left invariant by the infinitesimal symmetry generator that admits the invariant solution. This is not so. In this paper we incorporate the imposed initial value as a side condition to find 'infinitesimals' from which solutions satisfying the initial value can be recovered, along with the corresponding symmetry generator.

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Finding symmetries by incorporating initial conditions as side conditions

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It is generally believed that in order to solve initial value problems using Lie symmetry methods, the initial condition needs to be left invariant by the infinitesimal symmetry generator that admits the invariant solution. This is not so. In this paper we incorporate the imposed initial value as a side condition to find ‘infinitesimals’ from which solutions satisfying the initial value can be recovered, along with the corresponding symmetry generator.

1 Introduction

The most popular technique for finding exact solutions of the widest variety of differential equations (DEs) comes from Lie group analysis of differential equations, also known as the classical symmetries method. This was initiated by Sophus Lie (1881) over 120 years ago. The classical method for finding symmetries and reductions of partial differential equations (PDEs) is considered one of the most efficient methods for constructing solutions of non-linear equations. Essentially, for an n th order PDE in one dependent variable u and k independent variables $\mathbf{x} = (x_1, x_2, \dots, x_k)$,

$$\Delta(\mathbf{x}, u, u_{x_{i_1}}, u_{x_{i_1} x_{i_2}}, \dots, u_{x_{i_1} x_{i_2} \dots x_{i_n}}) = 0 \quad (1.1)$$

$$i_p = 1, \dots, k, \quad p = 1, \dots, n,$$

the method determines the transformations, in infinitesimal form,

$$\begin{aligned} x_i^* &= x_i + \epsilon X_i(\mathbf{x}, u) + O(\epsilon^2), \\ u^* &= u + \epsilon \eta(\mathbf{x}, u) + O(\epsilon^2) \end{aligned} \quad (1.2)$$

under which a differential equation is invariant. Invariant solutions may be obtained by solving a differential equation among a smaller set of variables consisting of the invariants of a sub-group of the invariance group of the equation. The coefficients X_i and η of the infinitesimal symmetry are often referred to as ‘the infinitesimals’. The invariance requirement is determined by

$$\Gamma^{(n)} \Delta|_{\Delta=0} = 0, \quad (1.3)$$

where

$$\Gamma = \sum_{i=1}^k X_i(\mathbf{x}, u) \frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u) \frac{\partial}{\partial u} \quad (1.4)$$

are vector fields which span the associated Lie algebra, and are called the infinitesimal generators of the transformation equation (1.2), and $\Gamma^{(n)}$ is the n th extension of Γ extended to the n th jet space. Equation (1.3) leads to an over-determined linear system of equations for the infinitesimals $X_i(\mathbf{x}, u)$ and $\eta(\mathbf{x}, u)$.

Symmetry groups of differential equations have a huge list of important applications: classification of invariant equations, linearization theorems, determination of conservation laws, but most importantly, the computation of invariant solutions (see e.g. Bluman and Kumei, 1989; Hill, 1992; Ibragimov, 1994/1995/1996).

In practice, the boundary and initial conditions are just as important as the governing equation for determining the behaviour of a system. The classical linear integral transforms of Fourier and Laplace are favoured for linear boundary value problems (BVPs) but they are not well-adapted for non-linear problems. Even though the symmetry methods apply well to non-linear PDEs, the standard procedure of applying the classical method to boundary value problems (BVPs) and initial value problems (IVPs) is very restrictive as it requires that the given boundary and initial conditions, as well as the governing PDE remain invariant under the transformation. So for equation (1.1) subject to

$$B(\mathbf{x}, u, u_1, \dots, u_{k-1}) = 0, \quad w(\mathbf{x}) = 0,$$

requires

$$\Gamma w(\mathbf{x}) = 0 \quad \text{when} \quad w(\mathbf{x}) = 0 \quad (1.5)$$

and

$$\Gamma^{(k-1)} B(\mathbf{x}, u, \dots, u_{k-1}) = 0 \quad \text{when} \quad B(\mathbf{x}, u, \dots, u_{k-1}) = 0 \quad \text{on} \quad w(\mathbf{x}) = 0. \quad (1.6)$$

Thus this procedure necessarily yields a smaller set of symmetries for the BVP or IVP than the PDE and generally means that the symmetry method cannot be applied to BVPs or IVPs.

Since 1969, there have been several generalisations of the classical Lie group method for symmetry reductions, such as the non-classical method (Bluman and Cole, 1969), and the method of generalised conditional symmetries (Fokas and Liu, 1994; Zhdanov, 1995). These generalisations were made in an effort to obtain a wider class of symmetries which will lead to new solutions not obtainable via the classical method (see e.g. Clarkson and Kruskal, 1989). However, these generalisations have not dealt with boundary values and initial conditions, mainly because of a widespread belief that boundary conditions (or initial conditions) could be incorporated only if they satisfied the overly restrictive condition of being separately and totally invariant. Hence in solving BVPs and IVPs, the conventional symmetry methods are often in fact, useless.

Goard (2003) found less restrictive conditions to equations (1.5) and (1.6) that needed to be met by infinitesimal generators in order to leave the given PDE invariant, but not the imposed condition, in order that they can still be used to solve the BVP (or IVP).

However, the new imposed conditions were not used to find the generators (which for example could be found by the classical or non-classical methods), but rather used after the generators were found to determine which generators would be suitable.

Olver and Rosenau (1986) show how many reduction methods, including the classical and non-classical methods, are unified under a framework whereby the governing PDE is appended with 'side conditions', such that the resulting over-determined system satisfy some sort of compatibility condition. The general theory of the compatibility of the governing equation with the invariant surface condition (ISC), i.e.

$$\sum_i X_i(\mathbf{x}, u)u_{x_i} - \eta(\mathbf{x}, u) = 0, \quad (1.7)$$

for non-classical symmetries, is given by Pucci and Saccomandi (1992). This has further been discussed and extended for evolutionary equations by Pucci and Saccomandi (2000) for finding ISCs that lead to compatible systems.

In this paper, we show how an initial condition (IC) can be used as a side condition in a similar way as to how the ISC is for non-classical symmetries, in order to determine a wider set of symmetries for the IVP which will leave invariant only those solutions that satisfy the initial condition.

While in this paper we concentrate on evolution equations in two independent variables of the form

$$u_t = K(x, t, u, u_x, u_{xx}, \dots) \quad (1.8)$$

the analysis applies equally well to PDEs with more independent variables.

Once we determine a suitable generator, we will as usual, solve the corresponding ISC (1.7) to get the functional form of the similarity solution. However rather than substituting this functional form into equation (1.8) to find our arbitrary function, as approached by Goard (2003) we can now simply compare the functional form at $t = 0$ with the IC to determine our arbitrary function.

In Section 2 we present the method for non-constant initial conditions and provide examples for this case in Section 3. In Section 4 we deal with the constant initial condition and then illustrate this case in Section 5. Finally in Section 6 we present our conclusion and also expound the limitations and further research work in the method.

2 Non-constant initial condition

Suppose we wish to solve the IVP, (1.8) subject to the IC

$$u(x, 0) = F(x) \quad (2.1)$$

when $F'(x) \neq 0$. We then seek infinitesimals $X_i(\mathbf{x}, u)$ and $\eta(\mathbf{x}, u)$ so that solutions to the IVP also satisfy the ISC

$$\sum_i X_i(\mathbf{x}, u)u_{x_i} = \eta(\mathbf{x}, u). \quad (2.2)$$

To simplify matters here we will only consider infinitesimals X_i independent of u .

From equations (2.1), (2.2) and (1.8), the infinitesimals need to satisfy at $t = 0$, $u = F$, the relationship

$$X(x, 0) = \frac{\eta(x, 0, F) - T(x, 0)K(x, 0, F, F', F'', \dots)}{F'}. \quad (2.3)$$

Differentiating equation (2.2) with respect to x and differentiating equation (2.2) with respect to t we get respectively that the infinitesimals need to satisfy at $t = 0$, $u = F$:

$$X_x(x, 0) = \frac{\eta_x + \eta_u F' - X F'' - T_x K - T(K^{1,0})}{F'} \quad (2.4)$$

and

$$X_t(x, 0) = \frac{\eta_t + \eta_u K - X(K^{1,0}) - T_t K - T(K^{0,1})}{F'}, \quad (2.5)$$

where $K = K(x, 0, F, F', F'', \dots)$,

$$K^{1,0} = D_x K(x, t, u(x, t), u_x(x, t), \dots) \Big|_{\substack{t=0 \\ u=F}},$$

i.e. the total derivative of K with respect to x , at $t = 0$, $u = F$,

$$K^{0,1} = D_t K(x, t, u(x, t), u_x(x, t), \dots) \Big|_{\substack{t=0 \\ u=F}},$$

i.e. the total derivative of K with respect to t , at $t = 0$, $u = F$, and in general we will use the notation

$$K^{i,j} = D_{\underbrace{x \dots x}_{i \text{ times}}} D_{\underbrace{t \dots t}_{j \text{ times}}} K(x, t, u(x, t), \dots) \Big|_{\substack{t=0 \\ u=F}}$$

for the total derivative of K with respect to x (i times) and t (j times).

From repeated differentiation of equation (2.2) we get the following relations at $t = 0$, $u = F$:

$$\begin{aligned} X_{xx}(x, 0) &= \frac{[\eta_{xx} + 2\eta_{xu}F' + \eta_{uu}(F')^2 + \eta_u F'' - 2X_x F'' - X F''' - T_{xx}K - 2T_x(K^{1,0}) - T(K^{2,0})]}{F'} \end{aligned} \quad (2.6)$$

$$\begin{aligned} X_{tt}(x, 0) &= \frac{[\eta_{tt} + 2\eta_{tu}K + \eta_{uu}(K^2) + \eta_u(K^{0,1}) - 2X_t(K^{1,0}) - X(K^{1,1}) - T_{tt}K - 2T_t(K^{0,1}) - T(K^{0,2})]}{F'} \end{aligned} \quad (2.7)$$

$$\begin{aligned} X_{xt}(x, 0) &= \frac{1}{F'} [\eta_{xt} + \eta_{xu}K + \eta_{ut}F' + \eta_{uu}KF' + \eta_u(K^{1,0}) - X_x(K^{1,0}) - X_t(F'') \\ &\quad - X(K^{2,0}) - T_{xt}K - T_x(K^{0,1}) - T_t K^{1,0} - T(K^{1,1})] \end{aligned} \quad (2.8)$$

$$X_{ttt}(x, 0) = \frac{1}{F'} \{ [\eta_{ttt} + 3\eta_{ttu}K + 3\eta_{tuu}K^2 + \eta_{uuu}K^3 + 3\eta_{tu}(K^{0,1}) + 3\eta_{uu}K(K^{0,1}) + \eta_u(K^{0,2})] \\ - [3X_{tt}K^{1,0} + 3X_t(K^{1,1}) + X(K^{1,2}) + T_{ttt}K + 3T_{tt}(K^{0,1}) + 3T_t(K^{0,2}) + T(K^{0,3})] \} \quad (2.9)$$

$$X_{tttt}(x, 0) = \frac{1}{F'} \{ [\eta_{tttt} + 3\eta_{ttt}(K^{0,1})^2 + 4\eta_{ttu}(K^{0,2}) + 6\eta_{tuu}(K^{0,1}) + 6\eta_{ttuu}K^2 + 4\eta_{ttuuu}K^3 \\ + \eta_{uuuu}K^4 + \eta_u(K^{0,3}) + 6\eta_{uuu}K^2(K^{0,1}) + 4\eta_{uu}K(K^{0,2}) + 12\eta_{tuu}K(K^{0,1}) + 4\eta_{tttu}K] \\ - [4X_{ttt}K^{1,0} + 6X_{tt}(K^{1,1}) + 4X_t(K^{1,2}) + X(K^{1,3}) + T_{tttt}K + 4T_{ttt}K^{0,1} \\ + 6T_{tt}(K^{0,2}) + 4T_t(K^{0,3}) + T(K^{0,4})] \}. \quad (2.10)$$

The proposed method for finding new symmetries for IVPs is then as follows:

Method

- (1) Select an ansatz for $T(x, t)$ and $\eta(x, t, u)$ (see Note 1 below).
- (2) Find $X(x, 0)$ from equation (2.3).
- (3) Select an ansatz for $X(x, t)$ using the information for $X(x, 0)$ from Step 2.
- (4) The values for the derivatives of X with respect to x at $t = 0$, i.e. $X_x(x, 0)$, $X_{xx}(x, 0) \dots$ from equations (2.4), (2.6) and subsequent equations, will automatically agree with the corresponding values using the $X(x, t)$ from Step 3.
- (5) Formulate the determining equations by equating the values for the derivatives of X with respect to t at $t = 0$, i.e. $X_t(x, 0)$, $X_{tt}(x, 0)$, $X_{ttt}(x, 0)$ and so on, found from the expressions as in equations (2.5), (2.7), (2.9), (2.10) and subsequent equations, with their values from using the $X(x, t)$ found in Step 3.
- (6) Solve the determining equations in Step 5 for the arbitrary functions in the ansatz.
- (7) Once the infinitesimals are found, they can be substituted into the ISC (2.2) that can then be solved to find the functional form of the similarity solution. Then this functional form can be found by substituting in the IC.
- (8) As a final check, substitute the solution into the governing equation (see remark below).

Note 1 After imposing initial data as well as the invariant surface condition, the allowable solutions are effectively contained in a very small set, which notionally has a very large symmetry group. This large group would be difficult to specify explicitly, so that a reasonable simplifying ansatz (Step 1) is necessary in practice. The selection of the ansatz may be guided by the forms of the infinitesimals for the PDE found by the direct method (see Clarkson and Kruskal, 1989) or non-classical method (see Bluman and Cole, 1969). For example, the direct method is found to produce a variety of new reductions involving reduced variables, one of which is of the form $\phi = f(t)u + g(x, t)$ and the other of which is of the form $z = \theta(t)x + \sigma(t)$. These are invariants of symmetry operators taking the general form

$$\frac{X}{T} = x\Omega(t) + \psi(t) \quad \text{and} \quad \frac{\eta}{T} = F(t)u + G(x, t).$$

As the method described in this paper derives determining equations at $t = 0$, one may try related straightforward ansatz such as

$$\begin{aligned}\eta &= (A(x)t^2 + B(x)t + C(x))u + D(x)t^2 + E(x)t + F(x), \\ T &= h(x)t^2 + f(x)t + g(x)\end{aligned}$$

or even a subset of this.

Note 2 The solutions will be invariant under the symmetry group subject to the governing equation, the ISC and the IC, i.e.

$$\Gamma^{(2)}(u_t - K(x, t, u, u_x, \dots)) \Big|_{\substack{u_t=K \\ Xu_x + Tu_t = \eta \\ u(x,0)=F(x)}} = 0$$

Remark For $T(x, 0) \neq 0$:

If equation (2.3) is satisfied by (X, T, η) at $t = 0$, $u = F$, then IVPs of the form

$$u_t = K(x, t, u, u_x, u_{xx}, \dots) + G(x, t); \quad u(x, 0) = F(x), \quad (2.11)$$

with $G(x, 0) = 0$ will also satisfy equation (2.3).

Similarly, if only the first determining equation (using $X_t(x, 0)$) is used to determine the infinitesimals, then IVPs of the form (2.11) with $G(x, 0) = 0$ and $G_t(x, 0) = 0$ will also satisfy the determining equation. Hence the infinitesimals may lead to a solution of equations (2.11) with $G(x, 0) = G_t(x, 0) = 0$ rather than equation (1.8).

In general if the first n determining equations (i.e. using $X_t(x, 0), \dots, \underbrace{X_{t..t}}_{n \text{ times}}(x, 0)$) are satisfied for equation (1.8), they will also be satisfied by equations (2.11) with $G(x, 0) = 0, \dots, G_t(x, 0) = 0, \dots, \underbrace{G_{t..t}}_{n \text{ times}}(x, 0) = 0$. Hence the more determining equations verified, the more chance $G(x, t) = 0$, and in fact $G(x, t) = o(t^n)$, a small- t approximation to zero. In practice, rarely more than four or five determining equations were necessary.

For $T(x, 0) = 0$, it is not as straightforward to state from the determining equations, the conditions on $G(x, t)$ in equations (2.11) at $t = 0$ so that equations (2.11) will satisfy the same determining equations.

Infinitesimals satisfying the first determining equation (i.e. using equation (2.5)) with $\eta_u \neq T_t$ at $t = 0$, $u = F$ will mean equations (2.11) satisfy the determining equation if $G(x, 0) = 0$. Then the i th, $i = 2, 3, \dots$ determining equation with $\eta_u \neq iT_t$ and $X \neq 0$ at $t = 0$, $u = F$ will be satisfied by equations (2.11) only if $\underbrace{G_{t..t}}_{i-1 \text{ times}}(x, 0) = 0$.

Hence more determining equations may be necessary especially if in the i th equation $X = 0$ or $\eta_u = iT_t$ at $t = 0$, $u = F$.

3 Examples

Note that in the following examples, reference to determining equation i refers to equating the equation for $\underbrace{X_{t..t}}_{i \text{ times}}(x, 0)$ as in equations (2.5), (2.7) etc, with its corresponding value using the form chosen for $X(x, t)$.

Example 1

$$u_t = u_{xx} + u_x^2 + \frac{2}{x}u_x; \quad u(x, 0) = \ln(1 + x). \quad (3.1)$$

Assuming the ansatz

$$\eta(x, t) = A(x)t + B(x) \quad (3.2)$$

and

$$T(x, t) = g(t) + j(x), \quad (3.3)$$

from equation (2.3) we require

$$X(x, 0) = B(x) + B(x)x - \frac{2}{x}g(0) - \frac{2}{x}j(x)$$

so we choose

$$X(x, t) = B(x) + B(x)x - \frac{2}{x}g(0) - \frac{2}{x}j(x) + tf(x). \quad (3.4)$$

We now solve for the arbitrary functions in the infinitesimals. From the first determining equation we find that $f(x)$ in the ansatz for $X(x, t)$ needs to be of the form

$$f(x) = A(x)x + A(x) + \frac{4}{x}B(x) - \frac{2g'(0)}{x} + \frac{2B(x)}{x^2} - \frac{4}{x^3}j(x) - \frac{4g(0)}{x^3}. \quad (3.5)$$

Then from the second determining equation we require

$$A(x)[10x^6 + 4x^7 + 8x^5 + 2x^4] + B(x)[8x^5 + 20x^4 + 16x^3 + 4x^2] + j(x)[-8x^3 - 16x^2 - 8x] \\ - g''(0)[2x^6 + x^5 + x^7] - g'(0)[4x^5 + 4x^3 + 8x^4] - g(0)[16x^2 + 8x^3 + 8x] = 0 \quad (3.6)$$

Solving for $A(x)$ from equation (3.6), then the third determining equation requires $g'''(0) = 0$, and so we assume $g(t) = at^2 + bt + c$.

Using the forms for the functions as given so far, we find that the fourth and fifth determining equations are exactly identified.

Hence we have from equation (3.5)

$$f(x) = \frac{ax(1+x)}{(1+2x)} - \frac{2b}{(2x+1)} + \frac{2B(x)}{x} - \frac{4c}{x^2(2x+1)} - \frac{4j(x)}{x^2(2x+1)} \quad (3.7)$$

and from equation (3.6)

$$A(x) = \frac{ax}{(2x+1)} - \frac{2}{x^2}B(x) + \frac{2b}{x(2x+1)} + \frac{4c}{x^3(2x+1)} + \frac{4j(x)}{x^3(2x+1)} \quad (3.8)$$

so that from equations (3.2), (3.3) and (3.4),

$$X(x, t) = B(x)(1+x) - \frac{2c}{x} - \frac{2}{x}j(x) + tf(x), \\ T(x, t) = (at^2 + bt + c) + j(x), \\ \eta(x, t) = A(x)t + B(x). \quad (3.9)$$

With $a = c = 0$, $b = 2$, $j(x) = x + x^2$ and $B(x) = x^2 + 2$ then from equations (3.7) and (3.8)

$$f(x) = 2x \text{ and } A(x) = -2 \text{ so that}$$

equation (3.9) simplifies to

$$\begin{aligned} X(x, t) &= x^2 + x^3 + 2xt, \\ T(x, t) &= x + x^2 + 2t, \\ \eta(x, t) &= x^2 + 2 - 2t. \end{aligned} \quad (3.10)$$

Solving the corresponding ISC (2.2) gives

$$u = -t + \ln(x + 2t + x^2) + \phi(xe^{-t}). \quad (3.11)$$

Then on substituting the IC $u(x, 0) = \ln(1 + x)$, we find $\phi(x) = -\ln x$ so that the solution to the IVP (3.1) is

$$u = \ln(x + 2t + x^2) - \ln x.$$

This solution is then formally verified.

Note 1 The invariance requirement

$$\Gamma^{(2)} \left(u_t - u_{xx} - u_x^2 - \frac{2u_x}{x} \right) \bigg|_{\substack{u_t = u_{xx} + u_x^2 + \frac{2}{x}u_x \\ Xu_t + Tu_x = \eta}} = 0$$

is satisfied only if $u_x = \frac{x^2 - 2t}{x(x + 2t + x^2)}$, i.e. if the solution satisfies the IC.

Note 2 The substitution of equation (3.11) into equation (3.1) gives

$$\begin{aligned} &2x^2te^{-t}\phi'(z) + 2xte^{-2t}((\phi'(z))^2 + \phi''(z)) + 2xt + 4te^{-t}\phi'(z) + x^2e^{-2t}((\phi'(z))^2 + \phi''(z)) \\ &+ x^3e^{-2t}((\phi'(z))^2 + \phi''(z)) + 6x^2e^{-t}\phi'(z) + x^4e^{-t}\phi'(z) + x^2 + 2 + x^3e^{-t}\phi'(z) + x^3 \\ &+ 4x + 4xe^{-t}\phi'(z) = 0, \end{aligned} \quad (3.12)$$

where $z = xe^{-t}$. Equation (3.12) cannot be reduced to an ordinary differential equation (ODE), but with $x = ze^t$ becomes

$$\begin{aligned} &(4z\phi'(z) + 2 + z^2(\phi'(z))^2 + z^2\phi''(z)) + te^t(2z^2\phi'(z) + 2z) + te^{-t}(2z(\phi'(z))^2 + 2z\phi''(z) + 4\phi'(z)) \\ &+ e^t(z^3\phi''(z) + 6z^2\phi'(z) + 4z + z^3(\phi'(z))^2) + e^{3t}(z^4\phi'(z) + z^3) + e^{2t}(z^2 + z^3\phi'(z)) = 0, \end{aligned}$$

which then reduces to a system of six compatible ODEs for $\phi(z)$. Hence equation (3.10) lead to partial invariants of equation (3.1).

Example 2

$$u_t = x \ln(u_{xx}) + \frac{2xt}{u_{xx}} + x^2 - x \ln 2; \quad u(x, 0) = x^2. \quad (3.13)$$

Assuming ansatz for T and η of the form

$$\begin{aligned} T &= at^2 + bt + k, \\ \eta &= t^2A(x) + tB(x) + C(x), \end{aligned} \quad (3.14)$$

from equation (2.3) we require

$$X(x, 0) = \frac{-kx}{2} + \frac{C(x)}{2x},$$

so we choose

$$X(x, t) = \frac{-kx}{2} + \frac{C(x)}{2x} + tf(x). \quad (3.15)$$

From the first determining equation we find the form necessary for $f(x)$ in the ansatz for X , namely,

$$f(x) = \frac{B(x) - C(x)}{2} + \frac{(k - b)}{2}x - k \quad (3.16)$$

and from the second determining equation we require

$$A(x) = B(x) - C(x) + \frac{C(x)}{2x} + x^2(k - b + a) + x(-4k + 2b). \quad (3.17)$$

Continuing, from the third and fourth determining equations we require respectively

$$B(x) = \frac{3}{2}C(x) - \left(\frac{25}{6}k + 4a - 4b\right)x^2 + 2kx \quad (3.18)$$

and

$$C(x) = (15k - 14b + 12a)x^2. \quad (3.19)$$

The next determining equation is exactly identified.

So we have from equations (3.14)–(3.19)

$$\begin{aligned} X(x, t) &= (7k - 7b + 6a)x + \frac{xt}{6}(13k - 12b + 6a), \\ T(x, t) &= at^2 + bt + k, \\ \eta(x, t) &= t^2 \left(\frac{13kx^2}{3} - 4bx^2 + 3ax^2 + \frac{11}{2}kx - 5bx + 6ax \right) \\ &\quad + t \left(\frac{55}{3}kx^2 - 17bx^2 + 14ax^2 + 2kx \right) + (15kx^2 - 14bx^2 + 12ax^2). \end{aligned} \quad (3.20)$$

Selecting $a = -1$, $b = -1$, $k = 0$ we have

$$X = x + xt, \quad T = -t^2 - t, \quad \eta = (x^2 - x)t^2 + 3x^2t + 2x^2.$$

Solving the corresponding ISC (2.2) gives

$$u = x^2(t + 1) + xt \ln(t + 1) + \phi(xt)$$

which on comparison at $t = 0$ with the IC $u(x, 0) = x^2$ gives $\phi = 0$, so that the solution of the IVP is

$$u = x^2t + x^2 + xt \ln(t + 1).$$

This is checked by direct substitution in equation (3.13).

Example 3

$$u_t = -u_{xxxx} - u_{xx} + \frac{4}{5}u_x^2 + \frac{u_{xx}^2}{5}; \quad u(x, 0) = \cos(2x). \quad (3.21)$$

Equation (3.21) is a modified Kuramoto–Sivashinsky equation (see Bernoff and Bertozzi, 1995) which describes near planar interfaces that are marginally long-wave unstable.

Letting

$$T = h(x)t + j(x) \quad (3.22)$$

and

$$\eta = A(x)u + B(x), \quad (3.23)$$

from equation (2.3) we require

$$X(x, 0) = -\frac{\cot(2x)}{2}A(x) - \frac{B(x)}{2\sin(2x)} - 6\cot(2x)j(x) + \frac{8j(x)}{5\sin(2x)}, \quad (3.24)$$

which we also choose for $X(x, t)$, so that X is simply a function of x . From the first determining equation we find

$$j(x) = -\frac{A(x)}{12} - \frac{5}{16}B(x) - \frac{5}{16}h(x)\cos(2x) + \frac{h(x)}{12}. \quad (3.25)$$

Then from the second and third determining equations we find respectively that

$$A(x) = -\frac{15}{2}B(x) - \frac{15}{4}h(x)\cos(2x) - h(x) \quad (3.26)$$

and

$$h(x) = 0. \quad (3.27)$$

Further determining equations were exactly identified. Hence from equations (3.22)–(3.27)

$$X = \frac{15}{8}\cot(2x)B(x), \quad T = \frac{5}{16}B(x), \quad \eta = \left(-\frac{15}{2}u + 1\right)B(x)$$

so that with $B(x) = \frac{16}{5}$,

$$X = 6\cot(2x), \quad T = 1, \quad \eta = -24u + \frac{16}{5}. \quad (3.28)$$

Solving the ISC (2.2) with equation (3.28) we find

$$u = \frac{2}{15} + \phi(z)e^{-24t}; \quad z = e^{12t}\cos(2x).$$

Using the IC $u(x, 0) = \cos(2x)$, we find $\phi(z) = z - \frac{2}{15}$ so that

$$u = \frac{2}{15} - \frac{2}{15}e^{-24t} + e^{-12t}\cos(2x).$$

4 Constant initial condition with $T(x, 0) \neq 0$

With a constant initial condition $u(x, 0) = c$, from equation (2.2) we require the infinitesimals to satisfy at $t = 0$, $u = c$:

$$T(x, 0) = \frac{\eta(x, 0, c)}{K}. \quad (4.1)$$

Differentiating the ISC (2.2) with respect to the independent variables, as in Section 2, we form a set of conditions for the infinitesimals to satisfy at $t = 0$, $u = c$. This time the derivatives of the infinitesimal T are made the subject. In particular we require at $t = 0$, $u = c$:

$$T_x(x, 0) = \frac{\eta_x - T(K^{1,0})}{K}, \quad (4.2)$$

$$T_t(x, 0) = \frac{\eta_t + \eta_u K - T(K^{0,1}) - X(K^{1,0})}{K}, \quad (4.3)$$

$$T_{xx}(x, 0) = \frac{\eta_{xx} - 2T_x(K^{1,0}) - T(K^{2,0})}{K}, \quad (4.4)$$

$$T_{tt}(x, 0) = \frac{[\eta_{tt} + 2\eta_{tu}K + \eta_{uu}K^2 + \eta_u(K^{0,1}) - 2T_t(K^{0,1}) - T(K^{0,2}) - 2X_t(K^{1,0}) - X(K^{1,1})]}{K} \quad (4.5)$$

$$T_{ttt}(x, 0) = \frac{1}{K} [\eta_{ttt} + 3\eta_{ttu}K + 3\eta_{tuu}K^2 + \eta_{uuu}K^3 + 3\eta_{tu}(K^{0,1}) + 3\eta_{uu}K(K^{0,1}) + \eta_u(K^{0,2}) - 3T_{tt}(K^{0,1}) - 3T_t(K^{0,2}) - T(K^{0,3}) - 3X_{tt}K^{1,0} - 3X_t(K^{1,1}) - X(K^{1,2})] \quad (4.6)$$

and

$$\begin{aligned} T_{ttt}(x, 0) = & -\frac{1}{K} [-4\eta_{tttu}K + 4X_{ttt}K^{1,0} + 6X_{tt}(K^{1,1}) + 4X_t(K^{1,2}) + X(K^{1,3}) + 4T_{ttt}K^{0,1} \\ & + 6T_{tt}(K^{0,2}) + 4T_t(K^{0,3}) + T(K^{0,4}) - 12K\eta_{ttu}K^{0,1} - 4\eta_{uu}K(K^{0,2}) \\ & - 6K^2\eta_{uuu}K^{0,1} - \eta_u(K^{0,3}) - 6\eta_{ttu}K^{0,1} - 4\eta_{tu}(K^{0,2}) - \eta_{ttt} \\ & - 3\eta_{uu}(K^{0,1})^2 - 6\eta_{ttuu}K^2 - 4\eta_{ttuu}K^3 - \eta_{uuuu}K^4]. \end{aligned} \quad (4.7)$$

As for non-constant initial conditions, we can select an ansatz for the infinitesimals. For $T(x, t)$ this depends on using the information from equation (4.1). Then we derive the determining equations by setting the values for the derivatives of T with respect to t at $t = 0$, i.e. $T_t(x, 0)$, $T_{tt}(x, 0)$, ... found from equations (4.3), (4.5), (4.6), (4.7) and further such equations, equal to their corresponding values using the ansatz for $T(x, t)$. The more determining equations checked the better, as the remark in Section 2 for $T(x, t) \neq 0$ is equally valid here. Then steps 6–8 from Section 2 should be applied.

5 Examples

In the following examples, reference to determining equation i refers to equating $\underbrace{T_{t\dots t}}_{i \text{ times}}(x, 0)$ as in equations (4.3), (4.5) and subsequent equations, with its corresponding value using the form chosen for $T(x, t)$.

Example 1

$$u_t = u_{xx} - \frac{u_x^2}{u} + t^2 u \ln u - u \cos(x)(t^3 - 1 - t) + x \cos(xt) u; \quad u(x, 0) = 1. \quad (5.1)$$

Assuming ansatz for η and X as

$$\begin{aligned}\eta &= (A(x)t + B(x))u, \\ X &= h(x) + tf(x),\end{aligned}\tag{5.2}$$

from equation (4.1), $T(x, 0) = \frac{B(x)}{\cos(x) + x}$,
so we let

$$T(x, t) = \frac{B(x)}{\cos(x) + x} + tg(x).\tag{5.3}$$

From the first three determining equations, we get respectively that we require

$$\begin{aligned}g(x) &= \frac{A(x) + h(x)(\sin(x) - 1)}{\cos(x) + x}, \\ B(x) &= \frac{-2f(x)}{x^3} [\cos(x) \sin(x) + x \sin(x) - \cos(x) - x]\end{aligned}$$

and

$$A(x) = -\frac{h(x)}{x} [\cos(x) + x \sin(x)].$$

The next two determining equations respectively give

$$f(x) = 0 \quad \text{and} \quad h(x) = x.$$

Hence we have from equations (5.2) and (5.3)

$$\begin{aligned}X &= x, \\ T &= -t, \\ \eta &= -(x \sin(x) + \cos(x)) tu.\end{aligned}\tag{5.4}$$

Solving the corresponding ISC gives

$$u = \exp(t \cos(x)) \phi(z); \quad z = xt.\tag{5.5}$$

Substituting the IC $u(x, 0) = 1$ into equation (5.5) does not give information on $\phi(z)$ so we substitute the functional form equation (5.5) into the governing PDE (5.1). This gives an equation for $\phi(z)$ which is not an ODE in z , but the resultant equation

$$t^2 \left[-\phi'' + \frac{(\phi')^2}{\phi} - \phi \ln \phi \right] + x[\phi' - \phi \cos(xt)] = 0\tag{5.6}$$

can be split into a system of two compatible ODEs

$$\begin{aligned}-\phi'' + \frac{(\phi')^2}{\phi} - \phi \ln \phi &= 0, \\ \phi' - \phi \cos(z) &= 0; \quad z = xt.\end{aligned}\tag{5.7}$$

Solving equation (5.7) gives $\phi(z) = \exp(\sin(z))$ so that

$$u = \exp(t \cos(x) + \sin(xt)).$$

The solution is verified by direct substitution.

Note 1 In general, as the method derives determining equations at $t = 0$, the functions of t in the infinitesimals need to be assumed. However it may be possible to find the solution with unknown functions of t using the forms of the infinitesimals suggested by the method. The next example illustrates this.

Example 2 The PDE to price zero-coupon bonds can be written as an IVP

$$u_t = \frac{w(x,t)^2}{2} u_{xx} + v(x,t) u_x - xu \quad (5.8)$$

subject to $u(x, 0) = 1$, where the interest rate x follows the stochastic differential equation

$$dx = v(x, t) dt + w(x, t) dX \quad (5.9)$$

with dX an increment in a Wiener process (see Wilmott, 1998).

Taking $v(x, t) = a - bx$ and $w(x, t) = cx^{\frac{1}{2}}$ so that equation (5.9) describes the Cox, Ingersoll and Ross process (see Wilmott, 1998), PDE (5.8) becomes

$$u_t = \frac{c^2 x}{2} u_{xx} + (a - bx) u_x - xu. \quad (5.10)$$

Assuming ansatz $\eta = u(F(t) + G(x))$, $X = h(x)$, from equation (4.1) we require

$$T(x, 0) = \frac{G(x) + F(0)}{-x}$$

which we will also assume for $T(x, t)$, so that T is simply a function of x . The determining equations imply that $G(x)$ is constant with

$$G(x) = -k_1 - F(0), \text{ where } k_1 = \frac{F''(0) + F'(0)b}{c^2}$$

so that $T(x, t) = \frac{k_1}{x}$ and $X = H(x) = \frac{-ak_1}{x} + (bk_1 - F'(0))$.

With $k_1 = 0$, the forms $X = 1$, $T = 0$, $\eta = uf(t)$ are implied. Then the determining equations are satisfied if

$$f(0) = 0, \quad f'(0) = -1, \quad f''(0) = b, \quad f'''(0) = c^2 - b^2, \quad f''''(0) = -4c^2b + b^3.$$

Note that if the form of the infinitesimals was incorrect, we would have a contradiction with at least one of the derivatives at zero needing to be a function of x .

It is not possible to formulate the function $f(t)$ simply from the five values at $t = 0$. However the functional form of the solution is

$$u = \phi(t) e^{f(t)x}. \quad (5.11)$$

This can then be substituted into the PDE (5.10) which then on equating coefficients of x gives

$$\begin{aligned} \phi'(t) - a\phi(t)f(t) &= 0, \\ f'(t) - \frac{c^2}{2}f(t)^2 + bf(t) + 1 &= 0. \end{aligned}$$

Hence we find the solution to equation (5.10) with $u(x, 0) = 1$ is equation (5.11) with

$$f(t) = \frac{-2}{c^2} \left\{ \frac{K m_1 e^{m_1 t} + m_2 e^{m_2 t}}{K e^{m_1 t} + e^{m_2 t}} \right\}$$

where $m_{1,2} = \frac{-b \pm \sqrt{b^2 + 2c^2}}{2}$, $K = \frac{-m_2}{m_1}$ and $\phi(t) = \exp\{a \int f(t) dt\}$.

6 Conclusion

We have demonstrated in this paper a new and simple method to incorporate an initial condition to determine a much wider class of ‘symmetries’ for an IVP than what can currently be found using conventional symmetry methods. That is, these symmetries are related to non-classical symmetries, or more generally to conditional symmetries because they leave invariant a system consisting of not only of the governing PDE and ISC, but also an initial condition. The reduced equation is no longer necessarily an ODE in the invariants and may be a system of ODEs of any order. This approach is also more general than that of the earlier paper of the author in which less restrictive conditions to equations (1.5) and (1.6) were found for *known* symmetries (from the classical or non-classical approach).

One drawback of the new method however, is that because the determining equations are at $t = 0$, the forms of general functions of t in the infinitesimals may not be obvious, and would need to be surmised from the values of the function and their derivatives at $t = 0$. Alternatively their forms can be assumed. As shown, this approach has proved to provide new ‘symmetries’ and a greater scope for finding solutions to IVP. The method can also be adapted to boundary conditions and currently further work is being done in this area.

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