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Abstract

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SELF-MATCHING BANDS IN THE PAPERFOLDING SEQUENCE

BRUCE BATES, MARTIN BUNDER, AND KEITH TOGNETTI

ABSTRACT. We compare term by term the paperfolding sequence with a copy displaced by d terms to obtain the matching fraction $M(d)$. It is shown that $M(d)$ has an interesting structure in that if $d = 2^b(1+2s)$, then $M(d) = \left|1 - \frac{3}{2^{b+1}}\right|$ thereby generating horizontal bands for each value of b . That is, $M(d)$ depends only on b .

1. INTRODUCTION

Consider two binary sequences: $S = f_1 f_2 f_3 \dots$ and S displaced by d , that is, the sequence $f_{d+1} f_{d+2} f_{d+3} \dots$. As the terms can differ only by a unit, we look at the expression $|f_{d+i} - f_i|$ for $i \in \mathbb{N}$. If this is zero we have a **match** at the i^{th} term; otherwise it is unity and we have a **mismatch**.

Example 1. Let $S = 1101100111\dots$ be displaced by 3 terms. Then $|f_{3+i} - f_i|$ can be represented pictorially as follows.

$$\begin{array}{cccccccccccc} 1 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & \dots \\ & & & & & 1 & 1 & 0 & 1 & 1 & 0 & 0 & \dots \\ \hline |f_{3+i} - f_i| : & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & \dots \end{array}$$

This suggests the following definition.

Definition 1. (The self-matching function). Let S be an infinite binary sequence. The proportion of matches for S , with S displaced by d , is given by:

$$M(d) = \lim_{m \rightarrow \infty} \left(\frac{m - \sum_{i=1}^m |f_{d+i} - f_i|}{m} \right).$$

Recently, Tognetti [4] described a surprisingly simple matching pattern for Bernoulli sequences for which $f_i = \lfloor (i+1)\alpha \rfloor - \lfloor i\alpha \rfloor$. This represents the difference sequence for the integer parts sequence. It was shown that the graph of $M(d)$ against d exhibited a Moiré pattern and that unexpectedly this pattern was obtained by simply folding the fractional parts graph about its middle.

This paper examines the self-similarity within the paperfolding sequence and reveals yet another interesting pattern within the graph of a paperfolding $M(d)$ against d . We show that the graph forms horizontal bands.

2. THE PAPERFOLDING OPERATION

There have been many studies on the paperfolding sequence, $S = 11011001110\dots$, since the seminal paper by Davis and Knuth [2]. It is based on the following simple operation: repeatedly fold a piece of paper, right over left, i times. When unfolded,

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the paper contains v-shaped and inverted v-shaped creases. If we represent a v-shape by a 1 and an inverted v-shape by a 0, we obtain the following paperfolding subsequence after i folds (containing $2^i - 1$ creases):

$$S_i = f_1 f_2 f_3 \dots f_{2^i-1} = 110 \dots 100.$$

For example,

$$\begin{aligned} S_1 &= 1, \\ S_2 &= 110, \\ S_3 &= 1101100. \end{aligned}$$

As i becomes unbounded we have the infinite sequence, S . A comprehensive treatment of various paperfolding properties as well as a survey of the development of the paperfolding sequence can be found in Bates et al [1]. There it was shown that S can be represented by the *interleaving* of two sequences, as follows.

Definition 2. (Interleave operator). The interleave operator $\#$ acting on the two sequences $U = u_1 u_2 \dots u_k$ and $V = v_1 v_2 \dots v_n$ where $k > n$, generates the following interleaved sequence:

$$U \# V = u_1 \dots u_p v_1 u_{p+1} \dots u_{2p} v_2 u_{2p+1} \dots u_{np} v_n u_{np+1} \dots u_k,$$

where $p = \left\lfloor \frac{k}{n+1} \right\rfloor$.

Definition 3. (Alternating sequence). The alternating sequence of length $2r$ is given by $A_{2r} = 1010 \dots 10$.

Definition 4. (Interleaving expression for paperfolding) For $i \geq 2$, the paperfolding sequence of length $2^i - 1$, S_i , is defined as

$$S_i = A_{2^{i-1}} \# S_{i-1} \text{ where } S_1 = 1.$$

S can also be represented through *mirroring*.

Definition 5. (Mirror paperfolding sequence). The mirror paperfolding sequence of length $2^i - 1$, $\overline{S_i^R}$, is defined as the reversal of S_i combined with each 1 being replaced by 0 and each 0 being replaced by 1.

The following results are found in Bates et al [1].

Theorem 1. $S_{i+1} = S_i \ 1 \ \overline{S_i^R}$ and $\overline{S_{i+1}^R} = S_i \ 0 \ \overline{S_i^R}$ where $S_1 = 1$.

Corollary 1. $S_i = A_{2^{i-1}} \# A_{2^{i-2}} \# \dots \# A_2 \# 1$ and $\overline{S_i^R} = A_{2^{i-1}} \# A_{2^{i-2}} \# \dots \# A_2 \# 0$.

Corollary 1 tells us that the paperfolding sequence is equivalent to a series of successive interleaves of alternating sequences applied to the term $S_1 = 1$; and the mirror paperfolding sequence is equivalent to a series of successive interleaves of alternating sequences applied to the term $\overline{S_1^R} = 0$.

Theorem 2. S_i contains $2^{i-1} - 1$ instances of 0 and 2^{i-1} instances of 1.

We now demonstrate a more general result: the paperfolding sequence is an interleave of smaller paperfolding sequences.

Definition 6. (Alternating paperfolding sequence). The alternating paperfolding sequence of length $2^i - 2^n$, $0 < n < i$, is given by

$$A_{i,n} = S_{i-n} \overline{S_{i-n}^R} S_{i-n} \overline{S_{i-n}^R} \dots S_{i-n} \overline{S_{i-n}^R},$$

where the right hand side consists of 2^{n-1} copies of $S_{i-n} \overline{S_{i-n}^R}$.

Theorem 3. $S_i = A_{i,n} \# S_n$ and $\overline{S_i^R} = A_{i,n} \# \overline{S_n^R}$.

Note that particular values of n yield familiar expressions for S_i . That is,

i) For $n = 1$, $S_i = A_{i,1} \# S_1 = S_{i-1} \overline{S_{i-1}^R}$ and

ii) For $n = i - 1$, $S_i = A_{i,i-1} \# S_{i-1} = A_{2^{i-1}} \# S_{i-1}$.

In order to evaluate f_i , we represent i as $2^k(2r+1)$ where $k, r \geq 0$. This representation is characteristic of many folding structures apart from paperfolding, such as with the stickbreaking sequence, the Stern-Brocot tree and the Sarkovsky ordering of cycles in chaos (See Devaney [3]). It follows that i in binary is the binary number r , followed by a 1 and then k 0s.

The following two results for f_i are found in Bates et al [1].

Theorem 4. For $i = 2^k(2r+1)$, $f_i = 1 + r \bmod 2$.

We use the fact that $2r+1$ can be partitioned into $4h+1$, for $r = 2h$; and $4h+3$, for $r = 2h+1$ in the formulation of the following result.

Theorem 5. For $k, h \geq 0$,

$$f_i = \begin{cases} 1, & \text{if } i = 2^k(4h+1) \\ 0, & \text{if } i = 2^k(4h+3) \end{cases}.$$

Corollary 2. For $i = 2^k(4h+a)$ and $s = 2^b(4l+t)$ where $a, t \in \{1, 3\}$,

i) $f_i = \frac{1}{2}(3-a)$

ii) $f_i = f_s$, if and only if $a = t$.

Theorem 6. For $i = 2^k(4h+a)$ and $s = 2^b(4l+t)$ where $a, t \in \{1, 3\}$,

i) if $b < k-1$,

(a) $f_{i+s} = f_s$,

(b) $f_{i+s} = f_i$, if and only if $a = t$,

ii) if $b = k-1$,

(a) $f_{i+s} \neq f_s$,

(b) $f_{i+s} = f_i$, if and only if $a \neq t$,

iii) if $b = k$,

(a) $f_{i+s} = f_i$, if and only if $a = t$ and $2 \mid (h+l)$; or,

$a \neq t$ and $h+l+1 = 2^u(4v+a)$ for some $u, v \geq 0$,

(b) $f_{i+s} = f_s$, if and only if $a = t$ and $2 \mid (h+l)$; or,

$a \neq t$ and $h+l+1 = 2^u(4v+t)$ for some $u, v \geq 0$.

Proof. We have $i = 2^k(4h+a)$ and $s = 2^b(4l+t)$ where $a, t \in \{1, 3\}$. We examine each case.

i) Since $i+s = 2^b(4(2^{k-b}h+l+2^{k-b-2}a)+t)$, (a) and (b) follow from Corollary 2 ii).

ii) Since $i+s = 2^{k-1}(4(l+2h)+(2a+t))$, as $t \neq (2a+t) \bmod 4$ for any a and t and $a = (2a+t) \bmod 4$, if and only if $a \neq t$, (a) and (b) follow by Corollary 2 ii).

iii) (a) For $a = t$, $i+s = 2^{k+1}(2(h+l)+a)$. Also by Corollary 2 ii),

- if $2 \mid (h+l)$, then $i+s = 2^{k+1} \left(4 \left(\frac{h+l}{2}\right) + a\right)$ so $f_{i+s} = f_i = f_s$,
 - if $2 \nmid (h+l)$, and $a = 3$, then $i+s = 2^{k+1} \left(4 \left(\frac{h+l+1}{2}\right) + 1\right)$ so $f_{i+s} \neq f_i, f_s$,
 - if $2 \nmid (h+l)$, and $a = 1$, then $i+s = 2^{k+1} \left(4 \left(\frac{h+l-1}{2}\right) + 3\right)$ so $f_{i+s} \neq f_i, f_s$.
- (b) For $a \neq t$, $i+s = 2^{k+2} (h+l+1)$. Accordingly,
- if $h+l+1 = 2^u (4v+a)$ for some $u, v \geq 0$, $f_{i+s} = f_i$,
 - if $h+l+1 = 2^u (4v+t)$ for some $u, v \geq 0$, $f_{i+s} = f_s$.

□

In the special case where $s = 1$, by i), ii) and iii) for $u \geq 0$ and $h' = 4 - h$,

$$f_{i+1} = f_i \quad \text{if and only if} \quad i = \begin{cases} 2^{u+2} (4h+1), \text{ or} \\ 2 (4h+3), \text{ or} \\ 8h'+1, \text{ or} \\ 2^{u+2} (4v+3) - 1. \end{cases}$$

3. THE GRAPH OF THE SELF-MATCHING FUNCTION, $M(d)$

We now state our main result.

Theorem 7. *Let $d = 2^b (2r+1)$. Then $M(d) = \left|1 - \frac{3}{2^{b+1}}\right|$.*

Proof. There are two cases to consider:

i) d is odd, that is, $b = 0$. There are two sub-cases:

(a) $d = 4l+1$.

(I) Consider $l = 0$, that is, $d = 1$. From Definition 4, S is the interleave of the sequences in 3.1.1 and 3.1.2 while S displaced by 1, is the interleave of 3.1.3 and 3.1.4. Corresponding matched

or mismatched entries in the overlay are shown by :

$$\begin{array}{ll}
 (3.1.1) & \lim_{i \rightarrow \infty} A_{2^{i-1}} : \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad \dots \\
 (3.1.2) & S : \quad \quad \quad 1 \quad \vdots \quad 1 \quad \vdots \quad 0 \quad \vdots \quad 1 \quad \vdots \quad 1 \quad \dots \\
 (3.1) & \quad \quad \quad \quad \quad \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \quad \vdots \\
 (3.1.3) & \lim_{i \rightarrow \infty} A_{2^{i-1}} : \quad 1 \quad \vdots \quad 0 \quad \vdots \quad 1 \quad \vdots \quad 0 \quad \vdots \quad 1 \quad \dots \\
 (3.1.4) & S : \quad \quad \quad 1 \quad 1 \quad 0 \quad 1 \quad \dots
 \end{array}$$

Consider (3.1.3) :

- Every odd entry is a 1. Each is aligned with odd entries in S in (3.1.2) which by Definition 4 are consecutive values of an infinite alternating sequence. Thus half of these alignments match.
- Every even entry is a 0. Each is aligned with even entries in S in (3.1.2) which by Definition 4 are consecutive values of S . By Theorem 2, the ratio of matching 0s in (3.1.3) is $\lim_{i \rightarrow \infty} \frac{2^{i-1}-1}{2^i-1} = \frac{1}{2}$. Thus half of these alignments match.

Consider (3.1.4) :

- Consecutive odd entries form an infinite alternating sequence. Each is aligned to even entries in (3.1.1) which are all 0s. Thus half of these alignments match.

- Consecutive even entries form S . Each is aligned with a 1 from (3.1.1). By Theorem 2, the ratio of matching 1s in (3.1.4) is $\lim_{i \rightarrow \infty} \frac{2^{i-1}}{2^i - 1} = \frac{1}{2}$. Thus half of these alignments match. It follows that $M(1) = \frac{1}{2}$.
 - (II) Consider $l > 0$. Each entry in 3.1.3 and 3.1.4 moves $4l$ spaces to the right. Despite this move, each entry in 3.1.1 and 3.1.2 (except the leftmost d entries which are now unaligned) is aligned to a value identical to that found in the case for $l = 0$. Thus $M(4l + 1) = M(1) = \frac{1}{2}$, $l \in \mathbb{N}$.
- (b) $d = 4l + 3$.
- (I) Consider $l = 0$, that is, $d = 3$. As with (a), S overlaid with itself, with displacement 3, can be broken down into the following four subsequences:

$$\begin{array}{ll}
 (3.2.1) & \lim_{i \rightarrow \infty} A_{2^{i-1}} : \quad 1 \quad 0 \quad 1 \quad 0 \quad 1 \quad \dots \\
 (3.2.2) & S : \quad \quad \quad 1 \quad 1 \quad \vdots \quad 0 \quad \vdots \quad 1 \quad \vdots \quad 1 \quad \dots \\
 (3.2.3) & \lim_{i \rightarrow \infty} A_{2^{i-1}} : \quad \quad \quad 1 \quad \vdots \quad 0 \quad \vdots \quad 1 \quad \vdots \quad 0 \quad \dots \\
 (3.2.4) & S : \quad \quad \quad 1 \quad 1 \quad 0 \quad \dots
 \end{array}$$

- Consider (3.2.3) :
- Every odd entry is a 1. Each is aligned with even entries in S in (3.2.2) which by Definition 4 are consecutive values of S . By Theorem 2, the ratio of matching 1s in (3.2.3) is $\lim_{i \rightarrow \infty} \frac{2^{i-1}}{2^i - 1} = \frac{1}{2}$. Thus half of these alignments match.
 - Every even entry is a 0. Each is aligned with odd entries in (3.2.2) which form an infinite alternating sequence. Thus half of these alignments match.
- Consider (3.2.4) :
- Consecutive odd entries form an infinite alternating sequence. Each is aligned to odd entries in (3.2.1) which are all 1s. Thus half of these alignments match.
 - Consecutive even entries form S . Each is aligned with a 0 from (3.2.1). By Theorem 2, the ratio of matching 0s in (3.2.4) is $\lim_{i \rightarrow \infty} \frac{2^{i-1} - 1}{2^i - 1} = \frac{1}{2}$. Thus half of these alignments match.
- It follows that $M(3) = \frac{1}{2}$.
- (II) Consider $l > 0$. Each entry in 3.2.3 and 3.2.4 moves $4l$ spaces to the right. Despite this move, each entry in 3.2.1 and 3.2.2 (except the leftmost d entries which are now unaligned) is aligned to a value identical to that found in the case for $l = 0$. Thus $M(4l + 3) = \frac{1}{2}$, $l \in \mathbb{N}$.
- Combining (a) and (b), for $b = 0$, $M(d) = \frac{1}{2}$.
- ii) d is even, that is, $d = 2^b(4l + t)$ where $t \in \{1, 3\}$, $b > 0$.

From Theorem 3, taking limits, $S = S_b f_1 \overline{S_b^R} f_2 S_b f_3 \overline{S_b^R} f_4 \dots$. Since each $S_b f_i$ and $\overline{S_b^R} f_i$ is of length 2^b , we also have

$$S = S_b f_{2^b} \overline{S_b^R} f_{2 \cdot 2^b} S_b f_{3 \cdot 2^b} \overline{S_b^R} f_{4 \cdot 2^b} \dots$$

So S overlaid with itself with displacement d can be viewed as

$$(3.3) \quad \begin{array}{cccccccc} S_b & 1 & \overline{S_b^R} & 1 & \dots & S_b & f_d & \overline{S_b^R} & f_{d+2^b} & S_b & \dots \\ & & & & & & & S_b & 1 & \overline{S_b^R} & 1 & \dots \end{array}$$

where after the $(\lceil \frac{4L+t}{2} \rceil)$ th instance of S_b in the first line, S_b entries are overlaid with $\overline{S_b^R}$, and $\overline{S_b^R}$ entries are overlaid with S_b . Consider these overlays of S_b and $\overline{S_b^R}$ entries in (3.3). By Theorem 1, each middle term is mismatched, thereby generating mismatches every 2^b spacings in (3.3). Thus for large m , $\frac{m}{2^b}$ terms are mismatched. Now consider the overlay of the other entries in (3.3). These occur every 2^b spacings and represent S overlaid with itself with odd displacement. By i), half of these entries mismatch and so for large m , there are $\frac{m}{2^{b+1}}$ of these mismatches. Since these overlays are mutually exclusive, we can add the mismatches. That is, for large m , there are $\frac{3m}{2^{b+1}}$ mismatches. Thus $M(d) = 1 - \frac{3}{2^{b+1}}$ for d even. $\square$

From Theorem 7, as $M(d)$ is a function of b only, then $M(d)$ is constant for constant b . Hence the graph of $M(d)$ consists of horizontal bands based on b such that each band has height $|1 - \frac{3}{2^{b+1}}|$ as shown in Figure 1. We note that although the matching band for $2(2r+1)$ is below the band for odd numbers ($b=0$) all the other bands are above the odd band. That is,

$$\begin{array}{llll} \text{Band 0, } (b=0), & M(d) = \frac{1}{2}, & d \text{ is odd,} \\ \text{Band 1, } (b=1), & M(d) = \frac{1}{4}, & d = 2 + 4s = 2, 6, 10, \dots, \\ \text{Band 2, } (b=2), & M(d) = \frac{5}{8}, & d = 4 + 8s = 4, 12, 20, \dots, \\ & \vdots & \\ \text{Band } n, (b=n), & M(d) = |1 - \frac{3}{2^{n+1}}|, & d = 2^n + 2^{n+1} \cdot s = 2^n, 2^n \cdot 3, 2^n \cdot 5, \dots \end{array}$$

Theorem 8. For $k > 0$, and $1 \leq d < 2^k$,

$$M(d) = M(2^k \pm d).$$

Proof. If $d = 2^b(2r+1) < 2^k$, then $b < k$, and $2^k \pm d = 2^b(2(2^{k-b-1} \pm r) \pm 1)$. By Theorem 7, $M(2^k \pm d) = M(d)$. $\square$

Theorem 8 tells us that if we have the section of the graph up to $d = 2^b - 1$, we can generate the graph up to $2^{b+1} - 1$ by adding the point $(2^b, M(2^b))$ and then translating the earlier section to the right of 2^b .

4. THE EXPECTED VALUE OF $M(d)$

The terms associated with band b for $b > 0$ have period 2^{b+1} . Hence the proportion of these terms that possess this matching is $\frac{1}{2^{b+1}}$. Band 0 makes the largest contribution to the expected value of $M(d)$, $E(M(d))$, of any band. It contains

Figure 1: Self Matching $M(d)$ versus d

half the total number of points, each with value $\frac{1}{2}$, making its total contribution $\frac{1}{4}$. It contributes half of $E(M(d))$ as shown below.

$$\begin{aligned}
 E(M(d)) &= \sum_{b=0}^{\infty} \left| 1 - \frac{3}{2^{b+1}} \right| \frac{1}{2^{b+1}} \\
 &= \frac{1}{4} + \sum_{b=1}^{\infty} \left(1 - \frac{3}{2^{b+1}} \right) \frac{1}{2^{b+1}} \\
 &= \frac{1}{2}.
 \end{aligned}$$

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