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Peter Nickolas
University of Wollongong, peter@uow.edu.au

Susan Andima

Ralph Kopperman
City University of New York, rdkcc@ccny.cuny.edu

S Popvassilev

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Abstract

Bouziad in 1996 generalized theorems of Montgomery (1936) and Ellis (1957), to prove that every Cech complete space with a separately continuous group operation must be a topological group. We generalize these results in a new direction, by dropping the requirement that the spaces be T2 or even T1. Our theorems then become applicable to groups with asymmetric topologies, such as the group of real numbers with the upper topology, whose open sets are the open upper rays. We first show a generic Ellis-type theorem for groups with a Hausdor k-bitopological structure whose symmetrization belongs to a class of k-spaces for which a classical Ellis-type theorem is known. We then develop a number of specific cases, including the following: Let (G T) be a group with a topology making multiplication separately continuous, whose k dual T k makes (G T Tk) a Hausdor k-bispace such that T T k is Cech complete. Then multiplication is jointly continuous with respect to both T and T k, and inversion is a homeomorphism between (G T) and (G T k).

Keywords

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A FAMILY OF ASYMMETRIC ELLIS-TYPE THEOREMS

S. ANDIMA, R. KOPPERMAN, P. NICKOLAS, S. POPVASSILEV

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Abstract. Bouziad in 1996 generalized theorems of Montgomery (1936) and Ellis (1957), to prove that every Čech-complete space with a separately continuous group operation must be a topological group. We generalize these results in a new direction, by dropping the requirement that the spaces be T_2 or even T_1. Our theorems then become applicable to groups with "asymmetric" topologies, such as the group of real numbers with the upper topology, whose open sets are the open upper rays.

We first show a generic Ellis-type theorem for groups with a Hausdorff k-bitopological structure whose symmetrization belongs to a class of k-spaces for which a classical Ellis-type theorem is known. We then develop a number of specific cases, including the following: Let $(G, \cdot, \mathcal{T})$ be a group with a topology making multiplication separately continuous, whose k-dual $\mathcal{T}^k$ makes $(G, \mathcal{T}, \mathcal{T}^k)$ a Hausdorff k-bispace such that $\mathcal{T} \vee \mathcal{T}^k$ is Čech-complete. Then multiplication is jointly continuous with respect to both $\mathcal{T}$ and $\mathcal{T}^k$, and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.

1. Introduction and background

Deane Montgomery [18] in 1936 proved that every completely metrizable space with a separately continuous group multiplication must have jointly continuous multiplication, and Zelasko [25] in 1960 went on to show that such a structure must also have continuous inversion and so be a topological group. In the meantime, Robert Ellis [7], [8] showed in 1957 that every locally compact Hausdorff space with a separately continuous group multiplication is a topological group. Since

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then, many mathematicians have worked to obtain similar results in more general spaces by varying the assumption of local compactness, but always requiring that the spaces be at least Tychonoff.

Recall that a group $(X, \cdot)$ with a topology $\mathcal{T}$ is called a *semitopological group* if multiplication is separately continuous, a *paratopological group* if multiplication is jointly continuous and a *topological group* if inversion is continuous as well. There was particular interest in showing that each Čech-complete semitopological group is a topological group, because Čech-complete spaces generalize both completely metrizable and locally compact Hausdorff spaces. A number of mathematicians achieved partial results, including Wu [24], Namioka [20], Christensen [6], Brand [5], Pfister [21], and Reznichenko [22]. Then in 1996 Ahmed Bouziad proved the complete result in two different papers, [3] and [4], as corollaries to two different theorems.

In this paper, we generalize some Ellis-type theorems in a different direction, by dropping the requirement that the spaces be T_2 or even T_1, so that our theorems apply to such spaces as the real numbers with the upper topology. The results here also generalize our earlier work in [1]. Both there and here, we have two sets of parallel results: bitopological results for groups with two suitably related topologies, and asymmetric results for groups with a single topology for which a suitable second "dual" topology exists. The dual in question here is the k-dual $\mathcal{T}^k$ of a topological space $(X, \mathcal{T})$ (see Note 1.5 below). In [1, Theorem 4.6], our asymmetric result stated that if $(X, \cdot, \mathcal{T})$ is a semitopological group such that $(X, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace (see Definition 1.3) and the join $\mathcal{T} \vee \mathcal{T}^k$ is locally compact, then $(X, \cdot, \mathcal{T})$ and $(X, \cdot, \mathcal{T}^k)$ are both paratopological groups and inversion is a homeomorphism from $(X, \mathcal{T})$ to $(X, \mathcal{T}^k)$. Here we give analogous but stronger asymmetric results, in which we require only that $\mathcal{T} \vee \mathcal{T}^k$ belongs to one of several classes of k-spaces, such as the class of Čech-complete spaces or that of Baire p-spaces. These results generalize the classical theorems cited above, because in Corollary 2.9 of [1], we have shown that $\mathcal{T} = \mathcal{T}^k$ whenever $(X, \mathcal{T})$ is a Hausdorff k-space. Similar comparisons apply between the bitopological result (Theorem 4.3) of [1] and the bitopological results of the current paper.

In the remainder of this section we describe the basic concepts that are needed below.

An important tool in the study of a non-Hausdorff space $(X, \mathcal{T})$ is the *specialization order* $\leq_{\mathcal{T}}$, which is the preorder on X defined by $x \leq_{\mathcal{T}} y$ if $x \in cl(y)$, or, equivalently, if $\mathcal{N}_x \subseteq \mathcal{N}_y$, where $\mathcal{N}_z$ denotes the neighborhood system of z for each $z \in X$. The specialization order is preserved by continuous functions. It is a partial order when the topology is T_0 and is equality when the topology is T_1. A space is *asymmetric* when its specialization order is not symmetric. Given a subset A of X, the *saturation* of A is the set $\uparrow A = \{b : a \leq b \text{ for some } a \in A\}$. If $\uparrow A = A$, then A is called *saturated* or an *upper set*. The dual expressions $\downarrow A$, *cosaturation*, *cosaturated* and *lower set* are defined similarly in terms of the inverse relation $\geq$. Intersections and unions of saturated sets are saturated while those of cosaturated sets are always cosaturated. Open sets are always saturated while closed sets are cosaturated.

Definition 1.1. *Bitopological spaces* (introduced by Kelly [14]) are sets with two topologies, $\mathcal{T}$ and $\mathcal{T}^*$, which need not be related in any way. Given a bitopological space $\mathcal{X} = (X, \mathcal{T}, \mathcal{T}^*)$, its *bitopological dual* is $\mathcal{X}^* = (X, \mathcal{T}^*, \mathcal{T})$, and its *symmetrization topology* is $\mathcal{T}^s = \mathcal{T} \vee \mathcal{T}^*$. For any topological property Q, a bitopological space is *pairwise* Q if both it and its bitopological dual are Q. A map $f : X \to Y$ is *pairwise continuous* from $(X, \mathcal{T}_X, \mathcal{T}_X^*)$ to $(Y, \mathcal{T}_Y, \mathcal{T}_Y^*)$ if it is continuous from $\mathcal{T}_X$ to $\mathcal{T}_Y$ and from $\mathcal{T}_X^*$ to $\mathcal{T}_Y^*$. We use notation from [15], since it emphasizes concepts central to this paper. In particular, a topological term refers to $\mathcal{T}$ if undecorated, to $\mathcal{T}^*$ if given a *, and to $\mathcal{T}^s$ if given an ^{s}; for example, "T is open" means $T \in \mathcal{T}$, "C is *-closed" means C is closed in $\mathcal{T}^*$, and "f is ^{s}-continuous from $\mathcal{X} = (X, \mathcal{T}_X, \mathcal{T}_X^*)$ to $\mathcal{Y} = (Y, \mathcal{T}_Y, \mathcal{T}_Y^*)$" means f is continuous from $(X, \mathcal{T}_X^s)$ to $(Y, \mathcal{T}_Y^s)$.

A bitopological space $\mathcal{X} = (X, \mathcal{T}, \mathcal{T}^*)$ is

(a) *pseudo-Hausdorff (pH)* if for $x \notin \mathrm{cl}\{y\}$ there are disjoint $T \in \mathcal{T},\ T^* \in \mathcal{T}^*$ such that $x \in T$ and $y \in T^*$,
(b) T_0 if $\mathcal{T}^s$ is a T_0 topology, and
(c) *Hausdorff (T_2)* if T_0 and pseudo-Hausdorff.

In our major theorems we require that the bispace $(X, \mathcal{T}, \mathcal{T}^*)$ be pairwise Hausdorff. Then $(X, \mathcal{T}^s)$ is a Hausdorff topological space [15, Theorem 2.4]. Further, $\mathcal{T}$ and $\mathcal{T}^*$ have opposite specialization orders, so that $\leq_{\mathcal{T}^*} = \geq_{\mathcal{T}}$ [15, Lemma 2.5 (a)] and *-saturated is equivalent to cosaturated.

Note 1.2. For a collection $\{\mathcal{X}_i = (X_i, \mathcal{T}_i, \mathcal{T}_i^*) : i \in I\}$ of bitopological spaces, the *product* is $\prod_I \mathcal{X}_i = (\prod_I X_i, \prod_I \mathcal{T}_i, \prod_I \mathcal{T}_i^*)$, with the projections $\pi_j : \prod_I X_i \to X_j$. The symmetrization of the product is the same as the product of the symmetrizations, because the subbasic open sets of $(\prod_I \mathcal{T}_i)^s$ are those of the form $\pi_i^{-1}(T_i) \cap \pi_i^{-1}(U_i)$, for $T_i \in \mathcal{T}_i,\ U_i \in \mathcal{T}_i^*$, while those of $\prod_I (\mathcal{T}_i^s)$ are of the form $\pi_i^{-1}(T_i \cap U_i)$, and these are the same sets. It is also clear that the specialization order of the product topology $\prod_I \mathcal{T}_i$ is the same as the product of the specialization orders $\leq_{\mathcal{T}_i}$, which is the componentwise ordering inherited from the factors. (That is, for $x, y \in \prod_I X_i$, $x \leq_{\Pi_I \mathcal{T}_i} y \iff x_i \leq_{\mathcal{T}_i} y_i$ for each $i \in I$.) Thus a product of sets is saturated if and only if its factors are saturated, and the saturation of a product is the product of the saturations of its factors. Similar statements hold for "cosaturated" and "cosaturation".

Note that the product of a collection of pseudo-Hausdorff bitopological spaces is pseudo-Hausdorff and that the product of a collection of T_0 bitopological spaces is T_0, and so the product of a collection of pairwise Hausdorff bitopological spaces is pairwise Hausdorff.

Definition 1.3. Given a pairwise Hausdorff bitopological space, $\mathcal{X} = (X, \mathcal{T}, \mathcal{T}^*)$, its *kb-coreflection* is the bispace

$$KB(\mathcal{X}) = (X, k(\mathcal{X}), k^*(\mathcal{X})),$$

where $k(\mathcal{X})$ (resp., $k^*(\mathcal{X})$) consists of those sets whose intersection with each ^{s}-compact subspace K is open in $\mathcal{T}|_K$ (resp., $\mathcal{T}^*|_K$). $\mathcal{X}$ is a *k-bispace* if $KB(\mathcal{X}) = \mathcal{X}$.

By Lemma 1.2 of [16] $KB(\mathcal{X})$ is a k-bispace which has the same ^{s}-compact subspaces as $\mathcal{X}$ as well as the same bitopological restriction to these subspaces. $k(\mathcal{X})$ and $k^*(\mathcal{X})$ have the same specialization orders as $\mathcal{T}$ and $\mathcal{T}^*$.

Kopperman and Lawson [16, Theorem 2.3] proved that the category of pairwise Hausdorff k-bispaces and pairwise continuous maps is isomorphic to that of strongly T_2 ordered k-spaces and order-preserving continuous maps of Nachbin [19]. (For precise definitions and results, see [19], [17] and [16].) This isomorphism allows us to study bitopological spaces by looking at the join of their two topologies, and is key to our work. As a consequence, we have the following (see [1, Note 1.8]) which enables us to pull information back from $\mathcal{T}^s$ to $\mathcal{T}$.

Note 1.4. If $(X, \mathcal{T}, \mathcal{T}^*)$ is a pairwise Hausdorff k-bispace, then the open sets of $\mathcal{T}$ are precisely the ^{s}-open saturated sets and the open sets of $\mathcal{T}^*$ are precisely the ^{s}-open cosaturated sets. Closed sets are characterized similarly.

Note 1.5. *Topological duals* (discussed in [15]) are a way to get from a (usually asymmetric) topology $\mathcal{T}$ a second topology $\mathcal{T}^d$ for which the bitopological space $(X, \mathcal{T}, \mathcal{T}^d)$ has good properties. In this paper, we are interested in two duals: given $(X, \mathcal{T})$, its *de Groot dual* $\mathcal{T}^g$ (also called its *cocompact topology*) is the topology whose closed sets are generated by $\{K : K$ is compact in $\mathcal{T}$ and upper with respect to $\leq_{\mathcal{T}}\}$. Its *k-dual* is the topology $\mathcal{T}^k$ for which C is closed iff $C \cap K$ is closed in $\mathcal{T}^g|_K$, for every set K compact in $\mathcal{T}^{gs} = \mathcal{T} \vee \mathcal{T}^g$. Both $\mathcal{T}^g$

and $\mathcal{T}^k$ have specialization orders opposite to that of $\mathcal{T}$, so that whenever the bispace $(X, \mathcal{T}, \mathcal{T}^g)$ or $(X, \mathcal{T}, \mathcal{T}^k)$ is (pseudo-) Hausdorff, it is pairwise (pseudo-) Hausdorff as well [15, Lemma 2.5 (b)].

When the bitopological space $\mathcal{X}^g = (X, \mathcal{T}, \mathcal{T}^g)$ is pairwise Hausdorff, we may alternatively describe $\mathcal{T}^k$ as $k^*(\mathcal{X}^g)$.

2. The Bitopological Theory

Definition 2.1. Let $(G, \cdot)$ be a group with topologies $\mathcal{T}$ and $\mathcal{T}^*$. $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a *bisemitopological group* if $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^*)$ are both semitopological groups, a *biparatopological group* if $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^*)$ are both paratopological groups, and a *bitopological group* if it is a biparatopological group such that inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$.

When bitopological terminology is applied to a group with two topologies $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$, it refers to its bitopological space, $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$.

One might wonder why our definition of bitopological group did not instead require that $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^*)$ be topological groups. But in semitopological groups, inversion is specialization-reversing, for if $x \leq_{\mathcal{T}} y$ then, since translations are continuous and thus order-preserving, it follows that $y^{-1} = x^{-1}xy^{-1} \leq_{\mathcal{T}} x^{-1}yy^{-1} = x^{-1}$. Since continuous maps are specialization-*preserving*, inversion can be continuous from $(G, \mathcal{T})$ to $(G, \mathcal{T})$ only when $\geq_{\mathcal{T}} \supseteq \leq_{\mathcal{T}}$, in which case $\geq_{\mathcal{T}} = \leq_{\mathcal{T}}$ and the specialization is symmetric. It is thus natural to require that inversion be continuous from $(G, \mathcal{T})$ to $(G, \mathcal{T}^*)$ and from $(G, \mathcal{T}^*)$ to $(G, \mathcal{T})$, as we do; this in turn forces $\leq_{\mathcal{T}^*} = \geq_{\mathcal{T}}$.

The classes of bisemitopological, biparatopological and bitopological groups are all closed with respect to arbitrary products.

A pairwise continuous function is clearly continuous with respect to the symmetrization topologies, and so continuity of group operations with respect to the individual topologies $\mathcal{T}$ and $\mathcal{T}^*$ always implies continuity with respect to their symmetrization $\mathcal{T} \vee \mathcal{T}^*$. Under the "right" conditions, the reverse is true as well, so that we can study group properties of the bispace by looking at group properties of the symmetrization. Lemma 2.2 and Theorem 2.3 make this precise.

Lemma 2.2. *Let $(G, \cdot)$ be a group with topologies $\mathcal{T}$ and $\mathcal{T}^*$, and let $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$.*

(a) *If inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$, then $\mathcal{T}^* = \mathcal{T}^{-1}$ and inversion is an autohomeomorphism of $(G, \mathcal{T}^s)$.*

(b) *If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group then $(G, \cdot, \mathcal{T}^s)$ is a semitopological group.*

(c) *If* $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ *is a biparatopological group then* $(G, \cdot, \mathcal{T}^s)$ *is a paratopological group.*

Proof. (a) Since inversion is always a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^{-1})$, it is clear that $\mathcal{T}^* = \mathcal{T}^{-1}$. Since inversion is its own inverse, it is then pairwise continuous from $(G, \mathcal{T}, \mathcal{T}^*)$ to $(G, \mathcal{T}^*, \mathcal{T})$, and thus from $(G, \mathcal{T} \vee \mathcal{T}^*)$ to $(G, \mathcal{T}^* \vee \mathcal{T})$. That is, inversion is an autohomeomorphism of $(G, \mathcal{T}^s)$.

(b) This is immediate, because translations are pairwise continuous on $(G, \mathcal{T}, \mathcal{T}^*)$ and so also continuous on $(G, \mathcal{T}^s)$.

(c) Since $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^*)$ are paratopological groups, multiplication is pairwise continuous as a function from $\mathcal{X}^2 = (G^2, \mathcal{T}^2, (\mathcal{T}^*)^2)$ to $(G, \mathcal{T}, \mathcal{T}^*)$ and so also from $(G^2, (\mathcal{T}^2)^s)$ to $(G, \mathcal{T}^s)$. But $(\mathcal{T}^2)^s = (\mathcal{T}^s)^2$ (Note 1.2), so multiplication is continuous from $(G^2, (\mathcal{T}^s)^2)$ to $(G, \mathcal{T}^s)$. That is, $(G, \cdot, \mathcal{T}^s)$ is a paratopological group. □

Theorem 2.3. *Let* $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ *be a bisemitopological group for which* $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$ *is a pairwise Hausdorff k-bispace.*

(a) *The following are equivalent.*
  (i) *Inversion is a homeomorphism between* $(G, \mathcal{T})$ *and* $(G, \mathcal{T}^*)$.
  (ii) $\mathcal{T}^* = \mathcal{T}^{-1}$.
  (iii) *Inversion is an autohomeomorphism of* $(G, \mathcal{T}^s)$.

(b) *If* $\mathcal{X}^2$ *is a k-bispace, then* $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ *is a biparatopological group if and only if* $(G, \cdot, \mathcal{T}^s)$ *is a paratopological group.*

(c) *If* $\mathcal{X}^2$ *is a k-bispace, then* $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ *is a bitopological group if and only if* $(G, \cdot, \mathcal{T}^s)$ *is a topological group.*

Proof. (a) That (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) is immediate from Lemma 2.2 (a). Assume inversion is a homeomorphism on $(G, \mathcal{T}^s)$. To show that inversion is continuous as a function from $(G, \mathcal{T})$ to $(G, \mathcal{T}^*)$, we use the representation of open sets as ^{s}-open saturated sets and that of *-open sets as ^{s}-open cosaturated sets in pairwise Hausdorff k-bispaces. (See Note 1.4.) Since inversion is a homeomorphism on $(G, \mathcal{T}^s)$, $(\mathcal{T}^s)^{-1} = \mathcal{T}^s$. As noted in the paragraph following Definition 2.1, inversion is specialization-reversing, so that H is *-open $\Leftrightarrow$ H is cosaturated and ^{s}-open $\Leftrightarrow$ H^{-1} is saturated and ^{s}-open $\Leftrightarrow$ H^{-1} is open in $\mathcal{T}$ $\Leftrightarrow$ H is open in $\mathcal{T}^{-1}$. Thus $\mathcal{T}^* = \mathcal{T}^{-1}$. It follows that inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$.

(b) Sufficiency is an immediate consequence of Lemma 2.2 (c). For necessity, assume $(G, \cdot, \mathcal{T}^s)$ is a paratopological group, so that multiplication, which we call m, is continuous from $(G \times G, \mathcal{T}^s \times \mathcal{T}^s)$ to $(G, \mathcal{T}^s)$. To show that m is also continuous from $(G \times G, \mathcal{T} \times \mathcal{T})$ to $(G, \mathcal{T})$, we again use the equivalence between open sets and ^{s}-open saturated sets. The specialization order of the product topology $\mathcal{T}^2$ is $\leq^2$ (Note 1.2), and it is straightforward to verify that m is order-preserving from $(G^2, \leq^2)$ to $(G, \leq)$, so inverse images of saturated sets are saturated. Now, let $U \in \mathcal{T}$. Then U is ^{s}-open and saturated, so that $m^{-1}(U)$ is open in $(\mathcal{T}^s)^2$, which equals $(\mathcal{T}^2)^s$ (Note 1.2). Further, $m^{-1}(U)$ is saturated with respect to $\leq^2$, and so open in $\mathcal{T}^2$, since $\mathcal{X}^2 = (G^2, \mathcal{T}^2, (\mathcal{T}^*)^2)$ is a k-bispace. That is, m is continuous from $(G \times G, \mathcal{T} \times \mathcal{T})$ to $(G, \mathcal{T})$, and $(G, \cdot, \mathcal{T})$ is a paratopological group. Similarly, $(G, \cdot, \mathcal{T}^*)$ is a paratopological group, and so $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a biparatopological group.

(c) follows immediately from (a) and (b). □

We apply this, in Theorems 2.5, 2.9 and 2.10, to k-bisemitopological groups whose symmetrizations belong to a class of topological spaces with a classical Ellis-type theorem.

Convention 2.4. For the duration of this paper, $\mathcal{C}$ will always represent a class of topological spaces closed under homeomorphism. We call $(G, \cdot, \mathcal{T})$ a *$\mathcal{C}$-group* if $(G, \mathcal{T})$ is in $\mathcal{C}$.

Theorem 2.5. *Let $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace whose symmetrization belongs to $\mathcal{C}$ and let $\cdot$ be a group operation on G.*

(a) *If every $\mathcal{C}$-semitopological group has continuous inversion and $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then $\mathcal{T}^* = \mathcal{T}^{-1}$, so that inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$.*

(b) *If each $\mathcal{C}$-paratopological group is a topological group and $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a biparatopological group, then it is a bitopological group.*

Proof. We are assuming that $(G, \mathcal{T}^s)$ is a $\mathcal{C}$-space.

(a) Since $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, $(G, \cdot, \mathcal{T}^s)$ is a semitopological group, as noted in Lemma 2.2 (b). By assumption, inversion is continuous on $(G, \mathcal{T}^s)$, and so by Theorem 2.3 (a), $\mathcal{T}^* = \mathcal{T}^{-1}$, and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$.

(b) Since $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a biparatopological group, $(G, \cdot, \mathcal{T}^s)$ is a paratopological group by Lemma 2.2 (c). By assumption, $(G, \cdot, \mathcal{T}^s)$ is a topological group, so that inversion is continuous on $(G, \mathcal{T}^s)$. Then inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$ by Theorem 2.3 (a), and so $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bitopological group by definition. □

We would also like to use the topological Ellis-type theorems to conclude that some classes of bisemitopological groups must be biparatopological and, in some cases, bitopological groups. But to use Theorem 2.3 for this purpose requires that the square of the bispace remain a k-bispace. The property "k-bispace" is not preserved by finite products in general. We deal with this in the following three lemmas, which are followed by our general Ellis-type theorem for a bisemitopological group to be biparatopological.

Lemma 2.6. *Let $\mathcal{X} = (X, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff bispace such that $(X, \mathcal{T}^s)$ is a k-space. Then $(X, \mathcal{T}, \mathcal{T}^*)$ is a k-bispace if and only if every ^{s}-open saturated set is open and every ^{s}-open cosaturated set is *-open.*

Proof. Sufficiency is clear from Note 1.4. For necessity, assume the condition. Note that, since $\mathcal{X}$ is pairwise Hausdorff, $\leq_{\mathcal{T}^*} = \geq_{\mathcal{T}}$. To show that $k(\mathcal{X}) = \mathcal{T}$, let $A \in k(\mathcal{X})$. Let K be ^{s}-compact. Since $A \cap K$ is open in $\mathcal{T}|_K$, $A \cap K = B \cap K$, for some $B \in \mathcal{T}$. Then $B \in \mathcal{T}^s$, and so $A \cap K$ is open in $\mathcal{T}^s|_K$. Since K was arbitrary, $A \in k(\mathcal{T}^s)$. But $k(\mathcal{T}^s) = \mathcal{T}^s$, so that $A \in \mathcal{T}^s$. Now, $k(\mathcal{X})$ and $\mathcal{T}$ have the same specialization order [16, Lemma 1.2], so that A is saturated and thus $A \in \mathcal{T}$ by our assumption. It follows that $k(\mathcal{X}) = \mathcal{T}$. In like manner, $k^*(\mathcal{X}) = \mathcal{T}^*$, and therefore $(X, \mathcal{T}, \mathcal{T}^*)$ is a k-bispace. □

Lemma 2.7 below gives us conditions sufficient to guarantee that finite products of k-bispaces are indeed k-bispaces again. (The lemma is essentially a generalization of Theorem 3.6 of [1]; we note that the bispaces in that theorem should have been assumed pairwise Hausdorff, not merely Hausdorff.) Notice that, whenever a pairwise Hausdorff bispace is a k-bispace and both saturations and cosaturations of ^{s}-open sets are ^{s}-open, it follows that every ^{s}-open set has open saturation and *-open cosaturation; when the symmetrization is a k-space, the two conditions are equivalent by Lemma 2.6 above.

Lemma 2.7. *Assume that finite products of $\mathcal{C}$-spaces are k-spaces. Let $\{\mathcal{X}_i = (X_i, \mathcal{T}_i, \mathcal{T}_i^*) : i \in I\}$ be a finite collection of pairwise Hausdorff k-bispaces such that, for each i, the symmetrization $\mathcal{T}_i^s$ is in $\mathcal{C}$ and both saturations and cosaturations of $\mathcal{T}_i^s$-open sets remain open in $\mathcal{T}_i^s$. Then $\prod_I \mathcal{X}_i$ is a k-bispace.*

Proof. We apply Lemma 2.6 to the bispace $\prod_I \mathcal{X}_i = (\prod_I X_i, \prod_I \mathcal{T}_i, \prod_I \mathcal{T}_i^*)$ and make use of properties of products discussed in Note 1.2. $(\prod_I \mathcal{T}_i)^s$ is the same as $\prod_I (\mathcal{T}_i^s)$, which is a k-topology, because finite products of $\mathcal{C}$-spaces are k-spaces. Let W be a set in $(\prod_I \mathcal{T}_i)^s$ which is saturated with respect to the specialization order of $\prod_I \mathcal{T}_i$, and let $x \in W$. We wish to show that $W \in \prod_I \mathcal{T}_i$. Since $(\prod_I \mathcal{T}_i)^s = \prod_I (\mathcal{T}_i^s)$, there are sets U_i open in $\mathcal{T}_i^s$ such that $x \in \prod_I U_i \subseteq W$. By our assumption, $\uparrow U_i$ is in $\mathcal{T}_i^s$ for each i and thus in $\mathcal{T}_i$ as well, because it is a saturated ^{s}-open set in a k-bispace (Note 1.4). Then $x \in \prod_I \uparrow U_i \in \prod_I \mathcal{T}_i$, while $\prod_I \uparrow U_i = \uparrow \prod_i U_i \subseteq W$, since W is saturated. So W is open in $\prod_I \mathcal{T}_i$. In like manner, cosaturated sets open in $(\prod_I \mathcal{T}_i)^s$ are open in $\prod_I \mathcal{T}_i^*$. It follows from Lemma 2.6 that $\prod_I \mathcal{X}_i$ is a k-bispace. □

The first sentence of the next lemma appeared, in a slightly more general setting, as Lemma 4.2 (b) of [1] and makes it clear that Lemma 2.7 applies to the square of any pairwise Hausdorff k-bisemitopological group whose symmetrization belongs to a class of k-spaces closed under finite products.

Lemma 2.8. *If $(G, \mathcal{T}, \mathcal{T}^*)$ is a pairwise Hausdorff bispace with an operation $\cdot$ making $(G, \cdot, \mathcal{T}^s)$ a semitopological group, then both saturations and cosaturations of ^{s}-open sets are ^{s}-open. If, in addition, $(G, \mathcal{T}, \mathcal{T}^*)$ is a k-bispace, then every ^{s}-open set has open saturation and *-open cosaturation.*

Proof. For $U \in \mathcal{T}^s$, $\uparrow U = (\uparrow\{e\})U = \bigcup\{vU : e \leq_{\mathcal{T}} v\}$, which is a union of ^{s}-open sets and thus ^{s}-open. In like manner, so also is $\downarrow U$. The second sentence then follows immediately from Note 1.4. □

Theorem 2.9. *Assume finite products of $\mathcal{C}$-spaces are k-spaces and every $\mathcal{C}$-semitopological group must also be a paratopological group. Let $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace with symmetrization in $\mathcal{C}$. If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a biparatopological group.*

Proof. Since $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, $(G, \cdot, \mathcal{T}^s)$ is a semitopological group by Lemma 2.2 (b), and it follows from Lemma 2.8 that saturations and cosaturations of ^{s}-open sets are ^{s}-open. Since $\mathcal{T}^s$ is in $\mathcal{C}$, $\mathcal{X}^2$ is a k-bispace by Lemma 2.7. By our assumption, $(G, \cdot, \mathcal{T}^s)$ is a paratopological group, and then $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a biparatopological group by Theorem 2.3 (b). □

Theorem 2.10. *Assume finite products of $\mathcal{C}$-spaces are k-spaces and every $\mathcal{C}$-semitopological group must also be a topological group. Let $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace with symmetrization in $\mathcal{C}$. If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a bitopological group.*

Proof. Since every topological group is a paratopological group and has continuous inversion, the conclusions follow immediately from Theorems 2.9 and 2.5 (a). □

3. Asymmetric Versions

Our theory applies to non-Hausdorff topological spaces $(X, \mathcal{T})$ by considering the bitopological space $(X, \mathcal{T}, \mathcal{T}^k)$, where $\mathcal{T}^k$ is the k-dual of Note 1.5. The resulting theorems are topological in nature and use no bitopological concepts in their statements, except for the requirement that $(G, \mathcal{T}, \mathcal{T}^k)$ be a Hausdorff k-bispace. (Recall from Note 1.5 that since $\mathcal{T}$ and $\mathcal{T}^k$ have opposite specialization orders, whenever the bispace $(X, \mathcal{T}, \mathcal{T}^k)$ is Hausdorff, it is pairwise Hausdorff as well.)

In [1], we observed that, whenever $f : (X, \mathcal{T}) \to (Y, \mathcal{U})$ is a homeomorphism, f is also a homeomorphism from $(X, \mathcal{T}^g)$ to $(Y, \mathcal{U}^g)$, $(X, \mathcal{T} \vee \mathcal{T}^g)$ to $(Y, \mathcal{U} \vee \mathcal{U}^g)$, $(X, \mathcal{T}^k)$ to $(Y, \mathcal{U}^k)$, and $(X, \mathcal{T} \vee \mathcal{T}^k)$ to $(Y, \mathcal{U} \vee \mathcal{U}^k)$. In particular, if $(G, \cdot)$ is a group and $\mathcal{T}$ is a topology on G such that all translations are continuous with respect to $\mathcal{T}$, then all translations are autohomeomorphisms with respect to $\mathcal{T}$, and thus with respect to $\mathcal{T}^g$, $\mathcal{T} \vee \mathcal{T}^g$, $\mathcal{T}^k$, and $\mathcal{T} \vee \mathcal{T}^k$ as well. So whenever $(G, \cdot, \mathcal{T})$ is a semitopological group, so also is $(G, \cdot, \mathcal{T}^k)$ [1, Theorem 4.4]. We use this to prove the next theorem, which gives the asymmetric versions of Theorems 2.5 (a), 2.9 and 2.10.

Theorem 3.1. *Let $(G, \cdot, \mathcal{T})$ be a semitopological group such that $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace with symmetrization in $\mathcal{C}$. Then $(G, \cdot, \mathcal{T}^k)$ is also a semitopological group, and*

- (a) *if every $\mathcal{C}$-semitopological group has continuous inversion, then inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$,*
- (b) *if finite products of $\mathcal{C}$-spaces are k-spaces and each $\mathcal{C}$-semitopological group must be a paratopological group, then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are both paratopological groups, and*
- (c) *if finite products of $\mathcal{C}$-spaces are k-spaces and each $\mathcal{C}$-semitopological group must be a topological group, then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are paratopological groups and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.*

Proof. Since $\mathcal{T}$ and $\mathcal{T}^k$ have opposite specialization orders, $(G, \mathcal{T}, \mathcal{T}^k)$ is a pairwise Hausdorff k-bispace [15, Lemma 2.5 (b)] with $(G, \mathcal{T} \vee \mathcal{T}^k) \in \mathcal{C}$. Since translations are autohomeomorphisms of $(G, \mathcal{T})$, they are of $(G, \mathcal{T}^k)$ as well, and so $(G, \cdot, \mathcal{T}^k)$ is a semitopological group. The conclusions then follow from Theorems 2.5 (a), 2.9 and 2.10, respectively. □

These asymmetric versions cover many cases in which a non-Hausdorff semitopological group (like $(\mathbb{R}, +, \mathcal{U})$, where $\mathcal{U} = \{(a, \infty) : a \in \mathbb{R}\} \cup \{\emptyset, \mathbb{R}\}$) has a second topology such that the resulting group with bitopological space is a bitopological group. In practice, asymmetric applications of the theory result from corresponding bitopological results.

4. Applications

In this section, we apply the results of the previous two sections to a sequence of classes of bisemitopological groups, as well as to an associated sequence of classes of asymmetric semitopological groups. These classes include those pairwise Hausdorff k-bisemitopological groups whose symmetrizations are (1) Čech-complete, (2) completely metrizable, (3) locally compact Hausdorff, (4) Baire p-, (5) Baire and p-σ-fragmentable by a complete sequence of covers, or (6) Čech-analytic Baire. In each case there is a corresponding theorem for those asymmetric spaces whose k-symmetrization has the appropriate property. The results for locally compact Hausdorff spaces were first developed in Theorems 4.3 and 4.6 of [1], but they are also immediate corollaries of our theorems for Čech-complete spaces.

Examples 4.8 and 4.9 demonstrate that our major results, Theorems 2.10 and 3.1, as well as their applications to the Čech-complete setting in Theorems 4.2 and 4.3, are genuine extensions of those in [1]. Example 4.8 is of a bispace whose symmetrization is Čech-complete, but neither (completely) metrizable nor locally compact, and to which our theory applies; Example 4.9 is of an asymmetric space whose k-symmetrization behaves in a similar way.

Definition 4.1. A Tychonoff space X is *Čech-complete* if it is a G_δ in βX. Čech-completeness also has an "internal" characterization; a Tychonoff space is Čech-complete if and only if it has a sequence $(\mathcal{U}_n)_{n\in\omega}$ of open covers with the following property: if $\mathcal{F}$ is a family of closed subsets of X with the finite intersection property and for each $n \in \omega$ there is an $F_n \in \mathcal{F}$ and a $U_n \in \mathcal{U}_n$ with $F_n \subseteq U_n$, then $\bigcap \mathcal{F} \neq \emptyset$.

Čech-complete spaces are k-spaces [2, Theorem 1] and the class of Čech-complete spaces is closed under the formation of countable products [9, Theorem 3.9.8]. Bouziad proved in 1996 [3, Corollary 3.4] that each Čech-complete semitopological group is a topological group. The next result then follows immediately from Theorem 2.10.

Theorem 4.2. *Let $(G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace whose symmetrization is Čech-complete. If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a bitopological group.*

We apply this to non-Hausdorff topological spaces in the Theorem below. Its proof using Theorem 4.2 is shown, but it could also be proved using Theorem 3.1 (c).

Theorem 4.3. *If $(G, \cdot, \mathcal{T})$ is a semitopological group and $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace whose symmetrization is Čech-complete, then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are both paratopological groups and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.*

Proof. Since $\mathcal{T}$ and $\mathcal{T}^k$ have opposite specialization orders, $(G, \mathcal{T}, \mathcal{T}^k)$ is a pairwise Hausdorff k-bispace. Since translations are homeomorphisms in $(G, \mathcal{T})$, they are homeomorphisms in $(G, \mathcal{T}^k)$ as well, and so $(G, \cdot, \mathcal{T}, \mathcal{T}^k)$ is a bisemitopological group. The conclusions are then an immediate consequence of Theorem 4.2. □

Since locally compact Hausdorff spaces are Čech-complete, we have the bitopological and asymmetric Ellis theorems of [1] as immediate corollaries, and since completely metrizable spaces are Čech-complete, we also have bitopological and asymmetric versions of Montgomery's theorem, as follows. (The asymmetric Ellis theorem [1, Theorem 4.6] was originally stated for locally skew compact topological spaces, which are those for which $(X, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace and $\mathcal{T} \vee \mathcal{T}^k$ is locally compact.)

Corollary 4.4. *Let $(G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace whose symmetrization is completely metrizable. If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a bitopological group.*

Corollary 4.5. *Let $(G, \cdot, \mathcal{T})$ be a semitopological group such that $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace whose symmetrization is completely metrizable. Then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are both paratopological groups and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.*

Corollary 4.6 ([1, Theorem 4.3])**.** *Let $(G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace with locally compact symmetrization. If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a bitopological group.*

Corollary 4.7 ([1, Theorem 4.6])**.** *If $(G, \cdot, \mathcal{T})$ is a semitopological group and $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace with locally compact symmetrization, then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are paratopological groups, and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.*

We now give an example of a pairwise Hausdorff k-bisemitopological group whose symmetrization is Čech-complete but neither locally compact nor metrizable. This shows that our bitopological Theorem 2.10, as well as Theorem 4.2, properly extends the bitopological Ellis theorem, Theorem 4.3, of [1].

Example 4.8. Let $\mathcal{U}$ and $\mathcal{L}$ denote the usual upper and lower topologies on $\mathbb{R}$, generated by rays of the form (a, ∞) and $(-\infty, b)$, respectively, let $\mathcal{E} = \mathcal{U} \vee \mathcal{L}$ (the usual Euclidean topology), and let $(G, \mathcal{T})$ be any non-metrizable locally compact Hausdorff topological group. Set $\mathcal{X}_1 = (\mathbb{R}, \mathcal{U}, \mathcal{L})^{\aleph_0}$ and $\mathcal{X}_2 = (G, \mathcal{T}, \mathcal{T})$, and then define $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$. By our observations in Note 1.2, $\mathcal{X}_1$, $\mathcal{X}_2$ and $\mathcal{X}$ are pairwise Hausdorff, and since each factor is a bisemitopological group, so is the product $\mathcal{X}$. The symmetrization of $\mathcal{X}_1$ is the usual product $\mathcal{E}^{\aleph_0}$ which is (completely) metrizable but not locally compact. Since the symmetrization of $(G, \mathcal{T}, \mathcal{T})$ is just $(G, \mathcal{T})$, which is locally compact but not metrizable, it follows that the symmetrization of $\mathcal{X}$ is neither locally compact nor metrizable. Both the symmetrized factors are, however, Čech-complete, and so the symmetrization of $\mathcal{X}$ is also Čech-complete, and then a k-space as well.

To confirm that Theorem 4.2 applies to $\mathcal{X}$, it suffices to verify that $\mathcal{X}$ is a k-bispace. For this, we apply Lemma 2.7 to $\mathcal{X}_1$ and $\mathcal{X}_2$, and it remains only to check that saturations and cosaturations of sets open in the symmetrizations are open in the first and second topologies, respectively, for both $\mathcal{X}_1$ and $\mathcal{X}_2$. The case of $\mathcal{X}_2$ is immediate, since $\mathcal{X}_2$ is symmetric. For $\mathcal{X}_1$, suppose that S is open in $\mathcal{E}^{\aleph_0}$, the symmetrization of $\mathcal{X}_1$, and consider $\uparrow S$, the saturation of S with respect to $\mathcal{U}^{\aleph_0}$. Let $y \in \uparrow S$. Then there is an $x \in S$ with $x \leq y$ and a family $\{U_i\}$ of basic open sets in $\mathcal{E}$ such that $x \in \prod_I U_i \subseteq S$, where finitely many of the U_i are open intervals and the remainder are $\mathbb{R}$ itself. Then $y \in \uparrow \prod_I U_i \subseteq \uparrow S$, where $\uparrow \prod_I U_i = \prod_I \uparrow U_i$, which is open in $\mathcal{U}^{\aleph_0}$, and so $\uparrow S$ is open in $\mathcal{U}^{\aleph_0}$. A similar argument applies to cosaturations of ^{s}-open sets. Hence, by Lemma 2.7, $\mathcal{X} = \mathcal{X}_1 \times \mathcal{X}_2$ is a k-bispace.

Example 4.9. We now adapt the example above to show that our asymmetric theorems, 3.1 and 4.2, are likewise proper extensions of the asymmetric Ellis theorem, Theorem 4.6, of [1]. In fact, let

$$\mathcal{X} = (\mathbb{R}, \mathcal{U}, \mathcal{L})^{\aleph_0} \times (G, \mathcal{T}, \mathcal{T}) = (\mathbb{R}^{\aleph_0} \times G,\ \mathcal{U}^{\aleph_0} \times \mathcal{T},\ \mathcal{L}^{\aleph_0} \times \mathcal{T})$$

be as in Example 4.8, where now the topological group $(G, \mathcal{T})$ is compact (as well as Hausdorff and non-metrizable). Then, writing $H = \mathbb{R}^{\aleph_0} \times G$ and $\mathcal{V} = \mathcal{U}^{\aleph_0} \times \mathcal{T}$, we claim that $\mathcal{V}^k = \mathcal{L}^{\aleph_0} \times \mathcal{T}$, which is all that is required to show that Theorem 3.1 applies to the semitopological group $(H, \mathcal{V})$.

Theorem 2.7 of [1] says that, for a collection of topological spaces $(X_i, \mathcal{T}_i)$ such that each $(X_i, \mathcal{T}_i, \mathcal{T}_i^g)$ is pseudo-Hausdorff, the de Groot dual of the product is the same as the product of the de Groot duals. Therefore, since $\mathcal{U}^g = \mathcal{L}$ (see Example 1.2 of [1]) and $(\mathbb{R}, \mathcal{U}, \mathcal{L})$ is pseudo-Hausdorff, it follows that $(\mathcal{U}^{\aleph_0})^g = \mathcal{L}^{\aleph_0}$. Also, since $(G, \mathcal{T})$ is compact Hausdorff, we have $\mathcal{T}^g = \mathcal{T}$, and again by Theorem 2.7

of [1], we have $\mathcal{V}^g = \mathcal{L}^{\aleph_0} \times \mathcal{T}$. Therefore, $\mathcal{X} = (H, \mathcal{V}, \mathcal{V}^g)$. Now, since $\mathcal{X}$ is a k-bispace, the observation at the end of Note 1.5 implies that $\mathcal{V}^k = k^*(\mathcal{X}) = \mathcal{V}^g = \mathcal{L}^{\aleph_0} \times \mathcal{T}$, as claimed.

Application of Theorem 3.1 to $(H, \mathcal{V})$ is therefore possible. On the other hand, the symmetrization of $\mathcal{X}$ is not locally compact, and so the asymmetric Ellis theorem of [1] cannot be applied to $(H, \mathcal{V})$.

Our technique also applies when the symmetrization is a Baire p-space, a property which generalizes that of Čech-complete. In this context, however, we cannot conclude that a bisemitopological group must be bitopological, but only that it must be biparatopological. Bouziad [4, Corollary 5] proved that every semitopological group which is a Baire p-space is a paratopological group. We extend this to bitopological spaces and non-Hausdorff topological spaces in the two theorems that follow.

Definition 4.10. A Tychonoff space X is a *p-space* if in βX there is a sequence $\{\mathcal{U}_n : n \in \omega\}$ of open covers of X such that $\bigcap_{n\in\omega} St_{\mathcal{U}_n}(x) \subseteq X$, for every $x \in X$ [2]. (Here $St_{\mathcal{U}_n}(x)$ denotes the union of all sets in $\mathcal{U}_n$ that contain x.)

Theorem 4.11. *Let $\mathcal{X} = (G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace whose symmetrization is a Baire p-space. If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a biparatopological group.*

Proof. Baire p-spaces are p-spaces, which, in turn, are k-spaces, and the class of p-spaces is closed under the formation of countable products [2]. By Corollary 5 of [4], every semitopological group which is a Baire p-space is a paratopological group. The conclusion then follows immediately from Theorem 2.9. □

Theorem 4.12. *If $(G, \cdot, \mathcal{T})$ is a semitopological group and $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace whose symmetrization is a Baire p-space, then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are both paratopological groups.*

Although Bouziad proved that every Baire p-semitopological group must be paratopological, we do not know (although others may) whether it must also be topological. Thus, we also do not know whether every bisemitopological group satisfying the conditions of Theorem 4.11 must also be a bitopological group or whether, under the conditions of Theorem 4.12, inversion must be a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.

Čech-complete spaces are Čech-analytic Baire spaces [10], which in turn are Baire spaces p-σ-fragmentable by a complete sequence of covers [3]. (For definitions, see [3], [11] and [12].) Bouziad actually proved that semitopological groups in this last class must be topological groups, and obtained the corresponding theorems for Čech-analytic Baire spaces and Čech-complete spaces as corollaries. It is natural to ask whether we could formulate our applications for these broader classes. But a Čech-analytic Baire space need not be a k-space, as we will see in Example 4.15, so that Theorems 2.9 and 2.10 do not apply without further assumptions. In part (a) of the following theorem, we cannot, with our current methods, conclude that $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a biparatopological group, but only that inversion must be a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$. The proof of part (a) is immediate from Theorem 3.2 of [3] and our Theorem 2.5 (a); then part (b) is an immediate corollary, and part (c) follows from Theorem 4.11.

Theorem 4.13. *Let $(G, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff k-bispace whose symmetrization is a Baire space p-σ-fragmentable by a complete sequence of covers.*

(a) *If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^*)$.*
(b) *If $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a biparatopological group, then it is a bitopological group.*
(c) *If the symmetrization is a p-space and $(G, \cdot, \mathcal{T}, \mathcal{T}^*)$ is a bisemitopological group, then it is a bitopological group.*

Versions of parts (a) and (c) for non-Hausdorff topological spaces then follow in the same manner that Theorem 4.3 followed from Theorem 4.2. We do not, however, easily obtain an asymmetric version for paratopological groups corresponding to part (b) of the previous theorem. (See Questions 5.3 and 5.4.)

Theorem 4.14. *If $(G, \cdot, \mathcal{T})$ is a semitopological group and $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace whose symmetrization is a Baire space p-σ-fragmentable by a complete sequence of covers, then inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$. If, in addition, the symmetrization is a p-space, then $(G, \cdot, \mathcal{T})$ and $(G, \cdot, \mathcal{T}^k)$ are both paratopological groups.*

As obvious corollaries, both Theorems 4.13 and 4.14 hold if the symmetrization is assumed to be Čech-analytic Baire instead of Baire and p-σ-fragmentable by a complete sequence of covers.

Notice that our asymmetric results in Theorems 4.3 and 4.12 generalize the classical theorems for Čech-complete spaces and Baire p-spaces, because both are k-spaces and by [1, Corollary 2.9], $\mathcal{T} = \mathcal{T}^k$ whenever $(X, \mathcal{T})$ is a Hausdorff k-space. We cannot easily, however, make the same claim for Theorem 4.14 without first showing that the spaces under discussion are k-spaces. We conclude with an example to demonstrate that a Čech-analytic Baire space need not be a

k-space, and so, at least with our current methods, we cannot prove Theorem 4.2 with Čech-analytic Baire in place of Čech-complete.

Example 4.15. Let $X = \mathbb{N} \cup \{x\}$ where $x \in \beta\mathbb{N} \setminus \mathbb{N}$. As indicated in [23], pp.102–105, X is not a k-space. Indeed since every compact subset is finite, its k-extension is discrete. The space is analytic (and hence Čech analytic) since it is the continuous image of the countable discrete space. It is Baire since the only nowhere dense set is $\{x\}$. Another familiar example with similar properties is the so-called Arens space; see e.g. Example E, chapter 2, p.77 of Kelley [13].

5. Questions

Question 5.1. Is the symmetrization of a pairwise Hausdorff k-bispace always a k-space? If so, the theory would be considerably simplified; for example, Lemma 2.6 should then read:

> Let $\mathcal{X} = (X, \mathcal{T}, \mathcal{T}^*)$ be a pairwise Hausdorff bispace. Then $(X, \mathcal{T}, \mathcal{T}^*)$ is a k-bispace if and only if $(X, \mathcal{T}^s)$ is a k-space, every ^{s}-open saturated set is $\mathcal{T}$-open and every ^{s}-open cosaturated set is *-open.

If not, when $(G, \cdot, \mathcal{T})$ is a semitopological group and $(G, \mathcal{T}, \mathcal{T}^*)$ is a pairwise Hausdorff k-bispace must $(G, \mathcal{T}^s)$ be a k-space?

Question 5.2. Does the conclusion of Lemma 2.7 hold if we remove the condition that both saturations and cosaturations of $\mathcal{T}_i^s$-open sets remain open in $\mathcal{T}_i^s$? That is, will it hold if we only assume that finite products of $\mathcal{C}$-spaces are k-spaces and that the $\mathcal{X}_i$ are pairwise Hausdorff k-bispaces with symmetrizations in $\mathcal{C}$?

Question 5.3. Whenever $(G, \cdot, \mathcal{T})$ is a semitopological group, then $(G, \cdot, \mathcal{T}^k)$ is also a semitopological group. When $(G, \cdot, \mathcal{T})$ is a paratopological group, it is a semitopological group, and so $(G, \cdot, \mathcal{T}^k)$ is a semitopological group. But must $(G, \cdot, \mathcal{T}^k)$ also be a paratopological group? If not, what is a useful set of conditions that would make it a paratopological group?

Question 5.4. At the moment, we have no asymmetric analogue for Theorem 2.5 (b). If Question 5.3 were to have an affirmative answer, then we would have the following analogue as a theorem.

> Assume that each $\mathcal{C}$-paratopological group is a topological group. If $(G, \cdot, \mathcal{T})$ is a paratopological group and $(G, \mathcal{T}, \mathcal{T}^k)$ is a Hausdorff k-bispace with symmetrization in $\mathcal{C}$, then $(G, \cdot, \mathcal{T}^k)$ is a paratopological group and inversion is a homeomorphism between $(G, \mathcal{T})$ and $(G, \mathcal{T}^k)$.

Does this hold even in the absence of a solution for Question 5.3? If not, would it hold with reasonable additional conditions?

Question 5.5. Example 4.15 shows that a Čech-analytic Baire topological space need not be a k-space. But must every Čech-analytic Baire semitopological group be a k-space?

Question 5.6. What are other classes of k-spaces with a topological Ellis-type theorem to which our theory applies?

Question 5.7. To what extent do similar methods apply to theorems about group actions and to theorems about separate versus joint continuity in general?

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(Andima) Long Island University - C.W. Post Campus, Department of Mathematics, 720 Northern Blvd., Brookville, NY 11548, USA
E-mail address: `SAndima@liu.edu`

(Kopperman) City College of CUNY, Department of Mathematics, 160 Convent Ave., New York, NY 10031, USA
E-mail address: `rdkcc@ccny.cuny.edu`

(Nickolas) University of Wollongong, School of Mathematics and Applied Statistics, NSW 2522, Australia
E-mail address: `peter@uow.edu.au`

(Popvassilev) City College of CUNY, Department of Mathematics, 160 Convent Ave., New York, NY 10031, USA
E-mail address: `strash.pop@gmail.com`