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### Abstract

Searching for a closed-form exact solution for American put options under the Black-Scholes framework has been a long standing problem in the past; many researchers believe that it is impossible to find such a solution. In this paper, a closed-form exact solution, in the form of a Taylor's series expansion, of the well-known Black-Scholes equation is presented for the first time. As a result of this analytic solution, the optimal exercise boundary, which is the main difficulty of the problem, is found as an explicit function of the risk-free interest rate, the volatility and the time to expiration.

### Keywords

form, solution, value, closed, exercise, optimal, its, american, boundary, put, exact

### Disciplines

Physical Sciences and Mathematics

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# A Closed-form Exact Solution for the Value of American Put and its Optimal Exercise Boundary

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## ABSTRACT

Searching for a closed-form exact solution for American put options under the Black-Scholes framework has been a long standing problem in the past; many researchers believe that it is impossible to find such a solution. In this paper, a closed-form exact solution, in the form of a Taylor's series expansion, of the well-known Black-Scholes<sup>1</sup> equation is presented for the first time. As a result of this analytic solution, the optimal exercise boundary, which is the main difficulty of the problem, is found as an explicit function of the risk-free interest rate, the volatility and the time to expiration.

**Keywords:** American Options, Closed-form Analytical Formulae, Homotopy-analysis Method

## 1. INTRODUCTION

Most traded options in today's financial markets are American Options. However, for a long time, it has been widely acknowledged that the valuation of American options is a much more intriguing problem (see Ref. 2–4), and “an analytical formula does not exist for the value of an American option where an early exercise may be optimal” (see Ref. 5). In this paper, an explicit closed-form exact solution for the value of American put and its optimal exercise boundary is presented for the first time.

For American options, the essential difficulty lies in the problem that they are allowed to be exercised at any time before the expiry and, mathematically, such an early exercise right, purchased by the holder of the option, changes the problem into a so-called free boundary problem, since the optimal exercise boundary (or early exercise boundary) prior to the expiration of the option is now time-dependent and is part of the solution. As a result of the unknown boundary being part of the solution, the valuation of American options becomes a highly nonlinear problem like any other free boundary problems. This is very different from the valuation of European options as the latter is only a linear problem if the well-known Black-Scholes equation is solved. Therefore, it is really this nonlinear feature that has hindered the search for an analytical solution for American options.

The valuation problem of American put options can be traced back to McKean<sup>6</sup> and Merton<sup>7</sup> who first suggested that the valuation of American options should be treated as a free boundary problem. This has drawn considerable research interests in this area. In the literature, there have been two types of approximate approaches, numerical solutions and analytical approximations, for the valuation of American options. Each type has its own advantages and limitations.

Of all numerical approaches, there are two subcategories. In the first subcategory, the Black-Scholes equation is directly solved with both time and stock price being discretized. Typical approaches are the finite difference method<sup>5</sup> and the finite element method.<sup>8</sup> Approaches in the second subcategory are based on the risk-neutral valuation at each time step. The binomial method<sup>9</sup> and the Monte Carlo simulation method<sup>10</sup> are the typical examples in this subcategory. Many of these methods still require intensive computation before a solution of reasonable accuracy can be obtained and in some cases, such as the explicit finite-difference scheme, the method may not even converge, as pointed out by Huang *et al.*<sup>2</sup> For American options, it can be shown that the solution

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near maturity is of singular behavior.<sup>11</sup> Naturally, it is difficult for most numerical methods to calculate the option price accurately in the neighborhood of maturity.

In comparison with numerical methods, analytical approximations are usually of simple form and can be easily computed. Typical methods in this category include the compound-option approximation method,<sup>12</sup> the quadratic approximation method,<sup>13</sup> the capped option approximation,<sup>14</sup> the integral-equation method<sup>2,3,15</sup> and the simple approximation through a nonlinear algebraic equation derived by Bunch and Johnson.<sup>16</sup> Recently, based on the pseudo-steady-state approximation, Zhu<sup>17</sup> proposed a new analytical approximation formula for the optimal exercise price. His formula seems to be simple and versatile in the parameter space. It must be pointed out that all these analytical approximation methods still require a certain degree of computation at the end. However, unlike the numerical approaches mentioned earlier, various approximations are made in order to reduce the intensity of the final numerical computation, a feature that distinguishes these methods from purely numerical approaches.

When the optimal exercise boundary is expressed in integral equation forms, some authors referred to them as “exact solutions”. They are indeed exact, in the sense that the differential equation as well as all boundary and initial conditions have been exactly satisfied. But, they are not in an explicit **solution** form yet as integral equations are still need to be solved. Sometimes, finding the solution of an integral equation, especially when it is nonlinear, is not any easier than finding the solution of the original differential equation.

Our closed-form solution is constructed, using the the homotopy-analysis method, which was initially suggested by Ortega and Rheinboldt<sup>18</sup> and has been successfully used to solve a number of heat transfer problems<sup>19,20</sup> and fluid-flow problems.<sup>21</sup> A Taylor’s series expansion of the unknown option price and the unknown optimal exercise boundary in terms of a parameter,  $p$ , is obtained as a result of the homotopy deformation constructed when the parameter  $p$  is varied continuously in the domain  $[0, 1]$ . The new series solution is a closed-form exact solution because all the differential equations and boundary conditions can be satisfied exactly and it is an explicit analytical solution because not only the option price, but also the optimal exercise boundary are determined explicitly as a function of all the input variables such as risk-free interest rate, the volatility and the time to expiration.

This paper is organized into four sections. In Section 2, a detailed description of the newly-found series solution of the Black-Scholes equation is provided. In Section 3, some examples are given, and our conclusions are stated in Section 4.

## 2. CLOSED-FORM EXACT SOLUTION

Since one can easily show that without dividends, American call options would be equivalent to their European counterparts, i.e, it is always optimal to hold an American call to expiration, we shall concentrate on solving the Black-Scholes equation for an American put option.

Let  $V(S, t)$  denote the value of an American put option, with  $S$  being the price of the underlying asset and  $t$  being the current time. Under the Black and Scholes framework,<sup>1</sup> the dimensionless differential system that governs the price of an American put option can be written as

$$\left\{ \begin{array}{l} -\frac{\partial V}{\partial \tau} + S^2 \frac{\partial^2 V}{\partial S^2} + \gamma S \frac{\partial V}{\partial S} - \gamma V = 0, \\ V(S_f(\tau), \tau) = 1 - S_f(\tau), \\ \frac{\partial V}{\partial S}(S_f(\tau), \tau) = -1, \\ \lim_{S \rightarrow \infty} V(S, \tau) = 0, \\ V(S, 0) = \max\{1 - S, 0\}, \end{array} \right. \quad (1)$$

in which  $S_f(\tau)$  is the optimal exercise boundary,  $\tau = (T - t) \cdot \frac{\sigma^2}{2}$  is the dimensionless time to expiry (i.e., the difference between the expiration time  $T$  and the current time  $t$ ),  $\gamma \equiv \frac{2r}{\sigma^2}$  can be viewed as an interest rate

relative to the volatility of the underlying asset price with  $r$  being the risk-free interest rate and  $\sigma$  being the volatility of the underlying asset price.

The initial condition can be further simplified. Since the optimal exercise price is equal to the strike price at the expiration time  $T$  as shown in Ref. 2, i.e.,  $S_f(0) = 1$  in the dimensionless system,  $1 - S < 0$  when  $S$  is in the range of  $S_f \leq S < \infty$ . Consequently, the initial condition in (1) can be simplified as  $V(S, 0) = 0$ .

If we introduce the Landau transform

$$x = \ln \frac{S}{S_f(\tau)}, \quad (2)$$

the differential system (1) can be written as

$$\left\{ \begin{array}{l} \frac{\partial V}{\partial \tau} - \frac{\partial^2 V}{\partial x^2} - (\gamma - 1) \frac{\partial V}{\partial x} + \gamma V = \frac{1}{S_f(\tau)} \frac{dS_f}{d\tau} \frac{\partial V}{\partial x}, \\ V(x, 0) = 0, \\ V(0, \tau) = 1 - S_f(\tau), \\ \frac{\partial V}{\partial x}(0, \tau) = -S_f(\tau), \\ \lim_{x \rightarrow \infty} V(x, \tau) = 0, \end{array} \right. \quad (3)$$

in which the nonlinearity concentrates explicitly in the nonhomogeneous term of the governing differential equation, whereas the boundary conditions defined on a moving boundary have now disappeared, an advantage we can use when the homotopy-analysis method is applied to solve a nonlinear system with fixed boundary conditions.

The homotopy-analysis method originated from the homotopic deformation in topology (e.g., see Ref. 22), and was initially suggested by Ortega and Rheinboldt.<sup>18</sup> The essential concept of the method is to construct a continuous “homotopic deformation” through a series expansion of the unknown function. The series solution of the unknown function is of infinitely many terms, but is nevertheless of a closed form according to the definition given by Gukhal.<sup>23</sup> To calculate the actual values of the unknown function at a point in space (the underlying asset price here) and time, one needs to truncate the infinite series to a finite one, just like in the calculation performed for many other standard mathematical functions. When each term (sometimes it is referred to as the “order”) in the series expansion can be calculated analytically rather than numerically (e.g., see Ref. 19, 20), a truly closed-form analytical solution is obtained. Theoretically speaking, one should be able to achieve machine accuracy if a sufficient number of terms is included in the summation process, for then all the numerical errors will result only from “truncation errors” when real numbers are stored in a computer with a finite number of digits.

Let’s now construct two new unknown functions  $\bar{V}(x, \tau, p)$  and  $\bar{S}_f(\tau, p)$  that satisfy the following differential system,

$$\left\{ \begin{array}{l} (1 - p)\mathcal{L}[\bar{V}(x, \tau, p) - \bar{V}_0(x, \tau)] = -p\{\mathcal{A}[\bar{V}(x, \tau, p), \bar{S}_f(\tau, p)]\}, \\ \bar{V}(x, 0, p) = (1 - p)\bar{V}_0(x, 0), \\ \bar{V}(0, \tau, p) + \bar{S}_f(\tau, p) = 1, \\ \frac{\partial \bar{V}}{\partial x}(0, \tau, p) + \bar{S}_f(\tau, p) = (1 - p) \left[ 1 + \frac{\partial \bar{V}_0}{\partial x}(0, \tau) - \bar{V}_0(0, \tau) \right], \\ \lim_{x \rightarrow \infty} \bar{V}(x, \tau, p) = 0, \end{array} \right. \quad (4)$$

where  $\mathcal{L}$  is a differential operator defined as

$$\mathcal{L} = \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial x^2} - (\gamma - 1) \frac{\partial}{\partial x} + \gamma,$$

and  $\mathcal{A}$  is a functional defined as

$$\mathcal{A}[\bar{V}(x, \tau, p), \bar{S}_f(\tau, p)] = \mathcal{L}(\bar{V}) - \frac{1}{\bar{S}_f(\tau, p)} \frac{\partial \bar{S}_f}{\partial \tau}(\tau, p) \frac{\partial \bar{V}}{\partial x}(x, \tau, p).$$

It can be easily verified that when  $p = 0$ , the solution of Eq. (4) is

$$\begin{cases} \bar{V}(x, \tau, 0) = \bar{V}_0(x, \tau), \\ \bar{S}_f(\tau, 0) = 1 - \bar{V}_0(0, \tau) = \bar{S}_0(\tau), \end{cases} \quad (5)$$

so long as the initial guess  $\bar{V}_0(x, \tau)$  satisfies the condition

$$\lim_{x \rightarrow \infty} \bar{V}_0(x, \tau) = 0.$$

On the other hand, when  $p = 1$ , the solution of Eq. (4) is exactly the solution we are seeking, i.e.,

$$\begin{cases} V(x, \tau) = \bar{V}(x, \tau, 1), \\ S_f(\tau) = \bar{S}_f(\tau, 1). \end{cases} \quad (6)$$

Therefore, Eq. (4) constitutes a continuous map (a homotopy, see Ref. 22) such that the solution we are seeking becomes the result of a continuous deformation from an initial and known function. This initial guess  $\bar{V}_0(x, \tau)$  in theory can be any continuous  $C^1$  function. However, if we choose a function that has already satisfied an additional condition  $\mathcal{L}\bar{V}_0(x, \tau) = 0$ , we should expect a faster convergence of the series.

To find the values of  $\bar{V}(x, \tau, 1)$  and  $\bar{S}_f(\tau, 1)$ , we can now expand the functions  $\bar{V}(x, \tau, p)$  and  $\bar{S}_f(\tau, p)$  as a Taylor's series expansion of  $p$

$$\bar{V}(x, \tau, p) = \sum_{m=0}^{\infty} \frac{\bar{V}_m(x, \tau)}{m!} p^m, \quad (7)$$

$$\bar{S}_f(\tau, p) = \sum_{m=0}^{\infty} \frac{\bar{S}_m(\tau)}{m!} p^m, \quad (8)$$

where  $\bar{V}_m$  and  $\bar{S}_m$  are the  $m$ th-order partial derivative of  $\bar{V}(x, \tau, p)$  and  $\bar{S}_f(\tau, p)$ , respectively, with respect to  $p$  and then evaluated at  $p = 0$ ,

$$\bar{V}_m(x, \tau) = \left. \frac{\partial^m}{\partial p^m} \bar{V}(x, \tau, p) \right|_{p=0}, \quad \bar{S}_m(\tau) = \left. \frac{\partial^m}{\partial p^m} \bar{S}_f(\tau, p) \right|_{p=0}.$$

To find all the coefficients in the above Taylor's expansions, we need to derive a set of governing partial differential equations and appropriate boundary and initial conditions for the unknown functions  $\bar{V}_m(x, \tau)$  and  $\bar{S}_m(\tau)$ . They can be derived from differentiating each equation in system (4) with respect to  $p$  and then setting  $p$  equal to zero. After this process, we obtain

$$\begin{cases} \mathcal{L}[\bar{V}_1(x, \tau)] = -\mathcal{L}[\bar{V}_0(x, \tau)] + \mathcal{A}'(x, \tau, 0), \\ \bar{V}_1(x, 0) = -\bar{V}_0(x, 0), \\ \bar{V}_1(0, \tau) + \bar{S}_1(\tau) = 0, \\ \frac{\partial \bar{V}_1}{\partial x}(0, \tau) + \bar{S}_1(\tau) = \bar{V}_0(0, \tau) - \frac{\partial \bar{V}_0}{\partial x}(0, \tau) - 1, \\ \lim_{x \rightarrow \infty} \bar{V}_1(x, \tau) = 0, \end{cases} \quad (9)$$

and

$$\left\{ \begin{array}{l} \mathcal{L}[\bar{V}_n(x, \tau)] = n \frac{\partial^{n-1} \mathcal{A}'}{\partial p^{n-1}} \Big|_{p=0}, \\ \bar{V}_n(x, 0) = 0, \\ \bar{V}_n(0, \tau) + \bar{S}_n(\tau) = 0, \\ \frac{\partial \bar{V}_n}{\partial x}(0, \tau) + \bar{S}_n(\tau) = 0, \\ \lim_{x \rightarrow \infty} \bar{V}_n(x, \tau) = 0. \end{array} \right. \quad \text{if } n \geq 2, \quad (10)$$

In Eqs. (9) and (10),  $\mathcal{A}'(x, \tau, p)$  is the second term of  $\mathcal{A}(x, \tau, p)$  without the negative sign.

After eliminating  $\bar{S}_n(\tau)$  from the two boundary conditions at  $x = 0$  in Eqs. (9) and (10), we can write Eqs. (9) and (10) in a general form

$$\left\{ \begin{array}{l} \mathcal{L}[\bar{V}_n(x, \tau)] = f_n(x, \tau), \\ \bar{V}_n(x, 0) = \psi_n(x), \\ \frac{\partial \bar{V}_n}{\partial x}(0, \tau) - \bar{V}_n(0, \tau) = \phi_n(\tau), \\ \bar{V}_n(\infty, \tau) = 0, \end{array} \right. \quad (11)$$

with  $f_n(x, \tau)$ ,  $\psi_n(x)$  and  $\phi_n(\tau)$  being expressed respectively as

$$f_n(x, \tau) = \begin{cases} -\mathcal{L}[\bar{V}_0(x, \tau)] + \mathcal{A}'(x, \tau, 0), & \text{if } n = 1, \\ n \frac{\partial^{n-1} \mathcal{A}'}{\partial p^{n-1}} \Big|_{p=0}, & \text{if } n \geq 2, \end{cases} \quad (12)$$

$$\psi_n(x) = \begin{cases} -\bar{V}_0(x, 0), & \text{if } n = 1, \\ 0, & \text{if } n \geq 2, \end{cases} \quad (13)$$

$$\phi_n(\tau) = \begin{cases} \bar{V}_0(0, \tau) - \frac{\partial \bar{V}_0}{\partial x}(0, \tau) - 1, & \text{if } n = 1, \\ 0, & \text{if } n \geq 2. \end{cases} \quad (14)$$

The elimination of  $\bar{S}_n(\tau)$  is the key to the success that a fully explicit analytical solution can be eventually worked out for this highly nonlinear problem. Eq. (11) can now be solved so that analytical solution for each  $\bar{V}_n(x, \tau)$  is expressed in the form of explicit integrals (note: not integral equations), which can be easily evaluated if numerical values of the solution is needed. Therefore, the elimination of  $\bar{S}_n(\tau)$  has finally paved the way for us to obtain a solution in an explicit and closed form with no approximation whatsoever. It is also starting from this point that the current solution deviates from other previous work based on the homotopy-analysis method. For example, Liao<sup>19</sup> and Liao and Zhu<sup>20</sup> had to resort to the boundary element method to find numerical solutions at each order  $n$  in their homotopy series solutions.

Now, after introducing another simple transformation and carrying out some lengthy and cumbersome derivation, an exact solution of Eq. (11) can be worked out analytically for each order as

$$\begin{aligned}
\bar{V}_n(x, \tau) = & e^{-\frac{1}{2}(\gamma-1)x - \frac{1}{4}(\gamma+1)^2\tau} \left\{ \frac{1}{\sqrt{\pi}} \left\{ e^{\frac{1}{2}(\gamma-1)x} \int_{-\frac{x}{2\sqrt{\tau}}}^{\frac{x}{2\sqrt{\tau}}} \psi_n(2\sqrt{\tau}\xi + x) e^{(\gamma-1)\sqrt{\tau}\xi - \xi^2} d\xi \right. \right. \\
& + \int_{\frac{x}{2\sqrt{\tau}}}^{\infty} \left[ e^{\frac{1}{2}(\gamma-1)x} \psi_n(2\sqrt{\tau}\xi + x) + e^{-\frac{1}{2}(\gamma-1)x} \psi_n(2\sqrt{\tau}\xi - x) \right] e^{(\gamma-1)\sqrt{\tau}\xi - \xi^2} d\xi \left. \right\} \\
& - (\gamma+1)\sqrt{\tau} e^{-\frac{1}{2}(\gamma-1)x + \frac{(\gamma+1)^2}{4}\tau} \int_{\frac{x}{2\sqrt{\tau}}}^{\infty} \psi_n(2\sqrt{\tau}\xi - x) e^{2\gamma\sqrt{\tau}\xi} \\
& \cdot \operatorname{erfc}\left(\xi + \frac{(\gamma+1)}{2}\sqrt{\tau}\right) d\xi \\
& - \frac{2}{\sqrt{\pi}} e^{\frac{(\gamma+1)^2}{4}\tau} \int_0^{\infty} e^{-\frac{(\gamma+1)}{2}\eta} \int_{\frac{x+\eta}{2\sqrt{\tau}}}^{\infty} \phi_n\left(\tau - \frac{(x+\eta)^2}{4\xi^2}\right) e^{-\left[\frac{(\gamma+1)(x+\eta)}{4\xi}\right]^2 - \xi^2} d\xi d\eta \\
& + \int_0^{\tau} \left\{ \frac{e^{\frac{(\gamma+1)^2}{4}\eta}}{\sqrt{\pi}} \left[ e^{\frac{1}{2}(\gamma-1)x} \int_{-\frac{x}{2\sqrt{\tau-\eta}}}^{\frac{x}{2\sqrt{\tau-\eta}}} f_n(2\sqrt{\tau-\eta}\xi + x, \eta) e^{(\gamma-1)\sqrt{\tau-\eta}\xi - \xi^2} d\xi \right. \right. \\
& + \int_{\frac{x}{2\sqrt{\tau-\eta}}}^{\infty} \left[ e^{\frac{1}{2}(\gamma-1)x} f_n(2\sqrt{\tau-\eta}\xi + x, \eta) \right. \\
& \left. \left. + e^{-\frac{1}{2}(\gamma-1)x} f_n(2\sqrt{\tau-\eta}\xi - x, \eta) \right] e^{(\gamma-1)\sqrt{\tau-\eta}\xi - \xi^2} d\xi \right\} \\
& - (\gamma+1)\sqrt{\tau-\eta} e^{-\frac{1}{2}(\gamma-1)x + \frac{(\gamma+1)^2}{4}\tau} \int_{\frac{x}{2\sqrt{\tau-\eta}}}^{\infty} f_n(2\sqrt{\tau-\eta}\xi - x, \eta) \\
& \cdot e^{2\gamma\sqrt{\tau-\eta}\xi} \operatorname{erfc}\left(\xi + \frac{(\gamma+1)}{2}\sqrt{\tau-\eta}\right) d\xi \left. \right\} d\eta, \tag{15}
\end{aligned}$$

where  $\operatorname{erfc}(x)$  denotes the complementary error function.

Once  $\bar{V}_n(x, \tau)$  is found at each order,  $\bar{S}_n(\tau)$  can be easily found from the third equation of Eqs. (9) and Eq. (10), i.e.,

$$\bar{S}_n(\tau) = -\bar{V}_n(0, \tau). \tag{16}$$

Upon finding the coefficients  $\bar{V}_n(x, \tau)$  and  $\bar{S}_n(\tau)$  from Eqs. (15) and (16), the final solution of our original problem Eq. (3) can be written, by virtue of Eqs. (7), and (8), in terms of a series of infinitely many terms as

$$\begin{cases} V(x, \tau) = \bar{V}(x, \tau, 1) = \sum_{m=0}^{\infty} \frac{\bar{V}_m(x, \tau)}{m!}, \\ S_f(\tau) = \bar{S}_f(\tau, 1) = \sum_{m=0}^{\infty} \frac{\bar{S}_m(\tau)}{m!}. \end{cases} \tag{17}$$

While we know that a nonlinear problem can never be equivalent to a **finite** sum of linear problems, the fact that the solution of a nonlinear problem here can be eventually written in terms of a sum of **infinitely** many linear problems is not surprising at all; the limiting process of approaching infinity has changed the nature of summing up a sequence of linear problems. As a result, both the optimal exercise boundary,  $S_f(\tau)$  and the option price  $V(x, \tau)$  can be written in terms of an analytical and closed-form solution.

The summation process begins with an initial guess  $V_0(x, \tau)$ , which can be virtually any continuous function defined on  $[0, \infty)$ . However, for the present American option problem, the corresponding European option value is chosen as the initial guess with three apparent merits: a) Although the boundary conditions at the moving boundary  $S_f(\tau)$  are not satisfied by the corresponding European option value, the boundary condition at  $x = \infty$



is automatically satisfied, and thus such an initial guess is expected to be very close to the final solution for large  $x$  values; b)  $f_1(x, \tau)$  in Eq. (12) is further simplified because the first term on the righthand side vanishes if  $V_0(x, \tau)$  is chosen as the value of the corresponding European option; c) Because the initial value of the corresponding European option is zero,  $\psi_1(x)$  in Eq. (13) vanishes as well. The integral involving  $\psi_n$  in Eq. (15) is thus entirely eliminated. These advantages have led to a fast convergence; about 30 terms are needed to reach a convergent solution with an accuracy up to the 5th decimal place. This is about one third of the terms needed when Liao<sup>19</sup> combined the homotopy-analysis method with the boundary element techniques to solve a nonlinear heat-transfer problem.

For exact solutions in the form of infinite series, quite often it is difficult to theoretically show the convergence of the series. Although sometimes the convergency is assumed (e.g., see Ref. 24), it is more reasonable to at least show the convergency through numerical evidence. Here we provide a semi-analytical argument about the convergence of the series solution. For the solutions presented in Eqs. (7) and (8), they are convergent if we can show that

$$\lim_{m \rightarrow \infty} \left( \frac{m}{m+1} \right) \left| \frac{\bar{V}_{m+1}}{\bar{V}_m} \right|^p < 1, \quad (18)$$

for  $p \in [0, 1]$ , according to the d'Alembert's ratio test. This is thus equivalent to showing that

$$\lim_{m \rightarrow \infty} \left| \frac{\bar{V}_{m+1}}{\bar{V}_m} \right| < 1. \quad (19)$$

However, due to the recursive nature of calculating  $\bar{V}_m$  in the solution procedure, this can only be demonstrated through some numerical examples presented in the next section.

### 3. EXAMPLES AND DISCUSSIONS

To help readers who may not be used to discussing financial problems with dimensionless quantities, all results, unless otherwise stated, are now converted back to dimensional quantities in this section before they are graphed and presented.

#### 3.1. Example 1

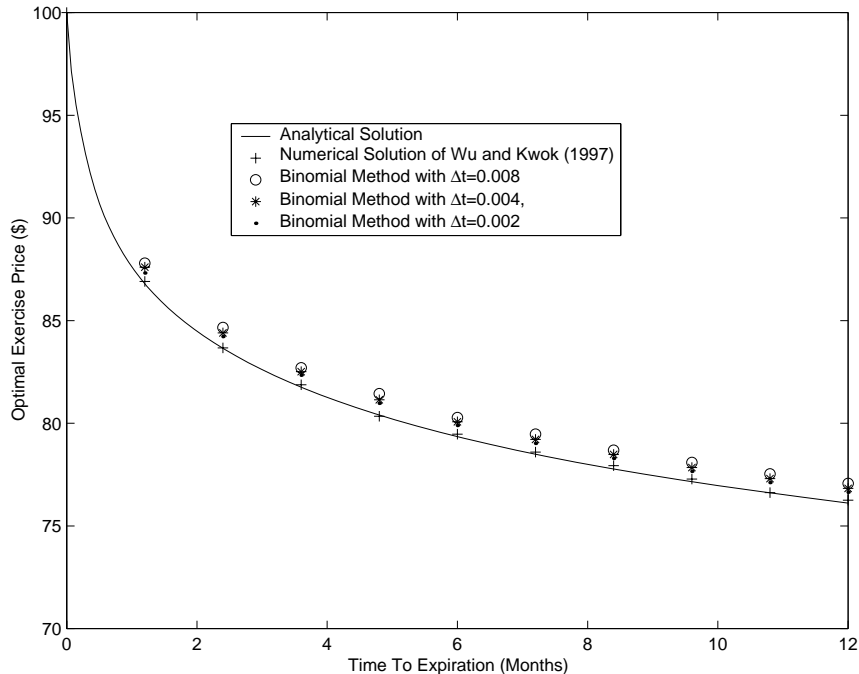
This is a sample case used by Wu and Kwok.<sup>5</sup> The parameters used by them are

- Strike price  $X = \$100$ ,
- Risk-free interest rate  $r = 0.1$ ,
- Volatility  $\sigma = 0.3$ ,
- Time to expiration  $T = 1$  (year).

In terms of the dimensionless variables, the two parameters involved are  $\gamma = 2.222$  and  $\tau_{exp} = 0.045$ .

There are many choices for the numerical computation of the integrals involved in the closed-form analytical solution Eqs. (15) and (16). All the results presented in this paper were calculated with a variable grid spacing in time and equal grid spacing in the dimensionless stock price. Symbolic calculation package Maple 9 was used to carry out the recursive computation of  $f_n(x, \tau)$  in (12) for  $n \geq 2$ . Numerical integration with a compound Simpson's rule was performed for the spatial integration and the simple trapezoidal rule was used for the temporal integration. Because the integrals involving an infinite upper limit converge extremely fast, only a small number is needed to replace the infinite upper limit; beyond this finite limit the integrand is virtually zero and contributes almost nothing to the result of the integration.

The numerical results of the analytical series solution were obtained when the solution became convergent after 29 terms were summed up. Depicted in Fig. 1 are three sets of results of the optimal exercise boundary,  $S_f$ . The first set, marked by the solid line, was calculated by the current analytical formula. The plus symbols



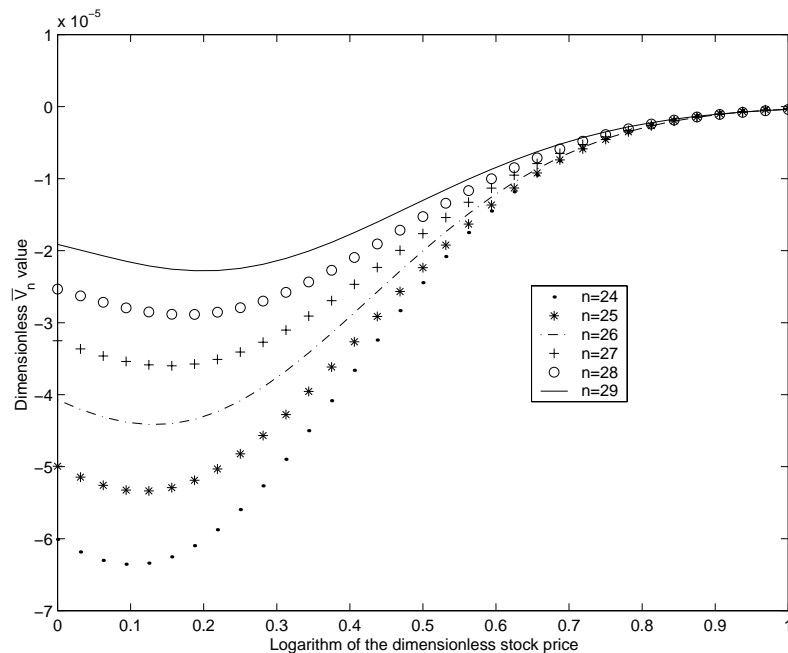
**Figure 1.** A Comparison of Optimal Exercise Prices in Example 1.

show the results presented in Figure 1 of Ref. 5. The circles, stars and dots indicate the results obtained with the binomial method<sup>9</sup> with  $\Delta t$  being equal to 0.008, 0.004 and 0.002, respectively.

The two sets of results obtained with current analytical formula and those obtained by Wu and Kwok's<sup>5</sup> appear to agree amazingly well. At the expiration time,  $t = T = 1$  (year), the optimal exercise boundary in Wu and Kwok's<sup>5</sup> numerical solution is  $B(T) = \$76.25$  whereas it is  $S_f(T) = \$76.11$  in our calculation based on the analytical solution presented in Eqs. (15) and (16). The difference between these two sets of results is less than 0.2%.

On the other hand, the results produced with the binomial method all seem to have over-estimated the optimal exercise price, although we can see the over-estimation becomes less and less when the grid spacing in time is further refined down from  $\Delta t = 0.008$  to 0.004 and eventually to 0.002. This convergent behavior of the Binomial method is in line with the calculation carried out by Bunch and Johnson.<sup>16</sup> While we have observed that, as  $\Delta t$  becomes smaller, the calculated optimal exercise boundary with the binomial method slowly approaches the exact solution curve and Wu and Kwok's<sup>5</sup> numerical solutions, we have also noticed, in Fig. 1, that there seems to be still quite a large difference between the results produced with the finest grid spacing in the binomial method and the other two sets of results. However, with the limited computational resources, we could only carry out our computation with  $\Delta t = 0.002$ ; the grid generated in the binomial method grows exponentially and even the results obtained with  $\Delta t = 0.002$  took nearly 50 hours on a PC.

As clearly shown in Fig. 1, the optimal exercise price is a monotonically decreasing function of  $T - t$  or a monotonically increasing function of  $t$ . When the time approaches the expiration time  $T$  of the option, the optimal exercise price rises sharply towards the strike price  $X = \$100$ . At  $t = T$ ,  $S_f(T) = X$  as we expected. Fig. 1 also exhibits that the rate of change of  $S_f$  is much larger near the expiration time than when the option contract is far from expiration. In fact, it is because of this large rate of change of the optimal exercise price near the expiration time, most numerical algorithms have difficulties dealing with the singular behavior of the optimal exercise price near  $t = T$  or  $\tau = 0$ . In our analytical solution, all we needed to do was to place more grid points in the neighborhood of  $\tau = 0$  to resolve the sharp change of the optimal exercise price near  $\tau = 0$ . No other numerical difficulties were experienced. The main reason for this is, of course, because the optimal exercise



**Figure 2.** Dimensionless Option Prices With Six Different Total Summation Numbers in Example 1.

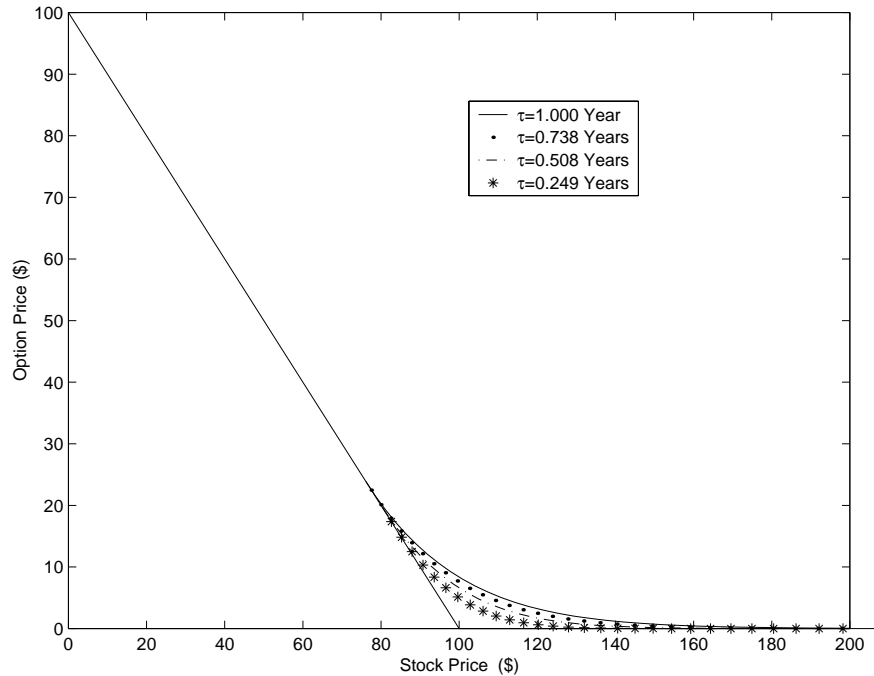
price has been written in an explicitly analytical and closed form, and it makes no difference in the numerical calculation no matter how close grid points are to each other and how close they are to the singular point at  $\tau = 0$  as long as the convergence has been achieved with enough terms included in the summations in (17).

In this example, the summations in (17) were carried out to 29 terms, when a convergent optimal exercise price was found; inclusion of more terms in the series solution resulted in a contribution in the order of  $10^{-5}$ ; the results presented in Fig. 1 and those with more terms included were hardly distinguishable at all. The uniform convergence of our results as  $n$  is increased can be clearly seen in Fig. 2, in which the dimensionless  $\bar{V}_n(x, \tau)$  values are plotted for  $n = 24$  to  $n = 29$ . As  $n$  increases, the magnitude of  $\bar{V}_n(x, \tau)$  monotonically and uniformly decreases. Therefore, condition (19) is clearly satisfied as suggested by the numerical evidence associated with this example.

With the closed-form analytic solution, we can graph the option value vs. the stock price at a fixed time. Depicted in Fig. 3 are the option prices  $V(S, t)$  as a function of  $S$  at four instants,  $\tau = T - t = 1$  (year),  $\tau = T - t = 0.738$  (years),  $\tau = T - t = 0.508$  (years), and  $\tau = T - t = 0.249$  (years), respectively. Clearly, the option price is a decreasing function of stock price. As it gets closer to the expiration of the option, i.e.  $\tau = 0$  or  $t = 1$  (year), the option price becomes closer to the payoff function  $\max\{X - S, 0\}$ , which is represented by the straight line and part of the abscissa from  $S = X = \$100$  and beyond.

Theoretically,  $S$  needs to become infinite before a put option becomes worthless. But, from our analytical formula, one can observe from Fig. 3 that if, for any  $\tau < 1$  (year), the stock price becomes about 1.8 times larger than the strike price, the option price becomes almost worthless for the interest rate given in this example.

With an analytical solution in hand, one can now find the analytical expressions for the hedge parameters  $\Delta$ ,  $\Gamma$ ,  $\Theta$ , *Vega* and *Rho*, known as the Greeks, by taking derivatives with respect to different variables. All one needs to do is to differentiate Eq. (15) with respect to the relevant parameters to produce the closed-form analytical formulae for the Greeks. Such closed-form analytical formulae for these important hedge parameters are certainly useful in trading practice as if they have to be calculated numerically based on a numerical solution for the option price, large errors are often observed if the basis functions adopted in the numerical procedure do not have enough degree of differentiability (see Tavella and Randall<sup>25</sup>). The actual differentiation process could



**Figure 3.** Option Prices At Different Times to Expiry in Example 1.

be involving and tedious as Eq. (15) is already quite a lengthy expression. However, with symbolic calculation packages, such as Maple, readily available today, such a task is quite trivial although the final expressions may be quite lengthy and cumbersome. Therefore, the final expressions for the Greeks are omitted in this paper.

### 3.2. Example 2

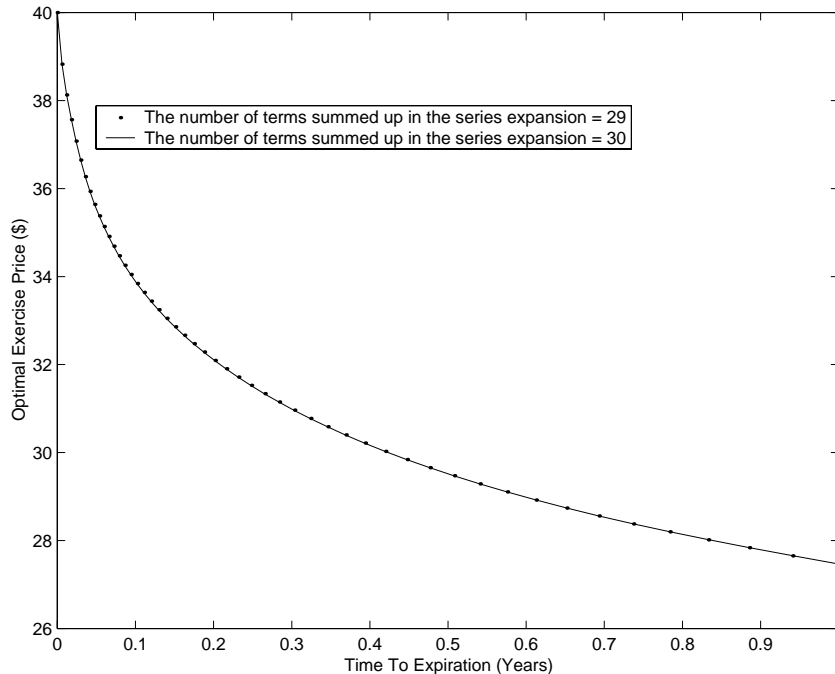
In the next example, we take the same example used by Bunch and Johnson,<sup>16</sup> who presented some simple approximations to the valuation of American put options and their critical stock price. In their example, the dimensional parameters are

- Strike price  $X = \$40$ ,
- Risk-free interest rate  $r = 0.0488$ ,
- Volatility  $\sigma = 0.3$ ,
- Time to expiration  $T = 1$  (year),

and the two dimensionless parameters can be easily calculated as

- Relative risk-free interest rate  $\gamma = 1.084$ ,
- Dimensionless time to expiration  $\tau_{exp} = 0.045$ .

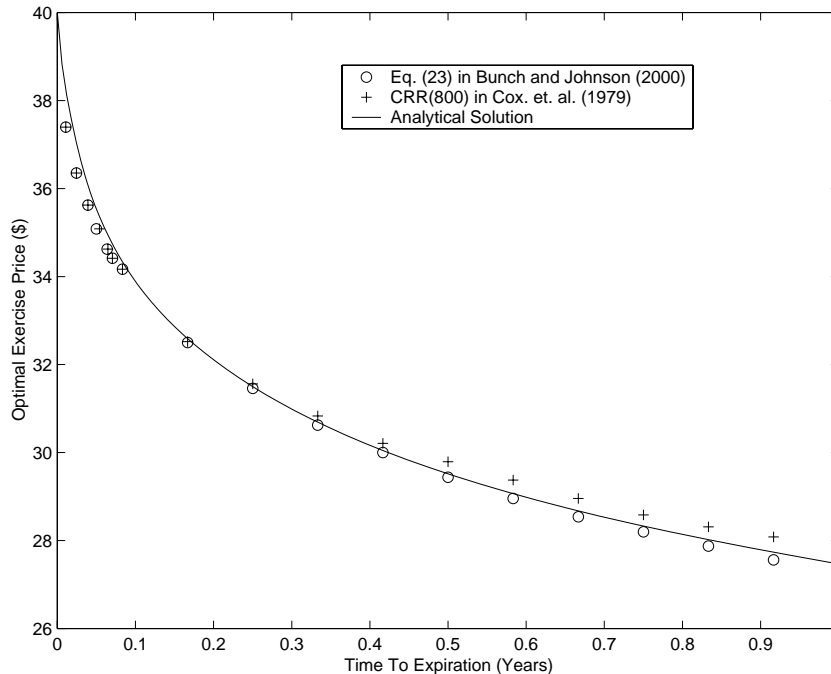
In this example, a convergent optimal exercise price was found when  $n$  reached 30 (the convergence of the option price is always reached before a convergent optimal exercise price is reached). Fig. 4 displays two optimal exercise prices corresponding to the cases with 29 terms included in the summations in Eq. (17) ( $n = 29$ ) and with 30 terms included in the summations ( $n = 30$ ), respectively. Clearly, with a difference of the order of  $10^{-5}$ , the two curves are indistinguishable in the figure.



**Figure 4.** Convergence Of the Optimal Exercise Price in Example 2

In their Fig. 1, Bunch and Johnson<sup>16</sup> compared the results from two of their approximation formulae (Eq. (23) and Eq. (29) in their paper) and the numerical results of Cox *et al.*,<sup>9</sup> based on the binomial method. Clearly, from that figure, results based on Eq. (23) and Cox *et al.*'s results with 800 time steps are close to each other. On the one hand, based on the derivation of two approximation formulae, Bunch and Johnson<sup>16</sup> believed that the results produced by their Eq. (23) “ought to have smaller errors” than those produced by their Eq. (29). On the other hand, we have every reason to believe that Cox *et al.*'s results with 800 time steps are also more accurate (or closer to the exact solution) than those with 150 steps through our binomial calculations carried out for the previous example. Between these two more accurate solutions, which one is more accurate then? Bunch and Johnson seemed to believe that “the CRR curves move toward the equation (23) curve” without any numerical evidence presented in their paper. But, it could be that the CRR(800) curve is already very close to the true solution curve and the approximation formula based on Eq. (23) gives a larger error. Or it could also be that the exact solution is somewhere in the middle of the gap between the two and is closer to neither the curve based on Eq. (23) nor the CRR(800) curve. Since there is still quite a bit of a gap between the Eq. (23) curve and the CRR(800) curve, especially when the time to expiration becomes large, it really makes readers wonder which scenario is true.

We can now answer these questions with the newly-found closed-form analytical solution. Fig. 5 shows the comparison of the current exact solution of the optimal exercise price with those produced by using the approximate solution Eq. (23) in Bunch and Johnson<sup>16</sup> and the numerical solution of Cox *et al.*<sup>9</sup> based on the binomial method with 800 time steps. As clearly shown in Fig. 5, the results produced by the current closed-form analytical solution are in excellent agreement with those produced by Bunch and Johnson's Equation (23). When the time to expiration becomes larger, the curve corresponding to the analytical solution appears to move a little bit more above the Eq. (23) curve. But, the simple approximation formula of Eq. (23) in conjunction with Eq. (A10) proposed by Bunch and Johnson<sup>16</sup> nevertheless has produced amazingly accurate results. As far as the numerical efficiency is concerned, their approximation formula certainly has its value, as all the computational effort concentrates on the calculation of the  $\alpha$  value iteratively from their Eq. (A10), whereas some double integrals in the closed-form solution presented here still need to be evaluated, though they converge very fast. However, the current closed-form solution has its value, too, as this is the first time such a closed-form analytical



**Figure 5.** Optimal exercise prices for the case in Example 2

solution has been found for this highly nonlinear problem. In fact, its usefulness can be seen as we can readily answer the questions posed in the previous paragraph.

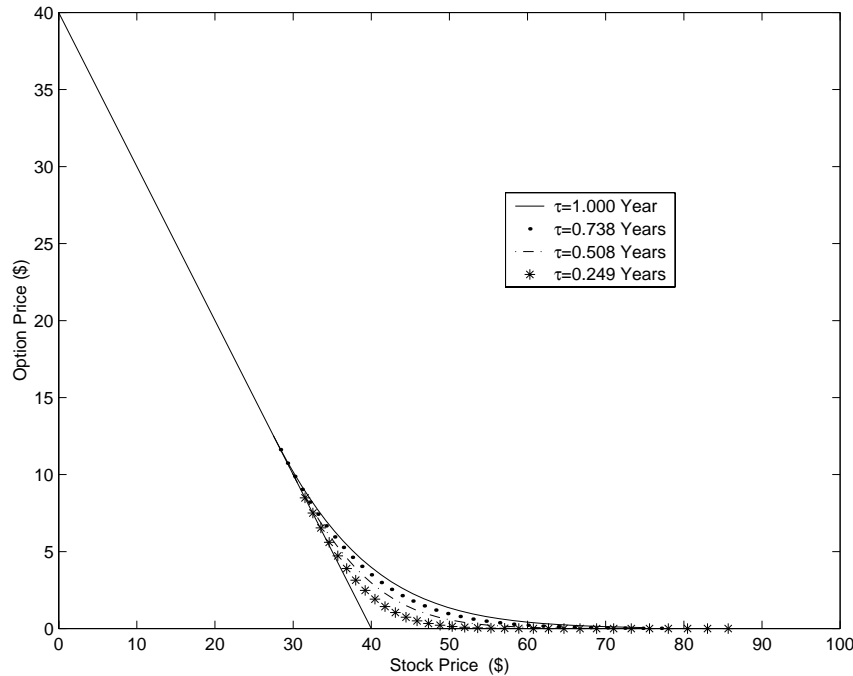
One should also notice that in contrast to the very smooth and monotonically decreasing curves produced by the current analytical formulae for both the optimal exercise price and the option price, curves produced by some numerical solutions do not appear to be smooth and truly monotonic (e.g., Ref. 10, 26). This could be due to the total number spatial grids limited by a particular numerical approach. For example, when radial basis functions are used (see Ref. 26), a great difficulty would be encountered in the inversion of the final solution matrix when the grid-spacing refinement is taken beyond a certain point. This problem also exists in the finite-difference approach (see Ref. 25); localized oscillations have been observed when grid spacing is refined beyond a point.

The option price as a function of  $S$  and  $t$  can be easily calculated as well, using the new analytical formula. Depicted in Fig. 6 are the option prices at four instants of time to expiration. Once again, one can see that the implementation of the moving boundary conditions appears to be excellent; all the option-price curves smoothly land on the intrinsic-value curve, or the pay-off function,  $\max\{X - S, 0\}$ .

#### 4. CONCLUSIONS

In this paper, the nonlinear problem of valuing American options is analytically solved with the homotopy-analysis method, and a closed-form solution of the well-known Black-Scholes equation is obtained for the first time. It is shown that the optimal exercise boundary, which is the key difficulty in the valuation of American options, can be expressed explicitly in a closed form in terms of the input parameters such as the risk-free interest rate, the volatility and the time to expiration.

Two examples are presented to compare the analytical solutions with some previously published numerical solutions or approximate solutions. The convergence of the new series solution has been demonstrated through the numerical evidence. The new analytical solution can be now used to validate other numerical solutions or approximate solutions. More importantly, the Greeks can be calculated, in closed-form too, without localized oscillation problems that have been observed in some numerical approaches.



**Figure 6.** Option prices at different times to expiration in Example 2

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