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Keywords

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HOMOLOGY FOR HIGHER-RANK GRAPHS AND TWISTED C^*-ALGEBRAS

ALEX KUMJIAN, DAVID PASK, AND AIDAN SIMS

Abstract. We introduce a homology theory for k-graphs and explore its fundamental properties. We establish connections with algebraic topology by showing that the homology of a k-graph coincides with the homology of its topological realisation as described by Kaliszewski et al. We exhibit combinatorial versions of a number of standard topological constructions, and show that they are compatible, from a homological point of view, with their topological counterparts. We show how to twist the C^*-algebra of a k-graph by a $\mathbb{T}$-valued 2-cocycle and demonstrate that examples include all noncommutative tori. In the appendices, we construct a cubical set $\widetilde{Q}(\Lambda)$ from a k-graph Λ and demonstrate that the homology and topological realisation of Λ coincide with those of $\widetilde{Q}(\Lambda)$ as defined by Grandis.

1. Introduction

In this paper we initiate the study of homology for higher-rank graphs. We develop a suite of fundamental results and techniques, and also establish connections with a number of related areas: Via the topological realisations of k-graphs introduced in [21], we establish connections with the cubical approach to algebraic topology used in [30]. We also show in an appendix how our approach connects the theory of k-graphs to the theory of cubical sets discussed in, for example, [5, 13, 14, 15, 19]. Our key motivation, however, is that our homology theory and in particular the associated cohomology theory promises to have an interesting application to C^*-algebras. We discuss this application in Section 7: we introduce the cohomology theory corresponding to our homology and show that $\mathbb{T}$-valued 2-cocycles on a k-graph can be used to twist its C^*-algebra. As examples we obtain all noncommutative tori and the Heegaard-type quantum 3-spheres of Baum, Hajac, Matthes and Szymański (see [1]). A more detailed study of the cohomology of k-graphs and the structure theory of the associated C^*-algebras will be the subject of future work.

Higher-rank graphs, or k-graphs, were introduced by the first two authors in [25] as a combinatorial model for the higher-rank Cuntz-Krieger algebras discovered and analysed by Robertson and Steger [38], and to unify the constructions of many other interesting C^*-algebras [24]. The C^*-algebras of higher-rank graphs have been studied by numerous authors over the last decade (see, for example, [6, 7, 10, 11, 40, 41, 43]).

The combinatorial properties of a k-graph suggest a sort of k-dimensional directed graph, and this point of view has been borne out in numerous ways in the study of k-graph C^*-algebras. More recently, however, it has begun to suggest relationships with topology.

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These connections first arose in [33, 34] where a theory of coverings and a notion of fundamental group for k-graphs was developed. These notions closely parallel the topological theory, but were motivated by C^*-algebraic considerations: the authors demonstrated that coverings of k-graphs correspond to relative skew products which in turn correspond to coaction crossed products and crossed products by homogeneous spaces.

The topological flavour of some of the results of [33, 34] suggest that each k-graph should have a topological realisation, which would be a k-dimensional CW complex, and that the k-graph could profitably be viewed as a combinatorial version of its topological realisation [33, Section 6]. Current work of the first and third authors with Kaliszewski and Quigg [21] bears this idea out, showing in particular that the fundamental groups of a k-graph and of its topological realisation are isomorphic and that many well-known k-graph constructions are well-behaved with respect to fundamental groups.

In the current paper, we expand on this idea further by commencing the study of homology of higher-rank graphs. After recalling basic definitions and notation in Section 2, we proceed in Section 3 to define our homology, prove that it is a functor, show that we can measure connectedness by the 0^{th} homology group, and show that the 1-cycles correspond naturally to integer combinations of undirected cycles in the k-graph.

In Section 4, we prove analogs of a number of standard theorems in algebraic topology for our homology. For example we show that the Künneth formula holds for the homology of a cartesian product of higher-rank graphs, and that the homology of the quotient of an acyclic k-graph by a free action of a discrete group G is isomorphic to the homology of G. We also show that every automorphism of a k-graph induces a long exact sequence in homology which corresponds exactly to the long exact sequence for a mapping torus.

In Section 5, we use a combination of these results and direct calculation to describe examples of 2-graphs whose homology is identical to that of the sphere, the torus, the Klein bottle and the projective plane respectively; we also present these examples in a way which indicates that their topological realisations should coincide with these four spaces. Details of these homeomorphisms will appear in [21]. In Section 6, we use an argument based on that given by Hatcher for simplicial complexes and singular homology [17], to show that our homology for a k-graph agrees with the singular homology of its topological realisation. This suggests strongly that our homology theory is a reasonable one for k-graphs.

Section 7 gives a taste of the C^*-algebraic application which motivates our study of homology for k-graphs: twisted k-graph C^*-algebras. We briefly discuss the cohomology of a higher-rank graph and check that it satisfies the Universal Coefficient Theorem. We introduce the notion of the C^*-algebra of higher-rank graph twisted by a $\mathbb{T}$-valued 2-cocycle, and show that the isomorphism class of the C^*-algebra depends only on the cohomology class of the cocycle. We then consider some basic examples of finite k-graphs whose twisted C^*-algebras capture the noncommutative tori and the Heegaard-type quantum 3-spheres of [1].

Our homology is modeled on the cubical version of singular homology in [30] and is closely related to the homology of a cubical set introduced by Grandis [14]. We establish in Appendix A that a k-graph Λ determines a cubical set $\widetilde{Q}(\Lambda)$, and that our homology of Λ is isomorphic to Grandis' homology of $\widetilde{Q}(\Lambda)$. Hence, in principle, some of our earlier results (Theorem 4.9 and part of the statement of Theorem 4.3) could be recovered from

Grandis'. However we provide a self-contained treatment avoiding unnecessary complications involving degeneracy maps: we believe that the resulting simplicity of presentation justifies our approach. We demonstrate in Appendix B, that the topological realisation of a k-graph as described in [21] is homeomorphic to the topological realisation, outlined in [14], of the associated cubical set.

Acknowledgements. The idea that homology of k-graphs might be of interest first arose from the study of topological realizations (see [21, 33]), which was suggested by John Quigg. We thank Mike Whittaker for a number of helpful discussions and in particular for his contributions to Examples 5.7 and 5.6. The second author thanks his coauthors for their hospitality.

2. Preliminaries

As in [27], in our definition of a k-graph we will allow for the possibility of 0-graphs with the convention that $\mathbb{N}^0$ is the trivial semigroup $\{0\}$. We insist that all k-graphs are nonempty.

We adopt the conventions of [27, 33] for k-graphs. Given a nonnegative integer k, a *k-graph* is a nonempty countable small category Λ equipped with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the *factorisation property*: for all $\lambda \in \Lambda$ and $m, n \in \mathbb{N}^k$ such that $d(\lambda) = m + n$ there exist unique $\mu, \nu \in \Lambda$ such that $d(\mu) = m$, $d(\nu) = n$, and $\lambda = \mu\nu$. When $d(\lambda) = n$ we say λ has *degree* n. We often use the same symbol d to denote the degree functor in all k-graphs in this paper.

For $k \geq 1$, the standard generators of $\mathbb{N}^k$ are denoted $e_1, \dots, e_k$, and for $n \in \mathbb{N}^k$ and $1 \leq i \leq k$ we write n_i for the i^{th} coordinate of n. For $n = (n_1, \dots, n_k) \in \mathbb{N}^k$ let $|n| = \sum_{i=1}^k n_i$. If Λ is a k-graph, then for $\lambda \in \Lambda$, we write $|\lambda|$ for $|d(\lambda)|$. For $m, n \in \mathbb{N}^k$ we write $m \leq n$ if $m_i \leq n_i$ for all $i \leq k$. We often implicitly identify $\mathbb{N}^{k_1+k_2} = \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$.

Given a k-graph Λ and $n \in \mathbb{N}^k$, we write Λ^n for $d^{-1}(n)$. The *vertices* of Λ are the elements of Λ^0. The factorisation property implies that $o \mapsto \mathrm{id}_o$ is a bijection from the objects of Λ to Λ^0. The domain and codomain maps in the category Λ therefore determine maps $s, r : \Lambda \to \Lambda^0$: for $\alpha \in \Lambda$, the *source* $s(\alpha)$ of α is the identity morphism associated with the object $\mathrm{dom}(\alpha)$ and similarly, $r(\alpha) = \mathrm{id}_{\mathrm{cod}(\alpha)}$. An *edge* in a k-graph is a morphism f with $d(f) = e_i$ for some $i = 1, \dots, k$. In keeping with graph terminology an element $\lambda \in \Lambda$ is often called a *path*.

A 0-graph is then a countable category whose only morphisms are the identity morphisms, which we regard as a collection of isolated vertices.

Each 1-graph Λ is the path-category of the directed graph with vertices Λ^0 and edges Λ^1 and range and source maps inherited from Λ. Conversely, if E is a directed graph, then its path-category E^* is a 1-graph under the length function. This leads to the unusual convention that a path in E is a sequence of edges $\alpha_1 \cdots \alpha_n$ such that $s(\alpha_i) = r(\alpha_{i+1})$ for all i, and we write $r(\alpha) = r(\alpha_1)$ and $s(\alpha) = s(\alpha_n)$.

Let λ be an element of a k-graph Λ and suppose that $0 \leq m \leq n \leq d(\lambda)$. By the factorisation property there exist unique elements $\alpha \in \Lambda^m$, $\beta \in \Lambda^{n-m}$ and $\gamma \in \Lambda^{d(\lambda)-n}$ such that $\lambda = \alpha\beta\gamma$. We define $\lambda(m, n) := \beta$. We then have $\lambda(0, m) = \alpha$ and $\lambda(n, d(\lambda)) = \gamma$. In particular, for $0 \leq m \leq d(\lambda)$,

$$\lambda = \lambda(0, m)\lambda(m, d(\lambda)).$$

For $v \in \Lambda^0$ and $E \subset \Lambda$, we write vE for $E \cap r^{-1}(v)$ and Ev for $E \cap s^{-1}(v)$.

Definition 2.1 ([25, Definition 5.1] (see also [34])). Let G be a discrete group, (Λ, d) a k-graph and $c : \Lambda \to G$ a functor. The *skew product* k-graph $\Lambda \times_c G$ is defined as follows: as a set $\Lambda \times_c G$ is the cartesian product $\Lambda \times G$ and $d(\lambda, g) = d(\lambda)$ (so $(\Lambda \times_c G)^0 = \Lambda^0 \times G$) with

$$s(\lambda, g) = (s(\lambda), gc(\lambda)) \quad \text{and} \quad r(\lambda, g) = (r(\lambda), g).$$

If $s(\lambda) = r(\mu)$ then (λ, g) and $(\mu, gc(\lambda))$ are composable in $\Lambda \times_c G$ and

$$(\lambda, g)(\mu, gc(\lambda)) = (\lambda\mu, g). \tag{2.1}$$

Examples 2.2.
(1) For $k \geq 0$ let $T_k = \mathbb{N}^k$ regarded as a k-graph with $d : T_k \to \mathbb{N}^k$ the identity map. So T_k has exactly one morphism of degree n for each $n \in \mathbb{N}^k$, and in particular a single vertex 0. For $k \geq 1$, T_k is generated by the k commuting elements, $e_1, \ldots, e_k$.
(2) For $n \geq 1$ let B_n be the path category of the directed graph with one vertex and n distinct edges $f_1, \ldots, f_n$. We refer to B_n as the 1-graph associated to the bouquet of n-circles (see Example 4.11(1)).
(3) For $n \geq 2$ let $\mathbb{F}_n$ be the free group on n generators $\{h_1, \ldots, h_n\}$ and define the functor $c : B_n \to \mathbb{F}_n$ by $c(f_i) = h_i$ for $i = 1, \ldots, n$. Let A_n denote the skew product 1-graph $B_n \times_c \mathbb{F}_n$. The underlying directed graph associated to A_n is the (right) Cayley graph of $\mathbb{F}_n$ and may be visualised as a uniform n-ary tree.
(4) For $k \geq 1$ and $m \in (\mathbb{N} \cup \{\infty\})^k$, we write $\Omega_{k,m}$ for the k-graph with

$$\Omega_{k,m} := \{(p, q) \in \mathbb{N}^k \times \mathbb{N}^k : p \leq q \leq m\}$$

and with structure maps $r(p, q) := (p, p)$, $s(p, q) := (q, q)$, $d(p, q) := q - p$ and $(p, q)(q, r) := (p, r)$. Define $\Omega_0 := \{0\}$ and for $k \geq 1$ let $\Omega_k := \Omega_{k,(\infty,\ldots,\infty)}$.
(5) For $k \geq 1$, let Δ_k be the k-graph with $\Delta_k := \{(p, q) \in \mathbb{Z}^k \times \mathbb{Z}^k : p \leq q\}$ and structure maps as in $\Omega_{k,m}$.
(6) Let (Λ_i, d_i) be a k-graph for $i = 1, 2$. The disjoint union $\Lambda_1 \sqcup \Lambda_2$ may be regarded as a k-graph with $d(\lambda) = d_i(\lambda)$ if $\lambda \in \Lambda_i$ and with other structure maps likewise inherited from the Λ_i.
(7) Let (Λ_i, d_i) be a k_i-graph for $i = 1, 2$. Then $(\Lambda_1 \times \Lambda_2, d_1 \times d_2)$ is a $(k_1 + k_2)$-graph where $\Lambda_1 \times \Lambda_2$ is the product category and $d_1 \times d_2 : \Lambda_1 \times \Lambda_2 \to \mathbb{N}^{k_1+k_2}$ is given by $(d_1 \times d_2)(\lambda_1, \lambda_2) = (d_1(\lambda_1), d_2(\lambda_2)) \in \mathbb{N}^{k_1} \times \mathbb{N}^{k_2}$ for $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$.

Let $k_1, k_2 \geq 1$. Let $\pi_1 : \mathbb{Z}^{k_1+k_2} \to \mathbb{Z}^{k_1}$ denote the projection onto the first k_1 coordinates and $\pi_2 : \mathbb{Z}^{k_1+k_2} \to \mathbb{Z}^{k_2}$ denote the projection onto the last k_2 coordinates. We frequently regard π_i as a homomorphism from $\mathbb{N}^{k_1+k_2}$ to $\mathbb{N}^{k_i}$.

A *k-graph morphism* between k-graphs is a degree-preserving functor. There is a category whose objects are k-graphs and whose morphisms are k-graph morphisms. Whenever we regard k-graphs as objects of a category in this paper, it will be this one.

Examples 2.3.
(1) For $k_1, k_2 \geq 1$ we have $T_{k_1+k_2} = \mathbb{N}^{k_1+k_2} = \mathbb{N}^{k_1} \times \mathbb{N}^{k_2} = T_{k_1} \times T_{k_2}$.
(2) For $k_1, k_2 \geq 1$ we have $\Delta_{k_1+k_2} \cong \Delta_{k_1} \times \Delta_{k_2}$. One checks that the map $(m, n) \mapsto ((\pi_1(m), \pi_1(n)), (\pi_2(m), \pi_2(n)))$ gives the desired isomorphism of k-graphs.

It is sometimes useful to consider morphisms between higher-rank graphs which do not preserve degree. The following definition is from [27, §2].

Definition 2.4. Let (Λ, d) be a k-graph and (Γ, d') be an ℓ-graph. A functor $\psi : \Lambda \to \Gamma$ is called a *quasimorphism* if there is a homomorphism $\pi : \mathbb{N}^k \to \mathbb{N}^\ell$ such that for all $\lambda \in \Lambda$ we have $\pi(d(\lambda)) = d'(\psi(\lambda))$.

Example 2.5. For $i = 1, 2$, let (Λ_i, d_i) be a k_i-graph. Let $\Lambda_1 \times \Lambda_2$ the associated cartesian product $(k_1 + k_2)$-graph. Since every element $\lambda \in \Lambda_1 \times \Lambda_2$ is of the form $\lambda = (\lambda_1, \lambda_2)$ where $\lambda_1 \in \Lambda_1$ and $\lambda_2 \in \Lambda_2$, for $i = 1, 2$ there is a natural functor $\psi_i : \Lambda_1 \times \Lambda_2 \to \Lambda_i$ given by $(\lambda_1, \lambda_2) \mapsto \lambda_i$; note that ψ_i is a quasimorphism with $d_i \circ \psi_i = \pi_i \circ (d_1 \times d_2)$.

Definition 2.6. Let $f : \mathbb{N}^k \to \mathbb{N}^l$ be a homomorphism and let Γ be an l-graph. The pullback $f^*\Gamma$ is the k-graph $\{(\gamma, n) \in \Gamma \times \mathbb{N}^k : f(n) = d(\gamma)\}$ with degree map $d(\gamma, n) = n$ (see [25, Definition 1.9]). The structure maps are given by $r(\gamma, n) = (r(\gamma), 0)$ and $s(\gamma, n) = (s(\gamma), 0)$. If $s(\lambda) = r(\mu)$ in Γ then (λ, n) and (μ, m) are composable in $f^*\Gamma$, and

$$(\lambda, n)(\mu, m) = (\lambda\mu, m + n). \tag{2.2}$$

All of the above is standard notation for k-graphs. In the remainder of this section we introduce some new notation related to k-graphs as a preliminary to the definition and basic properties of homology for k-graphs in Section 3

Definition 2.7. Let Λ be a k-graph where $k \geq 1$. For $\lambda \in \Lambda$ and $m \in \{1, -1\}$, we define

$$s(\lambda, m) := \begin{cases} s(\lambda) & \text{if } m = 1 \\ r(\lambda) & \text{if } m = -1 \end{cases} \qquad \text{and} \qquad r(\lambda, m) := s(\lambda, -m).$$

An *undirected path* is a pair (g, m) where $g = (g_1, \dots, g_n)$ is a sequence of edges in Λ and $m = (m_1, \dots, m_n)$ is a sequence of orientations, $m_i \in \{1, -1\}$ such that $s(g_i, m_i) = r(g_{i+1}, m_{i+1})$ for all i. If (g, m) is an undirected path, we define $s(g, m) := s(g_n, m_n)$ and $r(g, m) := r(g_1, m_1)$. If $r(g, m) = s(g, m)$, then we say that the undirected path (g, m) is *closed*.

A closed undirected path (g, m) is called *simple* if $s(g_i, m_i) \neq s(g_j, m_j)$ for $i \neq j$.

Definition 2.8. (cf. [33, §3]**)** A k-graph Λ is *connected* if the equivalence relation on Λ^0 generated by $\{(r(\lambda), s(\lambda)) : \lambda \in \Lambda\}$ is $\Lambda^0 \times \Lambda^0$.

Remark 2.9. A k-graph Λ is connected if and only if for all $u, v \in \Lambda^0$ there is an undirected path with source u and range v.

For each equivalence class $X \subseteq \Lambda^0$ from Definition 2.8, the k-graph $X\Lambda X$ is a connected component of Λ. Each k-graph is the disjoint union of its connected components.

For $k \geq 0$ define $\mathbf{1}_k := \sum_{i=1}^k e_i \in \mathbb{N}^k$. By convention $\mathbf{1}_0 = 0 \in \mathbb{N}^0$.

Definition 2.10. Let Λ be a k-graph. For $r \geq 0$ let

$$Q_r(\Lambda) = \{\lambda \in \Lambda : d(\lambda) \leq \mathbf{1}_k, |\lambda| = r\}.$$

Let $Q(\Lambda) = \cup_{r \geq 0} Q_r(\Lambda)$.

We have $Q_0(\Lambda) = \Lambda^0$, and $Q_r(\Lambda) = \emptyset$ if $r > k$. Let $0 < r \leq k$. The set $Q_r(\Lambda)$ consists of the morphisms in Λ which may be expressed as the composition of a sequence of r edges with distinct degrees. We regard elements of $Q_r(\Lambda)$ as unit r-cubes in the sense that each one gives rise to a commuting diagram of edges in Λ shaped like an r-cube. In particular, when $r \geq 1$, each element of $Q_r(\Lambda)$ has $2r$ faces in $Q_{r-1}(\Lambda)$ defined as follows.

Definition 2.11. Fix $\lambda \in Q_r(\Lambda)$ and write $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$ where $i_1 < \cdots < i_r$. For $1 \leq j \leq r$, define $F_j^0(\lambda)$ and $F_j^1(\lambda)$ to be the unique elements of $Q_{r-1}(\Lambda)$ such that there exist $\alpha, \beta \in \Lambda^{e_{i_j}}$ satisfying

$$F_j^0(\lambda)\beta = \lambda = \alpha F_j^1(\lambda).$$

Remark 2.12. Equivalently, $F_j^0(\lambda) = \lambda(0, d(\lambda) - e_{i_j})$ and $F_j^1(\lambda) = \lambda(e_{i_j}, d(\lambda))$. If $1 \leq i < j \leq r$, then $F_i^\ell \circ F_j^m = F_{j-1}^m \circ F_i^\ell$ for $\ell, m \in \{0, 1\}$.

Notation 2.13. Let X be a set. We write $\mathbb{Z}X$ for the free abelian group generated by X (so $\mathbb{Z}\emptyset = \{0\}$).

Remark 2.14. Let X and Y be sets. Then every function $f : X \to Y$ extends uniquely to a homomorphism $f : \mathbb{Z}X \to \mathbb{Z}Y$. In particular, the inclusion maps induce an isomorphism $\mathbb{Z}(X \sqcup Y) \cong \mathbb{Z}X \oplus \mathbb{Z}Y$. Moreover there is an isomorphism $\mathbb{Z}(X \times Y) \cong \mathbb{Z}X \otimes \mathbb{Z}Y$ determined by $(x, y) \mapsto x \otimes y$.

3. The homology of a k-graph

In this section we define the homology of a k-graph, compute some basic examples and provide descriptions of the first two homology groups. Throughout this paper, we use r (for rank) for the indexing subscript in complexes and in homology groups because n is more commonly used for a generic element of $\mathbb{N}^k$.

Definitions 3.1. For $r \in \mathbb{N}$ let $C_r(\Lambda) = \mathbb{Z}Q_r(\Lambda)$. For $r \geq 1$, define $\partial_r : C_r(\Lambda) \to C_{r-1}(\Lambda)$ to be the unique homomorphism such that

$$\partial_r(\lambda) = \sum_{\ell=0}^{1} \sum_{i=1}^{r} (-1)^{i+\ell} F_i^\ell(\lambda) \quad \text{for all } \lambda \in Q_r(\Lambda). \tag{3.1}$$

We write ∂_0 for the zero homomorphism $C_0(\Lambda) \to \{0\}$.

Remarks 3.2. For $f \in Q_1(\Lambda)$ we have $F_1^1(f) = s(f)$ and $F_1^0(f) = r(f)$ and so $\partial_1(f) = s(f) - r(f)$.

Fix $\lambda \in Q_2(\Lambda)$. Write $d(\lambda) = e_{j_1} + e_{j_2}$ with $j_1 < j_2$. Factorise $\lambda = f_1 g_1 = g_2 f_2$ where $d(f_i) = e_{j_1}$ and $d(g_i) = e_{j_2}$ for $i = 1, 2$. Then $F_2^0(\lambda) = \lambda(0, e_{j_1}) = f_1$, $F_2^1(\lambda) = \lambda(e_{j_2}, e_{j_1} + e_{j_2}) = f_2$, $F_1^0(\lambda) = \lambda(0, e_{j_2}) = g_2$ and $F_1^1(\lambda) = \lambda(e_{j_1}, e_{j_1} + e_{j_2}) = g_1$. Hence

$$\partial_2(\lambda) = g_1 + f_1 - f_2 - g_2. \tag{3.2}$$

For $r \geq 0$, ∂_r is a homomorphism and $\partial_r \circ \partial_{r+1} = 0$ by Remark 2.12. Hence we have the following.

Lemma 3.3. *Let Λ be a k-graph, then $(C_*(\Lambda), \partial_*)$ is a chain complex.*

We define the homology of Λ to be the homology of the chain complex $C_*(\Lambda)$.

Definition 3.4. For $r \in \mathbb{N}$ define $H_r(\Lambda) = \ker(\partial_r)/\operatorname{Im} \partial_{r+1}$. We call $H_r(\Lambda)$ the r^{th} *homology group of* Λ and we call $H_*(\Lambda)$ the *homology of* Λ.

Lemma 3.5. *Fix $n \in \mathbb{N}$. If $\psi : \Lambda_1 \to \Lambda_2$ is a k-graph morphism, then there is a homomorphism $\psi_* : H_r(\Lambda_1) \to H_r(\Lambda_2)$ determined by $\psi_*([\lambda]) = [\psi(\lambda)]$ for all $\lambda \in Q_r(\Lambda)$. Moreover, the assignments $\Lambda \mapsto H_r(\Lambda)$ and $\psi \mapsto \psi_*$ comprise a covariant functor from the category of k-graphs with k-graph morphisms to the category of abelian groups with homomorphisms.*

Proof. For $\lambda \in Q_r(\Lambda_1)$ we have $\psi(\lambda) \in Q_r(\Lambda_2)$ as ψ is degree preserving. Since it preserves factorisations, ψ intertwines the face maps on $Q_r(\Lambda_1)$ and $Q_r(\Lambda_2)$, so it intertwines the boundary maps ∂_r and therefore defines a homomorphism $\psi_* : H_r(\Lambda_1) \to H_r(\Lambda_2)$.

For the second assertion of the Lemma, we just have to check that $\psi \mapsto \psi_*$ preserves composition. This follows immediately from the definition. □

Remark 3.6. For a k-graph Λ and $r > k$, we have $Q_r(\Lambda) = \emptyset$, so $C_r(\Lambda)$ and $H_r(\Lambda)$ are trivial.

Remark 3.7. Let Λ_i be k-graphs for $i = 1, 2$. Then the chain complex $C_*(\Lambda_1 \sqcup \Lambda_2)$ decomposes as the direct sum of the complexes $C_*(\Lambda_1)$ and $C_*(\Lambda_2)$. Thus the canonical inclusions of Λ_1, Λ_2 into $\Lambda_1 \sqcup \Lambda_2$ induce an isomorphism $H_*(\Lambda_1) \oplus H_*(\Lambda_2) \cong H_*(\Lambda_1 \sqcup \Lambda_2)$. Indeed, this isomorphism holds for countable disjoint unions of k-graphs.

Remark 3.8. Let Λ be a k-graph and let Λ^{op} be the opposite category, which is a k-graph under the same degree map. We write λ^{op} for an element $\lambda \in \Lambda$ when regarded as an element of Λ^{op}. For each r, the assignment $\lambda \mapsto (-1)^r \lambda^{\mathrm{op}}$ induces an isomorphism $\phi_r : C_r(\Lambda) \to C_r(\Lambda^{\mathrm{op}})$. Using that $F_i^l(\lambda^{\mathrm{op}}) = F_i^{1-l}(\lambda)^{\mathrm{op}}$ for all $\lambda \in Q_r(\Lambda)$, a calculation shows that $\partial_{r+1} \circ \phi_{r+1} = \phi_r \circ \partial_{r+1}$ for all r. So ϕ_* is an isomorphism of complexes and hence induces an isomorphism $H_*(\Lambda) \cong H_*(\Lambda^{\mathrm{op}})$.

Examples 3.9.

(1) Let T_0 be the 0-graph of Examples 2.2 (1). Then $Q_0(T_0) = \{0\}$ and $Q_r(T_0) = \emptyset$ for all $r \geq 1$. Hence $C_0(T_0) = \mathbb{Z}\{0\}$ and $C_r(T_0) = \{0\}$ for all $r \geq 1$. Since $\partial_r = 0$ for all $r \geq 0$, we have $H_0(T_0) = \mathbb{Z}\{0\} \cong \mathbb{Z}$ and $H_r(T_0) = \{0\}$ for $r \geq 1$.

(2) More generally, for $k \geq 1$, we have $Q_0(T_k) = \{0\}$, $Q_r(T_k) = \emptyset$ for all $r > k$ and

$$Q_r(T_k) = \{e_{i_1} + \cdots + e_{i_r} \mid 1 \leq i_1 < \cdots < i_r \leq k\}$$

for $1 \leq r \leq k$. Thus $|Q_r(T_k)| = \binom{k}{r}$ for $0 \leq r \leq k$.

For $1 \leq j \leq r \leq k$, we have $F_j^0 = F_j^1$, so $\partial_r = 0$. Hence

$$H_r(T_k) = \mathbb{Z}Q_r(T_k) \cong \mathbb{Z}^{\binom{k}{r}} \qquad \text{for } 0 \leq r \leq k,$$

and $H_r(T_k) = \{0\}$ for $r > k$. In particular T_k has the same homology as the k-torus $\mathbb{T}^k$.

Definition 3.10. Let Λ be a k-graph and let (g, m) be an undirected path in Λ (see Definition 2.7). Then

$$h = \sum_{i=1}^{n} m_i g_i \in C_1(\Lambda)$$

is called the *trail* associated to (g, m). If (g, m) is closed, then h is said to be a *closed trail.* If in addition (g, m) is simple, then h is called a *simple closed trail.*

Remark 3.11. Let (g, m) be an undirected path in Λ with source u and range v. A straightforward computation shows that $\partial_1(h) = u - v$ where h is the trail associated to (g, m). Hence, if h is a closed trail then $\partial_1(h) = 0$. If h is a closed trail and $a \in \mathbb{Z}$ is nonzero then ah is also a closed trail.

Proposition 3.12. *Let Λ be a connected k-graph, then $H_0(\Lambda) \cong \mathbb{Z}$.*

Proof. Define a homomorphism $\theta : C_0(\Lambda) \to \mathbb{Z}$ by $\theta(v) = 1$ for all $v \in \Lambda^0$. It suffices to show that $\ker(\theta) \subset \mathrm{Im}(\partial_1)$, as the reverse inclusion is clear.

Fix distinct $u, v \in \Lambda^0$. Since Λ is connected there is an undirected path (g, m) from u to v. By Remark 3.11 $\partial_1(h) = u - v$ where h is the trail associated to (g, m). In particular, $u - v \in \mathrm{Im}(\partial_1)$.

Let $a = \sum_{i=1}^n m_i v_i \in \ker(\theta)$ with distinct v_i and $m_i \neq 0$ for all i. We prove by induction on $n \geq 2$ that $\sum_{i=1}^n m_i v_i \in \operatorname{Im}(\partial_1)$. When $n = 2$ we must have $m_1 + m_2 = 0$. The preceding paragraph yields a trail h such that $\partial_1(h) = v_1 - v_2$, and then $a = \partial_1(m_1 h) \in \operatorname{Im}(\partial_1)$.

Fix $n \geq 3$ and suppose the result holds for all ℓ with $n > \ell \geq 2$. Relabeling if necessary, we may assume that m_1 and m_2 have opposite sign, and $|m_1| \leq |m_2|$. We give a proof for the case $m_1 > 0$, the case $m_1 < 0$ being similar. Since Λ is connected there is an undirected path (g_1, m_1) from v_1 to v_2. Let $h_1 \in C_1(\Lambda)$ be the associated trail. Then $\partial_1(h_1) = v_1 - v_2$ and

$$a_1 = a - \partial_1(m_1 h_1) = (m_2 + m_1)v_2 + \sum_{i=3}^n m_i v_i.$$

By the inductive hypothesis $a_1 \in \operatorname{Im}(\partial_1)$ and so $a = a_1 + \partial(m_1 g_1) \in \operatorname{Im}(\partial_1)$. □

Combining Proposition 3.12, Remark 3.7 and Remark 2.9 gives the following.

Corollary 3.13. *Let Λ be a k-graph with p connected components (where $p \in \{1, 2, \dots\} \cup \{\infty\}$). Then $H_0(\Lambda) \cong \mathbb{Z}^p$. In particular Λ is connected if and only if $H_0(\Lambda) \cong \mathbb{Z}$.*

Example 3.14. Since Δ_1 is connected we have $H_0(\Delta_1) \cong \mathbb{Z}$ by Proposition 3.12. We claim that $H_r(\Delta_1) = 0$ for all $r \geq 1$. By Remark 3.6 it suffices to check that $H_1(\Delta_1) = \{0\}$. To see this fix $f \in C_1(\Delta_1) \setminus \{0\}$. Then we may express $f = \sum_{i=\ell}^m a_i(i, i+1)$, where $a_i \in \mathbb{Z}$ and $a_m \neq 0$. Then

$$\begin{aligned}\partial_1(f) &= \sum_{i=\ell}^m a_i\left((i+1, i+1) - (i, i)\right)\\ &= a_m(m+1, m+1) - a_\ell(\ell, \ell) + \sum_{i=\ell+1}^m (a_{i-1} - a_i)(i, i).\end{aligned}$$

Since $a_m \neq 0$ it follows that $\partial_1(f) \neq 0$. So ∂_1 is injective and hence $H_1(\Delta_1) = \ker(\partial_1)$ is trivial.

Proposition 3.15. *Let Λ be a k-graph. For each $a \in \ker \partial_1$, there exist simple closed trails $h_1, \dots, h_n$ in $C_1(\Lambda)$ such that $a = \sum_{i=1}^n m_i h_i$.*

Proof. For $a = \sum_{i=1}^n a_i f_i \in \ker \partial_1$ where the f_i are distinct elements of $Q_1(\Lambda)$, set $N(a) := \sum_{i=1}^n |a_i|$. We proceed by induction on $N(a)$. If $N(a) = 0$, the result is trivial. Fix $N > 0$ and suppose as an inductive hypothesis that whenever $N(a) < N$, there are simple closed trails h_i and integers m_i such that $a = \sum_{i=1}^n m_i h_i$. Fix a with $N(a) = N$. It suffices to show that there is a simple closed trail $h \in C_1(\Lambda)$ such that $N(a - h) < N(a)$.

Recall from Definition 2.7 that if $p \in \{1, -1\}$ and $f \in Q_1(\Lambda)$, then $s(f, p)$ means $s(f)$ if $p = 1$ and $r(f)$ if $p = -1$; and $r(f, p) = s(f, -p)$.

Express $a = \sum_{i=1}^n a_i f_i$ where the f_i are distinct elements of $Q_1(\Lambda)$, and each $a_i \neq 0$. Let $i_1 := 1$, let $p_1 := \operatorname{sign}(a_1)$. If $s(f_1) = r(f_1)$, then $h := p_1 f_1$ has the desired property. Otherwise, let $v_0 = r(f_1, p_1)$ and $v_1 = s(f_1, p_1)$. Since the coefficient of v_1 in $\partial_1(a)$ is zero, there must exist i_2 such that the coefficient of v_1 in $\partial_1(a_{i_2} f_{i_2})$ is nonzero with the opposite sign to that in $\partial_1(p_1 f_{i_1})$; let $p_2 := \operatorname{sign}(a_{i_2})$ and let $v_2 = s(f_{i_2}, p_2)$. Observe that $r(f_{i_2}, p_2) = s(f_{i_1}, p_1)$. We may continue iteratively, as long as the v_i are all distinct, to chose an index i_j such that $p_j := \operatorname{sign}(a_{i_j})$ has the property that the coefficient of v_{j-1} in $\partial_1(p_j f_{i_j})$ has the opposite sign to that in $\partial_1(p_{j-1} f_{i_{j-1}})$ for each j. We then set

$v_j := s(f_{i_j}, p_j)$, and observe that $r(f_{i_j}, p_j) = v_{j-1}$. Since there are only finitely many nonzero coefficients in a, this process must terminate: there is a first l such that $v_l \in \{v_0, v_1, \ldots, v_{l-1}\}$; say $v_l = v_q$ where $q < l$. Then $h := \sum_{j=q+1}^{l} p_j f_{i_j}$ is a simple closed trail. Since $p_j = \operatorname{sign}(a_{i_j})$ for each j, we have $N(a-h) = N(a)-(l-q) < N(a)$ as required. □

4. Fundamental results

In this section we prove versions of a number of standard results in homology theory which suggest that our notion of homology for k-graphs is a reasonable one. In Appendix A, we will show that each k-graph determines in a fairly natural way a cubical set, and that our homology then agrees with that of Grandis [14]. So a number of results in this section could be recovered from Grandis' work. However, it seems worthwhile to present self-contained proofs which are consistent with the notation and conventions associated with k-graphs.

We begin with a version of the Künneth formula for our homology (see Theorem 4.3). In order to do this we must show how our chain complexes behave with respect to cartesian product of k-graphs.

Recall from Example 2.5 that given a cartesian product graph $\Lambda_1 \times \Lambda_2$ there are quasi-morphisms $\psi_i : \Lambda_1 \times \Lambda_2 \to \Lambda_i$ consistent with the projections $\pi_i : \mathbb{N}^{k_1+k_2} \to \mathbb{N}^{k_i}$.

Lemma 4.1. *Let (Λ_i, d_i) be a k_i-graph for $i = 1, 2$ and $\Lambda_1 \times \Lambda_2$ the associated cartesian product $(k_1 + k_2)$-graph. Then for $r \geq 0$, we have $Q_r(\Lambda) = \bigsqcup_{r_1+r_2=r} Q_{r_1}(\Lambda_1) \times Q_{r_2}(\Lambda_2)$. Hence there is an isomorphism*

$$\Psi_r : C_r(\Lambda_1 \times \Lambda_2) \cong \bigoplus_{r_1+r_2=r} C_{r_1}(\Lambda_1) \otimes C_{r_2}(\Lambda_2) \tag{4.1}$$

given by $\Psi_r(\lambda_1, \lambda_2) = \lambda_1 \otimes \lambda_2$.

Proof. For the first assertion, just note that $(d_1 \times d_2)(\lambda_1, \lambda_2) \leq \mathbf{1}_{k_1+k_2}$ if and only if $d_i(\lambda_i) \leq \mathbf{1}_{k_i}$ for $i = 1, 2$. So

$$\begin{aligned} Q_r(\Lambda_1 \times \Lambda_2) &= \{(\lambda_1, \lambda_2) : (d_1 \times d_2)(\lambda_1, \lambda_2) \leq \mathbf{1}_{k_1+k_2}, |\lambda_1| + |\lambda_2| = r\} \\ &= \bigsqcup_{r_1+r_2=r} \{(\lambda_1, \lambda_2) : d_i(\lambda_i) \leq \mathbf{1}_{k_i}, |\lambda_i| = r_i \text{ for } i = 1, 2\} \\ &= \bigsqcup_{r_1+r_2=r} Q_{r_1}(\Lambda_1) \times Q_{r_2}(\Lambda_2). \end{aligned}$$

The second assertion follows from Remark 2.14. □

Recall from [29, V.9] that if K and L are chain complexes with boundary maps $\partial_r^K : K_r \to K_{r-1}$ and $\partial_r^L : L_r \to L_{r-1}$, then the tensor complex $K \otimes L$ is given by

$$(K \otimes L)_r = \bigoplus_{r_1+r_2=r} K_{r_1} \otimes L_{r_2},$$

with boundary maps

$$\partial_{r_1+r_2}^{K\otimes L}(k \otimes l) := \partial_{r_1}^K(k) \otimes l + (-1)^{r_1} k \otimes \partial_{r_2}^L(l) \quad \text{for all } k \in K_{r_1} \text{ and } l \in L_{r_2}. \tag{4.2}$$

The following is an analog of [14, Theorem 2.7].

Proposition 4.2. *Let Λ_i be a k_i-graph for $i = 1, 2$. The isomorphisms Ψ_r of Lemma 4.1 induce an isomorphism of complexes $\Psi : C_*(\Lambda_1 \times \Lambda_2) \to C_*(\Lambda_1) \otimes C_*(\Lambda_2)$.*

Proof. Fix r_1, r_2 such that $0 \le r_i \le k_i$ for $i = 1, 2$ and set $r = r_1 + r_2$. Let $\lambda_i \in Q_{r_i}(\Lambda_i)$ $(i = 1, 2)$. Then for each $0 \le j \le k_1 + k_2$ and $\ell \in \{0, 1\}$,

$$F_j^\ell(\lambda_1, \lambda_2) = \begin{cases} (F_j^\ell(\lambda_1), \lambda_2) & \text{if } 1 \le j \le r_1 \\ (\lambda_1, F_{j-r_1}^\ell(\lambda_2)) & \text{if } r_1 + 1 \le j \le r_1 + r_2. \end{cases}$$

Hence by (3.1) we may calculate:

$$\begin{aligned} \partial_r(\lambda_1, \lambda_2) &= \sum_{\ell=0}^{1} \sum_{j=1}^{r} (-1)^{\ell+j} F_j^\ell(\lambda_1, \lambda_2) \\ &= \sum_{\ell=0}^{1} \Big(\sum_{j=1}^{r_1} (-1)^{\ell+j} (F_j^\ell(\lambda_1), \lambda_2) + \sum_{j=r_1+1}^{r_1+r_2} (-1)^{\ell+j} (\lambda_1, F_{j-r_1}^\ell(\lambda_2)) \Big) \\ &= \sum_{\ell=0}^{1} \sum_{j=1}^{r_1} (-1)^{\ell+j} (F_j^\ell(\lambda_1), \lambda_2) + \sum_{\ell=0}^{1} \sum_{h=1}^{r_2} (-1)^{\ell+h+r_1} (\lambda_1, F_h^\ell(\lambda_2)) \\ &= (\partial_{r_1}(\lambda_1), \lambda_2) + (-1)^{r_1} (\lambda_1, \partial_{r_2}(\lambda_2)). \end{aligned} \tag{4.3}$$

It remains to show that for all r,

$$\partial_r(\Psi_r(\lambda_1, \lambda_2)) = \Psi_{r-1}(\partial_r(\lambda_1, \lambda_2)).$$

By definition of the boundary map ∂_r on $C_{r_1}(\Lambda_1) \otimes C_{r_2}(\Lambda_2)$ (see (4.2)), we have

$$\begin{aligned} \partial_r(\Psi_r(\lambda_1, \lambda_2)) &= \partial_r(\lambda_1 \otimes \lambda_2) \\ &= \partial_{r_1}(\lambda_1) \otimes \lambda_2 + (-1)^{r_1} \lambda_1 \otimes \partial_{r_2}(\lambda_2) \\ &= \Psi_{r-1}(\partial_{r_1}(\lambda_1), \lambda_2) + (-1)^{r_1} (\lambda_1, \partial_{r_2}(\lambda_2)), \end{aligned}$$

and this is equal to $\Psi_{r-1}(\partial_r(\lambda_1, \lambda_2))$ by (4.3). □

We may now state a Künneth formula for our homology. The map α was considered in [14, Theorem 2.7].

Theorem 4.3. *Let Λ_i be a k_i-graph for $i = 1, 2$. Then there is a split exact sequence*

$$0 \to \bigoplus_{r_1+r_2=r} H_{r_1}(\Lambda_1) \otimes H_{r_2}(\Lambda_2) \xrightarrow{\alpha} H_r(\Lambda_1 \times \Lambda_2) \xrightarrow{\beta} \bigoplus_{r_1+r_2=r-1} \operatorname{Tor}\big(H_{r_1}(\Lambda_1), H_{r_2}(\Lambda_2)\big) \to 0.$$

The homomorphisms α and β are natural with respect to maps induced by k-graph morphisms, but the splitting is not natural.

Proof. The result follows from Proposition 4.2 and [29, Theorem V.10.4] using the fact that $C_r(\Lambda)$ is torsion free for each r. □

Corollary 4.4. *Let Λ_i be a k_i-graph for $i = 1, 2$. Suppose that for some i the groups $H_r(\Lambda_i)$ are all torsion-free. Then the map α in Theorem 4.3 is an isomorphism, so*

$$H_r(\Lambda_1 \times \Lambda_2) \cong \bigoplus_{r_1+r_2=r} H_{r_1}(\Lambda_1) \otimes H_{r_2}(\Lambda_2).$$

Example 4.5. For $k \ge 2$, we have $T_k \cong T_1 \times \cdots \times T_1$ by Examples 2.3 (1). We claim that for $0 \le r \le k$ we have

$$H_r(T_k) \cong \mathbb{Z}^{\binom{k}{r}}.$$

For $k = 0, 1$ this follows by Examples 3.9. The general case follows by induction on k using Corollary 4.4.

Definition 4.6. We say that a k-graph Λ is *acyclic* if $H_0(\Lambda) \cong \mathbb{Z}$ and $H_r(\Lambda) = 0$ for all $r \geq 1$.

Remark 4.7. Let Λ_i be an acyclic k_i-graph for $i = 1, 2$. Then by Corollary 4.4 it follows that $\Lambda_1 \times \Lambda_2$ is an acyclic $k_1 + k_2$-graph.

Examples 4.8.

(1) Note that by Examples 2.3 (2) we have $\Delta_k \cong \Delta_1 \times \cdots \times \Delta_1$ for $k \geq 2$. By Example 3.14 Δ_1 is acyclic, and so by Remark 4.7 it follows that Δ_k is acyclic for all k. Indeed for $k \geq 1$ the k-graph Δ_k has the same homology as $\mathbb{R}^k$.

(2) Let Λ be a connected 1-graph which is a tree. By Proposition 3.12 we have $H_0(\Lambda) \cong \mathbb{Z}$. Since Λ contains no closed undirected paths, $C_1(\Lambda)$ has no closed trails. Thus by Proposition 3.15 $\ker(\partial_1) = 0$ and hence $H_1(\Lambda) = 0$. Since $H_r(\Lambda) = 0$ for $r > 1$, it follows that Λ is acyclic.

The proof of the next result follows the argument used in [4, II.4.1]. This result may also be deduced from [14, Theorem 3.3] using the identification of our homology with that of the corresponding cubical set established in Theorem A.9.

Theorem 4.9. *Suppose that Λ is an acyclic k-graph. If G is a discrete group acting freely on Λ, then $H_*(\Lambda/G) \cong H_*(G, \mathbb{Z})$.*

Proof. If M is a G-module, then we write DM for the submodule of M generated by the elements $\{gm - m : m \in M, g \in G\}$. We write M_G for M/DM. Note that $M \mapsto M_G$ is a functor from the category of G-modules to the category of abelian groups (so it maps a complex of G-modules to a complex of abelian groups). If G acts on a set X then $\mathbb{Z}X$ may be regarded as a G-module and $\mathbb{Z}X_G \cong \mathbb{Z}(X/G)$ (see [4, § II.2]).

Since G acts freely on Λ, it acts freely on each $Q_r(\Lambda)$. Thus $C_r(\Lambda) = \mathbb{Z}Q_r(\Lambda)$ is a free G-module. We have

$$\mathbb{Z}Q_r(\Lambda)_G \cong \mathbb{Z}Q_r(\Lambda/G).$$

Moreover, this isomorphism is compatible with the boundary maps. So if $C_*(\Lambda)_G$ denotes the complex obtained from $C_*(\Lambda)$ by applying the functor $M \mapsto M_G$, then $C_*(\Lambda)_G \cong C_*(\Lambda/G)$. Since Λ is acyclic, the sequence

$$\cdots \xrightarrow{\partial_3} C_2(\Lambda) \xrightarrow{\partial_2} C_1(\Lambda) \xrightarrow{\partial_1} C_0(\Lambda) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

is a resolution of $\mathbb{Z}$ by free G-modules. Since the complex $C_*(\Lambda)_G$ is isomorphic to the complex $C_*(\Lambda/G)$, we have

$$H_*(C_*(\Lambda)_G) \cong H_*(C_*(\Lambda/G)) = H_*(\Lambda/G).$$

Therefore, $H_*(G, \mathbb{Z}) \cong H_*(\Lambda/G)$. □

Recall that the fundamental group $\pi_1(\Lambda)$ of a connected 1-graph Λ is free (see for example [42, §2.1.8] or [24, §4]) and the universal cover T is a tree. Thus Λ may be realised as the quotient of T by the action of $\pi_1(\Lambda)$; moreover, if Λ has finitely many vertices and edges, then $\pi_1(\Lambda) \cong \mathbb{F}_p$, where $\mathbb{F}_p$ is the free group on p generators and $p = |\Lambda^1| - |\Lambda^0| + 1$ (see [39, §I.3.3, Theorem 4]). Since T is acyclic, we obtain the following result.

Corollary 4.10. *Let Λ be a connected 1-graph. Then $H_1(\Lambda) \cong H_1(\pi_1(\Lambda), \mathbb{Z})$. In particular if Λ has finitely many vertices and edges, then $\pi_1(\Lambda) \cong \mathbb{F}_p$ where $p = |\Lambda^1| - |\Lambda^0| + 1$ and so*

$$H_1(\Lambda) \cong H_1(\mathbb{F}_p, \mathbb{Z}) \cong \mathbb{Z}^p.$$

Examples 4.11. (1) Recall from Examples 2.2 (2) that B_n is the path category of a directed graph with a single vertex and n edges, regarded as a 1-graph. The universal cover of B_n, which we denote A_n, is the skew-product $B_n \times_c \mathbb{F}_n$, and can be identified with the Cayley graph of $\mathbb{F}_n$, the free group on n generators. By [25, Remark 5.6] $\mathbb{F}_n$ acts freely on A_n with $A_n/\mathbb{F}_n \cong B_n$. By Corollary 4.10 we have $H_1(B_n) \cong \mathbb{Z}^n$. Hence B_n has the same homology as the wedge of n circles.

(2) Let H be a subgroup of $\mathbb{Z}^k$. Then as in [26, §6.4] H acts freely on Δ_k. Since Δ_k is acyclic, by Theorem 4.9 we have $H_*(\Delta_k/H) \cong H_*(H, \mathbb{Z})$. If $H \cong \mathbb{Z}^q$, then for $0 \le r \le k$ we have (cf. Example 4.5)

$$H_r(\Delta_k/H) \cong \mathbb{Z}^{\binom{q}{r}}.$$

Hence Δ_k/H has the same homology as the q-torus. If H has finite index then $q = k$ and the quotient graph Δ_k/H may be viewed as yet another k-graph analog of the k-torus (note $\Delta_k/H = T_k$ when $H = \mathbb{Z}^k$).

(3) The following example indicates that Theorem 4.9 is in practise less useful than it might appear because it is difficult to recognise acyclic k-graphs (short of explicitly computing their homology). In particular one might expect that a pullback of an acyclic k-graph by a full-rank endomorphism of $\mathbb{N}^k$ is itself acyclic, but this is not so.

Let Λ be the 2-graph with $\Lambda^0 = \{v\}$, $\Lambda^{e_1} = \{a_1, a_2\}$, $\Lambda^{e_2} = \{b_1, b_2\}$, and factorisation property determined by $a_i b_j = b_i a_j$ for $i, j = 1, 2$. Recall that we denote the generators of $\mathbb{F}_2$ by h_1 and h_2. There is a functor $\sigma : \Lambda \to \mathbb{F}_2 \times \mathbb{Z}$ determined by $\sigma(a_i) = (h_i, 0)$ and $\sigma(b_i) = (h_i, 1)$. Let $\Gamma := \Lambda \times_\sigma (\mathbb{F}_2 \times \mathbb{Z})$, and observe that by [25, Remark 5.6] $\mathbb{F}_2 \times \mathbb{Z}$ acts freely on Γ with quotient Λ.

Let $A_2 = B_2 \times_c \mathbb{F}_2$ as in (1) above. Define $g : \mathbb{N}^2 \to \mathbb{N}^2$ by $g(m, n) := (m+n, n)$. Tedious calculations show that Γ is isomorphic to the pullback $g^*(A_2 \times \Delta_1)$[1].

We claim that Γ is not acyclic. Suppose that it is. Then Theorem 4.9 implies that $H_*(\Lambda) \cong H_*(\mathbb{F}_2 \times \mathbb{Z})$. By the Künneth theorem for group homology, since both $H_r(\mathbb{F}_2)$ and $H_r(\mathbb{Z})$ are trivial for $r \ge 2$,

$$H_2(\mathbb{F}_2 \times \mathbb{Z}) = H_1(\mathbb{F}_2) \otimes H_1(\mathbb{Z}) \cong \mathbb{Z}^2.$$

A straightforward computation shows that $a_1 b_1$, $a_2 b_2$ and $a_1 b_2 + a_2 b_1$ all belong to $\ker(\partial_2) = H_2(\Lambda)$, so the latter has rank at least three, giving a contradiction.

So Γ is not acyclic, despite being a pull-back of the acyclic graph $A_2 \times \Delta_1$ (see Remark 4.7) by the full-rank endomorphism g.

We now turn our attention to exact sequences of homology groups associated to automorphisms of k-graphs. Recall from [12] that if Λ is a k-graph and α is an automorphism of Λ, then there is a $(k+1)$-graph $\Lambda \times_\alpha \mathbb{Z}$ with morphisms $\Lambda \times \mathbb{N}$, range and source maps given by $r(\lambda, n) = (r(\lambda), 0)$, $s(\lambda, n) = (\alpha^{-n}(s(\lambda)), 0)$, degree map given

[1]This is not meant to be obvious. After unraveling the definitions of Γ and of $g^*(A_2 \times \Delta_1)$, one can check that the formulas $(a_i, (h, n)) \mapsto \big(((f_i, h), (n, n)), (1, 0)\big)$ and $(b_i, (h, n)) \mapsto \big(((f_i, h), (n, n+1)), (0, 1)\big)$ for $i = 1, 2$ determine the desired isomorphism.

by $d(\lambda, n) = (d(\lambda), n)$ and composition given by $(\lambda, m)(\mu, n) := (\lambda\alpha^m(\mu), m+n)$. In particular $(\Lambda \times_\alpha \mathbb{Z})^0 = \Lambda^0 \times \{0\}$.

We may describe the cubes of $\Lambda \times_\alpha \mathbb{Z}$ in terms of those of Λ as follows: $Q_0(\Lambda \times_\alpha \mathbb{Z}) = Q_0(\Lambda) \times \{0\}$ and for $0 \le r \le k$ an element of $Q_{r+1}(\Lambda \times_\alpha \mathbb{Z})$ is of the form $(\lambda, 0)$ where $\lambda \in Q_{r+1}(\Lambda)$ or $(\lambda, 1)$ where $\lambda \in Q_r(\Lambda)$, so

$$Q_{r+1}(\Lambda \times_\alpha \mathbb{Z}) = (Q_{r+1}(\Lambda) \times \{0\}) \sqcup (Q_r(\Lambda) \times \{1\}). \tag{4.4}$$

Given an element $a = \sum a_\lambda \lambda \in C_r(\Lambda)$, we shall somewhat inaccurately write $(a, 0) := \sum a_\lambda(\lambda, 0)$ and $(a, 1) := \sum a_\lambda(\lambda, 1)$ for the corresponding elements of $C_r(\Lambda \times_\alpha \mathbb{Z})$ and $C_{r+1}(\Lambda \times_\alpha \mathbb{Z})$. With this notation, the boundary map on $C_{r+1}(\Lambda \times_\alpha \mathbb{Z})$ is given by

$$\begin{aligned} \partial_{r+1}(\lambda, 0) &= (\partial_{r+1}(\lambda), 0) \quad \text{and} \\ \partial_{r+1}(\mu, 1) &= (-1)^r\big((\alpha^{-1}(\mu), 0) - (\mu, 0)\big) + (\partial_r(\mu), 1). \end{aligned} \tag{4.5}$$

We will deduce our long exact sequence for the homology of $\Lambda \times_\alpha \mathbb{Z}$ from the long exact sequence associated to a mapping-cone complex arising from the chain map $\alpha^{-1} - 1$ (see [29, Proposition II.4.3]). So we recall the definition of the mapping cone complex. Given a chain map $f : A_* \to B_*$, define a complex $M_* = M(f)_*$ by $M_r := A_{r-1} \oplus B_r$ (with the convention that $A_{-1} = \{0\}$) with boundary map

$$\partial_r(a, b) := (-\partial_{r-1}(a), \partial_r(b) + f(a)). \tag{4.6}$$

If α is an automorphism of a k-graph Λ, then α^{-1} maps cubes to cubes and intertwines boundary maps, and so induces a chain map $\alpha^{-1} : C_*(\Lambda) \to C_*(\Lambda)$. Hence $\alpha^{-1} - 1$ is also a chain map from $C_*(\Lambda)$ to itself.

Lemma 4.12. *Let Λ be a k-graph and let α be an automorphism of Λ. Then there is an isomorphism of chain complexes $\psi : C_*(\Lambda \times_\alpha \mathbb{Z}) \to M(\alpha^{-1} - 1)_*$ such that*

$$\psi(\lambda, 0) = (0, \lambda) \quad \text{and} \quad \psi(\mu, 1) = ((-1)^r\mu, 0)$$

for all $(\lambda, 0), (\mu, 1) \in Q_{r+1}(\Lambda \times_\alpha \mathbb{Z})$. Hence, $\psi_* : H_*(\Lambda \times_\alpha \mathbb{Z}) \to H_*(M(\alpha^{-1} - 1)_*)$ is an isomorphism.

Proof. Write $M_* := M(\alpha^{-1} - 1)_*$ and $C_* := C_*(\Lambda \times_\alpha \mathbb{Z})$. It is clear that ψ determines isomorphisms of groups $C_r \cong M_r$. So to see that ψ is an isomorphism of complexes, it suffices to show that it intertwines the boundary maps on generators. We consider cubes of the form $(\lambda, 0)$ and those of the form $(\mu, 1)$ separately. Fix $\lambda \in Q_{r+1}(\Lambda)$. We have $\partial_{r+1}(\psi(\lambda, 0)) = \partial_{r+1}(0, \lambda) = (0, \partial_{r+1}(\lambda))$ by (4.6), and $\psi(\partial_{r+1}(\lambda, 0)) = \psi(\partial_{r+1}(\lambda), 0) = (0, \partial_{r+1}(\lambda))$ by (4.5). So $\partial_{r+1}(\psi(\lambda, 0)) = \psi(\partial_{r+1}(\lambda, 0))$ as required.

Now fix $\mu \in Q_r(\Lambda)$. Then we have

$$\begin{aligned} \psi(\partial_{r+1}(\mu, 1)) &= \psi\big((-1)^r\big((\alpha^{-1}(\mu), 0) - (\mu, 0)\big) + (\partial_r(\mu), 1)\big) \\ &= (-1)^r\big(\psi(\alpha^{-1}(\mu), 0) - \psi(\mu, 0)\big) + \psi(\partial_r(\mu), 1) \\ &= (-1)^r(0, (\alpha^{-1} - 1)(\mu)) + (-1)^{r-1}(\partial_r(\mu), 0) \\ &= (-1)^r(-\partial_r(\mu), (\alpha^{-1} - 1)(\mu)). \end{aligned}$$

On the other hand,

$$\begin{aligned}\partial_{r+1}\psi((\mu,1)) &= (-1)^r\partial_{r+1}(\mu,0)\\ &= (-1)^r(-\partial_r(\mu),\partial_{r+1}(0)+(\alpha^{-1}-1)(\mu))\\ &= (-1)^r(-\partial_r(\mu),(\alpha^{-1}-1)(\mu)).\end{aligned}$$

□

Now recall from [29, Proposition 4.3], that a chain map $f : A_* \to B_*$ determines a long exact sequence

$$\cdots \to H_r(B_*) \xrightarrow{\iota_*} H_r(M(f)_*) \xrightarrow{\pi_*} H_{r-1}(A_*) \xrightarrow{f_*} H_{r-1}(B_*) \to \cdots \tag{4.7}$$

where $\iota_* : H_r(B_*) \to H_r(M(f)_*)$ is induced by the inclusion map $\iota : B_r \to M(f)_r$, and $\pi_* : H_r(M(f)_*) \to H_{r-1}(A_*)$ is induced by the projection $\pi : M(f)_r \to A_{r-1}$.

The following result gives an exact sequence which may be regarded as an analog of the Pimsner-Voiculescu sequence for crossed products of C^*-algebras (cf. [35, Theorem 2.4], [3, Theorem 10.2.1]).

Theorem 4.13. *Let Λ be a k-graph, and let α be an automorphism of Λ. Then there is an exact sequence*

$$\begin{aligned}0 \longrightarrow H_{k+1}(\Lambda\times_\alpha\mathbb{Z}) \xrightarrow{\pi_*} H_k(\Lambda) \xrightarrow{1-\alpha_*} H_k(\Lambda) \xrightarrow{\iota_*} H_k(\Lambda\times_\alpha\mathbb{Z}) \longrightarrow \cdots\\ \cdots \longrightarrow H_1(\Lambda\times_\alpha\mathbb{Z}) \xrightarrow{\pi_*} H_0(\Lambda) \xrightarrow{1-\alpha_*} H_0(\Lambda) \xrightarrow{\iota_*} H_0(\Lambda\times_\alpha\mathbb{Z}) \longrightarrow 0.\end{aligned}$$

Proof. The long exact sequence (4.7) applied with $f = \alpha^{-1}-1$ together with Lemma 4.12 (and identifying $H_*(\Lambda\times_\alpha\mathbb{Z}) \cong H_*(M(\alpha^{-1}-1)_*)$) gives a long exact sequence

$$\begin{aligned}0 \longrightarrow H_{k+1}(\Lambda\times_\alpha\mathbb{Z}) \xrightarrow{\pi_*} H_k(\Lambda) \xrightarrow{\alpha_*^{-1}-1} H_k(\Lambda) \xrightarrow{\iota_*} H_k(\Lambda\times_\alpha\mathbb{Z}) \longrightarrow \cdots\\ \cdots \longrightarrow H_1(\Lambda\times_\alpha\mathbb{Z}) \xrightarrow{\pi_*} H_0(\Lambda) \xrightarrow{\alpha_*^{-1}-1} H_0(\Lambda) \xrightarrow{\iota_*} H_0(\Lambda\times_\alpha\mathbb{Z}) \longrightarrow 0.\end{aligned}$$

Since α_* is an automorphism of $H_r(\Lambda)$ which commutes with $\alpha_*^{-1}-1$, both $\ker(\alpha_*^{-1}-1)$ and $\operatorname{Im}(\alpha_*^{-1}-1)$ are α_*-invariant. Therefore

$$\ker(\alpha_*^{-1}-1) = \ker(\alpha_*(\alpha_*^{-1}-1)) = \ker(1-\alpha_*)$$

and similarly, $\operatorname{Im}(\alpha_*^{-1}-1) = \operatorname{Im}(1-\alpha_*)$. □

Remark 4.14. Theorem 4.13 may also be proved using the topological realizations, introduced in [21] (see also Section 6), of Λ and $\Lambda\times_\alpha\mathbb{Z}$. To see how, recall from [21, Lemma 2.23] that α induces a homeomorphism $\tilde{\alpha}$ of the topological realisation X_Λ of Λ, and that $X_{\Lambda\times_\alpha\mathbb{Z}}$ is homeomorphic to the mapping torus $M(\tilde{\alpha})$. Combining this with Theorem 6.3 and the long exact sequence of [17, Example 2.48] yields the result.

5. Examples

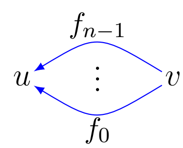
In this section we present some examples. We describe them using skeletons, so we first indicate what this means. Our examples are all 2-graphs (since there are already a number of interesting examples in this case), so we restrict ourselves to a discussion of skeletons for 2-graphs.

A 2-coloured graph is a directed graph E together with a map $c : E^1 \to \{1,2\}$. A complete collection of squares in E is a collection of relations of the form $ef \sim f'e'$ where $ef, f'e' \in E^2$ with $c(e) = c(e') = 1$ and $c(f) = c(f') = 2$ such that each bi-coloured path of length two appears in exactly one such relation[2]. It follows from [25, Section 6] (see also [18, Theorems 4.4 and 4.5]) that each pair consisting of a 2-coloured graph and a complete collection of pairs uniquely determines a 2-graph, and also that each 2-graph arises from such a pair $(E_\Lambda, \mathcal{C}_\Lambda)$. It is standard to refer to the equalities $ef = f'e'$ in Λ determined by the squares $ef \sim f'e'$ in $\mathcal{C}$ as the *factorisation rules.* We refer to E as the *skeleton* of Λ.

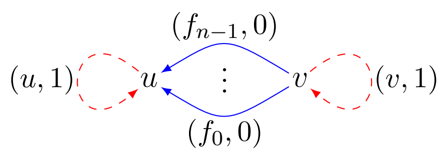
In our diagrams, edges of colour 1 are blue and solid, and edges of colour 2 are red and dashed.

Our first example is a 2-graph whose first homology group contains torsion. Combined with Example 5.2, it also demonstrates that the homology of a k-graph depends on the factorisation rules and not just on the skeleton.

Example 5.1. Fix $n > 1$ and consider the 1-graph Λ with skeleton



Define $\alpha \in \mathrm{Aut}(\Lambda)$ by $\alpha(f_i) = f_{i+1}$, where addition is modulo n (so α fixes vertices). Then $\Lambda \times_\alpha \mathbb{Z}$ (see page 12) is the 2-graph with skeleton



and factorisation rules $(f_i, 0)(v, 1) = (u, 1)(f_{i+1}, 0)$ for $i = 0, \dots, n-1$, where addition is modulo n.

We claim that

$$H_0(\Lambda \times_\alpha \mathbb{Z}) \cong \mathbb{Z}, \quad H_1(\Lambda \times_\alpha \mathbb{Z}) \cong \mathbb{Z} \oplus \mathbb{Z}/n\mathbb{Z}, \quad \text{and} \quad H_2(\Lambda \times_\alpha \mathbb{Z}) = \{0\}.$$

By Proposition 3.12 we have $H_0(\Lambda \times_\alpha \mathbb{Z}) \cong \mathbb{Z}$ and $H_0(\Lambda) \cong \mathbb{Z}$. Since α fixes vertices it follows that $\alpha_* : H_0(\Lambda) \to H_0(\Lambda)$ is the identity map. Hence $\ker(1 - \alpha_*) = H_0(\Lambda) \cong \mathbb{Z}$.

We next calculate $H_1(\Lambda)$. Since $C_2(\Lambda) = \{0\}$, we have $H_1(\Lambda) = \ker(\partial_1)$. Since $\partial_1(f_i) = u - v$ for all $0 \le i \le n-1$, and since $C_1(\Lambda) = \mathbb{Z}\{f_0, \dots, f_{n-1}\}$, we have

$$\{f_i - f_{i+1} : 0 \le i \le n-2\} \text{ is a basis for the } \mathbb{Z}\text{-module } H_1(\Lambda). \tag{5.1}$$

Let $b_i := f_i - f_{i+1}$ for $0 \le i \le n-2$ then $\alpha_*(b_i) = b_{i+1}$ for $0 \le i < n-2$, and

$$\alpha_*(b_{n-2}) = f_{n-1} - f_0 = -\sum_{i=0}^{n-2} b_i.$$

[2]Strictly speaking, in [18], a complete collection of squares is defined to be a collection $\mathcal{C}$ of coloured-graph morphisms from model coloured graphs E_{k,e_i+e_j} into Λ, and the relation $\sim$ is defined by $ef \sim f'e'$ if and only if the two paths traverse a common element of $\mathcal{C}$. But we can recover the collection of coloured-graph morphisms as in [18] from the relation $\sim$, so the two formalisms are equivalent.

Hence, regarded as an endomorphism of $\mathbb{Z}^{n-1}$, the map $1-\alpha_*$ is implemented by the $(n-1)\times(n-1)$ matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ -1 & 1 & 0 & \cdots & 0 & 1 \\ 0 & -1 & 1 & \cdots & 0 & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}.$$

Thus $\operatorname{Im}(1-\alpha_*)$ is spanned by the elements $b_i - b_{i+1}$ for $0 \le i \le n-3$ together with the element $b_{n-2} + \sum_{i=0}^{n-2} b_i$. Using this one checks that

$$\{b_0 - b_{n-2}, b_1 - b_{n-2}, \dots, b_{n-3} - b_{n-2}, nb_{n-2}\} \text{ is a basis for } \operatorname{Im}(1-\alpha_*). \tag{5.2}$$

From (5.1) one sees that

$$\{b_0 - b_{n-2}, b_1 - b_{n-2}, \dots, b_{n-3} - b_{n-2}, b_{n-2}\} \text{ is a basis for } H_1(\Lambda). \tag{5.3}$$

In particular, $\operatorname{rank}(\operatorname{Im}(1-\alpha_*)) = \operatorname{rank}(H_1(\Lambda))$, forcing $\ker(1-\alpha_*) = \{0\}$. Moreover, combining (5.3) with (5.2) shows that $\operatorname{coker}(1-\alpha_*) \cong \mathbb{Z}/n\mathbb{Z}$. Thus Theorem 4.13 implies that $H_2(\Lambda \times_\alpha \mathbb{Z}) = \{0\} = \ker(1-\alpha_*) = \{0\}$, and that $H_1(\Lambda \times_\alpha \mathbb{Z})$ is an extension of $\mathbb{Z}$ by $\mathbb{Z}/n\mathbb{Z}$ and hence is equal to $\mathbb{Z} \oplus (\mathbb{Z}/n\mathbb{Z})$. In particular, for $n = 2$, the graph $\Lambda \times_\alpha \mathbb{Z}$ has the same homology as the Klein bottle.

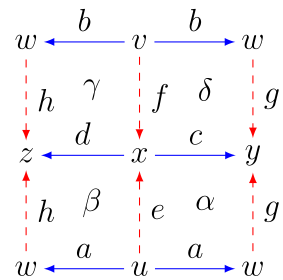
Example 5.2. Let T_1 be the 1-graph with a single vertex and a single edge as in Example 2.2(1), and let Λ and $\Gamma = \Lambda \times_\alpha \mathbb{Z}$ be as in Example 5.1 with $n = 2$. Then $\Lambda \times T_1$ has the same skeleton as Γ. To compute the homology of $\Lambda \times T_1$, we can use the Künneth theorem (Theorem 4.3): each of T_1 and Λ consists of a single simple closed undirected path, so it is routine to verify that $H_i(T_1) = H_i(\Lambda) = \mathbb{Z}$ for each of $i = 0, 1$. Hence $H_i(\Lambda \times T_1) = \mathbb{Z}^{\binom{2}{i}}$ for all i. So the homology of $\Lambda \times T_1$ is the same as that of the 2-torus (see Example 4.5), and in particular is not equal to that of Γ, even though they have the same skeleton.

We next describe a suite of examples of 2-graphs whose homology mirrors that of the sphere, the torus, the Klein bottle and the projective plane. We have presented examples matching the Klein bottle and the torus previously (see Examples 5.1 and 4.5), but we provide presentations here which suggest standard planar diagrams for these four spaces.

Remark 5.3. For a number of the following examples, we give a non-standard presentation of the skeleton and factorisation rules. Specifically for Examples 5.4–5.7, we present a commuting diagram (in the category Λ) which includes all 2-cubes as commuting squares. These diagrams are not the same as the skeletons because they involve some repeated vertices and edges. We present our examples this way to suggest planar diagrams for their topological realisations (see Section 6); indeed, we will sometimes refer to these commuting diagrams, very imprecisely, as planar diagrams for the associated 2-graphs.

When using this presentation of a 2-graph, one must check that the collection of squares specified in the diagram is complete: since vertices may be repeated in a planar diagram, it is possible that there are some bi-coloured paths in the skeleton which do not appear as the sides of a square in the diagram, and in this case, the diagram may not completely specify a 2-graph, and is in any case not a planar diagram for the 2-graph in the sense just discussed.

Example 5.4. Let Λ be the 2-graph described by the following planar diagram (see Remark 5.3).

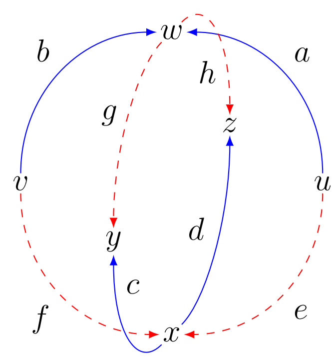


The skeleton of Λ is pictured in (5.4). The Greek letters in the centres of the commuting squares in the above diagram are the morphisms of degree $(1,1)$. So $\alpha = ce = ga$, $\beta = de = ha$, etc.

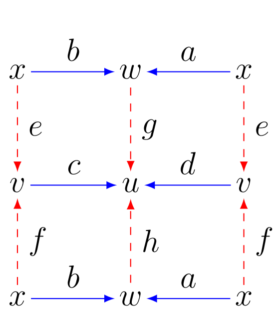
Since Λ is connected, $H_0(\Lambda) \cong \mathbb{Z}$ by Proposition 3.12. We have $\partial_2(\alpha - \beta + \gamma - \delta) = 0$ by a straightforward calculation and one can check that $\partial_2(n_1\alpha + n_2\beta + n_3\gamma) = 0$ implies $n_1 = n_2 = n_3 = 0$, so $H_2(\Lambda) = \ker(\partial_2) \cong \mathbb{Z}$. Moreover, $\partial_2(C_2(\Lambda))$ is spanned by $\partial_2(\alpha)$, $\partial_2(\beta)$ and $\partial_2(\gamma)$.

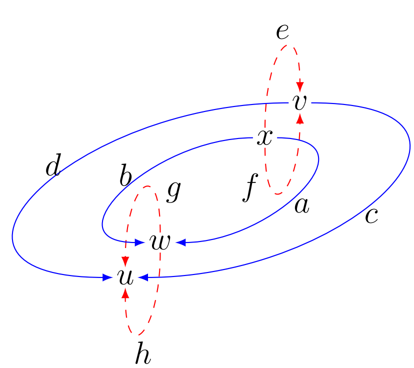
One checks that the set $\{\partial_2(\alpha), \partial_2(\beta), \partial_2(\gamma), d, e, f, g, h\}$ forms a basis for $C_1(\Lambda)$. So $C_1(\Lambda) = \partial_2(C_2(\Lambda)) \oplus \mathbb{Z}\{d, e, f, g, h\}$. Since $H_0(\Lambda) = \mathbb{Z}$ and $C_0(\Lambda)$ has rank 6, the image of ∂_1 has rank 5. It follows that $H_1(\Lambda) = \{0\}$. Hence Λ has the same homology as the sphere S^2. If we draw its skeleton as follows, the resemblance between Λ and a combinatorial sphere is striking.

(5.4)



Example 5.5. Consider the 2-graph Σ with planar diagram (see Remark 5.3) on the left and skeleton on the right in the following diagram.

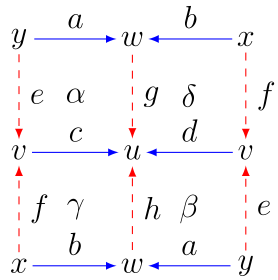


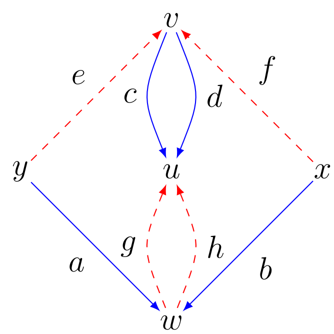


Let Λ be the 1-graph with two vertices connected by two parallel edges used in Example 5.2; we observed in the same example that the homology of Λ is that of the circle. Then Σ is isomorphic to $\Lambda \times \Lambda$, so by the Künneth theorem it has the homology of the 2-torus as in Example 5.2.

Example 5.6. We thank Mike Whittaker for his contributions to the construction and analysis of this example.

Let Λ be the 2-graph with planar diagram (see Remark 5.3) on the left and skeleton on the right in the following diagram. As above, the Greek letters in the centres of squares denote the morphisms in $\Lambda^{(1,1)}$ — so $\alpha = ga = ce$ etc.





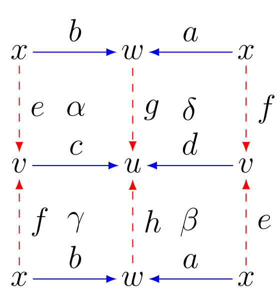
We claim that $H_0(\Lambda) \cong \mathbb{Z}$, $H_1(\Lambda) \cong \mathbb{Z}/2\mathbb{Z}$ and $H_2(\Lambda) = \{0\}$. Indeed, $C_0(\Lambda) = \mathbb{Z}\{u, v, w, x, y\}$, $C_1(\Lambda) = \mathbb{Z}\{a, b, c, d, e, f, g, h\}$ and $C_2(\Lambda) = \mathbb{Z}\{\alpha, \beta, \gamma, \delta\}$. Since Λ is connected, $H_0(\Lambda) \cong \mathbb{Z}$ which implies that $\partial_1(C_1(\Lambda))$ has rank 4. Since $\operatorname{rank} C_1(\Lambda) = 8$, $\operatorname{rank}\ker(\partial_1) = 4$ also. If $\partial_2(n_1\alpha + n_2\beta + n_3\gamma + n_4\delta) = 0$, then consideration of the coefficients of c and h forces $n_1 = -n_3 = n_2$, and then that the coefficient of a is zero forces $n_1 = n_2 = 0$, and hence $n_3 = 0$ also. Now considering the coefficient of d shows that $n_4 = -n_2 = 0$. So ∂_2 is injective, forcing $H_2(\Lambda) = \{0\}$, and also that $\operatorname{rank}\partial_2(C_2(\Lambda)) = 4$. We observed above that $\operatorname{rank}\ker(\partial_1) = 4$; hence

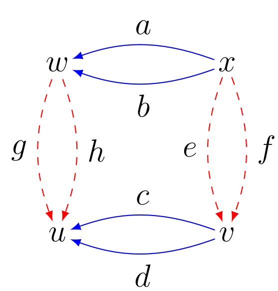
$$\operatorname{rank}(H_1(\Lambda)) = \operatorname{rank}(\ker(\partial_1)) - \operatorname{rank}(\partial_2(C_2(\Lambda))) = 0.$$

It is routine to check that $\{c-d, g-h, c+f-b-h, d+e-a-h\}$ is a basis for $\ker(\partial_1)$. To determine the image of ∂_2, first note that $c+f-b-h = \partial_2(\gamma)$ and $d+e-a-h = \partial_2(\beta)$. Moreover $(c-d)+(g-h)$ is the image of $\gamma-\delta$, which implies that $H_1(\Lambda)$ is generated by the class of $c-d$. Finally, $2(c-d) = \partial_2(\alpha-\beta+\gamma-\delta)$, and since $\{\alpha, \beta, \gamma, \alpha-\beta+\gamma-\delta)$ is a basis for $C_2(\Lambda)$, it follows that $H_1(\Lambda) \cong \mathbb{Z}/2\mathbb{Z}$ as required.

These homology groups are the same as those of the projective plane.

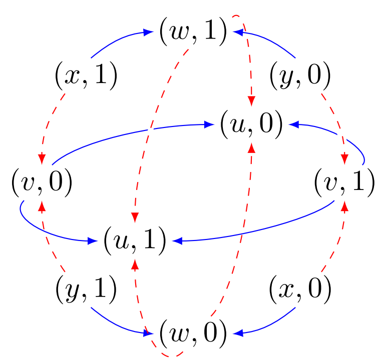
Example 5.7. Consider the 2-graph Λ with planar diagram (see Remark 5.3) on the left and skeleton on the right in the following diagram.





One can check, by calculating with bare hands, that the homology of this 2-graph is the same as that of the 2-graph $\Lambda \times_\alpha \mathbb{Z}$ of Example 5.1 with $n = 2$; that is, the same homology as the Klein bottle. Alternatively, one can deduce this from the topological realisation (see Remark 5.9 below).

Example 5.8. In Example 5.6, we realised the homology of the projective plane using a 2-graph Λ. This suggests that there ought to be a 2-graph with the homology of the sphere carrying a free action of $\mathbb{Z}/2\mathbb{Z}$ such that the quotient is isomorphic to Λ. By [25, Remark 5.6] (see also [34]), such a 2-graph must be a skew product of Λ by a functor taking values in $\mathbb{Z}/2\mathbb{Z}$. Here we present such an example. There is a functor $c : \Lambda \to \mathbb{Z}/2\mathbb{Z}$ determined by $c^{-1}(0) = \{b, c, g\}$ and $c^{-1}(1) = \{a, d, e, f, h\}$, and the skew-product graph $\Lambda \times_c \mathbb{Z}/2\mathbb{Z}$ has the desired property. The visual intuition that has pervaded this section appears again: one can check without too much difficulty that the skeleton of $\Lambda \times_c (\mathbb{Z}/2\mathbb{Z})$ can be drawn as follows (we have not labeled the edges since their labels can be deduced from the definition of the skew product and the labels of the vertices).



This picture suggests how to view the action of $\mathbb{Z}/2\mathbb{Z}$ on the skew-product graph as the action of the antipodal map on the sphere.

A similar situation arises for the Klein bottle and torus. Let Γ denote the crossed product graph $\Lambda \times_\alpha \mathbb{Z}$ of Example 5.1 with $n = 2$, so that the homology of Γ coincides with that of the Klein bottle. Let $c : \Gamma \to \mathbb{Z}/2\mathbb{Z}$ be the functor $c(\lambda, n) = n \pmod 2$. One can check that $\Gamma \times_c (\mathbb{Z}/2\mathbb{Z})$ is isomorphic to $\Lambda \times C_2$ where Λ is the 1-graph from Example 5.1 (with $n = 2$), and C_2 is the path category of the simple directed cycle of length 2. In particular, by the Künneth theorem, the homology of $\Gamma \times_c (\mathbb{Z}/2\mathbb{Z})$ is isomorphic to that of the torus. So our 2-graph representative Γ of the Klein bottle can be realised as a quotient of a 2-graph representative of a torus by a free $\mathbb{Z}/2\mathbb{Z}$ action.

Remark 5.9. As observed in [21], the topological realisations of the 2-graphs of Examples 5.4–5.7 (see Section 6) are indeed homeomorphic to each of the sphere, the torus, the projective plane and the Klein bottle as their homology suggests. In particular, Theorem 6.3 below combined with the descriptions of their topological realisations in [21] provide an alternative proof that these 2-graphs have the homology we have claimed for them.

6. Connection with homology of topological spaces

In this section, we show that the homology of the topological realisation X_Λ of a k-graph as defined in [21] agrees with the homology of Λ defined in §2. The corresponding fact for a cubical set was known already to Grandis: he indicates at the end of [14, Section 1.8] that the result is well known, with a reference to [32] for the simplicial case.

However, we have been unable to locate the details for cubical sets in the literature, so we include a proof of our result based on that given for simplicial complexes by Hatcher [17]. We prove in Appendix B that the topological realisation of a k-graph we define here is homeomorphic to the topological realisation $\mathcal{R}\widetilde{Q}(\Lambda)$ of the associated cubical set $\widetilde{Q}(\Lambda)$ (see Appendix A).

In [21], the topological realisation of a k-graph Λ is defined as follows. For $n \in \mathbb{N}^k$, let $[0, n] := \{t \in \mathbb{R}^k : 0 \leq t \leq n\}$. For $t \in \mathbb{R}^k$, let $\lfloor t \rfloor$ be the element of $\mathbb{Z}^k$ such that $\lfloor t \rfloor_i = \lfloor t_i \rfloor = \max\{n \in \mathbb{Z} : n \leq t_i\}$ for all $i \leq k$. Similarly, define $\lceil t \rceil$ by $\lceil t \rceil_i = \min\{n \in \mathbb{Z} : t_i \leq n\}$ for $i \leq k$. Consider the following equivalence relation on $\bigsqcup_{\lambda \in \Lambda} \big(\{\lambda\} \times [0, d(\lambda)]\big)$: for $\mu, \nu \in \Lambda$ and $s, t \in \mathbb{R}^k$ with $0 \leq s \leq d(\mu)$ and $0 \leq t \leq d(\nu)$, we define

$$(\mu, s) \sim (\nu, t) \iff s - \lfloor s \rfloor = t - \lfloor t \rfloor \text{ and } \mu(\lfloor s \rfloor, \lceil s \rceil) = \nu(\lfloor t \rfloor, \lceil t \rceil). \tag{6.1}$$

The topological realisation X_Λ is the quotient space $\Big(\bigsqcup_{\lambda \in \Lambda} \{\lambda\} \times [0, d(\lambda)]\Big)/\sim$. As in [21] we let $[\lambda, t]$ denote the equivalence class of the point (λ, t).

Definition 6.1. For $r \in \mathbb{N}$, let $\mathbf{I}^r$ denote the unit cube $[0, 1]^r$ in $\mathbb{R}^r$. Fix an r-cube $\lambda \in Q_r(\Lambda)$. Express $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$ where $i_1 < \cdots < i_r$. Let $\iota_\lambda : \mathbf{I}^r \to X_\Lambda$ denote the map $(t_1, \ldots, t_r) \mapsto \big[\lambda, \sum_{m=1}^r t_m e_{i_m}\big]$. Then $\Phi(\lambda) := \iota_\lambda$ defines a homomorphism $\Phi : C_r(\Lambda) \to C_r^{\mathrm{top}}(X_\Lambda)$.

Remark 6.2. The map Φ intertwines the boundary maps, so is a chain map. It therefore induces a homomorphism $\Phi_* : H_*(\Lambda) \to H_*^{\mathrm{top}}(X_\Lambda)$.

It will be shown in [21] that each k-graph morphism $\theta : \Lambda \to \Gamma$ induces a continuous map $\widetilde{\theta} : X_\Lambda \to X_\Gamma$ such that $\widetilde{\theta} \circ \iota_\lambda = \iota_{\theta(\lambda)}$ for all $\lambda \in Q(\Lambda)$. Hence both the chain map Φ and the homomorphism Φ_* of homology are natural in Λ. (with respect to k-graph morphisms).

Theorem 6.3. *Let Λ be a k-graph. For each $r \geq 0$, the map $\Phi_* : H_r(\Lambda) \to H_r^{\mathrm{top}}(X_\Lambda)$ is an isomorphism. Moreover this isomorphism is natural in Λ.*

Our proof parallels the argument of the first three paragraphs of [17, Theorem 2.27] where it is shown that the singular homology of a Δ-complex (see [17, page 103]) is the same as its simplicial homology. We first need to do some setting up.

Remark 6.4. We claim that Massey's definition of singular homology, which is based on cubes, is equivalent to the usual one based on simplices. By the uniqueness theorem of [31], if X has the homotopy type of a CW-complex, then any homology theory on X which satisfies the Eilenberg-Steenrod axioms [8] and which is additive in the sense that it carries disjoint unions to direct sums is naturally isomorphic to the usual singular homology. The Eilenberg-Steenrod axioms and additivity are all verified for Massey's singular homology in [30, Chapter VII]: Axiom 1 is (3.4), Axiom 2 is (3.5), Axiom 3 is (7.6.1), Axiom 4 is Theorem 5.1, Axiom 5 is Theorem 6.1, Axiom 6 is Theorem 6.2, Axiom 7 is Example 2.1, and additivity is Proposition 2.7. Alternatively that Massey's homology agrees with the simplicial formulation also follows from the original uniqueness theorem [8, Theorem 10.1] since we can triangulate X_Λ by adding a vertex at the centre of each cube (thereby dividing each r-cube into $2^r r!$ r-simplices).

To run Hatcher's argument, we use the cellular structure of X_Λ regarded as a CW-complex. For $0 \leq m \leq k$ let X_m denote the union of the images of the ι_λ where λ ranges over all r-cubes with $r \leq m$. We formally define $C_r^\Lambda(X_m) = C_r(\Lambda)$ if $m \geq r$ and to be zero otherwise. We obtain a nested sequence

$$C_*^\Lambda(X_0) \subseteq C_*^\Lambda(X_1) \subseteq \cdots \subseteq C_*^\Lambda(X_k) = C_*(\Lambda)$$

of complexes. In particular, for $l \leq m$ we may form the quotient complex

$$C_*^\Lambda(X_m, X_l) := C_*^\Lambda(X_m)/C_*^\Lambda(X_l),$$

which has relative homology groups $H_*^\Lambda(X_m, X_l)$. Then

$$H_r^\Lambda(X_m, X_{m-1}) \cong C_r^\Lambda(X_m, X_{m-1}) = \begin{cases} C_r(\Lambda) & \text{if } m = r, \\ \{0\} & \text{otherwise.} \end{cases} \tag{6.2}$$

Since every short exact sequence of complexes induces a long exact sequence in homology (see [17, Theorem 2.16]), we obtain a long exact sequence

$$\begin{aligned} \cdots \longrightarrow H_{r+1}^\Lambda(X_m, X_{m-1}) \longrightarrow H_r^\Lambda(X_{m-1}) \longrightarrow H_r^\Lambda(X_m) \longrightarrow H_r^\Lambda(X_m, X_{m-1}) \\ \longrightarrow H_{r-1}^\Lambda(X_{m-1}) \longrightarrow \cdots \longrightarrow H_0^\Lambda(X_m, X_{m-1}). \end{aligned} \tag{6.3}$$

The map $\Phi : C_*(\Lambda) \to C_*^{\mathrm{top}}(X_\Lambda)$ induces a map from $C_*(X_m)$ to $C_*^{\mathrm{top}}(X_m)$ for each m. Hence, it induces a map, also called Φ, from $C_*^\Lambda(X_m, X_{m-1})$ to $C_*^{\mathrm{top}}(X_m, X_{m-1})$.

The crucial step in Hatcher's proof of [17, Theorem 2.27] is the following isomorphism.

Lemma 6.5. *With notation as above, the induced map*

$$\Phi_* : H_r^\Lambda(X_m, X_{m-1}) \to H_r^{\mathrm{top}}(X_m, X_{m-1})$$

is an isomorphism for each r, m.

Proof. Suppose that $r \neq m$. Then $H_r^\Lambda(X_m, X_{m-1}) = \{0\}$ by (6.2) and $H_r^{\mathrm{top}}(X_m, X_{m-1}) = \{0\}$ by [17, Lemma 2.3.4 (a)]. Hence $\Phi_* : H_r^\Lambda(X_m, X_{m-1}) \to H_r^{\mathrm{top}}(X_m, X_{m-1})$ is an isomorphism for $m \neq r$. Since

$$H_r^\Lambda(X_r, X_{r-1}) \cong C_r(\Lambda) = \mathbb{Z}Q_r(\Lambda) \cong H_r^{\mathrm{top}}(Q_r(\Lambda) \times \mathbf{I}^r, Q_r(\Lambda) \times \partial\mathbf{I}^r),$$

it suffices to show that the canonical map $Q_r(\Lambda) \times \mathbf{I}^r \to X_r$ given by $(\lambda, t) \mapsto \iota_\lambda(t)$ induces an isomorphism

$$H_r^{\mathrm{top}}(Q_r(\Lambda) \times \mathbf{I}^r, Q_r(\Lambda) \times \partial\mathbf{I}^r) \cong H_r^{\mathrm{top}}(X_r, X_{r-1}).$$

To see this, observe that (X_r, X_{r-1}) is a good pair (see [17, p. 114]) in the sense that X_{r-1} is a nonempty closed subset of X_r which is a deformation retract of the open set

$$X_{r-1} \cup \{[\lambda, t] : \lambda \in Q_r(\Lambda), \min\{t_i, 1 - t_i\} < 1/3 \text{ for } 1 \leq i \leq r\}.$$

Let X_r/X_{r-1} be the quotient of X_r obtained by identifying X_{r-1} to a point. That (X_r, X_{r-1}) is a good pair combines with [17, Proposition 2.22] and Remark 6.4 to show that

$$H_r^{\mathrm{top}}(X_r, X_{r-1}) \cong H_r^{\mathrm{top}}(X_r/X_{r-1}).$$

Moreover, Φ_r induces a homeomorphism of $(Q_r(\Lambda) \times \mathbf{I}^r)/(Q_r(\Lambda) \times \partial\mathbf{I}^n)$ with X_r/X_{r-1}. Since $(Q_r(\Lambda) \times \mathbf{I}^r, Q_r(\Lambda) \times \partial\mathbf{I}^r)$ is also a good pair, the result follows from another application of [17, Proposition 2.22]. $\square$

Proof of Theorem 6.3. The naturality of Φ_* was observed in Remark 6.2. So we just need to show that Φ_* is an isomorphism.

Both $H_r(\Lambda)$ and $H_r(X_\Lambda)$ are trivial for $r > k$, so we may assume that $0 \leq r \leq k$. Fix $m \in \mathbb{N}$. If $r \leq m$ then we may regard the map $\Phi : C_r(\Lambda) \to C_r^{\mathrm{top}}(X_\Lambda)$ given in Definition 6.1 as a map from $C_r^\Lambda(X_m)$ to $C_r^{\mathrm{top}}(X_m)$; whereas if $r > m$ then both $C_r^\Lambda(X_m)$ and $C_r^{\mathrm{top}}(X_m)$ are trivial, and we define $\Phi : C_r^\Lambda(X_m) \to C_r^{\mathrm{top}}(X_m)$ to be the trivial map between trivial groups. As in Remark 6.2, Φ intertwines the boundary maps, and so induces a homomorphism $\Phi_* : H_*^\Lambda(X_m) \to H_*^{\mathrm{top}}(X_m)$.

We claim that these maps are all isomorphisms. We proceed by induction on m. Our base case is $m = 0$. Since X_0 is equal to the discrete space Λ^0, each of $H_0^\Lambda(X_0)$ and $H_0^{\mathrm{top}}(X_0)$ is canonically isomorphic to $\mathbb{Z}\Lambda^0$, and Φ_* is the identity map. Moreover, for $r \geq 1$, we have $H_r^\Lambda(X_0) = H_r^{\mathrm{top}}(X_0) = \{0\}$, so Φ_* is trivially an isomorphism. Now fix $m \geq 1$ and suppose as an inductive hypothesis that Φ_* is an isomorphism between $H_*^\Lambda(X_{m-1})$ and $H_*^{\mathrm{top}}(X_{m-1})$. Fix $r \geq 0$. Since Φ_* induces a map of short exact sequences of complexes, the naturality of the connecting map in the long exact sequence arising from a short exact sequence of complexes yields the following commuting diagram.

$$\begin{array}{ccccccccc}
H_r^\Lambda(X_m, X_{m-1}) & \longrightarrow & H_r^\Lambda(X_{m-1}) & \longrightarrow & H_r^\Lambda(X_m) & \longrightarrow & H_{r-1}^\Lambda(X_m, X_{m-1}) & \longrightarrow & H_{r-1}^\Lambda(X_{m-1}) \\
\downarrow \Phi_* & & \downarrow \Phi_* & & \downarrow \Phi_* & & \downarrow \Phi_* & & \downarrow \Phi_* \\
H_r^{\mathrm{top}}(X_m, X_{m-1}) & \longrightarrow & H_r^{\mathrm{top}}(X_{m-1}) & \longrightarrow & H_r^{\mathrm{top}}(X_m) & \longrightarrow & H_{r-1}^{\mathrm{top}}(X_m, X_{m-1}) & \longrightarrow & H_{r-1}^{\mathrm{top}}(X_{m-1})
\end{array}$$

The inductive hypothesis ensures that the second and fifth vertical maps are isomorphisms, and the first and fourth maps are isomorphisms by Lemma 6.5. Thus the Five Lemma (see, for example, [17, p 129]) implies that the middle vertical map is also an isomorphism, completing the induction. Hence, $\Phi_* : H_r^\Lambda(X_m) \to H_r^{\mathrm{top}}(X_m)$ is an isomorphism for all m. Since $H_r(\Lambda) = H_r^\Lambda(X_k)$ and $H_r^{\mathrm{top}}(X) = H_r^{\mathrm{top}}(X_k)$ for all $r \geq 0$ the desired result follows. □

7. Cohomology and twisted k-graph C^*-algebras

In this section we introduce cohomology for k-graphs and indicate how a $\mathbb{T}$-valued 2-cocycle may be used to twist a k-graph C^*-algebra. We first define the cohomology of a k-graph and provide a Universal Coefficient Theorem. We then show how to associate to each $\mathbb{T}$-valued 2-cocycle ϕ on Λ a twisted C^*-algebra $C^*_\phi(\Lambda)$. We obtain as relatively elementary examples all noncommutative tori and the Heegaard-type quantum 3-spheres of [1]. We will study cohomology for k-graphs and the structure of twisted k-graph C^*-algebras in greater detail in [28].

Notation 7.1. Let Λ be a k-graph and let A be an abelian group. For $r \in \mathbb{N}$, we write $C^r(\Lambda, A)$ for the collection of all functions $f : Q_r(\Lambda) \to A$. We identify $C^r(\Lambda, A)$ with $\mathrm{Hom}(C_r(\Lambda), A)$ in the usual way. Define maps $\delta^r : C^r(\Lambda, A) \to C^{r+1}(\Lambda, A)$ by

$$\delta^r(f)(\lambda) := f(\partial_{r+1}(\lambda)) = \sum_{i=1}^{r+1} \sum_{l=0}^{1} (-1)^{i+l} f(F_i^l(\lambda)).$$

Then $(C^*(\Lambda, A), \delta^*)$ is a cochain complex.

Mac Lane [29, Chapter II, Equation (3.1)] associates a cochain complex to a chain complex and an abelian group in a similar way, but with a slightly different sign convention for the boundary map. The resulting cohomology is isomorphic to the following.

Definition 7.2. We define the *cohomology* $H^*(\Lambda, A)$ of the k-graph Λ with coefficients in A to be the cohomology of the complex $C^*(\Lambda, A)$; that is $H^r(\Lambda, A) := \ker(\delta^r)/\operatorname{Im}(\delta^{r-1})$. For $r \geq 0$, we write $Z^r(\Lambda, A) := \ker(\delta^r)$ for the group of r-cocycles, and for $r > 0$, we write $B^r(\Lambda, A) = \operatorname{Im}(\delta^{r-1})$ for the group of r-coboundaries.

Theorem 7.3 (Universal Coefficient Theorem)**.** *Let Λ be a k-graph, and let A be an abelian group. For each $r \geq 0$, there is a short exact sequence*

$$0 \longrightarrow \operatorname{Ext}(H_{r-1}(\Lambda), A) \xrightarrow{\alpha} H^r(\Lambda, A) \xrightarrow{\beta} \operatorname{Hom}(H_r(\Lambda), A) \longrightarrow 0,$$

and the maps α and β are natural in A and Λ.

Proof. This follows directly from Mac Lane's theorem [29, Theorem III.4.1] applied to the complex $C_*(\Lambda)$. □

Recall from [25] that a k-graph Λ is *row-finite* if $v\Lambda^n$ is finite for all $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and is *locally convex* if, whenever $1 \leq i \neq j \leq k$ and $\lambda \in \Lambda^{e_i}$ with $r(\lambda)\Lambda^{e_j} \neq \emptyset$, we have $s(\lambda)\Lambda^{e_j} \neq \emptyset$ also.

We will follow the usual convention of writing the binary operation in an abelian group A additively, except when $A = \mathbb{T}$ where it is written multiplicatively.

Definition 7.4 (cf. [36, Equation (3.1)] and [37, Theorem C.1(i)–(ii)])**.** Let Λ be a row-finite locally convex k-graph and fix $\phi \in Z^2(\Lambda, \mathbb{T})$. A Cuntz-Krieger ϕ-representation of Λ in a C^*-algebra A is a set $\{p_v : v \in \Lambda^0\} \subseteq A$ of mutually orthogonal projections and a set $\{s_\lambda : \lambda \in \bigcup_{i=1}^k \Lambda^{e_i}\} \subseteq A$ satisfying

(1) for all $\lambda \in \Lambda^{e_i}$, $s_\lambda^* s_\lambda = p_{s(\lambda)}$;
(2) for all $1 \leq i < j \leq k$ and $\mu, \mu' \in \Lambda^{e_i}$, $\nu, \nu' \in \Lambda^{e_j}$ such that $\mu\nu = \nu'\mu'$,
$$s_{\nu'} s_{\mu'} = \phi(\mu\nu) s_\mu s_\nu; \text{ and}$$
(3) for all $v \in \Lambda^0$ and all $i = 1, \ldots, k$ such that $v\Lambda^{e_i} \neq \emptyset$,
$$p_v = \sum_{\lambda \in v\Lambda^{e_i}} s_\lambda s_\lambda^*.$$

The condition that a set $\{p_v : v \in \Lambda^0\}$ consists of mutually orthogonal projections is characterised by the algebraic relations $p_v^* = p_v^2 = p_v$ and $p_v p_w = \delta_{v,w} p_v$ for all $v, w \in \Lambda^0$. Given any collection $\{p_v : v \in \Lambda^0\}$ in a *-algebra satisfying these relations, and given any family $\{s_\lambda : \lambda \in \bigcup_{i=1}^k \Lambda^{e_i}\}$ in the same *-algebra satisfying relation (1), the norm of the image of each p_v and of each s_λ under any representation on Hilbert space is at most 1. So as in [2, Definition 1.2], there is a universal C^*-algebra generated by a Cuntz-Krieger ϕ-representation of Λ. A priori, this could be the zero algebra; but we will exhibit some interesting examples (see Examples 7.7, 7.9, 7.10) where it is not, and we will show in the forthcoming article [28] that in fact there is always a Cuntz-Krieger ϕ-representation of Λ in which every generator is nonzero.

Definition 7.5. Let Λ be a row-finite locally convex k-graph. Let $\phi \in Z^2(\Lambda, \mathbb{T})$. We define $C^*_\phi(\Lambda)$ to be the universal C^*-algebra generated by a Cuntz-Krieger ϕ-representation of Λ.

Proposition 7.6. *Let Λ be a row-finite locally convex k-graph.*

(1) *Let 1 denote the identity element of $C_2(\Lambda, \mathbb{T})$. Then $C^*_1(\Lambda)$ is canonically isomorphic to the k-graph algebra $C^*(\Lambda)$ defined in* [36].

(2) *Let $\psi, \phi \in Z^2(\Lambda, \mathbb{T})$, and suppose that $\alpha \in C^1(\Lambda, \mathbb{T})$ satisfies $\phi = \delta^1(\alpha)\psi$ so that ϕ and ψ are cohomologous. Let $\{p^\psi_v : v \in \Lambda^0\}$, $\{s^\psi_\lambda : \lambda \in \bigsqcup_{i=1}^k \Lambda^{e_i}\}$ be the universal generating Cuntz-Krieger ψ-representation of Λ and similarly for ϕ. Then there is an isomorphism $\pi : C^*_\psi(\Lambda) \to C^*_\phi(\Lambda)$ such that $\pi(p^\psi_v) = p^\phi_v$ for all $v \in \Lambda^0$ and $\pi(s^\psi_\lambda) = \alpha(\lambda)s^\phi_\lambda$ for all $\lambda \in \bigcup_{i=1}^k \Lambda^{e_i}$.*

Proof. (1) The combination of [37, Theorem C.1 and Lemma B.4] shows that $C^*(\Lambda)$ is the universal C^*-algebra generated by elements satisfying the relations of Definition 7.4 with $\phi(\mu\nu) = 1$ for all $\mu\nu \in Q_2(\Lambda)$.

(2) For $\lambda \in \bigcup_{i=1}^k \Lambda^{e_i}$, let $t_\lambda := \alpha(\lambda)s^\phi_\lambda$. If $\mu\nu = \nu'\mu'$ where $\mu, \mu' \in \Lambda^{e_i}$, $\nu, \nu' \in \Lambda^{e_j}$ and $1 \le i < j \le k$, then $\delta^1(\alpha) = \alpha(\mu')^{-1}\alpha(\nu')^{-1}\alpha(\mu)\alpha(\nu)$. Hence

$$\alpha(\nu')\alpha(\mu')\phi(\mu\nu) = \alpha(\nu')\alpha(\mu')\delta^1(\alpha)(\mu\nu)\psi(\mu\nu) = \alpha(\mu)\alpha(\nu)\psi(\mu\nu).$$

Using this, we calculate:

$$t_{\nu'}t_{\mu'} = \alpha(\nu')\alpha(\mu')s_{\nu'}s_{\mu'} = \alpha(\nu')\alpha(\mu')\phi(\mu\nu)s_\mu s_\nu = \alpha(\mu)\alpha(\nu)\psi(\mu\nu)s_\mu s_\nu = \psi(\mu\nu)t_\mu t_\nu.$$

So $\{t_\lambda : \lambda \in \bigcup_{i=1}^k \Lambda^{e_i}\}$ satisfies Definition 7.4(2) for the cocycle ψ. Hence the collections $\{p^\phi_v : v \in \Lambda^0\}$ and $\{t_\lambda : \lambda \in \bigcup_{i=1}^k \Lambda^{e_i}\}$ in $C^*_\phi(\Lambda)$ constitute a Cuntz-Krieger ψ-representation of Λ. The universal property of $C^*_\psi(\Lambda)$ therefore gives a homomorphism $\pi : C^*_\psi(\Lambda) \to C^*_\phi(\Lambda)$ such that $\pi(p^\psi_v) = p^\phi_v$ for all $v \in \Lambda^0$ and $\pi(s^\psi_\lambda) = t_\lambda = \alpha(\lambda)s^\phi_\lambda$ for all $\lambda \in \bigcup_{i=1}^k \Lambda^{e_i}$. Reversing the roles of ψ and ϕ in the above calculation yields an inverse, so π is an isomorphism. □

Example 7.7. Let T_2 denote $\mathbb{N}^2$ regarded as a 2-graph with degree functor the identity map (see Examples 2.2(1)). Fix $\theta \in [0, 1)$. There is precisely one 2-cube in T_2, namely $(1, 1)$. Define $\phi \in Z^2(T_2, \mathbb{T})$ by $\phi(1, 1) = e^{2\pi i\theta}$. By definition, $C^*_\phi(T_2)$ is the universal C^*-algebra generated by unitaries S_{e_1} and S_{e_2} satisfying

$$S_{e_2}S_{e_1} = e^{2\pi i\theta}S_{e_1}S_{e_2}.$$

That is, $C^*_\phi(T_2)$ is the rotation algebra A_θ.

Remark 7.8. Theorem 2.1 of [22] says that the obstruction to a product system over $\mathbb{N}^2$ of $\mathbb{C}$-correspondences being the product system associated to the 2-graph T_2 is measured by the element $\omega \in \mathbb{T}$ which implements the module isomorphism $\mathbb{C} \otimes \mathbb{C} \to \mathbb{C} \otimes \mathbb{C}$ between $X_{(1,0)} \otimes X_{(0,1)}$ and $X_{(0,1)} \otimes X_{(1,0)}$. We may regard $H^2(T_2, \mathbb{T})$ as the receptacle for this obstruction.

Example 7.9. More generally consider the k-graph T_k for $k \ge 2$. Then the twisted k-graph C^*-algebras over T_k correspond exactly to the noncommutative tori (see for example [20], [9]; note that their sign conventions differ). Let θ be a skew-symmetric $k \times k$ real matrix, then the associated noncommutative torus A_θ is the universal C^*-algebra generated by k unitaries $u_1, \dots, u_k$, satisfying (see [20])

$$u_n u_m = e^{2\pi i\theta_{m,n}}u_m u_n \qquad \text{for all} \quad 1 \le m, n \le k. \tag{7.1}$$

Recall that $Q_2(T_k) = \{e_m + e_n \mid 1 \le m < n \le k\}$. Set $\phi_\theta(e_m + e_n) = e^{2\pi i \theta_{m,n}}$. Then $\phi(\theta)$ is a 2-cocycle. Moreover $C^*_{\phi(\theta)}(T_k)$ is the universal C^*-algebra generated by k unitaries $S_{e_1}, \dots, S_{e_k}$ satisfying (7.1). Hence, $A_\theta \cong C^*_{\phi(\theta)}(T_k)$.

Example 7.10. In [1] the authors describe C^*-algebras $C(S^3_{pq\theta})$ where p, q, θ are parameters in $[0, 1)$. They show that $C(S^3_{pq\theta}) \cong C(S^3_{00\theta})$ [1, Theorem 2.8] for all p, q, θ. By definition, $C(S^3_{00\theta})$ is the universal C^*-algebra generated by elements S and T satisfying

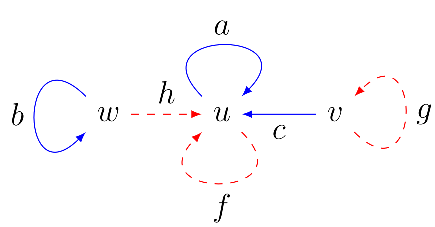
$$(1 - SS^*)(1 - TT^*) = 0, \tag{7.2}$$

$$S^*S = T^*T = 1, \tag{7.3}$$

$$ST = e^{2\pi i\theta} TS, \text{ and} \tag{7.4}$$

$$ST^* = e^{-2\pi i\theta} T^*S. \tag{7.5}$$

It was shown in [16, Remark 3.3] that $C(S^3_{000})$ is isomorphic to the Cuntz-Krieger algebra of the unique 2-graph Λ with skeleton E_Λ as pictured below.



Specifically, the isomorphism $C(S^3_{000}) \to C^*(\Lambda)$ carries S to $s_a + s_b + s_c$ and T to $s_f + s_g + s_h$.

Note that $T_2 = \mathbb{N}^2$ so the degree map on Λ yields a 2-graph morphism $f : \Lambda \to T_2$. A routine computation shows that f_* induces an isomorphism on homology. Hence by Theorem 7.3, f^* induces an isomorphism $H^2(T_2, \mathbb{T}) \cong H^2(\Lambda, \mathbb{T})$.

Let $\alpha = ah = hb$, $\beta = cg = fc$ and $\tau = af = fa$; so $Q_2(\Lambda) = \{\alpha, \beta, \tau\}$. For each $\theta \in [0, 1)$ the 2-cocycle on T_2 determined by $(1, 1) \mapsto e^{-2\pi i\theta}$ pulls back to a 2-cocycle ϕ_θ on Λ satisfying $\phi_\theta(\alpha) = \phi_\theta(\beta) = \phi_\theta(\tau) = e^{-2\pi i\theta}$ (the preceding paragraph shows that every 2-cocycle on Λ is cohomologous to one of this form). Fix $\theta \in [0, 1)$ and let $\{s_\lambda : \lambda \in \bigcup_{i=1}^k \Lambda^{e_i}\}$ and $\{p_v : v \in \Lambda^0\}$ be the generators of $C^*_{\phi(\theta)}(\Lambda)$. Define $\overline{S}, \overline{T} \in C^*_{\phi(\theta)}(\Lambda)$ by $\overline{S} := s_a + s_b + s_c$ and $\overline{T} = s_f + s_g + s_h$. We have

$$\overline{ST} = s_a s_f + s_c s_g + s_a s_h = e^{2\pi i\theta} s_f s_a + e^{2\pi i\theta} s_f s_c + e^{2\pi i\theta} s_h s_b = e^{2\pi i\theta} \overline{TS}.$$

So $\overline{S}, \overline{T}$ satisfy (7.4). Moreover

$$\begin{aligned}
\overline{T}^* \overline{S} &= \overline{T}^* p_u \overline{S} = (s_f^* + s_g^* + s_h^*)(s_\alpha s_\alpha^* + s_\beta s_\beta^* + s_\tau s_\tau^*)(s_a + s_b + s_c) \\
&= s_f^*(s_\beta s_\beta^*) s_c + s_f^*(s_\tau s_\tau^*) s_a + s_h^*(s_\alpha s_\alpha^*) s_a \\
&= s_f^*(e^{2\pi i\theta} s_f s_c)(s_g^* s_c^*) s_c + s_f^*(e^{2\pi i\theta} s_f s_a)(s_f^* s_a^*) s_a + s_h^*(e^{2\pi i\theta} s_h s_b)(s_h^* s_a^*) s_a \\
&= e^{2\pi i\theta}(s_c s_g^* + s_a s_f^* + s_b s_h^*) = e^{2\pi i\theta}(s_a + s_b + s_c)(s_f^* + s_g^* + s_h^*) \\
&= e^{2\pi i\theta} \overline{ST}^*,
\end{aligned}$$

which establishes (7.5). That $\overline{S}, \overline{T}$ also satisfy (7.2) and (7.3) is routine. Hence by the universal property of $C(S^3_{00\theta})$ the map $S \to \overline{S}$ and T to $\overline{T}$ extends to a homomorphism ρ from $C(S^3_{00\theta})$ to $C^*_{\phi(\theta)}(\Lambda)$.

Now let S and T be the generators of $C(S^3_{00\theta})$. Define

$$q_w = 1 - SS^*, \quad q_v = 1 - TT^*, \quad \text{and} \quad q_u = SS^*TT^*,$$

and

$$t_\eta = q_{r(\eta)} S q_{s(\eta)} \text{ for } \eta \in \Lambda^{e_1}, \quad \text{and } t_\eta = q_{r(\eta)} T q_{s(\eta)} \text{ for } \eta \in \Lambda^{e_2}.$$

It is routine to check that the pair $\{q_u, q_v, q_w\}$, $\{t_a, t_b, t_c, t_f, t_g, t_h\}$ is a Cuntz-Krieger $\phi(\theta)$-representation of Λ in $C(S^3_{00\theta})$. So the universal property of $C^*_{\phi(\theta)}(\Lambda)$ yields a homomorphism $\psi : C^*_{\phi(\theta)}(\Lambda) \to C(S^3_{00\theta})$ such that $\psi(p_x) = q_x$ for $x \in \Lambda^0$ and $\psi(s_\eta) = t_\eta$ for $\eta \in \Lambda^{e_1} \cup \Lambda^{e_2}$. One verifies that $\psi = \rho^{-1}$ and it follows that $C^*_{\phi(\theta)}(\Lambda) \cong C(S^3_{00\theta})$.

Our analysis of $H^2(\Lambda, \mathbb{T})$, together with Proposition 7.6, therefore shows that the collection of twisted 2-graph C^*-algebras associated to Λ is precisely the collection of algebras $C(S^3_{00\theta})$, and hence precisely the collection of algebras $C(S^3_{pq\theta})$ by [1, Theorem 2.8].

Appendix A. Connections with cubical homology

In this section we show that each k-graph determines a cubical set $\widetilde{Q}(\Lambda)$ and that our homology is isomorphic to that of $\widetilde{Q}(\Lambda)$ as defined by Grandis [14]. To define $\widetilde{Q}(\Lambda)$ we must make sense of degeneracy maps and degenerate cubes in a k-graph (see Definition A.1 below), and avoiding this was one motivation for providing a self-contained approach in Section 3 above. We could instead have made use of Khusainov's approach [23] using semicubical sets. This is in a sense more natural for k-graphs since it does not involve degeneracies: it is straightforward to show that the collection $Q_*(\Lambda)$ of cubes in a k-graph forms a semicubical set. However, the sign convention for the boundary maps in Khusainov's definition of homology differs from those of both Grandis and Massey [30].

Recall the following definition adapted from [14, §1.2]. In order to avoid a clash of notation we use f_i for the degeneracy maps; we also use $1, 0$ in place of $+, -$.

Definition A.1. A *cubical set* is a triple $X = (X_r, \partial_i^\ell, f_i)$ consisting of a sequence $(X_r)_{r=0}^\infty$ of sets, together with, for each $r \in \mathbb{N}$, maps

$$\partial_i^\ell : X_r \to X_{r-1} \quad \ell \in \{0,1\},\ 1 \le i \le r \quad \text{and} \quad f_i : X_{r-1} \to X_r \quad 1 \le i \le r$$

satisfying the *cubical relations*

$$\partial_i^\ell \partial_j^m = \partial_j^m \partial_{i+1}^\ell \text{ if } j \le i, \tag{A.1}$$

$$f_i f_j = f_{i+1} f_j \ \text{ if } j \le i, \tag{A.2}$$

$$\partial_i^\ell f_j = \begin{cases} f_j \partial_{i-1}^\ell & \text{if } j < i, \\ \text{id} & \text{if } j = i, \\ f_{j-1} \partial_i^\ell & \text{if } j > i. \end{cases} \tag{A.3}$$

The maps ∂_i^ℓ are called *faces* and the f_i are called *degeneracies*.

We now introduce the k-graph analog $\mathbb{1}$ of the model cocubical set $\mathbb{I}$ described in [14, §1.2] (that is, an object satisfying conditions dual to those set out in Definition A.1). Recall from Section 2 that for $r \ge 1$, $\mathbf{1}_r = \sum_{i=1}^r e_i$ (and $\mathbf{1}_0 := 0 \in \mathbb{N}^0$). We define (see Examples 2.2).

$$\mathbb{1}_r = \begin{cases} \Omega_{r,\mathbf{1_r}} & \text{if } \ r \ge 1; \\ \Omega_0 & \text{if } \ r = 0. \end{cases}$$

For $\ell = 0, 1$ define $\varepsilon_0^\ell : \mathbb{N}^0 \to \mathbb{N}^1$ by $\varepsilon_0^\ell(0) = \ell$. For $1 \le i \le r+1$ and $\ell \in \{0,1\}$ define $\varepsilon_i^\ell : \mathbb{N}^r \to \mathbb{N}^{r+1}$ by

$$\varepsilon_i^\ell(n_1, \dots, n_r) = (n_1, \dots, n_{i-1}, \ell, n_i \dots, n_r).$$

If $m \le n \le \mathbf{1}_r$ in $\mathbb{N}^r$, then $\varepsilon_i^\ell(m) \le \varepsilon_i^\ell(n) \le \mathbf{1}_{r+1}$ in $\mathbb{N}^{r+1}$; so we may extend ε_i^ℓ to a quasimorphism from $\mathbb{1}_r$ to $\mathbb{1}_{r+1}$ by setting $\varepsilon_i^\ell(m,n) := (\varepsilon_i^\ell(m), \varepsilon_i^\ell(n))$.

Define $\eta_1 : \mathbb{N}^1 \to \mathbb{N}^0$ by $\eta_1(n) = 0$ for all $n \in \mathbb{N}$. For $r \ge 2$ and $1 \le i \le r$ we define $\eta_i : \mathbb{N}^r \to \mathbb{N}^{r-1}$ by deleting the i^{th} coordinate:

$$\eta_i(n_1, \dots, n_r) := (n_1, \dots n_{i-1}, n_{i+1}, \dots n_r).$$

If $m \le n \le \mathbf{1}_r$ in $\mathbb{N}^r$, then $\eta_i(m) \le \eta_i(n) \le \mathbf{1}_{r-1}$ in $\mathbb{N}^{r-1}$; so η_i extends to a quasimorphism from $\mathbb{1}_r$ to $\mathbb{1}_{r-1}$ such that $\eta_i(m,n) = (\eta_i(m), \eta_i(n))$.

Proposition A.2. *The collection $\mathbb{1} = (\mathbb{1}_n, \varepsilon_i^\ell, \eta_i)$ forms a cocubical set.*

Proof. It is routine but tedious to check that the duals of the relations (A.1), (A.2) and (A.3) hold. □

Now we build a cubical set $\widetilde{Q}(\Lambda)$ from a k-graph Λ by considering collections of maps from $\mathbb{1}$ into Λ: Given $t, r, k \in \mathbb{N}$, a homomorphism $h : \mathbb{N}^r \to \mathbb{N}^k$ is called an *admissible map of rank t*, or just an *admissible map*, if there exist $1 \le i_1 < \cdots < i_t \le r$ and $1 \le j_1 < \cdots < j_t \le k$ such that

$$\text{(A.4)} \qquad h(e_{i_p}) = e_{j_p} \text{ for } p \le t \qquad \text{and} \qquad h(e_i) = 0 \text{ if } i \notin \{i_1, \dots, i_t\}.$$

Let Λ be a k-graph and fix $r \in \mathbb{N}$. A quasimorphism $\varphi : \mathbb{1}_r \to \Lambda$ is said to be an *r-cube* if there is an admissible map $h : \mathbb{N}^r \to \mathbb{N}^k$ such that $d_\Lambda \circ \varphi = h \circ d_{\mathbb{1}_r}$. We say that an r-cube φ has rank t if the associated admissible map has rank t. For $r \ge 0$ let

$$\widetilde{Q}_r(\Lambda) = \{\varphi : \mathbb{1}_r \to \Lambda : \ \varphi \text{ is an } r\text{-cube}\}.$$

For $1 \le i \le r+1$ and $\ell \in \{0,1\}$, define $\overline{\varepsilon}_i^\ell : \widetilde{Q}_{r+1}(\Lambda) \to \widetilde{Q}_r(\Lambda)$ by

$$\overline{\varepsilon}_i^\ell(\varphi) := \varphi \circ \varepsilon_i^\ell$$

and for $1 \le i \le r$, define $\overline{\eta}_i : \widetilde{Q}_{r-1}(\Lambda) \to \widetilde{Q}_r(\Lambda)$ by

$$\overline{\eta}_i(\varphi) := \varphi \circ \eta_i.$$

Remark A.3. Let φ be an $(r+1)$-cube of rank t with admissible map $h : \mathbb{N}^{r+1} \to \mathbb{N}^k$ given as in equation (A.4) above. If $j = i_p$ for some p, then $\overline{\varepsilon}_j^\ell(\varphi)$ is an r-cube whose rank is $t-1$. Otherwise it is an r-cube of rank t. In either case, the associated admissible map $h' : \mathbb{N}^r \to \mathbb{N}^k$ is given by

$$\text{(A.5)} \qquad h'(e_i) = \begin{cases} e_{j_p} & \text{if } i < j \text{ and } i = i_p \text{ for some } p \\ e_{j_p} & \text{if } i \ge j \text{ and } i = i_p - 1 \text{ for some } p \\ 0 & \text{otherwise.} \end{cases}$$

So $h'(t_1, \dots, t_{r-1}) = h(t_1, \dots, t_{j-1}, 0, t_j, \dots, t_{r-1})$.

Similarly, if φ is an r-cube of rank t with admissible map $h : \mathbb{N}^r \to \mathbb{N}^k$ given in equation (A.4) above, then $\overline{\eta}_j(\varphi)$ is an $(r+1)$-cube of rank t whose admissible map is given by

$$h''(e_i) = \begin{cases} e_{j_p} & \text{if } i < j \text{ and } i = i_p \text{ for some } p \\ e_{j_p} & \text{if } i > j \text{ and } i = i_p + 1 \text{ for some } p \\ 0 & \text{otherwise.} \end{cases} \tag{A.6}$$

So $h''(t_1, \dots, t_{r+1}) = h(t_1, \dots, t_{j-1}, t_{j+1}, \dots, t_{r+1})$.

Theorem A.4. *Let Λ be a k-graph. Then $\widetilde{Q}(\Lambda) = \big(\widetilde{Q}_r(\Lambda), \overline{\varepsilon}_i^\ell, \overline{\eta}_i\big)$ is a cubical set.*

Proof. This follows from Proposition A.2. □

In [14, §2.1] the homology of a cubical set is defined as follows: Let $X = (X_r, \partial_i^\ell, f_i)$ be a cubical set, then for $n \geq 1$ we define

$$\mathrm{Deg}_r(X) = \textstyle\bigcup_{i=1}^r \mathrm{Im}(f_i : X_{r-1} \to X_r) \subseteq X_r$$

and set $\mathrm{Deg}_0(X) = \emptyset$. The (normalised) chain complex $(C_*(X), \partial_*)$ is defined by

$$C_r(X) = \mathbb{Z}X_r / \mathbb{Z}\,\mathrm{Deg}_r(X) = \mathbb{Z}\overline{X}_r \text{ where } \overline{X}_r = X_r \setminus \mathrm{Deg}_r(X)$$
$$\partial_r(x) = \sum_{i,\ell} (-1)^{i+\ell}\, \partial_i^\ell x \quad \text{where } x \in \overline{X}_r.$$

The homology of X is then the homology of the complex $(C_*(X), \partial_*)$, so that

$$H_r(X) = \ker \partial_r / \operatorname{Im} \partial_{r+1}.$$

An r-cube $\varphi : \mathbb{1}_r \to \Lambda$ is called *degenerate* if its rank is strictly less than r. Otherwise it is said to be *nondegenerate*. We define

$$\overline{Q}_r(\Lambda) = \{\varphi : \mathbb{1}_r \to \Lambda : \ \varphi \text{ is a nondegenerate } r\text{-cube}\}$$
$$D_r(\Lambda) = \{\varphi : \mathbb{1}_r \to \Lambda : \ \varphi \text{ is a degenerate } r\text{-cube}\},$$

so $\widetilde{Q}_r(\Lambda) = \overline{Q}_r(\Lambda) \sqcup D_r(\Lambda)$.

Lemma A.5. *Let Λ be a k-graph. Then*

1. *for $1 \leq i \leq r$ and $\ell = 0, 1$, $\overline{\varepsilon}_i^\ell : \widetilde{Q}_{r+1}(\Lambda) \to \widetilde{Q}_r(\Lambda)$ preserves nondegenerate cubes, that is for $\varphi \in \overline{Q}_{r+1}(\Lambda)$ we have $\overline{\varepsilon}_i^\ell(\varphi) \in \overline{Q}_r(\Lambda)$;*
2. *for $1 \leq i \leq r$ and any $\varphi \in \widetilde{Q}_{r-1}(\Lambda)$ we have $\overline{\eta}_i(\varphi) \in D_r(\Lambda)$;*
3. *for all $r \geq 1$ we have $D_r(\Lambda) = \bigcup_{i=1}^r \overline{\eta}_i\big(\widetilde{Q}_{r-1}(\Lambda)\big)$.*

Proof. For (1), suppose that $\varphi : \mathbb{1}_{r+1} \to \Lambda$ has rank $r+1$. Then $\overline{\varepsilon}_i^\ell(\varphi) : \mathbb{1}_r \to \Lambda$ has rank r; so $\overline{\varepsilon}_i^\ell(\varphi) \in \overline{Q}_r(\Lambda)$.

For (2), suppose that $\varphi : \mathbb{1}_{r-1} \to \Lambda$ has rank $t \leq r-1$. Then $\overline{\eta}_i(\varphi) : \mathbb{1}_r \to \Lambda$ has rank $t < r$; so $\overline{\eta}_i(\varphi) \in D_r(\Lambda)$.

For (3), suppose that $\varphi \in D_r(\Lambda)$, that is $\varphi : \mathbb{1}_r \to \Lambda$ has rank $t < r$. Then there is an admissible map $h : \mathbb{N}^r \to \mathbb{N}^k$ of rank t such that $d_\Lambda \circ \varphi = h \circ d_{\mathbb{1}_r}$. Let $1 \leq i \leq r$ be such that $h(e_i) = 0$. Since φ does not depend on the i^{th} coordinate, we have $\varphi = \overline{\eta}_i \overline{\varepsilon}_i^0(\varphi)$; hence, $\varphi = \overline{\eta}_i(\varphi')$ where $\varphi' = \overline{\varepsilon}_i^0(\varphi) \in \widetilde{Q}_{r-1}(\Lambda)$. □

Grandis builds his directed homology from the complex given in the following lemma (see [14, §2.1]).

Lemma A.6. *Let Λ be a k-graph. Let*

$$\overline{C}_r(\Lambda) = \mathbb{Z}\overline{Q}_r(\Lambda)$$

$$\overline{\partial}_r(\lambda) = \sum_{\ell=0}^{1}\sum_{i=1}^{r}(-1)^{i+\ell}\,\overline{\varepsilon}_i^{\ell}(\lambda) \quad \lambda \in \overline{Q}_r(\Lambda) \tag{A.7}$$

Then $(\overline{C}(\Lambda)_*, \overline{\partial}_*)$ is a chain complex.

Proof. Theorem A.4 implies that $\widetilde{Q}(\Lambda) = (\widetilde{Q}_r(\Lambda), \overline{\varepsilon}_i^{\ell}, \overline{\eta}_i)$ is a cubical set. By Lemma A.5 (1) we see that $\overline{\varepsilon}_i^{\ell}\big(\overline{Q}_r(\Lambda)\big) \subset \overline{Q}_{r-1}(\Lambda)$ and so $\overline{\partial}_r$ is well defined. That $\overline{\partial}_r \circ \overline{\partial}_{r+1} = 0$ follows from the property (A.1) of $\overline{\varepsilon}_i^{\ell}$. Hence, $(\overline{C}_*(\Lambda), \overline{\partial}_*)$ is a complex. □

Our aim is to show that the homology $\overline{H}_*(\Lambda)$ defined by the complex $(\overline{C}_*(\Lambda), \overline{\partial}_*)$ is the same as the homology of the complex $(C_*(\Lambda), \partial_*)$ described in §1. We do this in Theorem A.9 by showing that the complexes are isomorphic. Recall the definition of $Q_r(\Lambda)$ given in §2:

$$Q_r(\Lambda) = \{\lambda \in \Lambda : d(\lambda) \le \mathbf{1}_k, |d(\lambda)| = r\}.$$

Lemma A.7. *Let Λ be a k-graph. For $r \ge 0$ and $\lambda \in Q_r(\Lambda)$ there is a unique $\varphi_\lambda \in \overline{Q}_r(\Lambda)$ such that $\varphi_\lambda(0, \mathbf{1}_r) = \lambda$. Conversely, given $\varphi \in \overline{Q}_r(\Lambda)$, the path $\lambda = \varphi(0, \mathbf{1}_r) \in Q_r(\Lambda)$ satisfies $\varphi_\lambda = \varphi$. The map $\lambda \mapsto \varphi_\lambda$ is a bijection from $Q_r(\Lambda)$ to $\overline{Q}_r(\Lambda)$ with inverse $\varphi \mapsto \varphi(0, \mathbf{1}_r)$.*

Proof. The result is trivial when $r = 0$ because $\mathbb{1}_0 = \{0\}$.

Fix $r \ge 1$ and $\lambda \in Q_r(\Lambda)$. Let $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$, and define an admissible map $h : \mathbb{N}^r \to \mathbb{N}^k$ by $h(e_j) = e_{i_j}$ for $j = 1, \ldots, r$. Define $\varphi_\lambda : \mathbb{1}_r \to \Lambda$ by

$$\varphi_\lambda(m, n) = \lambda(h(m), h(n))$$

Then $\varphi_\lambda : \mathbb{1}_r \to \Lambda$ is a nondegenerate r-cube with $\varphi_\lambda(0, \mathbf{1}_r) = \lambda$. The factorisation property ensures that there is only one nondegenerate cube with range λ.

Now fix $\varphi \in \overline{Q}_r(\Lambda)$. Suppose that $d(\varphi(0, \mathbf{1}_r)) = e_{i_1} + \cdots + e_{i_r}$ with $1 \le i_1 < \cdots < i_r \le k$. Let $\lambda = \varphi(0, \mathbf{1}_r)$ and define $h : \mathbb{N}^r \to \mathbb{N}^k$ by $h(e_j) = e_{i_j}$. Then for $(m, n) \in \mathbb{1}_r$ we have

$$\varphi_\lambda(m, n) = \lambda(h(m), h(n)) = \varphi(m, n);$$

so $\varphi_\lambda = \varphi$ as required. □

Recall from Section 2 that for $\lambda \in Q_r(\Lambda)$, if we express $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$ with $1 \le i_1 < \cdots < i_r \le k$, then

$$F_j^0(\lambda) = \lambda(0, d(\lambda) - e_{i_j}) \text{ and } F_j^1(\lambda) = \lambda(e_{i_j}, d(\lambda)).$$

Lemma A.8. *Let Λ be a k-graph and $r \ge 1$. Then for $\lambda \in Q_r(\Lambda)$ we have*

$$\overline{\varepsilon}_j^{\ell}(\varphi_\lambda)(0, \mathbf{1}_{r-1}) = F_j^{\ell}(\lambda) \text{ in } Q_{r-1}(\Lambda). \tag{A.8}$$

Proof. Let $d(\lambda) = e_{i_1} + \cdots + e_{i_r}$ and define $h : \mathbb{N}^r \to \mathbb{N}^k$ by $h(e_j) = e_{i_j}$ for $j \le r$. Then

$$\begin{aligned}\overline{\varepsilon}^\ell_j(\varphi_\lambda)(0, \mathbf{1}_{r-1}) &= \varphi_\lambda(\varepsilon^\ell_j(0), \varepsilon^\ell_j(\mathbf{1}_{r-1})) \\ &= \lambda(h(\varepsilon^\ell_j(0)), h(\varepsilon^\ell_j(\mathbf{1}_{r-1}))) \\ &= \begin{cases} \lambda(0, d(\lambda) - e_{i_j}) & \text{if } \ell = 0 \\ \lambda(e_{i_j}, d(\lambda)) & \text{if } \ell = 1 \end{cases} \\ &= F^\ell_j(\lambda).\end{aligned}$$

□

Theorem A.9. *Let Λ be a k-graph then the bijection of Lemma A.7 induces an isomorphism of complexes $(C_*(\Lambda), \partial_*) \cong (\overline{C}_*(\Lambda), \overline{\partial}_*)$. Hence $\overline{H}_*(\Lambda) \cong H_*(\Lambda)$.*

Proof. By Lemma A.7 the map $\lambda \mapsto \varphi_\lambda$ induces an isomorphism $\theta_r : C_r(\Lambda) \to \overline{C}_r(\Lambda)$. Let $\lambda \in Q_r(\Lambda)$. By Lemma A.8 we have $\theta_{r-1}(F^\ell_i(\lambda)) = \overline{\varepsilon}^\ell_i(\varphi_\lambda)$ for $i = 1, \dots, r$ and $\ell = 0, 1$. Hence, by (3.1) and (A.7) we have

$$\overline{\partial}_r \theta_r(\lambda) = \theta_{r-1} \partial_r(\lambda)$$

and the result follows. □

Appendix B. Topological realisations

Given a k-graph Λ we show that the topological realisation X_Λ of Λ is homeomorphic to the topological realisation $\mathcal{R}\widetilde{Q}(\Lambda)$ of the associated cubical set $\widetilde{Q}(\Lambda)$ as defined in [14, §1.8]. We define the cocubical set $\mathbf{I}^* = (\mathbf{I}^r, \dot{\varepsilon}^\ell_i, \dot{\eta}_i)$ of [14] as follows (we modify Grandis' notation to align with ours from Appendix A). For $r \ge 0$ let $\mathbf{I}^r$ be the unit cube in $\mathbb{R}^r$. For $1 \le i \le r+1$ and $\ell \in \{0, 1\}$ define the coface maps $\dot{\varepsilon}^\ell_i : \mathbf{I}^r \to \mathbf{I}^{r+1}$ and for $1 \le i \le r$ define codegeneracy maps $\dot{\eta}_i : \mathbf{I}^r \to \mathbf{I}^{r-1}$ by

$$\dot{\varepsilon}^\ell_i(t)_j = \begin{cases} t_j & \text{if } j < i \\ \ell & \text{if } j = i \\ t_{j-1} & \text{if } j > i \end{cases} \qquad \text{and} \qquad \dot{\eta}_i(t)_j = \begin{cases} t_j & \text{if } j < i \\ t_{j+1} & \text{if } j \ge i. \end{cases}.$$

Recall from [14] that $\mathcal{R}\widetilde{Q}(\Lambda)$ is a topological space endowed with maps $\widehat{\varphi} : \mathbf{I}^r \to \mathcal{R}\widetilde{Q}(\Lambda)$ for each $\varphi \in \widetilde{Q}_r(\Lambda)$ satisfying

$$\widehat{\varphi} \circ \dot{\varepsilon}^\ell_i = (\overline{\varepsilon}^\ell_i(\varphi))^\wedge \qquad \text{and} \qquad \widehat{\varphi} \circ \dot{\eta}_i = (\overline{\eta}_i(\varphi))^\wedge, \tag{B.1}$$

and is uniquely determined by the property that for any topological space X and any collection of continuous maps $\{\widetilde{\varphi} : \mathbf{I}^r \to X \mid 1 \le r, \varphi \in \widetilde{Q}_r(\Lambda)\}$ satisfying

$$\widetilde{\varphi} \circ \dot{\varepsilon}^\ell_i = (\overline{\varepsilon}^\ell_i(\varphi))^\sim \qquad \text{and} \qquad \widetilde{\varphi} \circ \dot{\eta}_i = (\overline{\eta}_i(\varphi))^\sim, \tag{B.2}$$

there is a unique continuous map $\pi : \mathcal{R}\widetilde{Q}(\Lambda) \to X$ satisfying $\pi \circ \widehat{\varphi} = \widetilde{\varphi}$ for all $\varphi \in \widetilde{Q}(\Lambda)$.

Fix $\varphi \in \widetilde{Q}_r(\Lambda)$ and let $h : \mathbb{N}^r \to \mathbb{N}^k$ be the associated admissible map. As in [21] extend h to a map from $\mathbb{R}^r$ to $\mathbb{R}^k$ by setting $h(t) := \sum_{i=1}^r t_i h(e_i)$. We define a map $\widetilde{\varphi} : \mathbf{I}_r \to X_\Lambda$ by

$$\widetilde{\varphi}(t) = [\varphi(0, \mathbf{1}_r), h(t)]. \tag{B.3}$$

Lemma B.1. *Let Λ be a k-graph. The maps $\widetilde{\varphi} : \mathbf{I}_r \to X_\Lambda$ of* (B.3) *are continuous, and satisfy* (B.2).

Proof. Fix $\varphi \in \widetilde{Q}_r(\Lambda)$ with associated admissible map h. Since $t \mapsto (\varphi(0, \mathbf{1}_r), h(t))$ is continuous from $\mathbf{I}^r$ to $\{\varphi(0, \mathbf{1}_r)\} \times [0, h(\mathbf{1}_r)]$, and since the quotient map from $\bigsqcup_{\lambda \in Q(\Lambda)} \{\lambda\} \times [0, d(\lambda)]$ to X_Λ is also continuous, the map $\widetilde{\varphi}$ is continuous. We check the identities (B.2). The calculations are routine but tedious so we only give a detailed proof of the first identity $\widetilde{\varphi} \circ \dot{\varepsilon}_i^\ell = (\overline{\varepsilon}_i^\ell(\varphi))^\sim$, this being the more complicated of the two calculations. The second identity follows from similar calculations. Define $h' : \mathbb{R}^{r-1} \to \mathbb{R}^k$ as in Remark A.3 by $h'(t_1, \dots, t_{r-1}) = h(t_1, \dots, t_{i-1}, 0, t_i, \dots, t_{r-1})$. For $t \in \mathbf{I}_{r-1}$

$$(\widetilde{\varphi} \circ \dot{\varepsilon}_i^\ell)(t) = \widetilde{\varphi}(\dot{\varepsilon}_i^\ell(t)) = [\varphi(0, \mathbf{1}_r), h(\dot{\varepsilon}_i^\ell(t))] = [\varphi(0, \mathbf{1}_r), h'(t) + \ell h(e_i)]. \tag{B.4}$$

Since h' is the admissible map associated to $\overline{\varepsilon}_i^\ell(\varphi)$, we also have

$$(\overline{\varepsilon}_i^\ell(\varphi))^\sim(t) = [\overline{\varepsilon}_i^\ell(\varphi)(0, \mathbf{1}_{r-1}), h'(t)] = [\varphi(\varepsilon_i^\ell(0, \mathbf{1}_{r-1})), h'(t)] \tag{B.5}$$

Since ℓ is an integer, $h'(t) + \ell h(e_i) - \lfloor h'(t) + \ell h(e_i) \rfloor = h'(t) - \lfloor h'(t) \rfloor$. Moreover, by the factorisation property, we have

$$\varphi(0, \mathbf{1}_r) = \varphi(0, \varepsilon_i^\ell(0))\varphi(\varepsilon_i^\ell(0, \mathbf{1}_{r-1}))\varphi(\varepsilon_i^\ell(\mathbf{1}_{r-1}), \mathbf{1}_r).$$

Hence, considering separately the cases $\ell = 0$ and $\ell = 1$, one can verify that

$$\varphi(0, \mathbf{1}_r)(\lfloor h'(t) + \ell h(e_i) \rfloor, \lceil h'(t) + \ell h(e_i) \rceil) = \varphi(\varepsilon_i^\ell(0, \mathbf{1}_{r-1}))(\lfloor h'(t) \rfloor, \lceil h'(t) \rceil)).$$

The definition (6.1) of the equivalence relation $\sim$ then gives

$$(\varphi(0, \mathbf{1}_r), h'(t) + \ell h(e_i)) \sim (\varphi(\varepsilon_i^\ell(0, \mathbf{1}_{r-1})), h'(t)).$$

Combining this with (B.4) and (B.5) establishes the first identity in (B.2). □

By Lemma B.1 and the defining property of $\mathcal{R}\widetilde{Q}(\Lambda)$, there is a unique continuous map $\pi : \mathcal{R}\widetilde{Q}(\Lambda) \to X_\Lambda$ such that $\pi \circ \widehat{\varphi} = \widetilde{\varphi}$ for all $\varphi \in \widetilde{Q}(\Lambda)$.

Theorem B.2. *Let Λ be a k-graph. The map $\pi : \mathcal{R}\widetilde{Q}(\Lambda) \to X_\Lambda$ is a homeomorphism.*

Proof. We construct a continuous inverse ψ for π. Define $\psi_0 : \bigsqcup_{d(\lambda) \le \mathbf{1}_k} \{\lambda\} \times [0, d(\lambda)] \to \mathcal{R}\widetilde{Q}(\Lambda)$ by

$$\psi_0(\lambda, t) := \widehat{\varphi}_\lambda(t),$$

where $\varphi_\lambda : \mathbb{1}_{|\lambda|} \to \Lambda$ is the k-graph quasimorphism canonically associated to λ. The map ψ_0 is clearly continuous.

If $\psi([\mu, s]) := \psi_0(\mu, s)$ determines a well-defined map $\psi : X_\Lambda \to \mathcal{R}\widetilde{Q}(\Lambda)$, then it will be continuous by definition of the topology on X_Λ, and will be an inverse for π. So suppose that $(\mu, s) \sim (\nu, t)$ where $\mu, \nu \in Q(\Lambda)$. Let $I_{(\mu,s)} := \{j : d(\mu)_j = 1 \text{ and } s_j \in \{0, 1\}\}$, and define $I_{(\mu,t)}$ similarly. List $I_{(\mu,s)} = \{j_1, \dots, j_p\}$ where $j_1 < \dots < j_p$. Define $F_{(\mu,s)}$ to be the composition of face maps $F_{(\mu,s)} = F_{j_1}^{s_{j_1}} \circ \dots \circ F_{j_p}^{s_{j_p}}$ (with the convention that if $I_{(\mu,s)} = \emptyset$, then $F_{(\mu,s)}$ is the identity map), and define $F_{(\nu,t)}$ similarly. Then

$$F_{(\mu,s)}(\mu) = \mu(\lfloor s \rfloor, \lceil s \rceil) = \nu(\lfloor t \rfloor, \lceil t \rceil) = F_{(\nu,t)}(\nu)$$

because $[\mu, s] = [\nu, t]$. Let $s' := s - \lfloor s \rfloor$ and $t' := t - \lfloor t \rfloor$. Then

$$(\mu, s) \sim (F_{(\mu,s)}(\mu), s') = (F_{(\nu,t)}(\nu), t') \sim (\nu, t),$$

so it suffices to show that $\psi_0(\mu, s) = \psi_0(F_{(\mu,s)}(\mu), s')$. Let $\dot{\varepsilon}_{(\mu,s)} : \mathbf{I}_{|\mu|-|I_{(\mu,s)}|} \to \mathbf{I}_{|\mu|}$ be the composition $\dot{\varepsilon}_{j_p}^{s_{j_p}} \circ \cdots \circ \dot{\varepsilon}_{j_1}^{s_{j_1}}$. Let $\overline{\varepsilon}_{(\mu,s)}$ be the composition of face maps in $\widetilde{Q}(\Lambda)$ corresponding to $F_{(\mu,s)}$. It is routine to see that

$$\varphi_{F_{(\mu,s)}(\mu)} = \overline{\varepsilon}_{(\mu,s)}(\varphi_\mu).$$

Hence the identities (B.1) imply that

$$\widehat{\varphi}_{F_{(\mu,s)}(\mu)} = (\overline{\varepsilon}_{(\mu,s)}(\varphi_\mu))\widehat{\ } = \widehat{\varphi}_\mu \circ \dot{\varepsilon}_{(\mu,s)}.$$

In particular, if $\overline{s}$ and $\overline{s}'$ are the elements of $\mathbf{I}_{|\mu|}$ and $\mathbf{I}_{|\mu|-|I_{(\mu,s)}|}$ which map to s and s' under the associated admissible maps, then

$$\psi_0(F_{(\mu,s)}(\mu), s') = \widehat{\varphi}_{F_{(\mu,s)}(\mu)}(\overline{s}') = \widehat{\varphi}_\mu \circ \dot{\varepsilon}_{(\mu,s)}(\overline{s}') = \widehat{\varphi}_\mu(\overline{s}) = \psi_0(\mu, s). \qquad \square$$

References

[1] P. F. Baum, P. M. Hajac, R. Matthes, and W. Szymański, *The K-theory of Heegaard-type quantum 3-spheres*, K-Theory **35** (2005), 159–186.

[2] B. Blackadar, *Shape theory for C^*-algebras*, Math. Scand. **56** (1985), 249–275.

[3] B. Blackadar, *K-theory for operator algebras.* MSRI Publications vol. 5, Cambridge University Press, 1998.

[4] K. Brown, *Cohomology of groups*, Graduate Texts in Mathematics, **87**, Springer-Verlag, New York-Berlin, 1982.

[5] R. Brown and P. J. Higgins, *The equivalence of ω-groupoids and cubical T-complexes*, Cahiers Topologie Géom. Différentielle **22** (1981), 349–370.

[6] K. R. Davidson and D. Yang, *Periodicity in rank 2 graph algebras*, Canad. J. Math. **61** (2009), 1239–1261.

[7] K. R. Davidson and D. Yang, *Representations of higher rank graph algebras*, New York J. Math. **15** (2009), 169–198.

[8] S. Eilenberg and N. Steenrod, Foundations of algebraic topology, Princeton University Press, Princeton, New Jersey, 1952, xv+328.

[9] G. Elliott and H. Li, *Strong Morita equivalence of higher-dimensional noncommutative tori. II*, Math. Ann. **341** (2008), 825–844.

[10] D. G. Evans, *On the K-theory of higher-rank graph C^*-algebras*, New York J. Math. **14** (2008), 1–31.

[11] C. Farthing, P. S. Muhly and T. Yeend, *Higher-rank graph C^*-algebras: an inverse semigroup and groupoid approach*, Semigroup Forum **71** (2005), 159–187.

[12] C. Farthing, D. Pask and A. Sims, *Crossed products of k-graph C^*-algebras by $\mathbb{Z}^l$*, Houston J. Math. **35** (2009), 903–933.

[13] L. Fajstrup, M. Raußen, E. Goubault, *Algebraic topology and concurrency*, Theoret. Comput. Sci. **357** (2006), 241–278.

[14] M. Grandis *Directed combinatorial homology and noncommutative tori (The breaking of symmetries in algebraic topology)*, Math. Proc. Cambridge Philosophical Soc. **138** (2005), 233–262.

[15] M. Grandis and L. Mauri, *Cubical sets and their site*, Theory Appl. Categ. **11** (2003), 185–211.

[16] P. M. Hajac, R. Matthes, W. Szymanski, Wojciech, *A locally trivial quantum Hopf fibration*, Algebr. Represent. Theory **9** (2006), 121–146.

[17] A. Hatcher, Algebraic topology, Cambridge University Press, Cambridge, 2002, xii+544.

[18] R. Hazelwood, I. Raeburn, A. Sims and S. B. G. Webster, *On some fundamental results about higher-rank graphs and their C^*-algebras*, in preparation.

[19] S. B. Isaacson, *Symmetric cubical sets*, J. Pure Appl. Algebra **215** (2011), 1146–1173.

[20] B. Itzá-Ortiz and N. C. Phillips, *Realization of a simple higher-dimensional noncommutative torus as a transformation group C*-algebra*, Bull. London Math. Soc. **40** (2008) 217-226.

[21] S. Kaliszewski, A. Kumjian, J. Quigg and A. Sims, *Topological realisations of higher-rank graphs*, in preparation.

[22] S. Kaliszewski, N. Patani and J. Quigg, *Obstructions to a general characterisation of graph correspondences*, preprint 2010 (arXiv:1010.3185v1 [math.OA]).

[23] A. A. Khusainov, *Homology groups of semi-cubical sets*, Sibirsk. Mat. Zh. **49** (2008), 224–237.

[24] A. Kumjian and D. Pask, *C^*-algebras of directed graphs and group actions*, Ergodic Theory Dynam. Systems **19** (1999), 1503–1519.

[25] A. Kumjian and D. Pask, *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.

[26] A. Kumjian, D. Pask and A. Sims, *C^*-algebras associated to coverings of k-graphs*, Doc. Math. **13** (2008), 161–205.

[27] A. Kumjian, D. Pask and A. Sims, *Generalised morphisms of k-graphs: k-morphs*, Trans. Amer. Math. Soc. **363** (2011), 2599–2626.

[28] A. Kumjian, D. Pask and A. Sims, *Cohomology of k-graphs and twisted C^*-algebras*, in preparation.

[29] S. Mac Lane. *Homology.* Die Grundlehren der mathematischen Wissenschaften, Band 114. Springer-Verlag, Berlin-New York, 1967.

[30] W. Massey, *A basic course in algebraic topology*, Graduate Texts in Mathematics **127** Springer-Verlag, Berlin-New York, 1991.

[31] J. Milnor, *On axiomatic homology theory*, Pacific J. Math. **12** (1962), 337–341.

[32] J. R. Munkres, Elements of algebraic topology, Addison-Wesley Publishing Company, Menlo Park, CA, 1984, ix+454.

[33] D. Pask, J. Quigg and I. Raeburn, *Fundamental groupoids of k-graphs*, New York J. Math. **10** (2004), 195–207.

[34] D. Pask, J. Quigg and I. Raeburn, *Coverings of k-graphs*, J. Algebra **289** (2005), 161–191.

[35] M. Pimsner and D. Voiculescu, *Exact sequences for K-groups and Ext-groups of certain cross-product C^*-algebras*, J. Operator Theory **4** (1980), 93–118.

[36] I. Raeburn, A. Sims and T. Yeend, *Higher-rank graphs and their C^*-algebras*, Proc. Edinb. Math. Soc. (2) **46** (2003), 99–115.

[37] I. Raeburn, A. Sims and T. Yeend, *The C^*-algebras of finitely aligned higher-rank graphs*, J. Funct. Anal. **213** (2004), 206–240.

[38] G. Robertson and T. Steger, *Affine buildings, tiling systems and higher rank Cuntz-Krieger algebras*, J. reine angew. Math. **513** (1999), 115–144.

[39] J-P. Serre. *Trees.* Springer-Verlag, Berlin-New York, 1980.

[40] A. Skalski and J. Zacharias, *Entropy of shifts on higher-rank graph C^*-algebras*, Houston J. Math. **34** (2008), 269–282.

[41] A. Skalski and J. Zacharias, *Poisson transform for higher-rank graph algebras and its applications*, J. Operator Theory **63** (2010), 425–454.

[42] J. Stillwell, *Classical Topology and Combinatorial Group Theory*, Graduate Texts in Mathematics **72** Springer-Verlag, Berlin-New York, 1980.

[43] S. Yamashita, *Some results on product system C^*-algebras and topological higher-rank graphs*, preprint 2009 (arXiv:0911.2978v1 [math.OA]).

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