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© 2020 John Wiley & Sons Ltd This study, under zero initial condition, aims to characterize the reachable set bound for a class of neutral Markovian jump systems (NMJSs) with interval time-varying delays and bounded disturbances. To begin with, the time-delays are considered to be mode-dependent while delay mode and system mode are different. By utilizing free-weighting matrix method and reciprocally convex combination technique, an ellipsoid-like bound is characterized for the concerned NMJS with completely known transition probabilities. Based on the provided analytical framework, the case of same delay mode and system mode is also handled. Then, benefitting from a group of free-connection weighting matrices, the reachable set estimation issue is tackled for the NMJS involving mode-independent time-varying delays and partially known transition probabilities. The theoretical analysis is confirmed by numerical simulations.

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Reachable set estimation for neutral Markovian jump systems with mode-dependent time-varying delays

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Abstract

This study, under zero initial condition, aims to characterize the reachable set bound for a class of neutral Markovian jump systems (NMJSs) with interval time-varying delays and bounded disturbances. To begin with, the time-delays are considered to be mode-dependent while delay mode and system mode are different. By utilizing free-weighting matrix method and reciprocally convex combination technique, an ellipsoid-like bound is characterized for the concerned NMJS with completely known transition probabilities. Based on the provided analytical framework, the case of same delay mode and system mode is also handled. Then, benefitting from a group of free-connection weighting matrices, the reachable set estimation issue is tackled for the NMJS involving mode-independent time-varying delays and partially known transition probabilities. The theoretical analysis is confirmed by numerical simulations.

Keywords: Markovian jump systems, Neutral systems, Reachable set estimation, Time-varying delays

1. Introduction

Time-delay, a provenance of deteriorating the system performance or even destabilizing the system, unavoidably exists in a variety of practical processes, such as chemical reactions, data transmissions, population migrations [1, 2, 3], and so forth. For a system involving time-delay, increasing research interest has been aroused in studying its dynamic properties, including controllability [4], stability [5] and dissipativity [6], to name a few.

Recently, motions of some practical systems (e.g., heat exchangers, distributed networks and car-like robots) have been found to preferably evolve with differential equations of neutral-type, where time-delay is encountered in both the states and their derivatives [7, 8, 9]. Accordingly, a more general class of time-delay systems has been promoted, which is known as neutral time-delay systems. It is inevitable that, caused by the existences of abruptly changing environments, unknown noise inputs and sudden executor faults, system structures may randomly jump in a finite mode set, which can be naturally governed by a Markov chain [10, 11, 12]. This gives rise to Markovian jump systems, whose diverse applications involve in, for instance, social dynamics, biological economics and neural networks [13]. By simultaneously taking into account delayed nature and random jumping characteristic, the NMJS is thus proposed and has received substantial attention. More specifically, for a class of neutral systems with uncertain Markovian jump modes, the stability and H_∞ performance analysis have been conducted in [14] by exploiting techniques of Lyapunov-Krasovskii functionals (LKFs) and linear matrix inequalities (LMIs). In [15], with the aid of a non-fragile observer, the sliding mode control has been designed for NMJSs with nonlinear perturbations. In [16], to estimate the state for NMJSs with parametric uncertainties, a state estimator has been characterized. It should be noted that, in above-reviewed literature, the time-delay is considered to be of constant-type. In

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practice, some nature processes and systems inherently involve time-varying delay, which is more common and suitable in describing the delay phenomenon [17].

It has been defined in [18] that, for a dynamic system with bounded disturbances, the reachable set consists of all state trajectories that are capable to be reached from the origin. Indeed, characterizing a bound to contain the reachable set makes practical senses in, for instance, estimating parameters [19], recognizing safe regions [20] and avoiding unknown collisions [21]. This stirs up the research interest in reachable set estimation, which has been concerned for a variety of systems, such as fuzzy systems [22], positive systems [23] and singular systems [24]. It is known that, owing to the existences of time-delay and Markovian jump parameters, delayed Markovian jump systems may exhibit certain dynamic behaviors (e.g., oscillation, divergence and destabilization) which take bad effects on performance analysis and control synthesis [25, 26]. In this regard, it is of significant importance to bound the reachable set for Markovian jump systems with time-delay. And a few promising results have been reported. For example, with consideration given to delayed Markovian jump systems of discrete-time type, the reachable set estimation problem has been investigated in [27]. In order to reduce the conservatism, a novel LKF containing triple summation terms has been suitably constructed in [28]. For a class of discrete-time Markovian jump neural networks, the analytical framework of bounding the reachable set has been provided in [29]. It is noteworthy that, as is extensively studied in above-mentioned scientific research, the delayed Markovian jump systems are described by discrete-time models. Although bounding reachable set for continuous-time counterparts also makes theoretical and engineering senses, few results have been reported in the existing literature. Note also that,up to date, the reachable set estimation for neutral time-delay systems has been rarely researched. Just in [30], an ellipsoid bound has been determined for linear neutral time-delay systems. By choosing an augmented LKF, sufficient conditions on tackling the concerned reachable set estimation issue are formulated within LMIs framework. However, the reported results may not take effects when bounding the reachable set for continuous-time NMJSs. This is due not only to some inherently existent phenomena (e.g., finite mode jumping), but also to distinct analysis frameworks of deterministic systems and Markovian jump systems. Unfortunately, the reachable set estimation has not been concerned for NMJSs with time-varying delays, which serves as the crucial motivation of this study.

On the basis of above discussions, we aim to tackle the reachable set estimation problem for a calss of continuous-time NMJSs with interval time-varying delays. According to whether time-varying delays depend on Markovian jump modes, this study is basically developed as follows. For NMJSs with mode-dependent time-delays, we first consider the case that delay mode and system mode are different. By employing free-weighting matrix method and reciprocally convex combination technique, an ellipsoid-like bound is characterized for the concerned NMJSs with completely known transition probabilities. Based on the provided analytical framework, the case of same delay mode and system mode is also handled. For NMJSs with mode-independent time-delays, the reachable set estimation is given by considering cases of completely known and partially known transition probabilities. To highlight main contributions of this study, we organize the following points. First, the interval time-varying delays concerned in this study are subject to Markovian jump modes, which may be more common and suitable in describing certain real-world delay phenomena than the constant time-delays in [14, 15, 16]. Second, as is commonly assumed in [8], the variation rate of time-varying delay is less than one, which may run counter to the practical case of larger variation rate. To relax this assumption, a group of free-weighting matrices is promisingly introduced. Third, as counterparts to the discrete-time Markovian jump systems in [27, 28, 29], this study bounds reachable set for the continuous-time ones, which also makes theoretical and engineering senses while has been rarely investigated. Fourth, compared to the deterministic neutral system in [30], this study gives the first attempt to bound reachable set for NMJSs with mode-dependent time-varying delays, where delay mode and system mode are different.

Notations: For an $n \times m$-dimensional real matrix (or an n-dimensional vector), it lies in the space $\mathbb{R}^{n\times m}$ ($\mathbb{R}^n$). Let I_n and $0_{n\times m}$ denotes, respectively, the $n \times n$ identity matrix and $n \times m$ zero matrix. For real matrix X, its transpose is X^{T}. $X > 0$ ($X < 0$) implies real matrix X is symmetric and positive (negative) definite. Denote $\mathrm{col}(X_1, X_2) = [X_1^{\mathrm{T}}\, X_2^{\mathrm{T}}]^{\mathrm{T}}$, where X_1 and X_2 are compatibly dimensional matrices (or vectors). A block diagonal matrix is $\mathrm{diag}\{\cdot\}$. The symmetric term in a symmetric matrix is represented by $*$. $\mathbb{E}[\cdot]$ denotes the expectation operator. For an event, $\mathbb{P}\{\cdot\}$ is the probability.

2. Problem formulation and preliminaries

Given two finite state spaces $\mathcal{U} = \{1, 2, \cdots, M\}$ and $\mathcal{V} = \{1, 2, \cdots, N\}$, right-continuous Markovian processes $\{u_t, t \geq 0\}$ and $\{v_t, t \geq 0\}$ are respectively generated from $\mathcal{G}_u = [\pi_{rm}] \in \mathbb{R}^{M\times M}$ and $\mathcal{G}_v = [\sigma_{sn}] \in \mathbb{R}^{N\times N}$ with

$$\mathbb{P}\{u_{t+\Delta t} = m|u_t = r\} = \begin{cases} \pi_{rm}\Delta t + \mathrm{o}(\Delta t), & r \neq m, \\ 1 + \pi_{rr}\Delta t + \mathrm{o}(\Delta t), & r = m, \end{cases} \tag{1}$$

$$\mathbb{P}\{v_{t+\Delta t} = n|v_t = s\} = \begin{cases} \sigma_{sn}\Delta t + \mathrm{o}(\Delta t), & s \neq n, \\ 1 + \sigma_{ss}\Delta t + \mathrm{o}(\Delta t), & s = n, \end{cases} \tag{2}$$

where $\Delta t > 0$ and $\mathrm{o}(\Delta t) \to 0$ $(\Delta t \to 0)$. For scalars π_{rm} $(r, m \in \mathcal{U})$ and σ_{sn} $(s, n \in \mathcal{V})$, one defines that: i) $\pi_{rm} \geq 0$ $(r \neq m)$ and $\pi_{rr} = -\sum_{m=1,m\neq r}^{M} \pi_{rm}$ $(r = m)$; ii) $\sigma_{sn} \geq 0$ $(s \neq n)$ and $\sigma_{ss} = -\sum_{n=1,n\neq s}^{N} \sigma_{sn}$ $(s = n)$.

To model NMJSs with mode-dependent time-varying delays, the following differential equation is employed:

$$\begin{cases} \dot{x}(t) - C(u_t)\dot{x}(t - d_{v_t}(t)) = A(u_t)x(t) + B(u_t)x(t - \tau_{v_t}(t)) + D(u_t)w(t), \\ x(t) = 0, t \in [-\tau, 0], \end{cases} \tag{3}$$

where $x(t) \in \mathbb{R}^p$ is the state. $w(t) \in \mathbb{R}^q$ is the disturbance input constrained by

$$\mathbb{E}\left[w^{\mathrm{T}}(t)w(t)\right] \leq \bar{w}^2, \tag{4}$$

where $\bar{w} > 0$. For $u_t = r \in \mathcal{U}$, A_r, B_r, C_r and D_r are appropriately dimensional real matrices. For $v_t = s \in \mathcal{V}$, let $d_{v_t}(t) = d_s(t)$ and $\tau_{v_t}(t) = \tau_s(t)$. With constants d_{as} and τ_{as} $(a = 1, 2, 3)$, time-varying delays $d_s(t)$ and $\tau_s(t)$ satisfy

$$\begin{array}{l} 0 \leq d_{1s} \leq d_s(t) \leq d_{2s}, \dot{d}_s(t) \leq d_{3s}, \\ 0 \leq \tau_{1s} \leq \tau_s(t) \leq \tau_{2s}, \dot{\tau}_s(t) \leq \tau_{3s}. \end{array} \tag{5}$$

Based on (5), one further denotes

$$\begin{array}{l} d_1 = \min\{d_{1s}|s \in \mathcal{V}\}, d_2 = \max\{d_{2s}|s \in \mathcal{V}\}, d_3 = \max\{d_{3s}|s \in \mathcal{V}\}, \\ \tau_1 = \min\{\tau_{1s}|s \in \mathcal{V}\}, \tau_2 = \max\{\tau_{2s}|s \in \mathcal{V}\}, \tau_3 = \max\{\tau_{3s}|s \in \mathcal{V}\}. \end{array} \tag{6}$$

Accordingly, the constant τ in (3) is reasonably defined as $\tau = \max\{d_2, \tau_2\}$, where d_2 and τ_2 represent, respectively, the maximum upper bound of neutral and discrete time-varying delays.

Under zero initial condition, all reachable state trajectories of delayed NMJSs (3) constitute the following set:

$$\mathcal{X} = \{x(t) \in \mathbb{R}^p|x(t) \,\text{and}\, w(t) \,\text{satisfy}\, (3) - (5)\}. \tag{7}$$

To give estimation on the reachable set (7), we employ an ellipsoid-like bound of the following form:

$$\mathcal{E}(P) = \{z \in \mathbb{R}^p|\mathbb{E}\left[z^{\mathrm{T}}Pz\right] \leq 1\}, \tag{8}$$

where matrix $P > 0$.

Remark 1. *In some engineering systems (e.g., heat exchangers, distributed networks and car-like robots), time-delay is encountered in both the states and their derivatives [31]. This promotes the scientific research on neutral time-delay systems. As such, increasing interest has been aroused in stability analysis and control synthesis. It should be noted that, on the topic of reachable set estimation for neutral time-delay systems, little attention has been paid. This partly motivates us for this study.*

Remark 2. *It is noteworthy that, in dynamic system (3), the neutral and discrete time-delay are subject to Markovian jump modes which are different from those of system parameters. This makes practical senses in modeling certain real-world systems since more generality and flexibility can be provided [32]. In practice, the time-delay encountered in some nature processes and systems is inherently time-varying. It has been reported in [8] that, compared to constant time-delay, the time-varying type is more common and suitable in describing the delay phenomenon.*

Remark 3. *In view of (5), one can obviously get that mode-dependent time-varying delays $d_s(t)$ and $\tau_s(t)$ may preserve nonzero lower bounds. This leads to the so-called interval time-varying delay, which is extensively encountered in some practical systems (e.g., networked control systems [33]). It has been assumed in [8] that, moreover, the delay variation rate is less than one. Such assumption is a prerequisite of ensuring that some delay-derivative-dependent terms are negative definite while may result infeasible solutions to the larger variation rate case. To relax this assumption, the free-weighting matrix method is promisingly employed in this study.*

Remark 4. *As is referred in [32], $\chi(t) = \{(x(t), u_t, v_t), t \geq 0\}$ is not a Markov process. To tackle the concerned reachable set estimation issue within the framework of Markovian systems, we first define $x_t(\epsilon) = x(t+\epsilon)$, $\forall \epsilon \in [-\tau, 0]$. Then, denote $\bar{\chi}(t) = \{(x_t, u_t, v_t), t \geq 0\}$. As such, it is obvious that $\bar{\chi}(t)$ is a Markov process with the weak infinitesimal generator $\mathcal{L}$.*

The objective of this paper is to characterize the ellipsoid-like bound (8) to contain the reachable set (7) for delayed NMJSs (3) with time-varying delays and bounded disturbance. The cases of mode-dependent and mode-independent time-delays are both concerned. To proceed the subsequent analysis, we review some lemmas as follows.

Lemma 1. *[34] Given a continuously differentiable function $\nu : [a, b] \to \mathbb{R}^p$, one denotes that $\nu_1 = \nu(b) - \nu(a)$ and $\nu_2 = \nu(b) + \nu(a) - \frac{2}{b-a}\int_a^b \nu(\alpha)\mathrm{d}\alpha$. Then the following inequality*

$$\int_a^b \dot{\nu}^{\mathrm{T}}(\alpha)\mathcal{Q}\dot{\nu}(\alpha)\mathrm{d}\alpha \geq \frac{1}{b-a}\left[\begin{array}{c} \nu_1 \\ \nu_2 \end{array}\right]^{\mathrm{T}}\left[\begin{array}{cc} \mathcal{Q} & * \\ 0 & 3\mathcal{Q} \end{array}\right]\left[\begin{array}{c} \nu_1 \\ \nu_2 \end{array}\right]$$

holds for a matrix $\mathcal{Q}(\in \mathbb{R}^{p\times p}) > 0$.

Lemma 2. *[35] Given a open subset $\mathcal{O}$ of $\mathbb{R}^n$, let $h_1, h_2, \cdots, h_N : \mathbb{R}^n \to \mathbb{R}$ have positive values in $\mathcal{O}$. Over $\mathcal{O}$, one has*

$$\min_{\{\alpha_i|\alpha_i>0,\sum_i \alpha_i=1\}} \sum_i \frac{1}{\alpha_i}h_i(t) = \sum_i h_i(t) + \max_{g_{i,j}(t)} \sum_{i\neq j} g_{i,j}(t)$$

subject to

$$\left\{g_{i,j} : \mathbb{R}^n \to \mathbb{R}, g_{j,i}(t) = g_{i,j}(t), \left[\begin{array}{cc} h_i(t) & g_{i,j}(t) \\ g_{j,i}(t) & h_j(t) \end{array}\right] \geq 0\right\}.$$

3. Main results

This section, by considering cases of mode-dependent and mode-independent time-delays, is basically organized to derive main results on reachable set estimation for delayed NMJSs with bounded disturbance.

3.1. Mode-dependent time-varying delays

This subsection begins with characterizing an ellipsoid-like bound to contain the reachable set of NMJSs with mode-dependent time-varying delays, where delay mode and system mode are different. Then, based on the provided analytical framework, consideration is given to the case of same delay mode and system mode.

By denoting

$$\begin{array}{rcl} \eta_1(t) & = & \mathrm{col}(x(t), \dot{x}(t), x(t-\tau_1), x(t-\tau_2), x(t-\tau_s(t))), \\ \eta_2(t) & = & \mathrm{col}(x(t-d_1), x(t-d_2), \dot{x}(t-d_s(t)), x(t-d_s(t))), \\ \eta_3(t) & = & \mathrm{col}(\int_{t-d_s(t)}^{t-d_1} \frac{x(\alpha)}{d_s(t)-d_1}\mathrm{d}\alpha, \int_{t-d_2}^{t-d_s(t)} \frac{x(\alpha)}{d_2-d_s(t)}\mathrm{d}\alpha), \\ \eta(t) & = & \mathrm{col}(\eta_1(t), \eta_2(t), \eta_3(t), w(t)), \\ \delta_i & = & [0_{p\times\{(i-1)p\}}\ I_p\ 0_{p\times\{(11-i)p+q\}}], i = 1, 2, \cdots, 11, \\ \delta_{12} & = & [0_{q\times 11p}\ I_q], \tau_{12} = \tau_2 - \tau_1, d_{12} = d_2 - d_1, \end{array}$$

the following theorem is presented to give reachable set estimation for NMJSs (3) with different delay mode and system mode.

Theorem 1. *For $r \in \mathcal{U}$, $s \in \mathcal{V}$ and given constants $d_a > 0$ and $\tau_a > 0$ $(a = 1,2,3)$, if there exist $p \times p$ matrices $P_{rs} > 0$, $Q_a > 0$, $Q_{b,rs} > 0$ $(b = 1,2)$, $R_a > 0$, $R_{b,rs} > 0$, $S_{1,rs} > 0$, $S_2 > 0$, N_c $(c = 1,2,3,4)$, $2p \times 2p$ matrix W and a scalar $\beta > 0$ such that the following LMIs hold:*

$$\Theta_{rs} < 0, \tag{9}$$

$$\mathcal{W}_{rs} = \begin{bmatrix} \mathcal{S}_{rs} & * \\ W^{\mathrm{T}} & \mathcal{S}_{rs} \end{bmatrix} > 0, \tag{10}$$

$$\sum_{m=1}^{M} \pi_{rm} Q_{b,ms} + \sum_{n=1}^{N} \sigma_{sn} Q_{b,rn} - Q_b < 0, \tag{11}$$

$$\sum_{m=1}^{M} \pi_{rm} R_{b,ms} + \sum_{n=1}^{N} \sigma_{sn} R_{b,rn} - R_b < 0, \tag{12}$$

$$\sum_{m=1}^{M} \pi_{rm} S_{1,ms} + \sum_{n=1}^{N} \sigma_{sn} S_{1,rn} - S_2 < 0, \tag{13}$$

with

$$\begin{aligned}
\Theta_{rs} &= \Theta_{1,rs} + \Theta_{2,rs} + \Theta_3 + \Theta_{4,rs} + \Theta_5 + \Theta_{6,rs} + \Theta_{7,rs} - \tfrac{\beta}{\bar{w}^2}\delta_{12}^{\mathrm{T}}\delta_{12}, \\
\Theta_{1,rs} &= \delta_2^{\mathrm{T}} P_{rs}\delta_1 + \delta_1^{\mathrm{T}} P_{rs}\delta_2 + \delta_1^{\mathrm{T}}\left(\textstyle\sum_{m=1}^{M}\pi_{rm}P_{ms} + \sum_{n=1}^{N}\sigma_{sn}P_{rn} + \beta P_{rs}\right)\delta_1, \\
\Theta_{2,rs} &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2} Q_{b,rs} + Q_3\right)\delta_1 - \sum_{b=1}^{2}\lambda_b\delta_{b+2}^{\mathrm{T}}Q_{b,rs}\delta_{b+2} - \lambda_2(1-\tau_3)\delta_5^{\mathrm{T}}Q_3\delta_5, \\
\Theta_3 &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2}\tau_b Q_b + \bar{\sigma}\tau_{12}Q_3\right)\delta_1, \\
\Theta_{4,rs} &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2} R_{b,rs}\right)\delta_1 + \delta_2^{\mathrm{T}}R_3\delta_2 - \sum_{b=1}^{2}\lambda_{b+2}\delta_{b+5}^{\mathrm{T}}R_{b,rs}\delta_{b+5} - \lambda_4(1-d_3)\delta_8^{\mathrm{T}}R_3\delta_8, \\
\Theta_5 &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2} d_b R_b\right)\delta_1 + \bar{\sigma}d_{12}\delta_2^{\mathrm{T}}R_3\delta_2, \\
\Theta_{6,rs} &= \delta_2^{\mathrm{T}}\left(d_{12}S_{1,rs} + \tfrac{d_{12}(d_1+d_2)}{2}S_2\right)\delta_2 - \tfrac{\lambda_4}{d_{12}}\mathcal{G}^{\mathrm{T}}\mathcal{W}_{rs}\mathcal{G}, \\
\Theta_{7,rs} &= \mathcal{M}_1^{\mathrm{T}}\mathcal{N}\mathcal{M}_{2,rs} + \mathcal{M}_{2,rs}^{\mathrm{T}}\mathcal{N}^{\mathrm{T}}\mathcal{M}_1,
\end{aligned}$$

where

$$\begin{aligned}
\lambda_1 &= e^{-\beta\tau_1}, \lambda_2 = e^{-\beta\tau_2}, \lambda_3 = e^{-\beta d_1}, \lambda_4 = e^{-\beta d_2}, \bar{\sigma} = \max\{-\sigma_{ss}\}, \mathcal{S}_{rs} = \mathrm{diag}\{S_{1,rs}, 3S_{1,rs}\}, \\
G_1 &= \mathrm{col}(\delta_6 - \delta_9, \delta_6 + \delta_9 - 2\delta_{10}), G_2 = \mathrm{col}(\delta_9 - \delta_7, \delta_9 + \delta_7 - 2\delta_{11}), \mathcal{G} = \mathrm{col}(G_1, G_2), \\
\mathcal{M}_1 &= \mathrm{col}(\delta_1, \delta_2, \delta_5, \delta_8), \mathcal{M}_{2,rs} = \delta_2 - C_r\delta_8 - A_r\delta_1 - B_r\delta_5 - D_r\delta_{12}, \mathcal{N} = \mathrm{col}(N_1, N_2, N_3, N_4).
\end{aligned}$$

Then, for delayed NMJSs (3) with different delay mode and system mode, the reachable set (7) is mean-square constrained within the following ellipsoid-like bound:

$$\mathcal{E}(P_{rs}) = \{x(t) \in \mathbb{R}^p | \mathbb{E}\left[x^{\mathrm{T}}(t)P_{rs}x(t)\right] \le 1\}. \tag{14}$$

Proof. For NMJSs (3), construct the following LKF candidate:

$$V(x_t, u_t, v_t) = \sum_{i=1}^{6} V_i(x_t, u_t, v_t), \tag{15}$$

where

$$\begin{aligned}
V_1(x_t,u_t,v_t) &= x^{\mathrm{T}}(t)P(u_t,v_t)x(t), \\
V_2(x_t,u_t,v_t) &= \textstyle\sum_{b=1}^{2}\int_{t-\tau_b}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_b(u_t,v_t)x(\alpha)\mathrm{d}\alpha + \int_{t-\tau_{v_t}(t)}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha, \\
V_3(x_t,u_t,v_t) &= \textstyle\sum_{b=1}^{2}\int_{-\tau_b}^{0}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_bx(\alpha)\mathrm{d}\alpha\mathrm{d}\theta + \bar{\sigma}\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\mathrm{d}\theta, \\
V_4(x_t,u_t,v_t) &= \textstyle\sum_{b=1}^{2}\int_{t-d_b}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_b(u_t,v_t)x(\alpha)\mathrm{d}\alpha + \int_{t-d_{v_t}(t)}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha, \\
V_5(x_t,u_t,v_t) &= \textstyle\sum_{b=1}^{2}\int_{-d_b}^{0}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_bx(\alpha)\mathrm{d}\alpha\mathrm{d}\theta + \bar{\sigma}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta, \\
V_6(x_t,u_t,v_t) &= \textstyle\int_{-d_2}^{-d_1}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)S_1(u_t,v_t)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta + \int_{-d_2}^{-d_1}\int_{\theta}^{0}\int_{t+\rho}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)S_2\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\rho\mathrm{d}\theta.
\end{aligned}$$

By utilizing the weak infinitesimal generator $\mathcal{L}$ in Remark 4, one can easily get that

$$\begin{aligned}
\mathcal{L}V_1(x_t,r,s) &= \dot{x}^{\mathrm{T}}(t)P_{rs}x(t)+x^{\mathrm{T}}(t)P_{rs}\dot{x}(t)+x^{\mathrm{T}}(t)\left(\sum_{m=1}^{M}\pi_{rm}P_{ms}+\sum_{n=1}^{N}\sigma_{sn}P_{rn}\right)x(t)\\
&= -\beta V_1(x_t,r,s)+\eta^{\mathrm{T}}(t)\Theta_{1,rs}\eta(t), \qquad (16)\\
\mathcal{L}V_2(x_t,r,s) &= -\beta V_2(x_t,r,s)+x^{\mathrm{T}}(t)\left(\sum_{b=1}^{2}Q_{b,rs}\right)x(t)+x^{\mathrm{T}}(t)Q_3x(t)-\sum_{b=1}^{2}\lambda_b x^{\mathrm{T}}(t-\tau_b)Q_{b,rs}x(t-\tau_b)\\
&- (1-\dot{\tau}_s(t))e^{-\beta\tau_s(t)}x^{\mathrm{T}}(t-\tau_s(t))Q_3x(t-\tau_s(t))\\
&+ \sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}Q_{b,ms}+\sum_{n=1}^{N}\sigma_{sn}Q_{b,rn}\right)x(\alpha)\mathrm{d}\alpha\\
&+ \sum_{n=1}^{N}\sigma_{sn}\int_{t-\tau_n(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\\
&\leq -\beta V_2(x_t,r,s)+\eta^{\mathrm{T}}(t)\Theta_{2,rs}\eta(t)\\
&+ \sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}Q_{b,ms}+\sum_{n=1}^{N}\sigma_{sn}Q_{b,rn}\right)x(\alpha)\mathrm{d}\alpha\\
&+ \sum_{n=1}^{N}\sigma_{sn}\int_{t-\tau_n(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha, \qquad (17)\\
\mathcal{L}V_3(x_t,r,s) &= -\beta V_3(x_t,r,s)+x^{\mathrm{T}}(t)\left(\sum_{b=1}^{2}\tau_bQ_b\right)x(t)+\bar{\sigma}\tau_{12}x^{\mathrm{T}}(t)Q_3x(t)\\
&- \sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_bx(\alpha)\mathrm{d}\alpha-\bar{\sigma}\int_{t-\tau_2}^{t-\tau_1}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\\
&= -\beta V_3(x_t,r,s)+\eta^{\mathrm{T}}(t)\Theta_3\eta(t)\\
&+ \sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_bx(\alpha)\mathrm{d}\alpha-\bar{\sigma}\int_{t-\tau_2}^{t-\tau_1}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha, \qquad (18)\\
\mathcal{L}V_4(x_t,r,s) &= -\beta V_4(x_t,r,s)+x^{\mathrm{T}}(t)\left(\sum_{b=1}^{2}R_{b,rs}\right)x(t)+\dot{x}^{\mathrm{T}}(t)R_3\dot{x}(t)-\sum_{b=1}^{2}\lambda_{b+2}x^{\mathrm{T}}(t-d_b)R_{b,rs}x(t-d_b)\\
&- (1-\dot{d}_s(t))e^{-\beta d_s(t)}\dot{x}^{\mathrm{T}}(t-d_s(t))R_3x(t-d_s(t))\\
&+ \sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}R_{b,ms}+\sum_{n=1}^{N}\sigma_{sn}R_{b,rn}\right)x(\alpha)\mathrm{d}\alpha\\
&+ \sum_{n=1}^{N}\sigma_{sn}\int_{t-d_n(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha\\
&\leq -\beta V_4(x_t,r,s)+\eta^{\mathrm{T}}(t)\Theta_{4,rs}\eta(t)\\
&+ \sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}R_{b,ms}+\sum_{n=1}^{N}\sigma_{sn}R_{b,rn}\right)x(\alpha)\mathrm{d}\alpha\\
&+ \sum_{n=1}^{N}\sigma_{sn}\int_{t-d_n(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha, \qquad (19)
\end{aligned}$$

$$
\begin{aligned}
\mathcal{L}V_5(x_t,r,s) &= -\beta V_5(x_t,r,s)+x^{\mathrm{T}}(t)\left(\sum_{b=1}^{2}d_bR_b\right)x(t)+\bar{\sigma}d_{12}\dot{x}^{\mathrm{T}}(t)R_3\dot{x}(t)\\
&- \sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_bx(\alpha)\mathrm{d}\alpha-\bar{\sigma}\int_{t-d_2}^{t-d_1}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha\\
&= -\beta V_5(x_t,r,s)+\eta^{\mathrm{T}}(t)\Theta_5\eta(t)\\
&- \sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_bx(\alpha)\mathrm{d}\alpha-\bar{\sigma}\int_{t-d_2}^{t-d_1}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha, \qquad (20)\\
\mathcal{L}V_6(x_t,r,s) &= -\beta V_6(x_t,r,s)+\dot{x}^{\mathrm{T}}(\alpha)\left(d_{12}S_{1,rs}+\frac{d_{12}(d_1+d_2)}{2}S_2\right)\dot{x}(\alpha)\\
&+ \int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}S_{1,ms}+\sum_{n=1}^{N}\sigma_{sn}S_{1,rn}-S_2\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta\\
&- \int_{t-d_2}^{t-d_1}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)S_{1,rs}\dot{x}(\alpha)\mathrm{d}\alpha\\
&\le -\beta V_6(x_t,r,s)+\dot{x}^{\mathrm{T}}(\alpha)\left(d_{12}S_{1,rs}+\frac{d_{12}(d_1+d_2)}{2}S_2\right)\dot{x}(\alpha)\\
&+ \int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}S_{1,ms}+\sum_{n=1}^{N}\sigma_{sn}S_{1,rn}-S_2\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta\\
&- \lambda_4\int_{t-d_2}^{t-d_1}\dot{x}^{\mathrm{T}}(\alpha)S_{1,rs}\dot{x}(\alpha)\mathrm{d}\alpha. \qquad (21)
\end{aligned}
$$

In accordance with Lemma 1, a bound of the $S_{1,rs}$-dependent integral term in (21) is estimated as

$$
\begin{aligned}
-\int_{t-d_2}^{t-d_1}\dot{x}^{\mathrm{T}}(\alpha)S_{1,rs}\dot{x}(\alpha)\mathrm{d}\alpha &= -\int_{t-d_s(t)}^{t-d_1}\dot{x}^{\mathrm{T}}(\alpha)S_{1,rs}\dot{x}(\alpha)\mathrm{d}\alpha-\int_{t-d_2}^{t-d_s(t)}\dot{x}^{\mathrm{T}}(\alpha)S_{1,rs}\dot{x}(\alpha)\mathrm{d}\alpha\\
&\le -\tfrac{1}{d_s(t)-d_1}\varrho_1^{\mathrm{T}}\mathcal{S}_{rs}\varrho_1-\tfrac{1}{d_2-d_s(t)}\varrho_2^{\mathrm{T}}\mathcal{S}_{rs}\varrho_2,
\end{aligned} \qquad (22)
$$

where $\varrho_1=G_1\eta(t)$ and $\varrho_2=G_2\eta(t)$.

For any matrix $W\in\mathbb{R}^{2p\times 2p}$ satisfying (10), conducting Lemma 2 on (22) yields

$$
-\int_{t-d_2}^{t-d_1}\dot{x}^{\mathrm{T}}(\alpha)S_{1,rs}\dot{x}(\alpha)\mathrm{d}\alpha\le-\tfrac{1}{d_{12}}\eta^{\mathrm{T}}(t)\left(\mathcal{G}^{\mathrm{T}}\mathcal{W}_{rs}\mathcal{G}\right)\eta(t), \qquad (23)
$$

where $\mathcal{G}=\mathrm{col}(G_1,G_2)$.

Accordingly, it follows from (21) and (23) that

$$
\begin{aligned}
\mathcal{L}V_6(x_t,r,s) &\le -\beta V_6(x_t,r,s)+\eta^{\mathrm{T}}(t)\Theta_{6,rs}\eta(t)\\
&+ \int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}S_{1,ms}+\sum_{n=1}^{N}\sigma_{sn}S_{1,rn}-S_2\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta.
\end{aligned} \qquad (24)
$$

From (3), one can obviously get that the following zero equation holds for any matrices $N_c\in\mathbb{R}^{p\times p}$:

$$
2\varsigma^{\mathrm{T}}(t)\mathcal{N}\Big(\dot{x}(t)-C_r\dot{x}(t-d_s(t))-A_rx(t)-B_rx(t-\tau_s(t))-D_rw(t)\Big)=0, \qquad (25)
$$

where $\varsigma(t)=\mathrm{col}(x(t),\dot{x}(t),x(t-\tau_s(t)),\dot{x}(t-d_s(t)))$ and $\mathcal{N}=\mathrm{col}(N_1,N_2,N_3,N_4)$.

When giving consideration to the Q_3-dependent integral term in (17), one further derives

$$
\begin{aligned}
&\sum_{n=1}^{N}\sigma_{sn}\int_{t-\tau_n(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\\
\le\ &\sum_{n=1,n\neq s}^{N}\sigma_{sn}\int_{t-\tau_2}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha+\sigma_{ss}\int_{t-\tau_1}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\\
=\ &-\sigma_{ss}\int_{t-\tau_2}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha+\sigma_{ss}\int_{t-\tau_1}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\\
\le\ &\bar{\sigma}\int_{t-\tau_2}^{t-\tau_1}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha.
\end{aligned} \qquad (26)
$$

Similarly, the R_3-dependent integral term in (19) can be bounded as

$$\sum_{n=1}^{N}\sigma_{sn}\int_{t-d_n(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha\leq\bar{\sigma}\int_{t-d_2}^{t-d_1}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha. \tag{27}$$

By summarizing (16)-(27), it is easy to get that

$$\begin{aligned}
&\mathcal{L}V(x_t,r,s)+\beta V(x_t,r,s)-\tfrac{\beta}{\bar{w}^2}w^{\mathrm{T}}(t)w(t)\\
\leq\;&\eta^{\mathrm{T}}(t)\Theta_{rs}\eta(t)+\textstyle\sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}Q_{b,ms}+\sum_{n=1}^{N}\sigma_{sn}Q_{b,rn}-Q_b\right)x(\alpha)\mathrm{d}\alpha\\
+\;&\textstyle\sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}R_{b,ms}+\sum_{n=1}^{N}\sigma_{sn}R_{b,rn}-R_b\right)x(\alpha)\mathrm{d}\alpha\\
+\;&\textstyle\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{m=1}^{M}\pi_{rm}S_{1,ms}+\sum_{n=1}^{N}\sigma_{sn}S_{1,rn}-S_2\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta.
\end{aligned} \tag{28}$$

In view of LMIs (9)-(13), we have

$$\mathcal{L}V(x_t,r,s)+\beta V(x_t,r,s)-\frac{\beta}{\bar{w}^2}w^{\mathrm{T}}(t)w(t)\leq 0. \tag{29}$$

Taking advantage of Dynkin's formula, it deduces that $\mathbb{E}\left[V_1(x_t,r,s)\right]\leq\mathbb{E}\left[V(x_t,r,s)\right]\leq 1$, which implies the ellipsoid-like bound (14) is a reachable set estimation of delayed NMJSs (3) with different delay mode and system mode. Hence, we accomplish the proof of Theorem 1. □

Remark 5. *In [30], when estimating some integral terms (e.g., $\int_{t-\tau}^{t}\dot{x}^{\mathrm{T}}(s)G\dot{x}(s)\mathrm{d}s$) in the derivative of the constructed LKF, the well-known Jensen inequality has been utilized. It has been reported in [34] that, however, the conservatism may be inherently entailed by employing such integral inequality. To obtain the conservatism reduction, some useful integral inequalities have been proposed, such as Wirtinger-based integral inequality [34] and free-matrix-based integral inequality [36]. It is recognized that, for integral inequality technique, the conservatism is reduced at the cost of increasing calculation burden. To achieve the tradeoff between conservatism reduction and calculation complexity, this paper promisingly formulates results within the framework of Wirtinger integral inequality associated with reciprocally convex combination approach.*

By considering that evolutions of delay mode and system mode are governed by same Markov chain $\{v_t,t\geq 0\}$, the NMJSs (3) can be thus written as

$$\begin{cases}\dot{x}(t)-C(v_t)\dot{x}(t-d_{v_t}(t))=A(v_t)x(t)+B(v_t)x(t-\tau_{v_t}(t))+D(v_t)w(t),\\ x(t)=0,t\in[-\tau,0],\end{cases} \tag{30}$$

Then, the reachable set estimation issue for delayed NMJSs (30) can be tackled by the following theorem.

Theorem 2. *For $s\in\mathcal{V}$ and given constants $d_a>0$ and $\tau_a>0$ $(a=1,2,3)$, if there exist $p\times p$ matrices $P_s>0$, $Q_a>0$, $Q_{bs}>0$ $(b=1,2)$, $R_a>0$, $R_{bs}>0$, $S_{1s}>0$, $S_2>0$, N_c $(c=1,2,3,4)$, $2p\times 2p$ matrix W and a scalar $\beta>0$ such that the following LMIs hold:*

$$\Pi_s<0, \tag{31}$$

$$\mathcal{W}_s=\begin{bmatrix}\mathcal{S}_s & *\\ W^{\mathrm{T}} & \mathcal{S}_s\end{bmatrix}>0, \tag{32}$$

$$\sum_{n=1}^{N}\sigma_{sn}Q_{bn}-Q_b<0, \tag{33}$$

$$\sum_{n=1}^{N}\sigma_{sn}R_{bn}-R_b<0, \tag{34}$$

$$\sum_{n=1}^{N}\sigma_{sn}S_{1n}-S_2<0, \tag{35}$$

with

$$\begin{array}{rcl}\Pi_s &=& \Pi_{1s}+\Pi_{2s}+\Pi_3+\Pi_{4s}+\Pi_5+\Pi_{6s}+\Pi_{7s}-\frac{\beta}{\bar{w}^2}\delta_{12}^{\mathrm{T}}\delta_{12},\\ \Pi_{1s} &=& \delta_2^{\mathrm{T}}P_s\delta_1+\delta_1^{\mathrm{T}}P_s\delta_2+\delta_1^{\mathrm{T}}\left(\sum_{n=1}^{N}\sigma_{sn}P_n+\beta P_s\right)\delta_1,\\ \Pi_{2s} &=& \delta_1^{\mathrm{T}}\left(\sum_{b=1}^{2}Q_{bs}+Q_3\right)\delta_1-\sum_{b=1}^{2}\lambda_b\delta_{b+2}^{\mathrm{T}}Q_{bs}\delta_{b+2}-\lambda_2(1-\tau_3)\delta_5^{\mathrm{T}}Q_3\delta_5, \Pi_3=\Theta_3,\\ \Pi_{4s} &=& \delta_1^{\mathrm{T}}\left(\sum_{b=1}^{2}R_{bs}\right)\delta_1+\delta_2^{\mathrm{T}}R_3\delta_2-\sum_{b=1}^{2}\lambda_{b+2}\delta_{b+5}^{\mathrm{T}}R_{bs}\delta_{b+5}-\lambda_4(1-d_3)\delta_8^{\mathrm{T}}R_3\delta_8, \Pi_5=\Theta_5,\\ \Pi_{6s} &=& \delta_2^{\mathrm{T}}\left(d_{12}S_{1s}+\frac{d_{12}(d_1+d_2)}{2}S_2\right)\delta_2-\frac{\lambda_4}{d_{12}}\mathcal{G}^{\mathrm{T}}\mathcal{W}_s\mathcal{G},\\ \Pi_{7s} &=& \mathcal{M}_1^{\mathrm{T}}\mathcal{N}\mathcal{M}_{2s}+\mathcal{M}_{2s}^{\mathrm{T}}\mathcal{N}^{\mathrm{T}}\mathcal{M}_1,\end{array}$$

where $\mathcal{S}_s=\mathrm{diag}\{S_{1s},3S_{1s}\}$, $\mathcal{M}_{2s}=\delta_2-C_s\delta_8-A_s\delta_1-B_s\delta_5-D_s\delta_{12}$ *and other elements are defined in Theorem 1. Then, for delayed NMJSs (30) with same delay mode and system mode, the reachable set (7) is mean-square constrained within the following ellipsoid-like bound:*

$$\mathcal{E}(P_s)=\{x(t)\in\mathbb{R}^p|\mathbb{E}\left[x^{\mathrm{T}}(t)P_sx(t)\right]\le 1\}. \tag{36}$$

Proof. For NMJSs (30), choose a LKF candidate as follows:

$$V(x_t,v_t)=\sum_{i=1}^{6}V_i(x_t,v_t), \tag{37}$$

where

$$\begin{array}{rcl}V_1(x_t,v_t) &=& x^{\mathrm{T}}(t)P(v_t)x(t),\\ V_2(x_t,v_t) &=& \sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_b(v_t)x(\alpha)\mathrm{d}\alpha+\int_{t-\tau_{v_t}(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha,\\ V_3(x_t,v_t) &=& \sum_{b=1}^{2}\int_{-\tau_b}^{0}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_bx(\alpha)\mathrm{d}\alpha\mathrm{d}\theta+\bar{\sigma}\int_{-\tau_2}^{-\tau_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha\mathrm{d}\theta,\\ V_4(x_t,v_t) &=& \sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_b(v_t)x(\alpha)\mathrm{d}\alpha+\int_{t-d_{v_t}(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha,\\ V_5(x_t,v_t) &=& \sum_{b=1}^{2}\int_{-d_b}^{0}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_bx(\alpha)\mathrm{d}\alpha\mathrm{d}\theta+\bar{\sigma}\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta,\\ V_6(x_t,v_t) &=& \int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)S_1(v_t)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta+\int_{-d_2}^{-d_1}\int_{\theta}^{0}\int_{t+\rho}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)S_2\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\rho\mathrm{d}\theta.\end{array}$$

Based on the derivation process of Theorem 1, the conclusion of Theorem 2 can be easily obtained. In this regard, details are omitted. □

3.2. Mode-independent time-varying delays

In this subsection, the mathematical model of NMJSs with mode-independent time-delays takes the following form:

$$\begin{cases}\dot{x}(t)-C(v_t)\dot{x}(t-d(t))=A(v_t)x(t)+B(v_t)x(t-\tau(t))+D(v_t)w(t),\\ x(t)=0, t\in[-\tau,0],\end{cases} \tag{38}$$

where $d(t)$ and $\tau(t)$ are mode-independent time-varying delays and satisfy

$$\begin{array}{c}0\le d_1\le d(t)\le d_2, \dot{d}(t)\le d_3,\\ 0\le\tau_1\le\tau(t)\le\tau_2, \dot{\tau}(t)\le\tau_3.\end{array} \tag{39}$$

By considering cases of completely known and partially known transition probabilities, we are going to bound the reachable set for delayed NMJSs (38).

Give some notations as

$$\begin{array}{rcl}\bar{\eta}_1(t) &=& \mathrm{col}(x(t),\dot{x}(t),x(t-\tau_1),x(t-\tau_2),x(t-\tau(t))),\\ \bar{\eta}_2(t) &=& \mathrm{col}(x(t-d_1),x(t-d_2),\dot{x}(t-d(t)),x(t-d(t))),\\ \bar{\eta}_3(t) &=& \mathrm{col}(\int_{t-d(t)}^{t-d_1}\frac{x(\alpha)}{d(t)-d_1}\mathrm{d}\alpha,\int_{t-d_2}^{t-d(t)}\frac{x(\alpha)}{d_2-d(t)}\mathrm{d}\alpha),\\ \bar{\eta}(t) &=& \mathrm{col}(\bar{\eta}_1(t),\bar{\eta}_2(t),\bar{\eta}_3(t),w(t)).\end{array}$$

Remark 6. *With the same definition of x_t in Remark 4, one can obviously get that $\{(x_t, v_t), t \geq 0\}$ is a Markovian process, whose weak infinitesimal generator is $\mathcal{D}$.*

For reachable set estimation for delayed NMJSs (38) with completely known transition probabilities, the following theorem is formulated.

Theorem 3. *For $s \in \mathcal{V}$ and given constants $d_a > 0$ and $\tau_a > 0$ $(a = 1, 2, 3)$, if there exist $p \times p$ matrices $P_s > 0$, $Q_a > 0$, $Q_{as} > 0$, $R_a > 0$, $R_{as} > 0$, $S_{1s} > 0$, $S_2 > 0$, N_c $(c = 1, 2, 3, 4)$, $2p \times 2p$ matrix W and a scalar $\beta > 0$ such that LMIs (32), (35) and*

$$\Xi_s < 0, \tag{40}$$

$$\sum_{n=1}^{N} \sigma_{sn} Q_{an} - Q_a < 0, \tag{41}$$

$$\sum_{n=1}^{N} \sigma_{sn} R_{an} - R_a < 0, \tag{42}$$

where

$$\begin{aligned}
\Xi_s &= \Xi_{1s} + \Xi_{2s} + \Xi_3 + \Xi_{4s} + \Xi_5 + \Xi_{6s} + \Xi_{7s} - \tfrac{\beta}{\bar{w}^2}\delta_{12}^{\mathrm{T}}\delta_{12},\\
\Xi_{1s} &= \Pi_{1s}, \Xi_{2s} = \delta_1^{\mathrm{T}}\left(\textstyle\sum_{a=1}^{3} Q_{as}\right)\delta_1 - \textstyle\sum_{b=1}^{2}\lambda_b \delta_{b+2}^{\mathrm{T}} Q_{bs}\delta_{b+2} - \lambda_2(1-\tau_3)\delta_5^{\mathrm{T}} Q_{3s}\delta_5,\\
\Xi_3 &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2}\tau_b Q_b + \tau_2 Q_3\right)\delta_1,\\
\Xi_{4s} &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2} R_{bs}\right)\delta_1 + \delta_2^{\mathrm{T}} R_{3s}\delta_2 - \textstyle\sum_{b=1}^{2}\lambda_{b+2}\delta_{b+5}^{\mathrm{T}} R_{bs}\delta_{b+5} - \lambda_4(1-d_3)\delta_8^{\mathrm{T}} R_{3s}\delta_8,\\
\Xi_5 &= \delta_1^{\mathrm{T}}\left(\textstyle\sum_{b=1}^{2} d_b R_b\right)\delta_1 + d_2\delta_2^{\mathrm{T}} R_3\delta_2, \Xi_{6s} = \Pi_{6s}, \Xi_{7s} = \Pi_{7s}
\end{aligned}$$

and other elements are defined in Theorem 2. Then, for delayed NMJSs (38) with completely known transition probabilities, the reachable set (7) is mean-square constrained within the ellipsoid-like bound (36).

Proof. Considering the following LKF candidate for delayed NMJSs (38):

$$\bar{V}(x_t, v_t) = \sum_{i=1}^{6} \bar{V}_i(x_t, v_t), \tag{43}$$

where $\bar{V}_1(x_t, v_t) = V_1(x_t, v_t)$, $\bar{V}_6(x_t, v_t) = V_6(x_t, v_t)$ and

$$\begin{aligned}
\bar{V}_2(x_t, v_t) &= \textstyle\sum_{b=1}^{2}\int_{t-\tau_b}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_b(v_t)x(\alpha)\mathrm{d}\alpha + \int_{t-\tau(t)}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3(v_t)x(\alpha)\mathrm{d}\alpha,\\
\bar{V}_3(x_t, v_t) &= \textstyle\sum_{b=1}^{2}\int_{-\tau_b}^{0}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_b x(\alpha)\mathrm{d}\alpha\mathrm{d}\theta + \int_{-\tau_2}^{0}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3 x(\alpha)\mathrm{d}\alpha\mathrm{d}\theta,\\
\bar{V}_4(x_t, v_t) &= \textstyle\sum_{b=1}^{2}\int_{t-d_b}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_b(v_t)x(\alpha)\mathrm{d}\alpha + \int_{t-d(t)}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3(v_t)\dot{x}(\alpha)\mathrm{d}\alpha,\\
\bar{V}_5(x_t, v_t) &= \textstyle\sum_{b=1}^{2}\int_{-d_b}^{0}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)R_b x(\alpha)\mathrm{d}\alpha\mathrm{d}\theta + \int_{-d_2}^{0}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta.
\end{aligned}$$

With the aid of Lemmas 1 and 2, conducting similar proof line of Theorem 1 yields

$$\begin{aligned}
&\mathcal{D}\bar{V}(x_t, s) + \beta\bar{V}(x_t, s) - \tfrac{\beta}{\bar{w}^2}w^{\mathrm{T}}(t)w(t)\\
\leq\ & \bar{\eta}^{\mathrm{T}}(t)\Xi_s\bar{\eta}(t) + \textstyle\sum_{b=1}^{2}\int_{t-\tau_b}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\textstyle\sum_{n=1}^{N}\sigma_{sn}Q_{bn} - Q_b\right)x(\alpha)\mathrm{d}\alpha\\
+\ & \textstyle\int_{t-\tau(t)}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\textstyle\sum_{n=1}^{N}\sigma_{sn}Q_{3n} - Q_3\right)x(\alpha)\mathrm{d}\alpha\\
+\ & \textstyle\sum_{b=1}^{2}\int_{t-d_b}^{t} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\textstyle\sum_{n=1}^{N}\sigma_{sn}R_{bn} - R_b\right)x(\alpha)\mathrm{d}\alpha\\
+\ & \textstyle\int_{t-d(t)}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\textstyle\sum_{n=1}^{N}\sigma_{sn}R_{3n} - R_3\right)\dot{x}(\alpha)\mathrm{d}\alpha\\
+\ & \textstyle\int_{-d_2}^{-d_1}\int_{t+\theta}^{t} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\textstyle\sum_{n=1}^{N}\sigma_{sn}S_{1n} - S_2\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta\\
-\ & \textstyle\int_{t-\tau_2}^{t-\tau(t)} e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3 x(\alpha)\mathrm{d}\alpha - \int_{t-d_2}^{t-d(t)} e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha.
\end{aligned} \tag{44}$$

Then, it can be easily deduced from LMIs (32), (35) and (40)-(42) that

$$\mathcal{D}\bar{V}(x_t, s) + \beta\bar{V}(x_t, s) - \frac{\beta}{\bar{w}^2} w^{\mathrm{T}}(t)w(t) \leq 0. \tag{45}$$

In accordance with Dynkin's formula, one thus gets the conclusion of Theorem 3. This completes the proof. □

Remark 7. *Owing to the complex and costly measurement process, the information of transition probabilities may not be completely grasped. As such, it is more practical and general to consider the case of partially known transition probabilities. Define*

$$\mathcal{V} = \mathcal{V}_1^s \bigcup \mathcal{V}_2^s, \tag{46}$$

where $\mathcal{V}_1^s = \{n|\sigma_{sn}\,\text{is known}, n \in \mathcal{V}\}$ *and* $\mathcal{V}_2^s = \{n|\sigma_{sn}\,\text{is unknown}, n \in \mathcal{V}\}$*.*

Then, we present the following theorem to give reachable set estimation for delayed NMJSs (38) with partially known transition probabilities.

Theorem 4. *For* $s \in \mathcal{V}$ *and given constants* $d_a > 0$ *and* $\tau_a > 0$ $(a = 1, 2, 3)$*, if there exist* $p \times p$ *matrices* $P_s > 0$*,* $Q_a > 0$*,* $Q_{as} > 0$*,* $R_a > 0$*,* $R_{as} > 0$*,* $S_{1s} > 0$*,* $S_2 > 0$*,* N_c $(c = 1, 2, 3, 4)$*,* $T_s = T_s^{\mathrm{T}}$*,* $T_{as} = T_{as}^{\mathrm{T}}$*,* $F_s = F_s^{\mathrm{T}}$*,* $F_{as} = F_{as}^{\mathrm{T}}$*,* $2p \times 2p$ *matrix* W *and a scalar* $\beta > 0$ *such that LMIs (32) and*

$$\hat{\Xi}_s < 0, \tag{47}$$

$$\sum_{n\in\mathcal{V}_1^s} \sigma_{sn}(Q_{an} - T_{as}) - Q_a < 0, \tag{48}$$

$$\sum_{n\in\mathcal{V}_1^s} \sigma_{sn}(R_{an} - F_{as}) - R_a < 0, \tag{49}$$

$$\sum_{n\in\mathcal{V}_1^s} \sigma_{sn}(S_{1n} - F_s) - S_2 < 0, \tag{50}$$

$$P_n - T_s \leq 0, n \in \mathcal{V}_2^s, n \neq s, \tag{51}$$

$$P_n - T_s \geq 0, n \in \mathcal{V}_2^s, n = s, \tag{52}$$

$$Q_{an} - T_{as} \leq 0, n \in \mathcal{V}_2^s, n \neq s, \tag{53}$$

$$Q_{an} - T_{as} \geq 0, n \in \mathcal{V}_2^s, n = s, \tag{54}$$

$$R_{an} - F_{as} \leq 0, n \in \mathcal{V}_2^s, n \neq s, \tag{55}$$

$$R_{an} - F_{as} \geq 0, n \in \mathcal{V}_2^s, n = s, \tag{56}$$

$$S_{1n} - F_s \leq 0, n \in \mathcal{V}_2^s, n \neq s, \tag{57}$$

$$S_{1n} - F_s \geq 0, n \in \mathcal{V}_2^s, n = s, \tag{58}$$

with

$$\hat{\Xi}_s = \hat{\Xi}_{1s} + \Xi_{2s} + \Xi_3 + \Xi_{4s} + \Xi_5 + \Xi_{6s} + \Xi_{7s} - \frac{\beta}{\bar{w}^2}\delta_{12}^{\mathrm{T}}\delta_{12},$$

where

$$\hat{\Xi}_{1s} = \delta_2^{\mathrm{T}} P_s \delta_1 + \delta_1^{\mathrm{T}} P_s \delta_2 + \delta_1^{\mathrm{T}} \left(\sum_{n\in\mathcal{V}_1^s} \sigma_{sn}(P_n - T_s) + \beta P_s \right) \delta_1$$

and other elements are defined in Theorem 3. Then, for delayed NMJSs (38) with partially known transition probabilities, the reachable set (7) is mean-square constrained within the ellipsoid-like bound (36).

Proof. It is easy to get from $\sum_{n=1}^{N}\sigma_{sn}=0$ that, for any appropriately dimensional matrices $T_s=T_s^{\mathrm{T}}$, $T_{as}=T_{as}^{\mathrm{T}}$, $F_s=F_s^{\mathrm{T}}$ and $F_{as}=F_{as}^{\mathrm{T}}$, the following zero equations hold:

$$
\begin{aligned}
-x^{\mathrm{T}}(t)\left(\sum_{n=1}^{N}\sigma_{sn}T_s\right)x(t)=0,\\
-\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n=1}^{N}\sigma_{sn}T_{bs}\right)x(\alpha)\mathrm{d}\alpha=0,\\
-\int_{t-\tau(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n=1}^{N}\sigma_{sn}T_{3s}\right)x(\alpha)\mathrm{d}\alpha=0,\\
-\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n=1}^{N}\sigma_{sn}F_{bs}\right)x(\alpha)\mathrm{d}\alpha=0,\\
-\int_{t-d(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{n=1}^{N}\sigma_{sn}F_{3s}\right)\dot{x}(\alpha)\mathrm{d}\alpha=0,\\
-\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{n=1}^{N}\sigma_{sn}F_s\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta=0.
\end{aligned}
\tag{59}
$$

Simple algebraic manipulations on (44) and (59) yield:

$$
\begin{aligned}
&\mathcal{D}\bar{V}(x_t,s)+\beta\bar{V}(x_t,s)-\tfrac{\beta}{\bar{w}^2}w^{\mathrm{T}}(t)w(t)\\
\le\ &\bar{\eta}^{\mathrm{T}}(t)\hat{\Xi}_s\bar{\eta}(t)+x^{\mathrm{T}}(t)\left(\textstyle\sum_{n\in\mathcal{V}_2^s}\sigma_{sn}(P_n-T_s)\right)x(t)\\
+\ &\textstyle\sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_1^s}\sigma_{sn}(Q_{bn}-T_{bs})-Q_b\right)x(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\int_{t-\tau(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_1^s}\sigma_{sn}(Q_{3n}-T_{3s})-Q_3\right)x(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_1^s}\sigma_{sn}(R_{bn}-F_{bs})-R_b\right)x(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\int_{t-d(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_1^s}\sigma_{sn}(R_{3n}-F_{3s})-R_3\right)\dot{x}(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_1^s}\sigma_{sn}(S_{1n}-F_s)-S_2\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta\\
+\ &\textstyle\sum_{b=1}^{2}\int_{t-\tau_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_2^s}\sigma_{sn}(Q_{bn}-T_{bs})\right)x(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\int_{t-\tau(t)}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_2^s}\sigma_{sn}(Q_{3n}-T_{3s})\right)x(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\sum_{b=1}^{2}\int_{t-d_b}^{t}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_2^s}\sigma_{sn}(R_{bn}-F_{bs})\right)x(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\int_{t-d(t)}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_2^s}\sigma_{sn}(R_{3n}-F_{3s})\right)\dot{x}(\alpha)\mathrm{d}\alpha\\
+\ &\textstyle\int_{-d_2}^{-d_1}\int_{t+\theta}^{t}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)\left(\sum_{n\in\mathcal{V}_2^s}\sigma_{sn}(S_{1n}-F_s)\right)\dot{x}(\alpha)\mathrm{d}\alpha\mathrm{d}\theta\\
-\ &\textstyle\int_{t-\tau_2}^{t-\tau(t)}e^{\beta(\alpha-t)}x^{\mathrm{T}}(\alpha)Q_3x(\alpha)\mathrm{d}\alpha-\int_{t-d_2}^{t-d(t)}e^{\beta(\alpha-t)}\dot{x}^{\mathrm{T}}(\alpha)R_3\dot{x}(\alpha)\mathrm{d}\alpha.
\end{aligned}
\tag{60}
$$

By considering LMIs (32) and (47)-(58), we have

$$
\mathcal{D}\bar{V}(x_t,s)+\beta\bar{V}(x_t,s)-\frac{\beta}{\bar{w}^2}w^{\mathrm{T}}(t)w(t)\le 0. \tag{61}
$$

According to Dynkin's formula, one can get the conclusion of Theorem 4. This completes the proof. □

Remark 8. *As is widely used in [18, 19, 22], the following optimization algorithm is given to characterize the smallest possible reachable set bound:*

$$
\begin{aligned}
&\text{maximize}\,\gamma\\
&\text{s.t.}\begin{cases}\gamma I\le P_{rs},\\ (9)-(13),\end{cases}
\end{aligned}
$$

which is equivalent to

$$\begin{aligned}&\text{minimize}\,\bar{\gamma}\;(\bar{\gamma}=\frac{1}{\gamma})\\&\text{s.t.}\left\{\begin{array}{l}\begin{bmatrix}\bar{\gamma}I & I\\ I & P_{rs}\end{bmatrix}\geq 0,\\(9)-(13).\end{array}\right.\end{aligned}$$

It should be noted that the above-mentioned optimization algorithm takes Theorem 1 for example while is also applicable to Theorems 2-4.

4. Numerical examples

This section, by three numerical simulations, is organized to validate the theoretical results.

Example 1. This example concerns with the reachable set estimation for delayed NMJSs (3) with two delay modes and two system modes. Prescribe

$$\mathcal{G}_v=\begin{bmatrix}-2 & 2\\ 1 & -1\end{bmatrix},\mathcal{G}_u=\begin{bmatrix}-0.6 & 0.6\\ 0.2 & -0.2\end{bmatrix},$$

then evolutions of delay mode and system mode are depicted in Fig. 1. The system parameters are

$$\begin{aligned}A1&=\begin{bmatrix}-6.15 & 0.05\\ -0.15 & -1.50\end{bmatrix},A_2=\begin{bmatrix}-3.15 & -0.09\\ 0.21 & -2.10\end{bmatrix},B1=\begin{bmatrix}-1.27 & -0.12\\ 0.24 & -0.25\end{bmatrix},\\B2&=\begin{bmatrix}1.45 & -0.16\\ 0.47 & -1.57\end{bmatrix},C1=\begin{bmatrix}-0.08 & -0.06\\ 0.03 & 0.04\end{bmatrix},C2=\begin{bmatrix}0.05 & 0.01\\ 0.05 & -0.06\end{bmatrix},\\D1&=\begin{bmatrix}0.6\\ -0.4\end{bmatrix},D2=\begin{bmatrix}0.4\\ -0.2\end{bmatrix}.\end{aligned}$$

Set $d_1=\tau_1=0.1$, $d_2=\tau_2=0.9$, $d_3=0.4$ and $\bar{w}=1$. For different values of τ_3, Table 1 presents the maximum allowable β which can guarantee feasibility of LMIs (9)-(13) in Theorem 1.

τ_3	0.2	0.4	0.6	0.8	1.0	1.2
β	0.9476	0.8187	0.6912	0.5733	0.4092	0.2674

Table 1: Maximum allowable β for various τ_3 in Example 1.

It is obviously shown in Table 1 that, for a larger variation rate τ_3, the obtained maximum allowable β is smaller. Note that, when the variation rate $\tau_3=1.2$ ($\tau_3>1$), the solvability of the concerned reachable set estimation problem has also been ensured by our proposed approach. This relaxes the assumption $\tau_3<1$ in [8] and thereby validates the discussions in Remark 3. By solving the optimization problem in Remark 8 with $\beta=0.6943$, one gets the minimum allowable $\bar{\gamma}$ for various variation rate τ_3. And Table 2 is organized to present the obtained results.

τ_3	0.2	0.4	0.6	0.8	1.0	1.2
$\bar{\gamma}$	0.7613	0.9741	1.2929	1.9183	2.4732	3.8213

Table 2: Minimum allowable $\bar{\gamma}$ for various τ_3 in Example 1.

From Table 2, one can get that the minimum allowable $\bar{\gamma}$ increases proportionally to the variation rate τ_3. It is deduced from Remark 8 that, for a smaller τ_3, the obtained reachable set bound is tighter.

Selecting the mode-dependent time-varying delays $d_s(t)$ and $\tau_s(t)$ $(s=1,2)$ in (3) as $d_1(t)=\tau_1(t)=0.4+0.1\sin(t)$ and $d_2(t)=\tau_2(t)=0.5+0.4\sin(t)$, it follows from (5) and (6) that $d_1=\tau_1=0.1$, $d_2=\tau_2=0.9$ and $d_3=\tau_3=0.4$. For the disturbance signal $w(t)$, choose it as $w(t)=\sin(t)$. Accordingly, the constraint

(4) holds for a constant $\bar{w}=1$. By prescribing $\beta=0.8187$ to solve the optimization problem in Remark 8, we obtain some feasible solutions as

$$P_{11}=\begin{bmatrix} 1.9735 & -0.0684 \\ -0.0684 & 0.8800 \end{bmatrix}, P_{12}=\begin{bmatrix} 1.6066 & -0.0453 \\ -0.0453 & 0.8352 \end{bmatrix},$$
$$P_{21}=\begin{bmatrix} 1.2310 & -0.0304 \\ -0.0304 & 0.7513 \end{bmatrix}, P_{22}=\begin{bmatrix} 1.1690 & -0.0271 \\ -0.0271 & 0.7397 \end{bmatrix}.$$

Under zero initial condition, Fig. 2 visualizes the obtained result on reachable set estimation of NMJSs (3), where all state trajectories have been successfully contained within the characterized ellipsoid-like bound $\mathcal{E}(P_{rs})$ $(r,s=1,2)$. In this regard, we validate the conclusion of Theorem 1.

Example 2. By considering the following two cases, this example aims to give reachable set estimation for a class of delayed NMJSs, whose system parameters are given as

$$\begin{aligned} A_1 &= \begin{bmatrix} -1.15 & -0.75 \\ 1.50 & -0.45 \end{bmatrix}, A_2=\begin{bmatrix} -3.46 & -0.34 \\ 0.57 & -1.65 \end{bmatrix}, B_1=\begin{bmatrix} -1.20 & 0.12 \\ 0.24 & -0.25 \end{bmatrix}, \\ B_2 &= \begin{bmatrix} -0.67 & -1.50 \\ 1.39 & 1.23 \end{bmatrix}, C_1=\begin{bmatrix} -0.15 & -0.06 \\ 0.50 & -0.50 \end{bmatrix}, C_2=\begin{bmatrix} -0.23 & 0.16 \\ 0.02 & -0.57 \end{bmatrix}, \\ D_1 &= \begin{bmatrix} -0.26 \\ -0.20 \end{bmatrix}, D_2=\begin{bmatrix} 1.06 \\ -0.65 \end{bmatrix}. \end{aligned}$$

Case I. The NMJSs (30) concerned in this case is subject to mode-dependent time-varying delays, where delay mode and system mode are regulated by

$$\mathcal{G}_v=\begin{bmatrix} -0.6 & 0.6 \\ 0.2 & -0.2 \end{bmatrix}.$$

And the mode evolution is shown in Fig. 3.

Choosing $d_1(t)=\tau_1(t)=0.3+0.2\sin(t)$ and $d_2(t)=\tau_2(t)=0.7+0.5\sin(t)$ and $w(t)=\sin(t)$, it thus follows from (4)-(6) that $d_1=\tau_1=0.1$, $d_2=\tau_2=1.2$, $d_3=\tau_3=0.5$ and $\bar{w}=1$. By solving the optimization problem in Remark 8 with $\beta=0.5339$, one gets the minimum allowable $\bar{\gamma}=0.9416$. And some feasible solutions are calculated as

$$P_1=\begin{bmatrix} 0.7852 & -0.0301 \\ -0.0301 & 0.7856 \end{bmatrix}, P_2=\begin{bmatrix} 0.9156 & -0.0964 \\ -0.0964 & 0.9987 \end{bmatrix}.$$

Starting from the origin, all state trajectories of delayed NMJSs (30) are depicted in Fig. 4, which have been constrained by the characterized ellipsoid-like bound $\mathcal{E}(P_s)$ $(s=1,2)$.

Case II. This case concerns the NMJSs (38) with mode-independent time-delays. The evolution of system mode is regulated by $\mathcal{G}_v$, which is same to that in Case I.

For mode-independent time-varying delays $d(t)$ and $\tau(t)$, one prescribes that $d(t)=\tau(t)=0.6+0.3\sin(t)$. The bounded disturbance $w(t)$ is chosen as $w(t)=\sin(t)$. From (4) and (39), one has $d_1=\tau_1=0.3$, $d_2=\tau_2=0.9$, $d_3=\tau_3=0.3$ and $\bar{w}=1$. Then, solving the optimization problem in Remark 8 yields the minimum allowable $\bar{\gamma}$ and some feasible solutions as

$$\bar{\gamma}=0.9271, P_1=\begin{bmatrix} 0.6360 & -0.0091 \\ -0.0091 & 0.6187 \end{bmatrix}, P_2=\begin{bmatrix} 0.7259 & -0.0428 \\ -0.0428 & 0.6622 \end{bmatrix}.$$

More intuitively, Fig. 5 plots the characterized ellipsoid-like bound $\mathcal{E}(P_s)$ $(s=1,2)$, whose intersection contains the reachable set of delayed NMJSs (38). Hence, one validates the conclusions of Theorems 2 and 3.

Example 3. In this example, the reachable set estimation is conducted on delayed NMJSs (38) with partially known transition probabilities. As is shown in Fig. 6, the concerned NMJSs preserve three Markovian jump modes, whose evolution is governed by

$$\mathcal{G}_v=\begin{bmatrix} -0.5 & ? & ? \\ 0.2 & -0.4 & 0.2 \\ ? & 0.2 & ? \end{bmatrix}.$$

Prescribe the system parameters as

$$\begin{aligned}
A_1 &= \begin{bmatrix} -3 & 0 \\ 0.6 & -1 \end{bmatrix}, A_2 = \begin{bmatrix} -11 & 1 \\ -0.6 & -3.3 \end{bmatrix}, A_3 = \begin{bmatrix} -1.5 & 0.2 \\ 0.5 & -1.1 \end{bmatrix}, \\
B_1 &= \begin{bmatrix} -1.2 & 0.3 \\ -1 & -1.3 \end{bmatrix}, B_2 = \begin{bmatrix} -2.2 & -1.3 \\ 0.7 & -2.1 \end{bmatrix}, B_3 = \begin{bmatrix} -0.2 & 0.6 \\ -0.2 & -0.2 \end{bmatrix}, \\
C_1 &= \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, C_2 = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.2 \end{bmatrix}, C_3 = \begin{bmatrix} -0.5 & 0 \\ 0 & -0.5 \end{bmatrix}, \\
D_1 &= \begin{bmatrix} 0.7 \\ -0.1 \end{bmatrix}, D_2 = \begin{bmatrix} 0.9 \\ -0.9 \end{bmatrix}, D_3 = \begin{bmatrix} 0.8 \\ -0.2 \end{bmatrix}.
\end{aligned}$$

Choose $d(t) = \tau(t) = 0.7 + 0.2\sin(t)$ and $w(t) = \sin(t)$. Then, one can get from (4) and (39) that $d_1 = \tau_1 = 0.5$, $d_2 = \tau_2 = 0.9$, $d_3 = \tau_3 = 0.2$ and $\bar{w} = 1$. According to Remark 8, we conduct the optimization algorithm on Theorem 4 with $\beta = 0.8249$ such that the minimum allowable $\bar{\gamma}$ is obtained as $\bar{\gamma} = 0.7842$. And some feasible solutions can be calculated as

$$P_1 = \begin{bmatrix} 0.9960 & -0.0280 \\ -0.0280 & 0.6691 \end{bmatrix}, P_2 = \begin{bmatrix} 1.1292 & -0.1251 \\ -0.1251 & 0.7205 \end{bmatrix}, P_3 = \begin{bmatrix} 0.7122 & -0.0252 \\ -0.0252 & 0.7130 \end{bmatrix}.$$

Then, the reachable set of delayed NMJSs (38) is visualized in Fig. 7, which is contained within the characterized ellipsoid-like bound $\mathcal{E}(P_s)$ $(s = 1, 2, 3)$. As a consequence, one can conclude that the concerned reachable set estimation issue has been successfully tackled.

5. Conclusion

In this study, the reachable set estimation has been investigated for a class of NMJSs with interval time-varying delays and bounded disturbance. The time-delays are considered to be dependent or independent on Markovian jump modes. For NMJSs with mode-dependent time-delays, we first consider the case of different delay mode and system mode. By exploiting free-weighting matrix method and reciprocally convex combination technique, a sufficient condition has been derived to give reachable set estimation for the concerned NMJSs with completely known transition probabilities. Based on the provided analytical framework, another consideration is given to the case that delay mode and system mode are same. Moreover, the reachable set estimation has been conducted for NMJSs with mode-independent time-delays, where the system mode is subject to completely known transition probabilities. Thanks to a group of free-connection weighting matrices, the case of partially known transition probabilities has been also handled. By three numerical simulations, the effectiveness of the proposed approach has been demonstrated. Since the multi-agent systems have been extensively researched [37, 38, 39, 40], our future attention will be paid to estimate the reachable set for multi-agent systems.

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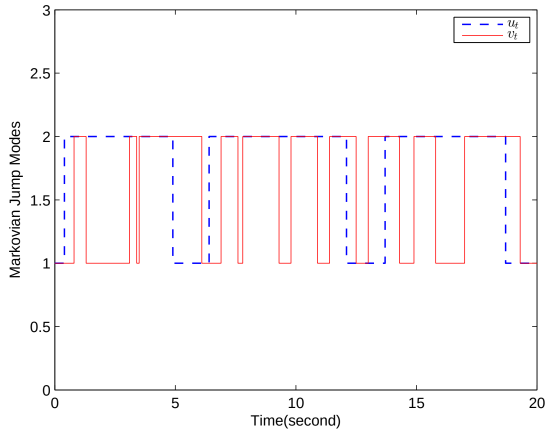

Figure 1: Modes evolution of NMJSs (3) in Example 1.

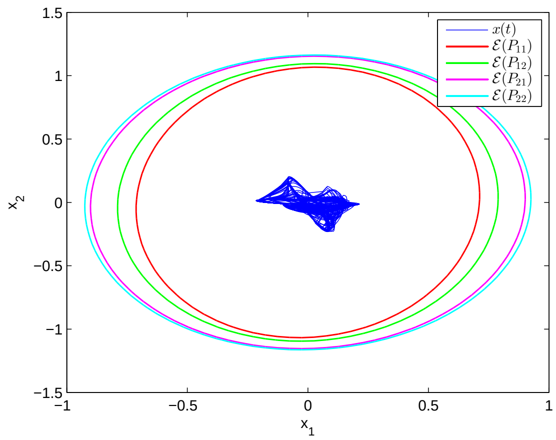

Figure 2: Result on reachable set estimation of NMJSs (3) in Example 1.

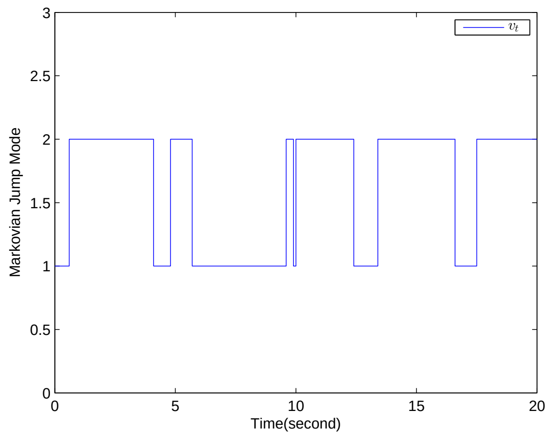

Figure 3: Mode evolution of NMJSs (30) in Example 2 for Case I.

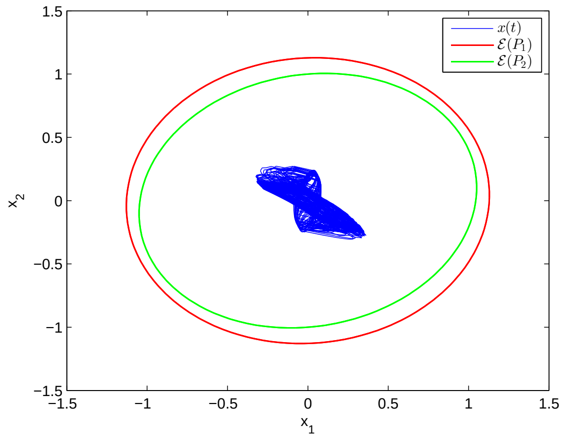

Figure 4: Result on reachable set estimation of NMJSs (30) in Example 2 for Case I.

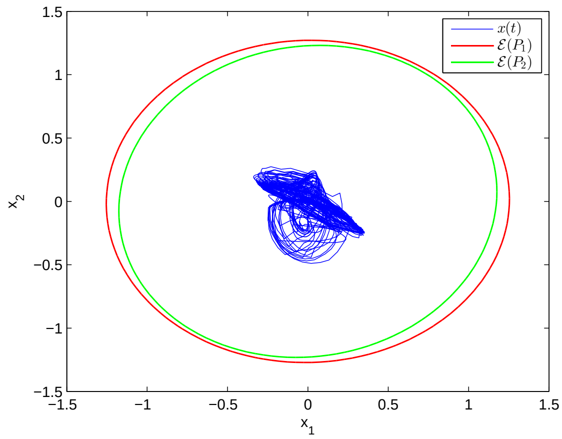

Figure 5: Result on reachable set estimation of NMJSs (30) in Example 2 for Case II.

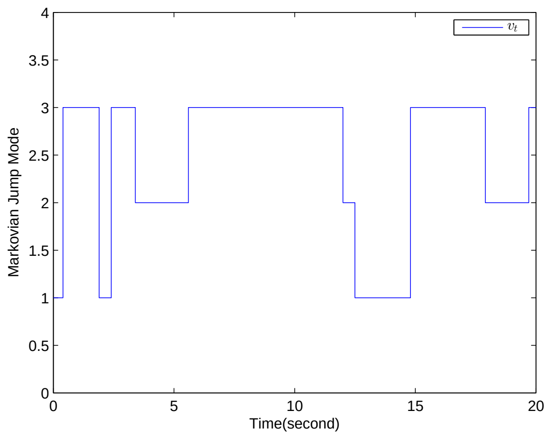

Figure 6: Mode evolution of NMJSs (38) in Example 3.

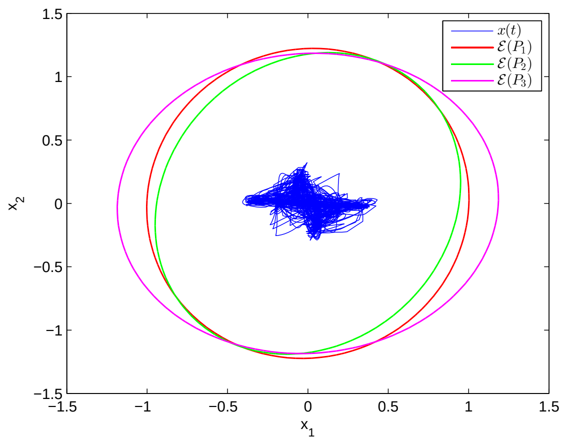

Figure 7: Result on reachable set estimation of NMJSs (38) in Example 3.