University of Wollongong

Research Online

Faculty of Engineering and Information
Sciences - Papers: Part A

Faculty of Engineering and Information
Sciences

1-1-2016

An accurate approximation formula for pricing European options with discrete dividend payments

Song-Ping Zhu

University of Wollongong, spz@uow.edu.au

Xinjiang He

University of Wollongong, xinjiang@uow.edu.au

Follow this and additional works at: <https://ro.uow.edu.au/eispapers>

Part of the [Engineering Commons](#), and the [Science and Technology Studies Commons](#)

Research Online is the open access institutional repository for the University of Wollongong. For further information contact the UOW Library: research-pubs@uow.edu.au

An accurate approximation formula for pricing European options with discrete dividend payments

Abstract

In this article, two relevant problems related to pricing European options with discrete dividend under the classic Black-Scholes framework are considered. For the case when a discrete dividend payment is proportional to the underlying asset value, we discuss an interesting phenomenon observed; the option price is independent of the dividend payment date. This appears to be at odds with one's intuition that dividend amount, as well as the dividend date, should both affect the price of a European call or put option. We reveal the fundamental reasons, from both mathematical and financial viewpoints, why this occurs. When the amount of the discrete dividend is fixed, we provide an approximation formula for European option prices, with only one-dimensional integrals involved. It should be noted that our formula is a general one since it can not only be applied when there is only a single dividend, but also be suitable for the case of multiple dividends.

Keywords

payments, discrete, options, dividend, european, accurate, pricing, formula, approximation

Disciplines

Engineering | Science and Technology Studies

Publication Details

Zhu, S. & He, X. (2018). An accurate approximation formula for pricing European options with discrete dividend payments. *IMA Journal of Management Mathematics*, 29 (2), 175-188.

An accurate approximation formula for pricing European options with discrete dividend payments

SONG-PING ZHU[†] AND XIN-JIANG HE

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, NSW 2522, Australia

[†]Corresponding author: Email: spz@uow.edu.au

[Received on 30 November 2015; accepted on 14 September 2016]

In this article, two relevant problems related to pricing European options with discrete dividend under the classic Black–Scholes framework are considered. For the case when a discrete dividend payment is proportional to the underlying asset value, we discuss an interesting phenomenon observed; the option price is independent of the dividend payment date. This appears to be at odds with one's intuition that dividend amount, as well as the dividend date, should both affect the price of a European call or put option. We reveal the fundamental reasons, from both mathematical and financial viewpoints, why this occurs. When the amount of the discrete dividend is fixed, we provide an approximation formula for European option prices, with only one-dimensional integrals involved. It should be noted that our formula is a general one since it can not only be applied when there is only a single dividend, but also be suitable for the case of multiple dividends.

Keywords: discrete dividend; approximation formula; multi-period.

1. Introduction

One of the incentives for people to invest in stock market is the attraction of receiving dividend payments, in addition to the potential gain of appreciation of the stock value itself. For options written on stocks with dividend payments, an interesting problem is always to find out the correct way to count in the dividend payment in pricing option values. There are generally two kinds of models for pricing these options, i.e., continuous dividend and discrete dividend models. The former is firstly used by Merton (1973) who incorporated the continuous dividend under the framework of the Black–Scholes (B–S) model (cf. Black & Scholes, 1973), which is a well-known model and has been adopted by a number of researchers (cf. Hyland *et al.*, 1999; Hui *et al.*, 2007), and provided a similar price formula. It is also adopted by a number of other authors, such as Krausz (1985), Chung & Shackleton (2003) and so on.

On the other hand, pricing options written on the underlying with discretely paid dividend is usually harder than the case with continuous dividend payment, simply because the latter does not involve any jump conditions in a partial differential equation (PDE) pricing system. However, the former is much closer to reality as hardly any underlying asset involved in exchange-traded options bears with a continuous dividend yield. The only exception would be foreign exchange (FX) options, in which the involved underlying is usually another currency, the interest of which could be viewed as a continuous dividend yield if the compounding period is sufficiently small, say daily compounding, which is a usual case in today's markets.

In this article, we therefore focus on discrete dividends, which can themselves be divided in two types. One is paid proportional to the underlying price at the payment day (denoted by the proportional dividend), while another is paid independent of the underlying price level (denoted by the fixed-amount

dividend). Firstly, we discuss an interesting phenomenon observed under the classic B–S framework; the option price is independent of the dividend payment date in the case of the proportional dividend. This appears to be at odds with one’s intuition that dividend amount, as well as the dividend date, should both affect the price of a European call or put option. Through a proper discussion, we shall show, both mathematically and financially, that while a European option price indeed depends up on not only the amount of the dividend, but also the ex-dividend date when the declared dividend is a fixed amount, it depends on the dividend yield rate only in the case of the discrete dividend payment being proportional to the underlying asset value. Merton (1973) seemed to be the first one to derive an analytical solution for European options with the proportional dividend under the B–S model.

However, when the discrete dividend is a fixed amount instead of being proportional to the underlying price, the simplicity and tractability cannot be preserved and no exact solution has been discovered, and thus the European option pricing problem with fixed-amount dividend, which is the second issue discussed in this article, has drawn plenty of attention among academic researchers. This has prompted the development of a number of such models, including the three popular ones, i.e. Escrowed model, Forward model and Piecewise log-normal model (cf. Frishling, 2002), among which the Piecewise log-normal model, assuming that the stock price jumps down at the date of dividend payment and follows a geometric Brownian motion during each non-dividend payment period, can yield most close-to-reality results, as pointed out by Frishling (2002) by comparing these three models. Unfortunately, there is no closed-form solution for this model, which makes it hard to be implemented in real markets since the calibration process is time consuming and the calculation of option prices with numerical methods can cost much more time. Therefore, we then derive a general approximation formula similar to the B–S formula with the Taylor series approximation technique, specified as the sum of one-dimensional integrals, for all the cases of the single and multiple dividends. Subsequently, numerical experiments are established to show the accuracy of this approximation.

The rest of the article is organized as follows. In Section 2, a detailed discussion on the reason why the dividend payment date has no effect on the option price in the case of the dividend yield is carried out. In Section 3, a good approximation formula for the European call option price with fixed-amount dividend is presented, followed by some numerical results demonstrating its accuracy. Concluding remarks are given in the last section.

2. European options with dividend yield

Unlike the materials presented in the next section, the key point of this section is to provide a convincing financial explanation, to a seemingly confusing conclusion that the option price of a European call or put with a discrete dividend payment is independent of the dividend payment date for the case of the proportional dividend to the underlying price at the ex-dividend day. Since such an explanation in this way is not well documented in the literature, we shall also provide a bit more detailed mathematical explanation for the completeness of the article as well as easiness for the readers.

In the classic B–S framework, the price of an option can be easily worked out by the well-known B–S formula (cf. Black & Scholes, 1973), which is the simplest case when there is no dividend payment between the current time t and the expiry date T . For the case with a continuous dividend yield paid to the underlying asset S , the formula can be easily modified to account for the influence of the continuous dividend yield to the option price evaluated at time t (cf. Wilmott *et al.*, 1993, pp. 91–93).

On the other hand, if the dividend payments are discrete, a jump condition must be imposed at each dividend payment date and the evaluation needs to be carried out from one dividend date to another, starting from the expiry time and moving backward until one reaches the current time t . Without loss of

generality, all one needs to consider is a special case with one known discrete dividend payment between t and T .

If the ex-dividend date is denoted as t_d , the jump condition of the option price across the dividend payment date is

$$V(S(t_d^-), t_d^-) = V(S(t_d^+), t_d^+), \quad (2.1)$$

where t_d^- and t_d^+ denote the time immediately before and after the dividend date, respectively. From (2.1), one may be puzzled where the ‘jump’ is, as the equation seems to suggest that the value of V does not change across the ex-dividend date. The right way to understand this jump condition is that it is a ‘horizontal jump’ in the sense that there is a jump in the independent variable S

$$S(t_d^+) = S(t_d^-) - D_{t_d^-} \quad (2.2)$$

rather a ‘vertical jump’ taking place in the dependent variable V . In (2.2), $D_{t_d^-}$ is the value of the dividend immediately before the dividend payment date.

For the case of the discrete dividend payment being declared as a given percentage of the underlying asset value right before the ex-dividend date, t_d , (i.e. $D_{t_d^-} = d_y S(t_d^-)$) with d_y being the given percentage, there is a closed-form pricing formula for a European call option $C_d(S, t; E, r, \sigma, T, t_d, d_y)$ and a European put option $P_d(S, t; E, r, \sigma, T, t_d, d_y)$, which are given as

$$C_d(S, t; E, r, \sigma, T, t_d, d_y) = (1 - d_y)C_E(S, t; E(1 - d_y)^{-1}, r, \sigma, T), \quad (2.3)$$

and

$$P_d(S, t; E, r, \sigma, T, t_d, d_y) = (1 - d_y)P_E(S, t; E(1 - d_y)^{-1}, r, \sigma, T), \quad (2.4)$$

the derivation of which can be found in (Wilmott *et al.*, 1993, pp. 93–97). Here, $C_E(S, t; E, r, \sigma, T)$ and $P_E(S, t; E, r, \sigma, T)$ represent the classic B–S formula for the corresponding call and put option without a dividend payment, respectively.

The disappearance of t_d in (2.3) and (2.4) implies that the option price at time t , prior to the dividend payment, has nothing to do with how long one has to wait until the dividend becomes ex-dividend. This may seem to be against one’s financial instinct and needs a financial explanation. The key to reveal the financial interpretation of the option price being independent of the time to ex-dividend date for the case of a discrete dividend being proportional to the underlying asset price right before the ex-dividend date is the main reason for the first part of this short article.

The fundamental reason is that in the classic B–S framework, the underlying asset is assumed to follow a geometric Brownian motion with a constant volatility rate $\sigma > 0$

$$dS_t = \mu S_t dt + \sigma S_t dW, \quad (2.5)$$

where μ is the drift rate and dW denotes increments of a standard Wiener process defined on a complete probability space $(\Omega, \mathcal{F}, Q)$. With this assumption, the discounted expectation of the underlying asset price $e^{-r(t_d-t)}E(S_{t_d})$ is nothing but S_t , which is independent from t_d and it is a known quantity at the time when an option contract needs to be priced. So, when an option needs to be priced at time t , knowing that there is a dividend payment at t_d but the dividend amount is not known at that moment (it is a random

number $d_y S_{t_d^-}$), the only ‘fair’ way, for both the buyer and the writer of the option, to account for the contribution of the dividend to the price of the option at time t is to count in its discounted expectation, just like we take the discounted expectation of the payoff function of an option as the ‘fair’ price of the option under the no arbitrage argument. When the discounted expectation of the discrete dividend payment is independent of the time to dividend date (S_t in this case), it is the fundamental reason why the option price evaluated at t is independent of t_d for the case when the dividend payment is assumed to be proportional to the underlying asset price right before the ex-dividend date as shown in (2.3) and (2.4).

Some people may try to understand the reason that a call option price does not change across the dividend payment with the argument that the option holder does not benefit from a dividend payment, such an argument certainly does not explain why a put option value does not change across a discrete dividend payment day, as the holder of a put option does pocket the dividend payment. The fundamental reason is that the effect of the dividend payment has already been factorized in formula (2.3) for a call or formula (2.4) for a put and the option holder cannot exercise their option until the expiry day, when he/she holds a European style option. For calls, it is straightforward to show that $C_d(S, t)$ is always less than its vanilla counterpart without dividend payment to count for the fact that the writer of the option will pocket the dividend. For puts, it can also be shown that $P_d(S, t)$ is always more than its vanilla counterpart without dividend payment to compensate the writer of the option as holder will receive the dividend. However, such a financial intuition is not so obvious mathematically. A put with a higher strike price is worth more. But, the less-than-one factor $(1 - d_y)$ in front of $P_E(S, t; E(1 - d_y)^{-1}, r, \sigma, T)$ in (2.4) could make the product smaller and thus the price of a put with a discrete dividend could be cheaper than its counterpart without any dividend payment at all. Fortunately, to mathematically prove that $aP_E(S, t; \frac{E}{a}, r, \sigma, T)$, with $0 < a \leq 1$, is a monotonically decreasing function for any $S > 0$ is not too difficult. The actual proof is thus omitted here.

It should be mentioned in passing that a very similar formula is given in Shreve (2004). However, no financial explanation was provided there, in terms of why the option price has no dependence on the dividend payment time. Also, the formula stated in (Shreve, 2004, pp. 238–240) is in a format that the ‘initial stock price’ has been replaced in the BS formula, in order to count in the effect of the discrete dividend payment. Mathematically, of course, there is nothing wrong there. But, financially, it is far better to adjust the strike price, as shown in Wilmott *et al.* (1993), because the underlying price (or the ‘initial stock price’) is usually a given value and the advantage of using (2.3) (or (2.4) for puts) is the interpretation that the reduced (or increased) option price is achieved through an equivalent option with a higher strike price.

Before leaving this section, it should be remarked that with discrete dividend payments, the put-call parity between European options is not of the same form as the case without the dividend payments. With no arbitrage argument, one can easily establish that the revised put-call parity for a single discrete dividend paid at t_d is

$$S_t - \text{PV}(D_{t_d^-}) + P_d(S, t) - C_d(S, t) = Ee^{-r(T-t)}, \quad (2.6)$$

as shown in Guo & Su (2006). In (2.6), $\text{PV}(D_{t_d^-})$ is the present value of the dividend value right before the ex-dividend date, $D_{t_d^-}$. For a discrete dividend proportional to the underlying price at t_d^- , $\text{PV}(D_{t_d^-}) = d_y S_t$, as discussed earlier. With the put-call parity (2.6), it is then consistent that both puts and calls do not change values, while there is a downward jump of the underlying price across the dividend payment date.

In contrast, as stated before that the European option price depends on not only the amount of the dividend, but also the dividend payment date when the amount of dividend is fixed, the price formula can

be quite difficult to be derived and there exists no closed-form formula for it, which makes our accurate approximation formula presented in the next section rather valuable.

3. European options with fixed-amount dividend

In this section, we will present an excellent approximation formula for the European call option price with fixed-amount dividend under the B–S framework. In particular, we will firstly derive a semi-closed-form pricing formula for options with single dividend and then extend it to a general case with multiple discrete dividends. Before we go any further, some notations used in this section should be introduced first. We let $f^{(n)}(S, t; t_{d_k}, 1 \leq k \leq n)$ be the European call option price paying n -period dividend, T stand for the expiry time, S and K represent the underlying price and the strike price, respectively. We also denote $C(S, t)$ to be the B–S European call option price at time t with no dividend paid in the life of the option contract.

3.1. Single fixed-amount dividend

In this subsection, a simple case is considered that a stock only pays one discrete dividend D at the time t_d before the expiry time T . Then it is not difficult to know that when $t > t_d$, the option price is exactly the same as the B–S price $C(S, t)$ since there is no dividend to be paid after time t_d . As for the case of $t < t_d$, we can also easily find the pricing PDE, which is exactly the B–S equation specified as

$$\frac{\partial f^{(1)}}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f^{(1)}}{\partial S^2} + rS \frac{\partial f^{(1)}}{\partial S} - rf^{(1)} = 0, \quad t < t_d, \quad (3.1)$$

where r is the risk-free interest rate and σ is the volatility. According to the continuity of option prices, the boundary condition for the PDE system (3.1) can be derived as

$$f^{(1)}(S_{t_d}, t_d^-; t_d) = \begin{cases} C(S_{t_d} - D, t_d^+), & S_{t_d} \geq D, \\ 0, & S_{t_d} < D. \end{cases}$$

After making the transformation of

$$S = Ke^x, \tau = \frac{1}{2}\sigma^2(t_d - t), f^{(1)} = Ke^{-\frac{1}{2}(m-1)x - \frac{1}{4}(m+1)^2\tau} u(x, \tau), m = \frac{2r}{\sigma^2}, \quad (3.2)$$

the PDE system (3.1) can be simplified to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2}, \quad \tau > 0, \quad (3.3)$$

with the boundary condition

$$u(x, 0) = \begin{cases} e^{\frac{1}{2}(m-1)x} \left[\left(e^x - \frac{D}{K}\right) N(z_1) - e^{-r(T-t_d)} N(z_2) \right], & x \geq \ln\left(\frac{D}{K}\right), \\ 0, & x < \ln\left(\frac{D}{K}\right), \end{cases}$$

where $z_1 = \frac{\ln\left(e^x - \frac{D}{K}\right) + \left(r + \frac{1}{2}\sigma^2\right)(T - t_d)}{\sigma\sqrt{T - t_d}}$, $z_2 = z_1 - \sigma\sqrt{T - t_d}$ and $N(\cdot)$ denotes the normal cumulative distribution function. The newly obtained PDE system is a heat equation with an initial condition,

which can be solved through a convolution of the initial condition with a fundamental solution (cf. [Friedman, 2013](#)) as follows

$$\begin{aligned} u(x, \tau) &= \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} u(s, 0) e^{\frac{(x-s)^2}{4\tau}} ds, \\ &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m-1)s - \frac{(x-s)^2}{4\tau}} \left[\left(e^s - \frac{D}{K} \right) N(z_1) - e^{-r(T-t_d)} N(z_2) \right] ds. \end{aligned} \quad (3.4)$$

Ideally, we want to derive a formula in a nice form as the case when the dividend is paid proportional to the underlying price discussed in the previous section. Unfortunately, the analytic simplicity could not be preserved when dividends are paid with fixed amount and no exact solution exists (cf. [Dai, 2009](#)). This could make the model hard to be implemented in real markets since the calculation of European option prices with n -period dividends would involve the numerical computation of a $(n+1)$ -dimensional integral, which is rather slow. As a result, some approximation methods have already been established to try to seek more efficient ways to price European options with discrete dividends. For example, [Amaro de Matos et al. \(2006\)](#) provided accurate bounds for the option price with a single dividend, while [Dai & Lyuu \(2006\)](#) approximated fixed-amount dividends with continuous dividend yields and provided their solution with the aid of recursive formulae. However, none of these are satisfactory since they are not general or efficient. Therefore, we aimed to find an approximation method that can not only be applied when there is only a single dividend, but also be extended to the case of multiple dividends, with a semi-closed-form pricing formula constructed by only one-dimensional integrals so that the calculation of this formula can be quite rapid, which is presented in the following.

Actually, it is not difficult to find that the existence of the term $\ln\left(e^s - \frac{D}{K}\right)$ is the main reason that leads to the difficulty in carrying out the integral (3.4) with respect to s . Thereby, what we will do is to provide an appropriate approximation for this term through a truncated Taylor series expansion. Given that $N(\cdot)$ is a bounded function, it is natural to notice that the integrand in Equation (3.4) will approach zero if the absolute value of s is large enough since e^{-s^2} decreases rapidly as s becomes larger, which implies that we should focus on the case that the absolute value of s is not very large when we make approximation. As a result, if we take into account the fact that $\frac{D}{K}$ should be quite small in real markets, we can obtain

$$\begin{aligned} \ln\left(e^s - \frac{D}{K}\right) &= \ln(e^s) - \frac{D}{K}e^{-s} + \mathbb{O}\left(\frac{D^2}{K^2}\right), \\ &= \ln(e^s) - \frac{D}{K}(1-s) + \mathbb{O}\left(\frac{D^2}{K^2}\right) + \frac{D}{K}\mathbb{O}(s^2), \\ &= s\left(1 + \frac{D}{K}\right) - \frac{D}{K} + \mathbb{O}\left(\frac{D}{K}\right). \end{aligned}$$

Hence, in the following, $\ln\left(e^s - \frac{D}{K}\right)$ will be replaced by $s\left(1 + \frac{D}{K}\right) - \frac{D}{K}$ in deriving the approximation formula. Actually, if we rewrite Equation (3.4) as

$$u(x, \tau) = A_1^{(1)} - A_2^{(1)} - A_3^{(1)}, \quad (3.5)$$

where

$$\begin{aligned} A_1^{(1)} &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m+1)s - \frac{(x-s)^2}{4\tau}} N(z_1) ds, \\ A_2^{(1)} &= \frac{D}{K} \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m-1)s - \frac{(x-s)^2}{4\tau}} N(z_1) ds, \\ A_3^{(1)} &= \frac{1}{2\sqrt{\pi\tau}} e^{-r(T-t_d)} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m-1)s - \frac{(x-s)^2}{4\tau}} N(z_2) ds, \end{aligned}$$

we need to work out the above three integrals, respectively. Then, applying the approximation $\ln\left(e^s - \frac{D}{K}\right) \doteq s\left(1 + \frac{D}{K}\right) - \frac{D}{K}$ yields

$$A_1^{(1)} \doteq \frac{1}{2\sqrt{\pi\tau}} e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} \int_{\ln(\frac{D}{K})}^{+\infty} e^{-\frac{[s-(m+1)\tau-x]^2}{4\tau}} N\left(\frac{s\left(1 + \frac{D}{K}\right) - \frac{D}{K} + (r + \frac{1}{2}\sigma^2)(T-t_d)}{\sigma\sqrt{T-t_d}}\right) ds, \quad (3.6)$$

which can be further approximated to

$$A_1^{(1)} \doteq \frac{1}{2\sqrt{\pi\tau}} e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} \int_{-\infty}^{+\infty} e^{-\frac{[s-(m+1)\tau-x]^2}{4\tau}} N\left(\frac{s\left(1 + \frac{D}{K}\right) - \frac{D}{K} + (r + \frac{1}{2}\sigma^2)(T-t_d)}{\sigma\sqrt{T-t_d}}\right) ds, \quad (3.7)$$

with the fact that $\ln\left(\frac{D}{K}\right)$ will approach $-\infty$ when $\frac{D}{K}$ is close to zero. Then, applying the following transformation $y = \frac{s - (m+1)\tau - x}{\sqrt{2\tau}}$ yields

$$\begin{aligned} A_1^{(1)} &= e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} N\left(\frac{\left(1 + \frac{D}{K}\right)[\sqrt{2\tau}y + x + (m+1)\tau] + (r + \frac{1}{2}\sigma^2)(T-t_d) - \frac{D}{K}}{\sigma\sqrt{T-t_d}}\right) dy, \\ &= e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} N\left(\frac{\left(1 + \frac{D}{K}\right)[x + (m+1)\tau] + (r + \frac{1}{2}\sigma^2)(T-t_d) - \frac{D}{K}}{\sigma\sqrt{T-t_d + \left(1 + \frac{D}{K}\right)^2(t_d - t)}}\right), \end{aligned} \quad (3.8)$$

the last step of which is obtained according to Lemma 2 in Azzalini (1985). In a similar fashion, the approximation of $A_2^{(1)}$ and $A_3^{(1)}$ can then be derived as

$$\begin{aligned} A_2^{(1)} &\doteq \frac{D}{K} e^{\frac{m-1}{2}x + \frac{(m-1)^2}{4}\tau} N\left(\frac{\left(1 + \frac{D}{K}\right)[x + (m-1)\tau] + (r + \frac{1}{2}\sigma^2)(T-t_d) - \frac{D}{K}}{\sigma\sqrt{T-t_d + \left(1 + \frac{D}{K}\right)^2(t_d - t)}}\right), \\ A_3^{(1)} &\doteq e^{-r(T-t_d)} e^{\frac{m-1}{2}x + \frac{(m-1)^2}{4}\tau} N\left(\frac{\left(1 + \frac{D}{K}\right)[x + (m-1)\tau] + (r - \frac{1}{2}\sigma^2)(T-t_d) - \frac{D}{K}}{\sigma\sqrt{T-t_d + \left(1 + \frac{D}{K}\right)^2(t_d - t)}}\right). \end{aligned}$$

Finally, to show the solution under the original parameters, we arrive at

$$f^{(1)}(S, t; t_d) = SN(d_0^{(1)}) - De^{-r(t_d-t)}N(d_1^{(1)}) - Ke^{-r(T-t)}N(d_2^{(1)}), \quad (3.9)$$

where

$$\begin{aligned} d_0^{(1)} &= \frac{(1 + \frac{D}{K}) \ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2) [T - t_d + (1 + \frac{D}{K})(t_d - t)] - \frac{D}{K}}{\sigma \sqrt{T - t_d + (1 + \frac{D}{K})^2 (t_d - t)}}, \\ d_1^{(1)} &= \frac{(1 + \frac{D}{K}) \ln(\frac{S}{K}) + (r + \frac{1}{2}\sigma^2) (T - t_d) + (r - \frac{1}{2}\sigma^2) (1 + \frac{D}{K})(t_d - t) - \frac{D}{K}}{\sigma \sqrt{T - t_d + (1 + \frac{D}{K})^2 (t_d - t)}}, \\ d_2^{(1)} &= \frac{(1 + \frac{D}{K}) \ln(\frac{S}{K}) + (r - \frac{1}{2}\sigma^2) [T - t_d + (1 + \frac{D}{K})(t_d - t)] - \frac{D}{K}}{\sigma \sqrt{T - t_d + (1 + \frac{D}{K})^2 (t_d - t)}}. \end{aligned}$$

By now, we have established the semi-closed-form formula for the case of the single fixed-amount dividend and we name it ‘one-period formula’. However, as known to us all that the stock could pay the dividend more than one time during the lifetime of an option in real markets, the one-period formula would become inadequate, and thus whether a simple formula could also be obtained for the case of multiple fixed-amount dividends with our approximation method is vital and will be presented in the next subsection.

3.2. Multiple fixed-amount dividends

In this subsection, we will show that our approximation approach in the last subsection can be extended to the case of multiple dividends, where the stock pays dividend D_n at time t_{d_n} with $t_{d_n} > t_{d_{n-1}}$. We will firstly consider a two-period case and then give a general formula for n -period case with the help of mathematical induction.

When the stock only pays D_1 and D_2 at time t_{d_1} and t_{d_2} , respectively, it is easy to deduce that the option price will equal to the B–S price $C(S, t)$ if $t > t_{d_2}$, while it can be calculated from our one-period formula if $t_{d_1} < t < t_{d_2}$. So the only period that we need to price is $[0, t_{d_1}]$, where the pricing PDE is still the B–S equation with the terminal condition

$$f^{(2)}(S_{t_{d_1}}, t_{d_1}^-; t_{d_1}, t_{d_2}) = \begin{cases} f^{(1)}(S_{t_{d_1}} - D_1, t_{d_1}^+; t_{d_2}), & S_{t_{d_1}} \geq D_1, \\ 0, & S_{t_{d_1}} < D_1, \end{cases}$$

With the same transformation (3.2) except that $f^{(1)}$ and t_d are replaced by $f^{(2)}$ and t_{d_1} , respectively, we can certainly obtain the following PDE

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2},$$

with the initial condition

$$u(x, 0) = \begin{cases} e^{\frac{1}{2}(m-1)x} \left[\left(e^x - \frac{D}{K} \right) N(d_{00}^{(1)}) - \frac{D}{K} e^{-r(t_{d_2}-t_{d_1})} N(d_{11}^{(1)}) + e^{-r(T-t_{d_1})} N(d_{22}^{(1)}) \right], & x \geq \ln\left(\frac{D_1}{K}\right), \\ 0, & x < \ln\left(\frac{D_1}{K}\right), \end{cases}$$

where $\{d_{kk}^{(1)}, k = 0, 1, 2\}$ are the same as $\{d_k^{(1)}, k = 0, 1, 2\}$ but t and t_d are replaced by t_{d_1} and t_{d_2} , respectively. To solve this heat equation, we can again apply the convolution and obtain

$$u(x, \tau) = A_1^{(2)} - A_2^{(2)} - A_3^{(2)} - A_4^{(2)}, \quad (3.10)$$

where

$$\begin{aligned} A_1^{(2)} &= \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m+1)s - \frac{(x-s)^2}{4\tau}} N(d_{00}^{(1)}) ds, \\ A_2^{(2)} &= \frac{D}{K} \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m-1)s - \frac{(x-s)^2}{4\tau}} N(d_{00}^{(1)}) ds, \\ A_3^{(2)} &= \frac{D}{K} e^{-r(t_{d_2} - t_{d_1})} \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m-1)s - \frac{(x-s)^2}{4\tau}} N(d_{11}^{(1)}) ds, \\ A_4^{(2)} &= e^{-r(T - t_{d_1})} \frac{1}{2\sqrt{\pi\tau}} \int_{\ln(\frac{D}{K})}^{+\infty} e^{\frac{1}{2}(m-1)s - \frac{(x-s)^2}{4\tau}} N(d_{22}^{(1)}) ds. \end{aligned}$$

It is obvious that all the above four integrals are to be worked out to find the final solution. When we apply the same approximation method as the last subsection, we would surely obtain

$$A_1^{(2)} \doteq \frac{1}{2\sqrt{\pi\tau}} e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} \int_{-\infty}^{+\infty} e^{-\frac{[s - (m+1)\tau - x]^2}{4\tau}} N(d_{00}^{(1)}) ds, \quad (3.11)$$

which can be further calculated as

$$\begin{aligned} A_1^{(2)} &= e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}} N\left(\frac{\left(1 + \frac{D_2}{K}\right)\left(1 + \frac{D_1}{K}\right)[\sqrt{2\tau}y + x + (m+1)\tau] + M}{\sigma\sqrt{T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)^2(t_{d_2} - t_{d_1})}}\right) dy, \\ &= e^{\frac{m+1}{2}x + \frac{(m+1)^2}{4}\tau} N\left(\frac{\left(1 + \frac{D_2}{K}\right)\left(1 + \frac{D_1}{K}\right)[x + (m+1)\tau] + M}{\sigma\sqrt{T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)^2(t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right)^2\left(1 + \frac{D_1}{K}\right)^2(t_{d_1} - t)}}\right), \end{aligned}$$

with the transformation of $y = \frac{s - (m+1)\tau - x}{\sqrt{2\tau}}$. Here,

$$M = (r + \frac{1}{2}\sigma^2)[T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)(t_{d_2} - t_{d_1})] - \frac{D_1}{K}\left(1 + \frac{D_2}{K}\right) - \frac{D_2}{K}.$$

With a similar manner, $A_2^{(2)}$, $A_3^{(2)}$ and $A_4^{(2)}$ could be figured out straightforwardly. Therefore, the final solution for this two-period case is derived in the form of original variables as

$$f^{(2)}(S, t; t_{d_1}, t_{d_2}) = SN(d_0^{(2)}) - D_1 e^{-r(t_{d_1} - t)} N(d_1^{(2)}) - D_2 e^{-r(t_{d_2} - t)} N(d_2^{(2)}) - K e^{-r(T - t)} N(d_3^{(2)}),$$

with $\{d_k^{(2)}, k = 1, 2, 3, 4\}$ being

$$\begin{aligned}
 d_0^{(2)} &= \frac{L + (r + \frac{1}{2}\sigma^2) \left[T - t_{d_2} + \left(1 + \frac{D_2}{K}\right) (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right) \left(1 + \frac{D_1}{K}\right) (t_{d_1} - t) \right]}{\sigma \sqrt{T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)^2 (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right)^2 \left(1 + \frac{D_1}{K}\right)^2 (t_{d_1} - t)}}, \\
 d_1^{(2)} &= \frac{L + (r + \frac{1}{2}\sigma^2) \left[T - t_{d_2} + \left(1 + \frac{D_2}{K}\right) (t_{d_2} - t_{d_1}) \right] + (r - \frac{1}{2}\sigma^2) \left(1 + \frac{D_2}{K}\right) \left(1 + \frac{D_1}{K}\right) (t_{d_1} - t)}{\sigma \sqrt{T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)^2 (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right)^2 \left(1 + \frac{D_1}{K}\right)^2 (t_{d_1} - t)}}, \\
 d_2^{(2)} &= \frac{L + (r + \frac{1}{2}\sigma^2) (T - t_{d_2}) + (r - \frac{1}{2}\sigma^2) \left[\left(1 + \frac{D_2}{K}\right) (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right) \left(1 + \frac{D_1}{K}\right) (t_{d_1} - t) \right]}{\sigma \sqrt{T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)^2 (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right)^2 \left(1 + \frac{D_1}{K}\right)^2 (t_{d_1} - t)}}, \\
 d_3^{(2)} &= \frac{L + (r - \frac{1}{2}\sigma^2) \left[T - t_{d_2} + \left(1 + \frac{D_2}{K}\right) (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right) \left(1 + \frac{D_1}{K}\right) (t_{d_1} - t) \right]}{\sigma \sqrt{T - t_{d_2} + \left(1 + \frac{D_2}{K}\right)^2 (t_{d_2} - t_{d_1}) + \left(1 + \frac{D_2}{K}\right)^2 \left(1 + \frac{D_1}{K}\right)^2 (t_{d_1} - t)}}, \\
 L &= \left(1 + \frac{D_2}{K}\right) \left(1 + \frac{D_1}{K}\right) \ln \left(\frac{S}{K}\right) - \frac{D_1}{K} \left(1 + \frac{D_2}{K}\right) - \frac{D_2}{K}.
 \end{aligned}$$

Although it seems that there is a long way to go before we can find a general solution for the n -period case, the derivation of this general case is actually similar to the two-period case if the time period $[0, T]$ is separated into two parts, i.e. $[0, t_{d_1}]$ and $[t_{d_1}, T]$. In this sense, what we only need is the solution of the $(n-1)$ -period case, which apparently depends on the $(n-2)$ -period case. As a result, with the essence of the mathematical induction method, we could eventually show that the general solution for the n -period case can be specified as

$$f^{(n)}(S, t; t_k, 1 \leq k \leq n) = SN(d_0^{(n)}) - \sum_{k=1}^n D_k e^{-r(t_{d_k} - t)} N(d_k^{(n)}) - K e^{-r(T-t)} N(d_{n+1}^{(n)}), \quad (3.12)$$

where

$$\begin{aligned}
 d_k^{(n)} &= \frac{B + C}{\sigma \sqrt{\sum_{j=1}^{n+1} \left\{ \left[\Pi_{i=j}^{n+1} \left(1 + \frac{D_i}{K}\right) \right]^2 (t_{d_j} - t_{d_{j-1}}) \right\}}}, \quad 0 \leq k \leq n+1, n \geq 0, \\
 B &= \left[\Pi_{j=1}^{n+1} \left(1 + \frac{D_j}{K}\right) \right] \ln \left(\frac{S}{K}\right) - \sum_{j=1}^{n+1} \left\{ \left[\Pi_{i=j+1}^{n+1} \left(1 + \frac{D_i}{K}\right) \right] \frac{D_j}{K} \right\}, \\
 C &= \left(r - \frac{1}{2}\sigma^2\right) \sum_{j=1}^k \left\{ \left[\Pi_{i=j}^{n+1} \left(1 + \frac{D_i}{K}\right) \right] (t_{d_j} - t_{d_{j-1}}) \right\} \\
 &\quad + \left(r + \frac{1}{2}\sigma^2\right) \sum_{j=k+1}^{n+1} \left\{ \left[\Pi_{i=j}^{n+1} \left(1 + \frac{D_i}{K}\right) \right] (t_{d_j} - t_{d_{j-1}}) \right\}, \\
 T &= t_{d_{n+1}}, \quad t = t_{d_0}, \quad D_{n+1} = 0.
 \end{aligned} \quad (3.13)$$

FIG. 1. Our approximation vs true price with different amount of dividend. Model parameters are: $\sigma = 0.4$, and $t_d = 0.3$. (a) Absolute error between our approximation and true price. (b) Relative error between our approximation and true price.

It should be particularly stressed that we have now approximated European option prices with n -period dividends, which are originally in the form of $(n + 1)$ -dimensional integrals, by the sum of $(n + 2)$ one-dimensional integrals. This could certainly save us much effort in the numerical calculation of option prices. It should also be noticed that this general formula also suits the non-dividend case ($n = 0$), where our formula becomes the B–S price formula, and this is consistent with our expectation. On the other hand, after an approximation formula is derived, it is natural for us to consider its accuracy, which will be presented in the next subsection.

3.3. Numerical verification

In this subsection, we will show the behaviour of our approximation by comparing our prices with those true values obtained through direct numerical integration of the exact solution. Firstly, the case of the single fixed-amount dividend will be used as a representative to show the comparison results. Moreover, in order to further demonstrate the superiority of our approximation over the true formula in the form of $(n + 1)$ -dimensional integrals, we also present the comparison results of the CPU time and relative errors with both approaches for n periods of dividends. In our numerical experiment, the expiry time of the European call option T is set to be 1 and the present time is 0. The risk-free interest rate and the initial stock price take the value of 0.03 and 100, respectively.

Depicted in Fig. 1 is the comparison of our price with the true price calculated from Equation (3.4) and results show that our approximation is quite accurate. To be more specific, what can be easily found in Fig. 1(a) is that no matter the European call option is ‘in the money’, ‘at the money’ or ‘out of the money’, the absolute error is a monotonic increasing function with respect to the amount of dividend, which is consistent with our expectation since our approximation is based on the fact that the dividend is relatively small according to the stock price level. Furthermore, the ‘in the money’ option price suffers from the largest absolute error, which is also reasonable if we take into consideration that its price is also the largest among the three kinds of options. When we turn to Fig. 1(b), a similar pattern could be witnessed that the relative error between the two prices is still an increasing function of the dividend level. However, if we take a close look, there exists some difference that the ‘out of the money’ options, instead of the ‘in the money’ ones, experience the most relative error. The main explanation for it can be that the price for this kind of option is quite low. To view these two sub-figures together, it is not difficult to find that when the

FIG. 2. Our approximation vs true price with different dividend payment time. Model parameters are: $\sigma = 0.6$, and $D = 10$. (a) Absolute error between our approximation and true price. (b) Relative error between our approximation and true price.

dividend rate reaches approximately 20%, the maximum absolute and relative error are only 0.028 and 0.33%, respectively, which are rather small, in contrast to the huge dividend rate.

A completely different picture is exhibited in Fig. 2, where the error is shown according to the dividend payment time. As shown in Fig. 2(a), with dividend payment time becoming closer to expiry, the absolute error will increase to some extent and then fall below the initial level for all the options included in different ‘moneyness’. Although it seems to be strange at first glance of this peculiar shape, it is not difficult to explain the behaviour of the errors as a function of the time to dividend payment, if we go back to the place where the approximation was made, i.e. Equation (3.6). If the dividend payment time is increased, the remaining time between this dividend payment time and expiry time is decreased, which would lead to more time range being exposed to approximation and thus has increased the total error. However, once the dividend payment time is increased beyond a critical point, a further increase does not contribute to more errors, as a result of the change of the slopes in the normal distribution function involved in approximation formula beyond this critical point. Furthermore, the absolute errors for the three kinds of options are quite close when the dividend payment time is near the present time with ‘out of money’ options possessing the largest error. When the payment time becomes closer to expiry, the distance between the three errors enlarges and the ‘in the money’ option price takes the place of possessing the largest error. In contrast, a different phenomenon could be observed in Fig. 2(b). Although the relative error experiences a similar increase and a subsequent decrease when the dividend payment time increase from 0.1 to 0.9, and it for the ‘out of money’ option price also changes from the highest to the lowest, an opposite phenomenon appears, showing that the gap among the relative errors for three kinds of options is relatively large if the dividend is paid early and it will be narrowed down when the dividend payment time is close to expiry. This is mainly because the price of options will increase if the stock pay the dividend later. Again, the whole Fig. 2 demonstrates the fact that although the dividend rate is as high as around 10%, the absolute and relative error are less than 0.005 and 0.024%, respectively, which reflects that our approximation is satisfactory.

It should be stressed that one great advantage of our approximation is the speed with which option price can be fast calculated. Such an advantage of an enormously faster calculation speed at the expenses of a minor sacrificed accuracy, compared with the case if the price has to be calculated with the exact solution procedure stage by stage (denoted by exact solution in the following), justifies the potential application of the newly proposed approximation formula in finance industry, particularly in current

TABLE 1. *The comparison of CPU time (seconds) with both approaches*

Number of dividends	Exact solution	Our approximation
One	2.94×10^{-3}	2.09×10^{-4}
Two	9.10×10^{-1}	2.48×10^{-4}
Three	$3.21 \times 10^{+2}$	3.17×10^{-4}

TABLE 2. *The relative errors between our approximation and exact solution*

Number of dividends	Case A	Case B
One	4.35×10^{-7}	2.10×10^{-4}
Two	1.14×10^{-6}	9.47×10^{-4}
Three	2.18×10^{-6}	2.6×10^{-3}

trend of algorithmic trading, since the B–S formulae are still heavily used in Finance industry today. To substantiate our claim, the CPU times needed to evaluate option prices with exact solution and our approximation, respectively, for the cases of one, two and three dividends are presented in Table 1. Clearly, it only takes very little time, which is around the order of 10^{-4} s when pricing options with our formula, and this order remains the same as the number of dividends is increased. This is expected since the increased number of discrete dividend payments only leads to more one-dimensional integrals to be calculated, which is not time-consuming at all. In contrast, it costs a lot more when we evaluate options with the exact solution procedure stage by stage. Even for the case of one dividend, the consumed CPU time with exact solution is already about 10 times more than that with our approximation. The consumed CPU time with exact solution has increased exponentially and by the time when we need to calculate the option price with only three dividend payments, the order of the consumed CPU has already reached 10^{+2} s or it is 10^6 times more than that of our formula!

Having already known the advantage of our approximation in terms of the CPU time, the accuracy of our formula should also be checked against the increase of the number of dividends, making sure that one does not lose too much in terms of accuracy while gaining speed. Table 2 displays the relative errors between our approximation and exact solution. To ensure that a meaningful discussion is carried out, with regard to accuracy as well as its sensitivity against dividend amount, we have presented here with two cases representing low and high dividends respectively. In Case A, a relatively small dividend of \$1 is set for each dividend payment, while in Case B we have increased the dividend amount 10-folds higher. From Table 2, we can observe that the relative error between our formula and exact solution will increase when the number of dividends is increased, which is also shown in previous figures. It should also be pointed out that the order of the relative error remains almost the same with the increased number of dividends, which implies that the relative error would still stay in a satisfactory range when the number of dividends is relatively large, considering the fact that the relative error is less than 10^{-5} when the dividend is around 1% of the underlying price. Even in the extreme cases with dividend level being around 10%, our formula can still be regarded as a good approximation of the true formula as the relative error is only 2.6×10^{-3} for the case of three-dividend payments.

4. Conclusion

In this article, two relevant issues regarding discrete dividends are considered. First of all, a paradox that the dividend payment date of a discrete dividend that is proportional to the value of the underlying asset value right before the dividend payment time appears to have no influence on the option price is explained mathematically and financially. The fundamental reason that the time of dividend payment is not involved in the pricing formula under the B–S framework is because this is the only ‘fair’ way to count in the effect of a discrete dividend payment under the risk-neutral assumption when an option is priced. Then, an approximation formula for the price of European options with fixed-amount dividends is presented and its accuracy is also numerically checked. Obviously, our formula could certainly facilitate the application in real markets since no matter how many times the underlying pay its dividends before the expiry of the option, the formula is always the sum of one-dimensional integrals, the calculation of which can be rather rapid.

Acknowledgments

The second author is indebted to the University of Wollongong for providing a full Ph.D. scholarship.

REFERENCES

- AMARO DE MATOS, J., DILAO, R. & FERREIRA, B. (2006) The exact value for European options on a stock paying a discrete dividend. Working paper, Munich Personal RePEc Archive.
- AZZALINI, A. (1985) A class of distributions which includes the normal ones. *Scand. J. Stat.*, **12**, 171–178.
- BLACK, F. & SCHOLES, M. (1973) The pricing of options and corporate liabilities. *J. Polit. Econ.*, **81**, 637–654.
- CHUNG, S.-L. & SHACKLETON, M. (2003) The simplest American and real option approximations: Geske–Johnson interpolation in maturity and yield. *Appl. Econ. Lett.*, **10**, 709–716.
- DAI, T.-S. (2009) Efficient option pricing on stocks paying discrete or path-dependent dividends with the stair tree. *Quant. Finance*, **9**, 827–838.
- DAI, T.-S. & LYUU, Y.-D. (2009) Accurate approximation formulas for stock options with discrete dividends. *Appl. Econ. Lett.*, **16**, 1657–1663.
- FRIEDMAN, A. (2013) *Partial Differential Equations of Parabolic Type*. New York, USA: Courier Corporation, pp. 14–22.
- FRISHLING, V. (2002) A discrete question. *Risk*, **15**, 115–116.
- GUO, W. & SU, T. (2006) Option put-call parity relations when the underlying security pays dividends. *Int. J. Bus. Econ.*, **5**, 225–230.
- HUI, C., LO, C.-F. & KU, K. (2007) Pricing vulnerable European options with stochastic default barriers. *IMA J. Manag. Math.*, **18**, 315–329.
- HYLAND, K., MCKEE, S. & WADDELL, C. (1999) Option pricing, Black-Scholes, and novel arbitrage possibilities. *IMA J. Manag. Math.*, **10**, 177–186.
- KRAUSZ, J. (1985) Option parameter analysis and market efficiency tests: a simultaneous solution approach. *Appl. Econ.*, **17**, 885–896.
- MERTON, R. C. (1973) Theory of rational option pricing. *Bell J. Econ. Manag. Sci.*, **4**, 141–183.
- SHREVE, S. E. (2004) *Stochastic Calculus for Finance II: Continuous-Time Models*, vol. 11. New York, USA: Springer Science & Business Media, pp. 238–240.
- WILMOTT, P., DEWYNNE, J. & HOWISON, S. (1993) *Option Pricing: Mathematical Models and Computation*. Oxford, England: Oxford Financial Press, pp. 91–97.