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Equivalence and stable isomorphism of groupoids, and diagonal-preserving stable isomorphisms of graph C*-algebras and Leavitt path algebras

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Abstract

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EQUIVALENCE AND STABLE ISOMORPHISM OF GROUPOIDS, AND DIAGONAL-PRESERVING STABLE ISOMORPHISMS OF GRAPH C^*-ALGEBRAS AND LEAVITT PATH ALGEBRAS

TOKE MEIER CARLSEN, EFREN RUIZ, AND AIDAN SIMS

Abstract. We prove that ample groupoids with σ-compact unit spaces are equivalent if and only if they are stably isomorphic in an appropriate sense, and relate this to Matui's notion of Kakutani equivalence. We use this result to show that diagonal-preserving stable isomorphisms of graph C^*-algebras or Leavitt path algebras give rise to isomorphisms of the groupoids of the associated stabilised graphs. We deduce that the Leavitt path algebras $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_{2-})$ are not stably *-isomorphic.

1. Introduction

A beautiful recent theorem of Matsumoto and Matui [19] relates diagonal-preserving isomorphism of Cuntz–Krieger algebras to the Bowen–Franks invariants of the corresponding shifts of finite type, and to isomorphism of the associated graph groupoids. As a result, diagonal-preserving isomorphism has become an important notion in structure theory for graph C^*-algebras and Leavitt path algebras [16, 11]. A key ingredient in Matsumoto and Matui's approach is the Weyl-groupoid construction, which reconstructs a groupoid from an associated algebra and diagonal subalgebra. This construction goes back to the work of Feldman and Moore [15] on von Neumann factors and was continued by Kumjian [17] and Renault [26] for C^*-algebras. More recently, it has been refined by Brownlowe–Carlsen–Whittaker [10] for graph C^*-algebras, by Brown–Clark–an Huef [7] for Leavitt path algebras, and by Ara–Bosa–Hazrat–Sims [4] for Steinberg algebras.

The Weyl-groupoid approach is well-suited to questions about isomorphisms of graph C^*-algebras or of Leavitt path algebras. But to use it to study stable isomorphism, one first needs a groupoid-theoretic analogue of the Brown–Green–Rieffel stable-isomorphism theorem for C^*-algebras. Here we supply such a theorem (Theorem 2.1), and explore its consequences for graph C^*-algebras and Leavitt path algebras (Section 4).

We begin in Section 2 by proving our Brown–Green–Rieffel theorem for ample groupoids with σ-compact unit spaces. We do not assume that our groupoids are Hausdorff or second countable. Our proof parallels Brown's proof that a full corner of a σ-unital C^*-algebra is stably isomorphic to the enveloping algebra. In Section 3 we digress to relate our results to Matui's definition [21] of Kakutani equivalence for ample groupoids with compact

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unit space. We extend this notion to ample groupoids with noncompact unit space and prove that it coincides with groupoid equivalence. We start Section 4 by checking that Tomforde's construction from a directed graph E of a graph SE satisfying $C^*(SE) \cong C^*(E) \otimes \mathcal{K}$ is compatible with stabilising the groupoid. We then explore the consequences of our Brown–Green–Rieffel theorem for groupoid C^*-algebras and Steinberg algebras, and particularly for graph C^*-algebras and Leavitt path algebras: Theorem 4.2 says, amongst other things, that there is a diagonal-preserving isomorphism $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ if and only if there is a diagonal-preserving isomorphism $C^*(SE) \cong C^*(SF)$, and likewise at the level of Leavitt path algebras. We deduce using results of Carlsen [11] that $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_{2-})$ are not stably *-isomorphic.

2. Groupoid equivalence and stable isomorphism

In this section we show that for ample groupoids, the Brown–Green–Rieffel stable-isomorphism theorem [9] works at the level of groupoids.

An ample groupoid is a groupoid G equipped with a topology with a basis of compact open sets such that inversion and composition in G are continuous, the unit space $G^{(0)}$ is Hausdorff, and the range and source maps $r, s : G \to G^{(0)}$ are local homeomorphisms. The unit space of an ample groupoid is automatically locally compact and totally disconnected.

For groupoids G and H, a G–H *equivalence* is a space Z with commuting free and proper actions of G on the left and H on the right such that $r : Z \to G^{(0)}$ induces a homeomorphism $Z/H \cong G^{(0)}$ and $s : Z \to H^{(0)}$ induces a homeomorphism of $G\backslash Z \cong H^{(0)}$; if such a Z exists, we say that G and H are *groupoid equivalent*. See [22, 25] for more detail.

We will write $\mathcal{R}$ for the full countable equivalence relation $\mathcal{R} = \mathbb{N} \times \mathbb{N}$, regarded as a discrete principal groupoid with unit space $\mathbb{N}$. A space is *σ-compact* if it has a countable cover by compact sets; if it is locally compact, totally disconnected and Hausdorff, it then has a countable cover by mutually disjoint compact open sets. Given an ample groupoid G, the product $G \times \mathcal{R}$ is an ample groupoid under the product topology and coordinatewise operations. We identify the unit space of $G \times \mathcal{R}$ with $G^{(0)} \times \mathbb{N}$.

Theorem 2.1. *Let G and H be ample groupoids. Suppose that $G^{(0)}$ and $H^{(0)}$ are σ-compact. Then G and H are groupoid equivalent if and only if $G \times \mathcal{R} \cong H \times \mathcal{R}$.*

The strategy is to prove that for any clopen $K \subseteq G^{(0)}$ that meets every G-orbit, $G \times \mathcal{R} \cong G|_K \times \mathcal{R}$, paralleling Brown's result about full corners of σ-unital C^*-algebras. Our proof follows Brown's very closely: Lemmas 2.2, 2.3 and 2.4 and their proofs are direct analogues of [8, Lemmas 2.3, 2.4 and 2.5].

We say that $U \subseteq G$ is an *open bisection* if U is open and r, s restrict to homeomorphisms of U onto $r(U), s(U)$ respectively. For $x \in G^{(0)}$, we denote $r^{-1}(x)$ by G^x and $s^{-1}(x)$ by G_x, and for $K \subseteq G^{(0)}$, we write $GK := s^{-1}(K)$, $KG := r^{-1}(K)$, and $G|_K := KG \cap GK$. A set $K \subseteq G^{(0)}$ is *G-full* if $r(GK) = G^{(0)}$.

Lemma 2.2. *Let G be an ample groupoid such that $G^{(0)}$ is σ-compact. Suppose that $K \subseteq G^{(0)}$ is clopen and G-full. Then there is a sequence of compact open bisections $V_i \subseteq GK$ with mutually disjoint ranges such that $\bigsqcup_i r(V_i) = G^{(0)}$.*

Proof. Choose a countable cover $\mathcal{U}$ of $G^{(0)}$ by compact open sets. Fix $U \in \mathcal{U}$. For $u \in U$, since K is G-full, there exists $\gamma_u \in G^u \cap GK$. Since K is open and G is ample, for each $u \in U$, there is a compact open bisection V_u such that $\gamma_u \in V_u \subseteq GK$. Each $r(V_u)$ is clopen in $G^{(0)}$ because $G^{(0)}$ is Hausdorff. Since U is compact, we can find $V_{u_1}, \dots V_{u_{j(U)}}$ with $U \subseteq \bigcup_{i=1}^{j(U)} r(V_{u_i})$. By choosing a finite collection like this for each $U \in \mathcal{U}$ and enumerating the union of these collections, we obtain a list $(V_i^0)_{i=1}^\infty$ of compact open bisections with $\bigcup r(V_i^0) = G^{(0)}$ and $\bigcup s(V_i^0) \subseteq K$. For each i, the set $X_i := r(V_i^0) \setminus \bigcup_{j<i} r(V_j^0)$ is compact open in $G^{(0)}$. Since $r^{-1} : r(V_i^0) \to V_i^0$ is a homeomorphism we deduce that $V_i := V_i^0 \cap X_i G$ is a compact open subset of V_i^0. These V_i suffice. □

Lemma 2.3. *Let G be an ample groupoid such that $G^{(0)}$ is σ-compact. Suppose that $K \subseteq G^{(0)}$ is clopen and G-full. Then there is an open bisection $W \subseteq G \times \mathcal{R}$ such that $r(W) = G^{(0)} \times \{1\}$ and $s(W) \subseteq K \times \mathbb{N}$ is clopen in $G^{(0)} \times \mathbb{N}$.*

Proof. Fix compact open bisections $(V_i)_{i=1}^\infty$ as in Lemma 2.2. Put $W := \bigcup_i V_i \times \{(1, i)\}$, which is open because the V_i are. The $r(V_i \times \{(1, i)\})$ are mutually disjoint because the $r(V_i)$ are; the $s(V_i \times \{(1, i)\})$ are clearly mutually disjoint. The maps s, r are homeomorphisms on W because they restrict to homeomorphisms on the relatively clopen subsets $V_i \times \{(1, i)\}$. Clearly $s(W) = \bigcup_i s(V_i) \times \{i\}$ is open. It is also closed because the $s(V_i)$ are closed in $G^{(0)}$, so $(G^{(0)} \times \mathbb{N}) \setminus s(W) = \bigcup_i \big((G^{(0)} \setminus s(V_i)) \times \{i\}\big)$ is open. □

Lemma 2.4. *Under the hypotheses of Lemma 2.3, there is an open bisection $Y \subseteq G \times \mathcal{R}$ such that $r(Y) = G^{(0)} \times \mathbb{N}$ and $s(Y) = K \times \mathbb{N}$.*

Proof. Write $\mathbb{N} = \bigsqcup_{i=1}^\infty N_i$ as a union of mutually disjoint infinite subsets. We claim that there exists a sequence Y_j of open bisections with mutually disjoint clopen ranges and mutually disjoint clopen sources such that for each $n \geq 0$, we have

$$\bigcup_{j=1}^{2n-1} r(Y_j) = \bigcup_{i=1}^{n} G^{(0)} \times N_i, \qquad \bigcup_{j=1}^{2n} r(Y_j) \subseteq \bigcup_{i=1}^{n+1} G^{(0)} \times N_i,$$
$$\bigcup_{j=1}^{2n-1} s(Y_j) \subseteq \bigcup_{i=1}^{n} K \times N_i, \qquad \text{and} \qquad \bigcup_{j=1}^{2n} s(Y_j) = \bigcup_{i=1}^{n} K \times N_i.$$

Suppose that $Y_1, \dots, Y_{2n}$ satisfy these equations (this is trivial when $n = 0$). To construct Y_{2n+1}, apply Lemma 2.3 to $G \times \big(\mathcal{R}|_{N_{n+1}}\big)$ and $K \times N_{n+1} \subseteq G^{(0)} \times N_{n+1}$ to obtain an open bisection $W \subseteq G \times \big(\mathcal{R}|_{N_{n+1}}\big) \times \mathcal{R}$ such that $r(W) = G^{(0)} \times N_{n+1} \times \{1\}$ and $s(W) \subseteq K \times N_{n+1} \times \mathbb{N}$ is clopen. Fix a bijection $\theta : N_{n+1} \times \mathbb{N} \to N_{n+1}$, and define

$$W' := \{(g, (p, \theta(q, m))) : (g, (p, q), (1, m)) \in W\} \subseteq G \times \big(\mathcal{R}|_{N_{n+1}}\big).$$

This W' is an open bisection with $r(W') = G^{(0)} \times N_{n+1}$ and $s(W') \subseteq K \times N_{n+1}$. Let

$$Y_{2n+1} := \big((G^{(0)} \times N_{n+1}) \setminus r(Y_{2n})\big)W'.$$

Since $r(Y_{2n})$ is clopen as part of the induction hypothesis, so is $(G^{(0)} \times N_{n+1}) \setminus r(Y_{2n})$; so Y_{2n+1} is open. Since r and s restrict to homeomorphisms on W', the set $s(Y_{2n+1})$ is clopen in $G^{(0)} \times \mathbb{N}$. We have $\bigcup_{j=1}^{2n+1} r(Y_j) = \bigcup_{i=1}^{n+1} G^{(0)} \times N_i$ by definition of Y_{2n+1}, and clearly $\bigcup_{j=1}^{2n+1} s(Y_j) \subseteq \bigcup_{i=1}^{n+1} K \times N_i$.

To construct Y_{2n+2}, choose a bijection $\phi : N_{n+1} \to N_{n+2}$, and define

$$\begin{aligned} Y_{2n+2} &:= \Big(\bigcup_{i \in N_{n+1}} G^{(0)} \times \{(\phi(i), i)\}\Big)\big((K \times N_{n+1}) \setminus s(Y_{2n+1})\big) \\ &= \big\{(u, \phi(n)) : (u, n) \in (K \times N_{n+1}) \setminus s(Y_{2n+1})\big\} \subseteq G^{(0)} \times \mathcal{R}. \end{aligned}$$

This is open because $s(Y_{2n+1})$ is closed. It is a bisection because ϕ is a bijection. Both $s(Y_{2n+2})$, $r(Y_{2n+2})$ are clopen in $G^{(0)} \times \mathbb{N}$ because s and r are homeomorphisms on

$\big(\bigcup_{i\in N_{n+1}} G^{(0)} \times \{(\phi(i), i)\}\big)$. We have $\bigcup_{j=1}^{2n+2} s(Y_j) = \bigcup_{i=1}^{n+1} K \times N_i$ and $\bigcup_{j=1}^{2n+2} r(Y_j) \subseteq G^{(0)} \times \bigcup_{i=1}^{n+2} N_i$ by construction. This proves the claim.

Let $Y := \bigcup_{i=1}^{\infty} Y_i$, which is open because the Y_i are open. Since the $s(Y_i)$ are mutually disjoint, s is injective on Y; and similarly for r. Since the Y_i are open and s, r restrict to homeomorphisms on the Y_i, we see that s, r are homeomorphisms on W. We have $r(Y) = \bigcup_n \bigcup_{j=1}^{2n-1} r(Y_j) = G^{(0)} \times \mathbb{N}$ and $s(Y) = \bigcup_n \bigcup_{j=1}^{2n} s(Y_j) = K \times \mathbb{N}$. □

We now obtain a groupoid version of [8, Corollary 2.6].

Proposition 2.5. *Let G be an ample groupoid such that $G^{(0)}$ is σ-compact. Suppose that $K \subseteq G^{(0)}$ is clopen and G-full. Then $G \times \mathcal{R} \cong G|_K \times \mathcal{R}$.*

Proof. Apply Lemma 2.4 to obtain an open bisection $Y \subseteq G\times\mathcal{R}$ such that $r(Y) = G^{(0)}\times\mathbb{N}$ and $s(Y) = K\times\mathbb{N}$. For $\gamma \in G\times\mathcal{R}$, we write $Y^{-1}\gamma Y$ for the element $\alpha^{-1}\gamma\beta$ obtained from the unique elements $\alpha, \beta \in Y$ with $r(\alpha) = r(\gamma)$ and $s(\beta) = s(\gamma)$. Since Y is a bisection, the map $\gamma \mapsto Y^{-1}\gamma Y$ is a groupoid homomorphism with range in $G|_K\times\mathcal{R}$. It is continuous because multiplication in $G\times\mathcal{R}$ is continuous. Since $\eta \mapsto Y\eta Y^{-1} : G|_K \times \mathcal{R} \to G \times \mathcal{R}$ is a continuous inverse, $\gamma \mapsto Y^{-1}\gamma Y$ is the desired isomorphism $G \times \mathcal{R} \to G|_K \times \mathcal{R}$. □

Proof of Theorem 2.1. Let Z be a G–H-equivalence. Consider the linking groupoid $L = G \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup H$ [27, Lemma 3]. By [13, Lemma 4.2], $G^{(0)}, H^{(0)} \subseteq L^{(0)}$ both satisfy the hypotheses of Proposition 2.5. So $G \times \mathcal{R} \cong L|_{G^{(0)}} \times \mathcal{R} \cong L \times \mathcal{R} \cong L|_{H^{(0)}} \times \mathcal{R} \cong H \times \mathcal{R}$.

Now suppose that $G\times\mathcal{R} \cong H\times\mathcal{R}$. The space $X := G\times\{(1, i) : i \in \mathbb{N}\}$ is a G–$(G\times\mathcal{R})$-equivalence, and similarly $Z := H \times \{(i, 1) : i \in \mathbb{N}\}$ is a $(H \times \mathcal{R})$–H-equivalence. Since $G \times \mathcal{R} \cong H \times \mathcal{R}$ and groupoid equivalence is an equivalence relation, we deduce that G and H are groupoid equivalent. □

3. Kakutani equivalence

Matui [21] defines Kakutani equivalence for ample groupoids G and H with compact unit spaces: G and H are Kakutani equivalent if there are full clopen subsets $X \subseteq G^{(0)}$ and $Y \subseteq H^{(0)}$ such that $G|_X \cong H|_Y$. We extend this notion to ample groupoids with non-compact unit spaces.

Definition 3.1. Let G and H be ample groupoids. Then G and H are *Kakutani equivalent* if there are a G-full clopen $X \subseteq G^{(0)}$ and an H-full clopen $Y \subseteq H^{(0)}$ such that $G|_X \cong H|_Y$.

Theorem 3.2. *Let G and H be ample groupoids with σ-compact unit spaces. The following are equivalent:*

(1) G and H are Kakutani equivalent;
(2) there exist full open sets $X \subseteq G^{(0)}$ and $Y \subseteq H^{(0)}$ such that $G|_X \cong H|_Y$;
(3) G and H are groupoid equivalent;
(4) $G \times \mathcal{R} \cong H \times \mathcal{R}$.

Proof. By Theorem 2.1, it suffices to show that (1)–(3) are equivalent. That (1) $\Longrightarrow$ (2) is obvious. Suppose that (2) holds. Then GX is a G–$G|_X$ equivalence under the actions determined by multiplication in G (see the argument of [13, Lemma 6.1]). Similarly, YH is a H–$H|_Y$ equivalence. Since groupoid equivalence is an equivalence relation, G and H are groupoid equivalent, giving (2) $\Longrightarrow$ (3).

Now suppose that Z is a G–H-equivalence. In this proof, for $K \subseteq G^{(0)}$ we write $[K]_G$ for the saturation $r(GK)$ of K in $G^{(0)}$; similarly, for $K' \subseteq H^{(0)}$, we write $[K']_H := r(HK')$.

Let $L = G \sqcup Z \sqcup Z^{\mathrm{op}} \sqcup H$ be the linking groupoid [27, Lemma 3]. Fix countable covers $G^{(0)} = \bigsqcup_{i=1}^{\infty} U_i$ and $H^{(0)} = \bigsqcup_{i=1}^{\infty} W_i$ by mutually disjoint compact open sets.

Claim. There exist $V_1, V_2 \dots \subseteq Z$ and $n_1 \leq n_2 \leq \cdots$ in $\mathbb{N}$ such that

(i) the V_i are compact open bisections with mutually disjoint ranges and sources;
(ii) $\bigcup_{l=1}^{j} U_l \subseteq \big[\bigcup_{i=1}^{n_j} r(V_i)\big]_G$ and $\bigcup_{l=1}^{j} W_l \subseteq \big[\bigcup_{i=1}^{n_j} s(V_i)\big]_H$ for all $j \in \mathbb{N}$; and
(iii) $r(V_i) \cap U_j = \emptyset$ and $s(V_i) \cap W_j = \emptyset$ for all $j \in \mathbb{N}$ and $i > n_j$.

We construct the V_i iteratively. Suppose either that $J = 0$, or that $J \geq 1$, $n_1 \leq n_2 \leq \cdots \leq n_J \in \mathbb{N}$, and $V_1, V_2, \dots, V_{n_J} \subseteq Z$ satisfy (i)–(iii) for all $j < J$.

The set $K := U_{J+1} \setminus \big[\bigcup_{i=1}^{n_J} r(V_i)\big]_G$ is compact. Fix $\Gamma \subseteq Z$ with $r(\Gamma) = K$. Suppose that $J \geq 1$ and $s(\Gamma) \cap \bigcup_{l=1}^{J} W_l \neq \emptyset$, say $\gamma \in \Gamma \cap s^{-1}\big(\bigcup_{l=1}^{J} W_l\big)$. Then $\bigcup_{l=1}^{J} W_l \subseteq \big[\bigcup_{i=1}^{n_J} s(V_i)\big]_H$ gives $s(\gamma) \in \big[\bigcup_{i=1}^{n_J} s(V_i)\big]_H$, so there exist $i \leq n_J$, $\alpha \in H^{s(\gamma)}$ and $\beta \in V_i$ with $s(\alpha) = s(\beta)$. But then $\beta\alpha^{-1}\gamma^{-1} \in GK \cap r(V_i)G$, contradicting the definition of K. So $s(\Gamma) \cap \bigcup_{l=1}^{J} W_l = \emptyset$. Similarly, if $s(\gamma) = s(\beta)$ for some $\gamma \in \Gamma$ and $\beta \in V_i$ where $i \leq n_J$, then $\gamma\beta^{-1} \in KG \cap Gr(V_i)$, which is impossible by definition of K; so $s(\Gamma) \cap s(V_i) = \emptyset$ for $i \leq n_J$. We also have $r(\Gamma) = K \subseteq \bigcup_{l>J} U_l \setminus \bigcup_{i=1}^{n_J} r(V_i)$, so for each $\gamma \in \Gamma$, there is a compact open bisection $V_\gamma^0 \subseteq Z$ containing γ with $r(V_\gamma^0) \cap U_l = \emptyset = s(V_\gamma^0) \cap W_l$ for $l \leq J$, and $r(V_\gamma^0) \cap r(V_i) = \emptyset = s(V_\gamma^0) \cap s(V_i)$ for $i \leq n_J$. Since K is compact, there are $V_1^0, \dots, V_m^0 \in \{V_\gamma : \gamma \in \Gamma\}$ with $K \subseteq \bigcup_{i=1}^{m} r(V_i^0)$. For $i \leq m$, let $V_i^1 := V_i^0 \setminus r^{-1}\big(\bigcup_{i'<i} r(V_{i'}^0)\big)$; so $K \subseteq \bigsqcup_{i=1}^{m} r(V_i^1)$. Let $V_{n_J+1} := V_1^1$ and iteratively put $V_{n_J+i} := V_i^1 \setminus s^{-1}\big(\bigcup_{i'<i} s(V_{n_J+i'})\big)$. Then $V_1, \dots, V_{n_J+m}$ are compact open bisections with mutually disjoint ranges and sources such that $r(V_i) \cap U_j = \emptyset$ for $j \leq J$ and $i > n_j$.

We claim that $K \subseteq \big[\bigcup_{i=1}^{m} r(V_{n_J+i})\big]_G$. Fix $x \in K$. Then there are $i \leq m$ and $\alpha \in V_i^1$ with $x = r(\alpha)$. By definition of V_{n_J+i} there exists $1 \leq i' \leq i$ and $\beta \in V_{n_J+i'}$ with $s(\beta) = s(\alpha)$. So $\alpha\beta^{-1} \in G^x \cap G \in r(V_{n_j+i'})$, forcing $x \in \big[\bigcup_{i=1}^{n_j} r(V_i)\big]_G$.

Now let $K' = W_{J+1} \setminus \big[\bigcup_{i=1}^{n_J+m} s(V_i)\big]_H$. We repeat the argument of the previous two paragraphs. Choose $\Lambda \subseteq Z$ with $s(\Lambda) = K'$. As above, $r(\Lambda) \cap \big(\bigcup_{l=1}^{J} U_l \cup \bigcup_{i=1}^{n_J+m} r(V_i)\big) = \emptyset$. For $\lambda \in \Lambda$ pick a compact open bisection $V_\lambda^0 \subseteq Z$ containing λ with $r(V_\lambda^0) \cap \big(\bigcup_{l=1}^{J} U_l \cup \bigcup_{i=1}^{n_J+m} r(V_i)\big) = \emptyset$, and $s(V_\lambda^0) \cap \big(\bigcup_{l=1}^{J} W_l \cup \bigcup_{i=1}^{n_J+m} s(V_i)\big) = \emptyset$. Use compactness and disjointify sources to obtain $V_{m+1}^1, \dots V_{m+m'}^1$ with $K' \subseteq \bigsqcup_{i=1}^{m'} s(V_{m+i}^1)$. Iteratively let $V_{n_J+m+i} := V_{m+i}^1 \setminus r^{-1}\big(\bigcup_{i'<i} r(V_{n_J+m+i'})\big)$. As for K above, $K' \subseteq \big[\bigcup_{i=1}^{n_J+m+p} s(V_i)\big]_H$. Let $n_{J+1} = n_J + m + m'$. Then $V_1, \dots V_{n_{J+1}}$ satisfy (i)–(iii) for $j < J+1$. The claim now follows by induction.

Now let $Y := \bigcup_{i=1}^{\infty} V_i$. Then (i) guarantees that Y is an open bisection. By (ii), $r(Y)$ is G-full and $s(Y)$ is H-full. Since each V_i is a compact open bisection, the $r(V_i)$ and $s(V_i)$ are clopen. So $r(Y)$ and $s(Y)$ are open. They are also closed: by (iii), each $U_j \setminus r(Y) = U_j \setminus \bigcup_{i=1}^{n_j} r(V_i)$ is open, and likewise and each $W_j \setminus s(Y) = W_j \setminus \bigcup_{i=1}^{n_j} s(V_i)$ is open; so $G^{(0)} \setminus r(Y) = \bigcup_j U_j \setminus r(Y)$ and $H^{(0)} \setminus s(Y) = \bigcup_j W_j \setminus s(Y)$ are open.

The map $\gamma \mapsto Y^{-1}\gamma Y$ from $G|_{r(Y)}$ to $H|_{s(Y)}$ is a groupoid isomorphism just as in the proof of Proposition 2.5. Hence G is Kakutani equivalent to H, giving (3) $\implies$ (1). $\square$

Corollary 3.3. *Let G and H be ample groupoids with σ-compact unit spaces. Suppose that there exists a G-full compact open subset of $G^{(0)}$. Then G and H are groupoid equivalent if and only if there are full compact open sets $X \subseteq G^{(0)}$ and $Y \subseteq H^{(0)}$ such that $G|_X \cong H|_Y$.*

Proof. Let $X_0 \subseteq G^{(0)}$ be a G-full compact open set. First suppose G and H are groupoid equivalent; say Z is a G–H equivalence. Following the construction of V_1 in the proof of Theorem 3.2—with $U_1 = X_0$—gives a compact open bisection $V \subseteq Z$ with $X_0 \subseteq [r(V)]$, and hence $G^{(0)} = [X_0] \subseteq [r(V)]$. Fix $y \in H^{(0)}$. Take $\gamma \in Z_y$. Take $\alpha \in G_{r(\gamma)}$ with $r(\alpha) \in r(V)$; say $\beta \in V \cap Z^{r(\alpha)}$. Then $\beta^{-1}\alpha\gamma \in s(V)H \cap H_y$. So $[s(V)] = H^{(0)}$. Now $X := r(V) \subseteq G^{(0)}$ and $Y := s(V) \subseteq H^{(0)}$ are full compact open sets and $\gamma \mapsto V^{-1}\gamma V : G|_X \to H_Y$ is an isomorphism. This proves the "$\Longrightarrow$" direction. The "$\Longleftarrow$" direction follows from (2) $\Longrightarrow$ (3) in Theorem 3.2. $\square$

4. Consequences for graph algebras

In this section we explore the consequences of Theorem 2.1 for stable isomorphism of graph C^*-algebras and of Leavitt path algebras. First we introduce some terminology to state our main result.

If E is a directed graph, then SE denotes the graph obtained by appending a head $\dots f_{3,v} f_{2,v} f_{1,v}$ at every vertex v. Theorem 4.2 of [30] shows that $C^*(SE) \cong C^*(E) \otimes \mathcal{K}$ (see also [2, Proposition 9.8]). We will show in Lemma 4.1 that this happens at the level of groupoids. First, we briefly describe the graph groupoid G_E: if E^∞ denotes the set of all infinite paths of E and E^* denotes the set of all finite paths of E, define

$$\partial E := E^\infty \cup \{x \in E^* : \ r(x) \text{ is a sink or an infinite emitter}\}.$$

For $\mu \in E^*$, define $Z(\mu) := \{\mu x : x \in \partial E, r(\mu) = s(x)\}$. Then the sets $Z(\mu \setminus F) := Z(\mu) \setminus \bigcup_{\nu \in F} Z(\mu\nu)$ indexed by $\mu \in E^*$ and finite subsets F of $r(\mu)E^1$ form a basis of compact open sets for a locally compact Hausdorff topology on ∂E. For each $n \geq 0$, the *shift map* $\sigma^n : \partial E^{\geq n} := \{x \in \partial E : |x| \geq n\} \to \partial E$ given by $\sigma^n(\mu x) = x$ for $\mu \in E^n$ and $x \in r(\mu)\partial E$ is a local homeomorphism.

We write G_E for the graph groupoid

$$G_E = \textstyle\bigcup_{m,n \in \mathbb{N}} \{(x, m-n, y) : x \in \partial E^{\geq m}, y \in \partial E^{\geq n} \text{ and } \sigma^m(x) = \sigma^n(y)\},$$

where $r(x,m,y) = (x,0,x)$, $s(x,m,y) = (y,0,y)$ and $(x,m,y)(y,n,z) = (x,m+n,z)$. This is an ample groupoid under the topology with basic open sets $Z(\alpha, \beta \setminus F) := \{(\alpha x, |\alpha| - |\beta|, \beta x) : x \in Z(r(\alpha) \setminus F)\}$ indexed by triples (α, β, F) where $\alpha, \beta \in E^*$, and $F \subseteq r(\alpha)E^1$ is finite.

Lemma 4.1. *Let E be a directed graph. Then $G_E \times \mathcal{R} \cong G_{SE}$ and $G_{SE} \cong G_{SE} \times \mathcal{R}$.*

Proof. For each $v \in E^0$, write $\mu_{0,v} := v$ and for $i \geq 1$ write $\mu_{i,v} := f_{i,v} f_{i-1,v} \dots f_{1,v}$. Then $\partial(SE) = \{\mu_{i,s(x)} x : x \in \partial E, i \in \mathbb{N}\}$. The map $\phi : \mu_{i,s(x)} x \mapsto (x,i)$ is a homeomorphism from $\partial(SE)$ to $\partial E \times \mathbb{N}$: cylinder sets of the form $Z(\mu_{i,s(\lambda)}\lambda \setminus F)$ are a basis of compact open sets for $\partial(SE)$, and ϕ restricts to a continuous bijection of each $Z(\mu_{i,s(\lambda)}\lambda \setminus F)$ onto the compact open set $Z(\lambda \setminus F) \times \{i\}$. It is routine to check that $\big((x,m,y),(i,j)\big) \mapsto \big(\phi^{-1}(x,i), m+i-j, \phi^{-1}(y,j)\big)$ is a groupoid isomorphism from $G_E \times \mathcal{R}$ to G_{SE}. Since $\mathcal{R} \times \mathcal{R} \cong \mathcal{R}$, we obtain $G_{SE} \cong G_{SE} \times \mathcal{R}$ as well. $\square$

Recall [10, 19] that graphs E and F are *orbit equivalent* if there exist a homeomorphism $h : \partial E \to \partial F$ and continuous functions $k, l : \partial E^{\geq 1} \to \mathbb{N}$ and $k', l' : \partial F^{\geq 1} \to \mathbb{N}$ such that $\sigma_F^{k(x)}(h(\sigma_E(x))) = \sigma_F^{l(x)}(h(x))$ and $\sigma_E^{k'(y)}(h^{-1}(\sigma_F(y))) = \sigma_E^{l'(y)}(h^{-1}(y))$ for all $x \in \partial E^{\geq 1}$ and $y \in \partial F^{\geq 1}$.

We assume familiarity with graph C^*-algebras and Leavitt path algebras; see [6] and [31] for the requisite background. Given a graph E, we call the abelian subalgebra $D(E) := \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\} \subseteq C^*(E)$ the *diagonal subalgebra* of the graph C^*-algebra, and for any commutative ring R with 1, we call the abelian subalgebra $D_R(E) := \operatorname{span}_R\{s_\mu s_{\mu^*} : \mu \in E^*\} \subseteq L_R(E)$ the diagonal subalgebra of the Leavitt path R-algebra. For an ample groupoid G, we write $C^*(G)$ for the (full) C^*-algebra of G (see for example [24] or [23]) and, for a commutative ring R with 1, we write $A_R(G)$ for the Steinberg algebra of G over R (see [29] and [13]). Any isomorphism of ample groupoids G and H induces an isomorphism $C^*(G) \cong C^*(H)$ carrying $C_0(G^{(0)})$ to $C_0(H^{(0)})$ and an isomorphism $A_R(G) \cong A_R(H)$ carrying $A_R(G^{(0)})$ to $A_R(H^{(0)})$. The canonical isomorphism $C^*(E) \cong C^*(G_E)$ carries $D(E)$ to the standard diagonal subalgebra $C_0(G_E^{(0)}) \subseteq C^*(G_E)$ (see the proof of [18, Proposition 4.1] and [10, Proposition 2.2]), and likewise at the level of Leavitt path algebras [13, Example 3.2]. We say that an isomorphism $\phi : C^*(E) \to C^*(F)$ of graph C^*-algebras is *diagonal preserving* if $\phi(D(E)) = D(F)$, and likewise for Leavitt path algebras.

We write $\mathcal{K}$ for the C^*-algebra of compact operators on $\ell^2(\mathbb{N})$, and $\mathcal{C}$ for the maximal abelian subalgebra of $\mathcal{K}$ consisting of diagonal operators. For a commutative ring R with 1 we write $M_\infty(R)$ for the ring of finitely supported, countably infinite square matrices over R and $D_\infty(R)$ for the abelian subring of $M_\infty(R)$ consisting of diagonal matrices. For any ample groupoid G, if $\mathcal{R}$ is the equivalence relation $\mathbb{N} \times \mathbb{N}$ of Section 2, there exist isomorphisms $C^*(G \times \mathcal{R}) \cong C^*(G) \otimes \mathcal{K}$ and $A_R(G \times \mathcal{R}) \cong A_R(G) \otimes M_\infty(R)$ that take $C_0(G^{(0)} \times \mathbb{N})$ to $C_0(G^{(0)}) \otimes \mathcal{C}$ and $A_R(G^{(0)} \times \mathbb{N})$ to $A_R(G^{(0)}) \otimes D_\infty(R)$.

An isomorphism $\phi : C^*(E) \otimes \mathcal{K} \to C^*(F) \otimes \mathcal{K}$ is *diagonal preserving* if $\phi(D(E) \otimes \mathcal{C}) = D(F) \otimes \mathcal{C}$, and similarly at the level of Leavitt path algebras.

Theorem 4.2. *Let E and F be directed graphs, and let R be a commutative integral domain with 1. The following are equivalent:*

(1) there is a diagonal-preserving isomorphism $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$;
(2) there is a diagonal-preserving *-ring isomorphism $L_R(E) \otimes M_\infty(R) \cong L_R(F) \otimes M_\infty(R)$;
(3) there is a diagonal-preserving isomorphism $C^*(SE) \cong C^*(SF)$;
(4) there is a diagonal-preserving *-ring isomorphism $L_R(SE) \cong L_R(SF)$;
(5) $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$;
(6) $G_{SE} \cong G_{SF}$.

These equivalent conditions imply each of

(7) SE and SF are orbit equivalent;
(8) there is a diagonal-preserving ring isomorphism $L_R(E) \otimes M_\infty(R) \cong L_R(F) \otimes M_\infty(R)$; and
(9) there is a diagonal-preserving ring isomorphism $L_R(SE) \cong L_R(SF)$.

The conditions (8) and (9) are equivalent. If every cycle in each of E and F has an exit, then (1)–(9) are all equivalent.

Our proof of Theorem 4.2 uses Crisp and Gow's collapsing procedure [14].

Lemma 4.3. *Let E be a directed graph, let T be a collapsible subgraph of E in the sense of Crisp and Gow, and let F be the graph obtained from E by collapsing T. Then $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$, and there are diagonal-preserving isomorphisms $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ and $L_R(E) \otimes M_\infty(R) \cong L_R(F) \otimes M_\infty(R)$ for every commutative unital ring R.*

Proof. Proposition 6.2 of [13] shows that G_E and G_F are equivalent groupoids. So Theorem 2.1 gives $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$. The result then follows because the canonical isomorphisms $C^*(E) \otimes \mathcal{K} \cong C^*(G_E) \otimes \mathcal{K}$ and $L_R(E) \otimes M_\infty(R) \cong A_R(G_E) \otimes M_\infty(R)$ are diagonal preserving. □

Proof of Theorem 4.2. Lemma 4.1 yields (5) $\iff$ (6); and also (1) $\iff$ (3) and (2) $\iff$ (4) since the canonical isomorphisms $C^*(E) \cong C^*(G_E)$ and $L_R(E) \cong A_R(G_E)$ are diagonal preserving. Theorem 5.1 of [10] gives (3) $\iff$ (6). Isomorphisms of groupoids induce diagonal-preserving *-ring isomorphisms of Steinberg algebras, giving (6) $\implies$ (4). To prove equivalence of (1)–(6), it now suffices to check (2) $\implies$ (5).

Suppose that (2) holds. The graphs E and F can be obtained from their Drinen–Tomforde desingularisations E' and F' by applications of Crisp and Gow's collapsing procedure [14]. So Lemma 4.3 gives $G_E \times \mathcal{R} \cong G_{E'} \times \mathcal{R} \cong G_{SE'}$, and similarly for F. Hence the diagonal-preserving *-ring isomorphism $L_R(E) \otimes M_\infty(R) \cong L_R(F) \otimes M_\infty(R)$ induces a diagonal-preserving *-ring isomorphism $L_R(SE') \cong L_R(SF')$. Since SE' and SF' are row-finite with no sinks, we can apply [7, Theorem 6.2][1] to obtain $G_{SE'} \cong G_{SF'}$. Since $G_E \times \mathcal{R} \cong G_{SE'}$ and $G_F \times \mathcal{R} \cong G_{SF'}$, this yields $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$.

That (6) $\implies$ (7) follows from [10, Theorem 5.1 (2) $\implies$ (4)]. Clearly (2) $\implies$ (8) and (4) $\implies$ (9). We have (8) $\iff$ (9) by another application of Lemma 4.1.

Now suppose that every cycle in each of E and F has an exit. Then [10, Theorem 5.1 (4) $\implies$ (2)] gives (7) $\implies$ (6), and [4, Corollary 4.4] gives (9) $\implies$ (6). □

We now deduce a "diagonal-preserving" version of [8, Corollary 2.6] for groupoid C^*-algebras and Steinberg algebras from Proposition 2.5. We do not require that G is Hausdorff (see for example [23] for the definition of the C^*-algebra of a non-Hausdorff groupoid, and [29] for the definition of the Steinberg algebra of a non-Hausdorff groupoid).

Lemma 4.4. *Let G be an ample groupoid such that $G^{(0)}$ is σ-compact and let R be a ring. Suppose that $K \subseteq G^{(0)}$ is clopen and G-full. Then*

(1) there is an isomorphism $\phi : C^*(G) \otimes \mathcal{K} \to C^*(G|_K) \otimes \mathcal{K}$ such that $\phi(C_0(G^{(0)}) \otimes \mathcal{C}) = C_0(K) \otimes \mathcal{C}$; and

(2) there is a *-ring isomorphism $\eta : A_R(G) \otimes M_\infty(R) \to A_R(G|_K) \otimes M_\infty(R)$ such that $\eta(A_R(G^{(0)}) \otimes D_\infty(R)) = A_R(K) \otimes D_\infty(R)$.

Proof. The canonical isomorphisms $C^*(G \times \mathcal{R}) \cong C^*(G) \otimes \mathcal{K}$ and $C^*(G|_K \times \mathcal{R}) \cong C^*(G|_K) \otimes \mathcal{K}$ carry $C_0(G^{(0)} \times \mathbb{N})$ to $C_0(G^{(0)}) \otimes \mathcal{C}$ and $C_0(K \times \mathbb{N})$ to $C_0(K) \otimes \mathcal{C}$. Similarly, the canonical *-ring isomorphisms $A_R(G \times \mathcal{R}) \cong A_R(G) \otimes M_\infty(R)$ and $A_R(G|_K \times \mathcal{R}) \cong A_R(G|_K) \otimes M_\infty(R)$ carry $A_R(G^{(0)} \times \mathbb{N})$ to $A_R(G^{(0)}) \otimes D_\infty(R)$ and $A_R(K \times \mathbb{N})$ to $A_R(K) \otimes D_\infty(R)$. Hence both statements follow from Proposition 2.5. □

From Lemma 4.4, Theorem 3.2 and Theorem 4.2 we obtain a version of [21, Theorem 5.4] for graph C^*-algebras and Leavitt path algebras. For a ring A, we denote by $M(A)$ the *multiplier ring* of A (see for example [5]).

Corollary 4.5. *Let E and F be directed graphs, and let R be a commutative integral domain with 1. The following are equivalent:*

(1) G_E and G_F are Kakutani equivalent;

[1] Theorem 6.2 of [7] says "no sources" rather than "no sinks" as they use the convention that $s_e^* s_e = p_{s(e)}$ rather than $s_e^* s_e = p_{r(e)}$.

(2) there exist projections $p_E \in M(D(E))$ *and* $p_F \in M(D(F))$ *and an isomorphism* $\phi : p_E C^*(E) p_E \to p_F C^*(F) p_F$ *such that* p_E *is full in* $C^*(E)$*,* p_F *is full in* $C^*(F)$*, and* $\phi(p_E D(E)) = p_F D(F)$*;*

(3) there exist projections $p_E \in M(D_R(E))$ *and* $p_F \in M(D_R(F))$ *and a* **-ring isomorphism* $\eta : p_E L_R(E) p_E \to p_F L_R(F) p_F$ *such that* p_E *is full in* $L_R(E)$*,* p_F *is full in* $L_R(F)$*, and* $\eta(p_E D_R(E)) = p_F D_R(F)$*.*

Proof. We prove (1) $\iff$ (2); the proof of (1) $\iff$ (3) is similar.

First suppose (1); say $X \subseteq G_E^{(0)}$ is a G_E-full clopen subset and $Y \subseteq G_F^{(0)}$ is a G_F-full clopen subset such that $(G_E)|_X \cong (G_F)|_Y$. Then the characteristic function of X corresponds to a projection $p_E \in M(D(E))$ which is full in $C^*(E)$ and such that $C^*((G_E)|_X) \cong p_E C^*(E) p_E$ by an isomorphism that maps $C_0(X)$ onto $p_E D(E)$. Similarly, the characteristic function of Y corresponds to a projection $p_F \in M(D(F))$ which is full in $C^*(F)$ and such that $C^*((G_F)|_Y) \cong p_F C^*(F) p_F$ by an isomorphism that maps $C_0(Y)$ onto $p_F D(F)$. The isomorphism $(G_E)|_X \cong (G_F)|_Y$ gives an isomorphism $C^*((G_E)|_X) \cong C^*((G_F)|_Y)$ that maps $C_0(X)$ onto $C_0(Y)$, which yields (2).

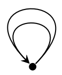
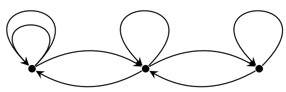
Now suppose (2). The projection $p_E \in M(D(E))$ corresponds to a G_E-full clopen subset X of $G_E^{(0)}$ such that there is an isomorphism $C^*((G_E)|_X) \cong p_E C^*(E) p_E$ that maps $C_0(X)$ onto $p_E D(E)$, and the projection $p_F \in M(D(F))$ corresponds to a G_F-full clopen subset Y of $G_F^{(0)}$ such that there is an isomorphism $C^*((G_F)|_Y) \cong p_F C^*(F) p_F$ that maps $C_0(Y)$ onto $p_F D(F)$. Lemma 4.4(1) gives a diagonal-preserving isomorphism $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$. So Theorem 4.2 implies that $G_E \times \mathcal{R}$ and $G_F \times \mathcal{R}$ are groupoid equivalent, and hence Theorem 3.2 implies that they are Kakutani equivalent. □

Theorem 3.2 implies that the equivalent conditions (1)–(3) of Corollary 4.5 are also equivalent to the equivalent conditions (1)–(6) of Theorem 4.2.

Corollary 4.6. *If* E *and* F *are directed graphs, then* $L_{\mathbb{Z}}(E) \otimes M_\infty(\mathbb{Z}) \cong L_{\mathbb{Z}}(F) \otimes M_\infty(\mathbb{Z})$ *as* **-rings if and only if there is a diagonal-preserving isomorphism* $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$*.*

Proof. First suppose that $L_{\mathbb{Z}}(E) \otimes M_\infty(\mathbb{Z}) \cong L_{\mathbb{Z}}(F) \otimes M_\infty(\mathbb{Z})$ as *-rings. Then $L_{\mathbb{Z}}(SE) \cong L_{\mathbb{Z}}(SF)$ as *-rings as well. By [11, Corollary 6], this *-isomorphism is diagonal preserving, so (2) $\Longrightarrow$ (1) of Theorem 4.2 gives a diagonal-preserving isomorphism $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$. The reverse implication follows from (1) $\Longrightarrow$ (2) of Theorem 4.2. □

An important question about Leavitt path algebras is whether the complex Leavitt path algebras of the graphs

$E_2 =$  and $E_{2-} =$ 

are isomorphic. This was recently answered in the negative for Leavitt path algebras over $\mathbb{Z}$ as *-rings [16]. We extend this to the question of stable *-isomorphism.

Corollary 4.7. *Let* E *and* F *be strongly connected finite graphs such that* $L_{\mathbb{Z}}(E) \otimes M_\infty(\mathbb{Z})$ *and* $L_{\mathbb{Z}}(F) \otimes M_\infty(\mathbb{Z})$ *are* **-isomorphic. Then* $\det(1 - A_E^t) = \det(1 - A_F^t)$*. In particular,* $L_{\mathbb{Z}}(E_2) \otimes M_\infty(\mathbb{Z})$ *and* $L_{\mathbb{Z}}(E_{2-}) \otimes M_\infty(\mathbb{Z})$ *are not* **-isomorphic.*

Proof. Corollary 4.6 gives a diagonal-preserving isomorphism $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$. If E and F have cycles with exits, then as discussed in the proof of [19, Corollary 3.8], the proof of [20, Theorem 4.1] combined with [19, Theorem 3.6] gives $\det(1 - A_E^t) =$

$\det(1 - A^t_F)$. If E and F have cycles without exists, then A_E and A_F are permutation matrices, so $\det(1 - A^t_E) = \det(1 - A^t_F) = 0$. To prove the final statement, one checks that $\det(1 - A^t_{E_2}) = -1$ and $\det(1 - A^t_{E_{2-}}) = 1$. □

Remark 4.8. It is natural to ask whether Corollary 4.7 can be used to decide whether $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_{2-})$ are Morita equivalent. Theorem 5 and part 2 of the remarks following Corollary 7 in [1] show that rings with enough idempotents are stably isomorphic if and only if they are Morita equivalent. But this is a result about ring isomorphisms, whereas Corollaries 4.6 and 4.7 are about *-ring isomorphisms (and the *-preserving hypothesis is crucial to the argument of [11, Corollary 6], upon which our results hinge). So the question remains open whether $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_{2-})$ are Morita equivalent. There is a notion of Morita *-equivalence for rings [3]. Though we were unable to locate a reference, it seems likely that an analogue of [1, Theorem 5] holds for stable *-isomorphism and Morita *-equivalence. If so, then such a result could be combined with Corollary 4.7 to prove that $L_{\mathbb{Z}}(E_2)$ and $L_{\mathbb{Z}}(E_{2-})$ are not Morita *-equivalent.

Theorem 4.2 has implications for the stable isomorphisms associated to Sørensen's move equivalences of graphs [28]. *Move equivalence* for graphs with finitely many vertices is the equivalence relation generated by four operations: deleting a regular source; collapsing a regular vertex; in-splitting at a regular vertex; and outsplitting. By [28, Theorem 4.3], if $C^*(E)$ and $C^*(F)$ are simple and E and F each contain at least one infinite emitter, then $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ if and only if E and F are move equivalent.

Corollary 4.9. *Let E and F be directed graphs with finitely many vertices. Suppose that E and F are move equivalent. Then $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$, and there are diagonal-preserving isomorphisms $C^*(E) \otimes \mathcal{K} \cong C^*(F) \otimes \mathcal{K}$ and $L_R(E) \otimes M_\infty(R) \cong L_R(F) \otimes M_\infty(R)$ for every commutative ring R with 1.*

Proof. Sørensen's moves (S), (R) and (I) are all examples of Crisp and Gow's collapsing procedure (see page 2070–2071 of [13]), so if F is obtained from E by applying any of these moves, then Lemma 4.3 shows that $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$. By [10, Theorem 6.1 and Corollary 6.2], if F is obtained from E by applying move (O), then $G_E \cong G_F$, so certainly $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$. Induction establishes that if E and F are move equivalent then $G_E \times \mathcal{R} \cong G_F \times \mathcal{R}$. The remaining statements follow from Theorem 4.2. □

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