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Multiple semiclassical states for coupled Schrödinger-Poisson equations with critical exponential growth

Zhisu Liu
Hunan University

Shangjiang Guo
Hunan University

Yanqin Fang
University of Wollongong, yf682@uowmail.edu.au

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Abstract

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Keywords

schrödinger-poisson, equations, critical, multiple, exponential, semiclassical, growth, states, coupled

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Multiple semiclassical states for coupled Schrödinger-Poisson equations with critical exponential growth

Zhisu Liu,^{a)} Shangjiang Guo,^{b)} and Yanqin Fang^{c)}

College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, People's Republic of China

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In this paper, we study the multiplicity of positive solutions for a class of Schrödinger-Poisson systems. Working in a variational setting, we prove the existence and multiplicity of positive solutions for the system when the Plank's constant is small and the potential satisfies some suitable conditions. We show that the number of positive solutions depends on the profile of the potential and that each solution concentrates around its corresponding global minimum point of the potential in the semiclassical limit. We also study the exponential decay. © 2015 AIP Publishing LLC. [<http://dx.doi.org/10.1063/1.4919543>]

I. INTRODUCTION

In the last years, there has been a great deal of works dealing with equations arising in quantum mechanics by means of variational tools, like the following electrostatic version of the Schrödinger-Poisson equation:

$$\begin{cases} -\frac{\hbar^2}{2m}\Delta u + ewu + \phi u = f(x,u), & \text{in } \mathbb{R}^3, \\ -\Delta \phi = 4\pi eu^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (1)$$

with $f \in C^1(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$. Here, m and e denote the mass and the charge of the particle, respectively, while $\hbar$ stands for the Plank's constant, and w denotes the phase. In the case where $f(x,u) = 0$, the wave function u satisfies the stationary solution of a quantum system proposed by Benci and Fortunato,⁷ describing the interaction of a particle with an electromagnetic field. In fact, the first equation in (1) is a nonlinear stationary Schrödinger equation coupled with a Poisson equation, where the time-independent ϕ depends on the charge of the wave function according to Maxwell's equations. The Schrödinger-Poisson equations are thus also called the Schrödinger-Maxwell equations. Moreover, system (1) containing an additional linear term $V(x)u$ which is used to describe the effect of an external potential has been considered in Ref. 15. Some results on multiplicity solutions of (1) can also be found in Refs. 7, 8, and 14.

Since the nonlinearity of the Schrödinger equation has its origin in the interaction among particles, many particle systems can be found, such as in the study of condensed matter and nonlinear optics problems, the nonlinear version of the 3D Schrödinger-Poisson equation, of the form similar to (1), has also been the object of interest for many authors. We refer the readers to Refs. 3, 5, and 34 and the references therein. In particular, for system (1) with $f(x,u) = u^p$, there are many results concerning the existence and multiplicity of solutions, radical solutions, and ground state solutions. For example, D'Aprile and Mugnai¹⁶ investigated the existence of a radical solution in the case where $p \in [3, 5)$. D'Aprile and Wei¹⁸ constructed positive radically symmetric bounded states for the case $p \in (1, \frac{11}{7})$. D'Aprile and Mugnai¹⁷ proved the nonexistence of nontrivial

^{a)}jiuzhisu183@sina.com

^{b)}shangjguo@hnu.edu.cn

^{c)}yanqingfang@126.com

solutions in the case where $p \in (0, 1)$ and $p \in [5, +\infty)$. Some multiplicity results about radical symmetric solutions can also be found in Refs. 15 and 34. Azzollini and Pomponio⁵ and Zhao and Zhao³⁹ studied the existence of ground state solutions when $p \in (3, 5)$ and $p \in (2, 5)$, respectively, which has been extended by Liu and Guo²⁸ to the critical case. For more existence results about Schrödinger-Poisson equations, we refer the reader to Refs. 11 and 38 and the references therein.

We note that if $\phi \equiv 0$ then (1) reduces to the Schrödinger equation of the form

$$-\frac{\hbar^2}{2m}\Delta u + V(x)u = f(x, u), \quad x \in \mathbb{R}^N. \quad (2)$$

In the past decade, there have been a large number of papers (see Refs. 23, 31, and 36) concerning with the semiclassical states of (1) with $\phi = 0$, that is, the behavior of solutions of (2) as $\hbar$ tends to zero. In some sense, this describes the transition between quantum mechanics and classical mechanics. More precisely, Pino and Felmer³¹ investigated the existence of spikes (solutions concentrating at a single point) on the nondegenerate critical points of the potential V . Gui²³ studied the existence of multi-bump solutions (solutions which concentrate around several different points). Recently, the existence of solutions concentrating around a whole sphere in $\mathbb{R}^N$ (for radial potentials) has been shown in Refs. 2 and 6.

Motivated by the above fact, in this paper, we are concerned with the following problem of semiclassical states:

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u + \phi u = f(u) + u^5 & \text{in } \mathbb{R}^3, \\ -\Delta \phi = u^2 & \text{in } \mathbb{R}^3, \end{cases} \quad (3)$$

where $V : \mathbb{R}^3 \rightarrow \mathbb{R}$ is a continuous function satisfying

(V) $V_\infty = \liminf_{n \rightarrow \infty} V(x) > V_0 = \inf_{x \in \mathbb{R}^3} V(x) > 0$ and each of $a^1, \dots, a^k$ is a point of a strict global minima of V in $\mathbb{R}^3$. That is, $V(a^i) = V_0, i = 1, \dots, k$,

and the nonlinearity $f : \mathbb{R} \rightarrow \mathbb{R}$ is C^1 and satisfies the following conditions:

- (f₁) $f(t) = 0$ for all $t < 0$ and $f(t) = o(t^3)$ as $t \rightarrow 0^+$;
 (f₂) There exists $4 < q < 2^* = 6$ and constant $D > 0$ such that

$$|f'(t)| \leq D(1 + |t|^{q-2});$$

- (f₃) The function $t \rightarrow \frac{f(t)}{t^3}$ is increasing in $(0, \infty)$;
 (f₄) $f(t) \geq \lambda t^\sigma$ for all $t > 0$ with some $\lambda > 0$ and $\sigma \in (3, 5)$.

Obviously, it follows from the conditions of (f₁)-(f₃) that

$$F(t) \geq 0, \quad 4F(t) \leq f(t)t, \quad \forall t \in \mathbb{R},$$

where $F(t) = \int_0^t f(s)ds$.

For (3) with $f(u) + u^5$ replaced by $u^p, p \in (1, 5)$, Ruiz and Vaira³⁵ recently proved the existence of multi-bump solutions whose bumps concentrate around a local minimum of the potential V , while Ianni and Vaira²⁵ considered the existence of positive bound state solutions concentrating on the local minimum of the potential V . Furthermore, Ianni and Vaira,²⁶ Ianni²⁷ investigated the existence of radially symmetric solutions concentrating on the spheres. The proof explored in Refs. 25–27 and 35 is based on a singular perturbation, essentially a Lyapunov-Schmitt reduction method.

Under the assumption (V) and the similar conditions as (f₁)-(f₄), He and Zou²⁴ considered the following Schrödinger-Poisson equation:

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u + \phi u = f(u) + u^5, & \text{in } \mathbb{R}^3, \\ -\epsilon^2 \Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (4)$$

and proved (4) has a ground state solution concentrating around global minimum of the potential V in the semi-classical limit and also studied the exponential decay of the ground state solution. More recently, Wang *et al.*³⁷ studied the existence and concentration of positive ground state solutions for

the semilinear Schrödinger-Poisson system,

$$\begin{cases} -\epsilon^2 \Delta u + V(x)u + \lambda \phi u = b(x)f(u), & \text{in } \mathbb{R}^3, \\ -\epsilon^2 \Delta \phi = u^2, & \text{in } \mathbb{R}^3, \end{cases} \quad (5)$$

where $\epsilon > 0$ is a small parameter and $\lambda \neq 0$ is a real parameter, f is a continuous superlinear and subcritical nonlinearity. Under some suitable assumptions on $V(x)$ and $b(x)$, Wang *et al.*³⁷ first proved the existence of least energy solution $(u_\epsilon, \phi_\epsilon)$ and then showed that u_ϵ converges to the least energy solution of the associated limit problem and concentrates to some set in $\mathbb{R}^3$ depending on the potentials V and b . Moreover, some properties for the least energy solution and some sufficient conditions for the nonexistence of positive ground state solutions are also considered in Ref. 37.

In present paper, our main interest is to study the multiplicity and concentration of positive solutions for the critical Schrödinger-Poisson equation (3) when ϵ is small. Our argument is different from those explored in the literature mentioned above. Indeed, our proof is based on the restriction of the associated energy functional to some subsets of Nehari solution manifold which depends on the strict global minimum point of V . One first finds a minimizing sequence in each subset and then uses the Enklend variational principle to obtain the Palais-Smale (PS) sequence from the corresponding abstract subset of Nehari solution manifold. Therefore, in the study of multiple positive solutions, we only need to prove that each Palais-Smale sequence contains a convergent subsequence in the restricted abstract set. Furthermore, motivated by the arguments of Wang,³⁶ we give some new estimates on Palais-Smale sequence to obtain that global maximum points of each positive solution concentrates around its corresponding strict global minimum point of V . However, we have to face two main difficulties. One is the critical nonlinearities and the lack of compactness of the embedding of $H^1(\mathbb{R}^3)$ into the space $L^q(\mathbb{R}^3)$, $q \in [2, 6)$, prevents us from using the variational methods in a standard way. The other is that we have to employ some technique to differentiate abstract subsets of Nehari solution manifold in order to obtain different Palais-Smale sequence.

Let us state our main results.

Theorem 1. Assume that (V) and (f_1) – (f_4) hold. Then there exists $\epsilon^* > 0$ such that for every $\epsilon \in (0, \epsilon^*)$ and each $j \in \{1, \dots, k\}$, problem (3) has a positive solution u_ϵ^j which satisfies

$$\lim_{\epsilon \rightarrow 0^+} V(\varrho_\epsilon^j) = V(a^j),$$

where $\varrho_\epsilon^j \in \mathbb{R}^3$ denotes the global maximum point of u_ϵ^j . Moreover, there are constants $C, c > 0$ such that

$$u_\epsilon^j(x) \leq C \exp\left(-c \frac{|x - \varrho_\epsilon^j|}{\epsilon}\right) \quad \text{for all } x \in \mathbb{R}^3 \text{ and } j \in \{1, 2, \dots, k\}.$$

Remark 1. The hypothesis (V) was first introduced by Rabinowitz in Refs. 32 and 33. Ianni and Varia^{25–27} studied the concentration behavior of positive solutions of (3) with $f(u) + u^5$ replaced by u^p , $p \in (1, 5)$, under the following assumption:

$$(V_1) \quad \inf_{x \in \mathbb{R}^3} V(x) > 0 \text{ and } V(x) \text{ is uniformly bounded for } x \in \mathbb{R}^3.$$

Here, however, we shall consider the two cases: $V_\infty < \infty$ and $V_\infty = \infty$. Therefore, Theorem 1 generalizes the relevant results of Refs. 25–27 and 35 in the sense that we treat more general potential V than that in Refs. 25–27 and that our arguments also work in the subcritical case. Moreover, in Refs. 25–27 and 35, the authors only studied the existence of positive solutions. In this paper, however, we are concerned with multiple positive solutions and see that the number of solutions depends on the strict global minimum points of V .

Furthermore, if the potential V has no other strict global minimum points than $a^1, \dots, a^k$, then we can prove that problem (3) has at least one ground state solution.

Theorem 2. In addition to conditions (V) and (f_1) – (f_4) , assume that the potential V has no other strict global minimum points than $a^1, \dots, a^k$. Then there exists $\bar{\epsilon}^* > 0$ such that for each $\epsilon \in (0, \bar{\epsilon}^*)$, we can find at least one ground state solution among the solutions of Theorem 1.

Remark 2. It is clear that the ground state solutions have the properties of concentration and exponential decay stated in Theorem 1. Moreover, the positive parameter $\bar{\epsilon}^*$ is smaller than ϵ^* .

Remark 3. As we mentioned before, He and Zou²⁴ studied the concentration behavior of the ground state solution for problem (4) under hypotheses (V), (f_1) – (f_4) , and

$$(f_5) \quad 0 < \theta F(t) = \theta \int_0^t f(\tau) d\tau \leq f(t)t \quad \text{for all } t > 0.$$

However, in view of Theorem 2, we see that hypotheses (V) and (f_1) – (f_4) are enough to obtain concentration behavior of multiple positive solutions for (3). Most importantly, we wish to take this opportunity to point out that our main results are still true for problem (4). Therefore, Theorems 1 and 2 are generalization of the relevant results of Ref. 24.

Remark 4. To our best knowledge, there have been many works concerning concentration behavior of multiple positive solutions for Schrödinger equations and Schrödinger-Poisson equations, such as Refs. 1, 4, 13, and 22 and so on. It has been proved in these papers that the number of positive solutions is related to the topology of the set of the global minimum points of the potential V and that these solutions concentrate around global minimum points of V as $\epsilon \rightarrow 0^+$. Their proofs are essentially based on the Ljusternik-Schnirelmann theory. In this paper, we employ complete different arguments but obtain similar results (Theorems 1 and 2). Moreover, we know that the global maximum point of each positive solution concentrates around the associated global minimum point of the potential V , which makes concentration phenomenon become more specific than the literature.

In this paper, C will be used to denote various different positive constants which may vary from line to line and are not essential to the problem.

The remainder of this paper is organized as follows. Some preliminary results are presented in Sec. II. Section III is devoted to the multiplicity proof of positive solutions. The proof of Theorems 1 and 2 will be given in Secs. IV and V, respectively.

II. NOTATIONS AND PRELIMINARIES

In this section, we first introduce the following notations. For any $\rho > 0$ and for any $z \in \mathbb{R}^3$, $B_\rho(z)$ denotes the ball of radius ρ centered at z and $|B_\rho(z)|$ denotes its Lebesgue measure. By $\|\cdot\|_s$, we denote the usual L^s -norm, where $1 \leq s \leq +\infty$. Define the function space

$$H^1(\mathbb{R}^3) := \{u \in L^2(\mathbb{R}^3) : |\nabla u| \in L^2(\mathbb{R}^3)\},$$

equipped with the usual norm

$$\|u\| := \left(\int_{\mathbb{R}^3} (|\nabla u|^2 + u^2) dx \right)^{\frac{1}{2}}.$$

Let $D^{1,2}(\mathbb{R}^3)$ be the completion of $C_0^\infty(\mathbb{R}^3)$ with respect to the Dirichlet norm,

$$\|u\|_{D^{1,2}} := \left(\int_{\mathbb{R}^3} |\nabla u|^2 dx \right)^{\frac{1}{2}}.$$

S denotes the best Sobolev constant,

$$S = \inf_{u \in D^{1,2}(\mathbb{R}^3) \setminus \{0\}} \frac{\int |\nabla u|^2 dx}{\left(\int_{\mathbb{R}^3} u^6 dx \right)^{1/3}}.$$

Recall that by the Lax-Milgram theorem, for every $u \in H^1(\mathbb{R}^3)$, the Poisson equation $-\Delta \phi = u^2$ has a unique positive solution $\phi = \phi_u \in D^{1,2}(\mathbb{R}^3)$ given by

$$\phi_u(x) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|} dy. \quad (6)$$

Substituting (6) into (3) and making the change of variable $\epsilon z = x$, we can rewrite (3) as the following equivalent equation:

$$-\Delta u + V(\epsilon x)u + \epsilon^2 \phi u = f(u) + u^5 \text{ in } \mathbb{R}^3. \quad (7)$$

The function $\phi : H^1(\mathbb{R}^3) \rightarrow D^{1,2}(\mathbb{R}^3)$, $\phi[v] = \phi_v$ has the following properties (see for Refs. 11 and 34).

Lemma 1. For every $u \in H^1(\mathbb{R}^3)$, we have

- (1) ϕ is continuous and maps bounded sets into bounded sets;
- (2) if $u_n \rightharpoonup u$ in $H^1(\mathbb{R}^3)$, then $\phi_{u_n} \rightharpoonup \phi_u$ in $D^{1,2}(\mathbb{R}^3)$;
- (3) $\phi_u \geq 0$, $\|\phi_u\|_{D^{1,2}} \leq C\|u\|^2$ and $\int_{\mathbb{R}^3} \phi_u u^2 dx \leq C\|u\|^4$;
- (4) $\phi_{tu}(x) = t^2 \phi_u(x) \geq 0$ for all $t > 0$ and $x \in \mathbb{R}^3$.

For every $\epsilon > 0$, let $E_\epsilon = \{u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(\epsilon x)u^2(x)dx < \infty\}$ be the Sobolev space endowed with the inner product and norm,

$$(u, v) = \int_{\mathbb{R}^3} (\nabla u \cdot \nabla v + V(\epsilon x)uv)dx, \quad \|u\|_\epsilon = (u, u)^{\frac{1}{2}}.$$

Hereafter, we omit the symbol dx in the integrals over $\mathbb{R}^3$ when no confusion can arise.

At this step, we see that (7) is variational and its solutions are the critical points of the functional

$$I_{V_\epsilon, \epsilon^2}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\epsilon x)u^2) + \frac{\epsilon^2}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{6} \int_{\mathbb{R}^3} u^6, \quad (8)$$

where $I_{V_\epsilon, \epsilon^2}$ belongs to $C^1(E_\epsilon, \mathbb{R})$ and $V_\epsilon = V(\epsilon x)$.

Next, we define the Nehari manifold associated with $I_{V_\epsilon, \epsilon^2}$ by

$$\mathcal{N}_{V_\epsilon, \epsilon^2} = \left\{ u \in E_\epsilon \setminus \{0\} : \int_{\mathbb{R}^3} (|\nabla u|^2 + V(\epsilon x)u^2 + \epsilon^2 \phi_u u^2) = \int_{\mathbb{R}^3} (f(u)u + u^6) \right\}$$

and consider the following minimization problem,

$$c_{V_\epsilon, \epsilon^2} = \inf_{u \in \mathcal{N}_{V_\epsilon, \epsilon^2}} I_{V_\epsilon, \epsilon^2}(u).$$

Similarly, we can define I_{V_∞, ϵ^2} , c_{V_∞, ϵ^2} , $I_{V_\infty, 0}$, $c_{V_\infty, 0}$. Define

$$J_\epsilon(u) = \langle I'_{V_\epsilon, \epsilon^2}(u), u \rangle = \|u\|_\epsilon^2 + \epsilon^2 \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} (f(u)u + u^6).$$

Then, for $u \in \mathcal{N}_{V_\epsilon, \epsilon^2}$,

$$\begin{aligned} \langle J'_\epsilon(u), u \rangle &= 2\|u\|_\epsilon^2 + 4\epsilon^2 \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} (f(u)u + f'(u)u^2 + 6u^6) \\ &\leq 3 \int_{\mathbb{R}^3} f(u)u - \int_{\mathbb{R}^3} f'(u)u^2 - 2 \int_{\mathbb{R}^3} u^6. \end{aligned} \quad (9)$$

From (f₃), we deduce that $3f(t) - f'(t)t \leq 0$ for $t > 0$. Therefore, it follows from (9) that

$$\langle J'_\epsilon(u), u \rangle \leq -2 \int_{\mathbb{R}^3} u^6 < 0 \quad \text{for } u \in \mathcal{N}_{V_\epsilon, \epsilon^2},$$

which implies that $\mathcal{N}_{V_\epsilon, \epsilon^2}$ is a C^1 manifold, and so the Nehari manifold $\mathcal{N}_{V_\epsilon, \epsilon^2}$ is a natural constraint for the functional $I_{V_\epsilon, \epsilon^2}$.

Let us recall the following fact for $\mathcal{N}_{V_\epsilon, \epsilon^2}$ and $c_{V_\epsilon, \epsilon^2}$ (see Ref. 24).

- Lemma 2.* (1) $\|u\|_\epsilon \geq C$ for all $u \in \mathcal{N}_{V_\epsilon, \epsilon^2}$.
 (2) For each $u \in E_\epsilon$, there exists a unique $t(u) > 0$ such that $t(u)u \in \mathcal{N}_{V_\epsilon, \epsilon^2}$. Moreover, $I_{V_\epsilon, \epsilon^2}(t(u)u) = \max_{t \geq 0} I_{V_\epsilon, \epsilon^2}(tu)$.
 (3) $c_{V_\epsilon, \epsilon^2} > 0$ for all $\epsilon > 0$.

III. MULTIPLICITY

In this section, we make use of the strict global minimum points of V to construct PS sequences, which play an important role in the proof of our main results. For $a > 0$, let $C_a(a^j)$ denote the hypercube $\Pi_{i=1}^3(a_i^j - a, a_i^j + a)$ centred at $a^j = (a_1^j, a_2^j, a_3^j)$, $j = 1, \dots, k$. Let $\overline{C_a(a^j)}$ and $\partial C_a(a^j)$ denote the closure and the boundary of $C_a(a^j)$, respectively. Let $\overline{C_a^j}$ denote $\overline{C_a(a^j)}$. By condition (V), we can choose numbers $L, l > 0$ so that $\overline{C_l(a^j)}$, $j = 1, \dots, k$ are disjoint and $V(x) > V(a^j)$ for all $x \in \partial C_l(a^j)$ and $j \in \{1, 2, \dots, k\}$, and $\cup_{j=1}^k C_l(a^j) \subset \Pi_{i=1}^3(-L, L)$. Define $\Psi_\epsilon \in C(\mathbb{R}, \mathbb{R})$, $h_\epsilon \in C(E_\epsilon, \mathbb{R}^3)$ by

$$\Psi_\epsilon(t) = \begin{cases} \frac{2L}{\epsilon}, & t > \frac{2L}{\epsilon}, \\ t, & \frac{2L}{\epsilon} \geq t \geq \frac{-2L}{\epsilon}, \\ -\frac{2L}{\epsilon}, & t < \frac{-2L}{\epsilon}, \end{cases} \quad (10)$$

$$h_\epsilon^i(u) = \frac{\int_{\mathbb{R}^3} \Psi_\epsilon(x_i) |u|^p}{\int_{\mathbb{R}^3} |u|^p}, \quad i = 1, 2, 3, \quad (11)$$

$$h_\epsilon(u) = (h_\epsilon^1(u), h_\epsilon^2(u), h_\epsilon^3(u)),$$

where $p \in [2, 6]$. Let $C_{l/\epsilon}^j \equiv C_{l/\epsilon}(a^j/\epsilon)$ and

$$N_\epsilon^j = \{u \in \mathcal{N}_{V_\epsilon, \epsilon^2} | u \geq 0 \text{ and } h_\epsilon(u) \in C_{l/\epsilon}^j\},$$

$$\partial N_\epsilon^j = \{u \in \mathcal{N}_{V_\epsilon, \epsilon^2} | u \geq 0 \text{ and } h_\epsilon(u) \in \partial C_{l/\epsilon}^j\}$$

for $j = 1, \dots, k$. It can be readily verified that N_ϵ^j and ∂N_ϵ^j are non-empty sets for all $j \in \{1, \dots, k\}$. Let us consider the minimization problems in N_ϵ^j and ∂N_ϵ^j for $I_{V_\epsilon, \epsilon^2}$ and set

$$\alpha_\epsilon^j = \inf_{u \in N_\epsilon^j} I_{V_\epsilon, \epsilon^2}(u) \quad \text{and} \quad \tilde{\alpha}_\epsilon^j = \inf_{u \in \partial N_\epsilon^j} I_{V_\epsilon, \epsilon^2}(u) \quad \text{for } j = 1, \dots, k.$$

It follows from Ref. 40 that we can take w be a positive radical ground state of

$$\begin{cases} -\Delta u + V_0 u = f(u) + u^5 & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (12)$$

such that $I_{V_0, 0}(w) = c_{V_0, 0} < \frac{S^{\frac{3}{2}}}{3}$ with assumptions (f₁)-(f₄), where $I_{V_0, 0}$ is defined by

$$I_{V_0, 0} = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + V_0 u^2) - \int_{\mathbb{R}^3} (F(u) + \frac{1}{6} u^6).$$

For small $\epsilon > 0$ satisfying $2\sqrt{\epsilon} < 1$, define $\chi_\epsilon \in C^1(\mathbb{R}^3, \mathbb{R}^+)$ by

$$\chi_\epsilon(x) = \begin{cases} 1, & |x| < \frac{1}{2\sqrt{\epsilon}} - 1, \\ 0, & |x| > \frac{1}{2\sqrt{\epsilon}}, \end{cases}$$

and $|\nabla \chi_\epsilon| \leq 2$ in $\mathbb{R}^3$. Take

$$v_\epsilon^j(x) = w \left(x - \frac{a^j}{\epsilon} \right) \chi_\epsilon \left(x - \frac{a^j}{\epsilon} \right)$$

for all $x \in \mathbb{R}^3$ and $j = 1, \dots, k$. Then, we have the following result.

Lemma 3. Assume that the conditions (V) and (f₁)-(f₄) hold. Then, for each positive number $\delta \leq c_{V_0, 0}$, there exists $\epsilon_\delta > 0$ such that, for any $\epsilon \in (0, \epsilon_\delta)$,

$$c_{V_\epsilon, \epsilon^2} \leq \alpha_\epsilon^j < c_{V_0, 0} + \delta \quad \text{for all } j = 1, \dots, k.$$

In particular, N_ϵ^j , $j = 1, \dots, k$, are non-empty sets.

Proof. For each fixed $j \in \{1, \dots, k\}$, it is easy to see that $v_\epsilon^j \in E_\epsilon$ and v_ϵ^j is bounded in $H^1(\mathbb{R}^3)$ uniformly for $\epsilon \in (0, \epsilon_\delta)$. Then there exists $t_\epsilon^j > 0$ such that $t_\epsilon^j v_\epsilon^j \in \mathcal{N}_{V_\epsilon, \epsilon^2}$. Therefore, we have

$$\frac{1}{(t_\epsilon^j)^2} \|v_\epsilon^j\|_\epsilon^2 = \int_{\mathbb{R}^3} \frac{f(t_\epsilon^j v_\epsilon^j)}{(t_\epsilon^j v_\epsilon^j)^3} (v_\epsilon^j)^4 - \epsilon^2 \int_{\mathbb{R}^3} \phi_{v_\epsilon^j} (v_\epsilon^j)^2 + (t_\epsilon^j)^2 \int_{\mathbb{R}^3} (v_\epsilon^j)^6. \quad (13)$$

Using Lebesgue's theorem and the conditions (f₁) and (f₂), one can verify that

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \|v_\epsilon^j\|_\epsilon^2 &= \int_{\mathbb{R}^3} (|\nabla w|^2 + V_0 w^2), \\ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} \phi_{v_\epsilon^j} (v_\epsilon^j)^2 &= \int_{\mathbb{R}^3} \phi_w w^2, \\ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} |v_\epsilon^j|^6 &= \int_{\mathbb{R}^3} w^6, \\ \lim_{\epsilon \rightarrow 0} \int_{\mathbb{R}^3} f(v_\epsilon^j) v_\epsilon^j &= \int_{\mathbb{R}^3} f(w) w. \end{aligned} \quad (14)$$

In view of (13) and Lemma 2, we can assume that $t_\epsilon^j \rightarrow A > 0$. Hence, it follows from (13) and (14) that

$$\frac{1}{A^2} \int_{\mathbb{R}^3} (|\nabla w|^2 + V_0 w^2) = \int_{\mathbb{R}^3} \frac{f(Aw)}{(Aw)^3} w^4 + A^2 \int_{\mathbb{R}^3} w^6. \quad (15)$$

Notice that

$$\int_{\mathbb{R}^3} (|\nabla w|^2 + V_0 w^2) = \int_{\mathbb{R}^3} \frac{f(w)}{w^3} w^4 + \int_{\mathbb{R}^3} w^6. \quad (16)$$

Combining (15), (16), and (f₃), we have that $A = 1$. We next claim that $h_\epsilon(t_\epsilon^j v_\epsilon^j) \in C_{l/\epsilon}^j$. Indeed,

$$h_\epsilon^i(t_\epsilon^j v_\epsilon^j) = \frac{\int_{\mathbb{R}^3} \Psi_\epsilon(x_i) u^p(x - \frac{a^j}{\epsilon}) \chi_\epsilon^p(x - \frac{a^j}{\epsilon})}{\int_{\mathbb{R}^3} u^p(x - \frac{a^j}{\epsilon}) \chi_\epsilon^p(x - \frac{a^j}{\epsilon})}, \quad i = 1, 2, 3.$$

If $|x_i - \frac{a^j}{\epsilon}| > \frac{1}{2\sqrt{\epsilon}}$, then $\chi_\epsilon(x_i - \frac{a^j}{\epsilon}) = 0$. From the definition of χ_ϵ , we have

$$\begin{aligned} h_\epsilon^i(t_\epsilon^j v_\epsilon^j) &= \frac{\int_{C_{1/2\sqrt{\epsilon}}^j(\frac{a^j}{\epsilon})} \Psi_\epsilon(x_i) u^p(x - \frac{a^j}{\epsilon}) \chi_\epsilon^p(x - \frac{a^j}{\epsilon})}{\int_{C_{1/2\sqrt{\epsilon}}^j(\frac{a^j}{\epsilon})} u^p(x - \frac{a^j}{\epsilon}) \chi_\epsilon^p(x - \frac{a^j}{\epsilon})} \\ &= \frac{\int_{C_{1/2\sqrt{\epsilon}}^j(0)} ((\frac{a^j}{\epsilon})_i + x_i) u^p(x) \chi_\epsilon^p(x)}{\int_{C_{1/2\sqrt{\epsilon}}^j(0)} u^p(x) \chi_\epsilon^p(x)}. \end{aligned}$$

In view of the earlier inequality, we see

$$(\frac{a^j}{\epsilon})_i - \frac{1}{2\sqrt{\epsilon}} \leq h_\epsilon^i(t_\epsilon^j v_\epsilon^j) \leq (\frac{a^j}{\epsilon})_i + \frac{1}{2\sqrt{\epsilon}}$$

for $i = 1, 2, 3$. By restricting $\frac{1}{2\sqrt{\epsilon}} < \frac{l}{\epsilon}$, then $h_\epsilon^i(t_\epsilon^j v_\epsilon^j) \in C_{l/\epsilon}^j$ and $t_\epsilon^j v_\epsilon^j \in \mathcal{N}_\epsilon^j$. Hence, our claim is true. Moreover, it follows from (13) and (14) that

$$\begin{aligned} I_{V_\epsilon, \epsilon^2}(t_\epsilon^j v_\epsilon^j) &= \frac{(t_\epsilon^j)^2}{2} \|v_\epsilon^j\|_\epsilon^2 + \frac{1}{4} \epsilon^2 (t_\epsilon^j)^4 \int_{\mathbb{R}^3} \phi_{v_\epsilon^j} (v_\epsilon^j)^2 - \frac{(t_\epsilon^j)^6}{6} \int_{\mathbb{R}^3} (v_\epsilon^j)^6 - \int_{\mathbb{R}^3} F(t_\epsilon^j v_\epsilon^j) \\ &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla w|^2 + V_0 w^2) - \int_{\mathbb{R}^3} F(w) - \frac{1}{6} \int_{\mathbb{R}^3} w^6 + o(\epsilon) \\ &= c_{V_0, 0} + o(\epsilon). \end{aligned} \quad (17)$$

Therefore, there exists $\epsilon_\delta > 0$ such that, for every $\epsilon \in (0, \epsilon_\delta)$,

$$\alpha_\epsilon^j < c_{V_0,0} + \delta$$

for all $j = 1, \dots, k$. This completes the proof. $\square$

In the following, we give a limit problem:

$$\begin{cases} -\Delta u + V_\infty u = f(u) + u^5, & \text{in } \mathbb{R}^3, \\ u \in H^1(\mathbb{R}^3), \end{cases} \quad (18)$$

where $V_\infty < \infty$. In view of Ref. 40, we can take w^* be a positive ground state solution and $I_{V_\infty,0}(w^*) = c_{V_\infty,0}$, where $I_{V_\infty,0}$ is the energy functional of (18). Then, we can obtain the following lemma.

Lemma 4. Assume that the conditions (V) and (f_1) – (f_4) hold. Then there exists $\epsilon_0 > 0$ such that

$$\alpha_\epsilon^j < \frac{1}{3} S^{\frac{3}{2}}$$

for all $j = 1, \dots, k$, and $\epsilon \in (0, \epsilon_0)$. Moreover, if $V_\infty < \infty$, then $\alpha_\epsilon^j < c_{V_\infty, \epsilon^2}$ for all $j = 1, \dots, k$, and $\epsilon \in (0, \epsilon_0)$.

Proof. It is easy to see that $c_{V_\infty, \epsilon^2} > c_{V_0,0}$. From Lemma 3, we deduce that $\alpha_\epsilon^j \rightarrow c_{V_0,0}$ as $\epsilon \rightarrow 0^+$ for all $j \in \{1, \dots, k\}$. Notice that $c_{V_0,0} < \frac{1}{3} S^{\frac{3}{2}}$, then there exists $\epsilon_1 > 0$ such that $\alpha_\epsilon^j < \frac{1}{3} S^{\frac{3}{2}}$ for all $\epsilon \in (0, \epsilon_1)$ and $j \in \{1, \dots, k\}$. Moreover, we can easily obtain $c_{V_0,0} < c_{V_\infty,0} < c_{V_\infty, \epsilon^2}$, where $V_\infty < \infty$. Therefore, there is also $\epsilon_2 > 0$ so that $\alpha_\epsilon^j < c_{V_\infty, \epsilon^2}$ for all $\epsilon \in (0, \epsilon_2)$ and $j \in \{1, \dots, k\}$. Let $\epsilon_0 = \min\{\epsilon_1, \epsilon_2\}$, then the conclusion follows immediately. The proof is complete. $\square$

The following concentration-compactness lemma will be often used in our arguments, which comes from Lions.²⁹

Lemma 5 (see Ref. 29). Let $r > 0$, $q \in [2, 2^*)$. If $\{u_n\}$ is bounded in $H^1(\mathbb{R}^N)$ and

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |u_n|^q = 0,$$

then we have $u_n \rightarrow 0$ in $L^p(\mathbb{R}^N)$ for $p \in (2, 2^*)$. Here, $2^* = \frac{2N}{N-2}$ if $N \geq 3$ and $2^* = \infty$ if $N = 1, 2$.

An estimate on minimizing level $\tilde{\alpha}_\epsilon^j$ is obtained as follows.

Lemma 6. Under conditions (V) and (f_1) – (f_4) , there exist positive numbers $\delta_1, \epsilon_{\delta_1}$ such that for any $j = 1, \dots, k$, the following inequality holds:

$$\tilde{\alpha}_\epsilon^j > c_{V_0,0} + \delta_1 \quad (19)$$

for all $\epsilon \in (0, \epsilon_{\delta_1})$.

Proof. Fix $j \in \{1, \dots, k\}$. Assume on the contrary that there exists a sequence $\{\epsilon_n\}$ with $\epsilon_n \rightarrow 0^+$ as $n \rightarrow \infty$ such that $\tilde{\alpha}_{\epsilon_n}^j \rightarrow c \leq c_{V_0,0}$. In our arguments, we denote $I_{V_{\epsilon_n}, \epsilon_n^2}$ by I_{ϵ_n} . Then, there exists $\{u_n\} \subset \partial N_{\epsilon_n}^j$ such that

$$\begin{aligned} \|u_n\|_{\epsilon_n}^2 + \epsilon_n^2 \int_{\mathbb{R}^3} \phi u_n u_n^2 &= \int_{\mathbb{R}^3} f(u_n) u_n + \int_{\mathbb{R}^3} u_n^6, \\ I_{\epsilon_n}(u_n) &\rightarrow c \leq c_{V_0,0} \end{aligned} \quad (20)$$

as $n \rightarrow \infty$. It follows from (20) that $\|u_n\|_{\epsilon_n} < C$ uniformly for $n \in \mathbb{N}$. Moreover, in view of Lemma 1, we have $\epsilon_n^2 \int_{\mathbb{R}^3} \phi u_n u_n^2 \rightarrow 0$ as $n \rightarrow \infty$, which implies that

$$\|u_n\|_{\epsilon_n}^2 = \int_{\mathbb{R}^3} f(u_n) u_n + \int_{\mathbb{R}^3} u_n^6 + o(1). \quad (21)$$

It is easy to conclude that $\mathcal{N}_{V_{\epsilon_n},0}$ contains the same properties as $\mathcal{N}_{V_{\epsilon_n},\epsilon_n^2}$. Therefore, for each $n \in \mathbb{N}$, there exists $t_n > 0$ such that $t_n u_n \in \mathcal{N}_{V_{\epsilon_n},0}$ satisfies

$$\|t_n u_n\|_{\epsilon_n}^2 = \int_{\mathbb{R}^3} f(t_n u_n) t_n u_n + t_n^6 \int_{\mathbb{R}^3} u_n^6 \quad (22)$$

and

$$I_{\epsilon_n}(t_n u_n) \geq c_{V_{\epsilon_n},0} \geq c_{V_0,0}. \quad (23)$$

We claim that $t_n \rightarrow 1$ as $n \rightarrow \infty$. Indeed, it follows from (22) and Lemma 2 that t_n is bounded uniformly on n . Up to subsequence, still denoted by t_n , we assume that $t_n \rightarrow A$ as $n \rightarrow \infty$. Combining (21) and (22), we have $A = 1$. Therefore, (20) and (23) imply that

$$I_{\epsilon_n}(u_n) \rightarrow c_{V_0,0} \text{ as } n \rightarrow \infty. \quad (24)$$

Next, we show that

$$\int_{\mathbb{R}^3} (V(\epsilon_n x) - V_0) u_n^2 \rightarrow 0 \quad \text{as } n \rightarrow \infty. \quad (25)$$

If not, we may assume that there exists a positive constant C such that, for large n ,

$$\int_{\mathbb{R}^3} (V(\epsilon_n x) - V_0) u_n^2 > C. \quad (26)$$

For each $n \in \mathbb{N}$, there exists $s_n \in \mathbb{R}^+$ such that $s_n u_n \in \mathcal{N}_{V_0,0}$ satisfies

$$s_n^2 \int_{\mathbb{R}^3} |\nabla u_n|^2 + s_n^2 \int_{\mathbb{R}^3} V_0 u_n^2 = \int_{\mathbb{R}^3} f(s_n u_n) s_n u_n + s_n^6 \int_{\mathbb{R}^3} u_n^6 \quad (27)$$

and it follows from (21) and (26) that

$$\int_{\mathbb{R}^3} |\nabla u_n|^2 + \int_{\mathbb{R}^3} V_0 u_n^2 + \frac{C}{2} < \int_{\mathbb{R}^3} f(u_n) u_n + \int_{\mathbb{R}^3} u_n^6 \quad (28)$$

for large n . Combining (27) and (28), we have

$$(1 - \frac{1}{s_n^2}) \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V_0 u_n^2) + \frac{C}{2} < \int_{\mathbb{R}^3} (\frac{f(u_n)}{u_n^3} - \frac{f(s_n u_n)}{s_n^3 u_n^3}) u_n^4 + (1 - s_n^2) \int_{\mathbb{R}^3} u_n^6$$

for large n . It is easy to see that there is a subsequence $\{s_n\}$ satisfying $s_n \rightarrow d$ as $n \rightarrow \infty$, where $d \in (0, 1)$. Therefore, there exists $c_1 > 0$ such that

$$s_n^2 < 1 - c_1 \quad \text{for } n \text{ sufficiently large.}$$

Let $g(t) := \frac{1}{4} f(t) t - F(t)$. Then from (f₃), we deduce that $g(t)$ is increasing for $t > 0$. Moreover, by (20) and (27), there exists $c_2 > 0$ such that

$$\begin{aligned} I_{\epsilon_n}(u_n) &= \frac{1}{4} \|u_n\|_{\epsilon_n}^2 + \int_{\mathbb{R}^3} (\frac{1}{4} f(u_n) u_n - F(u_n)) + \frac{1}{12} \int_{\mathbb{R}^3} u_n^6 \\ &> \frac{s_n^2}{4} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V_0 u_n^2) + \int_{\mathbb{R}^3} (\frac{1}{4} f(s_n u_n) s_n u_n - F(s_n u_n)) + \frac{s_n^6}{12} \int_{\mathbb{R}^3} |u_n|^6 + \frac{c_1}{4} \|u_n\|_{\epsilon_n}^2 \\ &= I_{V_0,0}(s_n u_n) + \frac{c_1}{4} \|u_n\|_{\epsilon_n}^2 \\ &\geq c_{V_0,0} + \frac{c_1 c_2}{4} \end{aligned}$$

for n sufficiently large, which contradicts with (24). Therefore, (25) holds. It then follows from (21) and (24) that

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V_0 u_n^2) &= \int_{\mathbb{R}^3} f(u_n) u_n + \int_{\mathbb{R}^3} u_n^6 + o(1), \\ I_{\epsilon_n}(u_n) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u_n|^2 + V_0 u_n^2) - \int_{\mathbb{R}^3} F(u_n) - \frac{1}{6} \int_{\mathbb{R}^3} u_n^6 + o(1) = c_{V_0,0} + o(1). \end{aligned} \quad (29)$$

In view of (29), we can deduce that there exists $s_n \rightarrow 1$ such that $\bar{u}_n := s_n u_n \in \mathcal{N}_{V_0,0}$ satisfies

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla \bar{u}_n|^2 + V_0 \bar{u}_n^2) &= \int_{\mathbb{R}^3} f(\bar{u}_n) \bar{u}_n + \int_{\mathbb{R}^3} \bar{u}_n^6, \\ I_{V_0,0}(\bar{u}_n) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \bar{u}_n|^2 + V_0 \bar{u}_n^2) - \int_{\mathbb{R}^3} F(\bar{u}_n) - \frac{1}{6} \int_{\mathbb{R}^3} \bar{u}_n^6 = c_{V_0,0} + o(1). \end{aligned} \quad (30)$$

Here, we claim that there exists a sequence $\bar{u}_n$ satisfying the following limits:

$$I_{V_0,0}(\bar{u}_n) \rightarrow c_{V_0,0} \quad \text{and} \quad I'_{V_0,0}(\bar{u}_n) \rightarrow 0 \text{ in } H^{-1}. \quad (31)$$

It is clear that $\{\bar{u}_n\}$ is a minimizing sequence in $\mathcal{N}_{V_0,0}$. Due to the Ekeland's variational principle in Refs. 19–21, there exists a sequence $\{\tilde{u}_n\} \subset \mathcal{N}_{V_0,0}$ such that

$$\|\tilde{u}_n - \bar{u}_n\| = o(1), \quad I_{V_0,0}(\tilde{u}_n) = c_{V_0,0} + o(1), \quad I'_{V_0,0}(\tilde{u}_n) - \gamma_n J'_{V_0,0}(\tilde{u}_n) = o(1),$$

where γ_n is a real number and $J_{V_0,0}(u) = \langle J'_{V_0,0}(u), u \rangle$ for all $u \in H^1(\mathbb{R}^3)$. We show that there is $b > 0$ such that $|\langle J'_{V_0,0}(\tilde{u}_n), \tilde{u}_n \rangle| \geq b$ for all $n \in \mathbb{N}$. Indeed, by (f₃) and (f₄), and the definition of $J_{V_0,0}$, we have

$$\begin{aligned} -\langle J'_{V_0,0}(\tilde{u}_n), \tilde{u}_n \rangle &= -2 \int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + V_0 \tilde{u}_n^2) + \int_{\mathbb{R}^3} (f(\tilde{u}_n) \tilde{u}_n + f'(\tilde{u}_n) \tilde{u}_n^2 + 6 \tilde{u}_n^6) \\ &\geq -1 \int_{\mathbb{R}^3} f(\tilde{u}_n) \tilde{u}_n + \int_{\mathbb{R}^3} f'(\tilde{u}_n) \tilde{u}_n^2 + 4 \int_{\mathbb{R}^3} \tilde{u}_n^6 \\ &\geq 4 \int_{\mathbb{R}^3} \tilde{u}_n^6, \end{aligned} \quad (32)$$

which follows from $3f(t) - f'(t)t \leq 0$ for $t \in \mathbb{R}$. Assume by contradiction that $\langle J'_{V_0,0}(\tilde{u}_n), \tilde{u}_n \rangle \rightarrow 0$, then one has $\int_{\mathbb{R}^3} \tilde{u}_n^6 \rightarrow 0$ as $n \rightarrow \infty$. By (f₁) and (f₂), we have $0 \leq \int_{\mathbb{R}^3} f(\tilde{u}_n) \tilde{u}_n \leq \varepsilon \int_{\mathbb{R}^3} \tilde{u}_n^4 + C_\varepsilon \int_{\mathbb{R}^3} |\tilde{u}_n|^6 \rightarrow 0$ since ε can be arbitrarily small. Consequently, we can deduce that $\tilde{u}_n \rightarrow 0$ in $H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$. However, this is impossible because $\{\tilde{u}_n\} \subset \mathcal{N}_{V_0,0}$. So $|\langle J'_{V_0,0}(\tilde{u}_n), \tilde{u}_n \rangle| \geq b$ for all $n \in \mathbb{N}$. In view of $\langle I'_{V_0,0}(\tilde{u}_n), \tilde{u}_n \rangle = 0$, we have $\langle \gamma_n J'_{V_0,0}(\tilde{u}_n), \tilde{u}_n \rangle = o(1)$. Therefore, from (32), we see that $\gamma_n = o(1)$, which yields that (31) holds. $\{\tilde{u}_n\}$ is thus a $(PS)_{c_{V_0,0}}$ sequence in $H^1(\mathbb{R}^3)$ for $I_{V_0,0}$.

From $\{\tilde{u}_n\} \subset \mathcal{N}_{V_0,0}$ and (31), we deduce that $C_1 \geq \|\tilde{u}_n\| \geq C > 0$ for all n and some $C_1 > 0$. We show that there exist positive constants R, θ , and $\{y_n\} \subset \mathbb{R}^3$ such that

$$\int_{B_R(y_n)} \tilde{u}_n^2 \geq \theta \quad \text{for all } n. \quad (33)$$

Otherwise, for every $R > 0$, we have

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_R(y)} \tilde{u}_n^2 = 0.$$

Applying the concentration-compactness principle of Lions²⁹ to $\tilde{u}_n^2$, we obtain that $\tilde{u}_n \rightarrow 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6)$. Therefore, it follows from (31) and (f₁)-(f₂) that

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + V_0 \tilde{u}_n^2) &= \int_{\mathbb{R}^3} \tilde{u}_n^6 + o(1), \\ I_{V_0,0}(\tilde{u}_n) &= \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + V_0 \tilde{u}_n^2) - \frac{1}{6} \int_{\mathbb{R}^3} \tilde{u}_n^6 + o(1) = c_{V_0,0}. \end{aligned} \quad (34)$$

We may assume that $\int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + V_0 \tilde{u}_n^2) \rightarrow l > 0$, then $\int_{\mathbb{R}^3} \tilde{u}_n^6 \rightarrow l$. Note that the Sobolev embedding theorem implies that

$$S \leq \frac{\int_{\mathbb{R}^3} (|\nabla \tilde{u}_n|^2 + V_0 \tilde{u}_n^2)}{(\int_{\mathbb{R}^3} \tilde{u}_n^6)^{\frac{1}{3}}},$$

from which we conclude that $l \geq S^{\frac{3}{2}}$. In view of (34), we have $c_{V_0,0} = \frac{1}{3}l \geq \frac{1}{3}S^{\frac{3}{2}}$, a contradiction. Therefore, (33) holds. Take $v_n = \tilde{u}_n(x + y_n)$. From the translation invariance of the functional $I_{V_0,0}$,

we conclude that $\{v_n\}$ is also a $(PS)_{c_{V_0,0}}$ sequence in $H^1(\mathbb{R}^3)$ of $I_{V_0,0}$. It is clear that $\{v_n\}$ is also a bounded sequence in $H^1(\mathbb{R}^3)$. By (33), there exist a subsequence $\{v_n\}$ and a non-zero $v_0 \in H^1(\mathbb{R}^3)$ such that

$$\begin{aligned} v_n &\rightharpoonup v_0 \quad \text{in } H^1(\mathbb{R}^3), \\ v_n &\rightarrow v_0 \quad \text{a.e. in } \mathbb{R}^3. \end{aligned}$$

This implies that v_0 is a non-trivial solution of (12). Using the Fatou's lemma,

$$\begin{aligned} c_{V_0,0} &\leq I_{V_0,0}(v_0) = \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla v_0|^2 + V_0 v_0^2) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(v_0) v_0 - F(v_0) \right) + \frac{1}{12} \int_{\mathbb{R}^3} v_0^6 \\ &\leq \liminf_{n \rightarrow \infty} \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla v_n|^2 + V_0 v_n^2) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(v_n) v_n - F(v_n) \right) + \frac{1}{12} \int_{\mathbb{R}^3} v_n^6 \\ &= c_{V_0,0} \end{aligned}$$

and so $I_{V_0,0}(v_0) = c_{V_0,0}$. Moreover, strong maximum principle implies that v_0 is a positive solution of (12). Set $w_n = v_n - v_0$. By the Brezis-Lieb lemma⁹ and $I_{V_0,0}(v_0) = c_{V_0,0}$, we obtain

$$\begin{aligned} \int_{\mathbb{R}^3} |\nabla v_n|^2 &= \int_{\mathbb{R}^3} |\nabla w_n|^2 + \int_{\mathbb{R}^3} |\nabla v_0|^2 + o(1), \\ \int_{\mathbb{R}^3} |v_n|^p &= \int_{\mathbb{R}^3} |w_n|^p + \int_{\mathbb{R}^3} |v_0|^p + o(1), \quad p \in [2, 6], \\ \int_{\mathbb{R}^3} f(v_n) v_n &= \int_{\mathbb{R}^3} f(v_0) v_0 + \int_{\mathbb{R}^3} f(w_n) w_n + o(1), \\ \int_{\mathbb{R}^3} F(v_n) &= \int_{\mathbb{R}^3} F(v_0) + \int_{\mathbb{R}^3} F(w_n) + o(1), \end{aligned}$$

which imply that

$$\begin{aligned} I_{V_0,0}(w_n) &= I_{V_0,0}(v_n) - I_{V_0,0}(v_0) + o(1) = c_{V_0,0} - c_{V_0,0} + o(1) = o(1), \\ I'_{V_0,0}(w_n) w_n &= I'_{V_0,0}(v_n) v_n - I'_{V_0,0}(v_0) v_0 + o(1) = o(1). \end{aligned}$$

It follows from the earlier equalities that $\int_{\mathbb{R}^3} (|\nabla w_n|^2 + w_n^2) \rightarrow 0$ as $n \rightarrow \infty$. Therefore, $v_n \rightarrow v_0$ in $H^1(\mathbb{R}^3)$; that is, $\bar{u}_n(x + y_n) \rightarrow v_0$ in $H^1(\mathbb{R}^3)$ as $n \rightarrow \infty$. It follows from $\|\bar{u}_n - \bar{u}_n\| = o(1)$ that $\bar{u}_n(x + y_n) \rightarrow v_0$ in $H^1(\mathbb{R}^3)$. By $s_n u_n(x + y_n) = \bar{u}_n(x + y_n)$ and $s_n \rightarrow 1$ as $n \rightarrow \infty$, we see that $\|u_n(x + y_n) - v_0\| = o(1)$ in $H^1(\mathbb{R}^3)$. Combining $p \in [2, 6]$ and Sobolev embedding inequality, we have $u_n(x + y_n) \rightarrow v_0$ in $L^p(\mathbb{R}^3)$ as $n \rightarrow \infty$. Therefore, for $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\frac{\int_{|x| \geq R_\varepsilon} |u_n(x + y_n)|^p}{\int_{\mathbb{R}^3} |u_n(x + y_n)|^p} \leq \varepsilon. \quad (35)$$

Moreover, by condition (V), we can choose $\nu > 0$ such that

$$V(x) > V_0 \quad \text{for } x \in \bar{C}_{l+\nu}^j \setminus C_{l-\nu}^j. \quad (36)$$

Formulas (10) and (11) imply that

$$\begin{aligned} |h_{\epsilon_n}^i(u_n)| &= \frac{\int_{\mathbb{R}^3} |\Psi_{\epsilon_n}((x + y_n)_i)| |u_n(x + y_n)|^p}{\int_{\mathbb{R}^3} |u_n(x + y_n)|^p} \\ &= \frac{\int_{|x| \geq R_\varepsilon} |\Psi_{\epsilon_n}((x + y_n)_i)| |u_n(x + y_n)|^p + \int_{|x| \leq R_\varepsilon} |\Psi_{\epsilon_n}((x + y_n)_i)| |u_n(x + y_n)|^p}{\int_{\mathbb{R}^3} |u_n(x + y_n)|^p}, \end{aligned} \quad (37)$$

where $i = 1, 2, 3$. Here, our aim is to prove that, up to subsequence, $\epsilon_n y_n \rightarrow y_0 \in \bar{C}_{l+\nu}^j \setminus C_{l-\nu}^j$. Assume on the contrary that one of the following three cases occurs:

- (1) $\{\epsilon_n y_n\} \subset \bar{C}_{l-\nu}^j$;
- (2) $\{\epsilon_n y_n\} \subset \mathbb{R}^N \setminus C_{l+\nu}^j$, and $\{\epsilon_n y_n\}$ is bounded;
- (3) $\epsilon_n(y_n)_i \rightarrow \infty$ as $n \rightarrow \infty$ for some $i \in \{1, 2, 3\}$.

Now, we consider case (1). It follows from (37) that, in the region $|x| \leq R_\varepsilon$, we have

$$(x + y_n)_i \in \left(\frac{a_i^j - (l - \nu)}{\epsilon_n} - R_\varepsilon, \frac{a_i^j + (l - \nu)}{\epsilon_n} + R_\varepsilon \right) \subset \left(-\frac{2L}{\epsilon_n}, \frac{2L}{\epsilon_n} \right)$$

for sufficiently large n . Furthermore, we have

$$\begin{aligned} h_{\epsilon_n}^i(u_n) &> \left(\frac{a_i^j - (l - \nu)}{\epsilon_n} - R_\varepsilon \right) - \frac{2L}{\epsilon_n} \varepsilon, \\ h_{\epsilon_n}^i(u_n) &< \left(\frac{a_i^j + (l - \nu)}{\epsilon_n} + R_\varepsilon \right) + \frac{2L}{\epsilon_n} \varepsilon. \end{aligned}$$

From the above inequalities, we can choose $\varepsilon > 0$, $\nu > \varepsilon$ sufficiently small such that

$$h_{\epsilon_n}^i(u_n) \in \left(\frac{a_i^j - l}{\epsilon_n}, \frac{a_i^j + l}{\epsilon_n} \right)$$

for sufficiently large n and $i = 1, 2, 3$, which contradicts $h_{\epsilon_n}(u_n) \in \partial C_{l/\epsilon_n}^j$.

If case (2) occurs, then we can assume that $\epsilon_n y_n \rightarrow \hat{y} \in \bar{C}_{l+\nu}^j$ as $n \rightarrow \infty$, and that $\hat{y}_i = a_i^j + l + \nu$ for some index i and $(y_n)_i > \frac{a_i^j + l + \frac{\nu}{2}}{\epsilon_n}$ for large n . From (37), we then have

$$h_{\epsilon_n}^i(u_n) > \left(\frac{a_i^j + (l + \frac{\nu}{2})}{\epsilon_n} - R_\varepsilon \right) - \frac{2L}{\epsilon_n} \varepsilon > \frac{a_i^j + l}{\epsilon_n}$$

for small ε and $\nu > \varepsilon$. This is also a contradiction.

Case (3) is also impossible by assuming $\epsilon_n(y_n)_i > 2L$ for some index i and large n and using the similar argument as that in Case (2).

Hence, $\epsilon_n y_n \rightarrow y_0 \in \bar{C}_{l+\nu}^j \setminus C_{l-\nu}^j$ as $n \rightarrow \infty$. By condition (36), $V_0 < V(y_0)$. Subsequently, using (31), we conclude that

$$\int_{\mathbb{R}^3} (|\nabla v_0|^2 + V(y_0)v_0^2) = \int_{\mathbb{R}^3} f(v_0)v_0 + \int_{\mathbb{R}^3} v_0^6,$$

which contradicts the earlier conclusion that v_0 is a positive solution of (12). This completes the proof. $\square$

Adopting the idea of Ni and Takagi³⁰ and Cao and Noussair,¹⁰ we obtain the following result.

Lemma 7. For every $u \in N_\epsilon^j$, $j \in \{1, \dots, k\}$ and $\epsilon \in (0, \epsilon_0)$, there exist $\delta > 0$ and a differentiable function $t = t(w) > 0$, where $w \in E_\epsilon$ with $\|w\|_\epsilon < \delta$, such that $t(0) = 1$, the function $z = t(w)(u - w) \in N_\epsilon^j$ for all $w \in E_\epsilon$ with $\|w\|_\epsilon < \delta$ and

$$\langle t'(0), \varphi \rangle = \frac{2 \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(\epsilon x) u \varphi) + 4\epsilon^2 \int_{\mathbb{R}^3} \phi_u u \varphi - \int_{\mathbb{R}^3} (f'(u)u + f(u))\varphi - 5 \int_{\mathbb{R}^3} u^5 \varphi}{\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\epsilon x) u^2) + 3\epsilon^2 \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} f'(u)u^2 - 5 \int_{\mathbb{R}^3} u^6}$$

for all $\varphi \in E_\epsilon$.

Proof. For $u \in N_\epsilon^j$, define a function $H_u : \mathbb{R} \times E_\epsilon \rightarrow \mathbb{R}$ by

$$H_u(t, w) = t \|u - w\|_\epsilon^2 + t^3 \epsilon^2 \int_{\mathbb{R}^3} \phi_{u-w} (u - w)^2 - \int_{\mathbb{R}^3} f(t(u - w))(u - w) - t^5 \int_{\mathbb{R}^3} (u - w)^6.$$

Then, $H_u(1, 0) = \langle I'_{V_\epsilon, \epsilon^2}(u), u \rangle = 0$ and

$$\begin{aligned} \frac{d}{dt} H_u(1, 0) &= \|u\|_\epsilon^2 + 3\epsilon^2 \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} f'(u)u^2 - 5 \int_{\mathbb{R}^3} u^6 \\ &= -2\|u\|_\epsilon^2 + \int_{\mathbb{R}^3} (3f(u) - f'(u)u)u - 2 \int_{\mathbb{R}^3} u^6 \\ &< 0. \end{aligned}$$

According to the implicit function theorem, there exist $\delta > 0$ and a differentiable function $t(w)$ defined for $w \in E_\epsilon(\mathbb{R}^3)$ with $\|w\|_\epsilon < \delta$ such that $t(0) = 1$,

$$\langle t'(0), \varphi \rangle = \frac{2 \int_{\mathbb{R}^3} (\nabla u \nabla \varphi + V(\epsilon x) u \varphi) + 4\epsilon^2 \int_{\mathbb{R}^3} \phi_u u \varphi - \int_{\mathbb{R}^3} (f'(u) u + f(u)) \varphi - 6 \int_{\mathbb{R}^3} u^5 \varphi}{\int_{\mathbb{R}^3} (|\nabla u|^2 + V(\epsilon x) u^2) + 3\epsilon^2 \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} f'(u) u^2 - 5 \int_{\mathbb{R}^3} u^6}$$

and $H_u(t(w), w) = 0$ for all $w \in E_\epsilon$ with $\|w\|_\epsilon < \delta$, which is equivalent to

$$\langle I'_{V_\epsilon \epsilon^2}(t(w)(u - w)), t(w)(u - w) \rangle = 0$$

for all $w \in E_\epsilon$ with $\|w\|_\epsilon < \delta$. Furthermore, it follows from the continuity of the maps h_ϵ and t that

$$h_\epsilon(t(w)(u - w)) \in C_{\frac{1}{\epsilon}}^j$$

for δ sufficiently small. Therefore, $t(w)(u - w) \in N_\epsilon^j$ for all $w \in E_\epsilon$ with $\|w\|_\epsilon < \delta$. The proof is complete. $\square$

In view of Lemmas 3-6, we can let a positive number $\delta \leq \min\{\delta_1, c_{V_0,0}\}$ and take $\epsilon^* = \min\{\epsilon_\delta, \epsilon_{\delta_1}, \epsilon_0\}$ such that, for every $\epsilon \in (0, \epsilon^*)$ and $j = 1, 2, \dots, k$,

$$c_{V_\epsilon \epsilon^2} \leq \alpha_\epsilon^j < c_{V_0,0} + \delta < \min\{2c_{V_0,0}, \tilde{\alpha}_\epsilon^j\} \leq \min\{2c_{V_\epsilon \epsilon^2}, \tilde{\alpha}_\epsilon^j\}. \quad (38)$$

The proof of the following lemma is similar to Chen *et al.*,¹² we now provide it for the completeness.

Proposition 1. Under conditions (V) and (f₁)-(f₄), there exists a minimizing sequence $\{u_n^j\} \subset N_\epsilon^j$ satisfying

$$I_{V_\epsilon \epsilon^2}(u_n^j) \rightarrow \alpha_\epsilon^j, \quad I'_{V_\epsilon \epsilon^2}(u_n^j) \rightarrow 0 \text{ in } E_\epsilon^{-1}$$

as $n \rightarrow \infty$ for every $\epsilon \in (0, \epsilon^*)$ and $j = 1, 2, \dots, k$.

Proof. Let $\bar{N}_\epsilon^j$ denote the closure of N_ϵ^j , then we notice that $\bar{N}_\epsilon^j = N_\epsilon^j \cup \partial N_\epsilon^j$ for each $j = 1, 2, \dots, k$. Hence, using (38), we see that there exists $\epsilon^* > 0$ such that $\alpha_\epsilon^j = \inf\{I_{V_\epsilon \epsilon^2}(u) | u \in \bar{N}_\epsilon^j\}$ for all $\epsilon \in (0, \epsilon^*)$ and $j = 1, 2, \dots, k$. Now, we fix $j \in \{1, 2, \dots, k\}$. Applying the Ekeland variational principle,²⁰ there exists a minimizing sequence $\{u_n\} \subset \bar{N}_\epsilon^j$ such that $I_{V_\epsilon \epsilon^2}(u_n) < \alpha_\epsilon^j + \frac{1}{n}$ and

$$I_{V_\epsilon \epsilon^2}(u_n) \leq I_{V_\epsilon \epsilon^2}(w) + \frac{1}{n} \|w - u_n\|_\epsilon \quad (39)$$

for all $w \in \bar{N}_\epsilon^j$. It follows from (38) that $\{u_n\} \subset N_\epsilon^j$ for n sufficiently large. Applying Lemma 7 with $u = u_n$, we obtain the function $t_n^*(w)$ defined for $w \in E_\epsilon$ and $\|w\|_\epsilon < \delta_n$ for some $\delta_n > 0$ such that $t_n^*(w)(u_n - w) \in N_\epsilon^j$. Choose $0 < \varepsilon < \delta_n$. Let $u \in E_\epsilon, u \neq 0$ and $w_\varepsilon = \frac{\varepsilon u}{\|u\|_\epsilon}$. Fix n and set $z_\varepsilon = t_n^*(w_\varepsilon)(u_n - w_\varepsilon)$. Since $z_\varepsilon \in N_\epsilon^j$, using (39), we obtain

$$I_{V_\epsilon \epsilon^2}(z_\varepsilon) - I_{V_\epsilon \epsilon^2}(u_n) \geq \frac{-1}{n} \|z_\varepsilon - u_n\|_\epsilon$$

and by the mean value theorem, we then have

$$\langle I'_{V_\epsilon \epsilon^2}(u_n), z_\varepsilon - u_n \rangle + o(\|z_\varepsilon - u_n\|_\epsilon) \geq \frac{-1}{n} \|z_\varepsilon - u_n\|_\epsilon.$$

Therefore,

$$\begin{aligned} & \langle I'_{V_\epsilon \epsilon^2}(u_n), -w_\varepsilon \rangle + (t_n^*(w_\varepsilon) - 1) \langle I'_{V_\epsilon \epsilon^2}(u_n), (u_n - w_\varepsilon) \rangle \\ & \geq \frac{-1}{n} \|z_\varepsilon - u_n\|_\epsilon + o(\|z_\varepsilon - u_n\|_\epsilon). \end{aligned} \quad (40)$$

We derive from $z_\varepsilon \in N_\varepsilon^j$ and (40) that

$$\begin{aligned} & -\varepsilon \langle I'_{V_\varepsilon, \varepsilon^2}(u_n), \frac{u}{\|u\|_\varepsilon} \rangle + \frac{(t_n^*(w_\varepsilon) - 1)}{t_n^*(w_\varepsilon)} \langle I'_{V_\varepsilon, \varepsilon^2}(z_\varepsilon), t_n^*(w_\varepsilon)(u_n - w_\varepsilon) \rangle \\ & + (t_n^*(w_\varepsilon) - 1) \langle I'_{V_\varepsilon, \varepsilon^2}(u_n) - I'_{V_\varepsilon, \varepsilon^2}(z_\varepsilon), (u_n - w_\varepsilon) \rangle \\ & \geq \frac{-1}{n} \|z_\varepsilon - u_n\|_\varepsilon + o(\|z_\varepsilon - u_n\|_\varepsilon). \end{aligned}$$

Hence,

$$\begin{aligned} \langle I'_{V_\varepsilon, \varepsilon^2}(u_n), \frac{u}{\|u\|_\varepsilon} \rangle & \leq \frac{1}{n\varepsilon} \|z_\varepsilon - u_n\|_\varepsilon + \frac{o(\|z_\varepsilon - u_n\|_\varepsilon)}{\varepsilon} \\ & + \frac{(t_n^*(w_\varepsilon) - 1)}{\varepsilon} \langle I'_{V_\varepsilon, \varepsilon^2}(u_n) - I'_{V_\varepsilon, \varepsilon^2}(z_\varepsilon), (u_n - w_\varepsilon) \rangle. \end{aligned} \quad (41)$$

It follows from the definition of z_ε that there exists a positive constant C independent of ε such that

$$\|z_\varepsilon - u_n\|_\varepsilon \leq \varepsilon + C(|t_n^*(w_\varepsilon) - 1|)$$

and by Lemma 7, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{|t_n^*(w_\varepsilon) - 1|}{\varepsilon} \leq \|(t_n^*)'(0)\|_{E_\varepsilon^{-1}} \leq C.$$

For fixed $n \in \mathbb{N}$, let $\varepsilon \rightarrow 0$ in (41) and using the fact that $\lim_{\varepsilon \rightarrow 0} \|z_\varepsilon - u_n\|_\varepsilon = 0$, we obtain

$$\langle I'_{V_\varepsilon, \varepsilon^2}(u_n), \frac{u}{\|u\|_\varepsilon} \rangle \leq \frac{C}{n}.$$

The above conclusions imply that

$$I_{V_\varepsilon, \varepsilon^2}(u_n) = \alpha_\varepsilon^j + o(1) \quad \text{and} \quad I'_{V_\varepsilon, \varepsilon^2}(u_n) = o(1) \quad \text{in } E_\varepsilon^{-1}.$$

This completes the proof. $\square$

For fixed $j \in \{1, 2, \dots, k\}$, we give the following compactness lemma.

Lemma 8. Assume that conditions (V) and (f₁)-(f₄) hold and that $\{u_n^j\} \subset N_\varepsilon^j$ is a sequence satisfying

$$I_{V_\varepsilon, \varepsilon^2}(u_n^j) \rightarrow \alpha_\varepsilon^j, \quad I'_{V_\varepsilon, \varepsilon^2}(u_n^j) \rightarrow 0 \quad \text{as } n \rightarrow \infty \quad (42)$$

for any $\varepsilon \in (0, \varepsilon^*)$, where ε^* is independent of j . Then $\{u_n^j\}$ has a subsequence, still denoted by $\{u_n^j\}$, satisfying $u_n^j \rightarrow u_0$ in E_ε as $n \rightarrow \infty$.

Proof. For the sake of simplicity, we denote u_n^j by u_n . In view of Proposition 1 and Lemma 4, we know assumption (42) is reasonable for $\alpha_\varepsilon^j < \frac{S_3^2}{3}$ with $\varepsilon \in (0, \varepsilon^*)$. Moreover, if $V_\infty < \infty$, we have $\alpha_\varepsilon^j < c_{V_\infty, \varepsilon^2}$ for $\varepsilon \in (0, \varepsilon^*)$. It is easy to see that $\{u_n\}$ is bounded in E_ε , we assume that, up to subsequence, $u_n \rightarrow u_0$ in E_ε and $u_n \rightarrow u_0$ a.e. in $\mathbb{R}^3$ as $n \rightarrow \infty$, for some $u_0 \in E_\varepsilon$. Therefore, $I'_{V_\varepsilon, \varepsilon^2}(u_0) = 0$. Now, we show that $u_0 \neq 0$. We have an observation that if there exist constants $\eta, R > 0$ such that

$$\liminf_{n \rightarrow \infty} \int_{B_R(0)} u_n^2 \geq \eta, \quad (43)$$

then $u_0 \neq 0$. Suppose on the contrary that $u_0 \equiv 0$. Then, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that

$$\limsup_{n \rightarrow \infty} \int_{B_{R_1}(0)} u_n^2 = 0 \quad (44)$$

for every $R_1 > 0$. We first assume that $V_\infty < \infty$. Let μ be such that

$$\inf_{x \in \mathbb{R}^3} V(x) < \mu < \liminf_{|x| \rightarrow \infty} V(x) = V_\infty$$

and for any fixed $\epsilon \in (0, \epsilon^*)$, take $R > 0$ such that

$$V(\epsilon x) \geq \mu, \quad \forall x \in \mathbb{R}^3 \setminus B_R(0).$$

Based on $u_n \in N_{\epsilon}^j$, we have $I_{V_{\epsilon}, \epsilon^2}(u_n) = \max_{t \geq 0} I_{V_{\epsilon}, \epsilon^2}(tu_n)$. Consequently,

$$\begin{aligned} I_{V_{\epsilon}, \epsilon^2}(u_n) &\geq I_{V_{\epsilon}, \epsilon^2}(tu_n) \\ &= \frac{t^2}{2} \|u_n\|_{\epsilon}^2 + \frac{\epsilon^2 t^4}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 - \int_{\mathbb{R}^3} F(tu_n) - \frac{t^6}{6} \int_{\mathbb{R}^3} u_n^6 \\ &\quad + I_{\mu, \epsilon^2}(tu_n) - I_{\mu, \epsilon^2}(u_n) \\ &\geq \frac{t^2}{2} \int_{B_R(0)} (V(\epsilon x) - \mu) u_n^2 + I_{\mu, \epsilon^2}(tu_n), \end{aligned} \quad (45)$$

where the function I_{μ, ϵ^2} is defined by

$$I_{\mu, \epsilon^2}(u) = \frac{1}{2} \int_{\mathbb{R}^3} (|\nabla u|^2 + \mu u^2) + \frac{\epsilon^2}{4} \int_{\mathbb{R}^3} \phi_u u^2 - \int_{\mathbb{R}^3} F(u) - \frac{1}{6} \int_{\mathbb{R}^3} u^6.$$

Similarly, to $c_{V_{\epsilon}, \epsilon^2}$, we define least energy level c_{μ, ϵ^2} . Consider s_n such that $I_{\mu, \epsilon^2}(s_n u_n) = \max_{t \geq 0} I_{\mu, \epsilon^2}(tu_n)$. We show the sequence $\{s_n\}$ is bounded. If not, there exists a subsequence of $\{s_n\}$, still denoted by $\{s_n\}$, such that $s_n \rightarrow \infty$ as $n \rightarrow \infty$. It follows from Lemma 2 that

$$s_n^2 \int_{\mathbb{R}^3} (|\nabla u_n|^2 + \mu u_n^2) + \epsilon^2 s_n^4 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \geq s_n^6 \int_{\mathbb{R}^3} u_n^6. \quad (46)$$

We claim that there exist $\delta > 0$ such that $\|u_n\|_6^6 \geq \delta > 0$. Otherwise, there exists a subsequence of $\{u_n\}$, still denoted by $\{u_n\}$, such that $\|u_n\|_6^6 \rightarrow 0$ as $n \rightarrow \infty$. Therefore, we have for any $r > 0$,

$$\lim_{n \rightarrow \infty} \sup_{y \in \mathbb{R}^3} \int_{B_r(y)} u_n^6 = 0.$$

Lemma 5 implies $u_n \rightarrow 0$ in $L^s(\mathbb{R}^3)$ for $s \in (2, 6]$. From (f₁)-(f₂), we deduce that $\int_{\mathbb{R}^3} F(Ku_n) \rightarrow 0$ as $n \rightarrow \infty$ for each $K > 0$. So we infer from the definition of $I_{V_{\epsilon}, \epsilon^2}$ and Lemma 2 that

$$\begin{aligned} \alpha_{\epsilon}^j + o(1) &= I_{V_{\epsilon}, \epsilon^2}(u_n) \geq I_{V_{\epsilon}, \epsilon^2}(Ku_n) \\ &= \frac{K^2}{2} \|u_n\|_{\epsilon}^2 + \frac{K^4}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 - \int_{\mathbb{R}^3} F(Ku_n) - \frac{K^6}{6} \int_{\mathbb{R}^3} u_n^6 \\ &\geq \frac{K^2 C}{2} - \int_{\mathbb{R}^3} F(Ku_n) - \frac{K^6}{6} \int_{\mathbb{R}^3} u_n^6 \rightarrow \frac{K^2 C}{2} \end{aligned}$$

as $n \rightarrow \infty$. Now, we arrive a contradiction if K is large enough. Therefore, our claim is true. Combining (46) with the boundedness of $\{u_n\}$ in E_{ϵ} , we obtain a contradiction. Consequently, the sequence $\{s_n\}$ is bounded. Letting $t = s_n$ in (45), we have

$$I_{V_{\epsilon}, \epsilon^2}(u_n) \geq \frac{s_n^2}{2} \int_{B_R(0)} (V(\epsilon x) - \mu) u_n^2 + c_{\mu, \epsilon^2} + o(1).$$

Taking the limit $n \rightarrow +\infty$, from (44) and Sobolev embedding inequality, we have $\alpha_{\epsilon}^j \geq c_{\mu, \epsilon^2}$. Next, letting $\mu \rightarrow V_{\infty}$, we obtain $\alpha_{\epsilon}^j \geq c_{V_{\infty}, \epsilon^2}$, which is a contradiction. If $V_{\infty} = \infty$ holds, then V is coercive and the continuous embedding $E_{\epsilon} \hookrightarrow L^s(\mathbb{R}^3)$ is compact for $2 \leq s < 6$. Hence, up to subsequence, $u_n \rightarrow 0$ in $L^s(\mathbb{R}^3)$ as $n \rightarrow \infty$. By Lemma 1, we conclude that

$$\int_{\mathbb{R}^3} \phi_{u_n} u_n^2 \leq C \|u_n\|_{\frac{12}{5}}^4 \rightarrow 0$$

as $n \rightarrow \infty$. Therefore, by (f₁)-(f₂) and the earlier inequality, we have

$$\|u_n\|_{\epsilon}^2 = \int_{\mathbb{R}^3} u_n^6 + o(1).$$

Since $\{u_n\} \subset N_\epsilon$ is bounded in E_ϵ , up to subsequence, we can assume $\|u_n\|_\epsilon^2 \rightarrow l > 0$ and $\|u_n\|_6^6 \rightarrow l$. In view of $I_{V_\epsilon, \epsilon^2}(u_n) = \alpha_\epsilon^j + o(1)$, we have $\frac{l}{3} = \alpha_\epsilon^j$. Notice that $l \geq S^{\frac{3}{2}}$. Hence, $S^{\frac{3}{2}} \leq l = 3\alpha_\epsilon^j < S^{\frac{3}{2}}$ which is a contradiction. Therefore, $u_0 \neq 0$. It remains to prove that, up to subsequence, $u_n \rightarrow u_0$ in E_ϵ as $n \rightarrow \infty$. Set $v_n = u_n - u_0$. Assume on the contrary that $\|v_n\|_\epsilon \geq \theta$ for large n and some constant $\theta > 0$. Moreover, $I'_{V_\epsilon, \epsilon^2}(u_0) = 0$ and the (PS) sequence $\{u_n\}$ imply that

$$\|u_0\|_\epsilon^2 + \epsilon^2 \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 = \int_{\mathbb{R}^3} f(u_0) u_0 + \int_{\mathbb{R}^3} u_0^6 \quad (47)$$

and

$$\|u_n\|_\epsilon^2 + \epsilon^2 \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 = \int_{\mathbb{R}^3} f(u_n) u_n + \int_{\mathbb{R}^3} u_n^6 + o(1). \quad (48)$$

By (47) and (48), the Brezis-Lieb lemma⁹ and Lemma 2.2 in Zhao,³⁹ we obtain

$$\|v_n\|_\epsilon^2 + \epsilon^2 \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 = \int_{\mathbb{R}^3} f(v_n) v_n + \int_{\mathbb{R}^3} v_n^6 + o(1).$$

Since $\|v_n\|_\epsilon \geq \theta$ for large n , it is straightforward to find a sequence $\{\tau_n\} \subset \mathbb{R}^+$ with $\tau_n \rightarrow 1$ as $n \rightarrow \infty$ such that

$$\|\tau_n v_n\|_\epsilon^2 + \epsilon^2 \int_{\mathbb{R}^3} \phi_{\tau_n v_n} (\tau_n v_n)^2 = \int_{\mathbb{R}^3} f(\tau_n v_n) \tau_n v_n + \int_{\mathbb{R}^3} (\tau_n v_n)^6,$$

which implies that $\tau_n v_n \in N_{V_\epsilon, \epsilon^2}$. And then

$$I_{V_\epsilon, \epsilon^2}(v_n) = I_{V_\epsilon, \epsilon^2}(\tau_n v_n) + o(1) \geq c_{V_\epsilon, \epsilon^2} + o(1). \quad (49)$$

Similarly, it follows from (47) that

$$I_{V_\epsilon, \epsilon^2}(u_0) \geq c_{V_\epsilon, \epsilon^2}. \quad (50)$$

Thus, by (49) and (50), the Brezis-Lieb lemma⁹ and Lemma 2.2 in Zhao,³⁹

$$\begin{aligned} I_{V_\epsilon, \epsilon^2}(u_n) &= \frac{1}{2} \|u_n\|_\epsilon^2 + \frac{\epsilon^2}{4} \int_{\mathbb{R}^3} \phi_{u_n} u_n^2 - \int_{\mathbb{R}^3} F(u_n) + \frac{1}{6} \int_{\mathbb{R}^3} u_n^6 \\ &= \frac{1}{2} \|u_0\|_\epsilon^2 + \frac{\epsilon^2}{4} \int_{\mathbb{R}^3} \phi_{u_0} u_0^2 - \int_{\mathbb{R}^3} F(u_0) + \frac{1}{6} \int_{\mathbb{R}^3} u_0^6 \\ &\quad + \frac{1}{2} \|v_n\|_\epsilon^2 + \frac{\epsilon^2}{4} \int_{\mathbb{R}^3} \phi_{v_n} v_n^2 - \int_{\mathbb{R}^3} F(v_n) + \frac{1}{6} \int_{\mathbb{R}^3} v_n^6 + o(1) \\ &\geq 2c_{V_\epsilon, \epsilon^2} + o(1), \end{aligned}$$

which implies that

$$\lim_{n \rightarrow \infty} I_{V_\epsilon, \epsilon^2}(u_n) = \alpha_\epsilon^j \geq 2c_{V_\epsilon, \epsilon^2},$$

which contradicts (38). Therefore, up to subsequence, $u_n \rightarrow u_0$ in E_ϵ as $n \rightarrow \infty$. $\square$

Proposition 2. Assume conditions (V) and (f₁)-(f₄) hold. Then, there exists $\epsilon^* > 0$ such that (3) has at least k positive solutions for each $\epsilon \in (0, \epsilon^*)$.

Proof. It follows from Proposition 1 that there exists $\epsilon^* > 0$ such that for $\epsilon \in (0, \epsilon^*)$ and each fixed $j \in \{1, \dots, k\}$, we can find minimizing sequence $\{u_n^j\}$ of $I_{V_\epsilon, \epsilon^2}$ satisfying $I_{V_\epsilon, \epsilon^2}(u_n^j) \rightarrow \alpha_\epsilon^j$ and $I'_{V_\epsilon, \epsilon^2}(u_n^j) \rightarrow 0$ as $n \rightarrow \infty$. By using Lemma 8, we have $u_n^j \rightarrow u_0^j$ in E_ϵ as $n \rightarrow \infty$. Thus, we have $I'_{V_\epsilon, \epsilon^2}(u_0^j) = 0$. Now, we show that $u_0^j \in N_\epsilon^j$. Since $\{u_n^j\} \subset N_\epsilon^j$, we have $u_0^j \in N_\epsilon^j \cup \partial N_\epsilon^j$. Moreover, $I_{V_\epsilon, \epsilon^2}(u_0^j) = \alpha_\epsilon^j < \bar{\alpha}_\epsilon^j$ and so $u_0^j \notin \partial N_\epsilon^j$. Thus, $u_0^j \in N_\epsilon^j$. It is clear that u_0^j is non-negative, and by the maximum principle, u_0^j is therefore positive for $j = 1, \dots, k$. By a standard regularity argument, we can show that $u_0^j \in C^2(\mathbb{R}^3)$, $j = 1, \dots, k$. Moreover, u_0^j ($j = 1, \dots, k$) are different since $C_{1/\epsilon}^j$ ($j = 1, \dots, k$) are disjoint. Hence, it is easy to see that there exists $(u_\epsilon^j, \phi_{u_\epsilon^j}) \in E_\epsilon \times D^{1,2}(\mathbb{R}^3)$ is positive solution of problem (3). We thus have proved Proposition 2. $\square$

IV. CONCENTRATION

In order to study the concentration behavior of positive solutions u_ϵ^j of problem (3) given by Proposition 2, where $j \in \{1, \dots, k\}$. We will consider the family $v_\epsilon^j(x) = u_\epsilon^j(\epsilon x)$ which is a family of positive solutions of problem (7). Adapting some arguments used in Ref. 36, we turn our attention to the behaviour of the maximum points of v_ϵ^j .

Lemma 9. Assume that conditions (V) and (f₁)-(f₄) hold. Then for every $j \in \{1, \dots, k\}$, there exist $\epsilon^ > 0$, a family $\{x_\epsilon^j\} \subset \mathbb{R}^3$ and constants $R, \theta > 0$ such that*

$$\int_{B_R(x_\epsilon^j)} (v_\epsilon^j)^2 \geq \theta \quad (51)$$

for all $\epsilon \in (0, \epsilon^*)$ and $j \in \{1, \dots, k\}$. Furthermore, the family $\{\epsilon x_\epsilon^j\}$ is bounded and $\epsilon x_\epsilon^j \rightarrow a^j$ as $\epsilon \rightarrow 0^+$, $j \in \{1, \dots, k\}$.

Proof. Fix $j \in \{1, \dots, k\}$. Moreover, for simplicity, here we denote v_ϵ^j and x_ϵ^j by v_ϵ and x_ϵ , respectively. Suppose by contradiction that (51) does not hold, then there is a sequence ϵ_n converging to zero, such that for all $R > 0$

$$\lim_{n \rightarrow \infty} \sup_{x \in \mathbb{R}^3} \int_{B_R(x)} v_{\epsilon_n}^2 = 0.$$

Using a similar argument as we have done in the proof of Lemma 6 and noticing the fact that $\alpha_{\epsilon_n}^j \rightarrow c_{V_0,0}$ as $n \rightarrow \infty$, which follows from Lemma 3, we have $c_{V_0,0} \geq \frac{1}{3}S^{\frac{3}{2}}$. This means a contradiction. Therefore, (51) holds. We next show that ϵx_ϵ is bounded. Otherwise, there exists a sequence $\epsilon_n \rightarrow 0^+$ such that $\epsilon_n x_{\epsilon_n} \rightarrow \infty$. We denote $x_n = x_{\epsilon_n}$ and $v_n = v_{\epsilon_n}$. Since

$$\frac{1}{4} \|v_n\|_{\epsilon_n}^2 + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(v_n) v_n - F(v_n) \right) + \frac{1}{12} \int_{\mathbb{R}^3} v_n^6 = I_{V_{\epsilon_n}, \epsilon_n^2}(v_n) = \alpha_{\epsilon_n}^j \rightarrow c_{V_0,0}$$

as $n \rightarrow \infty$, which follows from Lemma 3, $\|v_n\|_{\epsilon_n}$ is bounded for all n . Therefore, v_n is bounded in $H^1(\mathbb{R}^3)$; that is, $\|v_n\| \leq C$ uniformly for n . Now, let $w_n(x) = v_n(x + x_n)$. Then, $\{w_n\}$ is also bounded in $H^1(\mathbb{R}^3)$. Furthermore,

$$-\Delta w_n + V(\epsilon_n x + \epsilon_n x_n) w_n + \epsilon_n^2 \phi_{w_n} w_n = f(w_n) + w_n^5, \quad w_n > 0 \text{ in } \mathbb{R}^3. \quad (52)$$

On the other hand, by (51), we have

$$\int_{B_R(0)} w_n^2 \geq \theta$$

for all $n \in \mathbb{N}$. Considering first the case where $V_\infty = \infty$, we have

$$\int_{\mathbb{R}^3} V(\epsilon_n x + \epsilon_n x_n) w_n^2 \leq \int_{\mathbb{R}^3} f(w_n) w_n + \int_{\mathbb{R}^3} w_n^6.$$

Applying the Fatou's lemma to earlier inequality, we have $\infty = \liminf_{n \rightarrow \infty} \int_{\mathbb{R}^3} (f(w_n) w_n + w_n^6)$, which contradicts with the boundedness of $\{w_n\}$ in $H^1(\mathbb{R}^3)$. Now, let us consider the case where $V_\infty < \infty$. Note that by passing a subsequence if necessary, $w_n \rightharpoonup w_0 \geq 0$ in $H^1(\mathbb{R}^3)$ and $w_n \rightarrow w_0$ in $L_{loc}^q(\mathbb{R}^3)$ and $w_n \rightarrow w_0$ a.e. in $\mathbb{R}^3$. It is clear that $w_0 \not\equiv 0$. Since $\liminf_{|x| \rightarrow \infty} V(x) > V_0$, we can easily see from (52) in weak sense that

$$-\Delta w_n + V_0 w_n + \epsilon_n^2 \phi_{w_n} w_n < f(w_n) + w_n^5.$$

From $w_n \rightharpoonup w_0 \geq 0$ in $H^1(\mathbb{R}^3)$ and $w_0 \not\equiv 0$, we deduce that

$$\int_{\mathbb{R}^3} (|\nabla w_0|^2 + V_0 w_0^2) < \int_{\mathbb{R}^3} (f(w_0) w_0 + w_0^6).$$

Thus, the Fatou's lemma implies that there exists $0 < \beta < 1$ such that

$$\begin{aligned}
 c_{V_0,0} &= \inf_{u \in \mathcal{N}_{V_0,0}} I_{V_0,0}(u) \leq \max_{t \in \mathbb{R}^+} I_{V_0,0}(tw_0) = I_{V_0,0}(\beta w_0) \\
 &= \frac{\beta^2}{4} \int_{\mathbb{R}^3} (|\nabla w_0|^2 + V_0 w_0^2) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(\beta w_0) \beta w_0 - F(\beta w_0) \right) + \frac{\beta^6}{12} \int_{\mathbb{R}^3} w_0^6 \\
 &< \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla w_0|^2 + V_0 w_0^2) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(w_0) w_0 - F(w_0) \right) + \frac{1}{12} \int_{\mathbb{R}^3} w_0^6 \\
 &\leq \liminf_{n \rightarrow \infty} \left\{ \frac{1}{4} \int_{\mathbb{R}^3} (|\nabla w_n|^2 + V_0 w_n^2) + \int_{\mathbb{R}^3} \left(\frac{1}{4} f(w_n) w_n - F(w_n) \right) + \frac{1}{12} \int_{\mathbb{R}^3} w_n^6 \right\} \\
 &\leq \liminf_{n \rightarrow \infty} \left\{ I_{V_{\epsilon_n}, \epsilon_n^2}(v_n) - \frac{1}{4} \langle I'_{V_{\epsilon_n}, \epsilon_n^2}(v_n), v_n \rangle \right\} \\
 &= \lim_{n \rightarrow \infty} c_{V_{\epsilon_n}, \epsilon_n^2} = c_{V_0,0}.
 \end{aligned}$$

This is impossible and hence ϵ_n is bounded. Now for any sequence $\{\epsilon_n\}$ with $\epsilon_n \rightarrow 0$, we assume that $\epsilon_n x_n \rightarrow x_0$. We claim that x_0 is a global minimum point of V . Since we have known $\lim_{n \rightarrow \infty} I_{V_{\epsilon_n}, \epsilon_n^2}(v_n) = c_{V_0,0}$, similarly to the arguments in Lemma 6, we have $\int_{\mathbb{R}^3} (V(\epsilon_n x) - V_0) v_n^2 \rightarrow 0$ as $n \rightarrow \infty$. Moreover, $w_n \rightarrow w_0$ in $H^1(\mathbb{R}^3)$ and w_0 is a positive ground state solution of (12). Therefore, we have

$$\begin{aligned}
 \int_{\mathbb{R}^3} V_0 |w_0|^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V_0 |w_n|^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V_0 v_n^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_n x) v_n^2 \\
 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_n x + \epsilon_n x_n) |v_n(x + x_n)|^2 \geq \int_{\mathbb{R}^3} V(x_0) |w_0|^2,
 \end{aligned}$$

which implies $V(x_0) = V_0$. Recall that $w_n(x) = v_n(x + x_n)$ and $v_n \in N_{\epsilon_n}^j$. Moreover, $h_{\epsilon_n}(v_n) \in C_{l/\epsilon_n}^j$. Since $p \in [2, 6]$, we have $w_n \rightarrow w_0$ in $L^p(\mathbb{R}^3)$. Similarly to (35), for $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\frac{\int_{|x| \geq R_\varepsilon} |w_n|^p}{\int_{\mathbb{R}^3} |w_n|^p} \leq \varepsilon, \quad (53)$$

and (10) and (11) imply that

$$\begin{aligned}
 |h_{\epsilon_n}^i(v_n)| &= \frac{\int_{\mathbb{R}^3} |\Psi_{\epsilon_n}((x + x_n)_i)| |w_n|^p}{\int_{\mathbb{R}^3} |w_n|^p} \\
 &= \frac{\int_{|x| \geq R_\varepsilon} |\Psi_{\epsilon_n}((x + x_n)_i)| |w_n|^p + \int_{|x| \leq R_\varepsilon} |\Psi_{\epsilon_n}((x + x_n)_i)| |w_n|^p}{\int_{\mathbb{R}^3} |w_n|^p}.
 \end{aligned} \quad (54)$$

We claim that $x_0 \in C_{l+\nu}^j$ where ν has been given in (36). Assume on the contrary that $\{\epsilon_n x_n\} \subset \mathbb{R}^N \setminus C_{l+\nu}^j$. The similar argument as that in the proof of Lemma 6 yields a contradiction with the fact that $\epsilon_n h_{\epsilon_n}(v_n) \in C_l^j$. Therefore, our claim is true. It follows from (V), (36), $V(x_0) = V_0$ and $x_0 \in C_{l+\nu}^j$ that $x_0 = a^j$. The proof is complete. $\square$

Lemma 10. Assume that conditions (V) and (f₁)-(f₄) hold, then for any $j \in \{1, \dots, k\}$, there exists $\epsilon^* > 0$ such that $\lim_{|x| \rightarrow \infty} w_\epsilon(x) \rightarrow 0$ uniformly for $\epsilon \in (0, \epsilon^*)$, where w_ϵ is given in Lemma 9. Moreover, there exist constants $C, c > 0$ such that

$$w_\epsilon(x) \leq C e^{-c|x|} \quad \text{for all } x \in \mathbb{R}^3.$$

The proof is similar to Lemma 3.8 and 3.11 in Ref. 24, we omit it.

The proof of Theorem 1. Fix $j \in \{1, \dots, k\}$. Let $v_{\epsilon_n}^j$ be a solution of (7), still denoted by v_{ϵ_n} , then $w_n = v_{\epsilon_n}(x + x_n)$ is a solution of the problem

$$-\Delta w_n + V(\epsilon_n x + \epsilon_n x_n) w_n + \epsilon_n^2 \phi_{w_n} w_n = f(w_n) + w_n^5, \quad w_n > 0 \text{ in } \mathbb{R}^3,$$

where x_n is given in Lemma 9. Moreover, we know that $w_n \rightarrow w_0$ in $H^1(\mathbb{R}^3)$ and $\epsilon_n x_n \rightarrow x_0 = a^j$. We claim that there exists $\tau > 0$ such that $\|w_n\|_\infty \geq \tau$ for all n . Indeed, assume that, up to a subsequence, $\|w_n\|_\infty \rightarrow 0$. We deduce from (f₁) and (f₂) that for any $\varepsilon > 0$, there exists a $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$,

$$\frac{f(\|w_n\|_\infty)}{\|w_n\|_\infty^3} < \varepsilon \quad \text{and} \quad \|w_n\|_\infty \leq \varepsilon.$$

Hence, by (f₃), we have

$$\begin{aligned} \int_{\mathbb{R}^3} (|\nabla w_n|^2 + V_0 w_n^2) &< \int_{\mathbb{R}^3} \frac{f(\|w_n\|_\infty)}{\|w_n\|_\infty^3} w_n^4 + \int_{\mathbb{R}^3} w_n^6 \\ &\leq \varepsilon \int_{\mathbb{R}^3} w_n^4 + \|v_n\|_\infty \int_{\mathbb{R}^3} w_n^5 \leq C\varepsilon. \end{aligned}$$

This implies that $\|w_n\|^2 < C\varepsilon$ for $n \geq n_0$, which is impossible because $w_n \rightarrow w_0$ in $H^1(\mathbb{R}^3)$ and $w_0 \not\equiv 0$ by Lemma 9. Then, the claim is proved. Let h_n be the global maximum of w_n , then from Lemma 10, we infer that there exists some $R > 0$ such that $|h_n| \leq R$ uniformly for n . Now, we see that the global maximum of v_{ϵ_n} is $z_n = h_n + x_n$. Hence, it follows from the boundedness of $\{h_n\}$ that $\epsilon_n z_n \rightarrow x_0 = a^j$. Moreover, since the function $u_\epsilon(x) = v_\epsilon(\frac{x}{\epsilon})$ is a positive solution of (3), then the maximum point ϱ_ϵ^j of u_ϵ satisfies $\varrho_\epsilon^j = \epsilon z_\epsilon$. Therefore, we have $\lim_{\epsilon \rightarrow 0^+} V(\varrho_\epsilon^j) = V(a^j)$. It follows from Lemma 10 that

$$u_\epsilon(x) = v_\epsilon\left(\frac{x}{\epsilon}\right) = w_\epsilon\left(\frac{x - \epsilon z_\epsilon}{\epsilon}\right) = w_\epsilon\left(\frac{x - \epsilon z_\epsilon + \epsilon h_\epsilon}{\epsilon}\right) \leq C e^{-\frac{c|x - \varrho_\epsilon^j|}{\epsilon}}.$$

The proof is complete. $\square$

V. GROUND STATE SOLUTIONS

In this section, our aim is to prove that there exists a positive ground state among the solutions shown by Theorem 1. By Lemma 3, we always find a positive function $g_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0^+$ such that the set

$$\Theta_\epsilon(g_\epsilon) = \{u \in N_{V_\epsilon, \epsilon^2} | I_{V_\epsilon, \epsilon^2}(u) < c_{V_0, 0} + g_\epsilon\}$$

is non-empty for ϵ sufficiently small. Indeed, we can use Lemma 3 to deduce that $g_\epsilon = |I_{V_\epsilon, \epsilon^2}(v_\epsilon^j) - c_{V_0, 0}| \rightarrow 0$ as $\epsilon \rightarrow 0^+$, where v_ϵ^j is a positive solution of (7) given in Proposition 2, $j = 1, \dots, k$.

Lemma 11. Suppose that the conditions of Theorem 2 hold, then there exists $\bar{\epsilon}^ > 0$ such that*

$$\sup_{u \in \Theta_\epsilon(g_\epsilon)} \inf_{x \in \cup_{j=1}^k C_{l/2\epsilon}^j} |h_\epsilon(u) - x| < \frac{l}{2\epsilon} \quad (55)$$

for all $\epsilon \in (0, \bar{\epsilon}^*)$.

Proof. Assume on the contrary that there exists a sequence $\{\epsilon_n\}$ satisfying $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\sup_{u \in \Theta_{\epsilon_n}(g_{\epsilon_n})} \inf_{x \in \cup_{j=1}^k C_{l/2\epsilon_n}^j} |h_{\epsilon_n}(u) - x| \geq \frac{l}{2\epsilon_n}.$$

For each n , there exists $u_n \in \Theta_{\epsilon_n}(g_{\epsilon_n})$ such that

$$\inf_{x \in \cup_{j=1}^k C_{l/2\epsilon_n}^j} |h_{\epsilon_n}(u_n) - x| = \sup_{u \in \Theta_{\epsilon_n}(g_{\epsilon_n})} \inf_{x \in \cup_{j=1}^k C_{l/2\epsilon_n}^j} |h_{\epsilon_n}(u) - x| + o(1).$$

It is easy to see that $I_{V_{\epsilon_n}, \epsilon_n^2}(u_n) \rightarrow c_{V_0, 0}$ as $n \rightarrow \infty$. Using similar arguments as that in the proof of Lemma 6, we have $\int_{\mathbb{R}^3} (V(\epsilon_n x) - V_0) u_n^2 \rightarrow 0$ as $n \rightarrow \infty$. Moreover, there exists $\{x_n\} \subset \mathbb{R}^3$ such that $u_n(\cdot + x_n) \rightarrow u_0$ in $H^1(\mathbb{R}^3)$, where u_0 is a positive ground state solution of (12). We show that $\{\epsilon_n x_n\}$

is bounded in $\mathbb{R}^3$. Otherwise, we may assume that $|\epsilon_n x_n| \rightarrow \infty$ as $n \rightarrow \infty$. Then, by the Fatou's lemma, one has

$$\begin{aligned} \int_{\mathbb{R}^3} V_0 |u_0|^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V_0 |u_n(x + x_n)|^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V_0 u_n^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_n x) u_n^2 \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_n x + \epsilon_n x_n) |u_n(x + x_n)|^2 \geq \int_{\mathbb{R}^3} V_\infty |u_0|^2. \end{aligned}$$

This is impossible since $V_\infty > V_0$. Consequently, $\{\epsilon_n x_n\}$ is bounded and, up to subsequence, it converges to some x_0 . By the following fact that

$$\begin{aligned} \int_{\mathbb{R}^3} V_0 |u_0|^2 &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V_0 |u_n(x + x_n)|^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V_0 u_n^2 = \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_n x) u_n^2 \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^3} V(\epsilon_n x + \epsilon_n x_n) |u_n(x + x_n)|^2 \geq \int_{\mathbb{R}^3} V(x_0) |u_0|^2, \end{aligned}$$

we have $V(x_0) = V_0$, i.e.,

$$x_0 \in \{a^1, \dots, a^k\} \subset \cup_{j=1}^k C_{l/2}^j. \quad (56)$$

Therefore, for large n , we have $x_n \in \cup_{j=1}^k C_{l/2\epsilon_n}^j$. Moreover, let $w_n = u_n(\cdot + x_n)$, then $w_n \rightarrow u_0$ in $H^1(\mathbb{R}^3)$. Therefore, for any $\varepsilon > 0$, there exists $R_\varepsilon > 0$ such that

$$\frac{\int_{|x| \geq R_\varepsilon} |w_n|^p}{\int_{\mathbb{R}^3} |w_n|^p} \leq \varepsilon, \quad (57)$$

and (56), (10), and (11) imply that

$$\begin{aligned} |h_{\epsilon_n}^i(u_n) - (x_n)_i| &\leq \frac{\int_{|x_i| \geq R_\varepsilon} (|\Psi_{\epsilon_n}((x + x_n)_i)| + |(x_n)_i|) |w_n|^p + \int_{|x_i| \leq R_\varepsilon} |\Psi_{\epsilon_n}((x + x_n)_i) - (x_n)_i| |w_n|^p}{\int_{\mathbb{R}^3} |w_n|^p} \\ &< \varepsilon \frac{3L}{\epsilon_n} + \frac{\int_{|x_i| \leq R_\varepsilon} |\Psi_{\epsilon_n}(x_i)| |w_n|^p}{\int_{\mathbb{R}^3} |w_n|^p} \\ &< \varepsilon \frac{3L}{\epsilon_n} + R_\varepsilon, \quad i \in \{1, 2, 3\} \end{aligned} \quad (58)$$

for large n . Therefore, let $\varepsilon = \frac{l}{24L}$, then we have $\frac{l}{2\epsilon_n} \leq |h_{\epsilon_n}(u_n) - x_n| < \sqrt{3}(\frac{l}{8\epsilon_n} + R_{l/24L})$, which is impossible for large n . The proof is complete. $\square$

The proof of Theorem 2. In view of Lemma 11, we know that there exists a positive number $\bar{\epsilon}^*$ such that, for each $\epsilon \in (0, \bar{\epsilon}^*)$,

$$\sup_{u \in \Theta_\epsilon(g_\epsilon)} \inf_{x \in \cup_{j=1}^k C_{l/2\epsilon}^j} |h_\epsilon(u) - x| < \frac{l}{2\epsilon};$$

that is,

$$\text{dist} \left(h_\epsilon(u), \cup_{j=1}^k C_{l/2\epsilon}^j \right) < \frac{l}{2\epsilon} \quad \text{for all } u \in \Theta_\epsilon(g_\epsilon).$$

This means that

$$h_\epsilon(u) \in \cup_{j=1}^k C_{l/\epsilon}^j$$

for all $u \in \Theta_\epsilon(g_\epsilon)$. Hence, $\Theta_\epsilon(g_\epsilon) \subset \cup_{j=1}^k N_\epsilon^j$ and we can find at least one ground state solution $u_0 \in N_\epsilon^j$ of (7) for some $j \in \{1, \dots, k\}$. Therefore, problem (3) has at least a ground state solution. $\square$

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