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# Loss of derivatives in the infinite type

Tran Vu Khanh

*University of Wollongong, tkhanh@uow.edu.au*

Stefano Pinton

*Universta degli Studi di Padova*

Giuseppe Zampieri

*Universta degli Studi di Padova*

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# Loss of derivatives in the infinite type

## Abstract

We prove hypoellipticity with loss of  $\epsilon$  derivatives for a system of complex vector fields whose Lie-span has a superlogarithmic estimate. In  $\mathbb{C}^n$ , the model is  $(L, f^k L)$  where  $f^k = z^k \bar{h}^k = z^k \bar{h}^k$  for  $h \neq 0$  and  $L$  is the vector field tangential to the exponentially non-degenerate hypersurface of infinite type defined by  $|z|^2 = e^{-1} |z|^\alpha$  for  $\alpha < 1$ .

## Keywords

type, infinite, loss, derivatives

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# LOSS OF DERIVATIVES IN THE INFINITE TYPE

TRAN VU KHANH, STEFANO PINTON AND GIUSEPPE ZAMPIERI

**ABSTRACT.** **Abstract:** We prove hypoellipticity with loss of  $\epsilon$  derivatives for a system of complex vector fields whose Lie-span has a superlogarithmic estimate. In  $\mathbb{C} \times \mathbb{R}$ , the model is  $(\bar{L}, \bar{f}^k L)$  where  $\bar{f} = \bar{z}h$  for  $h \neq 0$  and  $L$  is the vector field tangential to the exponentially non-degenerate hypersurface of infinite type defined by  $x_2 = e^{-\frac{1}{|z|^\alpha}}$  for  $\alpha < 1$ .

**Keywords:** hypoellipticity, loss of derivatives, superlogarithmic estimate, infinite type.

## 1. INTRODUCTION

A system of vector fields  $(L_j)_j$  has a subelliptic estimate when it has a gain of  $\delta > 0$  derivatives in the sense that  $\|\Lambda^\delta u\|^2 \lesssim \sum_j \|L_j u\|^2 + \|u\|^2$ ,  $u \in C_c^\infty$ . Here  $\Lambda$  is the standard elliptic pseudodifferential operator of order 1. A system which has finite bracket type  $2m$  is a system whose commutators of order  $2m - 1$  span the whole tangent space. It is well known that finite type of order  $2m$  implies a  $\delta$ -subelliptic estimate for some  $\delta \leq \frac{1}{2m}$ . If  $(L, \bar{L})$ , in  $\mathbb{C} \times \mathbb{R}$ , are identified to the generators of the tangential bundle  $T^{1,0}M \oplus T^{0,1}M$  to a pseudoconvex hypersurface  $M \subset \mathbb{C}^2$ , then  $(L, \bar{L})$  has finite type  $2m$  if and only if the contact of a complex curve  $\gamma$  with  $M$  is at most  $2m$ . Let the hypersurface  $M$  be “rigid”, that is, graphed by  $\operatorname{Re} w = g(z)$  for a real  $C^\infty$  function  $g$ , and set  $g_1 = \partial_z g$ ,  $g_{1\bar{1}} = \partial_z \partial_{\bar{z}} g$  and  $t = \operatorname{Im} w$ . With this notation we have  $L = \partial_z - ig_1(z)\partial_t$  and  $[L, \bar{L}] = g_{1\bar{1}}\partial_t$ . It is assumed that  $M$  is pseudoconvex, that is,  $g_{1\bar{1}} \geq 0$  (which also motivates the choice of an even type  $2m$ ). In terms of  $g$ , the condition of finite type  $2m$  means that  $g_{1\bar{1}}$  has some non-vanishing derivative of order  $2(m - 1)$ . In particular, this happens if  $g_{1\bar{1}} \gtrsim |x|^{2(m-1)}$ ; in this case, according for example to [12], we have a  $\frac{1}{2m}$ -subelliptic estimate.

A system has a superlogarithmic estimate if it has logarithmic gain of derivatives with an arbitrarily large constant, that is, for any  $\delta$  and

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for suitable  $c_\delta$

$$(1.1) \quad \|\log(\Lambda)u\|^2 \lesssim \delta \sum_j \|L_j u\|^2 + c_\delta \|u\|_{-1}^2, \quad u \in C_c^\infty.$$

A system which satisfies (1.1) is “precisely  $H^s$ -hypoelliptic” for any  $s$ :  $u$  is  $H^s$  where the  $L_j u$ ’s are (Kohn [7]). In particular, the system is  $C^\infty$ -hypoelliptic. Let  $L = \partial_z - ig_1(z)\partial_t$  for  $g$  of infinite type but exponentially non-degenerate in the sense that for a real curve  $S \subset \mathbb{C}$  we have

$$(1.2) \quad d_S^\alpha |\log g_{1\bar{1}}| \searrow 0 \text{ as } d_S \searrow 0 \text{ for } \alpha \leq 1,$$

where  $d_S$  denotes the distance to  $S$ . Under this assumption, the system  $(L, \bar{L})$  has a superlogarithmic estimate (cf. [12]). If we consider the perturbed system  $(\bar{L}, \bar{f}^k L)$  for  $\bar{f} = \bar{z}h(z)$  with  $h \neq 0$  and  $k \geq 1$ , the system has no more a superlogarithmic estimate, in general; if  $k > 1$ , a logarithmic loss occurs (Proposition 1.4 below). However, it is worth noticing that  $\mathcal{L}ie(\bar{L}, \bar{f}^k L)$ , the span of commutators of order  $\leq k - 1$ , has a superlogarithmic estimate (since it produces  $L$  as a commutator of order  $k - 1$ ). We are able to prove below that, in the terminology of Kohn [8], the system  $(\bar{L}, \bar{f}^k L)$  has an arbitrarily small loss of  $\epsilon$  derivatives and thus, in particular, is  $C^\infty$ -, but not exactly  $H^s$ -, hypoelliptic. Let  $\zeta_0$  and  $\zeta_1$  be cut-off functions in a neighborhood of 0 with  $\zeta_0 \prec \zeta_1$  in the sense that  $\zeta_1 \equiv 1$  over a neighborhood of  $\text{supp } \zeta_0$ .

**Theorem 1.1.** *Let  $L = \partial_z - ig_1(z)\partial_t$  and assume that 0 is a point of infinite type, i.e.  $g_{1\bar{1}} = 0^\infty(|z|)$  but not exponentially degenerate, i.e. (1.2) is fulfilled. Then the system  $(\bar{L}, \bar{f}^k L)$  (any  $k$ ) has an arbitrarily small loss of  $\epsilon$  derivatives, that is,*

$$(1.3) \quad \|\zeta_0 u\|_s^2 \lesssim \|\zeta_1 \bar{L} u\|_{s+\epsilon}^2 + \|\zeta_1 \bar{f}^k L u\|_{s+\epsilon}^2 + \|\bar{f}^k u\|_\epsilon^2 + \|u\|_0^2.$$

The proof of this and of the two theorems below follows in Section 2. Generally, an estimate of type (1.3) for smooth  $u$  does not yield finiteness of  $\|\zeta_0 u\|_s$  for a  $H^\epsilon$ -solution  $u$  of  $\bar{L}u = f$ ,  $\bar{f}^k L u = g$  when  $\zeta_1 f$  and  $\zeta_1 g$  are in  $H^{s+\epsilon}$ . However,  $L$  has coefficient  $t$ -independent and therefore it commutes with the  $t$ -derivatives. On the other hand, the  $t$ -derivatives describe the full Sobolev norm on the “positively microlocalized” component  $u^+$  (cf. §2 below) which is the only one which needs to be controlled. For this reason, if we use a sequence of pseudodifferential smoothing operators in  $t$ ,  $\chi_\nu(\partial_t) \rightarrow \text{id}$  as in [8] and [1], and we remark that

$$\bar{L}(\chi_\nu(\partial_t)u^+) = \chi_\nu(\partial_t)(\bar{L}u^+) + \text{Order}_{-\infty},$$

then, (1.3) applied to  $\Lambda^s(\chi_\nu(\partial_t)u^+) = \chi_\nu(\partial_t)(\Lambda^s u^+)$  yields

**Corollary 1.2.** *In the situation of Theorem 1.1, the system  $(\bar{L}, \bar{f}^k Lu)$  is hypoelliptic with loss of  $\epsilon$  derivatives:  $(\bar{L}u, \bar{f}^k Lu) \in H^{s+\epsilon}$ ,  $u \in H^\epsilon$  implies  $u \in H^s$ .*

For  $k = 1$  we have an estimate for local regularity without loss

**Proposition 1.3.** *In the situation above, assume in addition*

$$(1.4) \quad |g_1| \lesssim g_{1\bar{1}}^{\frac{1}{2}};$$

then

$$(1.5) \quad \|\zeta_0 u\|_s^2 \lesssim \|\zeta_1 \bar{L}u\|_s^2 + \|\zeta_1 \bar{f}^k Lu\|_s^2 + \|u\|_0^2.$$

When  $k > 1$ , a loss must occur

**Proposition 1.4.** *Assume that  $g = e^{-\frac{1}{|z|^\alpha}}$ . If*

$$(1.6) \quad \|\zeta_0 u\|_s^2 \lesssim \|(\log \Lambda)^r \zeta_1 \bar{L}u\|_s^2 + \|(\log \Lambda)^r \zeta_1 \bar{f}^k Lu\|_s^2 + \|\bar{f}^k u\|_\epsilon^2 + \|u\|_0^2,$$

then we must have  $r \geq \frac{k-(\alpha+1)}{\alpha}$ .

As far as we know, this is the first time that the problem of hypoellipticity is discussed for degenerate vector fields  $(\bar{L}, \bar{f}^k L)$  obtained from  $L = \partial_z - ig_1(z)\partial_t$  of infinite type, that is, satisfying  $g_{1\bar{1}} = 0^\infty(|z|)$ . However, it is necessary to make further assumptions such as (1.2). This guarantees a superlogarithmic estimate ([12]), and in turn, hypoellipticity according to Kohn [7]. Hypoellipticity with loss of derivatives for  $L = \partial_z - i\bar{z}\partial_t$  was discovered by Kohn in [8]. In this case,  $L$  is the  $(1, 0)$  vector field tangential to the strictly pseudoconvex hypersurface  $\text{Re } w = |z|^2$  and the loss amounts to  $\frac{k-1}{2}$ . The problem was further discussed by Bove, Derridj, Kohn and Tartakoff in [1] essentially for the vector field  $L = \partial_z - i\bar{z}|z|^{2(m-1)}\partial_t$  tangential to the hypersurface  $\text{Re } w = |z|^{2m}$  and the resulting loss is  $\frac{k-1}{2m}$ . In both cases the conclusion extends to the sum of squares  $L\bar{L} + \bar{L}|z|^{2k}L$  and the loss doubles to  $\frac{k-1}{m}$ . Moreover, in [1], analytic hypoellipticity has been proved; notice that this cannot be discussed in our framework, since,  $g$  having infinite type, it cannot be real analytic. For the vector fields  $L = \partial_z - ig_1(z)\partial_t$  tangential to a general pseudoconvex hypersurface of finite type with  $g_{1\bar{1}}$  vanishing at order  $2(m-1)$  along a real curve), hypoellipticity with loss of  $\frac{k-1}{2m}$  derivatives has been proved by the authors in [11]. Under some additional conditions, the result also extends to sums of squares (with double loss  $\frac{k-1}{m}$ ). When the hypersurface has infinite type as in the present paper, it is therefore natural to expect an arbitrarily small loss of derivatives.

## 2. TECHNICAL PRELIMINARIES AND PROOF

Our ambient space is  $\mathbb{C} \times \mathbb{R}$  identified with  $\mathbb{R}^3$  endowed with coordinates  $(z, \bar{z}, t)$  or  $(\operatorname{Re} z, \operatorname{Im} z, t)$ . We denote by  $\xi = (\xi_{\operatorname{Re} z}, \xi_{\operatorname{Im} z}, \xi_t)$  the variables dual to  $(\operatorname{Re} z, \operatorname{Im} z, t)$ , by  $\Lambda_\xi^s$  the standard symbol  $(1 + |\xi|^2)^{\frac{s}{2}}$ , and by  $\Lambda^s$  the pseudodifferential operator with symbol  $\Lambda_\xi^s$ ; this is defined by  $\Lambda^s(u) = \mathcal{F}^{-1}(\Lambda_\xi^s \mathcal{F}(u))$  where  $\mathcal{F}$  is the Fourier transform. We also consider the partial symbol  $\Lambda_{\xi_t}^s$  and the associate pseudodifferential operator  $\Lambda_t^s$ . We denote by  $\|u\|_s := \|\Lambda^s u\|_0$  (resp.  $\|u\|_{\mathbb{R}, s} := \|\Lambda_t^s u\|_0$ ) the full (resp. totally real)  $s$ -Sobolev norm. We use the notation  $\gtrsim$  and  $\lesssim$  to denote inequalities up to multiplicative constants; we denote by  $\sim$  the combination of  $\gtrsim$  and  $\lesssim$ . In  $\mathbb{R}_\xi^3$ , we consider a conical partition of the unity  $1 = \psi^+ + \psi^- + \psi^0$  where  $\psi^\pm$  have support in a neighborhood of the axes  $\pm \xi_t$  and  $\psi^0$  in a neighborhood of the plane  $\xi_t = 0$ , and introduce a decomposition of the identity  $\operatorname{id} = \Psi^+ + \Psi^- + \Psi^0$  by means of  $\Psi^\pm$ , the pseudodifferential operators with symbols  $\psi^\pm$ ; we accordingly write  $u = u^+ + u^- + u^0$ . Since  $|\xi_{\operatorname{Re} z}| + |\xi_{\operatorname{Im} z}| \lesssim \xi_t$  over  $\operatorname{supp} \psi^+$ , then  $\|u^+\|_{\mathbb{R}, s} \sim \|u^+\|_s$ .

We carry on the discussion by describing the properties of commutation of the vector fields  $L$  and  $\bar{L}$  for  $L = \partial_z - i g_1(z) \partial_t$ . The crucial equality is

$$(2.1) \quad \|Lu\|^2 = ([L, \bar{L}]u, u) + \|\bar{L}u\|^2, \quad u \in C_c^\infty,$$

which is readily verified by integration by parts. Since  $\sigma(\partial_t)$ , the symbol of  $\partial_t$ , is dominated by  $\sigma(L)$  and  $\sigma(\bar{L})$  in the “elliptic region” (the support of  $\psi^0$ ) and since  $L$  can be controlled by  $\bar{L}$  with an additional  $\epsilon \partial_t$  (because of (2.1)), then  $\|u^0\|_1^2 \lesssim \|\bar{L}u\|_0^2 + \|u\|_0^2$ . As for  $u^-$ , recall that  $[L, \bar{L}] = g_{1\bar{1}} \partial_t$  and hence  $g_{1\bar{1}} \sigma(\partial_t) \leq 0$  over  $\operatorname{supp} \psi^-$ . Thus (2.1) yields  $\|Lu\|^2 \lesssim \|\bar{L}u\|^2$ . It follows that, if  $L$  and  $\bar{L}$  have superlogarithmic estimate as in our application, then

$$\|\log(\Lambda)u^-\|^2 \leq \delta \|\bar{L}u^-\|^2 + c_\delta \|u\|^2.$$

In conclusion, only estimating  $u^+$  is relevant. We note here that, over  $\operatorname{supp} \Psi^+$ , we have  $g_{1\bar{1}} \xi_t \geq 0$ ; thus

$$(2.2) \quad \begin{aligned} \|g_{1\bar{1}}^{\frac{1}{2}} u^+\|_{\frac{1}{2}}^2 &= |([L, \bar{L}]u^+, u^+)| \\ &\leq \|Lu^+\|^2 + \|\bar{L}u^+\|^2. \end{aligned}$$

Following Kohn [7], we introduce a microlocal modification of  $\Lambda^s$ , denoted by  $R^s$ ; this is the pseudodifferential operator with symbol

$R_\xi^s := (1 + |\xi|^2)^{\frac{s\sigma(x)}{2}}$ ,  $\sigma \in C_c^\infty$ , that is,  $R^s(u) = \mathcal{F}^{-1}(R_\xi^s \mathcal{F}(u))$ . Often, what is used is in fact the partial operator in  $t$  with symbol  $R_{\xi_t}^s$ . We denote it by the same symbol  $R^s$  and observe that,  $f$  being independent of  $t$ , we have

$$(2.3) \quad [R^s, f] = 0.$$

The relevant property of  $R^s$  is

$$\|\Lambda^s \zeta_0 u\|^2 \underset{\sim}{\leq} \|R^s \zeta_0 u\|^2 + \|\zeta_0 u\|^2 \quad \text{if } \zeta_0 \prec \sigma.$$

Thus,  $R^s$  is equivalent to  $\Lambda^s$  over functions supported in the region where  $\sigma \equiv 1$ . In addition,  $\zeta_1 R^s$  better behaves with respect to commutation with  $L$ ; in fact, Jacobi equality yields

$$(2.4) \quad [\zeta_1 R^s, L] \sim \dot{\zeta}_1 R^s + \zeta_1 \log(\Lambda) R^s.$$

Thus, on one hand we have the disadvantage of the additional  $\log(\Lambda)$  in the second term, but we gain much in the cut-off because

$$(2.5) \quad \dot{\zeta}_1 R^s \text{ is of order } 0 \text{ if } \text{supp } \dot{\zeta}_1 \cap \text{supp } \sigma = \emptyset.$$

Property (2.5) is crucial in localizing regularity in presence of super-logarithmic estimate.

*Proof of Theorem 1.1.* As it has already been noticed, it suffices to prove (1.3) only for  $u^+$  and for  $\|\cdot\|_{\mathbb{R}, s}$ ; thus we write for simplicity  $u$  and  $\|\cdot\|_s$  but mean  $u^+$  and  $\|\cdot\|_{\mathbb{R}, s}$ . Moreover, we can use a cut-off  $\zeta = \zeta(t)$  in  $t$  only. In fact, for a cut-off  $\zeta = \zeta(z)$  we have  $[L, \zeta(z)] = \dot{\zeta}$  and  $\dot{\zeta} \equiv 0$  at  $z = 0$ . On the other hand,  $\bar{f}^k L \sim L$  outside  $z = 0$  which yields gain of derivatives, instead of loss. We call ‘‘good’’ a term in the right side (upper bound) of an estimate we wish to prove and ‘‘absorbable’’ a term which comes as a fraction (small constant or sc) of a formerly encountered term. We take cut-off functions in a neighborhood of 0:  $\zeta_0 \prec \sigma \prec \zeta_1$ ; we have for  $u \in C^\infty$

$$(2.6) \quad \begin{aligned} \|\zeta_0 u\|_s^2 &= \|\zeta_0 \zeta_1 u\|_s^2 \\ &\leq \|R^s \zeta_0 \zeta_1 u\|^2 + \|u\|_0^2 \\ &\underset{\sim}{\leq} \|\zeta_0 R^s \zeta_1 u\|_0^2 + \|[R^s, \zeta_0] \zeta_1 u\|_0^2 + \|u\|_0^2 \\ &\leq \|R^s \zeta_1 u\|_0^2 + \|u\|_0^2 \\ &\underset{\sim}{\leq} \|\zeta_1 R^s \zeta_1 u\|_0^2 + \|u\|_0^2, \end{aligned}$$

where the inequality in the fourth line follows from interpolation in Sobolev spaces and the last from  $\text{supp}(1 - \zeta_1) \cap \text{supp}\sigma = \emptyset$ . We have

$$(2.7) \quad \begin{aligned} \|\zeta_0 u\|_s^2 &\underset{\sim}{\leq} \underbrace{\|\zeta_1 R^s \zeta_1 u\|_0^2}_{(a)} + \|u\|^2 \\ &\underset{(*)}{\leq} \delta \left( \|L(\zeta_1 R^s \zeta_1)u\|^2 + \|\bar{L}(\zeta_1 R^s \zeta_1)u\|^2 \right) + c_\delta \|u\|^2, \end{aligned}$$

where the inequality marked by  $(*)$  follows from compactness which is a byproduct of superlogarithmic estimate. In the last line, we leave aside the central term and attack the first. Using integration by parts, we have

$$(2.8) \quad L \underset{\sim}{\leq} \bar{L} + [L, \bar{L}] \quad \text{microlocally on } \text{supp } \psi^+.$$

We rewrite the commutator. For this we recall an easy result about interpolation in Sobolev spaces. For positive  $\epsilon$ ,  $r$ ,  $n_1$ ,  $n_2$  with  $n_1$  and  $n_2$  integers satisfying  $0 < n_1 \leq r$  and  $n_2 > 0$ ,

$$(2.9) \quad \|h^r u\|_{\frac{1}{2}}^2 \leq sc \|h^{r-n_1} u\|_{\frac{1}{2}-n_1\epsilon}^2 + lc \|h^{r+n_2} u\|_{\frac{1}{2}+n_2\epsilon}^2.$$

We apply (2.9) for  $h = g_{11}^{\frac{1}{2r}}$ ,  $n_1 = r$ ,  $\epsilon = \frac{1}{2r}$ ,  $n_2 = 1$  (and note that  $h$  needs not to be smooth because  $h$  is a function of  $z$  whereas Sobolev norms are taken with respect to  $t$ ). We observe that, since  $g$  has infinite order, then  $g_{11}^{\frac{1}{2r}} \underset{\sim}{\leq} |f|^k$  for any  $r$  and any  $k$ . It follows

$$(2.10) \quad \begin{aligned} \| [L, \bar{L}]^{\frac{1}{2}} \zeta_1 R^s \zeta_1 u \|^2 &= \| g_{11}^{\frac{1}{2}} \zeta_1 R^s \zeta_1 u \|^2_{\frac{1}{2}} \\ &\underset{\sim}{\leq} sc \|\zeta_1 R^s \zeta_1 u\|_0^2 + lc \|g_{11}^{\frac{1}{2}} g_{11}^{\frac{1}{2r}} \zeta_1 R^s \zeta_1 u\|_{\frac{1}{2} + \frac{1}{2r}}^2 \\ &\underset{\sim}{\leq} sc \|\zeta_1 R^s \zeta_1 u\|_0^2 + lc \|g_{11}^{\frac{1}{2}} \Lambda^{\frac{1}{2}} \bar{f}^k \zeta_1 R^s \zeta_1 u\|_\epsilon^2 \\ &= sc \|\zeta_1 R^s \zeta_1 u\|_0^2 + lc \| [L, \bar{L}]^{\frac{1}{2}} \bar{f}^k \zeta_1 R^s \zeta_1 u \|_\epsilon^2 \\ &\underset{\sim}{\leq} \|\zeta_1 R^s \zeta_1 u\|_0^2 + \|L \bar{f}^k (\zeta_1 R^s \zeta_1)u\|_\epsilon^2 + \|\bar{L} \bar{f}^k (\zeta_1 R^s \zeta_1)u\|_\epsilon^2. \end{aligned}$$

We wish to first estimate the second term in the bottom of (2.10) in which we also replace  $L \bar{f}^k$  by  $\bar{f}^k L$ . In doing so, we encounter an error term  $\|\bar{f}^k (\zeta_1 R^s \zeta_1)u\|_\epsilon^2$  that we will estimate later on; (in fact,  $[L, \bar{f}^k] \sim \bar{f}^k$  since  $\bar{f} = \bar{z}h$  and  $[L, \bar{z}] = 0$ ). After this, we recall Jacobi identity, observe that  $[\bar{f}^k L, \zeta_1 R^s \zeta_1]$  has order arbitrarily close to  $s - 1$  (because



of a logarithmic extra term), that is

$$(2.11) \quad \begin{aligned} [\bar{f}^k L, \zeta_1 R^s \zeta_1] &= [L, \zeta_1] R^s \zeta_1 \bar{f}^k + \zeta_1 [\bar{L}, R^s] \zeta_1 \bar{f}^k + \zeta_1 R^s [\bar{L}, \zeta_1] \bar{f}^k \\ &\sim \underbrace{\zeta_1 R^s \zeta_1}_{\text{0-order by (2.5)}} \bar{f}^k + \underbrace{\zeta_1 \log(\Lambda) R^s \zeta_1}_{\text{by (2.4)}} \bar{f}^k + \underbrace{\zeta_1 R^s \zeta_1}_{\text{0-order by (2.5)}} \bar{f}^k. \end{aligned}$$

Thus we can commute  $\bar{f}^k L$  with  $\zeta_1 R^s \zeta_1$  up to an error as described in (2.11) which yields

$$\begin{aligned} \|L \bar{f}^k (\zeta_1 R^s \zeta_1) u\|_\epsilon^2 &\lesssim \|\bar{f}^k L (\zeta_1 R^s \zeta_1) u\|_\epsilon^2 + \|\bar{f}^k (\zeta_1 R^s \zeta_1) u\|_\epsilon^2 \\ &\lesssim \|(\zeta_1 R^s \zeta_1) \bar{f}^k L u\|_\epsilon^2 + \|(\zeta_1 \log(\Lambda) R^s \zeta_1) \bar{f}^k u\|_\epsilon^2 \\ &\quad + \|\bar{f}^k u\|_\epsilon^2 + \|\bar{f}^k (\zeta_1 R^s \zeta_1) u\|_\epsilon^2. \end{aligned}$$

On the other hand, since  $[\zeta_1, \log(\Lambda)] R^s$  has order 0, then

$$\begin{aligned} \|(\zeta_1 \log(\Lambda) R^s \zeta_1) \bar{f}^k u\|_\epsilon^2 &\lesssim \|(\log(\Lambda) (\zeta_1 R^s \zeta_1) \bar{f}^k u)\|_\epsilon^2 + \|\bar{f}^k u\|_\epsilon^2 \\ &\lesssim \underbrace{\delta \left( \|L (\zeta_1 R^s \zeta_1) \bar{f}^k u\|_\epsilon^2 + \|\bar{L} (\zeta_1 R^s \zeta_1) \bar{f}^k u\|_\epsilon^2 \right)}_{\text{absorbed by the last line of (2.10)}} + \|\bar{f}^k u\|_\epsilon^2, \end{aligned}$$

suplog estimate

where we are using the equality  $[\Lambda_t^\epsilon, L] = 0$  as well as  $[\Lambda^\epsilon, \log(\Lambda)] = 0$ . In the same way, using again (2.11), we estimate the central term in the last line of (2.7) which was left aside, that is,  $\|\bar{L} (\zeta_1 R^s \zeta_1) u\|_\epsilon^2 \lesssim \|(\zeta_1 R^s \zeta_1) \bar{L} u\|_\epsilon^2 + \|u\|_\epsilon^2$ . What remains, is to estimate the last term in the bottom of (2.10) (together with the error term  $\|\bar{f}^k (\zeta_1 R^s \zeta_1) u\|_\epsilon^2$ ). First, from Jacobi identity we get

$$[\bar{L} \bar{f}^k, \zeta_1 R^s \zeta_1] \sim (0\text{-order}) \bar{f}^k + \zeta_1 \log(\Lambda) R^s \zeta_1 \bar{f}^k + (0\text{-order}) \bar{f}^k,$$

so that we are eventually reduced to estimate  $\|(\zeta_1 R^s \zeta_1) \bar{L} \bar{f}^k u\|_\epsilon^2$ . This is the most difficult operation. We have (by the identity  $[\bar{L}, \bar{f}^k] \sim \bar{f}^{k-1}$ )

$$\|(\zeta_1 R^s \zeta_1) \bar{L} \bar{f}^k u\|_\epsilon^2 \lesssim \underbrace{\|(\zeta_1 R^s \zeta_1) \bar{f}^k \bar{L} u\|_\epsilon^2}_{\text{good}} + \|(\zeta_1 R^s \zeta_1) \bar{f}^{k-1} u\|_\epsilon^2.$$

Next we estimate the last term in the line above which also serves as an estimate for the term  $\|\bar{f}^k (\zeta_1 R^s \zeta_1) u\|_\epsilon^2$  that was encountered before. We have

$$\begin{aligned} \underbrace{\|(\zeta_1 R^s \zeta_1) \bar{f}^{k-1} u\|_\epsilon^2}_{(c)} &\sim \underbrace{\left( (\zeta_1 R^s \zeta_1) \bar{f}^{k-1} u, (\zeta_1 R^s \zeta_1) [\bar{L}, \bar{f}^k] u \right)_\epsilon}_{*} \\ &= -(*, (\zeta_1 R^s \zeta_1) \bar{f}^k \bar{L} u)_\epsilon + (*, (\zeta_1 R^s \zeta_1) \bar{L} \bar{f}^k u)_\epsilon. \end{aligned}$$

Now,

$$\left\{ \begin{array}{l} \left| (*, (\zeta_1 R^s \zeta_1) \bar{f}^k \bar{L} u)_\epsilon \right| \lesssim sc \|*\|_\epsilon^2 + \underbrace{\|(\zeta_1 R^s \zeta_1) \bar{f}^k \bar{L} u\|_\epsilon^2}_{\text{good}} \\ \left| (*, (\zeta_1 R^s \zeta_1) \bar{L} \bar{f}^k u)_\epsilon \right| \lesssim \left| \underbrace{((\zeta_1 R^s \zeta_1) \bar{f}^{k-1} f L u, (\zeta_1 R^s \zeta_1) \bar{f}^{k-1} u)_\epsilon}_{\text{good}} \right| + \underbrace{\left| (\zeta_1 R^s \zeta_1) \bar{f}^{k-1} u \right|_\epsilon}_{\text{absorbed by (c)}} \\ + 2 \left| \underbrace{(*)}_{\text{absorbed by (c)}}, \underbrace{[\bar{L}, (\zeta_1 R^s \zeta_1)] \bar{f}^k u}_d \right|_\epsilon. \end{array} \right.$$

We estimate (d). We notice that

$$(2.12) \quad [\bar{L}, (\zeta_1 R^s \zeta_1)] \sim \zeta_1 \log(\Lambda) R^s \zeta_1 + (0\text{-order}).$$

We also remark that

$$(2.13) \quad \begin{cases} [\Lambda^\epsilon \zeta_1, \log(\Lambda)] R^s \text{ has order } 0 & (i) \\ [\zeta_1, \Lambda^\epsilon] R^s \text{ has order } 0 & (ii) \\ [L, \Lambda^\epsilon] = 0 & (iii). \end{cases}$$

Hence

$$(2.14) \quad \begin{aligned} \|(d)\|_\epsilon^2 &\lesssim \|(\zeta_1 \log(\Lambda) R^s \zeta_1) \bar{f}^k u\|_\epsilon^2 + \|\bar{f}^{k-1} u\|_\epsilon^2 + \|u\|_0^2 \\ &\stackrel{\text{by (2.12)}}{\lesssim} \|(\log(\Lambda) \zeta_1 \Lambda^\epsilon R^s \zeta_1) \bar{f}^k u\|_0^2 + \|\bar{f}^{k-1} u\|_\epsilon^2 + \|u\|_0^2 \\ &\stackrel{\text{by (2.13) (i) and (ii)}}{\leq} \delta \left( \|L(\zeta_1 \Lambda^\epsilon R^s \zeta_1) \bar{f}^k u\|^2 + \|\bar{L}(\zeta_1 \Lambda^\epsilon R^s \zeta_1) \bar{f}^k u\|^2 \right) \\ &\stackrel{\text{by suplog estimate}}{\leq} \delta \left( \|L(\zeta_1 \Lambda^\epsilon R^s \zeta_1) \bar{f}^k u\|^2 + \|\bar{L}(\zeta_1 \Lambda^\epsilon R^s \zeta_1) \bar{f}^k u\|^2 \right) \\ &\quad + \underbrace{c_\delta \|(\zeta_1 \Lambda^\epsilon R^s \zeta_1) \bar{f}^k u\|_\epsilon^2}_{\text{absorbed by (c)}} + \|\bar{f}^{k-1} u\|_\epsilon^2 + \|u\|_0^2, \end{aligned}$$

where the last absorption occurs because  $\bar{f}^k = sc \bar{f}^{k-1}$ . Finally, the term with  $\delta$  is absorbed by the last term in (2.10) (after we transform  $\Lambda^\epsilon$  into  $\|\cdot\|_\epsilon$  to fit into (2.10) and use the fact that  $[\bar{L} \zeta_1, \Lambda^\epsilon] \sim \dot{\zeta}_1 \Lambda^{\epsilon-1} \bar{L}$  and  $[L \zeta_1, \Lambda^\epsilon] \sim \dot{\zeta}_1 \Lambda^{\epsilon-1} L$ ). This concludes the proof of (1.3). □

*Proof of Proposition 1.3.* As above, we stay in the positive microlocal cone, the support of  $\psi^+$ , and consider only derivatives and cut-off with respect to  $t$ . From the trivial identity  $[L, f] \sim 1$ , and from  $[L, \zeta_0] \sim \dot{\zeta}_0 g_1$ , we get

$$\begin{aligned} \|\zeta_0 u\|_s^2 &= ([L, f] \zeta_0 u, \zeta_0 u)_s \\ &\lesssim \|\bar{f} \zeta_0 \bar{L} u\|_s^2 + \|\bar{f} \zeta_0 L u\|_s^2 + \|\bar{f} g_1 \zeta_1 u\|_s^2 + sc \|\zeta_0 u\|_s^2. \end{aligned}$$

Now, the last term is absorbed. As for the term before

$$\begin{aligned} \|\bar{f}g_1\zeta_1u\|_s^2 &\stackrel{\text{by (1.4)}}{\leq} \|\bar{f}g_{11}^{\frac{1}{2}}\Lambda^{\frac{1}{2}}\zeta_1u\|_{s-\frac{1}{2}}^2 \\ &\stackrel{\text{by (2.2)}}{\leq} \|\bar{f}L\zeta_1u\|_{s-\frac{1}{2}}^2 + \|\bar{f}\bar{L}\zeta_1u\|_{s-\frac{1}{2}}^2 + \|\zeta_1u\|_{s-\frac{1}{2}}^2 \\ &\lesssim \|\zeta_1\bar{f}Lu\|_{s-\frac{1}{2}}^2 + \|\bar{f}\bar{L}\zeta_1u\|_{s-\frac{1}{2}}^2 + \|\bar{f}\zeta_2u\|_{s-\frac{1}{2}}^2 + \|\zeta_1u\|_{s-\frac{1}{2}}^2 \quad \text{for } \zeta_2 \succ \zeta_1. \end{aligned}$$

Now,  $\|\bar{f}\zeta_2u\|_{s-\frac{1}{2}}^2$  and  $\|\zeta_1u\|_{s-\frac{1}{2}}^2$  are not absorbable by  $\|\zeta_0u\|_s^2$ , but can be estimated by the 0-norm using induction over  $j$  such that  $\frac{j}{2} \geq s$ . □

*Proof of Proposition 1.4.* As always, we stay in the positive microlocal cone and take derivatives and cut-off only in  $t$ . We prove the result for  $s$  replaced by 0 and  $\epsilon$  replaced by  $-\eta$ . The conclusion for general  $s$  follows from the fact that  $\partial_t$  commutes with  $L$  and  $\bar{L}$ . We define

$$v_\lambda = e^{-\lambda(e^{-\frac{1}{|z|^\alpha}} - it + (e^{-\frac{1}{|z|^\alpha}} - it)^2)} \quad \lambda \gg 0.$$

We denote by  $-\lambda A$  the term at exponent and note that  $\text{Re } \lambda A \sim \lambda(e^{-\frac{1}{|z|^\alpha}} + t^2)$ . For  $L = \partial_z + ig_1(z)\partial_t$ , we have  $\bar{L}v_\lambda = 0$  (which is the key point) and moreover, since  $|\bar{f}|^k \sim |z|^k$

$$|\bar{f}^k Lv_\lambda| \sim \lambda |z|^{k-(\alpha+1)} e^{-\lambda(e^{-\frac{1}{|z|^\alpha}} + t^2)} e^{-\frac{1}{|z|^\alpha}}.$$

We set

$$\lambda(e^{-\frac{1}{|z|^\alpha}}, t) = (\theta_1, \frac{1}{\sqrt{\lambda}}\theta_2).$$

Under this change we have, over  $\text{supp } \zeta_0$  and  $\text{supp } \zeta_1$  which implies  $\theta_1 \ll \lambda$ ,

$$|z|^{k-(\alpha+1)} = \frac{1}{(\log \lambda - \log \theta_1)^{\frac{k-(\alpha+1)}{\alpha}}}.$$

Hence we interchange

$$|\bar{f}^k Lv_\lambda| \dashrightarrow \frac{1}{(\log \lambda)^{\frac{k-(\alpha+1)}{\alpha}}} \left( \frac{\theta_1 + \theta_2^2}{\left(1 - \frac{\log \theta_1}{\log \lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}} \right) e^{-(\theta_1 + \theta_2^2)}.$$

Notice that  $\theta_1 \ll \lambda$  and hence, for suitable positive  $c_1$  and  $c_2$ , we have  $c_1 < \frac{\theta_1 + \theta_2^2}{\left(1 - \frac{\log \theta_1}{\log \lambda}\right)^{\frac{k-(\alpha+1)}{\alpha}}} < c_2$ , uniformly over  $\lambda$ . We also interchange

$$v_\lambda \dashrightarrow e^{-(\theta_1 + \theta_2^2)}.$$

Taking  $L^2$  norms yields

$$\|\bar{f}^k L v_\lambda\|^2 \sim \frac{1}{(\log \lambda)^{2\frac{k-(\alpha+1)}{\alpha}}} \|v_\lambda\|^2.$$

So, the effect on  $L^2$  norm of the action of  $\bar{f}^k L$  over  $v_\lambda$  is comparable to  $\frac{1}{(\log \lambda)^{\frac{k-(\alpha+1)}{\alpha}}}$ . We describe now the effect of the pseudodifferential operator  $\log(\Lambda_t)$ . We claim that

$$(2.15) \quad \|\log(\Lambda_t) e^{-\lambda t^2}\|^2 \sim (\log \lambda)^2 \|e^{-\lambda t^2}\|^2.$$

This is a consequence of

$$(2.16) \quad \log(\Lambda_t) e^{-\lambda t^2} \sim \log \lambda e^{-\lambda t^2} + \left( \log(\Lambda_{\tilde{t}}) e^{-\tilde{t}^2} \right) \Big|_{\tilde{t}=\sqrt{\lambda}t},$$

that we go to prove now. Using the coordinate change  $\tilde{\theta} = \sqrt{\lambda}\theta$ ,  $\tilde{\xi} = \frac{\xi}{\sqrt{\lambda}}$ , we get

$$\begin{aligned} & \int e^{it\xi} \log(\Lambda_\xi) \left( \int e^{-i\xi\theta} e^{-\lambda\theta^2} d\theta \right) d\xi \\ &= \int e^{it\sqrt{\lambda}\tilde{\xi}} \left( \log\left(\frac{1}{\lambda} + |\tilde{\xi}|^2\right)^{\frac{1}{2}} + \log(\sqrt{\lambda}) \right) \left( \int e^{i\tilde{\xi}\tilde{\theta} - \tilde{\theta}^2} d\tilde{\theta} \right) d\tilde{\xi} \\ &= \log(\sqrt{\lambda}) e^{-\lambda t^2} + \left( \log(\Lambda_{\tilde{t}}^\lambda) e^{-\tilde{t}^2} \right) \Big|_{\tilde{t}=\sqrt{\lambda}t}, \end{aligned}$$

where  $\log(\Lambda_{\tilde{t}}^\lambda)$  is the operator with symbol  $\log\left(\frac{1}{\lambda} + |\tilde{\xi}|^2\right)^{\frac{1}{2}}$ . This proves (2.16) and in turn the claim (2.15). In the same way, we can check that  $\|\Lambda_t^{-\eta} e^{-\lambda t^2}\|^2 \sim \lambda^{-2\eta} \|e^{-\lambda t^2}\|^2$ .

We combine now the effect over  $v_\lambda$  of  $\bar{f}^k L$  with that of  $\log(\Lambda_t)$ . If

$$\|\zeta_0 v_\lambda\|^2 \lesssim \|\zeta_1 (\log \Lambda_t)^r \bar{f}^k L v_\lambda\|^2 + \|v_\lambda\|_{-\eta}^2,$$

then, since the right side is estimated from above by

$$\left( (\log \lambda)^{2r} (\log \lambda)^{-2\frac{k-(\alpha+1)}{\alpha}} + \lambda^{-2\eta} \right) \|v_\lambda\|^2,$$

we must have that the logarithmic term is not infinitesimal which forces  $r \geq \frac{k-(\alpha+1)}{\alpha}$ .

□

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Tran Vu Khanh  
School of Mathematics and Applied Statistics,  
University of Wollongong,  
NSW, Australia, 2522  
E-mail: tkhanh@uow.edu.au

Stefano Pinton  
Dipartimento di Matematica,  
Università di Padova,  
via Trieste 63, 35121 Padova,  
Italy  
E-mail: pinton@math.unipd.it

Giuseppe Zampieri  
Dipartimento di Matematica,  
Università di Padova,  
via Trieste 63, 35121 Padova,  
Italy