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Scale-multiplicative semigroups and geometry: automorphism groups of trees

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Abstract

A scale-multiplicative semigroup in a totally disconnected, locally compact group G is one for which the restriction of the scale function on G is multiplicative The maximal scale-multiplicative semigroups in groups acting 2-transitively on the set of ends of trees without leaves are determined and shown to correspond to geometric features of the tree.

Keywords

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Scale-multiplicative semigroups and geometry: automorphism groups of trees

Udo Baumgartner, Jacqui Ramagge, and George A. Willis[1]

Abstract. A scale-multiplicative semigroup in a totally disconnected, locally compact group G is one for which the restriction of the scale function on G is multiplicative. The maximal scale-multiplicative semigroups in groups acting 2-transitively on the set of ends of trees without leaves are determined and shown to correspond to geometric features of the tree.

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Keywords. Scale function, tidy subgroup, homogeneous tree, tree ends.

1. Introduction

The scale function on a totally disconnected, locally compact group, G, was introduced in [16] as a tool for use in the proof of a conjecture made in [10]. The scale, $s(x)$, of an element x in G is a positive integer that by [17, Theorem 3.1] is equal to

$$s(x) := \min\left\{[xVx^{-1} : xVx^{-1} \cap V] : V \leqslant G,\ V \text{ compact and open}\right\}. \qquad (1)$$

The scale is attained in (1) because it is the minimum of a set of positive integers, which may be seen as follows. Compact open subgroups exist because G has a base of neighborhoods of the identity comprising such subgroups (see [8], [13, Theorem II.2.3] or [9, Theorem II.7.7]). Then, for V compact and open, $xVx^{-1} \cap V$ is an open subgroup of xVx^{-1}, which is compact, whence $[xVx^{-1} : xVx^{-1} \cap V]$ is a positive integer.

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Since the scale function takes positive integer values, it cannot be multiplicative on G, or on subgroups of G, unless $s(x) = 1$ for every x. This rarely occurs because $s(x) = 1 = s(x^{-1})$ only if x normalizes some compact, open subgroup of G. There is no obstruction to the scale being multiplicative on a semigroup, however, and it indeed is multiplicative on singly generated semigroups since $s(x^n) = s(x)^n$ for every $n \in \mathbb{Z}^+$ and x in G by [16, Corollary 3]. A semigroup on which the scale function is multiplicative will be called *scale-multiplicative* or *s-multiplicative*.

This paper investigates the maximal s-multiplicative semigroups in totally disconnected, locally compact groups. Our long-term goal in doing so is the creation of a combinatorial structure for each group that has s-multiplicative semigroups as its elements and relations between these elements derived from how the semigroups intersect. The proposed structure will encode information about the given group, and there will be a natural action of the group on it.

Two steps are taken towards this goal here. One step is a study of groups having only finitely many maximal s-multiplicative semigroups: those having only one or two are characterised and examples of groups having a finite even number of such semigroups are described. In these finite examples the maximal s-multiplicative semigroups are seen to form a combinatorial geometry, see Remark 3.4. The other step is that the maximal s-multiplicative semigroups in certain automorphism groups of regular trees are determined and related to the geometry of the tree. These examples are prototypes for the proposed construction, although that construction will need to produce combinatorial geometries considerably more general than the kind seen in the examples if they are to cover groups such as that in [3].

Our goal is inspired by work of J. Tits in which combinatorial geometries and group actions are constructed for various totally disconnected, locally compact groups. In particular, he shows in [15] that a tree may be recovered from its automorphism group by identifying maximal compact, open subgroups with the vertices and edges of the tree and defining the incidence relation in terms of how they intersect. It is seen below that these subgroups are maximal s-multiplicative semigroups, but additional (non-uniscalar) semigroups are found and are identified with ends of the tree and open sets of ends. Broadening attention from compact, open subgroups to s-multiplicative semigroups thus brings more geometry into view. This added information is not exclusive to s-multiplicative semigroups because the ends of a regular tree may also be identified with non-compact maximal elliptic subgroups of its automorphism group. However, s-multiplicative semigroups carry extra dynamical information, through the scale function and translation direction, that the static picture presented by subgroups does not and which is likely to be useful for building a structure space. Moreover, there are maximal s-multiplicative semigroups that do not correspond to any maximal elliptic subgroups and which may be identified with sets of ends forming a base for the topology on the space of ends, see Remark 4.12(4).

Future work will go beyond relating maximal s-multiplicative semigroups to Tits' ideas on automorphisms of trees. One part of this work will be to integrate the current attempt with other approaches that seek a geometric representation for general totally disconnected, locally compact groups such as: the space of directions (which also comes with a topology, see [5]); the (discrete) metric space of compact, open subgroups; contraction subgroups; the Chabauty space of closed subgroups; and, indeed, the set of maximal elliptic subgroups. Another part of the work will be describing the maximal s-multiplicative semigroups in automorphism groups of buildings, semisimple Lie groups over local fields (where an algebraic rather than geometric description of the semigroups is sought) and other groups such as Neretin's group and the group studied in [3], thus providing examples to guide the proposed construction.

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The following notational convention will be used: the set of positive integers will be denoted by $\mathbb{Z}^+$, and the set of natural numbers (including 0) by $\mathbb{N}$. The identity of a group, G, will be denoted e_G.

2. Multiplicative semigroups

This section defines what it means for a semigroup to be multiplicative over a compact, open subgroup V and to be scale-multiplicative. Semigroups satisfying either of these two conditions will be called *multiplicative*. We show that semigroups multiplicative over V are scale-multiplicative and open, from which it will follow that there exist open, maximal scale-multiplicative semigroups. If the scale function is not identically 1, there also exist such semigroups which are non-compact as well. That the group inverse produces an involution on the set of multiplicative semigroups is also shown.

The notion of a semigroup being multiplicative over a compact, open subgroup will be defined first.

Definition 2.1. Suppose that V is a compact, open subgroup of a totally disconnected, locally compact group G. A semigroup $\mathcal{S} \subseteq G$ is *multiplicative over V* if $V \subseteq \mathcal{S}$ and the map $s_V : \mathcal{S} \to \mathbb{Z}^+$ given by $s_V(x) := [xVx^{-1} : xVx^{-1} \cap V]$ satisfies $s_V(xy) = s_V(x)s_V(y)$ for all $x, y \in \mathcal{S}$.

Since the set of semigroups multiplicative over V is closed under increasing unions, each semigroup multiplicative over V is contained in a maximal such semigroup. This maximal semigroup is not unique in general.

Our first result relates multiplicativity over V to the notion of scale-multiplicativity to be defined later on. For the statement, recall that any compact, open subgroup V at which the minimum in (1) is attained is said to be *minimizing for x*.

Proposition 2.2. *Let the semigroup $\mathcal{S}$ be multiplicative over V. Then V is minimizing for every $x \in \mathcal{S}$ and $s(x) = s_V(x)$.*

Proof. Let $x \in \mathcal{S}$. Then, by the "spectral radius formula" for the scale, see [12, Theorem 7.7],

$$s(x) = \lim_{n\to\infty} [x^n V x^{-n} : x^n V x^{-n} \cap V]^{1/n} = \lim_{n\to\infty} s_V(x^n)^{1/n}.$$

Since $\mathcal{S}$ is multiplicative over V, the last is equal to $s_V(x)$ and it follows that V is minimizing for x. □

If x and x^{-1} both belong to a semigroup multiplicative over V, then

$$s_V(x)s_V(x^{-1}) = s_V(xx^{-1}) = s_V(e_G) = 1.$$

Hence $s_V(x) = s_V(x^{-1}) = 1$ and $xVx^{-1} = V$. The 'only if' direction of the following corollary is thus established. The 'if' direction is straightforward.

Corollary 2.3. *The subgroup $H \leqslant G$ is multiplicative over the compact, open subgroup V if and only if H normalizes V.*

It is shown next that each subgroup V that is minimizing for x gives rise to a semigroup multiplicative over V and containing x. An essential part of the proof is the fact that a compact, open subgroup V that is minimizing for x has a decomposition $V = V_+V_- = V_-V_+$ where $V_\pm$ are closed subgroups of V with $xV_+x^{-1} \geqslant V_+$ and $xV_-x^{-1} \leqslant V_-$. This follows from the fact that every minimizing subgroup is also *tidy* for x, see [16, Definition p.343] and [17, Theorem 3.1].

Lemma 2.4. *Suppose that V is a compact, open subgroup of G that is minimizing for $x \in G$ and let $n \in \mathbb{Z}^+$. Then $(VxV)^n = V_-x^nV_+$ and $s(y) = s(x)^n$ for every $y \in (VxV)^n$.*

Proof. We have

$$(VxV)^n = \underbrace{(VxV)\dots(VxV)}_{n} = V(xVx^{-1})\dots(x^{n-1}Vx^{1-n})x^nV.$$

The comment preceding the statement of the lemma shows that

$$V(xVx^{-1}) = V_-V_+(xV_+x^{-1})(xV_-x^{-1}) = V_-(xV_+x^{-1})(xV_-x^{-1}),$$

where the last is equal to $V_-(xV_-x^{-1})(xV_+x^{-1})$, and which is in turn equal to $V_-(xV_+x^{-1})$ because $xV_-x^{-1} \leqslant V_-$. Iterating this argument yields

$$V(xVx^{-1})\dots(x^{n-1}Vx^{1-n})x^nV = V_-(x^{n-1}V_+x^{1-n})x^nV.$$

Next, since $(x^{n-1}V_+x^{1-n})x^n = x^n(x^{-1}V_+x) \leqslant x^nV_+$, it follows that

$$V(xVx^{-1})\dots(x^{n-1}Vx^{1-n})x^nV = V_-x^nV.$$

Finally, using again that $V = V_-V_+$ and that $x^nV_-x^{-n} \leqslant V_-$, we obtain

$$(VxV)^n = V(xVx^{-1})\dots(x^{n-1}Vx^{1-n})x^nV = V_-x^nV_+\,, \tag{2}$$

our first claim. Hence $y \in Vx^nV$ and, by [16, Theorem 3], V is tidy and therefore minimizing for y, and $s(y) = s(x^n)$. Since $s(x^n) = s(x)^n$ for every $n \in \mathbb{Z}^+$ by [16, Corollary 3], it follows that $s(y) = s(x)^n$ as required. $\square$

Lemma 2.4 and continuity of the scale function [16, Corollary 4] imply:

Proposition 2.5. *Let $x \in G$ and suppose that V is a compact, open subgroup of G that is minimizing for x. Then the semigroup generated by x and V is multiplicative over V. If $s(x) \neq 1$, then this semigroup is not compact.*

Proof. To establish the first claim, note that the semigroup generated by x and V is $\mathcal{S} = V \cup \bigcup_{n\in\mathbb{Z}^+}(VxV)^n$. Given y and z in $\mathcal{S}$, consider two cases.

In the first case, assume that $y \in (VxV)^m$ and $z \in (VxV)^n$. Then yz belongs to $(VxV)^{m+n}$ and the claim follows because $s(y) = s(x)^m$, $s(z) = s(x)^n$ and $s(yz) = s(x)^{m+n}$ by Lemma 2.4.

In the second case, at least one of the elements y and z belongs to V and it may be supposed without loss that $y \in V$. Then $s(yz) = s(z)$ by [16, Theorem 3] and, since $s(V) = \{1\}$, it follows that $s(yz) = s(y)s(z)$.

For the second claim note that, if $s(x) \neq 1$, then $\{x^n : n \in \mathbb{N}\}$ is not contained in any compact subset of G by continuity of the scale function. $\square$

The previous discussion shows that there are open subsemigroups of G satisfying the next definition.

Definition 2.6. Let G be a totally disconnected locally compact group. A semigroup, $\mathcal{S} \subseteq G$ is *scale-multiplicative*, or *s-multiplicative*, if

$$s(xy) = s(x)\,s(y) \quad \text{for every } x,\ y \in \mathcal{S}.$$

As remarked in the introduction, every s-multiplicative semigroup is contained in a maximal one. Since the scale function is continuous the following is immediate.

Proposition 2.7. *Every s-multiplicative subsemigroup of G is contained in a maximal such semigroup. All maximal s-multiplicative semigroups are closed.*

In view of Proposition 2.5 and Theorem 4.11 below, maximal open s-multiplicative semigroups are of particular interest. Should $\mathcal{S}$ contain an open subgroup V, then $\mathcal{S} = \mathcal{S}V$ is open. Any semigroup containing a semigroup multiplicative over V is therefore open. In particular, any maximal s-multiplicative semigroup containing a semigroup as in Proposition 2.5 is open. So far as we know, this is the only circumstance in which open s-multiplicative semigroups occur.

Question 2.8. *Let $\mathcal{S}$ be a maximal s-multiplicative semigroup that contains an open semigroup. Does $\mathcal{S}$ contain an open subgroup of G?*

Note that, even when an s-multiplicative semigroup $\mathcal{S}$ contains an open subgroup, that subgroup need not be minimizing for all elements of $\mathcal{S}$. In particular, the semigroups in Theorem 4.11 contain a largest open subgroup but that subgroup is not minimizing for any element that does not belong to it.

The next result implies that there is a natural involution on the set of maximal s-multiplicative semigroups.

Proposition 2.9. *The semigroup $\mathcal{S}$ is s-multiplicative if and only if $\mathcal{S}^{-1}$ is s-multiplicative. Moreover, $\mathcal{S}$ is maximal s-multiplicative if and only if $\mathcal{S}^{-1}$ is maximal s-multiplicative.*

Proof. Suppose that $\mathcal{S}^{-1}$ is s-multiplicative and consider $x, y \in \mathcal{S}$. Denoting the modular function on G by $\Delta : G \to (\mathbb{R}^+, \times)$, [16, Corollary 1] implies that

$$s(xy) = \Delta(xy)s((xy)^{-1}) = \Delta(x)\Delta(y)s(y^{-1})s(x^{-1}),$$

because Δ is a homomorphism and $\mathcal{S}^{-1}$ is s-multiplicative. Since $s(x) = \Delta(x)s(x^{-1})$ and similarly for $s(y)$, it follows that $s(xy) = s(x)s(y)$ and that $\mathcal{S}$ is a s-multiplicative semigroup.

For the maximality statement, note that the inverse map is an involution preserving proper containment. □

For any s-multiplicative semigroup $\mathcal{S}$, the map $s : \mathcal{S} \to (\mathbb{Z}^+, \times)$ is a homomorphism whose range is a subsemigroup of $(\mathbb{Z}^+, \times)$. This semigroup is not equal to all of $\mathbb{Z}^+$ if G is compactly generated, because the scale has only finitely many prime divisors in that case, see [18].

To prepare a characterization of groups with a unique s-multiplicative semigroup, we need the following lemma. A subset will be called *uniscalar* if and only if the set of values of the scale function on it is $\{1\}$.

Lemma 2.10. *Let G be a totally disconnected, locally compact group and let $x \in G$ satisfy $s(x) \neq 1$. Then no s-multiplicative semigroup of G contains both x and x^{-1}. In particular, a s-multiplicative semigroup of G that is invariant under inversion is a uniscalar subgroup.*

Proof. Aiming to prove the contrapositive, suppose that $\mathcal{S}$ is a s-multiplicative semigroup of G that contains x and x^{-1}. Then $1 = s(e_G) = s(xx^{-1}) = s(x)s(x^{-1})$ and we must have $s(x) = s(x^{-1}) = 1$ in contradiction to the assumption $s(x) \neq 1$. This proves the first statement.

The first statement implies that the scale of every element of a s-multiplicative semigroup which is invariant under inversion is 1. Since a semigroup invariant under inversion is a group, the second statement follows as well. □

Corollary 2.11. *Let G be a totally disconnected, locally compact group. Then G is uniscalar if and only if it has a unique maximal s-multiplicative semigroup, which then necessarily is the whole of G.*

Proof. If G is uniscalar, then G is a scale-multiplicative semigroup which is necessarily maximal and unique with this property. If G is not uniscalar, it contains an element x such that $s(x) \neq 1$. By Lemma 2.10, x and x^{-1} can not be contained in a common s-multiplicative semigroup. Hence there must be at least two different maximal s-multiplicative semigroups. □

3. Groups with finitely many maximal s-multiplicative semigroups

We begin this section by constructing totally disconnected, locally compact groups with exactly two s-multiplicative semigroups. Thereafter we analyse the structure of such groups in general. We conclude the section with an example illustrating possible patterns of finite collections of s-multiplicative semigroups that can occur.

Let $H \rtimes_\alpha \mathbb{Z}$ be a totally disconnected, locally compact group, where α is an automorphism of H. By identifying the internal and external semi-direct products, we write $s(\alpha) = s(e_H, 1)$. Suppose that there is a compact, open subgroup $V \leqslant H$ with

$$\alpha(V) > V \quad \text{and} \quad \bigcup_{n\in\mathbb{Z}} \alpha^n(V) = H. \tag{3}$$

Then $s(\alpha) = [\alpha(V) : V]$ and $s(\alpha^{-1}) = [\alpha^{-1}(V) : \alpha^{-1}(V) \cap V] = 1$. Thus V is minimizing for α^{-1} and hence, by [17, Corollary 3.11] or [12, Corollary 5.3], for α.

Such semidirect products are of particular interest because every group G that is not uniscalar has subgroups of this form. If $s(x) > 1$ and V is tidy for x, then $V_{++} := \bigcup_{n\in\mathbb{Z}} x^n V_+ x^{-n}$ is a closed subgroup of G that is normalized by x, and $\langle x, V_{++}\rangle = V_{++} \rtimes \langle x\rangle$, see [16, Proposition 2]. Denote the inner automorphism $y \mapsto xyx^{-1}$ of G by α_x, so that $V_{++} \rtimes \langle x\rangle = V_{++} \rtimes_{\alpha_x} \langle x\rangle$. Multiplicative semigroups contained in $V_{++} \rtimes_{\alpha_x} \langle x\rangle$ will extend to maximal s-multiplicative semigroups in G. By way of illustration of Proposition 2.5, observe that the semigroup in H generated by V and x, respectively x^{-1}, is the disjoint union

$$\bigsqcup_{n\in\mathbb{N}} \alpha_x^n(V)x^n, \quad \text{respectively} \quad \bigsqcup_{n\in\mathbb{N}} Vx^{-n}.$$

These semigroups are multiplicative over V but are not maximal.

Proposition 3.1. *Let $G = H \rtimes_\alpha \mathbb{Z}$ with $V \leqslant H$ as in (3). Then*

$$\mathcal{S}_+ = \{(h,n) \in H \rtimes_\alpha \mathbb{Z} : n \geq 0\} \quad \textit{and} \quad \mathcal{S}_- = \{(h,n) \in H \rtimes_\alpha \mathbb{Z} : n \leq 0\}$$

are the only maximal s-multiplicative semigroups in G. These semigroups are multiplicative over V if and only if $V \lhd H$.

Proof. Put $q := [\alpha(V) : V] > 1$. We derive a formula for the scale of (h,n) in $H \rtimes_\alpha \mathbb{Z}$ in terms of q. For this, choose m such that $h \in \alpha^m(V)$. Then

$$(h,n)\alpha^{m-n}(V)(h,n)^{-1} = (h,0)\alpha^m(V)(h,0)^{-1} = \alpha^m(V).$$

Hence, if $n \geq 0$, then $(h,n)\alpha^{m-n}(V)(h,n)^{-1} \geqslant \alpha^{m-n}(V)$ and it follows that $\alpha^{m-n}(V)$ is minimizing for (h,n) and

$$s(h,n) = [\alpha^m(V) : \alpha^{m-n}(V)] = q^n.$$

If $n \leq 0$, then $(h,n)\alpha^{m-n}(V)(h,n)^{-1} \leqslant \alpha^{m-n}(V)$ and $s(h,n) = 1$. Therefore

$$s(h,n) = \begin{cases} q^n & \text{if } n \geq 0, \\ 1 & \text{if } n < 0. \end{cases}$$

Since the product of (g,m) and (h,n) in G is $(g,m)(h,n) = (g\alpha^m(h), m+n)$, and $q > 1$, it follows that

$$s((g,m)(h,n)) < s(g,m)s(h,n) \text{ if } m \text{ and } n \text{ have opposite signs} \tag{4}$$

and

$$s((g,m)(h,n)) = s(g,m)s(h,n) \text{ if } m \text{ and } n \text{ have the same sign.} \tag{5}$$

By (5), $\mathcal{S}_+$ and $\mathcal{S}_-$ are s-multiplicative semigroups. Since $G = \mathcal{S}_+ \cup \mathcal{S}_-$, (4) implies that every s-multiplicative semigroup is contained in one of $\mathcal{S}_+$ or $\mathcal{S}_-$. It follows that both $\mathcal{S}_+$ and $\mathcal{S}_-$ are maximal s-multiplicative semigroups, and that G contains no others.

We now prove that $\mathcal{S}_+$ and $\mathcal{S}_-$ are multiplicative over V if and only if $V \lhd H$. Assume that $\mathcal{S}_+$ is multiplicative over V. Then, in particular, the subgroup H is multiplicative over V and H normalizes V, by Corollary 2.3. That $V \lhd H$ if

$\mathcal{S}_-$ is multiplicative over V follows similarly. On the other hand, if $V \lhd H$, then $(h,n)V(h,n)^{-1} = \alpha^n(V)$ for every $(h,n) \in G$ and it follows that

$$\begin{aligned} s_V(h,n) &= [(h,n)V(h,n)^{-1} : (h,n)V(h,n)^{-1} \cap V] = [\alpha^n(V) : \alpha^n(V) \cap V] \\ &= s(h,n) \end{aligned} \tag{6}$$

for every $(h,n) \in H \rtimes_\alpha \mathbb{Z}$. Then s_V is multiplicative on $\mathcal{S}_+$ and $\mathcal{S}_-$ because s is, and $\mathcal{S}_+$ and $\mathcal{S}_-$ are multiplicative over V. □

As previously remarked, every group G that is not uniscalar has closed subgroups isomorphic to some $H \rtimes_\alpha \mathbb{Z}$. Multiplicative semigroups in each of these subgroups extend to maximal s-multiplicative semigroups in G. We recall in the next section a canonical action of $H \rtimes_\alpha \mathbb{Z}$ on a homogenous tree. The structure space of maximal s-multiplicative semigroups will consequently be related to this tree. It will be seen also that this relationship is not straightforward even when G itself acts on a tree.

We conclude this section with a few examples and preliminary results in the case when the number of s-multiplicative semigroups is finite. These will illustrate the way in which intersection relations between s-multiplicative semigroups correspond to incidence relations in a geometry in the higher rank case and in which structural information about a group might be recovered from knowledge of its s-multiplicative semigroups.

Corollary 2.11 characterizes groups having a unique maximal s-multiplicative semigroup. Extending an argument provided by the referee, we here treat the special case of groups with exactly two maximal s-multiplicative semigroups.

Proposition 3.2. *Let G be a totally disconnected, locally compact group with exactly two maximal s-multiplicative semigroups. Then these semigroups are inverses, their union is G, their intersection H equals $\{x \in G : s(x) = 1 = s(x^{-1})\}$ and is an open, normal subgroup of G. The quotient G/H is isomorphic to the infinite cyclic group* $(\mathbb{Z}, +)$.

Proof. In any totally disconnected, locally compact group, every element of G is contained in some maximal s-multiplicative semigroup. Therefore, the union of these semigroups always equals G.

We verify next that the two maximal s-multiplicative semigroups are inverses of each other. By Corollary 2.11, the group G is not uniscalar. Choose m in G with non-trivial scale, and, for further use below, with minimal scale having this property. Let $\mathcal{S}$ be a maximal s-multiplicative semigroup of G containing m. By Proposition 2.9, $\mathcal{S}^{-1}$ is another maximal s-multiplicative semigroup, which differs from $\mathcal{S}$ by Lemma 2.10. By assumption, there are exactly two maximal s-multiplicative semigroups of G. We therefore conclude that these are $\mathcal{S}$ and $\mathcal{S}^{-1}$ and that they are exchanged by the inverse map.

In order to verify that $\mathcal{S}$ and $\mathcal{S}^{-1}$ are open, choose a compact, open subgroup, V, of G. Then V is a s-multiplicative semigroup, hence contained in one of $\mathcal{S}$, $\mathcal{S}^{-1}$, and, being invariant under inversion, in both of them.

Hence $\mathcal{S} \cap \mathcal{S}^{-1}$ is open. Being invariant under inversion, $\mathcal{S} \cap \mathcal{S}^{-1}$ is also a subgroup of G, and it is normal, since conjugation by elements of G permutes the maximal s-multiplicative semigroups.

We show next that $\mathcal{S} \cap \mathcal{S}^{-1} = H := \{x \in G : s(x) = 1 = s(x^{-1})\}$. The left hand side is a s-multiplicative semigroup, that is invariant under inversion, hence is a uniscalar subgroup by Lemma 2.10. This implies $\mathcal{S} \cap \mathcal{S}^{-1} \subseteq H$ by the definition of H. Since the group generated by any element, x say, of H is s-multiplicative and invariant under inversion, we conclude $x \in \mathcal{S} \cap \mathcal{S}^{-1}$. Since $x \in H$ was arbitrary, we obtain $\mathcal{S} \cap \mathcal{S}^{-1} = H$ as claimed.

Finally, we prove that G/H is infinite cyclic. We will show that $G = \langle m \rangle H$, where m was chosen in the second paragraph of this proof. Our claim then follows, because by multiplicativity of the scale and the choice of m, no positive power of m has scale 1, thus, in particular, such a power is not in H. Hence we will have $G = \langle m \rangle \ltimes H$. Observe that $m \in \mathcal{S} \setminus H = \mathcal{S} \setminus \mathcal{S}^{-1}$.

The proof of $G = \langle m \rangle H$ requires two tools.

First, no element of $\mathcal{S} \setminus H$ has scale 1. To see this, let $x \in \mathcal{S}$ have $s(x) = 1$. We will show $s(x^{-1}) = 1$, which implies $x \in H$, as required. The element mx^{-n} belongs to $\mathcal{S}$ for every $n > 0$, otherwise $(mx^{-n})^{-1} = x^n m^{-1} \in \mathcal{S}$, leading to $1 = s(x^n) = s(x^n m^{-1})s(m)$, which contradicts $s(m) > 1$. Hence both x^{-1} and $x^n m^{-1}$ belong to $\mathcal{S}^{-1}$ for all $n > 0$ and it follows that $s(m^{-1}) = s(x^{-1})^n s(x^n m^{-1})$ for every $n > 0$. The last equation can only be satisfied if $s(x^{-1}) = 1$, so that $x^{-1} \in H$ and hence $x \in H$.

Our second tool is the following. For every $y \in \mathcal{S} \setminus H$ with $m^{-1}y \in \mathcal{S}^{-1}$ we have $s(y) = s(m)$ and $s(y^{-1}m) = 1$. To see this, observe that $y^{-1}m = (m^{-1}y)^{-1} \in \mathcal{S}$ and
$$s(m) = s(y)s(y^{-1}m). \tag{7}$$
By our first tool, $s(y) \neq 1$. Using the minimality of $s(m)$, equation (7) implies $s(y) = s(m)$ and $s(y^{-1}m) = 1$ as claimed.

With our tools in place, we are now ready to show that $\mathcal{S} \subset \langle m \rangle H$. To that end, pick $x \in \mathcal{S}$. We will show $x \in \langle m \rangle H$. This claim is obvious for $x \in H$, so we may suppose $x \in \mathcal{S} \setminus H$. The set of non-negative integers n such that $m^{-n}x \in \mathcal{S} \setminus H$ contains 0 and cannot be unbounded because for any such integer n
$$s(x) = s(m^n \cdot m^{-n}x) = s(m)^n s(m^{-n}x)$$
and $s(m) > 1$. Therefore, we can define $n(x)$ as the smallest non-negative integer such that
$$m^{-n(x)}x \in \mathcal{S} \setminus H \quad \text{and} \quad m^{-1}m^{-n(x)}x \in \mathcal{S}^{-1}. \tag{8}$$
Using our second tool with $y := m^{-n(x)}x$ we see that
$$s(m^{-n(x)}x) = s(m) \tag{9}$$

and

$$s(x^{-1}m^{n(x)+1}) = s(x^{-1}m^{n(x)} \cdot m) = s(y^{-1}m) = 1. \tag{10}$$

An aside: it follows from (9) that $s(x) = s(m)^{n(x)+1}$.

Equation (9) implies that $m^{-n(x)}x$ is an element of $y \in \mathcal{S} \setminus H$ with minimum scale and so we may take m to be $m^{-n(x)}x$ in (7) and y to be m and conclude that $s(m^{-n(x)-1}x) = 1$. Together with equation (10) and the characterization of H in terms of scales, we deduce that $m^{-n(x)-1}x \in H$ and therefore $x \in \langle m\rangle H$. Since x is a general element of $\mathcal{S}$, it follows that $\mathcal{S} \subset \langle m\rangle H$ and then, since H is normal, that $\mathcal{S}^{-1} \subset \langle m\rangle H$ as well. Therefore $G = \langle m\rangle H$ as claimed. □

The number of maximal s-multiplicative semigroups can be finite and bigger than 2 and they have a variety of possible intersection patterns, as illustrated by the following examples.

Example 3.3. Let W be $\mathbb{Q}_p^n$ and let H consist of groups of diagonal matrices over $\mathbb{Q}_p$ of the form $\mathrm{diag}(p^{\eta_1}, \dots, p^{\eta_n})$ with η_i in $\mathbb{Z}$ for all i. After discussing general properties, we will impose conditions on the exponents η_i below.

The group H acts on W by matrix multiplication. Put $G := H \ltimes W$.

(1) Let $h := \mathrm{diag}(p^{\eta_1}, \dots, p^{\eta_n})$ be in H and v in W. Then conjugation by hv acts by multiplication with p^{η_i} on

$$W_i := \{(w_j)_{1\le j\le n} \in W : w_j = 0 \text{ if } j \neq i\}.$$

(2) The subgroup $\mathbb{Z}_p^n$ of W is tidy for all g in G.

(3) With h and v as in (1), the value of the scale function at hv is $p^{\sum_{\eta_i\ge 0}\eta_i}$.

(4) Let $h := \mathrm{diag}(p^{\eta_1}, \dots, p^{\eta_n})$ and $k := \mathrm{diag}(p^{\kappa_1}, \dots, p^{\kappa_n})$ be in H and v, w be in W. Then hv and kw are in a common s-multiplicative semigroup if and only if for all indices i we have $\eta_i\kappa_i \ge 0$.

(5) There is a maximal s-multiplicative semigroup of G for every $\epsilon = (\epsilon_1, \dots, \epsilon_n)$ in $\{-1, 1\}^{\{1,\dots,n\}}$ given by

$$\mathcal{S}_\epsilon = \bigcup_{\eta_i\epsilon_i \ge 0 \text{ for } 1\le i\le n} \mathrm{diag}(p^{\eta_1}, \dots, p^{\eta_n})W.$$

Subgroups of G obtained by imposing restrictions on the vectors in H illustrate diverse intersection patterns for maximal s-multiplicative semigroups.

(1) With no restrictions, there are 2^n maximal s-multiplicative semigroups in G, one for each $\epsilon \in \{-1, 1\}^{\{1,\dots,n\}}$. The intersection of maximal s-multiplicative semigroups $\mathcal{S}_{\epsilon_1}$ and $\mathcal{S}_{\epsilon_2}$ comprises all $\mathrm{diag}(p^{\eta_1}, \dots, p^{\eta_n})W \subseteq G$ such that $\eta_i = 0$ if $\epsilon_1(i)\epsilon_2(i) = -1$ and $\eta_i\epsilon_1(i) = \eta_i\epsilon_2(i) \ge 0$ otherwise. There is thus one such semigroup for each closed orthant in $\mathbb{R}^n$ and two maximal semigroups intersect in a semigroup containing non-uniscalar elements if and only if the intersection of their corresponding orthants is larger than a point.

(2) Suppose instead that the η_i depend $\mathbb{Z}$-linearly on integral parameters x_1 and x_2, that is, $\eta_i = k_{i1}x_1 + k_{i2}x_2$ for some fixed $(k_{i1}, k_{i2}) \in \mathbb{Z}^n \times \mathbb{Z}^n$. Then the criterion in (4) above shows that the maximal s-multiplicative semigroups correspond to the closed sectors in the (x_1, x_2)-plane formed by the lines

$$k_{i1}x_1 + k_{i2}x_2 = 0, \quad 1 \le i \le n.$$

The number of maximal semigroups is therefore $2n$. The intersection of two maximal semigroups contains non-uniscalar elements if and only if their corresponding sectors are adjacent, that is, meet in a ray.

Remark 3.4. A graph, Γ_H, may be defined for each H in Example 3.3 in which $V(\Gamma_H)$ is the set of maximal s-multiplicative semigroups in $H \ltimes W$ and two semigroups $\mathcal{S}_1$ and $\mathcal{S}_2$ are adjacent if $\mathcal{S}_1 \cap \mathcal{S}_2$ is maximal in both $\mathcal{S}_1$ and $\mathcal{S}_2$. In (1) this graph is an n-dimensional hypercube and in (2) it is a $(2n)$-gon.

It is clear that groups with finitely many maximal s-multiplicative semigroups have a restricted structure but it is less clear whether Proposition 3.2 can be extended to show that they all have similar structure to that seen in Example 3.3. For instance, if the number of maximal s-multiplicative semigroups is odd, then at least one of them must be equal to its own inverse and therefore be a uniscalar subgroup of G. No groups with three maximal s-multiplicative semigroups are known however. It will be informative to investigate this class of groups.

4. Highly transitive automorphism groups of locally finite trees

In this section we determine the maximal s-multiplicative semigroups for groups of automorphisms of a tree, T, that act 2-transitively on the boundary ∂T. Such groups are discussed by F. Choucroun in [7]. The standing assumption in this section is that the group G has this property, and it will be seen in Theorem 4.11 that the maximal s-multiplicative semigroups are closely identified with features of T.

Automorphisms of a tree are classified into elliptic and hyperbolic types, see [15]. Terminology and results relating to this classification to be used below will be briefly reviewed before proceeding. The vertices of T will be denoted by $V(T)$ and its edges by $E(T)$. We adopt the convention that the edge between vertices v and w comprises the ordered pairs (v, w) and (w, v). Then (v, w) is the *oriented edge* from v to w and v is the *initial vertex of the edge* while w is the *the terminal vertex*. If e is an oriented edge, then $o(e)$ and $t(e)$ denote its initial and terminal vertices respectively, and $\bar{e}$ is the oppositely oriented edge with initial vertex $t(e)$ and terminal vertex $o(e)$.

Every automorphism of a tree either fixes a vertex, inverts an edge or induces a nontrivial translation along an infinite geodesic in the tree. The first two types are called *elliptic* and have at least one fixed point in the tree.[1] The last type is *hyperbolic*, in which case the geodesic of translation is unique and called the *axis*. Define the length, $\ell(g)$, of an automorphism g of the tree to be the minimum distance, in the usual graph metric, by which it moves points in the tree. The length of g is 0 for g elliptic, and is the length of translation along the axis for g hyperbolic.

Define the *minimal set for g* to be

$$\min(g) := \{p \in T : d(p, g.p) = \ell(g)\}.$$

Thus $\min(g)$ is the set of fixed points of g if g is elliptic and the axis of g if g is hyperbolic.

Two oriented edges in a tree are *coherent* if the distance between their respective initial vertices equals the distance between their terminal vertices and *incoherent* otherwise. A hyperbolic automorphism, h say, is said to *translate along* $(o(e), t(e))$ if and only if the oriented edges $(o(e), t(e))$ and $(h.o(e), h.t(e))$ are coherent and $d(o(e), h.t(e)) = d(t(e), h.o(e)) + 2$.

A semi-infinite geodesic in a tree is a *ray*. Two rays in a tree are said to belong to *the same end*, if and only if their intersection is a ray. This defines an equivalence relation on the set of rays in the tree, whose equivalence classes are called *the set of ends of the tree*. The automorphism group of the tree acts on its set of ends. Given a hyperbolic automorphism, h say, each ray $\mathfrak{r} \subseteq \min(h)$ satisfies either $h.\mathfrak{r} \subset \mathfrak{r}$ or $h.\mathfrak{r} \supset \mathfrak{r}$ and these conditions distinguish two distinct ends of the axis of h. The first is called *the attracting end of h* and denoted by $\epsilon_+(h)$, while the second is called *the repelling end of h* and denoted by $\epsilon_-(h)$

We introduce some notation for segments. Given two vertices v and w, by the *segment* $]v, w]$ we mean the set of vertices on the path from v to w not including v. By Lemma 31 from [5], the scale of a hyperbolic element h is the product

$$\prod_{v \in]w, h.w]} q(v), \tag{11}$$

where $q(v)$ is the valency of the vertex v minus 1 in the minimal invariant subtree of $G_{\epsilon_+(h)}$, and w is any vertex on the axis of h.

Having recalled basic results for groups acting on trees we now briefly return to the groups $H \rtimes_\alpha \mathbb{Z}$ considered in the last section, whose basic importance for s-multiplicative semigroups was explained there.

[1] Note that, in contrast with some parts of the literature, we consider tree automorphisms that invert edges, so-called *inversions*, to be elliptic.

The group $H \rtimes_\alpha \mathbb{Z}$ is an HNN-extension and consequently acts on a homogeneous tree T_{q+1} with valency $q+1$ and fixes an end ω, see [4, Section 4]. Denote the corresponding representation by $\rho\colon H \rtimes_\alpha \mathbb{Z} \to \mathrm{Aut}(T_{q+1})$. The kernel of ρ is the largest compact, normal subgroup of $H \rtimes_\alpha \mathbb{Z}$, the group $\rho(H)$ is contained in the set of elliptic automorphisms of the tree, and $\rho(\alpha)$ is a hyperbolic element that has ω as its attracting end. Then the maximal s-multiplicative semigroups in $H \rtimes_\alpha \mathbb{Z}$ have the following characterization.

$$\mathcal{S}_+ = \{x \in H \rtimes_\alpha \mathbb{Z}\colon \rho(x) \text{ is elliptic or } \rho(x) \text{ has } \omega \text{ as its attracting end}\}$$

and

$$\mathcal{S}_- = \{x \in H \rtimes_\alpha \mathbb{Z}\colon \rho(x) \text{ is elliptic or } \rho(x) \text{ has } \omega \text{ as its repelling end}\}\,.$$

Denote the axis of $\rho(\alpha)$ by $\mathfrak{l}$. Then $\rho(V)$ is the fixator of a ray, $[v_0, \omega)$ say, on $\mathfrak{l}$. Label the other vertices on $\mathfrak{l}$ as $v_n = \rho(\alpha^n).v_0$. Then the subset

$$(\alpha^{n-1}(V), n) := \big\{(x, n)\colon x \in \alpha^{n-1}(V)\big\} \subseteq H \rtimes_\alpha \mathbb{Z}$$

of the semigroup generated by V and α translates v_j to v_{j+n} for every $j \geq -1$, and the subset $(V, -n)$ of the semigroup generated by V and α^{-1} translates v_j to v_{j-n} for every $j \geq n$.

This explains how maximal s-multiplicative semigroups are represented geometrically for these examples. We next return to groups with a 2-transitive action on the set of ends of a tree.

The following lemma will be used on numerous occasions in this section. It appears in [11, Lemma 1.2] and [2, Lemma 6.8], and implicitly in [15, Lemme 3.1].

Lemma 4.1. *Suppose that an oriented edge, e, and its image under an automorphism g differ and are coherent. Then g is hyperbolic and the edges e and $g.e$ lie on the axis of g.*

Proposition 3.4 in [15] states that every group of elliptic automorphisms of a tree fixes either a vertex, inverts an edge, or fixes an end of the tree. The arguments to follow require a version of this result for semigroups. One way to establish this is to work through the proof in [2, Proposition 7.2], which uses [1], and observe that it also applies to semigroups. We take the alternative approach of deducing it from the corresponding statement for groups.

Lemma 4.2. *Every semigroup of elliptic automorphisms of a tree fixes either a vertex, inverts an edge, or fixes an end of the tree.*

Proof. The set of hyperbolic automorphisms of the tree is open because, given a hyperbolic automorphism x, any automorphism that agrees with x on two adjacent vertices on its axis is also hyperbolic, by Lemma 4.1. Hence the set of elliptic automorphisms of the tree is closed and it may be assumed that the given semigroup is closed. Since the closed semigroup generated by a single elliptic automorphism is compact, it is a group. Hence the given closed semigroup is in fact a group and Proposition 3.4 in [15] applies. □

The above proof in fact shows more than is asserted in the lemma.

Corollary 4.3. *Every closed semigroup of elliptic automorphisms of a tree is in fact a group. If this group does not fix an end of the tree, then it is compact.*

Elliptic elements normalize the stabilizer of a point of the tree, which is a compact, open subgroup. Hence the scale function of a closed subgroup, G, of the automorphism group of a locally finite tree takes the value 1 at each elliptic element. Although hyperbolic elements in G can also have scale 1, this does not occur under the hypothesis that G acts 2-transitively on ∂T, as the next result shows.

Proposition 4.4. *Let T be a tree that has no leaves and for which there is an edge $\{e, \bar{e}\}$ such that each component of $T \setminus \{e, \bar{e}\}$ has at least two ends. Let G be a group of automorphisms of T acting 2-transitively on ∂T. Then*:

(1) *for every distinct pair $\omega_1, \omega_2 \in \partial T$ the geodesic $]\omega_1, \omega_2[$ in T is the axis of some hyperbolic element in G and, moreover, a suitable conjugate of any hyperbolic element will have axis $]\omega_1, \omega_2[$*;

(2) *if v is a vertex with valency greater than 2, then G_v acts transitively on the sphere with centre v and radius r for every $r \geq 1$*;

(3) *the group G has at most two orbits, say O and E, on the set of vertices of T that have valency greater than 2*; *denoting by k the minimal distance between distinct elements of $O \cup E$, every vertex in T is within k of O and E*;

(4) *the vertices of every geodesic $]\omega_1, \omega_2[= (\dots, v_j, v_{j+1}, \dots)$ in T satisfy that $v_j \in E$ for some $j \in \mathbb{Z}$ and that*: *$v_{j+nk} \in E$ for all even n*; *$v_{j+nk} \in O$ for all odd n*; *and v_i has valency 2 otherwise*;

(5) *every hyperbolic element $h \in G$ translates $]\epsilon_-(h), \epsilon_+(h)[$ through a distance nk for some $n \in \mathbb{Z}$ and, if G is closed in* Aut(T)*, the scale function of G is given by*

$$s(g) = (q_O q_E)^{\ell(g)/2k} \tag{12}$$

where $q_E + 1$ and $q_O + 1$ are the valencies of vertices in E and O respectively;

(6) *for each end $\omega \in \partial T$ the group G_ω does not preserve any non-empty proper subtree.*

Proof. (1) By [2, Proposition 7.2], if G does not contain any hyperbolic element, then it fixes a vertex, inverts a a unique edge or fixes an end of T. The last is ruled out by the 2-transitivity of G on ∂T. It may be seen that G cannot fix a vertex either. For this, assume that v is fixed by G and let $\{f, \bar{f}\}$ be the edge adjacent to v that is closest to the edge $\{e, \bar{e}\}$ in the statement of the proposition. Then the component of $T \setminus \{f, \bar{f}\}$ containing v has at least one end, ω, because T has no leaves, and the component of $T \setminus \{f, \bar{f}\}$ to which v does not belong contains a component of $T \setminus \{e, \bar{e}\}$ and has at least two ends, ω_1 and ω_2. Then no element of G can fix ω_1 and move ω to ω_2, which contradicts the 2-transitivity of G on ∂T. Similarly, assuming that $\{f, \bar{f}\}$ is the unique edge inverted by G, at least one component of $T \setminus \{f, \bar{f}\}$ has two or more ends ω_1 and ω_2 and the other has at least one end ω. No element of G can fix ω_1 and send ω to ω_2, which again contradicts the 2-transitivity of G. Assuming that G contains only elliptic elements thus leads to a contradiction. Hence G contains a hyperbolic element, h say.

Since G is 2-transitive on ∂T, there is $g \in G$ that sends $\epsilon_+(h)$ to ω_1 and $\epsilon_-(h)$ to ω_2. Then ghg^{-1} is hyperbolic and $]\omega_1, \omega_2[$ is its axis.

(2) Let w_1 and w_2 be two vertices on the sphere with centre v and radius r. Then the 'hyperbolic triangle' with vertices v, w_1 and w_2 contains at most two edges incident on v and so there is an edge, $\{e, \bar{e}\}$ say, incident on v and not in this triangle. Choose rays $\mathfrak{r}$, $\mathfrak{r}_1$ and $\mathfrak{r}_2$ originating at v and passing through e, w_1 and w_2 respectively, and let ω, ω_1 and ω_2 be the corresponding ends. Since G acts 2-transitively on ∂T, there is $g \in G$ such that $g.\omega = \omega$ and $g.\omega_1 = \omega_2$. Then $g \in G_v$ and $g.w_1 = w_2$. Hence G_v acts transitively on the sphere as claimed.

(3) Let $w_0 \neq v_0$ be two vertices with valency greater than 2. That there is at least one such vertex, v_0 say, follows because T has at least three ends, and that there is a second follows because every vertex in $G.v_0 \ni h.v_0$ has this property. Because of (2), every vertex on the sphere with centre v_0 and radius $d(v_0, w_0)$ belongs to the orbit $G_{v_0}.w_0$ and every vertex on the sphere with centre w_0 and radius $d(v_0, w_0)$ belongs to the orbit $G_{w_0}.v_0$. It follows that all spheres with centre v_0 and radius $2nd(v_0, w_0)$, $n \geq 1$, are contained in the orbit $G.v_0$. Therefore every vertex is within distance $d(v_0, w_0)$ of $G.v_0$.

Choose v_0 and w_0 above so that $d(v_0, w_0)$ is minimized and let k be this minimum value. Let E be the set of vertices at distance nk from v_0 with n even and O be the set of vertices at distance nk with n odd. Then, as seen above, every vertex is within distance k of E and O, and $E \subseteq G.v_0$ and $O \subseteq G.w_0$.

(4) It will be shown first that there is $j \in \mathbb{Z}$ such that v_j belongs to E. Given a vertex u in $]\omega_1, \omega_2[$, there is $w' \in O$ within distance k from u. Hence u is on or inside the sphere with centre w' and radius k. This sphere is contained in E and $]\omega_1, \omega_2[$ must pass through it in order to escape to ω_1. Therefore there is a vertex of $]\omega_1, \omega_2[$ that belongs to E. All other claims follow immediately.

(5) The claim about translation length follows from the description of geodesics in (4). Since each of E and O is contained in a single G-orbit, all vertices in these sets have the same valency, which is denoted $q_E + 1$ and $q_O + 1$ as stated. If h translates through nk with n odd, then E and O are both contained in the same G-orbit and $q_E = q_O = q$. In this case each segment $]w, h.w]$, with w on the axis of h, contains exactly n vertices with valency $q+1 > 2$ and (11) yields the claimed value of $s(h)$. If h translates through nk with n even, then q_E and q_O may differ but each segment $]w, h.w]$, with w on the axis of h, contains exactly $n/2$ vertices with each valency $q_E + 1$ and $q_O + 1$ and (11) applies again.

(6) Every edge $\{e, \bar{e}\}$ lies on a geodesic $]\omega, \omega'[$ for some $\omega' \in \partial T$. Hence, by (1), $\{e, \bar{e}\}$ lies on the axis of a hyperbolic element in G_ω. Every such axis is contained any tree invariant under G_ω. $\square$

Remark 4.5. It has in fact been seen in the proof of Proposition 4.4 that the existence of a group of automorphisms of T acting 2-transitively on ∂T implies, under a weak hypothesis on T otherwise, that T may be obtained from a semi-homogeneous tree by subdividing each edge into k edges. This conclusion is similar to that of [7, Théorème 1.6.1].

Therefore for the remainder of this section the tree T may be assumed, without any loss of generality on the hypotheses of Proposition 4.4, to be semi-homogeneous with all vertices having valency at least 3.

It follows immediately from Proposition 4.4(4) that hyperbolic elements g and h satisfy $s(gh) = s(g)s(h)$ if and only if their translation lengths add, that is, $\ell(gh) = \ell(g) + \ell(h)$, and similarly if the scale of the product is less than (or greater than) the product of the scales. The next few results, which apply to all trees, describe the circumstances in which translation lengths of hyperbolic elements add.

Lemma 4.6. *Let t_1 and t_2 be hyperbolic automorphisms of a tree whose axes do not have any edge in common. Let $\mathfrak{d}$ the unique shortest path connecting the axes of t_1 and t_2, and let d be the number of edges of $\mathfrak{d}$. Then the following are true.*

(1) *The product t_2t_1 is hyperbolic and $\ell(t_2t_1) = \ell(t_2) + \ell(t_1) + 2d$.*

(2) *The axis of t_2t_1 contains the union of: $t_1^{-1}(\mathfrak{d})$; the last $\ell(t_1)$ edges preceding $\mathfrak{d}$ on the axis of t_1; $\mathfrak{d}$; the first $\ell(t_2)$ edges succeeding $\mathfrak{d}$ on the axis of t_2; and $t_2(\mathfrak{d})$.*

(3) *The direction of translation of t_2t_1 agrees with those of t_1 and t_2 on the common segments of their axes, and is in the direction from the axis of t_1 to that of t_2 on $\mathfrak{d}$.*

Proof. Let a be the vertex common to both $\mathfrak{d}$ and the axis of t_1, and b the vertex common to $\mathfrak{d}$ and the axis of t_2. The automorphism t_2t_1 maps $t_1^{-1}.\mathfrak{d}$ to $t_2.\mathfrak{d}$. Since $t_1^{-1}.a$ lies on the axis of t_1 and $t_2.b$ lies on the axis of t_2, the subtree spanned by $t_1^{-1}.\mathfrak{d}$ and $t_2.\mathfrak{d}$ is the path from $t_1^{-1}.b$ to $t_2.a$ of length $\ell(t_1)+\ell(t_2)+3d$, as shown in Figure 1.

Since each edge $e \in t_1^{-1}.\mathfrak{d}$ is coherent with $(t_2t_1).e \in t_2.\mathfrak{d}$, statements (1)–(3) follow by applying Lemma 4.1 and computing the distance from $t_1^{-1}.b$ to $t_2.b$. □

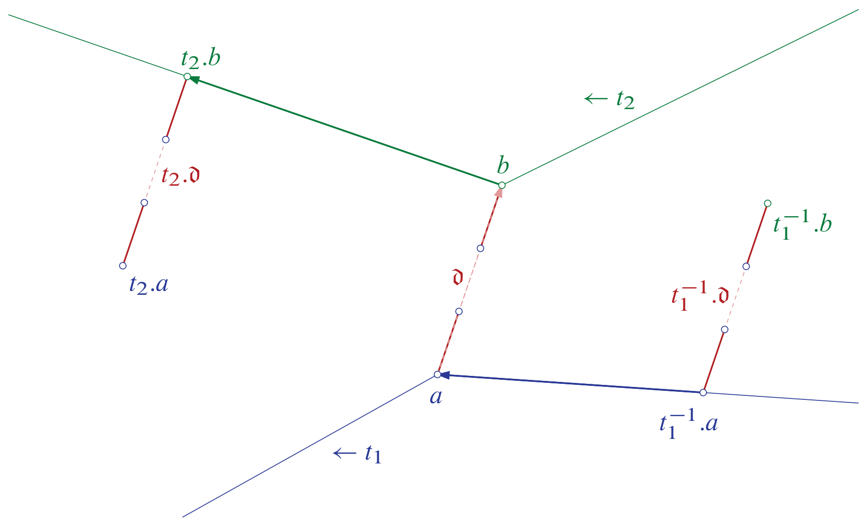

Figure 1. The axes of t_1 and t_2 do not intersect in an edge.

Lemma 4.7. *Let t_1 and t_2 be hyperbolic automorphisms of a tree whose axes intersect in a nontrivial path $\mathfrak{c}$ along which t_1 and t_2 translate in the same direction. Then*

(1) *the product t_2t_1 is hyperbolic and the direction of translation of t_2t_1 agrees with those of t_1 and t_2 on the common segments of their axes.*

(2) *the axis of t_2t_1 contains the union of: the edges in $\mathfrak{c}$, the last $\ell(t_1)$ edges preceding $\mathfrak{c}$ on the axis of t_1 and the first $\ell(t_2)$ edges succeeding $\mathfrak{c}$ on the axis of t_2.*

(3) $\ell(t_2t_1) = \ell(t_2) + \ell(t_1)$.

Proof. If e is any edge on $\mathfrak{c}$, then $t_1^{-1}.e$, e and $t_2.e$ are all coherent because t_1 and t_2 translate in the same direction and the coherence relation on edges is transitive. Parts (1) –(3) then follow by applying Lemma 4.1. □

Lemma 4.8. *Let t_1 and t_2 be hyperbolic automorphisms of a tree whose axes intersect in a nontrivial path $\mathfrak{c}$ along which t_1 and t_2 translate in opposite directions.*

Then the product t_2t_1 may be hyperbolic or elliptic and $\ell(t_2t_1) < \ell(t_2) + \ell(t_1)$.

Proof. Let $e = (v, w)$ be an edge on $\mathfrak{c}$ and suppose that t_1 translates along (v, w) and t_2 along (w, v). Then t_2t_1 maps the vertex $t_1^{-1}.w$ to $t_2.w$, and the path from $t_1^{-1}.w$ to w and then to $t_2.w$ includes the reversal from v to w and back to v. Hence $\ell(t_2t_1)$ is at most $d(t_1^{-1}.w, v) + d(v, t_2.w) = \ell(t_1) + \ell(t_2) - 2$. □

Lemmas proved up to this point allow the pairs of elements belonging to a semigroup on which translation distances add to be characterized in terms of their minimizing sets.

Lemma 4.9. *If two automorphisms of a locally finite tree are contained in a semigroup on which the translation length is an additive function, then their minimal sets intersect non-trivially.*

Proof. If both automorphisms are elliptic the claim follows from Lemma 4.2. When both both automorphisms are hyperbolic, it follows from Lemma 4.6.

If one of the automorphisms, h say, is hyperbolic while the other, r say, is elliptic, we give a contrapositive argument as follows.

By additivity, both h and rh are hyperbolic and of the same translation length. Suppose the minimal sets of r and h have trivial intersection. Then r does not fix any point of the axis of h. Thus the axes of h and rh do not intersect. By Lemma 4.6(4) we conclude that $\ell(rh) > \ell(r) + \ell(h)$, hence that r and h do not lie in a common additive semigroup and the claim is verified. □

Certain s-multiplicative semigroups in G (acting 2-transitively on ∂T) are described next in preparation for showing, in Theorem 4.11, that every such semigroup is contained in one of these. Additional notation is required for the characterization of the s-multiplicative semigroups. For each $v \in V(T)$, set

$$E(v) = \{e \in E(T) : t(e) = v\}$$

to be the set of edges incident on and directed to v. For each proper non-empty subset, $I \subset E(v)$, put

$$I^* = \{\bar{e} : e \in E(v) \setminus I\},$$

a set of edges incident on and directed away from v. We shall also write

$$G_{(v)} = \{g \in G : v \in \min(g)\},$$

the set of elements of G that fix v if they are elliptic or such that their axis passes through v if they are hyperbolic. If g is in $G_{(v)}$ and is hyperbolic, and if there are $e_1 \in I$ and $e_2 \in I^*$ on the axis of g and such that g translates in the direction of e_1 and e_2, we shall say that *g translates in through I and out through I^*.*

Proposition 4.10. *Let T be a semi-homogeneous tree in which each vertex has valency at least three. Let G be a group of automorphisms of the tree T that acts 2-transitively on ∂T. For v a vertex in T and I a proper, non-empty subset of $E(v)$ put*

$$\begin{aligned} G_{(v,I)} := \{g \in G_{(v)} \colon & g.I = I \text{ if } g \text{ is elliptic, and} \\ & g \text{ translates in through } I \text{ and out through } I^* \text{ if } g \text{ is hyperbolic}\} \end{aligned}$$

For an end ω of the tree put

$$G_{\omega+} := \{g \in G_\omega \colon g \text{ is elliptic, or } g \text{ is hyperbolic and } \omega = \epsilon_+(g)\}$$

and

$$G_{\omega-} := \{g \in G_\omega \colon g \text{ is elliptic, or } g \text{ is hyperbolic and } \omega = \epsilon_-(g)\}\,.$$

Then $G_{(v,I)}$, $G_{\omega+}$, and $G_{\omega-}$ are s-multiplicative subsemigroups of G. Moreover $G_{(v,I)} \subseteq G_{(w,J)}$ implies $G_{(v,I)} = G_{(w,J)}$ and either $(v,I) = (w,J)$ or v and w are the two endpoints of the same edge $\{e, \overline{e}\}$ and $\overline{I}^* = \{e\} = J$.

Proof. As was seen in Proposition 4.4, elements $g, h \in G$ satisfy $s(gh) = s(g)s(h)$ if and only if $\ell(gh) = \ell(g) + \ell(h)$, and the proof below refers to lemmas about additivity of translation length for statements about multiplicativity of the scale.

We begin by showing that $G_{(v,I)}$ is s-multiplicative and closed under multiplication. Consider $g, h \in G_{(v,I)}$. If both g and h are elliptic and stabilise I, then the product will again be elliptic and stabilise I, and hence back in $G_{(v,I)}$. If both g and h are hyperbolic, translating in through I and out through I^*, then we are either in the case of Lemma 4.6 with $d = 0$ or Lemma 4.7. Either way, $s(gh) = s(g)s(h)$ as required. Hence it remains only to prove that a product of an elliptic element and a hyperbolic element in $G_{(v,I)}$ is again back in $G_{(v,I)}$. Suppose g is elliptic stabilizing I and h is hyperbolic translating in through $e_1 \in I$ and out through $e_2 \in I^*$. Consider first hg. Then $g^{-1}.e_1 \in I$ and $hg.(g^{-1}.e_1) = h.e_1$ is coherent with and different from $g^{-1}.e_1$ because h is hyperbolic. Hence, by Lemma 4.1, hg is hyperbolic and the edges $g^{-1}.e_1$ and $h.e_1$ lie on the axis of hg. The path between these two edges contains v and $e_2 \in I^*$. Hence $hg \in G_{(v,I)}$. Moreover, $\ell(hg) = \ell(h)$ and hence $s(hg) = s(h) = s(h)s(g)$. Finally, consider gh. A similar argument shows that e_1 and $gh.e_1$ are coherent and different, proving that gh is hyperbolic by Lemma 4.1. Since $g.e_2 \in I^*$ is on the axis of gh, we conclude that $gh \in G_{(v,I)}$. Moreover, $\ell(gh) = \ell(h)$ and hence $s(gh) = s(h) = s(g)s(h)$. Hence $G_{(v,I)}$ is an s-multiplicative subsemigroup of G. The supplementary statement on the semigroups $G_{(v,I)}$ will we shown at the end to this proof.

Consider now $G_{\omega+}$. Given two hyperbolic elements in $G_{\omega+}$, their axes intersect in a ray in the equivalence class ω. Hence they satisfy the hypotheses of Lemma 4.7 and their product is again a hyperbolic element in $G_{\omega+}$ and the scale of the product is the product of their scales. All elements in $G_{\omega+}$ fix ω. Hence, given an elliptic element g and a hyperbolic element h in $G_{\omega+}$, there is some ray $\mathfrak{r}$ in ω that is fixed by g and translated towards ω by h. Hence $\mathfrak{r}$ is translated by both gh and hg, and $\ell(gh) = \ell(h) = \ell(hg)$ as required.

Since $G_{\omega-}$=$(G_{\omega+})^{-1}$, the result for $G_{\omega-}$ follows from the result for $G_{\omega+}$ and Proposition 2.9.

Towards proving the supplementary statement on the semigroups $G_{(v,I)}$, we now show the auxiliary result that for any pair of coherently oriented edges e_1 and e_2 of T, there is a hyperbolic element in G that translates in the direction of both e_1 and e_2. Choose a vertex m on the geodesic containing e_1 and e_2. There are ends ω_- and ω_+ of T such that e_1 lies on the ray $]\omega_1, m]$ and e_2 lies on the ray $[m, \omega_2[$. By Proposition 4.4(1), there is a hyperbolic element $h \in G$ with axis $]\omega_1, \omega_2[$. It may be supposed that h translates in the direction of e_1 and thus also in the direction of e_2, showing our auxiliary result.

Assume now $G_{(v,I)} \subseteq G_{(w,J)}$. Inverting $G_{(v,I)}$ and $G_{(w,J)}$ if necessary, we may suppose that the first oriented edge on $[v, w]$, $e = (v, v_1)$ say, is in $\overline{I}^*$.

We claim that $d(v, w)$ is at most 1. Choose an oriented edge e_1 with terminal vertex v and an oriented edge e_2 with initial vertex v_1 and terminal vertex not on the geodesic $[v, w]$. Since the edges e_1 and e_2 are coherently oriented, by our auxiliary result, there is a hyperbolic element, h say, that translates in the direction of both e_1 and e_2. Evidently the element h is in $G_{(v,I)}$ but not in $G_{(w,J)}$ if $d(v, w) > 1$. We conclude that indeed $d(v, w) \leq 1$.

Hence we either have $v = w$ or $e = (v, w)$ is an oriented edge.

In the first case, apply our auxiliary result to pairs of oriented edges $e_1 \in I$ and $e_2 \in I^*$ to deduce $I \subseteq J$ and $I^* \subseteq J^*$ from our assumption $G_{(v,I)} \subseteq G_{(w,J)}$, obtaining the sought-after supplementary statement on the semigroups $G_{(v,I)}$.

In the second case, $e = (v, v_1) = (v, w)$, which is in $\overline{I}^*$. If there were another oriented edge, say f, in $\overline{I}^*$, we could use our auxiliary result above with an arbitrary $e_1 \in I$ and $e_2 := f$ to obtain a hyperbolic element in $G_{(v,I)}$ that does not belong to $G_{(w,J)}$, a contradiction. Since $e \in \overline{I}^*$ and $G_{(v,I)} \subseteq G_{(w,J)}$, we have $e \in J$ as well. It remains to show that e is the only edge in J also, since the equality $G_{(v,I)} = G_{(w,J)}$ is a consequence.

The assumption that there be another oriented edge, l say, different from e in J implies, applying our auxiliary result to the coherently oriented edges e and $\overline{l}$, that there is a hyperbolic element $h \in G$ translating in the direction of e and $\overline{l}$. This element h belongs to $G_{(v,I)}$ because e is the only edge in $\overline{I}^*$. But that would mean that it belongs to $G_{(w,J)} \supseteq G_{(v,I)}$ as well, in contradiction to our assumption $l \in J$. The proof is complete. □

We finally come to the description of the maximal s-multiplicative semigroups in G.

Theorem 4.11. *Let T be a semi-homogeneous tree in which each vertex has valency at least three. Suppose that G is a closed subgroup of the automorphism group of T that acts 2-transitively on ∂T. Then every s-multiplicative semigroup in G is contained in one of the following types, which are hence maximal.*

(1) *The fixator of the midpoint of an edge that is inverted by G.*

(2) *The fixator of a vertex.*

(3) *A set of the form $G_{(v,I)}$ as defined in Proposition 4.10, with $I \subset E(v)$ proper and not empty.*

(4) *A set of the form $G_{\omega+}$ or $G_{\omega-}$ for an end ω as defined in Proposition 4.10.*

Proof. Let $\mathcal{S}$ be a s-multiplicative semigroup in G. We will show that $\mathcal{S}$ is contained in one of the s-multiplicative semigroups listed. For each subset, $\mathcal{F}$, of $\mathcal{S}$ put $\min(\mathcal{F}) = \bigcap_{g\in\mathcal{F}} \min(g)$. Then, since $\min(g)$ is a subtree and the intersection of subtrees is a subtree, Lemma 4.9 and [14, Lemma 10 in section 6.5] show that $\min(\mathcal{F}) \neq \emptyset$ for every finite $\mathcal{F}$.

Suppose first that $\min(\mathcal{S}) = \emptyset$. Then for each edge e, there is a finite subset $\mathcal{F} \subset \mathcal{S}$ such that $\min(\mathcal{F})$ is contained in one of the semitrees obtained by deleting $\{e, \bar{e}\}$. Otherwise, e would be in $\min(\mathcal{F})$ for every $\mathcal{F}$ and therefore in $\min(\mathcal{S})$. Hence there is a unique end, ω, of the tree such that every subtree $\min(\mathcal{F})$ contains a ray belonging to ω. It follows that each element of $\mathcal{S}$ is either elliptic and fixes a ray in ω or is hyperbolic and translates a ray in ω. By Lemma 4.8, no two hyperbolic elements can translate in opposite directions and so we are in case (4). Note that there do exist hyperbolic elements in G_ω, by Proposition 4.4(1), and so $G_{\omega+}$ and $G_{\omega-}$ are distinct semigroups, and the 2-transitivity of G on ∂T ensures, by Proposition 4.4 (1), that $\min(\mathcal{S})$ is indeed empty.

Suppose next that $\mathcal{S}$ contains an edge inversion g. Then $\min(g)$ is a singleton, namely, the midpoint, p, of the edge. Since, by Lemma 4.9, $\min(h)$ intersects $\{p\}$ for every $h \in \mathcal{S}$, it follows that $\{p\} = \min(\mathcal{S})$ and we are in case (1).

It remains to treat the case when $\min(\mathcal{S})$ is not empty and, since $\mathcal{S}$ does not invert any edge, contains a vertex v. If $\mathcal{S}$ contains only elliptic elements, then v is fixed by all elements of $\mathcal{S}$ and we are in case (2). Note that $\min(\mathcal{S})$ cannot be larger than $\{v\}$ in this case by Proposition 4.4 (2). If $\mathcal{S}$ contains hyperbolic elements, then the axis of each one passes through v. Put

$$I = \{(w, v)\colon \text{there is } h \in \mathcal{S} \text{ that translates along } (w, v)\}.$$

Then $I \neq \emptyset$ and, since any hyperbolic $h \in \mathcal{S}$ translates out along some edge f and $\bar{f}$ then cannot belong to I by Lemma 4.8, we also have that I is a proper subset of $E(v)$. Hence we are in case (3). Note that, in this case, 2-transitivity of the action of G on ∂T ensures, by Proposition 4.4 (1), that for every $e_1 \in I$ and $e_2 \in I^*$ there is a hyperbolic $h \in \mathcal{S}$ that translates along e_1 and e_2. Hence $\min(\mathcal{S}) = \{v\}$ unless I or I^* consists of a single edge e, in which case $\min(\mathcal{S}) = \{e, \bar{e}\}$.

To show that each of the sets listed is indeed a maximal s-multiplicative semigroup it suffices to show that none is contained in any of the others.

CASE (1). The semigroup G_p fixing the midpoint, p, of an edge cannot be contained in any semigroup G_v, $G_{(v,I)}$, $G_{\omega+}$ or $G_{\omega-}$ because G_p contains an inversion and the other semigroups do not. Hence G_p is maximal. (This case only occurs if G contains an inversion.)

CASE (2). The semigroup G_v cannot be contained in G_p, $G_{\omega+}$ or $G_{\omega-}$ because, if it were, G_v would fix a midpoint of an edge or an end in addition to v, which it does not. Similarly, if G_v were contained in $G_{(w,I)}$ for some w and I, then w would have to equal v but G_v is transitive on $E(v)$ while $G_{(v,I)}$ is not. Hence G_v is not contained in any other semigroup and is maximal.

CASE (3). The semigroup $G_{(v,I)}$ contains hyperbolic automorphisms of T and so is not contained in G_p or G_v for any midpoint p or vertex v. It is not contained in $G_{\omega+}$ or $G_{\omega-}$ for any $\omega \in \partial T$ because it does not fix ω. Hence $G_{(v,I)}$ is maximal.

CASE (4): The semigroups $G_{\omega+}$ and $G_{\omega-}$ cannot be contained in any of the others because $\min(G_{\omega+})$ and $\min(G_{\omega-})$ are empty while the minimizing sets of the other semigroups are not. Hence $G_{\omega+}$ and $G_{\omega-}$ are maximal. □

Remark 4.12.

(1) The maximal s-multiplicative semigroups of type (1)–(3) in Theorem 4.11 are open because they contain an open subgroup of G, while the maximal s-multiplicative semigroups of type (4) are not.

(2) The semigroups of type (3), despite being open, do not contain subgroups tidy for their hyperbolic elements unless either $|I| = 1$ or $|I^*| = 1$.

(3) The inverse map on maximal s-multiplicative semigroups, see Proposition 2.9, satisfies $G_{\omega+}^{-1} = G_{\omega-}$ for every $\omega \in \partial T$ and $G_{(v,I)}^{-1} = G_{(v,J)}$, where $J = \bar{I}^*$, for every $v \in V(T)$ and $I \subset E(v)$.

(4) The semigroup $G_{(v,I)} \cap G_{\omega+}$ is not uniscalar if and only if the initial edge in the path from v to ω belongs to I^*. The maximal s-multiplicative multiplicative semigroups of type (3) thus identify sets of ends that form a basis for the ends topology.

Remark 4.13. The condition that G act 2-transitively on ∂T is required at two places in the characterisation of the maximal s-multiplicative semigroups in G. The first is to derive the formula (12) for the scale found in Proposition 4.4, and the second is to guarantee that distinct sets $I, J \subset E(v)$ determine distinct semigroups $G_{(v,I)}$ and $G_{(v,J)}$.

Formula (12) relates the scale of h to translation distance but a formally weaker and more complicated condition suffices for this. It is enough to know, for each vertex v, that for every hyperbolic h with $v \in \min(h)$, the valency of v in the minimal invariant subtree of $G_{\epsilon_+(h)}$ does not depend on h. (This holds, for example, if for every hyperbolic h this invariant subtree is the whole tree.) However, it is possible that these weaker conditions, together with the hypothesis that G does not fix an end, already imply that G acts 2-transitively on ∂T.

Remark 4.14. Theorem 4.11 indicates a limitation on the information about G provided by its maximal s-multiplicative semigroups. The hypotheses are satisfied by both $\mathrm{Aut}(T_{p+1})$ and $\mathrm{PGL}(2, \mathbb{Q}_p)$ acting on T_{p+1}, which is its Bruhat-Tits tree (see [14, chapter II]), and the map $\mathcal{S} \mapsto \mathcal{S} \cap \mathrm{PGL}(2, \mathbb{Q}_p)$ is a bijection from the maximal s-multiplicative semigroups in $\mathrm{Aut}(T_{p+1})$ to those in $\mathrm{PGL}(2, \mathbb{Q}_p)$. Therefore any invariant constructed from maximal s-multiplicative semigroups will not distinguish between $\mathrm{PGL}(2, \mathbb{Q}_p)$ and $\mathrm{Aut}(T_{p+1})$.

The groups $\mathrm{PGL}(2, \mathbb{Q}_p)$ and $\mathrm{Aut}(T_{p+1})$ are distinguished however by the *local structure theory* studied in [6]. The information carried by s-multiplicative semigroups is global and complements this local theory.

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