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Abstract

Engineering-based edge detection techniques generally use local intensity information to identify whether a pixel location is part of a boundary. Boundaries are presumed present where sharp transitions in the observed intensities occur. Unfortunately, these approaches are sensitive to error and hidden partial boundaries, which hinders the determination of closed object boundaries. In this research, a method to obtain statistically optimal closed object boundaries is presented.

Keywords

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STOCHASTIC RECOGNITION OF CLOSED OBJECT BOUNDARIES IN IMAGES

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ABSTRACT

Engineering-based edge detection techniques generally use local intensity information to identify whether a pixel location is part of a boundary. Boundaries are presumed present where sharp transitions in the observed intensities occur. Unfortunately, these approaches are sensitive to error and hidden partial boundaries, which hinders the determination of closed object boundaries. In this research, a method to obtain statistically optimal closed object boundaries is presented.

1. INTRODUCTION

Statistical methods have been developed for the image restoration problem and often use Markov random field methodology. Geman and Geman¹ were among the first to take a Bayesian approach and consider Markov random fields as prior models for obtaining restored images. For an image with $N = n \times m$ pixels, the goal of restoration is to determine the underlying true pixel intensities at the N pixel locations. Thus, there are N unknown (unobserved) parameters to be estimated and N observations available for estimation. A statistical approach places a (joint) prior probability measure on the pixel locations of the image. This probability measure is often specified by q unknown parameters, yielding $N + q$ unknown parameters to be estimated. When the ratio of available data to unknown parameters is small, adopting a Bayesian approach to incorporate additional information is a necessity. Due to the large number of parameters, Monte Carlo Markov chain algorithms have been proposed for multi-parameter estimation in image models.

Though most of the statistical imaging literature has concentrated on image restoration, recent articles have appeared considering statistical approaches to the boundary identification problem. Object boundaries are important features in themselves and are also important in identifying gross nonstationarities in the image prior to considering the stochastic restoration techniques referred to above. Geman and Geman first considered the inclusion of an edge (boundary) process in the image restoration problem. Of particular interest is the identification of closed object boundaries in gray-scale images. Geman *et al.*² consider closed boundary detection by constrained optimization, where they choose to optimize according to a single criterion that combines the local detection of edges with their pruning, lining, and smoothing. More specifically, constraints are added to their Markov random field approach in order to disallow boundaries that suddenly stop.

Geman *et al.* incorporate the closed boundary constraints into their Monte Carlo Markov chain algorithm by gradually penalizing non closed boundaries more and more at each step of the algorithm. Theoretical results guarantee convergence of the constrained optimization algorithm to the optimal closed boundary configuration. Unfortunately, approximations to the annealing schedule (see Section 6) that ensures convergence are necessary and Geman *et al.* note that, in their examples, their algorithm never actually obtains a final boundary estimate that is indeed closed. For practical applications, such as automatic target recognition, it is important that boundary closure is guaranteed.

The underlying idea of this research is not to allow the current boundary estimate at any given stage of the Monte Carlo Markov chain algorithm to leave the appropriate parameter space, namely, the space of closed boundary configurations. This is in contrast to the Geman *et al.* approach. By defining a prior probability model on the space of closed boundary configurations and appropriately specifying transition probability functions on this space, an algorithm is constructed such that any approximation to the statistically optimal boundary estimate will have the necessary property of closure.

2. PROBLEM SPECIFICATION

2.1. Notation and Definitions

Let D denote the $n \times m$ lattice of pixel locations. Let $Y(s)$ denote the observed intensity at pixel location $s \equiv (s^1, s^2) \in D$ and assume there are a discrete number of possible intensities; i.e., $Y(s) \in \{1, \dots, k\}$, where k is a finite integer. Arbitrarily, the pixel location $(1, 1)$ is selected to be the upper left pixel in the lattice; then $s \equiv (s^1, s^2)$ denotes the center of a pixel with the first index indicating the center of the row s^1 and the second index indicating the center of the column s^2 .

Next a dual lattice for D is defined, which will be called the edge lattice E . Define $u \equiv (u^1, u^2)$ and

$$E \equiv \{u: u = s + (.5, .5) \text{ and } s \in D\}, \quad (1)$$

the (dual) lattice of potential edge sites. Define $W(u)$ to be the value of the boundary process at location $u \in E$, with $W(u) = 1$ if u is part of a boundary and $W(u) = 0$ otherwise. When $W(u) = 1$, the edge site u is referred to as being lit. Noise in the image means that typically the process $W(\cdot)$ is unobserved.

Figure 2.1. The squares denote pixels and the dots are the pixel locations in the 8×8 lattice D . The shaded circles correspond to sites on the dual edge lattice E .

Let Ω_E denote the set of all possible boundary configurations; the subscript E emphasizes that Ω_E is the set of all combinations of zero (not lit) and one (lit) on E . Thus the cardinality of Ω_E , denoted $|\Omega_E|$, is 2^{nm} .

Consider $u_0 \in E$ and define

$$N_r(u_0) \equiv \left\{ u \in E: u \neq u_0, (u^1 - u_0^1)^2 + (u^2 - u_0^2)^2 \leq r \right\}; \quad r = 1, 2, \dots \quad (2)$$

The set $N_r(u_0)$ is simply the neighboring potential edge sites within “checkerboard distance” r of u_0 .

Definition 2.1: Let $u_0 \in E$. The four-neighborhood or, equivalently, the set of four-neighbors of u_0 is $N_1(u_0)$. The eight-neighborhood or set of eight-neighbors of u_0 is $N_2(u_0)$.

Definition 2.2: Define $N_E(u_0) \equiv \{u \in E: u \in N_2(u_0) \text{ and } W(u) = 1\}$. This set is referred to as the eight-neighbor edge set for u_0 .

Definition 2.3: A four-path [eight-path] is a sequence of sites $u_1, u_2, \dots, u_k \in E$ such that $u_{i+1} \in N_1(u_i)$ [$u_{i+1} \in N_2(u_i)$], $i = 1, \dots, k-1$.

Definition 2.4: Two lit sites u_a and u_b are four-connected [eight-connected] in E if there exists a four-path [eight-path] $u_1, u_2, \dots, u_k$ of lit sites in E , such that $u_1 = u_a$, $u_k = u_b$.

Definition 2.5: Two lit sites u_a and u_b are said to be eight-connected but not four-connected in E if there exists an eight-path $u_1, u_2, \dots, u_k$, of lit sites in E , such that $u_1 = u_a$, $u_k = u_b$, and either, $u_{i+1} \notin N_1(u_i)$, or $u_{i-1} \notin N_1(u_i)$ for each $i = 2, \dots, k-1$.

Definition 2.6: A set $U \subseteq E$ is four-connected [eight-connected] in E if for every pair $u_a, u_b \in U$, u_a is four-connected [eight-connected] to u_b .

Definition 2.7: A set $U \subseteq E$ is eight-connected but not four-connected in E if for every pair $u_a, u_b \in U$, u_a is eight-connected but not four-connected to u_b .

Figure 2.2. Two ways to interpret three lit sites in this edge site configuration.

Definition 2.8: A boundary configuration $\omega \in \Omega_E$ is said to be closed if, for any $u_0 \in E$ such that $W(u_0, \omega) = 1$, $|N_E(u_0, \omega)| \geq 2$. The extra argument ω of W and N_E is included here to indicate clearly the dependence on the boundary configuration being considered. The boundary configuration space $\Omega_E^{CL} \equiv \{\omega \in \Omega_E : \omega \text{ is closed}\}$ is called the set of closed boundary configurations.

Definition 2.9: When $W(u_0) = 1$, the local boundary configuration $N_E(u_0)$ is said to be allowable if for each pair $u_1, u_2 \in N_E(u_0)$, $u_1 \notin N_E(u_2)$. These local configurations are thus eight-connected but not four-connected. Put more simply, neighbors u_1 and u_2 of u_0 are not allowed to be vertically, horizontally, or diagonally adjacent.

Definition 2.10: A boundary configuration $\omega \in \Omega_E$ is said to be restricted eight-connected if, for any $u_0 \in E$ such that $W(u_0, \omega) = 1$, $|N_E(u_0, \omega)| \geq 2$ and $N_E(u_0, \omega)$ is allowable. The boundary configuration space $\Omega_E^P \equiv \{\omega \in \Omega_E : \omega \text{ is restricted eight-connected}\}$ is called the set of permissible boundary configurations.

Though each $\omega \in \Omega_E^P$ is a closed boundary, it is clear that Ω_E^P is not the set of all closed boundary configurations Ω_E^{CL} ; there is an additional condition of allowability for all neighborhoods $\{N_E(u, \omega) : W(u, \omega) = 1\}$. However, considering the set Ω_E^P reduces the complexity of the boundary identification problem. For example, suppose three sites are lit in the configuration depicted in Figure 2.2. The question arises as to whether this configuration represents a closed triangle or a sharp right turn in the boundary. Any ω that includes three lit sites as in Figure 2.2 is not in Ω_E^P . Thus, if a boundary configuration ω is in the space Ω_E^P , there is no need to waste time determining whether a pair of adjacent lit edge sites are directly connected by a boundary; from the definition of $\omega \in \Omega_E^P$ this question is always answered in the affirmative. Restricting attention to the space Ω_E^P is not a weakness of the proposed method and a generalization of the proposed algorithm to the space of closed boundaries, Ω_E^{CL} , is straightforward.⁴

2.2. Bayesian Approach to Closed Boundary Identification

The goal of this research is to identify the optimal (in a statistical sense) closed boundary configuration $\omega^* \in \Omega_E^P$ given the available information, including observed pixel intensities as well as prior knowledge regarding object boundaries in the scene of interest. The Bayesian paradigm is used to construct a probability measure on the configuration space Ω_E^P , given the pixel intensities. Only the permissible boundary configurations are given non-zero prior probability. It is important to note that placing a probability measure on Ω_E^P is equivalent to placing a joint probability measure on $\{W(u) : u \in E\}$, and vice-versa.

For $\omega \in \Omega_E^P$ and $y \equiv (y(s) : s \in D)$, let $\Pr(y | \omega) \equiv \Pr(Y(s) = y(s) : s \in D | \omega)$ denote the intensity probability mass function (pmf), where $\Pr(A | B)$ denotes the pmf of A given B . Also for $\omega \in \Omega_E^P$, denote the prior boundary pmf by $\Pr(\omega)$. The distribution of interest is $\Pr(\omega | Y(s) : s \in D)$; $\omega \in \Omega_E^P$, which is referred to as the posterior boundary pmf.

Theorem 2.1: (A Consequence of Bayes Theorem) $\Pr(\omega | Y(s) : s \in D) \propto \Pr(Y(s) : s \in D | \omega) \cdot \Pr(\omega)$; $\omega \in \Omega_E^P$.

Our goal is to obtain the boundary configuration $\omega^* \in \Omega_E^P$ that maximizes the posterior boundary pmf. Such a choice is equivalent to a Bayes procedure when considering a 0–1 loss function and is called the maximum-a-posteriori boundary.

Definition 2.11: A boundary configuration $\omega^* \in \Omega_E^P$ is a maximum-a-posteriori boundary if

$$\Pr(\omega^* | y) = \max_{\omega \in \Omega_E^P} \Pr(\omega | y). \quad (3)$$

3. THE INTENSITY MODEL

The interaction between a permissible boundary configuration ω and the observed intensity values is governed by the intensity pmf. Usually the goal of obtaining underlying closed boundaries in a gray-scale image is motivated by the desire to identify

objects. In image analysis, objects are often characterized by regions displaying considerable homogeneity. That is, in an uncorrupted image, one may identify objects as regions where the intensity value is constant. It thus makes sense to model the intensity pmf based on the premise that objects are defined as regions of homogeneity. Notice that here the interest is in two-dimensional objects in two-dimensional space. The method to be described will ignore, deliberately, points and lines with end points inside the image domain.

Let $X(s)$ denote the true underlying intensity at pixel location $s \in D$. It is assumed that $X(s)$ is a random field with realizations *constant within connected regions* of D .

Definition 3.1: A connected region is a subset of pixel locations in D that are four-connected.

In what follows, an additive Gaussian model is assumed for the observed intensities,

$$Y(s) = X(s) + \epsilon(s); \quad s \in D, \quad (4)$$

where $\epsilon(s) \sim \text{NID}(0, \sigma^2)$; $s \in D$ (NID is an abbreviation for “normally and independently distributed”), and the ϵ -process is independent of the X -process. Note that each $\omega \in \Omega_E^P$ partitions D into disjoint connected regions. Let $d_i(\omega)$ denote a generic connected region for a particular ω . The number of disjoint connected regions depends on ω and will be denoted by $K(\omega)$. Each $\omega \in \Omega_E^P$ has disjoint connected regions $\{d_i(\omega) : i = 1, \dots, K(\omega)\}$ in D , where it is assumed that $X(s)$ is constant on connected regions. That is, $X(s) = X(t)$ if $s, t \in d_i(\omega)$; $i = 1, \dots, K(\omega)$. Define $\mu_i(\omega) \equiv X(s)$; $s \in d_i(\omega)$. For $\omega \in \Omega_E^P$, it follows from (4) that

$$\Pr(Y(s) : s \in D | \omega) = \prod_{i=1}^{K(\omega)} \Pr(Y(s) : s \in d_i(\omega)), \quad (5)$$

where $Y(s) | \omega \sim \text{NID}(\mu_i(\omega), \sigma^2)$; $s \in d_i(\omega)$. Further, let $n_i(\omega)$ denote the number of distinct sites $s_{ij} \in d_i(\omega) \subseteq D$. Thus,

$$\Pr(Y(s) : s \in D | \omega) = \prod_{i=1}^{K(\omega)} \prod_{j=1}^{n_i(\omega)} (2\pi\sigma^2)^{-\frac{1}{2}} \exp\left\{-\frac{(Y(s_{ij}) - \mu_i(\omega))^2}{2\sigma^2}\right\}, \quad (6)$$

where $\{\mu_i(\omega) : i = 1, \dots, K(\omega)\}$ and $\sigma^2 > 0$ are parameters. Specification or estimation of the parameters is necessary.⁴

4. THE PRIOR BOUNDARY MODEL

4.1. Notation and Definitions

Definition 4.1: The set $\mathcal{N} \equiv \{\mathcal{N}(u) : u \in E\}$ is a neighborhood system for E if $\mathcal{N}$ is a collection of subsets of E for which 1) $u \notin \mathcal{N}(u)$ and 2) $u_1 \in \mathcal{N}(u_2) \Leftrightarrow u_2 \in \mathcal{N}(u_1)$. The set $\mathcal{N}(u)$ is called the set of neighbors of u .

Definition 4.2: A subset $C \subseteq E$ is defined to be a clique if every pair of distinct sites in C are neighbors or C consists of a single site.

In this research, the prior boundary pmf is modeled as a Markov random field with support on the configuration space Ω_E^P .

Definition 4.3: A random field $\{W(u) : u \in E\}$ is a Markov random field (MRF) on Ω_E^P with neighborhood system $\mathcal{N}$ if the joint probability mass function $\Pr(W(u) : u \in E)$ (or, equivalently, $\Pr(\omega)$; $\omega \in \Omega_E^P$) is positive for all $\omega \in \Omega_E^P$, and the conditional probability of $W(u_0)$, given the values of $\{W(u) : u \in E, u \neq u_0\}$, equals the conditional probability of $W(u_0)$, given only the values of $\{W(u) : u \in \mathcal{N}(u_0)\}$.

Definition 4.4: A Gibbs Distribution relative to $\{E, \mathcal{N}\}$ is a probability measure $\Pr(\omega)$ on Ω_E^P with the following representation:

$$\Pr(\omega) = \frac{1}{Z_0} \exp(-U_0(\omega)); \quad \omega \in \Omega_E^P, \quad (7)$$

where $U_0(\omega) \equiv \sum_{C \in \mathcal{C}} V_C(\omega)$, $\mathcal{C}$ denotes the set of cliques for the neighborhood system $\mathcal{N}$, and Z_0 is the partition function (or normalizing constant), defined by $Z_0 \equiv \sum_{\omega \in \Omega_E^P} \exp(-U_0(\omega))$. The function $U_0(\omega)$ is referred to as the energy function.

and $V_C(\omega)$ is called a potential function. Each V_C is a function on Ω_E^P with the property that $V_C(\omega)$ is a function of only those values of $W(u)$ for which $u \in C$.

The Hammersly-Clifford Theorem identifies a Markov random field-Gibbsian equivalence that can be used to provide a simple, practical way of specifying Markov random fields by specifying the values of potential functions.³

4.2. Specification of the Prior Closed Boundary Model

The selection of a good prior boundary model should be based on any information regarding the scene of interest or on intuition regarding the genesis or behavior of boundaries in images. In what follows, an example of a Markov random field prior pmf is described that penalizes permissible boundary configurations with many disjoint connected regions of homogeneity. More general prior models are presented in Helterbrand, Cressie, and Davidson.⁴

For the closed-boundary prior model, we use the homogenous neighborhood system $\mathcal{N}_8 \equiv \{N_8(u) : u \in E\}$, where $N_r(u)$ is defined by (2). Those cliques with the largest order (cardinality) for this neighborhood system correspond to the set

$$C_2 \equiv \{N_2(u) \cup u : u \in E\}. \quad (8)$$

The potential functions of the Gibbs pmf can be defined to discourage boundary generation, as we now demonstrate.

Definition 4.5: Call $u_0 \in E$ a branch site of $\omega \in \Omega_E^P$ if $|N_E(u_0, \omega)| \geq 3$, where $N_E(u_0, \omega)$ is defined in Definition 2.2.

Consider Figure 4.1, where u_0 is the center edge site (enclosed with a box) in the lattice. If there are no branch sites in this subregion, the boundary through u_0 partitions the image domain into two disjoint connected sets. However, if u_0 is a branch site, the image domain is partitioned into three disjoint connected sets. Hence, a branching of a boundary introduces additional disjoint connected sets. To inhibit the inclusion of excessive boundaries it is therefore sensible to penalize permissible boundary configurations that have many branch sites.

Figure 4.1. A Branch site partitions the image domain into one more disjoint region.

The set C_2 defined in (8), which contains the largest-order cliques of the neighborhood system $\mathcal{N}_8$, can be used to quantify the penalties for branch sites.

Let $V_C(\omega) = \theta > 0$ for cliques $C \in C_2$ with clique configurations of the form shown in Figure 4.2. This is the set of all allowable clique configurations where the center pixel is a branch site of ω . For a boundary configuration $\omega \in \Omega_E^P$, denote the set of cliques with such configurations by $\mathcal{C}_y(\omega)$; the subscript “y” is chosen in reference to the rotated y shape of each configuration.

Figure 4.2. The allowable clique configurations where the center pixel is a branch site

Similarly, let $V_C(\omega) = \psi > 0$ for cliques $C \in C_2$ with configurations as shown in Figure 4.3a. Denote the set of cliques with this configuration by $\mathcal{C}_x(\omega)$; the subscript “x” is chosen in reference to the x shape of this configuration.

Figure 4.3. a) The allowable multiple-branch clique configuration. b) The allowable diamond clique configuration.

Figure 4.4. The outlawed clique configurations.

The number of disjoint connected sets associated with a permissible boundary configuration cannot be calculated by the number of branch sites alone. For example, disjoint connected sets can be generated that isolate a region of one pixel width. Let $V_C(\omega) = \zeta > 0$ for cliques $C \in C_2$ with configurations as shown in Figure 4.3b. Though the choice of ζ is situation dependent, in general ζ should be large to discourage small regions of unit area, unless coarse resolution edge lattices are used.⁴ The set of cliques with this local configuration are referred to as $C_d(\omega)$; the subscript “d” is chosen in reference to the diamond shape of this configuration.

One advantage of using cliques to assign prior probabilities is that the neighborhood system can ensure that ω is indeed a permissible boundary configuration, that is, that $\omega \in \Omega_E^P$. Since $|\Omega_E|$ is usually very large, it is important to have a simple method for guaranteeing $\omega \in \Omega_E^P$. As a matter of interest, we have illustrated in Figure 4.4 the clique configurations that are not allowable. Infinite penalties are placed on these to ensure that $\omega \in \Omega_E^P$. Call this set $C_o(\omega)$; the subscript “o” is chosen to indicate that these configurations are outlawed.

4.3. Final Prior Boundary Model

Note that θ , ζ , and ψ are the parameters for this prior model. If we further place a small penalty κ on each edge site of ω , and define $\mathcal{N}(\omega)$ as the set of all lit sites, the resulting prior energy function reduces to

$$U_0(\omega) = \kappa|\mathcal{N}(\omega)| + \theta|C_y(\omega)| + \psi|C_x(\omega)| + \zeta|C_d(\omega)| + \infty|C_o(\omega)|. \quad (9)$$

With implementation in Image Algebra FORTRAN, the sum of these potentials can be calculated quickly for a given boundary configuration. However, as discussed in Section 2, the main advantage of the Markov random field approach to modeling the prior pmf for $\omega \in \Omega_E^P$ is that the joint prior probability distribution of $\{W(u) : u \in E\}$ is uniquely determined by the conditional distributions $\Pr(W(u) | W(v) : v \in N_8(u))$.

5. THE POSTERIOR CLOSED BOUNDARY MODEL

In Section 3 the assumed form of the intensity model was described and in Section 4 a simple prior boundary pmf was constructed. By Bayes Theorem, for $\omega \in \Omega_E^P$,

$$\Pr(\omega | Y(s) : s \in D) \propto \exp \left\{ - \left(U_0(\omega) + \sum_{i=1}^{K(\omega)} \sum_{j=1}^{n_i(\omega)} (Y(s_{ij}) - \mu_i(\omega))^2 / 2\sigma^2 \right) \right\}. \quad (10)$$

Since $\{Y(s) : s \in D\}$ is given, define

$$U_1(\omega) \equiv U_0(\omega) + \sum_{i=1}^{K(\omega)} \sum_{j=1}^{n_i(\omega)} (Y(s_{ij}) - \mu_i(\omega))^2 / 2\sigma^2 \quad (11)$$

and note that the posterior boundary pmf,

$$\Pr(\omega | Y(s) : s \in D) = \frac{1}{Z_1} \exp(-U_1(\omega)), \quad (12)$$

is also a Markov random field, where Z_1 is the partition function that ensures the posterior sums to unity.

That the posterior pmf is also a Markov random field brings with it all the theoretical and computational advantages discussed in Section 4. More specifically, the joint (conditional on the observed intensities) posterior pmf of $\{W(u) : u \in E | Y(s) : s \in D\}$ (or, equivalently, the posterior pmf of $\omega \in \Omega_E^P$) is uniquely determined by the conditional distributions $\Pr(W(u) | W(v) : v \in N_8(u), Y(s) : s \in D)$.

6. MARKOV CHAIN ALGORITHMS TO ESTIMATE THE MAP CLOSED BOUNDARY

6.1. Introduction

In this section, methods to obtain an estimate of a maximum-a-posteriori (MAP) closed boundary are discussed. If κ , θ , ψ , ζ , and σ^2 are known, the posterior boundary pmf is known for all $\omega \in \Omega_E^P$. Therefore, conceptually speaking, it would be possible to search the entire space Ω_E^P to find a boundary configuration ω^* that maximizes the posterior distribution. However, the cardinality of the search space is usually astronomical and, therefore, for most imaging problems a direct search is infeasible.

Though a particular realization is desired, namely, a configuration from the set of configurations that attain the mode of the posterior distribution, it is first necessary to construct algorithms that will essentially allow one to sample from $\Pr(W(\mathbf{u}) : \mathbf{u} \in E | Y(\mathbf{s}) : \mathbf{s} \in D)$.

Definition 6.1: A discrete-time Markov process is a stochastic process with the property that, given the configuration $\omega^{(t)}$, the configurations $\omega^{(s)}$ for $s > t$ are not influenced by the values of $\omega^{(u)}$ for $u < t$. That is,

$$\Pr(\omega^{(s)} = \omega | \omega^{(t)}, \omega^{(t-1)}, \dots, \omega^{(0)}) = \Pr(\omega^{(s)} = \omega | \omega^{(t)}); s > t. \quad (13)$$

A Markov chain is a discrete-time Markov process whose state space is finite or countable.

Let Ω denote a generic configuration space and let $\omega^{(0)} \in \Omega$ denote an initial configuration. Let $T(\omega)$ denote the first time a Markov chain is in state ω .

Definition 6.2: A transition probability function is a mapping from $\Omega \times \Omega \rightarrow [0, 1]$, denoted $\Pr(\omega_1 | \omega_0)$; $\omega_0, \omega_1 \in \Omega$, with the property $\sum_{\omega_1 \in \Omega} \Pr(\omega_1 | \omega_0) = 1$, for each $\omega_0 \in \Omega$. We can further let $\Pr(\cdot | \cdot)$ change over time t and denote the transition probability function at time t as $\Pr^{(t)}(\omega_1 | \omega_0)$; $\omega_0, \omega_1 \in \Omega$.

Definition 6.3: A configuration $\omega \in \Omega$ is said to be accessible if for every $\omega_0 \in \Omega$, $\Pr(T(\omega) < \infty | \omega^{(0)} = \omega_0) > 0$.

Definition 6.4: A Markov chain is said to be irreducible if every $\omega \in \Omega$ is accessible.

Our goal in this section is to construct an irreducible Markov chain with state space Ω_E^P and limiting pmf equal to the posterior given by (12). Let $\omega^{(t)} \in \Omega_E^P$ denote the state (configuration) of the Markov chain at time t . Then, for properly specified transition probabilities between any two configurations $\omega_1, \omega_2 \in \Omega_E^P$, the Markov chain $\omega^{(0)}, \omega^{(1)}, \dots$, will converge in distribution to a random field with pmf $\Pr(W(\mathbf{u}) : \mathbf{u} \in E | Y(\mathbf{s}) : \mathbf{s} \in D)$.⁵ By running the chain sufficiently long, one can obtain a configuration $\omega^{(\nu)} \in \Omega_E^P$ (ν large) that can be considered a sample from $\Pr(\cdot | Y(\mathbf{s}) : \mathbf{s} \in D)$ on Ω_E^P .

Monte Carlo Markov chain algorithms for single-site replacement will be described first. A discussion of multiple-site replacement strategies will follow. Third, a properly specified transition probability function on Ω_E^P will be prescribed. Finally, a simple modification of the Monte Carlo Markov chain algorithm will be described that theoretically guarantees convergence to a MAP closed boundary $\omega^* \in \Omega_E^P$.

6.2. Single-Site Replacement Algorithms

A Monte Carlo Markov chain algorithm requires an appropriate choice of transition probabilities $\Pr^{(t)}(\omega_0 | \omega_1)$; $\omega_0, \omega_1 \in \Omega$ from which the sequence of configurations $\{\omega^{(t)} : t = 1, 2, \dots\}$ can be generated. Single-site replacement refers to updating $\omega^{(t-1)}$ with a configuration $\omega^{(t)}$ by changing the value of $\{W(\mathbf{u}) : \mathbf{u} \in E\}$ at no more than one site $\mathbf{u}_0 \in E$.

Let N denote the total number of potential edge sites on E ($N = n \times m$). A key requirement of a Markov chain algorithm is that all N potential edge sites must be visited infinitely often during boundary updating. Let $\mathbf{u}^{(t)}$ denote the site visited at time t .

A binary (0–1) single-site replacement algorithm can be described as follows. At time t , define $\omega_1^{(t)}$ to be the boundary configuration with

$$W(\mathbf{u}) = \begin{cases} 1 - W^{(t-1)}(\mathbf{u}) & \text{if } \mathbf{u} = \mathbf{u}^{(t)}, \\ W^{(t-1)}(\mathbf{u}) & \text{otherwise.} \end{cases} \quad (14)$$

Then, let $\omega^{(t)} = \omega_1^{(t)}$ with probability

$$\alpha^M(\omega^{(t-1)}, \omega_1^{(t)}) \equiv \min \left\{ 1, \frac{\Pr(\omega_1^{(t)} | Y(s) : s \in D)}{\Pr(\omega^{(t-1)} | Y(s) : s \in D)} \right\}, \quad (15)$$

and $\omega^{(t)} = \omega^{(t-1)}$ with probability $1 - \alpha^M(\omega^{(t-1)}, \omega_1^{(t)})$. The probability α^M in (15) is referred to as the selection probability function corresponding to the Metropolis algorithm.⁶

Since the posterior boundary pmf is a Markov random field, the Metropolis selection probability (15) can be written as $\alpha^M(\omega^{(t-1)}, \omega_1^{(t)}) \equiv \min \left\{ 1, \exp \left[\left(U_1(\omega^{(t-1)}) - U_1(\omega_1^{(t)}) \right) \right] \right\}$. Alternatively, Barker⁷ uses

$$\alpha^B(\omega^{(t-1)}, \omega_1^{(t)}) \equiv \frac{\exp \left[-U_1(\omega_1^{(t)}) \right]}{\exp \left[-U_1(\omega_1^{(t)}) \right] + \exp \left[-U_1(\omega^{(t-1)}) \right]}. \quad (16)$$

The selection probability of the Gibbs sampler¹ reduces to Barker's selection probability in the case of binary images. These selection probabilities yield appropriate transition probabilities for a Monte Carlo Markov chain algorithm to converge to the appropriate distribution if the Markov chain is irreducible.

It is clear that single-site replacement cannot be used for the closed boundary model described here. The problem is that a single-site replacement algorithm does not define an irreducible Markov chain for the permissible boundary configuration space Ω_E^P . If $W^{(0)}(\mathbf{u}) = 1$ and a transition to $W^{(1)}(\mathbf{u}) = 0$ is made, then connectedness may be lost. A requirement for the success of the Markov chain algorithm is that each permissible configuration be accessible from any other permissible configuration. However, if multiple site replacement is considered, a Markov chain algorithm can be constructed that is irreducible on Ω_E^P and will converge to the desired posterior boundary pmf.

6.3. Multiple-Site Replacement

Recall that $\mathbf{u}^{(t)}$ is the site visited at time t . As well as $\mathbf{u}^{(t)}$, we also choose $N_2(\mathbf{u}^{(t)})$ to be updated at time t . Hence nine sites, which we denote as $U(t)$ and refer to as the sweep area⁸, are to be simultaneously updated.

To implement the Gibbs sampler at time t , replace $\{W(\mathbf{u}) : \mathbf{u} \in U(t)\}$ with any configuration of values from the set $\{0, 1\}^9$; there are $2^9 = 512$ such combinations. Set $W^{(t)}(\mathbf{u}) = W^{(t-1)}(\mathbf{u})$ for those sites not to be updated. Let $\omega_p^{(t)}$; $p = 1, 2, \dots, 2^9$, denote the candidate boundaries resulting from the replacement. At time t , choose $\omega^{(t)} = \omega_p^{(t)}$ with probability

$$\alpha^G(\omega_p^{(t)}) \equiv \frac{\exp \left[-U_1(\omega_p^{(t)}) \right]}{\sum_{r=1}^{512} \exp \left[-U_1(\omega_r^{(t)}) \right]}, \quad (17)$$

by analogy with (16). Only a few of the 512 candidate configurations will be permissible and thus will receive positive probability of being selected. Second, notice the crucial role of σ^2 in the posterior energy function (11). A low value of σ^2 will insist on fidelity to the observed pixel intensities and promote the generation of boundaries while a larger σ^2 will inhibit boundaries.

The transition probability function corresponding to equation (17) can be written,

$$\Pr(\omega_1 | \omega_0) = \begin{cases} \Pr(W(\mathbf{u}, \omega_1) : \mathbf{u} \in U(t) | W(\mathbf{v}, \omega_0) : \mathbf{v} \notin U(t), Y(s) : s \in D), & \text{if } W(\mathbf{v}, \omega_1) = W(\mathbf{v}, \omega_0) \text{ for all } \mathbf{v} \notin U(t) \\ 0, & \text{otherwise.} \end{cases} \quad (18)$$

For multiple-site replacement, given a systematic site-visitation schedule, the Markov chain specified by this transition probability function has as its stationary distribution the desired posterior boundary pmf, and this chain converges to its

stationary distribution. In general, the systematic site visitation schedule can be dropped in favor of another visitation scheme as long as each site is visited infinitely often.

The proofs of the results in this section are variations of the standard Markov chain results for stationary Markov processes, except for two main differences. First, a multiple-site replacement strategy is presented as opposed to the standard single-site replacement strategy. As pointed out at the end of Section 6.2, multiple-site replacement is necessary in order to satisfy the accessibility requirement.

Second, for simplicity, consider a raster-scan visitation strategy. For the configuration space Ω_E^P , it is *not* true that the transition probability from any one configuration $\omega^{(t)} \in \Omega_E^P$ to any candidate configuration $\omega \in \Omega_E^P$ at time $(t+N)$ is positive, as is the case for standard Markov Chain results presented in the imaging literature. The problem is due to the constrained nature of the configuration space. A closed boundary completely enclosed by another closed boundary cannot be created in one pass through E ; two passes through E are required. These types of boundaries arise in images through objects that are embedded within other objects.

Let $g(n, m) \equiv \max\{\text{integers } i : i \leq \min(n, m)/4\}$. The value of $g(n, m)$ corresponds to the maximum number of sequentially embedded (i.e., object 1 is embedded in object 2, which in turn is embedded in object 3, etc.) boundaries in a configuration defined on a $n \times m$ lattice that can appear and yet retain restricted eight-connectedness. It can then be shown that the transition probability from any one configuration $\omega^{(t)} \in \Omega_E^P$ to any other configuration $\omega \in \Omega_E^P$ at time $[t+(N \cdot g(n, m))]$ is positive. Two lemmas and a theorem are presented with the posterior pmf's dependence on $\{Y(s) : s \in D\}$ suppressed. The proofs of these results can be found in Helterbrand, Cressie, and Davidson.⁴

Lemma 6.1: The posterior boundary pmf $\Pr(\cdot)$ given by (10) is the stationary distribution for a Markov chain on Ω_E^P with transition probability function defined by (18). That is,

$$\Pr(\omega_1) = \sum_{\omega_0 \in \Omega_E^P} \Pr(\omega_1 | \omega_0) \Pr(\omega_0) \quad (19)$$

for every $\omega_1 \in \Omega_E^P$.

Next, a boundedness result is given involving starting configurations. Without loss of generality a systematic site visitation schedule is assumed. Let $N^I \equiv \{u : 3.5 \leq u^1 \leq n - 1.5, 3.5 \leq u^2 \leq m - 1.5\}$. Let $T_0 = 0$ and define $T_k = k \times (|N^I| \times g(n, m))$; $k = 1, 2, \dots$. Let $K(t) = \max\{k : T_k < t\}$. Since every site is visited infinitely often it is clear that $K(t) \rightarrow \infty$ as $t \rightarrow \infty$.

Lemma 6.2: There exists a constant r , $0 \leq r < 1$, such that for every t ,

$$\max_{\omega, \eta_1, \eta_2} \left| \Pr(\omega^{(t)} = \omega | \omega^{(0)} = \eta_1) - \Pr(\omega^{(t)} = \omega | \omega^{(0)} = \eta_2) \right| \leq r^{K(t)}. \quad (20)$$

Given Lemma 6.1 and Lemma 6.2 the desired theorem can now be proved.

Theorem 6.1: Consider a systematic multiple-site visitation strategy and $\Pr(\omega_1 | \omega_0)$ as defined by (18), then for every starting configuration $\eta \in \Omega_E^P$ and every $\omega \in \Omega_E^P$,

$$\lim_{t \rightarrow \infty} \Pr(\omega^{(t)} = \omega | \omega^{(0)} = \eta) = \Pr(\omega), \quad (21)$$

where from (10) $\Pr(\omega)$ is the posterior boundary pmf on Ω_E^P .

6.4. Simulated Annealing

By running the chain described in Section 6.3 sufficiently long, the configuration $\omega^{(\nu)} \in \Omega_E^P$ for large ν can be considered as a sample from the posterior boundary pmf. However, the particular configuration $\omega^* \in \Omega_E^P$ that maximizes the posterior distribution is of interest. A modification of the Markov chain algorithm leads to a selection from the set of configurations

that achieve the posterior mode. This modification is the use of an annealing schedule.¹ The idea is to introduce a parameter T as a function of time t and consider

$$\Pr(\omega | Y(s) : s \in D)^{1/T(t)}. \quad (22)$$

The effect of $T(t)$ on the local transition probability function is as follows. At low temperatures (i.e., $T(t)$ small) the local transition probability function becomes concentrated at candidate configurations that minimize the local energy, whereas at high temperatures it is essentially uniform. Simulated annealing is a sequential procedure made up of a Monte Carlo Markov chain algorithm and an annealing schedule for $T(t)$ tending to zero that are combined to sample from (22). Geman and Geman (1984, Theorem B) provide a theorem prescribing conditions on the annealing schedule that guarantee theoretical convergence to minimal energy states.

Note that the required modification to the Monte Carlo Markov chain algorithm is trivial, and the local nature of the calculations is preserved. Combining the proposed transition probability function (18), a systematic site-visitation schedule, and a proper annealing schedule, a Markov-chain algorithm is constructed that can be used to obtain the maximum-a-posteriori closed boundary estimate. Unfortunately, the required annealing schedule is too slow for applications. Therefore, approximations are necessary.

6.5. Iterated Conditional Modes

In an attempt to approximate the maximum-a-posteriori (MAP) estimator for image reconstruction, Besag⁹ introduced the iterated conditional modes algorithm. In the context of the MAP closed boundary problem, instead of a random choice from the candidate boundaries at time t with respect to (18), the mode of the current candidate boundaries is automatically selected. Such a choice guarantees that $\Pr(\omega^{(t)}) \geq \Pr(\omega^{(t-1)})$; however, convergence may be to a local maximum. This method corresponds to an annealing schedule where $T(t) = 0$ for all t .

A permissible starting configuration is required for the closed-boundary identification algorithm. Because the required annealing schedule is too slow, iterated conditional modes is used as an approximation to the general algorithm and thus the choice of starting configuration is important.

Unfortunately, simple automatic methods that use available intensity data to obtain an initial closed-boundary estimate are lacking.¹⁰ One advantage of using a coarse to fine resolution updating algorithm is that the user is required to only input a closed boundary at the coarsest resolution to begin the updating algorithm. However, given a user-specified closed boundary, it is relatively straightforward to approximate the closed boundary with a permissible boundary at the same or finer resolution.⁴

It is necessary to identify sites with outlawed configurations. However, a lit site that is found to have an outlawed configuration cannot be removed immediately from the closed boundary estimate since it may be a branch site or a multiple branch site for the closed boundary estimate. If the site is removed, connectedness may be lost. A correction for lit sites with two or more lit neighbors and an outlawed configuration is necessary. A three-stage procedure is developed in Helterbrand, Cressie, and Davidson⁴ to remove disallowed local configurations from given closed boundary estimates.

7. IMPLEMENTATION OF THE CLOSED BOUNDARY IDENTIFICATION ALGORITHM USING IMAGE ALGEBRA

Given a specified posterior pmf on Ω_E^p and an appropriate Monte Carlo Markov chain algorithm to obtain a MAP boundary, it is necessary to identify the required image processing steps for implementation. It is advantageous to have a common mathematical environment to represent the image processing operations. Fortunately, Image Algebra provides such an environment.¹¹ This highly structured mathematical foundation is intended to be a unified algebraic approach for image processing, optimization, comparison, coding, and performance evaluation.

7.1. A Single Resolution Closed Boundary Identification Algorithm

In this subsection the image processing steps required to implement the closed boundary identification algorithm are outlined in image algebra pseudocode. For further details and generalizations, see Helterbrand, Cressie, and Davidson.⁴

To initialize the algorithm, let $W^{(0)}$ be an initial permissible boundary configuration and let Y be the observed image. A raster scan site visitation schedule will be assumed in this description.

Definitions of Images

Let $\{C(p): p = 1, 2, \dots, 194\}$, be the set of 3×3 images representing all those allowable configurations referred to in Section 6.3. Let H_{qr} denote a $q \times q$ image with pixel intensity equal to 0 for pixel locations in the $r \times r$ center of the image, and pixel intensity equal to 1 elsewhere. Let H_{qr}^{-1} denote a $q \times q$ image with pixel intensity equal to 1 for pixel locations in the $r \times r$ center of the image, and pixel intensity equal to 0 elsewhere. Let 0_q denote a $q \times q$ image with pixel intensity equal to 0 everywhere. The invariant templates necessary are presented in Figures 7.1 and 7.2.

Figure 7.1. The invariant templates temp(1) and temp(2).

Figure 7.2. The invariant templates temp(3), temp(4), temp(5), and temp(6), respectively.

Algorithm

Step 0: Set $t = 0$.

Step 1: Set $t = t + 1$, and set $T(t)$ according to Theorem B of Geman and Geman.¹

Step 1b: Choose site $u^{(t)}$ to update at iteration t .

Step 2: Let $W7^{(t-1)} = W^{(t-1)} | \{u: (u^1 - u^{1(t)})^2 + (u^2 - u^{2(t)})^2 \leq 18\}$. Comment: $W7$ is an image of the current boundary configuration in the 7×7 neighborhood of $u^{(t)}$.

Step 3a: Set $p = 0$.

Step 3b: Set $p = p + 1$; if $p = 195$, set $d=0$ and go to **Step 10a**.

Step 4: Let $W_p^{(t)} = (W7^{(t-1)} \cdot H_{73}) + C(p) | 0_7$. Comment: $W_p^{(t)}$ is an image with the candidate configuration replacing the current 3×3 local configuration.

Step 5a: Check permissibility: Let $LN_p = W_p^{(t)} \cdot (W_p^{(t)} \oplus \text{temp}(1)) \cdot H_{75}^{-1}$. Comment: LN_p is a 7×7 image with pixel intensities equal to the number of lit neighbors each pixel in the 5×5 neighborhood of $u^{(t)}$ possesses.

Step 5b: If $! \text{sum}(W_p^{(t)} \cdot \chi_{=0}(LN_p) + \chi_{=1}(LN_p)) > C$, go to **Step 3b**; else continue. Comment: This is a check that all lit sites have at least two lit neighbors.

Step 5c: If, for one of $j = 3, 4, 5, 6$, $! \max\{(W_p^{(t)} \oplus \text{temp}(j)) \cdot H_{75}^{-1}\} = 3$, go to **Step 3b**; else continue. Comment: This is a check that the local configurations for all pixels in the 5×5 neighborhood of $u^{(t)}$ are allowable.

Step 6: At this step, the candidate configuration is known to be permissible. Now calculate the prior energy,

$$U_p^{(t)} = ! \text{sum} \left((W7^{(t-1)} \cdot \kappa) + (+\chi_{=3}(LN_p)) \cdot \theta + (\chi_{=4}(LN_p)) \cdot \psi + \left\{ \chi_{=4}(W_p^{(t)} \oplus \text{temp}(2)) \right\} \cdot \eta \right). \quad (23)$$

Step 7a: Calculate the energy associated with the intensity model. Let $W_p = W_p^{(t)} | W^{(t-1)}$. Comment: This is the entire boundary configuration on E for candidate configuration p at time t .

Step 7b: Obtain the disjoint connected regions and their corresponding binary image representations $\{R_p(i): i = 1, 2, \dots, K(W_p)\}$. Comment: Method is given in Helterbrand, Cressie, and Davidson.⁴

Step 7c: Calculate the total sums of squares error. For region i , $i=1,2,\dots,K(C(p))$, let

$$SS_p(i) = \left[\text{sum}(Y \cdot R_p(i))^2 / (\text{sum}(R_p(i))) \right] - \left[\text{sum}(Y \cdot R_p(i)) / (\text{sum}(R_p(i))) \right]^2, \quad (24)$$

$$\text{and } SS_p = \sum_{i=1}^{K(\mathbf{W}_p)} SS_p(i).$$

Step 8: Calculate the posterior energy for the candidate configuration, $UP_p^{(t)} = \{U_p^{(t)} + SS_p/2\sigma^2\}/T(t)$.

Step 9: Go to Step 3b.

Step 10a: Gibbs Sampler: Set $d = d+1$. Generate a random number, rand, from a Uniform(0,1) distribution. Calculate, $F(d) = \left(\sum_{p=1}^d \exp(-UP_p^{(t)}) \right) / Z$, where Z is the normalizing constant defined in (17).

Step 10b: If $F(d) < \text{rand}$, go to Step 10a; else, set $UP^{(t)} = UP_d^{(t)}$, set $\mathbf{W}^{(t)} = \mathbf{W}_d^{(t)}$, and go to Step 1.

This algorithm uses simulated annealing (Section 6.4). If iterated conditional modes is used (Section 6.5), set $T(t) = 1$ for all t , and replace Step 10 with a step that considers whether the candidate's posterior energy is less than the current boundary configuration's posterior energy.

8. CONCLUSIONS

In this research a statistical approach to identify closed object boundaries is presented. The key aspect of this approach is that a prior model is defined with support only on the space of permissible closed boundary configurations. This is in contrast with previous stochastic methods described in the boundary-identification literature, which in practice do not yield closed boundary estimates. However, by incorporating appropriate penalties on the potential functions, we obtain a posterior model with support on the space of permissible boundary configurations that in turn yields closed boundary estimates. We also show that a multiple-site updating strategy is necessary in order to ensure theoretical convergence of the Monte Carlo Markov chain algorithm.

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