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Statistical image algebra: a Bayesian approach

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Abstract

A mathematical structure used to express image processing transforms, the AFATL image algebra has proved itself useful in a wide variety of applications. The theoretical foundation for the image algebra includes many important constructs for handling a wide variety of image processing problems: questions relating to linear and nonlinear transforms, including decomposition techniques [9], [5]; mapping of transformations to computer architectures [8], [4]; neural networks [11], [6]; recursive transforms [10]; and data manipulation on hexagonal arrays. However, statistical notions have been included only on a very elementary level in [12], and on a more sophisticated level in [2]. In this paper we present an extension of the current image algebra that includes a Bayesian statistical approach that is similar in spirit to parts of [2]. Here we show how images are modeled as random vectors, probability functions or mass functions are modeled as images, and conditional probability functions are modeled as templates. The remainder of the paper gives a brief discussion of the current image algebra, an example of the use of image algebra to express high-level image processing transforms, and the presentation of the statistical development of the image algebra.

Keywords

image, bayesian, statistical, algebra, approach

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Statistical Image Algebra: A Bayesian Approach

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ABSTRACT

A mathematical structure used to express image processing transforms, the AFATL image algebra has proved itself useful in a wide variety of applications. The theoretical foundation for the image algebra includes many important constructs for handling a wide variety of image processing problems: questions relating to linear and nonlinear transforms, including decomposition techniques [9], [5]; mapping of transformations to computer architectures [8], [4]; neural networks [11], [6]; recursive transforms [10]; and data manipulation on hexagonal arrays. However, statistical notions have been included only on a very elementary level in [12], and on a more sophisticated level in [2]. In this paper we present an extension of the current image algebra that includes a Bayesian statistical approach that is similar in spirit to parts of [2]. Here we show how images are modeled as random vectors, probability functions or mass functions are modeled as images, and conditional probability functions are modeled as templates.

The remainder of the paper gives a brief discussion of the current image algebra, an example of the use of image algebra to express high-level image processing transforms, and the presentation of the statistical development of the image algebra.

1. Image Algebra

In this section we present only basic image algebra concepts. An in-depth discussion of image algebra can be found in [12].

The four main operands are value sets, point sets, images and templates. A *value set* is a set F together with an associative and commutative operation. Value sets of interest in this paper are $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{R}_{\pm\infty} = \mathbb{R} \cup \{\infty, -\infty\}$, the set of integers, real numbers, complex numbers, and extended real numbers, respectively. The usual operations associated with these sets will be considered. A *point set* is a set $X \subset \mathbb{R}^n$, n -dimensional Euclidean space. An *F-valued image* a on X is an element $a \in F^X$, and has form

$$\{(x, a(x)) : x \in X, a(x) \in F\}.$$

For point sets X, Y , a *template* is an image whose values are images. In particular, an *F valued template* t is an element $t \in (F^X)^Y$. Thus, $t \in (F^X)^Y$ has form

$$\{(x, t_y(x)) : x \in X, t_y(x) \in F\}.$$

The point y is called the *target pixel location*.

Image algebra operations between images are the ones induced from the value set F . In particular, if $a, b \in \mathbf{R}^X$, then we have the following two operations:

$$\begin{aligned} a + b &\equiv c \in \mathbf{R}^X, c = \{(x, c(x)) : c(x) = a(x) + b(x), x \in X\} \\ a * b &\equiv c \in \mathbf{R}^X, c = \{(x, c(x)) : c(x) = a(x) * b(x), x \in X\} \end{aligned}$$

If $a, b \in \mathbf{R}_{\pm\infty}^X$, then we can define

$$a \vee b \equiv c \in \mathbf{R}_{\pm\infty}^X, c = \{(x, c(x)) : c(x) = a(x) \vee b(x), x \in X\}.$$

While the most fundamental operands of the image algebra are images, the most powerful tool are the templates and template operations. We start by describing the global reduce function, as this plays a key part in describing the relation between image algebra and neural nets. Suppose that $X \subset \mathbf{R}^n$ is a finite set, $X = \{x_1, \dots, x_p\}$. If γ is an associative and commutative binary operation on the value set F , the *global reduce operation* Γ on F^X induced by γ is defined by

$$\Gamma a = \Gamma_{x \in X}(a(x)) = a(x_1) \gamma a(x_2) \gamma \dots \gamma a(x_p),$$

where $a \in F^X$. Thus, Γ takes an image to a value. In particular, if $F = \mathbf{R}$ and $\gamma = +$, then $\Gamma = \sum$ and $\sum a = \sum_{x \in X} a(x) \in \mathbf{R}$.

Templates and images are combined by appropriate binary operations. Let F_1, F_2 , and F be three values sets and $o : F_1 \times F_2 \rightarrow F$ be a binary operation. If γ is an associative binary operation on F , $a \in F_1^X$, and $t \in (F_2^X)^Y$, then the *generalized template operation* or *generalized right product of a with t induced by γ and o* is the binary operation $\odot : F_1^X \times (F_2^X)^Y \rightarrow F^Y$ defined by

$$a \odot t \equiv \{(y, b(y)) : b(y) = \Gamma_{x \in X}(a(x) o t_y(x)), y \in Y\}.$$

Here the set of terms $a(x) o t_y(x)$ can be viewed as an image on X , say $c(x) = a(x) o t_y(x)$. Thus,

$$b(y) = \Gamma c = \Gamma_{x \in X} c(x) = c(x_1) \gamma c(x_2) \gamma \dots \gamma c(x_p).$$

The two main specific cases are the *generalized backward convolution* $\oplus$ and the *backward additive maximum* $\boxplus$. If $a \in \mathbf{R}^X$, $t \in (\mathbf{R}^X)^Y$, then the backward generalized convolution is

$$a \oplus t \equiv b \in \mathbf{R}^Y, b = \left\{ (y, b(y)) : b(y) = \sum_{x \in S(t_y)} a(x) t_y(x), y \in Y \right\}. \quad (1)$$

The backward additive maximum is

$$a \boxplus t \equiv b \in \mathbf{R}_{\pm\infty}^Y, b = \left\{ (y, b(y)) : b(y) = \bigvee_{x \in S_{-\infty}(t_y)} a(x) + t_y(x), y \in Y \right\}, \quad (2)$$

and the backward multiplicative maximum is

$$a \oslash t \equiv b \in (\mathbf{R}_{\infty}^{\geq 0})^Y, \quad b = \left\{ (y, b(y)) : b(y) = \bigvee_{x \in S(t_y)} a(x) * t_y(x), \quad y \in Y \right\}, \quad (3)$$

The value set and operations underlying the convolution $\oplus$ operation is $(\mathbf{R}, +, *)$, while the value set underlying the additive operation is $(\mathbf{R}_{\pm\infty}, \vee, +)$. For the $\oslash$ operation, the value set is $(\mathbf{R}_{\infty}^{\geq 0}, \vee, *)$. Here, $\mathbf{R}_{\pm\infty} = \mathbf{R} \cup \{-\infty, \infty\}$ and $\mathbf{R}_{\infty}^{\geq 0} = \{r \in \mathbf{R} : r \geq 0\} \cup \{\infty\}$. It turns out that since the value sets $\mathbf{R}_{\pm\infty}$ and $\mathbf{R}_{\infty}^{\geq 0}$ are isomorphic as lattice-ordered groups (in the sense of Birkhoff [1]), that the algebraic structures concerning the $\boxtimes$ and $\oslash$ are isomorphic [4]. The operation $\boxtimes$ is a generalization of the familiar mathematical morphology. For example, if $\mathbf{X} = \mathbf{Y}$, $\mathbf{X} = \{(i, j) : 1 \leq i \leq k, 1 \leq j \leq h\}$ is a rectangular array, and $t \in (\mathbf{R}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ is a translation invariant template (a template which is shift invariant), then equation (2) represents a morphological dilation.

In equations (1) and (3), the set $S(t_y) = \{x \in \mathbf{X} : t_y(x) \neq 0\}$ is defined to be the *support* of the template t at location $y \in \mathbf{Y}$ for the value set $\mathbf{R}$. For Eq. (2), the *infinite support* is defined to be the set $S_{-\infty}(t_y) = \{x \in \mathbf{X} : t_y(x) \neq -\infty\}$.

One commonly used image processing operation is the *characteristic function*. The image algebra has a generalization which is defined as follows. Let $2^{\mathbf{X}}$ denote the power set of $\mathbf{X}$. Let $(F, \circ, \times)$ be a value set with two binary operations $\circ$ and $\times$ with two unique elements I_1 and I_2 satisfying

$$\begin{aligned} r \circ I_1 &= I_1 \circ r = r \quad \forall r \in F, \text{ and} \\ r \times I_2 &= I_2 \times r = r \quad \forall r \in F. \end{aligned}$$

Given $a \in F^{\mathbf{X}}$ and $S \in (2^F)^{\mathbf{X}}$, we define the *characteristic function of $a \in F^{\mathbf{X}}$ with respect to S* to be

$$\chi_S(a) = b = \left\{ (x, b(x)) : b(x) = \begin{cases} I_2 & \text{if } a(x) \in S(x) \\ I_1 & \text{otherwise} \end{cases} \right\}.$$

Note that $(F, \circ, \times) = (\mathbf{R}, +, *)$ satisfies the above conditions with $I_1 = 0$ and $I_2 = 1$. This results in the usual characteristic function

$$\chi_S(a) = b, \text{ where } b(x) = \begin{cases} 1 & \text{if } a(x) \in S(x) \\ 0 & \text{otherwise} \end{cases}.$$

For example, image thresholding on real valued images can be expressed using the characteristic function with $S(x) = \{k\}$, k a constant, or $S(x)$ can vary at each pixel location to give variable thresholding. To distinguish between the value sets $\mathbf{R}$ and $\mathbf{R}_{\pm\infty}$, we use $\chi^{-\infty}$ for the characteristic function when $F = \mathbf{R}_{\pm\infty}$. In this case we have $\circ = \vee$ and $I_1 = -\infty$, and $\times = +$ and $I_2 = 0$.

A common concept in image processing is that of a *configuration function*. A configuration function $\mathcal{N}$ is defined to be a function $\mathcal{N} : \mathbf{Y} \rightarrow 2^{\mathbf{X}}$, where $\mathbf{X}$ and $\mathbf{Y}$ are point sets. If $\mathbf{X} = \mathbf{Y}$ and $\mathcal{N}$ is shift invariant, then we say that $\mathcal{N}$ is an *invariant configuration*.

Statistical notions such as moments have already been described using image algebra. Let $a \in \mathbf{R}^{\mathbf{X}}$, where $\mathbf{X} \subset \mathbf{Z}^2$. Then the moment of order pq is defined as

$$m_{pq} = \sum_i \sum_j i^p j^q a(i, j)$$

and the pq-th central moment as

$$\mu_{pq} = \sum_i \sum_j (i - \bar{i})^p (j - \bar{j})^q \mathbf{a}(i, j),$$

where $\bar{i} = \frac{m_{10}}{m_{00}}$, and $\bar{j} = \frac{m_{01}}{m_{00}}$. To express this in image algebra, we have

$$m_{pq} = \sum \sum (\mathbf{i}^p * \mathbf{j}^q * \mathbf{a}),$$

where $\mathbf{i} = \{(h, k, \mathbf{i}(h, k)) : \mathbf{i}(h, k) = h, (h, k) \in \mathbf{X}\}$, and $\mathbf{j} = \{(h, k, \mathbf{j}(h, k)) : \mathbf{j}(h, k) = k, (h, k) \in \mathbf{X}\}$. Thus,

$$m_{00} = \sum \mathbf{a}, \quad \bar{i} = \frac{\sum (\mathbf{i} * \mathbf{a})}{\sum \mathbf{a}}, \quad \bar{j} = \frac{\sum (\mathbf{j} * \mathbf{a})}{\sum \mathbf{a}}.$$

Define the mean images $\bar{\mathbf{i}}$ and $\bar{\mathbf{j}}$ by $\bar{\mathbf{i}} = \bar{i}$ and $\bar{\mathbf{j}} = \bar{j}$ (note that each mean image is a constant image). Then the nonzero central moments are given by

$$\mu_{pq} = \sum \sum [(\mathbf{i} - \bar{\mathbf{i}})^p * (\mathbf{j} - \bar{\mathbf{j}})^q * \mathbf{a}].$$

2. Examples of Image Algebra in Image Processing

This section presents two examples, one on artificial neural networks and the other on a high-level image processing algorithm. More details on neural networks can be found in [11] and [7].

Artificial Neural Networks. Artificial neural network models are specified mainly by the network topology, node characteristics, and training or learning rules. Let $\mathbf{X} = \{1, \dots, N\}$ denote the input neuron domain. Then $\mathbf{a} = (a_1, \dots, a_N) \in \mathbf{R}^{\mathbf{X}} = \mathbf{R}^N$ denotes the input to the neural net. For $\mathbf{Y} = \{1, \dots, M\}$ where M denotes the number of neurons at the upper layer, define $\mathbf{t} \in (\mathbf{R}^{\mathbf{X}})^{\mathbf{Y}}$ by $t_j(i) = w_{ji}$ for all $i \in \mathbf{X}$, $j \in \mathbf{Y}$. For ease of notation, we can simply denote the template by the symbol $\mathbf{w}$ instead of $\mathbf{t}$, so we view $\mathbf{w}$ as a template representing the weights. The following represents image algebra pseudocode for describing the Hopfield net.

```

repeat
     $\mathbf{a} = \mathbf{c}$ 
     $\mathbf{b} = \mathbf{a} \oplus \mathbf{w}$ 
     $\mathbf{c} = \chi_{>0}(\mathbf{b}) - \chi_{<0}(\mathbf{b}) + \mathbf{a} * \chi_{>0}(\mathbf{b})$ 
until  $\mathbf{c} = \mathbf{a}$ 

```

Image Algebra Pseudocode for the Hopfield Network

Here, $\mathbf{X}=\mathbf{Y}$, $\mathbf{T} = \mathbf{0}$, and f is the hardlimiter activation function, where $f(x) = \begin{cases} 1 & x \geq 0 \\ -1 & x < 0 \end{cases}$. The template $\mathbf{w}$ is defined by

$$\mathbf{w}_j(i) = \mathbf{w}_{ji} = \begin{cases} \sum_{k=1}^m p_j^k p_i^k & j \neq i \\ 0 & j = i \end{cases}$$

where $\mathbf{p}^k = (p_1^k, \dots, p_N^k)$ is the k -th exemplar, $k=1, \dots, m$ (m exemplars total), and $\mathbf{c}$ is the input pattern to be classified.

The image algebra as a mathematical structure allows a more general environment for describing possible relationships of neural connections and computing than those provided by the classical artificial neural networks. In particular, the general form of operations between sets of neurons, which are represented by images, and the weights, which are presented by templates, are furnished by the generalized matrix product [3].

A High-Level Image Processing Transform. In this example we show how value sets and operations γ and $\circ$ can be chosen to represent a high level image processing transform such as is used in computer vision or automated target recognition (ATR) problems. Let $\mathbf{X}$ be an array and suppose that in $\mathbf{X}$ we have h line segments. Here a line L_i consists of the pixel locations in $\mathbf{X}$ designating the line. Thus, we can view an image $\mathbf{L}$ of lines on $\mathbf{X}$ to be an F valued image on the set $H = \{1, \dots, h\}$, where F is the set of all possible lines on $\mathbf{X}$. Hence, $\mathbf{L} \in F^H$ implies that $\mathbf{L} = \{(i, L(i)) : L(i) = \{x_{j_1}, \dots, x_{j_{L(i)}}\} \text{ is a line on } \mathbf{X}, i \in H\}$. Suppose we wish to write an algorithm that identifies, relative to an arbitrary pixel j in $\mathbf{X}$, if there is a line that is at a particular orientation, a particular distance, and has a minimum length: say at horizontal, D pixels directly north, and is centered over j with a length of at least k pixels. Define $\mathbf{t} \in (F^H)^H$ by

$$t_j(i) = \begin{cases} \emptyset & \text{for } i \neq j \\ W_j & i = j \end{cases}, \quad i, j \in H$$

where W_j is the line, relative to j , centered above j , at a horizontal position, and k pixels long. Here $\emptyset$ represents the empty line. See Figure 1 for an example when $D = 3$ and $k = 5$.

Figure 1. W_j consists of the pixel locations marked by the shading relative to j .

Choosing $\gamma = \cup$ and $\circ = \cap$ we can use the above generalized product ⑦ to perform the desired operation. Calculating $\mathbf{b} = \mathbf{L} \textcircled{\gamma} \mathbf{t}$ produces the image $\mathbf{b} \in F^H$ which has a value, at location j , of

$$\mathbf{b}(j) = \bigcup_{i \in H} (L(i) \cap t_j(i)) = L(j) \cap t_j(j) = L(j) \cap W_j.$$

See Figure 2 for an example. If each line in the image $\mathbf{L}$ is at least k pixels in length, then the output $\mathbf{b}(j)$ will either consist of the line segment W_j or the empty set. This short algorithm may be used to identify a horizontal line segment associated with each j that is exactly D pixels to the north and at least k pixels in length, such as an edge of a simple bridge with no superstructures.

Figure 2. Computation of $b = L \otimes t$, resulting in horizontal line at location j .

This simple example shows how versatile the image algebra is at expressing transformations other than pixel-level ones. If the statistical image algebra is developed correctly, then transformations of this type describing high-level image processing algorithms of a statistical nature should be able to be expressed similarly. The power of being able to express transformations in this way is that the notation allows a concise methodology to represent pixel-level or higher level operations on a more global scale, avoiding indexing notation, do loops, and the like.

3. Statistical Image Algebra

The key to developing a statistical image algebra is in the notion of random variables, which has a firm mathematical footing. Viewing an image as a random vector, which is not a new concept, is the first step. The next step is to establish the concept of a template as a probability function. Then we may represent, as an example, the Bayesian relationship of a posterior distribution to its prior and model distributions. In this section we present research which has been accomplished to date in developing the statistical image algebra.

Let $S = \mathbf{R}$ be the sample space, $\mathcal{A}$ be the σ -field of Borel sets, and let P be a probability distribution on $(S, \mathcal{A})$. Let $a_i : S \rightarrow \mathbf{R}$ be a random variable for $i=1, \dots, N$. Let $\mathbf{X} = \{1, \dots, N\} \subset \mathbf{Z}$. Define a *random $\mathcal{R}$ image on $\mathbf{X}$ with respect to the probability space $(S, \mathcal{A}, P)$* to be a sequence of N random variables $a_1, \dots, a_N$, where each belongs to the set $\mathcal{R} = \{b : b : S \rightarrow \mathbf{R} \text{ is a r.v. wrt the probability space } (S, \mathcal{A}, P)\}$. Thus a random $\mathcal{R}$ image $\mathbf{a}$ has form

$$\mathbf{a} = \{(i, a_i) : i = 1, \dots, N\}$$

or simply $\mathbf{a} = (a_1, \dots, a_N)$, where a_i is a random variable (r.v.) for $i=1, \dots, N$. The set of all random $\mathcal{R}$ images on $\mathbf{X}$ is denoted by $\mathcal{R}^{\mathbf{X}}$.

All results and properties holding true for random vectors hold true for random images, as a random image is nothing more than a random vector. Thus, pointwise addition, pointwise maximum, and pointwise multiplication of two $\mathcal{R}$ images $\mathbf{a}$ and $\mathbf{b}$ are defined in the natural way:

$$\begin{aligned}\mathbf{a} + \mathbf{b} &\equiv \mathbf{c} \in \mathcal{R}^{\mathbf{X}}, \mathbf{c} = \{(x, c(x)) : c(x) = a(x) + b(x), x \in \mathbf{X}\} \\ \mathbf{a} * \mathbf{b} &\equiv \mathbf{c} \in \mathcal{R}^{\mathbf{X}}, \mathbf{c} = \{(x, c(x)) : c(x) = a(x) * b(x), x \in \mathbf{X}\} . \\ \mathbf{a} \vee \mathbf{b} &\equiv \mathbf{c} \in \mathcal{R}^{\mathbf{X}}, \mathbf{c} = \{(x, c(x)) : c(x) = a(x) \vee b(x), x \in \mathbf{X}\}\end{aligned}$$

Other operations, such as the characteristic function, and other unary and induced operations, remain defined the same as for the deterministic case, with the exception that the value set here is $\mathcal{R}$

To define a template, we let G be the set of all $\mathcal{R}$ images on $\mathbf{X}$. Then, as in the deterministic case, we say $\mathbf{t} \in G^{\mathbf{Y}} = (\mathcal{R}^{\mathbf{X}})^{\mathbf{Y}}$ is an $\mathcal{R}$ template from $\mathbf{Y}$ to $\mathbf{X}$, where

$$\mathbf{t}_{\mathbf{y}} = \{(x, t_{\mathbf{y}}(x)) : x \in \mathbf{X}, \text{ and } t_{\mathbf{y}}(x) \text{ is a r.v. wrt } (S, \mathcal{A}, P)\}.$$

The usual pointwise operations between templates, such as template addition, multiplication, and maximum, are straightforward: for $\mathbf{s}, \mathbf{t} \in (\mathcal{R}^{\mathbf{X}})^{\mathbf{Y}}$, we have:

$$\begin{aligned}\mathbf{s} + \mathbf{t} &\equiv \mathbf{r} \in (\mathcal{R}^{\mathbf{X}})^{\mathbf{Y}}, \mathbf{r}_{\mathbf{y}} = \{(x, r_{\mathbf{y}}(x)) : r_{\mathbf{y}}(x) = s_{\mathbf{y}}(x) + t_{\mathbf{y}}(x), x \in \mathbf{X}\} \\ \mathbf{s} * \mathbf{t} &\equiv \mathbf{r} \in (\mathcal{R}^{\mathbf{X}})^{\mathbf{Y}}, \mathbf{r}_{\mathbf{y}} = \{(x, r_{\mathbf{y}}(x)) : r_{\mathbf{y}}(x) = s_{\mathbf{y}}(x) * t_{\mathbf{y}}(x), x \in \mathbf{X}\} . \\ \mathbf{s} \vee \mathbf{t} &\equiv \mathbf{r} \in (\mathcal{R}^{\mathbf{X}})^{\mathbf{Y}}, \mathbf{r}_{\mathbf{y}} = \{(x, r_{\mathbf{y}}(x)) : r_{\mathbf{y}}(x) = s_{\mathbf{y}}(x) \vee t_{\mathbf{y}}(x), x \in \mathbf{X}\}\end{aligned}$$

Let γ be an associative and commutative binary operation on the set $\mathcal{R}$. Then the *global reduce operation* Γ on $\mathcal{R}^{\mathbf{X}}$ induced by γ is defined by

$$\Gamma \mathbf{a} = \Gamma_{x \in \mathbf{X}}(a(x)) = a(x_1) \gamma a(x_2) \gamma \cdots \gamma a(x_p),$$

where $\mathbf{a} \in \mathcal{R}^{\mathbf{X}}$. Thus, Γ takes an image to a random variable.

Combining an image and a template as in the deterministic case gives us a similar result. Let $\mathcal{R}_1, \mathcal{R}_2$, and $\mathcal{R}$ be three sets,

$$\begin{aligned}\mathcal{R} &= \{\mathbf{b} : \mathbf{b} : S \rightarrow \mathbf{R} \text{ is a r.v. wrt the probability space } (S, \mathcal{A}, P)\} \\ \mathcal{R}_1 &= \{\mathbf{b} : \mathbf{b} : S_1 \rightarrow \mathbf{R} \text{ is a r.v. wrt the probability space } (S_1, \mathcal{A}, P_1)\} . \\ \mathcal{R}_2 &= \{\mathbf{b} : \mathbf{b} : S_2 \rightarrow \mathbf{R} \text{ is a r.v. wrt the probability space } (S_2, \mathcal{A}, P_2)\}\end{aligned}$$

Let $o : \mathcal{R}_1 \times \mathcal{R}_2 \rightarrow \mathcal{R}$ be a binary operation. If γ is an associative binary operation on $\mathcal{R}$, $\mathbf{a} \in \mathcal{R}_1^{\mathbf{X}}$, and $\mathbf{t} \in (\mathcal{R}_2^{\mathbf{X}})^{\mathbf{Y}}$, then the *generalized right product of $\mathbf{a}$ with the template $\mathbf{t}$ induced by γ and o* is the binary operation $\odot : \mathcal{R}_1^{\mathbf{X}} \times (\mathcal{R}_2^{\mathbf{X}})^{\mathbf{Y}} \rightarrow \mathcal{R}^{\mathbf{Y}}$ defined by

$$\mathbf{a} \odot \mathbf{t} \equiv \{(y, b(y)) : b(y) = \Gamma_{x \in \mathbf{X}}(a(x) o t_{\mathbf{y}}(x)), y \in \mathbf{Y}\}.$$

Note that the result $\mathbf{b}$ is a random $\mathcal{R}$ image on $\mathbf{Y}$. Taking $\Gamma = \sum$, and $o = *$, we get a linear convolution:

$$\mathbf{a} \oplus \mathbf{t} \equiv \mathbf{b} \in \mathcal{R}^{\mathbf{Y}}, \mathbf{b} = \left\{ (y, b(y)) : b(y) = \sum_{x \in \mathbf{X}} a(x) t_{\mathbf{y}}(x), y \in \mathbf{Y} \right\}.$$

The operations $\boxtimes$ and $\odot$ are defined in the usual way, and the same definitions for support of a template can be made as in the deterministic case. As they are straightforward, we omit them here.

An Example. Let us consider the special case where each pixel is independently distributed, that is, there is no spatial dependence between pixels. Let $\mathbf{X} = \{1, \dots, N\}$, and let $F = \{0, 1, \dots, m-1\}$ represent the number of gray levels. Suppose we let

$$\mathcal{R} = \{\mathbf{b} : \mathbf{b}: S \rightarrow F \text{ is a r.v. wrt the probability space } (S, \mathcal{A}, P)\}.$$

Let θ be an $\mathcal{R}$ image on $\mathbf{X}$ representing the true values of the image at hand. Let $\theta_i \sim h_{\theta_i}, i = 1, \dots, n$. Here, h_{θ_i} can be viewed as a histogram (normalized) for each i , that is, $h_{\theta_i}(k) = \Pr\{\theta_i = k\}$. Let $\mathbf{a}$ be an $\mathcal{R}$ image on $\mathbf{X}$, $\mathbf{a}_i \sim h_{\mathbf{a}_i|\theta_i}, i = 1, \dots, N$. So $h_{\mathbf{a}_i|\theta_i}(u | k) = \Pr\{\mathbf{a}_i = u | \theta_i = k\}$ is the conditional probability of the observed data $\mathbf{a}_i$ given the true value θ_i . Also note that $\{h_{\mathbf{a}_i|\theta_i}(u | k)\}_{u=0}^{m-1}$ is a probability function for each k and for $i=1, \dots, N$.

Now, the distribution of interest is $\theta_i | \mathbf{a}_i \sim h_{\theta_i|\mathbf{a}_i}$ so $h_{\theta_i|\mathbf{a}_i}(k | u) = \Pr\{\theta_i = k | \mathbf{a}_i = u\}$ is the conditional probability of the true data given the observed data. Thus, $\{h_{\theta_i|\mathbf{a}_i}(k | u)\}_{k=0}^{m-1}$ is a probability function for each u and for $i=1, \dots, N$.

Thus, using the above constructs, we can state Bayes Theorem by

$$h_{\theta_i|\mathbf{a}_i}(k | u) = \frac{h_{\theta_i}(k) h_{\mathbf{a}_i|\theta_i}(u | k)}{r(\mathbf{a}_i)}$$

where

$$r(\mathbf{a}_i) = \sum_{k=1}^m h_{\mathbf{a}_i|\theta_i}(u | k) h_{\theta_i}(k). \quad (4)$$

Here, h_{θ_i} is the prior distribution of the image. Furthermore, Eq. (4) can be represented by a linear convolution such as

$$\mathbf{r} = \mathbf{h}_{\theta_i} \oplus \mathbf{h}_{\mathbf{a}_i|\theta_i},$$

where the $\oplus$ is defined by equation (1). The function $\mathbf{h}$ can be viewed as a template and the prior as an image, where $\mathbf{h}_{\theta_i} \in F^{\mathbf{X}}$, $h_{\theta_i}(k) = \Pr\{\theta_i = k\}$, and $\mathbf{h}_{\theta_i|\mathbf{a}_i} \in (F^{\mathbf{X}})^{\mathbf{Y}}$, $h_{\theta_i|\mathbf{a}_i}(k | u) = \Pr\{\theta_i = k | \mathbf{a}_i = u\}$.

The assumption of independent distribution of the data is unrealistic. However, this short discussion provides a starting point for developing more a practical statistical image algebra. The next step is to model $\mathbf{h}_{\mathbf{a}|\theta}$ as a Markov random field using a global linear type operation such as discussed above.

This research shows great promise to be a very useful notation tool that may suggest novel solutions to problems in statistical image processing. The concepts displayed here show that more research to develop this statistical image algebra will undoubtedly prove to be useful.

4. Conclusions

This paper has presented the basic concepts for a rigorous theoretical foundation of statistical image algebra. Research is ongoing to continue development of both the theoretical foundation and applications to image processing, with a current focus on applications to object recognition using a spectral representation.

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