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Hölder regularity of the solution to the complex Monge-Ampère equation with L p density

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Abstract

On a smooth domain ⊂⊂ Cn,we consider the Dirichlet problem for the complex Monge-Ampère equation ((ddcu)n = f dV, u|b ≡ φ). We state the Hölder regularity of the solution u when the boundary value φ is Hölder continuous and the density f is only L p, p > 1. Note that in former literature (Guedj-Kolodziej-Zeriahi) the weakness of the assumption f ∈ L p was balanced by taking φ ∈ C1,1 (in addition to assuming strongly pseudoconvex).

Keywords

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HÖLDER REGULARITY OF THE SOLUTION TO THE COMPLEX MONGE-AMPÈRE EQUATION WITH L^p DENSITY

LUCA BARACCO, TRAN VU KHANH, STEFANO PINTON AND GIUSEPPE ZAMPIERI

Abstract. On a smooth domain $\Omega \subset\subset \mathbb{C}^n$, we consider the Dirichlet problem for the complex Monge-Ampère equation ($(dd^c u)^n = f dV$, $u|_{b\Omega} \equiv \varphi$). We state the Hölder regularity of the solution u when the boundary value φ is Hölder continuous and the density f is only L^p, $p > 1$. Note that in former literature (Guedj-Kolodziej-Zeriahi) the weakness of the assumption $f \in L^p$ was balanced by taking $\varphi \in C^{1,1}$ (in addition to assuming Ω strongly pseudoconvex).
MSC: 32U05, 32U40, 53C55

1. Introduction

For a bounded pseudoconvex domain $\Omega \subset\subset \mathbb{C}^n$, the Dirichlet problem for the Monge-Ampère equation consists in

$$\begin{cases} (dd^c u)^n = f dV & \text{in } \Omega, \\ u = \varphi & \text{on } b\Omega. \end{cases} \tag{1.1}$$

In our discussion we take a density $0 \le f \in L^p(\Omega)$, $1 < p \le +\infty$, a boundary datum φ in some Hölder class and look for a plurisubharmonic solution $u \in C^\beta(\bar\Omega)$ for a certain β. Sometimes we use the notation $MA(\varphi, f)$ for the problem (1.1) and $u(\Omega, \varphi, f)$ for its solution. This problem has been extensively investigated in recent years under the assumption that Ω is strongly pseudoconvex. Bremermann [3], Walsh [15] and Bedford-Taylor[1] show that there is a solution $u \in C^0(\bar\Omega)$ if $\varphi \in C^0(b\Omega)$ and $f \in C^0(\bar\Omega)$. By the well known "comparison principle"' (cf. Kolodziej [12]), the solution is unique; what matters is to prove the Hölder continuity of this C^0-solution. In this direction, in [1] is proved that $u \in C^{\frac{\alpha}{2}}(\bar\Omega)$ if $\varphi \in C^\alpha(b\Omega)$, $f^{\frac{1}{n}} \in C^{\frac{\alpha}{2}}(\bar\Omega)$. A recent interest has been dedicated to the case when Ω is no longer strongly pseudoconvex but has a certain "finite type" m. Li proves in [14] that $u \in C^{\frac{\alpha}{m}}(\bar\Omega)$ if $\varphi \in C^\alpha(b\Omega)$ and $f^{\frac{1}{n}} \in C^{\frac{\alpha}{m}}(\bar\Omega)$. Ha and Khanh in [8] get the same conclusion with a more geometric notion of finite type (cf. (1.2) below) and have also a generalization for the infinite type. Coming back to the case of Ω strongly pseudoconvex, Caffarelli, Kohn and Nirenberg prove in [4] that $u \in C^\infty(\bar\Omega)$, for $\varphi \in C^\infty(b\Omega)$ and $f \in C^\infty(\bar\Omega)$, in case $f > 0$ in $\bar\Omega$. Lowering the smoothness of f gives the problem additional difficulty. Guedj, Kolodziej and Zeriahi prove in [7] that if $f \in L^p(\Omega)$ with $p > 1$ and $\varphi \in C^{1,1}(b\Omega)$ then $u \in C^\gamma(\bar\Omega)$ for any $\gamma < \gamma_p := \frac{2}{qn+1}$ where $\frac{1}{q} + \frac{1}{p} = 1$. Recently, Charabati has obtained in [6] that $u \in C^{\frac{\gamma}{2}}(\bar\Omega)$ for the same datum as in [7] on a bounded strongly hyperconvex Lipschitz domain i.e. on a domain for which there exists a Lipschitz plurisubharmonic defining function ρ such that $(dd^c\rho)^n \ge c dV$. Our purpose is twofold: to lower the regularity of φ and to allow a (geometric) finite type m for Ω with some $m \ge 2$. What we get is that if $f \in L^p(\Omega)$ with $p > 1$ and

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$\varphi \in C^\alpha(b\Omega)$ with $0 < \alpha \le 2$ then $u \in C^{\frac{\alpha}{m}}(\bar\Omega)$ if $\alpha < \gamma_p$ otherwise $u \in C^{\frac{\gamma}{m}}(\bar\Omega)$ for any $\gamma < \gamma_p$. To go into the detail of our geometric setting we consider a submanifold $S \subset b\Omega$ of CR dimension 0. Let d_S be the distance to S and $(L_{b\Omega})$ be the Levi form of $b\Omega$. We assume that $b\Omega$ has finite type m along S in the sense that

$$L_{b\Omega} \gtrsim d_S^{m-2}. \tag{1.2}$$

To convert (1.2) into a suitable property for our use, we need two basic results. First, from Khanh and Zampieri [10], we know that (1.2) implies the potential-theoretic "$t^{\frac{1}{m}}$-property". By [9] and [8] this implies in turn that there is an exhaustion function ρ which defines Ω by $\rho < 0$ such that

$$i\partial\bar\partial\rho \ge \mathrm{Id} \;\; \text{in } \Omega, \quad \rho \in C^{\frac{2}{m}}(\bar\Omega). \tag{1.3}$$

Remark 1.1. According to Catlin [5], if Ω has finite D'Angelo type D, then it has the "$t^{\frac{1}{m}}$-property" for $\frac{1}{m} := D^{-n^2 D^{n^2}}$; again, this implies the existence of the exhaustion $\rho \in C^{2D^{-n^2D^{n^2}}}(\bar\Omega)$ with $i\partial\bar\partial\rho \ge \mathrm{Id}$ in Ω.

It is (1.3) the property which rules many passages of this paper. Here is our result

Theorem 1.2. *Let $\Omega \subset\subset \mathbb{C}^n$ be a C^2-smooth pseudoconvex domain of finite type m with $m \ge 2$ in the sense of* (1.2) *and let $\varphi \in C^\alpha(b\Omega)$ with $0 < \alpha \le 2$ and $f \in L^p(\Omega)$ with $p > 1$. Then the unique solution u to $MA(\Omega, \varphi, f)$ is in $C^{\min(\frac{\alpha}{m}, \frac{\gamma}{m})}(\bar\Omega)$ with $\gamma < \gamma_p$ where $\gamma_p := \frac{2}{qn+1}$ and $\frac{1}{p} + \frac{1}{q}$=1.*

The proof follows in Section 3. Throughout the paper we use $\lesssim$ and $\gtrsim$ to denote an estimate up to a positive constant and $\sim$ for the combination of $\lesssim$ and $\gtrsim$. Finally, the indices m, α, p, γ and γ_p only take ranges as in Theorem 1.2.

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2. Hölder regularity of a subsolution

We say that $v \in C^0(\bar\Omega)$ is a subsolution to $MA(\Omega, \varphi, f)$ if v is plurisubharmonic, $v|_{b\Omega} = \varphi$ and $(dd^c v)^n \ge f$ in Ω.

Proposition 2.1. *Let ρ satisfy* (1.3)*. Then there is a subsolution $v \in C^0(\bar\Omega)$ to $MA(\Omega, \varphi, f)$ for $\varphi \in C^0(b\Omega)$ and $f \in L^p(\Omega)$.*

Proof. For a large ball $\mathbb{B}$ containing Ω, we define

$$\tilde f(z) := \begin{cases} f(z) & \text{if } z \in \Omega, \\ 0 & \text{if } z \in \mathbb{B} \setminus \Omega. \end{cases}$$

We consider the solutions

$$\begin{cases} u_1 = u(\mathbb{B}, 0, \tilde f) \in C^0(\bar{\mathbb{B}}) & \text{by Kolodziej on the ball } \mathbb{B} \text{ (strongly pseudoconvex) [13]}, \\ u_2 = u(\Omega, (-u_1)|_{b\Omega}, 0) \in C^0(\bar\Omega) & \text{by Blocki [2]}. \end{cases}$$

Taking summation u_1+u_2 we have a subsolution to $MA(\Omega,0,f)$ in $C^0(\bar\Omega)$. Using the solution $u(\Omega,\varphi,0)\in C^0(\bar\Omega)$ provided by [2] and putting

$$v=u_1+u_2+u(\Omega,\varphi,0),$$

we get the desired subsolution.

□

We change a little our setting and take $\varphi\in C^\alpha(b\Omega)$ and $f\in L^\infty(\Omega)$. If ζ is a general point of $b\Omega$ we set

$$v_\zeta(z):=\begin{cases}\varphi(\zeta)-c[-\rho(z)+|z-\zeta|^2]^{\frac{\alpha}{2}} & \text{if } 0<\alpha\le 1,\\ \varphi(\zeta)-\sum_j 2\mathsf{Re}\,\frac{\partial\varphi}{\partial z_j}(\zeta)(z_j-\zeta_j)-c[-\rho(z)+|z-\zeta|^2]^{\frac{\alpha}{2}} & \text{if } 1<\alpha\le 2.\end{cases}\tag{2.1}$$

If there is an exhaustion function $\rho\in C^{\frac{2}{m}}(\bar\Omega)$ such that $i\partial\bar\partial\rho\ge \text{Id}$ in Ω then we can find c, independent of ζ and only depending on $\|\varphi\|_{C^\alpha(\bar\Omega)}$ and $\|f\|_{L^\infty(\Omega)}$ such that (cf. [8, 14])

$$\begin{cases}v_\zeta(z)\le\varphi(z) & \text{if } z\in b\Omega,\\ v_\zeta(\zeta)=\varphi(\zeta),\\ (dd^cv_\zeta)^n\ge f & \text{in } \Omega,\\ v_\zeta\in C^{\frac{\alpha}{m}}(\bar\Omega).\end{cases}\tag{2.2}$$

Using the family $\{v_\zeta\}_{\zeta\in b\Omega}$ it is readily seen (cf. [8, 14]) that for any plurisubharmonic $C^0(\bar\Omega)$ solution to MA we have $u(\Omega,\varphi,f)\in C^{\frac{\alpha}{m}}(\bar\Omega)$ for $\varphi\in C^\alpha(b\Omega)$ and $f^{\frac{1}{n}}\in C^{\frac{\alpha}{m}}(\bar\Omega)$; in particular, $u(\Omega,\varphi,0)\in C^{\frac{\alpha}{m}}(\bar\Omega)$ for $\varphi\in C^\alpha(b\Omega)$. We lower the smoothness of f. We start from

Proposition 2.2. *Let ρ satisfy* (1.3). *Then there is a subsolution $v\in C^{\frac{\alpha}{m}}(\bar\Omega)$ to $MA(\,\Omega,\varphi,f)$ for $\varphi\in C^\alpha(b\Omega)$ and $f\in L^\infty(\Omega)$.*

Proof. We consider the solution $u(\Omega,\varphi,0)\in C^{\frac{\alpha}{m}}(\bar\Omega)$ by [14] and [8] and define

$$v=u(\Omega,\varphi,0)+c\rho.$$

For $c\gtrsim\|f\|^{\frac{1}{n}}_{L^\infty(\Omega)}$, v is a subsolution.

□

We now take $f\in L^p(\Omega)$.

Proposition 2.3. *Let ρ satisfy* (1.3).*Then there is a subsolution $v\in C^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}(\bar\Omega)$ to $MA(\,\Omega,\varphi,f)$ for $\varphi\in C^\alpha(b\Omega)$ and $f\in L^p(\Omega)$.*

Proof. We define $\mathbb{B}$ and $\tilde f$ as in the proof of Proposition 2.1. Since $\tilde f$ is bounded near the boundary, we consider the solutions

$$\begin{cases}u_1=u(\mathbb{B},0,\tilde f)\in C^\gamma(\bar{\mathbb{B}}) & \text{by [7]},\\ u_2=u(\Omega,(-u_1)|_{b\Omega},0)\in C^{\frac{\gamma}{m}}(\bar\Omega) & \text{by [14] and [8]}.\end{cases}$$

Taking the solution $u(\Omega,\varphi,0)\in C^{\frac{\alpha}{m}}(\bar\Omega)$ (cf. [8, 14]) and taking summation $v=u_1+u_2+u(\Omega,\varphi,0)$ we have the conclusion.

□

3. Hölder regularity of the solution - Proof of Thorem 1.2

We recall a crucial fact from [13]. For a general domain, not necessarily strongly pseudoconvex, the existence of $u(\Omega,\varphi,0)\in C^0(\bar\Omega)$ (which turns out to be equivalent to the existence of a maximal function with boundary datum φ), in addition to the existence of a subsolution $v\in C^0(\bar\Omega)$ for $\varphi\in C^0(b\Omega)$ and $f\in L^p(\Omega)$, implies the existence of a solution $u(\Omega,\varphi,f)\in L^\infty(\Omega)$. In particular,

Theorem 3.1. *(Kolodziej [13]) Assume Ω is defined by $\rho<0$ for $\rho\in C^0(\bar\Omega)$ such that $i\partial\bar\partial\rho\geq Id$ in Ω. Then for any $\varphi\in C^0(b\Omega)$, $f\in L^p(\Omega)$ there is a (unique) plurisubharmonic solution $u(\Omega,\varphi,f)\in L^\infty(\Omega)$.*

Proof. By the property of ρ, which implies b-regularity, there is a solution for continuous data, in particular for $f=0$, that is $u(\Omega,\varphi,0)$ (cf. [2]); thus there is a maximal function for the given boundary data. Again by the property of ρ, there is a subsolution for $\varphi\in C^0(b\Omega)$, $f\in L^p(\Omega)$ (Proposition 2.1 above). Then by [13] Thm. C p. 97 (3 lines after the statement) there is a solution in $L^\infty(\Omega)$.

□

Remark 3.2. The solution $u(\Omega,\varphi,f)$ for $\varphi\in C^0(b\Omega)$, $f\in L^p(\Omega)$ is in fact in $C^0(\bar\Omega)$ by Kolodziej [11]. Note that the paper makes the general assumption of pseudoconvexity of Ω but this is needless for this specific conclusion. This is confirmed by private communication with the author.

We assume from now $i\partial\bar\partial\rho\geq \mathrm{Id}$ in Ω for $\rho\in C^{\frac{2}{m}}(\bar\Omega)$. According to Proposition 2.3 above, when we take a smoother boundary datum $\varphi\in C^\alpha(b\Omega)$, there is a subsolution $v\in C^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}(\bar\Omega)$ for $f\in L^p$. What follows is dedicated to show that, in this situation, the L^∞ plurisubharmonic solution $u(\Omega,\varphi,f)$ is in fact in $C^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}(\bar\Omega)$.

Let $w:=u(\Omega,\varphi,0)\in C^{\frac{\alpha}{m}}(\bar\Omega)$ (cf. [8, 14]); comparison principle yields at once

$$v\leq u(\varphi,f)\leq w. \tag{3.1}$$

By (3.1) and by the $C^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}$ regularity of v and w we get

$$|u(z)-u(\zeta)|\lesssim|z-\zeta|^{\min(\frac{\alpha}{m},\frac{\gamma}{m})},\quad z\in\bar\Omega,\ \zeta\in b\Omega,$$

and therefore for δ suitably small

$$|u(z)-u(z')|\lesssim\delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})},\quad z,\,z'\in\Omega\setminus\Omega_\delta \text{ and } |z-z'|<\delta \tag{3.2}$$

where $\Omega_\delta:=\{z\in\mathbb{C}^n:\ r(z)<-\delta\}$ and r is a C^2 defining function for Ω with $|\nabla r|=1$ in a neighborhood of $b\Omega$. We have to prove that (3.2) also holds for $z,\,z'\in\Omega_\delta$. We use the notation

$$\begin{cases} u_{\frac{\delta}{2}}:=\sup\limits_{|\zeta|<\frac{\delta}{2}}u(z+\zeta),\quad z\in\bar\Omega_\delta,\\ \tilde u_{\frac{\delta}{2}}:=\frac{1}{\sigma_{2n-1}\left(\frac{\delta}{2}\right)^{2n-1}}\int_{b\mathbb{B}(z,\frac{\delta}{2})}u(\zeta)dS(\zeta),\quad z\in\bar\Omega_\delta,\end{cases} \tag{3.3}$$

where $\sigma_{2n-1}\left(\frac{\delta}{2}\right)^{2n-1}=Vol(b\mathbb{B}(z,\frac{\delta}{2}))$. It is a classical consequence of Riesz Theorem that for a general plurisubharmonic function u, not necessarily C^2, there is well defined Δu in the space of positive Borel measures. We use the notation $\|\Delta u\|^\Omega$ for the total mass of Δu on Ω.

Theorem 3.3. *Let $0<\epsilon<1$. We have*

$$\|\tilde{u}_{\frac{\delta}{2}}-u\|_{L^1(\Omega_\delta)}\lesssim \delta^{1-\epsilon}\|(-r)^{1+\epsilon}\Delta u\|^{\Omega_{\frac{\delta}{2}}}. \tag{3.4}$$

Proof. The proof is inspired by [7] Lemma 4.3; the novelty here consists in replacing δ^2 by $\delta^{1-\epsilon}(-r)^{1+\epsilon}$. We start from

$$\begin{aligned}\tilde{u}_{\frac{\delta}{2}}(z)-u(z)&\sim\frac{1}{\delta^{2n-1}}\int_{b\mathbb{B}(0,\frac{\delta}{2})}(u(z+\xi)-u(z))dS(\xi)\\&\sim\frac{1}{\delta^{2n-2}}\int_{b\mathbb{B}(0,\frac{\delta}{2})}dS(\xi)\int_0^1\nabla u(z+s\xi)\cdot\frac{\xi}{\delta}ds\\&\underset{\text{divergence thm.}}{=}\frac{1}{\delta^{2n-2}}\int_0^1 sds\int_{\mathbb{B}(0,\frac{\delta}{2})}\Delta u(z+s\xi)\\&\underset{s\xi=\zeta,\,s\delta=t}{\sim}\frac{1}{\delta^{2n-2}}\int_0^{\frac{\delta}{2}}\frac{t}{\delta^2}\frac{t^{-2n}}{\delta^{-2n}}dt\int_{\mathbb{B}(0,t)}\Delta u(z+\zeta).\end{aligned}\tag{3.5}$$

We denote by τ_ζ the translation by ζ and observe that $\tau_\zeta\Omega_\delta\subset\Omega_{\frac{\delta}{2}}\subset\subset\Omega$ for any $\zeta\in\mathbb{B}(0,t)$. Observing that the positive measure Δu has finite mass on compact subsets of Ω, in particular on $\Omega_{\frac{\delta}{2}}$, we get, for $t<\frac{\delta}{2}$

$$\int_{\Omega_\delta}dV(z)\int_{\mathbb{B}(0,t)}\Delta u(z+\zeta)\lesssim t^{2n}\int_{\Omega_{\frac{\delta}{2}}}\Delta u(z). \tag{3.6}$$

We now perform integration $\int_{\Omega_\delta}\cdot dV(z)$ in both sides of (3.5), apply (3.6) and end up with

$$\begin{aligned}\int_{\Omega_\delta}(\tilde{u}_{\frac{\delta}{2}}-u)(z)\,dV(z)&\lesssim\int_0^{\frac{\delta}{2}}t^{-2n+1}t^{2n}dt\int_{\Omega_{\frac{\delta}{2}}}\Delta u\\&\lesssim\int_0^{\frac{\delta}{2}}t\delta^{-(1+\epsilon)}dt\int_{\Omega_{\frac{\delta}{2}}}(-r)^{1+\epsilon}\Delta u\\&\sim\delta^{1-\epsilon}\|(-r)^{1+\epsilon}\Delta u\|^{\Omega_{\frac{\delta}{2}}}.\end{aligned}\tag{3.7}$$

□

At this point, the problem is to prove the boundedness of $\|(-r)^{1+\epsilon}\Delta u\|^{\Omega_{\frac{\delta}{2}}}$ uniformly in δ. This holds (cf. Theorem 3.4 below) because of the presence of the factor $(-r)^{1+\epsilon}$. In absence of this factor, one should suppose from the beginning that Δu has finite total mass on Ω; in turn, this would be a consequence of the hypothesis $\varphi\in C^{1,1}$ (cf. [7]).

Theorem 3.4. *We have*

$$\|(-r)^{1+\epsilon}\Delta u\|^{\Omega}\lesssim\|(-r)^{-1+\epsilon}u\|_{L^1(\Omega)}. \tag{3.8}$$

Proof. We take a system of smooth cut-off functions $\chi_\nu(|z|)\in C^\infty_c(\mathbb{B}^{2n}(0,\frac{1}{\nu}))$, $\|\chi_\nu\|_{L^1}\equiv1$, $\frac{1}{\nu}\to0$, and regularize

$$u_\nu:=\int_\Omega u(\tau)\chi_\nu(|z-\tau|)dV(\tau).$$

The u_ν's belong to $C^\infty(\Omega)$, converge to u on Ω, and satisfy

$$\begin{cases} \sup_{\Omega_{\frac{1}{\nu}}} |\nabla u_\nu| = \sup_{\Omega_{\frac{1}{\nu}}} |\nabla(u * \chi_\nu)| \le \nu \|u\|_{L^1(\Omega)} \\ \sup_{\Omega_{\frac{1}{\nu}}} u_\nu \le c \quad \text{independent of } \nu. \end{cases} \tag{3.9}$$

Now that the u_ν's are regular, the Δu_ν's are well defined functions and hence we use the notation $\Delta u_\nu dV$ for the associated measures. We have

$$\begin{aligned}
\int_{\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \Delta u_\nu dV(z) &= \int_{\Omega_{\frac{1}{\nu}}} \operatorname{div}((-r)^{1+\epsilon} \nabla u_\nu) dV(z) + (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \nabla r \cdot \nabla u_\nu dV(z) \\
&\underset{\text{Stokes}}{=} \int_{b\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu dS^{2n-1}(z) + (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \nabla r \cdot \nabla u_\nu dV(z) \\
&= \int_{b\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu dS^{2n-1}(z) + (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} \operatorname{div}((-r)^\epsilon (\nabla r\, u_\nu)) dV(z) \\
&\quad + \epsilon(1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^{\epsilon-1} \nabla r \cdot \nabla r u_\nu dV(z) - (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \Delta r u_\nu dV(z) \\
&\underset{\text{Stokes}}{=} \int_{b\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \nabla r \cdot \nabla u_\nu dS^{2n-1}(z) + (1+\epsilon) \int_{b\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \nabla r \cdot \nabla r\, u_\nu dV(z) \\
&\quad + \epsilon(1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^{\epsilon-1} \nabla r \cdot \nabla r u_\nu dV(z) - (1+\epsilon) \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon \Delta r u_\nu dV(z) \\
&\underset{(3.9)}{\lesssim} O(\nu^{-\epsilon}) + (1+\epsilon) O(\nu^{-\epsilon}) + \int_{\Omega_{\frac{1}{\nu}}} (-r)^{\epsilon-1} |u_\nu| dV(z) + \int_{\Omega_{\frac{1}{\nu}}} (-r)^\epsilon |u_\nu| dV(z) \\
&\lesssim O(\nu^{-\epsilon}) + \|(-r)^{-1+\epsilon} u\|_{L^1(\Omega)}.
\end{aligned} \tag{3.10}$$

On the other hand, since u is plurisubharmonic, then Δu is a measure on Ω and $\Delta u_\nu dV \underset{\text{weakly}}{\to} \Delta u$. The conclusion follows from the following elementary Lemma

Lemma 3.5. *Assume* $\Delta u_\nu \ge 0$ *and*

$$\begin{cases} \int_{\Omega_{\frac{1}{\nu}}} (-r)^{1+\epsilon} \Delta u_\nu dV \ \textit{are bounded} \\ \Delta u_\nu dV \underset{\textit{weakly}}{\to} \Delta u. \end{cases}$$

Then

$$\int_\Omega (-r)^{1+\epsilon} \Delta u \ \textit{is bounded.}$$

The proof is just a consequence of the dominated convergence theorem for the sequence $(-r)^{1+\epsilon} \psi_\nu \Delta u_\nu dV \to (-r)^{1+\epsilon} \Delta u$ where ψ_ν are the characteristic functions of the sets $\Omega_{\frac{1}{\nu}}$. With Lemma 3.5 in our hands, we get the conclusion of the proof of Theorem 3.4.

□

To end the proof of Theorem 1.2 we shall need the stability estimate (Theorem (1.1) in [7])

Theorem 3.6. *Fix $0 \leq f \in L^p(\Omega)$, $p > 1$. Let U, W be two bounded plurisubharmonic functions in Ω such that $(dd^cU)^n = fdV$ in Ω and let $U \geq W$ on $\partial\Omega$. Fix $s \geq 1$ and $0 \leq \eta < \frac{1}{nq+s}$, $\frac{1}{p} + \frac{1}{q} = 1$. Then there exists a uniform constant $C = C(\eta, \|f\|_{L^p(\Omega)}) > 0$ such that*

$$\sup_{\Omega}(W - U) \leq C\|(W - U)_+\|^{\eta}_{L^s(\Omega)},$$

where $(W - U)_+ := \max(W - U, 0)$.

End of Proof of Theorem 1.2. Again, we follow the guidelines of [7]. Along with $\tilde{u}_\delta$ defined by (3.3) we introduce $\hat{u}_\delta := \frac{1}{\sigma_{2n}(\delta)^{2n}} \int_{\mathbb{B}(z,\delta)} u(\zeta)dV(\zeta), \quad z \in \Omega_\delta$. We recall that Lemma 4.2 of [7] states the equivalence between

$$\sup_{\Omega_\delta}(u_{\frac{\delta}{2}} - u) \lesssim \delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})} \tag{3.11}$$

and

$$\sup_{\Omega_\delta}(\hat{u}_{\frac{\delta}{2}} - u) \lesssim \delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})} \tag{3.12}$$

On the other hand, on account of the obvious inequalities

$$\hat{u}_\delta \leq \tilde{u}_\delta \leq u_\delta,$$

we see that whatever of (3.11) and (3.12) is equivalent to

$$\sup_{\Omega_\delta}(\tilde{u}_{\frac{\delta}{2}} - u) \lesssim \delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}. \tag{3.13}$$

We have thus to prove (3.13). To see it, we remark that

$$\begin{aligned} \|\tilde{u}_{\frac{\delta}{2}} - u\|_{L^1(\Omega_\delta)} &\underset{\text{Theorem 3.3}}{\lesssim} \delta^{1-\epsilon}\|(-r)^{1+\epsilon}\Delta u\|^{\Omega_{\frac{\delta}{2}}} \\ &\underset{\text{Theorem 3.4}}{\lesssim} \delta^{1-\epsilon}. \end{aligned} \tag{3.14}$$

By (3.2), we have for a suitable c

$$\tilde{u}_{\frac{\delta}{2}} \leq u_{\frac{\delta}{2}} \leq u + c\delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})} \quad \text{in a neighborhood of } b\Omega_\delta.$$

We are going to apply Theorem 3.6 for Ω_δ with $U := u + c\delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}$, $W := \tilde{u}_{\frac{\delta}{2}}$ and $s := 1$; thus we get

$$\begin{aligned} \sup_{\Omega_\delta}\left(\tilde{u}_{\frac{\delta}{2}} - (u + c\delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})})\right) &\underset{\text{stability estimate}}{\lesssim} \|\left(\tilde{u}_{\frac{\delta}{2}} - (u + c\delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})})\right)_+\|^{\eta}_{L^1(\Omega_\delta)} \\ &\lesssim \|\tilde{u}_{\frac{\delta}{2}} - u\|^{\eta}_{L^1(\Omega_\delta)} \\ &\underset{(3.14)}{\lesssim} \delta^{(1-\epsilon)\eta}, \end{aligned} \tag{3.15}$$

for any $\eta < \frac{1}{2}\gamma_p = \frac{1}{np+1}$. It follows

$$\sup_{\Omega_\delta}\left(\tilde{u}_{\frac{\delta}{2}} - u\right) \lesssim \delta^{(1-\epsilon)\eta} + \delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})} \lesssim \delta^{\frac{\gamma}{2}} + \delta^{\min(\frac{\alpha}{m},\frac{\gamma}{m})}$$

and hence (3.13) is proved since $m \geq 2$. Here the last inequality follows by choosing $\epsilon = \frac{\gamma_p - \gamma}{\gamma_p + \gamma} > 0$ and $\eta = \frac{1}{4}(\gamma_p + \gamma) < \frac{1}{2}\gamma_p$ since $\gamma < \gamma_p$.

From (3.2) and (3.11) (which is equivalent to (3.13)), it is easy to prove that

$$|u(z) - u(z')| \lesssim |z - z'|^{\min(\frac{\alpha}{m}, \frac{\gamma}{m})} \quad \text{for any } z, z' \in \bar{\Omega};$$

thus the proof of Theorem 1.2 is complete. □

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