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Abstract
We consider the motion of convex surfaces with normal speed given by arbitrary strictly monotone, homogeneous degree one functions of the principal curvatures (with no further smoothness assumptions). We prove that such processes deform arbitrary uniformly convex initial surfaces to points in finite time, with spherical limiting shape. This result was known previously only for smooth speeds. The crucial new ingredient in the argument, used to prove convergence of the rescaled surfaces to a sphere without requiring smoothness of the speed, is a surprising hidden divergence form structure in the evolution of certain curvature quantities.

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CONTRACTION OF CONVEX SURFACES BY NON-SMOOTH FUNCTIONS OF CURVATURE

BEN ANDREWS AND JAMES MCCOY

ABSTRACT. We consider the motion of convex surfaces with normal speed given by arbitrary strictly monotone, homogeneous degree one functions of the principal curvatures (with no further smoothness assumptions). We prove that such processes deform arbitrary uniformly convex initial surfaces to points in finite time, with spherical limiting shape. This result was known previously only for smooth speeds. The crucial new ingredient in the argument, used to prove convergence of the rescaled surfaces to a sphere without requiring smoothness of the speed, is a surprising hidden divergence form structure in the evolution of certain curvature quantities.

1. INTRODUCTION

A natural class of geometric evolution equations for hypersurfaces consists of those in which each point moves with normal velocity equal to a monotone function f of the principal curvatures $\kappa_1, \cdot, \kappa_n$ of the hypersurface at that point. These include the motion by mean curvature with $f = H = \kappa_1 + \cdots + \kappa_n$, the Gauss curvature flow with $f = K = \kappa_1 \cdots \kappa_n$, and many other examples. In previous work it has been found that in many cases where f is a smooth homogeneous degree one function of the principal curvatures, the evolving hypersurfaces remain convex, contract to a point in finite time, and become asymptotically spherical in shape as the final time is approached: This was first shown for the mean curvature flow by Huisken [13], then for the nth root of the Gauss curvature (and, under additional assumptions on the curvature of the initial hypersurface, the square root of the scalar curvature) by Chow [10, 11]. The same result for arbitrary smooth convex f, or concave f vanishing on the boundary of the positive cone $\Gamma^+ = \{(\kappa_1, \cdots, \kappa_n) : \kappa_i > 0 \text{ for all } i = 1, \cdots, n\}$ was proved in [1], while smooth f which are concave on Γ^+ and have dual function $f_*(r_1, \ldots, r_n) = f(r_1^{-1}, \ldots, r_n^{-1})^{-1}$ also concave were dealt with in [4], while the case with f_* concave and zero on the boundary of the positive cone was included in [7]. Finally, in the case of two-dimensional surfaces in space, it was proved in [5] that a similar result holds for convex surfaces moving by arbitrary smooth, homogeneous degree one f, with no further concavity assumptions.

In [6], the authors with Holder, Wheeler, Wheeler and Williams began an investigation of the effects of reduced smoothness of f in such evolution equations. This complicates the analysis in several ways, not least of which is that functions of curvature no longer have well-defined second derivatives in general, so that maximum principle arguments cannot be directly applied. By approximating a non-smooth speed f using suitable smoothings, we showed that under nonsmooth, convex, degree-one homogeneous functions of the principal curvatures, solution hypersurfaces contract to round points in finite time. Under suitable

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rescaling the convergence is exponential in C^2 to the sphere. The restriction to convex speeds arose from two main issues: Firstly, this was required in our construction of suitably smoothed speed functions which satisfy monotonicity, convexity and homogeneity requirements; and secondly, because the argument for establishing decay of the traceless part of the second fundamental form independent of the smoothing parameter required application of Harnack estimates for fully nonlinear partial differential equations, and this required certain structure in the evolution equation for the second fundamental form which we could establish only in the case of convex f.

In this paper, we establish a similar result for arbitrary (non-smooth) strictly monotone homogeneous degree one functions of the principal curvatures in the case of convex surfaces evolving in space, thereby extending the main result of [5] to non-smooth speeds. This requires two substantial changes to the argument developed in [6]: First, we set up a different family of approximating speeds that maintain homogeneity without requiring convexity; and second, we observe that, for $n=2$, evolution equations for degree zero homogeneous functions of curvature can be rewritten in divergence form, facilitating the application of a weak Harnack inequality for divergence form operators to show exponential convergence to spheres without requiring the speed to satisfy any second order condition. Without this rather surprising divergence structure, the evolution equation for curvature quantities includes terms involving the second derivatives of f which are uncontrollable as the smoothing parameter approaches zero, and this prevents the application of Harnack or similar estimates.

In a forthcoming article we will consider hypersurfaces evolving by nonsmooth degree-one homogeneous concave speeds in higher dimensions, a situation which introduces different technical difficulties.

Let M_0 be a compact, strictly convex surface without boundary, embedded in $\mathbb{R}^3$ and represented by $X_0 : \mathbb{S}^2 \to X_0\left(\mathbb{S}^2\right) = M_0 \subset \mathbb{R}^3$. In this paper we require M_0 to be of class C^2, that is, M_0 can be represented locally as the graph of a C^2 function u. We consider the family of maps $X_t = X(\cdot,t)$ evolving according to

$$\begin{aligned} \frac{\partial}{\partial t} X(x,t) &= -F(\mathscr{W}(x,t))\,\nu(x,t) \;\; x\in\mathbb{S}^2,\; 0<t\le T<\infty \\ X(\cdot,0) &= X_0, \end{aligned} \tag{1}$$

where $\mathscr{W}(x,t)$ is the matrix of the Weingarten map of $M_t = X_t\left(\mathbb{S}^2\right)$ at the point $X_t(x)$ and $\nu(x,t)$ is the outer unit normal to M_t at $X_t(x)$. The function F is assumed to satisfy the following conditions:

Conditions 1.1.

a) *$F(\mathscr{W}) = f(\kappa(\mathscr{W}))$ where $\kappa(\mathscr{W})$ gives the eigenvalues κ_1, κ_2 of $\mathscr{W}$ and f is a symmetric function defined on the positive cone*
$$\Gamma^+ = \left\{\kappa = (\kappa_1,\kappa_2) \in \mathbb{R}^2 : \kappa_i > 0 \text{ for all } i = 1,2\right\};$$
b) *f is homogeneous of degree 1: $f(k\kappa) = kf(\kappa)$ for any $k>0$.*

c) *f is locally uniformly elliptic in the sense that for every compact subset D of Γ^+ there are constants $0 < \underline{C}_D \le \overline{C}_D$ such that, for every $\kappa \in D$ and $\kappa' \in D$ with $\kappa_i' \ge \kappa_i$ for each i,*
$$\underline{C}_D \sum_{i=1}^{2}\left(\kappa_i' - \kappa_i\right) \le f\left(\kappa'\right) - f(\kappa) \le \overline{C}_D \sum_{i=1}^{2}\left(\kappa_i' - \kappa_i\right).$$
d) *f is strictly positive on Γ^+ and $f(1,1) = 1$.*

Remark 1.1. It is a consequence of condition c) that f is locally Lipschitz in the positive cone: In each of the subsets $D_s = \{\kappa \in \Gamma_+ :\ s^{-1} \leq \kappa_i \leq s\}$ we have constants $\underline{C}_s$ and $\overline{C}_s$ as in Condition c). For any κ and κ' in D_s we can use condition c) to compare both $f(\kappa)$ and $f(\kappa')$ to $f(\tilde{\kappa})$, where $\tilde{\kappa} \in D_s$ is defined by $\tilde{\kappa}_i = \max\{\kappa_i, \kappa_i'\}$ for each i. We conclude that $|f(\kappa') - f(\kappa)| \leq \overline{C}_s \sum_{i=1}^{2} |\kappa_i' - \kappa_i|$ for each $\kappa, \kappa' \in D_s$.

Nonsmooth speeds may be constructed in several ways and are of interest in image processing applications of curvature flow (see [15], for example). Writing

$$\kappa_{\max} = \frac{1}{2}\left(\kappa_1 + \kappa_2 + |\kappa_1 - \kappa_2|\right) \text{ and } \kappa_{\min} = \frac{1}{2}\left(\kappa_1 + \kappa_2 - |\kappa_1 - \kappa_2|\right)$$

we may set

$$f(\kappa_1, \kappa_2) = p\kappa_{\min} + q\kappa_{\max},$$

where $p, q \in (0,1), p+q = 1, p \geq q$. When $p = q = \frac{1}{2}$ this corresponds to the mean curvature, but otherwise, f is not smooth due to the presence of the absolute value. It is easy to check that this example speed satisfies Conditions 1.1, in particular, c) is satisfied with $\underline{C}_D = q$ and $\overline{C}_D = p$.

Other nonsmooth speeds may be obtained by taking maxima or minima of other curvature functions satisfying Conditions 1.1. Examples include

$$f(\kappa) = \max\left(\frac{1}{2}H, a|A|\right) \text{ or } f(\kappa) = \min\left(\frac{1}{2}H, a|A|\right),$$

where $|A| = \sqrt{\kappa_1^2 + \kappa_2^2}$, for any constant a such that $\frac{1}{2} < a < \frac{1}{\sqrt{2}}$, or

$$f(\kappa) = \max\left(\frac{1}{2}H, bK^{\frac{1}{2}}\right) \text{ or } f(\kappa) = \min\left(\frac{1}{2}H, bK^{\frac{1}{2}}\right),$$

for any constant $b > 1$. That these functions satisfy Conditions 1.1 follows from the properties of H, $|A|$ and $K^{\frac{1}{2}}$.

Further nonsmooth speeds arise in Bellman type equations and parabolic Isaacs equations. In particular, the latter second order operators are uniformly elliptic but not convex or concave. We refer the interested reader to [9] for more information about these operators in the setting of elliptic equations.

Our main result in this article is the following:

Theorem 1.2. *Let X_0 be a uniformly convex C^2 immersion of $\mathbb{S}^2$ in $\mathbb{R}^3$, and suppose F satisfies Conditions 1.1. Then the evolution equation* (1) *has a unique solution in $C^2_{loc}(\mathbb{S}^2 \times [0,T)) \cap C^{2,\beta}_{loc}(\mathbb{S}^2 \times (0,T))$ for some finite maximal $T > 0$. The corresponding surfaces $M_t := X\left(\mathbb{S}^2, t\right)$ converge to a point $p \in \mathbb{R}^2$ as $t \to T$. The rescaled surfaces $\tilde{M}_t$ given by*

$$\tilde{X}(x,t) = \frac{X(x,t) - p}{\sqrt{2(T-t)}} \tag{2}$$

converge in the $C^{2,\beta'}$ topology as $t \to T$ to an embedding $\tilde{X}(\cdot, T)$ whose image is equal to the unit sphere centred at the origin, where $0 < \beta' < \beta$. The convergence of the rescaled curvatures to 1 *is exponential with respect to the natural time parameter. If the speed F is more regular then the solution and its exponential convergence to the sphere is correspondingly more regular.*

The structure of this article is as follows. In Section 2 we set up our smooth approximating speeds F^{ε} for any F satisfying Conditions 1.1. In Section 3 we establish estimates independent of ε, including an estimate on the curvature pinching ratio of the evolving surfaces and uniform parabolicity of all our flows. Together with an upper speed bound while the inradius of solution surfaces remains positive, we show that solutions to our flow equations converge in finite time to a point $p \in \mathbb{R}^2$. In Section 5 we show, by rewriting the evolution equation for a certain function of curvature in divergence form, that under appropriate scaling solution surfaces converge exponentially in C^2 to the unit sphere.

We remark that it is possible to relax the C^2 assumption on the initial surface slightly. However, since we require that the initial surface have bounded ratio of principal curvatures at each point and have positive speed F, our argument does not allow initial hypersurfaces less regular than $C^{1,1}$. The question of whether the same asymptotic behaviour holds for more irregular initial surfaces is delicate and depends strongly on the detailed structure of the speed function F, as illustrated by counterexamples constructed in [7, Section 11].

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2. Construction of approximating speeds

A slightly non-standard approach to mollifying nonsmooth speed functions is needed to ensure the mollified functions are all homogeneous of the same degree. It is different from the construction in [6] since we do not assume convexity of F. Since we are working with $\kappa \in \Gamma^+$ (a property that is preserved under (1)), we may first make a change of coordinates $z_i = \ln \kappa_i$. Now define $\psi : \mathbb{R}^2 \to \mathbb{R}$ via $f(\kappa) = e^{\psi(z)}$, that is, $\psi(z) = \ln f(\kappa)$. Conditions 1.1 correspond naturally to properties of ψ:

Lemma 2.1. *f satisfies Conditions 1.1 if and only if the function ψ defined as above has the following properties:*

a) *ψ is well-defined on $\mathbb{R}^2$, symmetric in z_1, z_2 and locally Lipschitz;*
b) *ψ is increasing in each argument, strictly in the sense that for any compact set $K \subset \mathbb{R}^2$ there exist constants $\underline{A}_K$ and $\overline{A}_K$ such that for and any (w_1, w_2) and (z_1, z_2) in K with $z_i \geq w_i$ for $i = 1, 2$,*

$$\underline{A}_K \sum_{i=1}^{2} (z_i - w_i) \leq \psi(z_1, z_2) - \psi(w_1, w_2) \leq \overline{A}_K \sum_{i=1}^{2} (z_i - w_i);$$

c) $\psi(0,0) = 0$;
d) *For any $\tilde{k} \in \mathbb{R}$,*

$$\psi\left(z + \tilde{k}\right) := \psi\left(z_1 + \tilde{k}, z_2 + \tilde{k}\right) = \psi(z) + \tilde{k}.$$

Proof:

a) That f is positive on Γ^+ ensures ψ is well defined. That $\kappa = (\kappa_1, \kappa_2) \in \Gamma^+$ means $z = (z_1, z_2) \in \mathbb{R}^2$. Since f is symmetric in the κ_i, it follows that ψ is symmetric in the z_i. That ψ is locally Lipschitz follows from the fact that f is locally Lipschitz, using the mean value theorem for the exponential function (a more detailed argument appears below).
b) If K is a compact subset of $\mathbb{R}^2$, then $D = \{(e^{z_1}, e^{z_2}) : (z_1, z_2) \in K\}$ is a compact subset of Γ_+. Then for (w_1, w_2) and (z_1, z_2) in K with $z_i > w_i$ we write

$$\psi(z_1, z_2) - \psi(w_1, w_2) = \log f(\kappa') - \log f(\kappa)$$

where $\kappa_i = \mathrm{e}^{w_i}$ and $\kappa_i' = \mathrm{e}^{z_i}$. Since $f(\kappa') > f(\kappa)$ we have

$$\frac{1}{f(\kappa')}(f(\kappa') - f(\kappa)) \leq \psi(z_1, z_2) - \psi(w_1, w_2) \leq \frac{1}{f(\kappa)}(f(\kappa') - f(\kappa)).$$

On the assumption that condition 1.1 c) holds, we deduce that

$$\frac{\underline{C}_D}{f(\kappa')}\sum_{i=1}^{2}(\kappa_i' - \kappa_i) \leq \psi(z_1, z_2) - \psi(w_1, w_2) \leq \frac{\overline{C}_D}{f(\kappa)}\sum_{i=1}^{2}(\kappa_i' - \kappa_i).$$

Finally, since $\kappa_i' - \kappa_i = \mathrm{e}^{z_i} - \mathrm{e}^{w_i}$, we have

$$\kappa_{\min}\sum_{i=1}^{2}(z_i - w_i) \leq \sum_{i=1}^{2}\kappa_i(z_i - w_i) \leq \sum_{i=1}^{2}(\kappa_i' - \kappa_i) \leq \sum_{i=1}^{2}\kappa_i'(z_i - w_i) \leq \kappa_{\max}'\sum_{i=1}^{2}(z_i - w_i).$$

Combining these, and observing that $\kappa_{\min}$, $\kappa_{\max}$ and $f(\kappa)$ are bounded above and below by positive constants on D, we deduce the required inequality. The converse implication is similar.

c) The normalisation $f(1,1) = 1$ corresponds precisely to $\psi(0,0) = 0$;

d) That f is homogeneous of degree 1 implies that for any $k > 0$,

$$\ln f(k\kappa) = \ln[kf(\kappa)] = \ln f(\kappa) + \ln k.$$

This corresponds exactly to the required statement in terms of ψ, where $\tilde{k} = \ln k$. □

Now we will obtain smooth approximations to f by mollifying the function ψ in a standard way. Specifically, for $\varepsilon > 0$, set

$$\psi_\varepsilon(z) = \int_{\mathbb{R}^2} j_\varepsilon(y)\,\psi(z-y)\,dy$$

and then $f_\varepsilon(\kappa) = e^{\psi_\varepsilon(z)}$. Here, as a mollifier, we are using a smooth non-negative function j_ε that vanishes outside $B_\varepsilon(O)$ and satisfies $\int_{\mathbb{R}^2} j_\varepsilon(y)\,dy = 1$. One suitable function j_ε may be defined via

$$j(x) = \begin{cases} c_2\,\mathrm{e}^{\frac{1}{|x|^2-1}} & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1, \end{cases}$$

where c_2 is a normalisation constant such that $\int_{\mathbb{R}^2} j(x)\,dx = 1$. For $\varepsilon > 0$ we then take

$$j_\varepsilon(x) = \varepsilon^{-2}\,j\left(\frac{x}{\varepsilon}\right).$$

For more details on mollifiers and their standard properties we refer the reader to [12, Chapter 7]. We next check that the smooth functions f_ε satisfy Conditions 1.1.

Lemma 2.2. *The smooth functions $f_\varepsilon : \Gamma^+ \to \mathbb{R}$ defined via the above process and appropriately normalised, satisfy Conditions 1.1, and the constants $\underline{C}_D$ and $\overline{C}_D$ can be chosen independent of ε for each compact $D \subset \Gamma_+$. In addition, $f_\varepsilon \to f$ uniformly on compact subsets of Γ^+ and we have*

$$\mathrm{e}^{-\varepsilon} f(\kappa) \leq f_\varepsilon(\kappa) \leq \mathrm{e}^{\varepsilon} f(\kappa). \tag{3}$$

Proof: Given $\varepsilon > 0$, that the function ψ_ε is smooth is a standard property of mollification. Since $\kappa_i = e^{z_i}$ it is clear that f_ε is defined on Γ^+ and since ψ_ε is smooth, so is f_ε. That $\psi_\varepsilon \to \psi$ uniformly on compact subsets of $\mathbb{R}^2$ is a standard property of mollification. Via the coordinate change and definitions of f_ε and f, it follows that $f_\varepsilon \to f$ uniformly on compact subsets of Γ^+.

We check Conditions 1.1 in order, using Lemma 2.1 where needed.

a) Since ψ is symmetric in z_1, z_2 (Lemma 2.1, a)), so is ψ_ε. Hence also f_ε is symmetric in κ_1, κ_2.
b) By Lemma 2.1d) we have for each $\varepsilon > 0$

$$\psi_\varepsilon\left(z+\tilde{k}\right) = \int_{\mathbb{R}^2} j_\varepsilon\left(y\right)\psi\left(z+\tilde{k}-y\right)dy = \int_{\mathbb{R}^2} j_\varepsilon\left(y\right)\left[\psi\left(z-y\right)+\tilde{k}\right]dy$$
$$= \int_{\mathbb{R}^2} j_\varepsilon\left(y\right)\psi\left(z-y\right)dy + \tilde{k} = \psi_\varepsilon(z)+\tilde{k}.$$

It follows that (writing $z_i = \log\kappa_i$)

$$f_\varepsilon(k\kappa) = \mathrm{e}^{\psi_\varepsilon(\log(k\kappa_1),\log(k\kappa_2))} = \mathrm{e}^{\psi_\varepsilon(z_1+\log k, z_2+\log k)} = \mathrm{e}^{\psi_\varepsilon(z)+\log k} = k\mathrm{e}^{\psi(z)} = kf_\varepsilon(\kappa).$$

c) Strict monotonicity (in the sense of Condition 1.1 c)) for f is equivalent to the strict monotonicity of ψ in the sense of Lemma 2.1 b). Applying the mollification, we find that for any compact subset K of $\mathbb{R}^2$ and any $w \in K$ and $z \in K\cap(w+\Gamma_+)$,

$$\underline{A}_{\hat{K}}\sum_{i=1}^{2}(z_i-w_i) \leq \psi_\varepsilon(z)-\psi_\varepsilon(w) \leq \overline{A}_{\hat{K}}\sum_{i=1}^{2}(z_i-w_i).$$

provided $\varepsilon \leq 1$, where $\hat{K} = \overline{B_1(K)}$, which is again a compact subset of $\mathbb{R}^2$. Thus ψ_ε, and hence also f_ε, satisfy the strict monotonicity conditions uniformly in ε.
d) We have

$$\psi_\varepsilon\left(0,0\right) = \int_{\mathbb{R}^2} j_\varepsilon\left(y\right)\psi\left(-y\right)dy = \int_{B_\varepsilon(O)} j_\varepsilon\left(y\right)\psi\left(-y\right)dy.$$

Enclosing $B_\varepsilon\left(O\right)$ by a cube centred at O and using monotonicity of ψ we estimate

$$-\varepsilon = \psi\left(-\varepsilon,-\varepsilon\right) \leq \psi\left(-y\right) \leq \psi\left(\varepsilon,\varepsilon\right) \leq \varepsilon,$$

so

$$-\varepsilon \leq \psi_\varepsilon\left(0,0\right) \leq \varepsilon$$

and therefore

$$\mathrm{e}^{-\varepsilon} \leq f_\varepsilon\left(1,1\right) \leq \mathrm{e}^{\varepsilon}.$$

It follows that each f_ε may be normalised.

Similarly, for any fixed $z \in \mathbb{R}^2$, we may estimate

$$\psi_\varepsilon\left(z\right) = \int_{\mathbb{R}^2} j_\varepsilon\left(y-z\right)\psi\left(y\right)dy \leq \max_{B_\varepsilon(z)}\psi \leq \psi(z+\varepsilon) = \psi(z)+\varepsilon,$$

and similarly $\psi_\varepsilon(z) \geq \psi(z)-\varepsilon$. Taking exponentials yields (3).

To the f_ε we associate F^ε by defining $F^\varepsilon\left(\mathscr{W}\right) := f_\varepsilon\left(\kappa_1,\kappa_2\right)$, again appealing to results of [4, 8] for properties of such functions and relationships between their first and second derivatives.

3. Estimates on curvature and geometry independent of ε

Since the functions f_ε satisfy Conditions 1.1, it is known from [5] that smooth convex initial surfaces contract in finite time to round points under the flows

$$\begin{aligned} &\frac{\partial}{\partial t}X^\varepsilon\left(x,t\right) = -F^\varepsilon\left(\mathscr{W}^\varepsilon\left(x,t\right)\right)\nu^\varepsilon\left(x,t\right)\ x\in\mathbb{S}^2,\ 0<t\leq T_\varepsilon<\infty \\ &X\left(\cdot,0\right) = X_0. \end{aligned} \tag{4}$$

Note that we use the same initial data for each flow. We want to establish estimates independent of ε to deduce the behaviour of the 'limit flow' (1). Because f may be no smoother than locally Lipschitz, the convergence of the rescaled solutions to the sphere will be in $C^{2,\beta}$, as was the case in [6].

Theorem 3.1. *The maximal time T_ε of existence of a solution to* (4) *satisfies*

$$\frac{\rho_-^2}{2} \le T_\varepsilon \le \frac{\rho_+^2}{2},$$

where ρ_- and ρ_+ denote the inradius and circumradius respectively of the initial surface M_0. In particular, these estimates on the maximal time are independent of ε.

Proof: The results of [5] give existence of a unique solution of (4) for each $\varepsilon > 0$, with the solution existing until they contract to a single point. The proof of the estimate on the maximal time follows by comparison of M_t^ε with enclosing and enclosed spheres as in [6, Lemma 5.1]. □

A lower bound on the speed, independent of ε, follows easily from the evolution equation for F^ε and the properties of the f_ε.

Lemma 3.2. *Under the flows* (4)*, if we restrict to $\varepsilon \le 1$, then as long as the solution exists,*

$$F^\varepsilon(\cdot,t) \ge \frac{1}{\mathrm{e}} \min_{M_0} F.$$

Proof: As in [6, Lemma 4.2, i)], under (4) the speed F_ε satisfies

$$\frac{\partial}{\partial t} F^\varepsilon = \mathscr{L}^\varepsilon F^\varepsilon + \dot{F}_\varepsilon^{kl} h_{km}^\varepsilon h_{\varepsilon\ l}^{m} F^\varepsilon, \tag{5}$$

where we denote with ε quantities associated to the evolving surface M_t^ε. In particular, $\mathscr{L}^\varepsilon := \dot{F}_\varepsilon^{ij} \nabla_i^\varepsilon \nabla_j^\varepsilon$ where ∇^ε denotes the covariant derivative on M_t^ε. In view of Lemma 2.2, $\mathscr{L}^\varepsilon$ is an elliptic operator and evolution equations such as (5) are parabolic. Applying the maximum principle to (5), we have

$$F_\varepsilon(x,t) \ge \min_{M_t^\varepsilon} F^\varepsilon \ge \min_{M_0} F^\varepsilon \ge \mathrm{e}^{-\varepsilon} \min_{M_0} F \ge \frac{1}{\mathrm{e}} \min_{M_0} F,$$

where we have used (3) and the restriction to $\varepsilon \le 1$. □

Examination of the proof in Section 3 of [5] shows that the following estimate on the pinching ratio of the principal curvatures holds under (1) for every $\varepsilon > 0$:

Lemma 3.3. *Let M_0 be a smooth, uniformly convex surface. As long as the solution to the flow* (4) *exists, the principal curvatures satisfy*

$$\frac{\kappa_{\max}^\varepsilon}{\kappa_{\min}^\varepsilon}(\cdot,t) \le \max_{M_0} \frac{\kappa_{\max}}{\kappa_{\min}} =: c_p.$$

Lemma 3.3 implies, in particular, that the set within Γ^+ in which the principal curvatures of M_t^ε lie under the flow (4) is a cone over a compact subset of $\{|A| = 1\}$.

Corollary 3.4. *There are constants $0 < \underline{C} \le \overline{C} < \infty$, independent of $\varepsilon \in (0,1]$, such that, for $i = 1,2$, while a solution to* (4) *exists, we have for all $x \in \mathbb{S}^2$*

$$\underline{C} \le \dot{f}_\varepsilon^i(\kappa_\varepsilon(x,t)) \le \overline{C}.$$

Proof: By the homogeneity of f_ε, the partial derivatives $\dot{f}_\varepsilon^i$ are homogeneous of degree zero, so we have $\dot{f}_\varepsilon^i(\kappa) = \dot{f}_\varepsilon^i\left(\frac{\kappa}{|\kappa|}\right)$ for any $\kappa \in \Gamma_+$. By Lemma 3.3 we have that for each $(x,t) \in \mathbb{S}^2 \times 0,T)$, and each $\varepsilon > 0$, we have $\frac{\kappa^\varepsilon_{\max}}{\kappa^\varepsilon_{\min}} \le c_p$, and hence $\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|}$ lies in the compact set $\left\{\kappa \in \Gamma_+ :\ |\kappa| = 1,\ \frac{\kappa_{\max}}{\kappa_{\min}} \le c_p\right\}$. For $\delta \in (0,1]$ we also have that $\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|}$ and $\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|} + \delta e_1$ both lie in the compact set

$$\tilde{D} := \left\{\kappa \in \Gamma_+ :\ 1 \le |\kappa| \le 2,\ \kappa_{\min} \ge \frac{1}{\sqrt{1+c_p^2}}\right\}.$$

Therefore by Lemma 2.2 we have

$$\underline{C}_{\tilde{D}}\delta \le f_\varepsilon\left(\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|} + \delta e_1\right) - f_\varepsilon\left(\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|}\right) \le \overline{C}_{\tilde{D}}\delta,$$

and consequently

$$\dot{f}^i_\varepsilon(\kappa_\varepsilon(x,t)) = \dot{f}^i_\varepsilon\left(\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|}\right) = \lim_{\delta\to 0_+} \frac{f_\varepsilon\left(\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|} + \delta e_i\right) - f_\varepsilon\left(\frac{\kappa_\varepsilon(x,t)}{|\kappa_\varepsilon(x,t)|}\right)}{\delta} \in \left[\underline{C}_{\tilde{D}}, \overline{C}_{\tilde{D}}\right].$$

□

The curvature pinching estimate of Lemma 3.3 has implications for the geometry of the evolving surfaces: According to [1, Theorem 5.1], this implies a bound on the ratio of the circumradius $\rho^+_{\varepsilon,t}$ to the inradius $\rho^-_{\varepsilon,t}$ of the surface M^ε_t:

Lemma 3.5. *For any $\varepsilon > 0$ and $t \in [0,T_\varepsilon)$ we have $\rho^+_{\varepsilon,t} \le 2c_p\rho^-_{\varepsilon,t}$.*

This has an immediate corollary relating the circumradius and inradius to the time of existence, by a simple refinement of Lemma 3.1:

Corollary 3.6. *For any $\varepsilon > 0$ and $t \in [0,T_\varepsilon)$ we have*

$$\frac{1}{2c_p}\sqrt{2(T_\varepsilon - t)} \le \rho^-_{\varepsilon,t} \le \rho^+_{\varepsilon,t} \le 2c_p\sqrt{2(T_\varepsilon - t)}.$$

Proof: Inner sphere barriers give lower bounds on the inradius for times $t' > t$: $\rho^-_{\varepsilon,t'} \ge \sqrt{(\rho^-_{\varepsilon,t})^2 - 2(t'-t)}$ while the right-hand side remains positive. Since $\rho^-_{\varepsilon,t'}$ approaches zero as $t' \to T_\varepsilon$, we deduce that $(T_\varepsilon - t) \ge \frac{1}{2}\left(\rho^-_{\varepsilon,t}\right)^2$. Similarly, using outer sphere barriers we deduce that $(T_\varepsilon - t) \le \frac{1}{2}\left(\rho^+_{\varepsilon,t}\right)^2$. The result follows by combining these inequalities with Lemma 3.5. □

From the construction of the smoothed speeds F_ε we can also control the time of existence T_ε and the final point p_ε as $\varepsilon \to 0$:

Lemma 3.7. *Suppose $\bar{\varepsilon} \in (0,1]$, and ε_1 and ε_2 are both in the interval $(0,\bar{\varepsilon}]$. Then*

$$\mathrm{e}^{-2\bar{\varepsilon}}T_{\varepsilon_1} \le T_{\varepsilon_2} \le \mathrm{e}^{2\bar{\varepsilon}}T_{\varepsilon_1}$$

and

$$|p_{\varepsilon_2} - p_{\varepsilon_1}| \le 4c_p\sqrt{\sinh(2\bar{\varepsilon})T_{\varepsilon_2}}.$$

Proof: Since $0 < \varepsilon_1, \varepsilon_2 \le \bar{\varepsilon}$, the estimates (3) imply that $\mathrm{e}^{-2\bar{\varepsilon}}F_{\varepsilon_2} \le F_{\varepsilon_1} \le \mathrm{e}^{2\bar{\varepsilon}}F_{\varepsilon_2}$. It follows that $\hat{X}(.,t) := X_{\varepsilon_2}(.,\mathrm{e}^{2\bar{\varepsilon}}t)$ evolves with inward normal speed equal to $\mathrm{e}^{2\bar{\varepsilon}}F_{\varepsilon_2} \ge F_{\varepsilon_1}$, so $\hat{X}$ is an inner barrier to the flow with speed F_{ε_1}. It follows that $T_{\varepsilon_1} \ge \mathrm{e}^{-2\bar{\varepsilon}}T_{\varepsilon_2}$. Interchanging ε_1 and ε_2 gives $T_{\varepsilon_2} \ge \mathrm{e}^{-2\bar{\varepsilon}}T_{\varepsilon_1}$, proving the first claimed inequality.

To prove the bound on $|p_{\varepsilon_2} - p_{\varepsilon_1}|$, we observe that by the argument above, p_{ε_2} is enclosed by $X_{\varepsilon_1}(\mathbb{S}^2, \mathrm{e}^{-2\bar{\varepsilon}} T_{\varepsilon_2})$. By Corollary 3.6 this implies that

$$|p_{\varepsilon_2} - p_{\varepsilon_1}| \leq 2c_p \sqrt{2(T_{\varepsilon_1} - \mathrm{e}^{-2\bar{\varepsilon}} T_{\varepsilon_2})}.$$

The result follows using the inequality $T_{\varepsilon_1} \leq \mathrm{e}^{2\bar{\varepsilon}} T_{\varepsilon_2}$. □

We now establish an upper bound on the speed, independent of ε, based upon an idea used by Tso [17] for Gauss curvature flow. The proof is similar to the case in [6] except that we use Lemma 3.3 instead of convexity of f to estimate one of the zero order terms in the relevant evolution equation.

Lemma 3.8. *Under the flow* (4) *for $\varepsilon \leq 1$, for $t \in [0, T_\varepsilon)$, we have the upper bound*

$$F^\varepsilon(x,t) \leq \max\left\{ 4\mathrm{e}c_p \frac{\max_{M_0} F}{\rho_0^-} \sqrt{2(T_\varepsilon - t)}, \frac{64\overline{C} c_p^3}{\sqrt{2(T_\varepsilon - t)}} \right\},$$

where ρ_0^- is the inradius of M_0.

Proof: Fix $\bar{t}$, and choose the origin at the centre of an insphere (of radius $r_- = \rho^-_{\varepsilon,\bar{t}}$) of $M^\varepsilon_{\bar{t}}$. Then the support function $s^\varepsilon := \langle X^\varepsilon, \nu^\varepsilon \rangle$ of M^ε_t satisfies $s^\varepsilon(x,t) \geq r_-$ for all $x \in \mathbb{S}^2$ and $t \in [0, \bar{t}]$. The function $Q^\varepsilon = \frac{F^\varepsilon}{2s^\varepsilon - r_-}$ evolves according to

$$\frac{\partial}{\partial t} Q^\varepsilon = \mathscr{L}^\varepsilon Q^\varepsilon + 4\dot{F}^{kl}_\varepsilon \frac{\nabla^\varepsilon_k s^\varepsilon}{2s^\varepsilon - \rho^-_{\varepsilon,t}} \nabla^\varepsilon_l Q^\varepsilon + Q^2_\varepsilon \left[4 - r_- \frac{\dot{F}^{kl}_\varepsilon h^\varepsilon_{km} h^{m}_{\varepsilon\ l}}{F^\varepsilon} \right].$$

We estimate the last term at a maximum point using Corollary 3.4: By the Cauchy-Schwarz inequality and the Euler homogeneity relation we have

$$\dot{F}^{kl}_\varepsilon h^\varepsilon_{km} h^{m}_{\varepsilon\ l} = \sum_{i=1}^2 \dot{f}^i_\varepsilon (\kappa^\varepsilon_i)^2 \geq \frac{\left(\sum_{i=1}^2 \dot{f}^i_\varepsilon \kappa^\varepsilon_i\right)^2}{\sum_{i=1}^2 \dot{f}^i_\varepsilon} \geq \frac{f^2_\varepsilon}{2\overline{C}}.$$

Therefore, noting also that $2s^\varepsilon - r_- \geq r_-$ we have

$$\frac{\partial}{\partial t} Q^\varepsilon \leq \mathscr{L}^\varepsilon Q^\varepsilon + 4\dot{F}^{kl}_\varepsilon \frac{\nabla^\varepsilon_k s^\varepsilon}{2s^\varepsilon - r_-} \nabla^\varepsilon_l Q^\varepsilon + Q^2_\varepsilon \left[4 - \frac{r_-^2}{2\overline{C}} Q_\varepsilon \right].$$

If a new maximum of Q^ε is attained, then the left-hand side is non-negative, while the first term on the right-hand side is non-positive and the second term is equal to zero, so the bracketed term must be non-negative. Therefore,

$$Q^\varepsilon(x,t) \leq \max\left\{ \max_{M_0} Q^\varepsilon, \frac{8\overline{C}}{r_-^2} \right\}$$

for $(x,t) \in \mathbb{S}^2 \times [0, \bar{t}]$, where we have again used (3) and assumed $\varepsilon \leq 1$. Now we apply this at $t = \bar{t}$, obtaining

$$F^\varepsilon(x, \bar{t}) \leq 2\rho^+_{\varepsilon,\bar{t}} \max\left\{ \frac{\mathrm{e} \max_{M_0} F}{\rho_0^-}, \frac{8\underline{C}}{r_-^2} \right\}.$$

The result now follows after applying the bounds on $\rho^\pm_{\varepsilon,\bar{t}}$ from Corollary 3.6. □

We can also deduce Hölder continuity estimates for curvatures, independent of ε:

Lemma 3.9. *The second fundamental form of solutions of* (4) *remains bounded in* $C^{0,\beta}$*, for some* $\beta \in (0,1)$ *depending only on* $\underline{C}$ *and* $\overline{C}$*: Precisely, there exists* C_2 *such that for each* $\varepsilon \in (0,1]$ *and* $t \in [0, T_\varepsilon)$ *we have*

$$|\kappa^{\varepsilon}_{\max}(x,t)| \leq \frac{C_2}{\sqrt{T_\varepsilon - t}}$$

and for $|t_1 - t_2| < \frac{1}{2}|T_\varepsilon - t_2|$ *we have*

$$\left| h^{\varepsilon}_{(x_1,t_1)}(v_1,v_1) - h^{\varepsilon}_{(x_2,t_2)}(v_2,v_2) \right| \leq \frac{C_2}{t_2^{\beta/2}} \frac{d_{t_2}((x_1,v_1),(x_2,v_2))^{\beta} + |t_2 - t_1|^{\beta/2}}{(T_\varepsilon - t_2)^{(1+\beta)/2}}.$$

This follows from the results of [3], applied to the evolution equation for the function which defines the evolving surface as a radial graph: Corollaries 3.4 and 3.6 imply that this evolution equation is uniformly parabolic on time intervals of the form $[T_\varepsilon - 2a, T_\varepsilon - a]$ for any $a \in (0, T_\varepsilon/2)$, so that the estimates of [3] can be applied.

4. Rescaled solutions

In order to investigate the behaviour of solutions near the final time we use an appropriate rescaling, as in [13]: We define

$$\tilde{X}_\varepsilon(x,t) = \frac{1}{\sqrt{2(T_\varepsilon - t)}} \left(X_\varepsilon(x,t) - p_\varepsilon \right).$$

We further define a new time parameter as

$$\tau = -\frac{1}{2} \ln \left(1 - \frac{t}{T_\varepsilon} \right)$$

and observe $t \in [0, T_\varepsilon)$ corresponds to $\tau \in [0, \infty)$. The rescaled immersions $\tilde{M}^{\varepsilon}_{\tau} = \tilde{X}^{\varepsilon}(\mathbb{S}^2, \tau)$ evolve according to

$$\frac{\partial}{\partial \tau} \tilde{X}_\varepsilon(x,\tau) = -F_\varepsilon\left(\tilde{\mathscr{W}}_\varepsilon(x,\tau) \right) \tilde{\nu}_\varepsilon(x,\tau) + \tilde{X}_\varepsilon(x,\tau), \tag{6}$$

with initial condition

$$\tilde{X}_\varepsilon(x,0) = \frac{1}{\sqrt{2T_\varepsilon}} \left[X_0(x) - p_\varepsilon \right].$$

Uniform bounds on the inner and outer radii of $\tilde{M}^{\varepsilon}_{\tau}$ follow by rescaling the result of Corollary 3.6, as in [1, Lemma 7.2]. A uniform upper bound on $\tilde{F}_\varepsilon$ holds, by translating the result of Lemma 3.8 to the rescaled solutions. Finally, the second fundamental form of $\tilde{M}^{\varepsilon}_{\tau}$ is uniformly bounded in $C^{2,\beta}$ (away from the initial time) by the result of Lemma 3.9.

5. Improving curvature pinching estimates

In this section we establish an exponential decay estimate for a curvature pinching quantity under (6), independent of the approximation parameter ε. Note that the uniform curvature pinching estimate of Lemma 3.3 continues to hold for solutions of (6), since this estimate is invariant under rescaling. The uniform parabolicity established in Corollary 3.4 also holds for (6), since $\dot{f}^{i}_{\varepsilon}$ is homogeneous of degree zero.

In previous work [6, Section 7], we established decay estimates on a curvature pinching quantity by applying a weak Harnack estimate for supersolutions of non-divergence form uniformly parabolic equations with measureable coefficients. In that setting the gradient terms arising in the evolution of second fundamental form had a favourable sign, due to the convexity of the speed F. In the present setting, we make no such convexity assumption, so the gradient terms which arise are uncontrollable as the approximation parameter ε approaches zero. However, we show that in the case $n = 2$, these terms can be absorbed into a remarkable divergence form structure, so that weak Harnack estimates for divergence form equations can be applied to obtain the required decay estimate.

To simplify notation we drop the tildes and the ε subscripts and superscripts.

Lemma 5.1. *Let* $Q(\mathscr{W}) := \frac{2|\mathring{A}|^2}{H^2} = \frac{(\kappa_1-\kappa_2)^2}{(\kappa_1+\kappa_2)^2} =: q(\kappa_1, \kappa_2)$. *Then under equation* (6), *the quantity* Q *evolves on* $\{Q \neq 0\}$ *according to*

$$\frac{\partial}{\partial t}Q = \nabla_k\left(\dot{F}^{kl}\nabla_l Q\right) - \frac{\dot{f}^1(2\kappa_2-\kappa_1)}{2\kappa_2 Q}(\nabla_1 Q)^2 - \frac{\dot{f}^2(2\kappa_1-\kappa_2)}{2\kappa_1 Q}(\nabla_2 Q)^2 + \left[2\frac{\dot{f}^1}{\kappa_2} - \frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\right]\nabla_1 Q\nabla_1 h_{22} + \left[2\frac{\dot{f}^2}{\kappa_1} - \frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\right]\nabla_2 Q\nabla_2 h_{11} - \frac{8f}{(\kappa_1+\kappa_2)^3}\left((\nabla_1 h_{22})^2 + (\nabla_2 h_{11})^2\right). \tag{7}$$

Here the terms on the right are computed at a point where $\kappa_1 \neq \kappa_2$, *in a frame which diagonalises the second fundamental form.*

Proof: As in [5, Equation 6], the quantity Q evolves under (6) according to

$$\frac{\partial}{\partial t}Q = \mathscr{L}Q + \left(\dot{Q}^{ij}\ddot{F}^{kl,rs} - \dot{F}^{ij}\ddot{Q}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs} + \left(\dot{F}^{kl}h_k{}^m h_{ml} - 1\right)\dot{Q}^{ij}h_{ij}, \tag{8}$$

where $\mathscr{L}Q = \dot{F}^{kl}\nabla_k\nabla_l Q$ and the -1 term arises from rescaling. Since Q is homogeneous of degree zero, $\dot{Q}^{ij}h_{ij} = 0$ and the last term vanishes. Next we rewrite some of the components $\nabla_i h_{jk}$ in terms of ∇Q: Since $\nabla_i Q = \dot{Q}^{kl}\nabla_i h_{kl}$, in coordinates diagonalising $\mathscr{W}$ we have

$$\nabla_i Q = \dot{q}^1\nabla_i h_{11} + \dot{q}^2\nabla_i h_{22}.$$

Observing that $\dot{q}^1 = \frac{4\kappa_2(\kappa_1-\kappa_2)}{(\kappa_1+\kappa_2)^3}$ and $\dot{q}^2 = \frac{4\kappa_1(\kappa_2-\kappa_1)}{(\kappa_1+\kappa_2)^3}$, we deduce that

$$\nabla_1 h_{11} = \frac{1}{\dot{q}^1}\left(\nabla_1 Q - \dot{q}^2\nabla_1 h_{22}\right) \text{ and } \nabla_2 h_{22} = \frac{1}{\dot{q}^2}\left(\nabla_2 Q - \dot{q}^1\nabla_2 h_{11}\right) \tag{9}$$

Using these, we may compute as in [5] that

(10)

$$
\begin{aligned}
&\left(\dot{Q}^{ij}\ddot{F}^{kl,rs}-\dot{F}^{ij}\ddot{Q}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs}\\
&=\left[\frac{\dot{q}^1\ddot{f}^{11}-\dot{f}^1\ddot{q}^{11}}{(\dot{q}^1)^2}\right](\nabla_1 Q)^2+\left[\frac{\dot{q}^2\ddot{f}^{22}-\dot{f}^2\ddot{q}^{22}}{(\dot{q}^2)^2}\right](\nabla_2 Q)^2\\
&\quad+2\left[\left(\ddot{f}^{12}-\frac{\dot{q}^2}{\dot{q}^1}\ddot{f}^{11}\right)-\frac{\dot{f}^1}{\dot{q}^1}\left(\ddot{q}^{12}-\frac{\dot{q}^2}{\dot{q}^1}\ddot{q}^{11}\right)\right]\nabla_1 Q\nabla_1 h_{22}\\
&\quad+2\left[\left(\ddot{f}^{12}-\frac{\dot{q}^1}{\dot{q}^2}\ddot{f}^{22}\right)-\frac{\dot{f}^2}{\dot{q}^2}\left(\ddot{q}^{12}-\frac{\dot{q}^1}{\dot{q}^2}\ddot{q}^{22}\right)\right]\nabla_2 Q\nabla_2 h_{11}\\
&\quad+\left\{\dot{q}^1\left[\left(\frac{\dot{q}^2}{\dot{q}^1}\right)^2\ddot{f}^{11}-2\left(\frac{\dot{q}^2}{\dot{q}^1}\right)\ddot{f}^{12}+\ddot{f}^{22}\right]-\dot{f}^1\left[\left(\frac{\dot{q}^2}{\dot{q}^1}\right)^2\ddot{q}^{11}-2\left(\frac{\dot{q}^2}{\dot{q}^1}\right)\ddot{q}^{12}+\ddot{q}^{22}\right]\right.\\
&\qquad\left.+2\left(\frac{\dot{f}^1\dot{q}^2-\dot{f}^2\dot{q}^1}{\kappa_1-\kappa_2}\right)\right\}(\nabla_1 h_{22})^2\\
&\quad+\left\{\dot{q}^2\left[\left(\frac{\dot{q}^1}{\dot{q}^2}\right)^2\ddot{f}^{22}-2\left(\frac{\dot{q}^1}{\dot{q}^2}\right)\ddot{f}^{12}+\ddot{f}^{11}\right]-\dot{f}^2\left[\left(\frac{\dot{q}^1}{\dot{q}^2}\right)^2\ddot{q}^{22}-2\left(\frac{\dot{q}^1}{\dot{q}^2}\right)\ddot{q}^{12}+\ddot{q}^{11}\right]\right.\\
&\qquad\left.+2\left(\frac{\dot{f}^1\dot{q}^2-\dot{f}^2\dot{q}^1}{\kappa_1-\kappa_2}\right)\right\}(\nabla_2 h_{11})^2.
\end{aligned}
$$

This expression can be simplified by observing that $\frac{\dot{q}^2}{\dot{q}^1}=-\frac{\kappa_1}{\kappa_2}$ and using the homogeneity of f and q: In particular, we have

$$
\ddot{f}^{12}-\frac{\dot{q}^2}{\dot{q}^1}\ddot{f}^{11}=\ddot{f}^{12}+\frac{\kappa_1}{\kappa_2}\ddot{f}^{11}=\frac{\ddot{f}^{12}\kappa_2+\ddot{f}^{11}\kappa_1}{\kappa_2}=0,
$$

since $\dot{f}^1$ is homogeneous of degree zero. Similarly $\ddot{f}^{12}-\frac{\dot{q}^1}{\dot{q}^2}\ddot{f}^{22}=0$, while $\frac{\ddot{q}^{12}-\frac{\dot{q}^2}{\dot{q}^1}\ddot{q}^{11}}{\dot{q}^1}=-\frac{1}{\kappa_2}$ and $\frac{\ddot{q}^{12}-\frac{\dot{q}^1}{\dot{q}^2}\ddot{q}^{22}}{\dot{q}^2}=-\frac{1}{\kappa_1}$ since $\dot{q}^i$ is homogeneous of degree -1. We also have

$$
\left(\frac{\dot{q}^1}{\dot{q}^2}\right)^2\ddot{f}^{22}-2\left(\frac{\dot{q}^1}{\dot{q}^2}\right)\ddot{f}^{12}+\ddot{f}^{11}=\left(\frac{\kappa_2}{\kappa_1}\right)^2\ddot{f}^{22}+2\left(\frac{\kappa_2}{\kappa_1}\right)\ddot{f}^{12}+\ddot{f}^{11}=\frac{1}{\kappa_1^2}D^2f(\kappa,\kappa)=0,
$$

since f is linear along radial lines. Similarly we have $\left(\frac{\dot{q}^2}{\dot{q}^1}\right)^2\ddot{q}^{11}-2\left(\frac{\dot{q}^2}{\dot{q}^1}\right)\ddot{q}^{12}+\ddot{q}^{22}=0$ since q is constant along radial lines. Finally, we note that

$$
\frac{\dot{f}^1\dot{q}^2-\dot{f}^2\dot{q}^1}{\kappa_1-\kappa_2}=-4\frac{\dot{f}^1\kappa_1+\dot{f}^2\kappa_2}{(\kappa_1+\kappa_2)^3}=-\frac{4f}{(\kappa_1+\kappa_2)^3}.
$$

Substituting these expressions in (10) yields

$$
\begin{aligned}
(11)\qquad &\left(\dot{Q}^{ij}\ddot{F}^{kl,rs}-\dot{F}^{ij}\ddot{Q}^{kl,rs}\right)\nabla_i h_{kl}\nabla_j h_{rs}\\
&=\left[\frac{\dot{q}^1\ddot{f}^{11}-\dot{f}^1\ddot{q}^{11}}{(\dot{q}^1)^2}\right](\nabla_1 Q)^2+\left[\frac{\dot{q}^2\ddot{f}^{22}-\dot{f}^2\ddot{q}^{22}}{(\dot{q}^2)^2}\right](\nabla_2 Q)^2\\
&\quad+\frac{2\dot{f}^1}{\kappa_2}\nabla_1 Q\nabla_1 h_{22}+\frac{2\dot{f}^2}{\kappa_1}\nabla_2 Q\nabla_2 h_{11}-\frac{8f}{(\kappa_1+\kappa_2)^3}\left((\nabla_1 h_{22})^2+(\nabla_2 h_{11})^2\right).
\end{aligned}
$$

Now we use the relationship between second derivatives of F and derivatives of f (see [1] or [4]) to rewrite the $\mathscr{L}$ term of (7): Writing $\mathscr{L}Q=\nabla_k\left(\dot{F}^{kl}\nabla_l Q\right)-\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l Q$ we compute

$$
\begin{aligned}
(12)\qquad &\ddot{F}^{kl,rs}\nabla_k h_{rs}\nabla_l Q\\
&=\frac{1}{2}\ddot{F}^{kl,rs}\nabla_i h_{rs}\left(\delta_{ik}\nabla_l Q+\delta_{il}\nabla_k Q\right)\\
&=\frac{1}{2}\left[\ddot{f}^{kr}\nabla_i h_{kk}\left(\delta_{ik}\nabla_k Q+\delta_{ik}\nabla_k Q\right)+2\sum_{k<r}\frac{\dot{f}^k-\dot{f}^r}{\kappa_k-\kappa_r}\nabla_i h_{kr}\left(\delta_{ik}\nabla_r Q+\delta_{ir}\nabla_k Q\right)\right]\\
&=\ddot{f}^{kr}\nabla_k h_{rr}\nabla_k Q+\frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\nabla_i h_{12}\left(\delta_{i1}\nabla_2 Q+\delta_{i2}\nabla_1 Q\right)\\
&=\ddot{f}^{11}\nabla_1 h_{11}\nabla_1 Q+\ddot{f}^{12}\nabla_1 h_{22}\nabla_1 Q+\ddot{f}^{21}\nabla_2 h_{11}\nabla_2 Q\\
&\quad+\frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\left(\nabla_1 h_{12}\nabla_2 Q+\nabla_2 h_{12}\nabla_1 Q\right)\\
&=\frac{\ddot{f}^{11}}{\dot{q}^1}(\nabla_1 Q)^2+\frac{\ddot{f}^{22}}{\dot{q}^2}(\nabla_2 Q)^2+\left(\ddot{f}^{12}-\frac{\dot{q}^2}{\dot{q}^1}\ddot{f}^{11}+\frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\right)\nabla_1 h_{22}\nabla_1 Q\\
&\quad+\left(\ddot{f}^{21}-\frac{\dot{q}^1}{\dot{q}^2}\ddot{f}^{22}+\frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\right)\nabla_2 h_{11}\nabla_2 Q\\
&=\frac{\ddot{f}^{11}}{\dot{q}^1}(\nabla_1 Q)^2+\frac{\ddot{f}^{22}}{\dot{q}^2}(\nabla_2 Q)^2+\frac{\dot{f}^1-\dot{f}^2}{\kappa_1-\kappa_2}\left(\nabla_1 h_{22}\nabla_1 Q+\nabla_2 h_{11}\nabla_2 Q\right).
\end{aligned}
$$

Substituting (12) and (11) into (8) yields (7), using the identities $\frac{\ddot{q}^{11}}{(\dot{q}^1)^2}=\frac{2\kappa_2-\kappa_1}{2\kappa_2 q}$ and $\frac{\ddot{q}^{22}}{(\dot{q}^2)^2}=\frac{2\kappa_1-\kappa_2}{2\kappa_1 q}$. □

Proposition 5.1. *There exist $\beta\in(0,1)$ and positive constants C and γ such that for every $\varepsilon\in(0,1]$,*

$$\|\tilde{\mathscr{W}}_\varepsilon(x,\tau)-I\|_{C^{0,\beta}(M_\tau)}\le C\mathrm{e}^{-\gamma\tau}$$

for all $\tau\in[0,\infty)$, where the left-hand side is understood in the sense of Lemma 3.9.

Proof: The result of Lemma 5.1, together with the bounds on $\frac{\kappa_2}{\kappa_1}$ and on $\dot{f}^i$ from Lemma 3.3 and Corollary 3.4, imply the following for the solution of (6) for $\varepsilon\in(0,1]$:

$$
\begin{aligned}
(13)\qquad \frac{\partial}{\partial\tau}Q\le\nabla_k\left(\dot{F}^{kl}\nabla_l Q\right)+a\frac{\dot{F}^{kl}\nabla_k Q\nabla_l Q}{Q}+\frac{b}{\sqrt{Q}(\kappa_1+\kappa_2)}|\nabla Q|\left(|\nabla_1 h_{22}|+|\nabla_2 h_{11}|\right)\\
-\frac{c}{(\kappa_1+\kappa_2)^2}\left((\nabla_1 h_{22})^2+(\nabla_2 h_{11})^2\right),
\end{aligned}
$$

where a, b and c are constants independent of ε, with c strictly positive. Applying the Cauchy-Schwarz inequality to the last term on the first line gives the following:

$$\frac{\partial}{\partial\tau}Q \le \nabla_k\left(\dot{F}^{kl}\nabla_l Q\right) + d\frac{\dot{F}^{kl}\nabla_k Q\nabla_l Q}{Q}, \tag{14}$$

for some constant $d \ge 0$, from which we conclude that where $Q \ne 0$ we have

$$\frac{\partial}{\partial\tau}Q^{1+d} \le \nabla_k\left(\dot{F}^{kl}\nabla_l Q^{1+d}\right). \tag{15}$$

Finally, we observe that for $\varepsilon > 0$, since F_ε is smooth, Schauder estimates imply that the solution $\tilde{X}^\varepsilon$ is smooth. Consequently at points with $Q = 0$, both sides of the above inequality are zero, so the inequality (15) holds everywhere.

Although we now have a subsolution of a divergence-form equation, the weak Harnack inequality for divergence form parabolic operators (see [16, Theorem 1.2]), requires a uniformly parabolic equation defined on Euclidean spacetime cylinder. In order to achieve this we proceed to write the evolving surfaces as graphs over suitable time intervals:

Fix any $\tau_0 \ge 0$, and take the origin to be at an incentre of $\tilde{M}^\varepsilon_{\tau_0}$. It follows from Corollary 3.6 that $\tilde{M}^\varepsilon_{\tau_0}$ encloses $B_{\frac{1}{2c_p}}(0)$ and is enclosed by $B_{4c_p}(0)$. By the comparison principle using spherical barriers, it follows that there exists $\delta > 0$ (independent of τ_0 and ε) such that $\tilde{M}^\varepsilon_\tau$ encloses $B_{\frac{1}{4c_p}}(0)$ and is enclosed by $B_{4c_p}(0)$ for $\tau_0 \le \tau \le \tau_0 + \delta$. It follows by convexity that we can write $\tilde{M}^\varepsilon_\tau = \{r(z,\tau)z : z \in \mathbb{S}^2\}$, with r a $C^{2,\beta}$ function on $\mathbb{S}^2 \times [\tau_0, \tau_0 + \delta]$ satisfying $\frac{1}{4c_p} \le r(z,\tau) \le 8c_p$ and $|\bar{\nabla} r| \le C$, where C is a constant which depends only on c_p and $\bar{\nabla}$ is the standard connection on $\mathbb{S}^2$. The map $\hat{X}(z,t) = r(z,t)z$ evolves according to the equation

$$\frac{\partial\hat{X}}{\partial\tau} = \frac{\partial\tilde{X}}{\partial\tau} - \frac{\bar{\nabla}_i r}{r\sqrt{r^2 + |\bar{\nabla} r|^2}} \cdot \frac{\partial\tilde{X}}{\partial z_1},$$

so, since the coefficients on the last term are bounded, Q evolves according to

$$\frac{\partial}{\partial\tau}Q^{1+d} \le \nabla_k\left(\dot{F}^{kl}\nabla_l Q^{1+d}\right) + a^i\nabla_i Q^{1+d},$$

where the coefficients a^i are bounded. Here ∇ is the connection on the evolving surface, which are related to the connection $\bar{\nabla}$ by terms involving derivatives up to second order of r, which are uniformly bounded. Thus we also have

$$\frac{\partial}{\partial\tau}Q^{1+d} \le \bar{\nabla}_k\left(\dot{F}^{kl}\bar{\nabla}_l Q^{1+d}\right) + \tilde{a}^i\bar{\nabla}_i Q^{1+d},$$

where the coefficients $\tilde{a}^i$ are again uniformly bounded. A similar inequality holds in suitable local coordinates (for example, graphical coordinates over suitably small balls) for $\mathbb{S}^2$ with respect to coordinate derivatives:

$$\frac{\partial}{\partial\tau}Q^{1+d} \le \partial_k\left(\dot{F}^{kl}\partial_l Q^{1+d}\right) + \hat{a}^i\partial_i Q^{1+d},$$

and we note that $\dot{F}^{kl}$ is uniformly bounded above and below compared to the Euclidean inner product in such local coordinates.

Now we apply the weak Harnack inequality: Let $Q_+ = \sup_{M_{\tau_0}} Q \in [0,1)$. Applying the weak Harnack estimate to $Z = Q_+^{1+d} - Q^{1+d}$ (which is a supersolution of a uniformly parabolic divergence form equation with bounded coefficients), using a chaining argument with suitable coordinate patches, to deduce that

$$Z(z,\tau_0+s)\geq C(s)\int_{\mathbb{S}^2\times[\tau_0,\tau_0+\delta/2]}Z. \tag{16}$$

We will show that the right hand side of (16) is bounded below by a positive multiple of Q_+^{1+d}, by using the following result which controls the Hölder norm of the second fundamental form in terms of a bound for Q, if we wait for a short time:

Lemma 5.2. *There exists a constant C (independent of ε and $\tau_0\geq 1$) such that*

$$\|\tilde{\mathscr{W}}\|_{C^{0,\beta}(\mathbb{S}^2\times[\tau_0+\delta/4,\tau_0+\delta])}\leq C\sqrt{Q_+}.$$

Proof: We begin by observing that a bound on Q implies strong geometric bounds, amounting to bounds on the Lipschitz difference from the unit sphere: The formula in [2, Lemma 5] gives the following:

$$\begin{aligned}\left|\langle X(x,t)-\bar{q}(t),\nu(x,t)\rangle-\frac{1}{8\pi}\int_{M_t}H\,d\mu\right| &\leq \frac{\sqrt{Q_+}}{8\pi}\int_{M_t}H(y)\left(1+\langle\nu(x,t),\nu(y,t)\rangle\right)d\mu(y)\\ &=\frac{\sqrt{Q_+}}{8\pi}\int_{M_t}H\,d\mu,\end{aligned} \tag{17}$$

where $\bar{q}(t)=\frac{1}{4\pi}\int_{M_t}K(x,t)X(x,t)\,d\mu$ is the Steiner point, and we used the identity $\int_{M_t}H\nu\,d\mu=0$. Note that we continue to suppress the dependence on ε.

It follows in particular that $\bar{r}(1-\sqrt{Q_+})\leq\rho_-(t)\leq\rho_+(t)\leq\bar{r}(1+\sqrt{Q_+})$, where $\bar{r}=\frac{1}{8\pi}\int H d\mu$ is half of the mean width. Using spherical barriers as in the proof of Corollary 3.6, we deduce that $\frac{\sqrt{2(T_\varepsilon-t)}}{1+\sqrt{Q_+}}\leq\bar{r}\leq\frac{\sqrt{2(T_\varepsilon-t)}}{1-\sqrt{Q_+}}$, and therefore

$$\frac{1-\sqrt{Q_+}}{1+\sqrt{Q_+}}\leq\tilde{\rho}_-^{\varepsilon}(\tau)\leq\tilde{\rho}_+^{\varepsilon}(\tau)\leq\frac{1+\sqrt{Q_+}}{1-\sqrt{Q_+}}. \tag{18}$$

Now we write the evolving surface $\tilde{M}_\tau$ as a graph over the Steiner point $\bar{q}$ of $\tilde{M}_{\tau_0}$, for $\tau_0\leq\tau\leq\tau_0+\delta$: Then $\tilde{M}_\tau=\{\bar{q}+r(z,\tau)z:\ z\in\mathbb{S}^2\}$, where r is a smooth function defined on $\mathbb{S}^2\times[\tau_0,\tau_0+\delta]$, satisfying $\frac{1-\sqrt{Q_+}}{1+\sqrt{Q_+}}\leq r(z,\tau)\leq\frac{1+\sqrt{Q_+}}{1-\sqrt{Q_+}}$. This implies in particular that $|\tilde{r}(z,\tau)-1|\leq C\sqrt{Q_+}$, and the convexity of $\tilde{M}_\tau$ implies that $|\bar{\nabla}r|\leq C\sqrt{Q_+}$ also.

Under (6), modified to rescale about the point $\bar{q}$, the function $r-1$ evolves according to a uniformly parabolic fully nonlinear parabolic equation, to which the estimates of [3] can be applied (in suitable coordinate patches), yielding an estimate of the form

$$\|r-1\|_{C^{2,\beta}(\mathbb{S}^2\times[\tau_1+s^2/2,\tau_1+s^2])}\leq Cs^{-\beta}\|r-1\|_{C^{1,1}(\mathbb{S}^2\times[\tau_1,\tau_1+s^2])},$$

for any $\tau_0\leq\tau_1\leq\tau_0+\delta-s^2$. This may be interpreted as an estimate in suitable weighted spaces:

$$\|r-1\|_{2,\beta,2+\beta}\leq C\|r-1\|_{1,1,2} \tag{19}$$

where $\|u\|_{k,\alpha,p}=\sup\left\{s^{-p}\|u\|_{C^{k,\alpha}(Q)}:\ Q\subset\mathbb{S}^2\times[\tau_0+s,\tau_0+\delta]\right\}$, where the supremum ranges over parabolic cylinders in suitable coordinate patches. An interpolation inequality holds between these norms (see [14, Proposition 4.2]):

$$\|r-1\|_{1,1,2}\leq C\|r-1\|_{2,\beta,2+\beta}^{\frac{2}{2+\beta}}\|r-1\|_{0,0,0}^{\frac{\beta}{2+\beta}}. \tag{20}$$

Combining (19) and (20) yields

$$\|r-1\|_{2,\beta,2+\beta} \leq C \sup_{\mathbb{S}^2\times[\tau_0,\tau_0+\delta]} |r-1| \leq C\sqrt{Q_+}. \tag{21}$$

In particular, this gives the estimate

$$\|r-1\|_{C^{2,\beta}(\mathbb{S}^2\times[\tau_0+\delta/4,\tau_0+\delta])} \leq C\|r-1\|_{C^0(\mathbb{S}^2\times[\tau_0,\tau_0+\delta])}. \tag{22}$$

The right hand side is bounded by $C\sqrt{Q_+}$ by the estimate (18), proving the Lemma. $\square$

We next combine the result of Lemma 5.2 with the estimate (16) to prove a decay estimate for Q: First note that for $\tau \in [\tau_0+\delta/4, \tau_0+\delta/2]$ we have $|Q|_{C^{0,\beta}(M_\tau)} \leq CQ_+$ by the Lemma. It follows that $|Z|_{C^{0,\beta}(M_\tau)} \leq CQ_+^{1+d}$. Also, since M_τ must have an umbilic point (where $Q=0$) and hence $Z = Q_+^{1+d}$, we have $\sup_{M_\tau} Z = Q_+^{1+d}$. The following interpolation inequality allows us to control the integral of Z from below:

$$Q_+^{1+d} = \sup_{M_\tau} Z \leq C\left(\int_{M_\tau} Z\right)^{\frac{\beta}{2+\beta}} \|Z\|_{C^{0,\beta}(M_\tau)}^{\frac{2}{2+\beta}} \leq C\left(\int_{M_\tau} Z\right)^{\frac{\beta}{2+\beta}} Q_+^{\frac{2(1+d)}{2+\beta}},$$

for a small positive constant C, from which it follows that $\int_{M_\tau} Z \geq CQ_+^{1+d}$ for each $\tau \in [\tau_0+\delta/4, \tau_0+\delta/2]$. The estimate (16) then implies that $\inf_{M_{\tau_0+\delta}} Z \geq CQ_+^{1+d}$, and therefore $Q_+(\tau_0+\delta) := \sup_{M_{\tau_0+\delta}} Q^{1+d} = Q_+^{1+d} - \inf_{M_{\tau_0+\delta}} Z \leq (1-C)Q_+(\tau_0)$. Since $Q_+(\tau)$ is non-increasing in τ, this implies an exponential decay of the form $Q_+(\tau) \leq Ce^{-\gamma\tau}$, where $-\gamma = \frac{1}{\delta}\log(1-C)$.

Finally, the exponential decay of $Q_+(\tau)$, together with the result of Lemma 5.2, implies the exponential decay of $|\tilde{\mathscr{W}} - I|_{C^{0,\beta}}$, proving Proposition 5.1. $\square$

6. Construction of a solution to (1) and proof of Theorem 1.2

The estimates obtained in the previous two sections allow us to construct a solution of (1) as a limit of the solutions of (4) as $\varepsilon \to 0$: The result of Proposition 5.1 implies that $\tilde{M}_\tau^\varepsilon$ has $C^{2,\beta}$ distance to a unit sphere bounded by $Ce^{-\gamma\tau}$. Combining this with an argument using spherical barriers implies that the centre of this sphere from the origin is also bounded by $Ce^{-\gamma\tau}$, and we conclude that the $C^{2,\beta}$ distance of $\tilde{M}_\tau^\varepsilon$ from the unit sphere about the origin is bounded by $Ce^{-\gamma\tau}$. It follows by the Arzela-Ascoli Theorem that for a subsequence of ε approaching zero, the hypersurfaces $\tilde{M}_\tau^\varepsilon$ converge in $C^{2,\beta'}$ to a family $\tilde{M}_\tau$ which converge exponentially to the unit sphere in $C^{2,\beta}$, and in particular have normal speed equal to F. The convergence of the parametrisations $\tilde{X}^\varepsilon$ to a limiting parametrisation satisfying (6) follows as in [1] or [13], and rescaling gives the required solution of (1).

7. Uniqueness of solutions

In this final section we address the question of uniqueness of the solution of (1): In the previous section we have constructed a solution, but did not yet prove that this is the only solution. As we will show, the uniqueness follows from a simple geometric barrier argument using the homogeneity of the flow and the comparison principle.

Choose the origin to be at an interior point of the region enclosed by M_0. Suppose $Y: \mathbb{S}^2\times[0,T) \to \mathbb{R}^3$ is any solution of (1) with initial data X_0, and let X be the solution constructed in the previous sections. Let Ω_t be the region enclosed by $M_t = X(\mathbb{S}^2,t)$, and $\hat{\Omega}_t$ the region enclosed by $\hat{M}_t = Y(\mathbb{S}^2,t)$. By assumption, $\hat{\Omega}_t$ and Ω_t converge in Hausdorff distance to Ω_0 as $t \to 0$. Let $\lambda > 0$, and observe that $X_\lambda^-(x,t) = e^{-\lambda}X(x, e^{2\lambda}t)$

and $X^+_\lambda(x,t) = \mathrm{e}^{\lambda} X(x, \mathrm{e}^{-2\lambda} t)$. Then both X^+ and X^- are solutions of (1), and by the comparison principle we have

$$\mathrm{e}^{-\lambda}\Omega_{\mathrm{e}^{2\lambda}t} \subset \hat{\Omega}_t \subset \mathrm{e}^{\lambda}\Omega_{\mathrm{e}^{-2\lambda}t},$$

for each $\lambda > 0$. Finally, we note that the solution X is bounded in C^2, and hence the Hausdorff distance from $\mathrm{e}^{-\lambda}\Omega_{\mathrm{e}^{2\lambda}t}$ or $\mathrm{e}^{\lambda}\Omega_{\mathrm{e}^{-2\lambda}t}$ to Ω_t converges to zero as λ approaches zero. Consequently, $\hat{\Omega}_t = \Omega_t$ for each t in the common interval of existence. The fact that Y agrees with X follows from the fact that $Y_0 = X_0$ and both have time derivative normal to the surface at each point and time.

Remark: The argument presented in our previous work [6] contains an error, specifically in the argument after equation (7.7) comparing an integral average to a maximum: This requires an interpolation inequality which yields a multiple of a sublinear power of the maximum rather than a multiple of the maximum itself, and this power has been omitted in the equation following (7.7). This omission does not significantly affect the result of that paper, merely resulting in a polynomial rate of convergence to a sphere rather than an exponential rate. The argument presented in the present paper (specifically, Lemma 5.2) can be adapted to that situation to prove the exponential decay. The required details will be provided elsewhere in a more general context.

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