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Abstract

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ITERATES OF HOLOMORPHIC SELF-MAPS ON PSEUDOCONVEX DOMAINS OF FINITE AND INFINITE TYPE IN $\mathbb{C}^n$

TRAN VU KHANH AND NINH VAN THU

Abstract. Using the lower bounds on the Kobayashi metric established by the first author [16], we prove a Wolff-Denjoy-type theorem for a very large class of pseudoconvex domains in $\mathbb{C}^n$. This class includes many pseudoconvex domains of finite type and infinite type.

1. Introduction

In 1926, Wolff [22] and Denjoy [9] established their famous theorem on the behavior of iterates of holomorphic self-mappings of the unit disk Δ of $\mathbb{C}$ that do not admit fixed points.

Theorem (Wolff-Denjoy [22, 9], 1926)**.** *Let $\phi : \Delta \to \Delta$ be a holomorphic self-map without fixed points. Then there exists a point x in the unit circle $\partial\Delta$ such that the sequence $\{\phi^k\}$ of iterates of ϕ converges, uniformly on any compact subsets of Δ, to the constant map taking the value x.*

The generalization of this theorem to domains in $\mathbb{C}^n$, $n \geq 2$, is the focus of this paper. This has been done in several cases:

- the unit ball (see [13]);
- strongly convex domains (see [2, 4, 5]);
- strongly pseudoconvex domains (see [3, 14]);
- pseudoconvex domains of strictly finite type in the sense of Range [20] (see [3]) ;
- pseudoconvex domains of finite type in $\mathbb{C}^2$ (see [15, 23]).

The main goal of this paper is to prove a Wolff-Denjoy-type theorem for a general class of bounded pseudoconvex domains in $\mathbb{C}^n$ that includes many pseudoconvex domains of both finite and infinite type. In particular, we shall prove the following (the definitions are given below).

Theorem 1. *Let $\Omega \subset \mathbb{C}^n$ be a bounded, pseudoconvex domain with C^2-smooth boundary $\partial\Omega$. Assume that*

(i) Ω has the f-property with f satisfying $\displaystyle\int_1^\infty \frac{\ln\alpha}{\alpha f(\alpha)}d\alpha < \infty$; and

(ii) the Kobayashi distance of Ω is complete.

If $\phi : \Omega \to \Omega$ is a holomorphic self-map such that the sequence of iterates $\{\phi^k\}$ is compactly divergent, then the sequence $\{\phi^k\}$ converges, uniformly on a compact set, to a point of the boundary.

We say that a Wolff-Denjoy-type theorem for Ω holds if the conclusion of Theorem 1 holds. We will prove Theorem 1 in Section 3 using the (known) estimates of the Kobayashi distance on domains of the f-property and the work by Abate [2, 3, 4].

We now recall some the definitions of the f-property (see also [16, 17]) and the Kobayashi distance.

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Definition 1. Let $f : \mathbb{R}^+ \to \mathbb{R}^+$ be a smooth, monotonically increasing functions so that $f(\alpha)\alpha^{-1/2}$ is decreasing. We say that $\Omega \subset \mathbb{C}^n$ has the f-property if there exists a family of functions $\{\psi_\eta\}$ such that

(i) the functions ψ_η are plurisubharmonic, $|\psi_\eta| \leq 1$, and C^2 on Ω;
(ii) $i\partial\bar\partial\psi_\eta \geq c_1 f(\eta^{-1})^2 \mathrm{Id}$ and $|D\psi_\eta| \leq c_2\eta^{-1}$ on $\{z \in \Omega : 0 < \delta_\Omega(z) < \delta\}$ for some constants $c_1, c_2 > 0$, where $\delta_\Omega(z)$ is the Euclidean distance from z to the boundary $\partial\Omega$.

This is an analytic condition where the function f reflects the geometric "type" of the boundary. For example, viewing Catlin's results on pseudoconvex domains of finite type through the lens of the f-property [6, 7], a domain is of finite type if and only if there exists an $\epsilon > 0$ such that the t^ϵ-property holds. If domain is convex and of finite type m, then the $t^{1/m}$-property holds [18]. Furthermore, there is a large class of infinite type pseudoconvex domains that satisfy an f-properties [17, 16]. For example (see [17]), the $\ln^{1/\alpha}$-property holds for both the complex ellipsoid of infinite type

$$\Omega = \left\{ z \in \mathbb{C}^n : \sum_{j=1}^n \exp\left(-\frac{1}{|z_j|^{\alpha_j}}\right) - e^{-1} < 0 \right\} \tag{1}$$

with $\alpha := \max_j\{\alpha_j\}$, and the real ellipsoid of infinite type

$$\tilde\Omega = \left\{ z = (x_1 + iy_1, \ldots, x_n + iy_n) \in \mathbb{C}^n : \sum_{j=1}^n \exp\left(-\frac{1}{|x_j|^{\alpha_j}}\right) + \exp\left(-\frac{1}{|y_j|^{\beta_j}}\right) - e^{-1} < 0 \right\} \tag{2}$$

with $\alpha := \max_j\{\min\{\alpha_j, \beta_j\}\}$, where $\alpha_j, \beta_j > 0$ for all $j = 1, 2, \ldots$. The influence of the f-property on estimates of the Kobayashi metric and distance will be given in Section 2.

On hyperbolic manifolds, completeness of the Kobayashi distance (or k-completeness for short) is a natural condition. For a bounded domain $\Omega \subset \mathbb{C}^n$, k-completeness of means

$$k_\Omega(z_0, z) \to \infty \quad \text{as} \quad z \to \partial\Omega$$

for any point $z_0 \in \Omega$ where $k_\Omega(z_0, z)$ is the Kobayashi distance from z_0 to z. It is well-known that this condition holds for strongly pseudoconvex domains [11], convex domains [19], pseudoconvex domains of finite type in $\mathbb{C}^2$ [23], pseudoconvex Reinhardt domains [21], domains enjoying a local holomorphic peak function at any boundary point [12]. We also remark that the domain defined by (1) (resp. (2)) is k-complete because it is a pseudoconvex Reinhardt domain (resp. convex domain). These remarks immediately lead to the following corollary.

Corollary 2. *Let Ω be a bounded domain in $\mathbb{C}^n$ with smooth boundary $\partial\Omega$. The Wolff-Denjoy-type theorem for Ω holds if Ω satisfies at least one of the following settings:*

(a) Ω is a strongly pseudoconvex domain;
(b) Ω is a pseudoconvex domain of finite type and $n = 2$;
(c) Ω is a convex domain of finite type;
(d) Ω is a pseudoconvex Reinhardt domain of finite type;
(e) Ω is a pseudoconvex domain of finite type (or of infinite type having the f-property with $f(t) \geq \ln^{2+\epsilon}(t)$ for any $\epsilon > 0$) such that Ω has a local, continuous, holomorphic peak function at each boundary point, i.e., for any $x \in \partial\Omega$ there exist a neighborhood U of x and a holomorphic function p on $\Omega \cap U$, continuous up to $\bar\Omega \cap U$, and satisfies

$$p(x) = 1, \quad p(z) < 1, \quad \text{for all } z \in \bar\Omega \cap U \setminus \{x\};$$

(f) Ω is defined by (1) or (2) with $\alpha < \frac{1}{2}$.

Finally, throughout the paper we use $\lesssim$ and $\gtrsim$ to denote inequalities up to a positive multiplicative constant, and $H(\Omega_1, \Omega_2)$ to denote the set of holomorphic maps from Ω_1 to Ω_2.

2. The Kobayashi metric and distance

We start this section by defining the Kobayashi metric.

Definition 2. Let Ω be a domain in $\mathbb{C}^n$, and $T^{1,0}\Omega$ be its holomorphic tangent bundle. The Kobayashi (pseudo)metric $K_\Omega : T^{1,0}\Omega \to \mathbb{R}$ is defined by

$$K_\Omega(z, X) = \inf\{\alpha > 0 \mid \exists\ \Psi \in H(\Delta, \Omega) : \Psi(0) = z, \Psi'(0) = \alpha^{-1}X\}, \tag{3}$$

for any $z \in \Omega$ and $X \in T^{1,0}\Omega$, where Δ be the unit open disk of $\mathbb{C}$.

In the case that Ω is a smoothly pseudoconvex bounded domain of finite type, it is known that there exists $\epsilon > 0$ such that the Kobayashi metric K_Ω has the lower bound $\delta_\Omega^{-\epsilon}(z)$ (see [8], [10]), in the sense that,

$$K_\Omega(z, X) \gtrsim \frac{\|X\|}{\delta_\Omega^\epsilon(z)},$$

where $\|X\|$ is the Euclidean length of X. Recently, the first author [16] obtained lower bounds on the Kobayashi metric for a general class of pseudoconvex domains in $\mathbb{C}^n$, that contains all domains of finite type and many domains of infinite type.

Theorem 3. *Let Ω be a pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary $\partial\Omega$. Assume that Ω has the f-property with f satisfying $\displaystyle\int_s^\infty \frac{d\alpha}{\alpha f(\alpha)} < \infty$ for $s \geq 1$, and denote by $(g(s))^{-1}$ this finite integral. Then,*

$$K(z, X) \gtrsim g(\delta_\Omega^{-1}(z))\|X\| \tag{4}$$

for any $z \in \Omega$ and $X \in T_z^{1,0}\Omega$.

The Kobayashi (pseudo)distance $k_\Omega : \Omega \times \Omega \to \mathbb{R}^+$ on Ω is the integrated form of K_Ω. k_Ω is given by

$$k_\Omega(z, w) = \inf\left\{\int_a^b K_\Omega(\gamma(t), \dot\gamma(t))dt \;\middle|\; \gamma : [a, b] \to \Omega, \text{piecwise } C^1\text{-smooth curve}, \gamma(a) = z, \gamma(b) = w\right\}$$

for any $z, w \in \Omega$. An essential property of k_Ω is that it is a *contraction* under holomorphic maps, i.e.,

$$\text{if} \quad \phi \in H(\Omega, \tilde\Omega) \quad \text{then} \quad k_{\tilde\Omega}(\phi(z), \phi(w)) \leq k_\Omega(z, w), \quad \text{for all} \quad z, w \in \Omega. \tag{5}$$

We need the following lemma from [1, 11].

Lemma 4. *Let Ω be a bounded C^2-smooth domain in $\mathbb{C}^n$ and $z_0 \in \Omega$. Then there exists a constant $c_0 > 0$ depending on Ω and z_0 such that*

$$k_\Omega(z_0, z) \leq c_0 - \frac{1}{2}\ln \delta_\Omega(z)$$

for any $z \in \Omega$.

We recall that the curve $\gamma : [a, b] \to \Omega$ is called a minimizing geodesic with respect to the Kobayashi metric between two points $z = \gamma(a)$ and $w = \gamma(b)$ if

$$k_\Omega(\gamma(s), \gamma(t)) = t - s = \int_s^t K_\Omega(\gamma(\tau), \dot\gamma(\tau))d\tau, \quad \text{for any } s, t \in [a, b], s \leq t.$$

This implies that

$$K(\gamma(t), \dot\gamma(t)) = 1, \quad \text{for any } t \in [a, b].$$

The relation between the Kobayashi distance $k_\Omega(z,w)$ and the Euclidean distance $\delta_\Omega(z,w)$ is contained in the following lemma, itself a generalization of [15, Lemma 36].

Lemma 5. *Let Ω be a bounded, pseudoconvex, C^2-smooth domain in $\mathbb{C}^n$ satisfying the f-property with $\int_1^\infty \frac{\ln\alpha}{\alpha f(\alpha)}d\alpha < \infty$ and $z_0 \in \Omega$. Then there exists a constant c only depending on z_0 and Ω such that*

$$\delta_\Omega(z,w) \le c\int_{e^{2k_\Omega(z_0,\gamma)}}^\infty \frac{c_0+\ln\alpha}{\alpha f(\alpha)}d\alpha, \tag{6}$$

for all $z,w\in\Omega$, where γ is a minimizing geodesic connecting z to w and c_0 is the constant given in Lemma 4. Here, $k_\Omega(z_0,\gamma)$ is the Kobayashi distance from z_0 to the curve γ.

Proof. We may assume that $z\neq w$. Let p be a point on γ of minimal distance to z_0. We can assume that $p\neq z$ (if not, we interchange z and w) and denote by $\gamma_1:[0,a]\to\Omega$ the reparametrized piece of γ going from p to z. By the minimality of $k_\Omega(z_0,\gamma)=k_\Omega(z_0,p)$ and the triangle inequality we have

$$k_\Omega(z_0,\gamma_1(t)) \ge k_\Omega(z_0,\gamma) \quad\text{and}\quad k_\Omega(z_0,\gamma_1(t)) \ge k_\Omega(p,\gamma_1(t)) - k_\Omega(z_0,p) = t - k_\Omega(z_0,\gamma) \tag{7}$$

for any $t\in[0,a]$. Substituting $z=\gamma_1(t)$ into the inequality in Lemma 4, it follows

$$\frac{1}{\delta_\Omega(\gamma_1(t))} \ge e^{2k_\Omega(z_0,\gamma_1(t))-2c_0}$$

for all $t\in[0,a]$. Since γ_1 is a unit speed curve with respect to K_Ω we have

$$\begin{aligned}\delta_\Omega(p,z) &\le \int_0^a \|\gamma_1'(t)\|dt \\ &\lesssim \int_0^a \left(g\left(\frac{1}{\delta_\Omega(\gamma_1(t))}\right)\right)^{-1} K_\Omega(\gamma_1(t),\gamma_1'(t))dt \\ &\lesssim \int_0^a \left(g\left(e^{2k_\Omega(z_0,\gamma_1(t))-2c_0}\right)\right)^{-1}dt.\end{aligned} \tag{8}$$

We now compare a with $2k_\Omega(z_0,\gamma)+c_0$. In the case $a>2k_\Omega(z_0,\gamma)+c_0$, we split the integral into two parts and use the inequalities (7) and the fact that g is increasing. We then have

$$\begin{aligned}\delta_\Omega(p,z) &\lesssim \int_0^{2k_\Omega(z_0,\gamma)+c_0}\left(g\left(e^{2k_\Omega(z_0,\gamma_1(t))-2c_0}\right)\right)^{-1}dt + \int_{2k_\Omega(z_0,\gamma)+c_0}^a \left(g\left(e^{2k_\Omega(z_0,\gamma_1(t))-2c_0}\right)\right)^{-1}dt \\ &\lesssim \int_0^{2k_\Omega(z_0,\gamma)+c_0}\left(g\left(e^{2k_\Omega(z_0,\gamma)-2c_0}\right)\right)^{-1}dt + \int_{2k_\Omega(z_0,\gamma)+c_0}^\infty \left(g\left(e^{2t-2k_\Omega(z_0,\gamma)-2c_0}\right)\right)^{-1}dt \\ &\lesssim \frac{2k_\Omega(z_0,\gamma)+c_0}{g\left(e^{2k_\Omega(z_0,\gamma)-2c_0}\right)} + \int_{e^{2k_\Omega(z_0,\gamma)}}^\infty \frac{d\beta}{\beta g(\beta)} \\ &\lesssim \left(\frac{c_0+\ln s}{g(se^{-2c_0})} + \int_s^\infty \frac{d\beta}{\beta g(\beta)}\right)\Big|_{s=e^{2k_\Omega(z_0,\gamma)}}.\end{aligned} \tag{9}$$

By the definition of $(g(s))^{-1}$ in Theorem 3 and the fact that $f(\alpha)\alpha^{-1/2}$ decreasing, it follows

$$\begin{aligned}\frac{1}{g(se^{-2c_0})} &= \int_{se^{-2c_0}}^\infty \frac{d\alpha}{\alpha f(\alpha)} = \int_s^\infty \frac{d\alpha}{\alpha f(\alpha e^{-2c_0})} \\ &= \int_s^\infty \frac{e^{c_0}d\alpha}{\alpha^{3/2}\left(\alpha e^{-2c_0}\right)^{-1/2} f(\alpha e^{-2c_0})} \le \int_s^\infty \frac{e^{c_0}d\alpha}{\alpha^{3/2}\alpha^{-1/2}f(\alpha)} = \frac{e^{c_0}}{g(s)},\end{aligned} \tag{10}$$

thus obtaining

$$\delta_\Omega(p,z) \le c\left(\frac{c_0+\ln s}{g(s)} + \int_s^\infty \frac{d\beta}{\beta g(\beta)}\right)\Big|_{s=e^{2k_\Omega(z_0,\gamma)}},$$

where c is the multiplication of e^{c_0} with a positive constant. We also notice that

$$\begin{aligned}\int_s^\infty \frac{d\beta}{\beta g(\beta)} &= \int_s^\infty \frac{1}{\beta}\left(\int_\beta^\infty \frac{d\alpha}{\alpha f(\alpha)}\right)d\beta = \iint_{\{(\alpha,\beta):\ \beta\le\alpha<\infty, s\le\beta<\infty\}} \frac{d\alpha d\beta}{\beta\alpha f(\alpha)}\\ &= \iint_{\{(\alpha,\beta):\ s\le\alpha<\infty, s\le\beta\le\alpha\}} \frac{d\alpha d\beta}{\beta\alpha f(\alpha)} = \int_s^\infty \frac{1}{\alpha f(\alpha)}\left(\int_s^\alpha \frac{d\beta}{\beta}\right)d\alpha\\ &= \int_s^\infty \frac{\ln\alpha - \ln s}{\alpha f(\alpha)}d\alpha = \int_s^\infty \frac{\ln\alpha}{\alpha f(\alpha)}d\alpha - \frac{\ln s}{g(s)}.\end{aligned}$$

Therefore, in this case we obtain

$$\delta_\Omega(p,z) \le c\left(\frac{c_0}{g(s)} + \int_s^\infty \frac{\ln\alpha}{\alpha f(\alpha)}d\alpha\right)\Big|_{s=e^{2k_\Omega(z_0,\gamma)}} = c\int_{e^{2k_\Omega(z_0,\gamma)}}^\infty \frac{c_0+\ln\alpha}{\alpha f(\alpha)}d\alpha.$$

In the case $a < 2k_\Omega(z_0,\gamma)+c_0$, we make the same estimate but without decomposing the integral. By a symmetric argument with w instead of z, we also have

$$\delta_\Omega(p,w) \le c\int_{e^{2k_\Omega(z_0,\gamma)}}^\infty \frac{c_0+\ln\alpha}{\alpha f(\alpha)}d\alpha.$$

The conclusion of this lemma now follows by the triangle inequality. □

Corollary 6. *Let Ω be a bounded, pseudoconvex domain in $\mathbb{C}^n$ with C^2-smooth boundary satisfying the f-property with $\int_1^\infty \frac{\ln\alpha}{\alpha f(\alpha)}d\alpha < \infty$. Furthermore, assume that Ω is k-complete. Let $\{w_n\},\{z_n\} \subset \Omega$ be two sequences such that $w_n \to x \in \partial\Omega$ and $z_n \to y \in \bar{\Omega}\setminus\{x\}$. Then $k_\Omega(w_n,z_n)\to\infty$.*

Proof. Fix a point $z_0\in\Omega$ and let $\gamma_n : [a_n,b_n]\to\Omega$ be a minimizing geodesic connecting $z_n=\gamma(a_n)$ and $w_n=\gamma(b_n)$. Since $x\ne y$, it follows $\delta(z_n,w_n)\gtrsim 1$. By Lemma 5, it follows

$$1 \lesssim c\int_{e^{2k_\Omega(z_0,\gamma_n)}}^\infty \frac{c_0+\ln\alpha}{\alpha f(\alpha)}d\alpha.$$

This inequality implies that $k_\Omega(z_0,\gamma_n)\lesssim 1$ because $\lim_{s\to\infty}\int_s^\infty \frac{c_0+\ln\alpha}{\alpha f(\alpha)}d\alpha = 0$. Consequently, there is a point $p_n\in\gamma_n$ such that $k_\Omega(z_0,p_n)=k_\Omega(z_0,\gamma_n)\lesssim 1$. Moreover,

$$\begin{aligned}k_\Omega(z_0,w_n) &\le k_\Omega(z_0,p_n)+k_\Omega(p_n,w_n)\\ &\le k_\Omega(z_0,p_n)+k_\Omega(w_n,z_n)\\ &\lesssim k_\Omega(w_n,z_n)+1.\end{aligned}$$

Since Ω is k-complete, it follows that $k_\Omega(z_0,w_n)\to\infty$ as $w_n\to x\in\partial\Omega$. This proves Corollary 6. □

3. Proof of Theorem 1

In order to prove Theorem 1, we recall the definition of small and big horospheres and F-convexity from [2, 3].

Definition 3. (see [2, p.228]) Let Ω be a domain in $\mathbb{C}^n$. Fix $z_0\in\Omega$, $x\in\partial\Omega$ and $R>0$. Then the small horosphere $E_{z_0}(x,R)$ and the big horosphere $F_{z_0}(x,R)$ of center x, pole z_0 and radius R are defined by

$$E_{z_0}(x,R) = \{z\in\Omega\colon \limsup_{\Omega\ni w\to x}[k_\Omega(z,w)-k_\Omega(z_0,w)] < \frac{1}{2}\ln R\}$$

and

$$F_{z_0}(x,R)=\{z\in\Omega\colon \liminf_{\Omega\ni w\to x}[k_\Omega(z,w)-k_\Omega(z_0,w)]<\frac{1}{2}\ln R\}.$$

Definition 4. (see [3, p.185]) A domain $\Omega\subset\mathbb{C}^n$ is called F-convex if for every $x\in\partial\Omega$

$$\overline{F_{z_0}(x,R)}\cap\partial\Omega\subseteq\{x\}$$

holds for every $R>0$ and for every $z_0\in\Omega$.

Remark 1. The bidisk Δ^2 in $\mathbb{C}^2$ is not F-convex. Indeed, since $d_{\Delta^2}((1/2,1-1/k),(0,1-1/k))-d_{\Delta^2}((0,0),(0,1-1/k))=d_\Delta(1/2,0)-d_\Delta(0,1-1/k)\to-\infty$ as $\mathbb{N}^*\ni k\to\infty$, $(1/2,1)\in\overline{F^{\Delta^2}_{(0,0)}((0,1),R)}\cap\partial(\Delta^2)$ for any $R>0$.

Remark 2. If Ω is either a strongly pseudoconvex domain in $\mathbb{C}^n$, or a pseudoconvex domain of finite type in $\mathbb{C}^2$, or a pseudoconvex domain of strict finite type in $\mathbb{C}^n$ then Ω is F-convex (see [2, 3, 23]).

Now, we prove that F-convexity holds on a larger class of pseudoconvex domains.

Proposition 7. *Let Ω be a domain satisfying the hypotheses of Theorem 1. Then Ω is F-convex.*

Proof. Let $R>0$ and $z_0\in\Omega$. Assume by contradiction that there exists $y\in\overline{F_{z_0}(x,R)}\cap\partial\Omega$ with $y\neq x$. Then we can find a sequence $\{z_n\}\subset\Omega$ with $z_n\to y\in\partial\Omega$ and a sequence $\{w_n\}\subset\Omega$ with $w_n\to x\in\partial\Omega$ such that

$$k_\Omega(z_n,w_n)-k_\Omega(z_0,w_n)\leq\frac{1}{2}\ln R. \tag{11}$$

Moreover, for each $n\in\mathbb{N}^*$ there exists a minimizing geodesic γ_n connecting z_n to w_n. Let p_n be a point on γ_n of minimal distance $k_\Omega(z_0,\gamma_n)=k_\Omega(z_0,p_n)$ to z_0. We consider the following two cases for the sequence $\{p_n\}$.

Case 1. There exists a subsequence $\{p_{n_k}\}$ of the sequence $\{p_n\}$ such that $p_{n_k}\to p_0\in\Omega$ as $k\to\infty$.

$$\begin{aligned}k_\Omega(w_{n_k},z_{n_k})&\geq k_\Omega(w_{n_k},p_{n_k})+k_\Omega(p_{n_k},z_{n_k})\\&\geq k_\Omega(w_{n_k},z_0)-k_\Omega(z_0,p_{n_k})+k_\Omega(p_{n_k},z_{n_k}).\end{aligned} \tag{12}$$

From (11) and (12), we obtain

$$k_\Omega(p_{n_k},z_{n_k})\leq k_\Omega(w_{n_k},z_{n_k})-k_\Omega(w_{n_k},z_0)+k_\Omega(z_0,p_{n_k})\leq\frac{1}{2}\ln R+k_\Omega(z_0,p_{n_k})\lesssim 1.$$

This is a contradiction since Ω is k-complete.

Case 2. Otherwise, $p_n\to\partial\Omega$ as $n\to\infty$. By Lemma 5, there are constants c and c_0 only depending on z_0 such that

$$\delta_\Omega(w_n,z_n)\leq c\int_{e^{2k_\Omega(z_0,\gamma_n)}}^\infty\frac{c_0+\ln\alpha}{\alpha f(\alpha)}d\alpha. \tag{13}$$

On the other hand, $\delta_\Omega(w_n,z_n)\gtrsim 1$ since $x\neq y$. Thus, the inequality (13) implies that

$$k_\Omega(z_0,\gamma_n)=k_\Omega(z_0,p_n)\lesssim 1. \tag{14}$$

Therefore,

$$\begin{aligned}k_\Omega(z_n,w_n)&\geq k_\Omega(z_n,p_n)+k_\Omega(p_n,w_n)\\&\geq k_\Omega(z_0,z_n)+k_\Omega(z_0,w_n)-2k_\Omega(z_0,p_n).\end{aligned} \tag{15}$$

Combining with (11) and (14), we get

$$k_\Omega(z_0,z_n)\leq k_\Omega(z_n,w_n)-k_\Omega(z_0,w_n)+2k_\Omega(z_0,p_n)\lesssim\ln R+1.$$

This is a contradiction since $z_n\to y\in\partial\Omega$ and hence the proof is complete. ☐

The following theorem is a generalization of Theorem 3.1 in [3].

Proposition 8. *Let Ω be a domain satisfying the hypothesis in Theorem 1 and fix $z_0 \in \Omega$. Let $\phi \in H(\Omega, \Omega)$ such that $\{\phi^k\}$ is compactly divergent. Then there is a point $x \in \partial\Omega$ such that for all $R > 0$ and for all $m \in \mathbb{N}$*

$$\phi^m(E_{z_0}(x, R)) \subset F_{z_0}(x, R).$$

Proof. Since $\{\phi^k\}$ is compactly divergent and Ω is k-complete,

$$\lim_{k\to+\infty} k_\Omega(z_0, \phi^k(z_0)) = \infty.$$

For every $\nu \in \mathbb{N}$, let k_ν be the largest integer k satisfying $k_\Omega(z_0, \phi^k(z_0)) \leq \nu$; then

$$k_\Omega(z_0, \phi^{k_\nu}(z_0)) \leq \nu < k_\Omega(z_0, \phi^{k_\nu+m}(z_0)) \ \forall \nu \in \mathbb{N},\ \forall m > 0. \tag{16}$$

Again, since $\{\phi^k\}$ is compactly divergent, up to a subsequence, we can assume that

$$\phi^{k_\nu}(z_0) \to x \in \partial\Omega.$$

Fix an integer $m \in \mathbb{N}$. Without loss of generality we may assume that $\phi^{k_\nu}(\phi^m(z_0)) \to y \in \partial\Omega$. Using Corollary 6 and the fact that

$$k_\Omega(\phi^{k_\nu}(\phi^m(z_0)), \phi^{k_\nu}(z_0)) \leq k_\Omega(\phi^m(z_0), z_0) \qquad \text{(by (5))}$$

it must hold that $x = y$.

Set $w_\nu = \phi^{k_\nu}(z_0)$. Then $w_\nu \to x$ and $\phi^m(w_\nu) = \phi^{k_\nu}(\phi^m(z_0)) \to x$. From (16), we also have for $m \geq 0$

$$\limsup_{\nu\to+\infty} [k_\Omega(z_0, w_\nu) - k_\Omega(z_0, \phi^m(w_\nu))] \leq 0. \tag{17}$$

Now, fix $m > 0,\ R > 0$ and take $z \in E_{z_0}(x, R)$. Then

$$\begin{aligned}
&\liminf_{\Omega\ni w\to x} [k_\Omega(\phi^m(z), w) - k_\Omega(z_0, w)] \\
&\leq \liminf_{\nu\to+\infty} [k_\Omega(\phi^m(z), \phi^m(w_\nu)) - k_\Omega(z_0, \phi^m(w_\nu))] \\
&\leq \liminf_{\nu\to+\infty} [k_\Omega(z, w_\nu) - k_\Omega(z_0, \phi^m(w_\nu))] \\
&\leq \liminf_{\nu\to+\infty} [k_\Omega(z, w_\nu) - k_\Omega(z_0, w_\nu)] \\
&\quad + \limsup_{\nu\to+\infty} [k_\Omega(z_0, w_\nu) - k_\Omega(z_0, \phi^m(w_\nu))] \\
&\leq \liminf_{\nu\to+\infty} [k_\Omega(z, w_\nu) - k_\Omega(z_0, w_\nu)] \\
&\leq \limsup_{\Omega\ni w\to x} [k_\Omega(z, w) - k_\Omega(z_0, w)] \\
&< \frac{1}{2}\ln R,
\end{aligned} \tag{18}$$

that is $\phi^m(z) \in F_{z_0}(x, R)$. Here, the first inequality follows by $\phi^m(w_\nu) \to x$, the second follows by (5), the fourth follows by (17), and the last one follows from the fact that $z \in E_{z_0}(x, R)$. □

Lemma 9. *Let Ω be a F-convex domain in $\mathbb{C}^n$. Then for any $x, y \in \partial\Omega$ with $x \neq y$ and for any $R > 0$, we have $\lim_{a\to y} E_a(x, R) = \Omega$, i.e., for each $z \in \Omega$, there exists a number $\epsilon > 0$ such that $z \in E_a(x, R)$ for all $a \in \Omega$ with $|a - y| < \epsilon$.*

Proof. Suppose that for some $z \in \Omega$ such that there exists a sequence $\{a_n\} \subset \Omega$ with $a_n \to y$ and $z \notin E_{a_n}(x, R)$. Then we have

$$\limsup_{w\to x} [k_\Omega(z, w) - k_\Omega(a_n, w)] \geq \frac{1}{2}\ln R.$$

This implies that

$$\liminf_{w\to x}[k_\Omega(a_n,w)-k_\Omega(z,w)]\le \frac{1}{2}\ln\frac{1}{R}.$$

Thus, $a_n\in \overline{F_z(x,1/R)}$, for all $n=1,2,\cdots$. Therefore, $y\in \overline{F_z(x,1/R)}\cap\partial\Omega=\{x\}$, which is absurd, and the proof is complete. □

Now we are ready to prove our main result.

Proof of Theorem 1. First we fix a point $z_0\in\Omega$. By Proposition 8 there is a point $x\in\partial\Omega$ such that for all $R>0$ and for all $m\in\mathbb{N}$

$$\phi^m(E_{z_0}(x,R))\subset F_{z_0}(x,R).$$

We need to show that for any $z\in\Omega$

$$\phi^m(z)\to x \quad \text{as} \quad m\to+\infty.$$

Indeed, let $\psi(z)$ be a limit point of $\{\phi^m(z)\}$. Since $\{\phi^m\}$ is compactly divergent, $\psi(z)\in\partial\Omega$. By Lemma 9, for any $R>0$ there is $a\in\Omega$ such that $z\in E_a(x,R)$. By Proposition 8, $\phi^m(z)\in F_a(x,R)$ for every $m\in\mathbb{N}^*$. Therefore,

$$\psi(z)\in\partial\Omega\cap\overline{F_a(x,R)}=\{x\}$$

by Proposition 7; thus the proof is complete. □

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References

[1] M. Abate, *Boundary behavior of invariant distances and complex geodesics*, Atti Accad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. 80 (1986), 100–106.

[2] M. Abate, *Horospheres and iterates of holomorphic maps*, Math. Z. 198 (1988), 225–238.

[3] M. Abate, *Iteration theory, compactly divergent sequences and commuting holomorphic maps*, Annali Scuola Norm. Sup. Pisa Cl. Sci. 18 (1991), 167–191.

[4] M. Abate, *Iteration theory of holomorphic maps on taut manifolds*, Mediterranean Press, Rende, Cosenza, 1989.

[5] M. Abate and J. Raissy, *Wolff-Denjoy theorems in non-smooth convex domains*, Ann. Mat. Pura Appl. (4) 193 (2014), no. 5, 1503–1518.

[6] D. Catlin, *Necessary conditions for subettipticity of the $\bar\partial$-Neumann problem*, Ann. of Math. (2) 117 (1983), no. 1, 147–171.

[7] D. Catlin, *Subelliptic estimates for the $\bar\partial$-Neumann problem on pseudoconvex domains*, Ann. of Math. (2) 126 (1987), no. 1, 131–191.

[8] S. Cho, *A lower bound on the Kobayashi metric near a point of finite type in $\mathbb{C}^n$*, J. Geom. Anal., 2-4 (1992), 317–325.

[9] A. Denjoy, *Sur l'itération des fonctions analytiques*, C. R. Acad. Sci. Paris 182(1926), 255–257.

[10] K. Diederich and J. E. Fornaess, *Proper holomorphic maps onto pseudoconvex domains with real-analytic boundary*, Ann. of Math. (2), 110 (3) (1979),575–592.

[11] F. Forstneric and J.-P. Rosay, *Localization of the Kobayashi metric and the boundary continuity of proper holomorphic mappings* , Math. Ann. 279 (2)(1987), 239–252.

[12] H. Gaussier, *Tautness and complete hyperbolicity of domains in $\mathbb{C}^n$*, Proc. Amer. Math. Soc. 127, 1 (1999), 105–116.

[13] M. Hervé, *Quelques propriétés des applications analytiques d'une boule à m dimensions dans elle-même*, J. Math. Pures Appl. 42 (1963), 117–147.

[14] X. J. Huang, *A non-degeneracy property of extremal mappings and iterates of holomorphic self-mappings*, Ann. Scuola Norm. Sup. Pisa 21 (1994), 399–419.

[15] A. Karlsson, *On the dynamics of isometries*, Geometry and Topology, Volume 9 (2005), 2359–2394.

[16] T. V. Khanh, *Lower bounds on the Kobayashi metric near a point of infinite type*, J. Geom. Anal. 26 (2016), 616–629.
[17] T. V. Khanh and G. Zampieri,*Regularity of the $\bar{\partial}$-Neumannn problem at infinity type*, J. Funct. Anal. 259 (2010), 2760–2775.
[18] J. D. McNeal, *Convex domains of finite type*, J. Funct. Anal., 108 (1992),361–373.
[19] P. Mercer, *Complex geodesics and iterates of holomorphic maps on convex domains in* $\mathbb{C}^n$, Trans. Amer. Math. Soc. 338 (1993), 201–211.
[20] R.M. Range, *The Carathéodory metric and holomorphic maps on a class of weakly pseudoconvex domains*, Pacific J. Math. 78 (1978), 173–189.
[21] T. Warszawski, *Boundary behavior of the Kobayashi distance in pseudoconvex Reinhardt domains*, Michigan Math. J. 61 (2012), 575–592.
[22] J. Wolff, *Sur l'iteration des fonctions bornes*, C. R. Acad. Sci. Paris 182(1926), 200–201.
[23] W. Zhang and F. Ren, *Dynamics on weakly pseudoconvex domains*, Chinese Ann. Math. Ser. B 16 (1995), no. 4, 467–476.

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