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An analytical solution for Parisian up-and-in calls

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Abstract
We derive an analytical solution for the value of Parisian up-and-in calls by using the "moving window" technique for pricing European-style Parisian up-and-out calls. Our pricing formula can be applied to both European-style and American-style Parisian up-and-in calls, due to the fact that with an "in" barrier, the option holder cannot do or decide on anything before the option is activated, and once the option is activated it is just a plain vanilla call, which could be of American style or European style.

Keywords
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An analytical solution for Parisian up-and-in calls

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Abstract

In this work, we derive an analytical solution for the value of Parisian up-and-in calls by using the "moving window" technique developed by Zhu and Chen [15] for pricing European-style Parisian up-and-out calls. Our pricing formula can be applied to both European-style and American-style Parisian up-and-in calls, due to the fact that with an "in" barrier, the option holder cannot do or decide on anything before the option is activated, and once the option is activated it is just a plain vanilla call, which could be of American style or European style.

Keywords. Parisian options, "moving window" technique, analytical solutions, coupled integral equations.

1 Introduction

Barrier options are cheaper alternatives of vanilla options for hedging and speculating, but the "one-touch" knock-in or knock-out feature is prone to market manipulations. To eliminate these manipulations, Parisian options are introduced, the underlying asset price has to continually stay above or below the asset barrier for a prescribed amount of time before the knock-out or knock-in feature is activated. However, the introduction of the "time barrier" means that the option valuation problem is now a three-dimensional problem, which is more complicated to solve. This is especially true in the case of American-style Parisian knock-out options, as the optimal exercise boundary is a three-dimensional surface.

Fortunately, this difficulty disappears in the valuation of American-style Parisian knock-in options. In fact, by definition, before the knock-in feature is activated, the option holder cannot do anything regardless of the exercise style of the option, and once the "knock-in" feature is activated, the value of the Parisian option takes on the value of the embedded vanilla American option. Therefore, the solution procedure for the valuation of an American-style Parisian knock-in option and that of its European-style counterpart should be very similar, and the only difference is that upon activation one becomes a vanilla American option, and the other becomes a vanilla European option. Thus, the technique proposed by Zhu and Chen [15] for their solution of European Parisian up-and-out calls could be applied to find analytical solutions for both American-style and European-style Parisian knock-in options. Recently, this technique was used to find a simple analytical solution for Parisian down-and-in calls [17]. This paper aims to apply the same technique again for the derivation of an analytical solution for Parisian up-and-in calls.

The paper is organized as follows. In Section 2, we introduce the PDE systems governing the price of a Parisian up-and-in call. The solution procedure is presented in Section 3, while Section 4 provides a numerical example to illustrate the implementation of our formulae. Conclusion is given in the last section.

2 The PDE systems

By definition, a Parisian up-and-in call will be knocked in and become the embedded vanilla call, which could be of American or European style, if the underlying asset price continually stays above the barrier $\bar{S}$ for a prescribed time period $\bar{J}$. Otherwise, the Parisian up-and-in call will expire worthless.

For some extreme values of $\bar{S}$ and $\bar{J}$, one can easily observe that a Parisian up-and-in call becomes worthless or degenerates to either a one-touch barrier option or a vanilla option. For other non-degenerate cases, the price of a Parisian up-and-in call depends on the underlying asset price S, the current time t and the barrier time J, in addition to other parameters such as the volatility rate σ, the risk-free interest rate r and the expiry time T.

We now assume that the underlying asset price S with a continuous dividend yield D follows a lognormal Brownian motion governed by

$$dS = (r - D)Sdt + \sigma SdZ, \tag{2.1}$$

where Z is a standard Brownian motion.

Based on similar financial arguments in [15], the pricing domains of those non-degenerated cases can be elegantly reduced as

$$\begin{aligned} &I: \ \{0 \le S \le \bar{S},\ 0 \le t \le T - \bar{J},\ J = 0\}, \\ &II: \ \{\bar{S} \le S < \infty,\ J \le t \le J + T - \bar{J},\ 0 \le J \le \bar{J}\}. \end{aligned}$$

Let $V_1(S,t)$ and $V_2(S,t,J)$ denote the option prices in the region I and II, respectively. Following the arguments in [6, 15], we can show that V_1 and V_2 should satisfy the following

PDE systems defined in domain I and domain II, respectively,

$$\mathcal{A}_1 \begin{cases} \dfrac{\partial V_1}{\partial t} + \mathbb{L}V_1 = 0, \\ V_1(S, T - \bar{J}) = 0, \\ V_1(0, t) = 0, \\ V_1(\bar{S}, t) = V_2(\bar{S}, t, 0), \end{cases} \qquad \mathcal{A}_2 \begin{cases} \dfrac{\partial V_2}{\partial t} + \dfrac{\partial V_2}{\partial J} + \mathbb{L}V_2 = 0, \\ V_2(S, t, \bar{J}) = C(S, t), \\ V_2(S, t, J) \backsim S \ as\ S \to +\infty, \\ V_2(\bar{S}, t, J) = V_2(\bar{S}, t, 0), \end{cases} \tag{2.2}$$

$$\text{connectivity condition} : \frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, t, 0), \tag{2.3}$$

where $C = C_A$ (the embedded vanilla American option) if the Parisian option is of American-style, or $C = C_E$ (the embedded vanilla European option) if the Parisian option is of European-style, and $\mathbb{L} = \dfrac{\sigma^2 S^2}{2} \dfrac{\partial^2}{\partial S^2} + (r - D)S\dfrac{\partial}{\partial S} - rI$, with I being the identity operator.

It should be pointed out first that the option will expire worthless if the asset price still stays below or at the asset barrier at $t = T - \bar{J}$ because there is not enough time left for J reaching $\bar{J}$. Therefore, $V_1(S, t) = 0$, for all $t \geq T - \bar{J},\ S \leq \bar{S}$. This fact explains the "terminal condition" in $\mathcal{A}_1$ at $t = T - \bar{J}$. Secondly, the "terminal condition", with respect to J, in $\mathcal{A}_2$ corresponds to the "knock-in" feature that the option price is equal to that of the embedded call, denoted by $C_A(S, t)$ or $C_E(S, t)$, at the time t the option is activated. Thirdly, we have the inhomogeneous boundary condition in $\mathcal{A}_2$ when S approaches infinity because in this case the knock-in feature will be surely triggered and thereby the knock-in option price would be the same with its embedded option price, which is equivalent with the asset price S. Finally, the last equation in $\mathcal{A}_2$ of (2.2) holds only for $0 \leq J < \bar{J}$, i.e, before the "knock in" feature is triggered.

The above coupled PDE systems resemble these in [15], so their "moving window" technique can be adopted to obtain the solution for our problem. In the next section, we shall discuss the solution procedure.

3 Solution of the coupled PDE systems

Following the method of [15] the three-dimensional system in (2.2-2.3) can be reduced to a two-dimensional system by replacing the sum of the partial derivatives of V_2, $\frac{\partial V_2}{\partial t}+\frac{\partial V_2}{\partial J}$, by its directional derivative $\sqrt{2}\frac{\partial V_2}{\partial l}$, in the direction of $(\sqrt{2},\sqrt{2})$. After a further change of variable by $l=\sqrt{2}l'$, the PDE systems in (2.2-2.3) is transformed to the following two-dimensional PDE systems:

$$\mathcal{A}_1\begin{cases}\dfrac{\partial V_1}{\partial t}+\mathbb{L}V_1=0,\\[2ex] V_1(S,T-\bar{J})=0,\\[2ex] V_1(0,t)=0,\\[2ex] V_1(\bar{S},t)=W(t),\end{cases}\qquad \mathcal{A}_2\begin{cases}\dfrac{\partial V_2}{\partial l'}+\mathbb{L}V_2=0,\\[2ex] V_2(S,\bar{J};t)=C(S,t+\bar{J}),\\[2ex] V_2(S,l';t)\backsim S\ as\ S\rightarrow+\infty,\\[2ex] V_2(\bar{S},l';t)=W(t+l'),\end{cases}\tag{3.4}$$

$$\text{connectivity condition}:\frac{\partial V_1}{\partial S}(\bar{S},t)=\frac{\partial V_2}{\partial S}(\bar{S},0;t),\tag{3.5}$$

where $\mathcal{A}_1$ is defined on $t\in[0,T-\bar{J}]$, $S\in[0,\bar{S}]$, $\mathcal{A}_2$ is defined on $l'\in[0,\bar{J}]$, $S\in[\bar{S},\infty)$, and $t\in[0,T-\bar{J}]$ is only a parameter as far as the PDE system in $\mathcal{A}_2$ is concerned. The unknown function $W(t)=V_2(\bar{S},0;t)$, which provides the coupling between the two PDE systems, needs to be solved as part of the solution.

To solve the newly established pricing system (3.4-3.5) effectively, we shall first non-dimensionalize all variables by introducing the following dimensionless variables:

$$S = \bar{S}e^x,\ \tau = (T - \bar{J} - t)\frac{\sigma^2}{2},\ \tilde{l} = \frac{\sigma^2}{2}(\bar{J} - l^{'}),\ \bar{J}^{'} = \frac{\sigma^2 \bar{J}}{2},\ T^{'} = \frac{\sigma^2 T}{2},\ W(t) = \bar{S}W^{'}(\tau),$$
$$V_1(S,t) = \bar{S}V_1^{'}(x,\tau),\ V_2(S,l';t) = \bar{S}V_2^{'}(x,\tilde{l};\tau),\ C(S,t) = \bar{S}C^{'}(x,\tau+\bar{J}'). \tag{3.6}$$

With all primes and tildes dropped from now on, the dimensionless coupled PDE systems read:

$$\mathcal{A}_1 \begin{cases} \dfrac{\partial V_1}{\partial \tau} = \mathbb{L}V_1, \\ V_1(x,0) = 0, \\ \lim\limits_{x\to-\infty} V_1(x,\tau) = 0, \\ V_1(0,\tau) = W(\tau), \end{cases} \qquad \mathcal{A}_2 \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathbb{L}V_2, \\ V_2(x,0;\tau) = C(x,\tau), \\ V_2(x,l;\tau) \backsim e^x \ as\ x \to +\infty, \\ V_2(0,l;\tau) = W(\tau - \bar{J} + l), \end{cases} \tag{3.7}$$

$$\text{connectivity condition}: \frac{\partial V_1}{\partial x}(0,\tau) = \frac{\partial V_2}{\partial x}(0,\bar{J};\tau), \tag{3.8}$$

where $\mathcal{A}_1$ is defined on $\tau \in [0, T-\bar{J}],\ x \in (-\infty, 0]$, $\mathcal{A}_2$ is defined on $l \in [0,\bar{J}],\ x \in [0,\infty)$, with the parameter $\tau \in [0, T-\bar{J}]$, $\mathbb{L} = \dfrac{\partial^2}{\partial x^2} + k\dfrac{\partial}{\partial x} - \gamma I$ with $k = \gamma - q - 1,\ \gamma = \dfrac{2r}{\sigma^2},\ q = \dfrac{2D}{\sigma^2}$. Note that the asset price and the option prices (V_1 and V_2) are non-dimensionalized by the asset barrier $\bar{S}$ here. As a result, the x-domains in $\mathcal{A}_1$ and $\mathcal{A}_2$ are semi-infinite.

By applying the Laplace transform technique, the solution of $\mathcal{A}_1$ can be easily found as

$$V_1(x,\tau) = \int_0^\tau W(s)g_1(x,\tau - s)ds, \quad \forall x \le 0, \tag{3.9}$$

where

$$g_1(x,\tau) = -\frac{x}{2\sqrt{\pi}\tau^{\frac{3}{2}}}e^{\alpha x + \beta\tau - \frac{x^2}{4\tau}}, \alpha = -\frac{k}{2}, \beta = -\frac{k^2}{4} - \gamma.$$

Since the PDE in $\mathcal{A}_2$ is linear, its solution can be found by superposition of the solutions of the following two systems :

$$\mathcal{B}_1 \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathbb{L}V_2, \\ \\ V_2(x,0;\tau) = 0, \\ \\ \lim\limits_{x\to+\infty} V_2(x,l;\tau) = 0, \\ \\ V_2(0,l;\tau) = W(\tau - \bar{J} + l), \end{cases} \qquad \mathcal{B}_2 \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathbb{L}V_2, \\ \\ V_2(x,0;\tau) = C(x,\tau), \\ \\ V_2(x,l;\tau) \backsim e^x \ as \ x \to +\infty, \\ \\ V_2(0,l;\tau) = 0. \end{cases}$$

System $\mathcal{B}_1$ is very similar to that of $\mathcal{A}_1$ so its solution can be easily found as

$$V_2^{(1)}(x,l;\tau) = \int_0^l W(\tau - \bar{J} + s) g_2(x, l-s) ds, \quad \forall x \geq 0,$$

where $g_2(x,l) = -g_1(x,l)$.

By using the variable transform $V_2(x,l;\tau) = e^{\alpha x + \beta\tau} u(x,l;\tau)$ (with α, β defined as above), $\mathcal{B}_2$ can be transferred to a standard Heat problem on a semi-infinite domain, whose solution can be found in [7]. As a result, the solution of $\mathcal{B}_2$ can be obtained as follows

$$V_2^{(2)}(x,l;\tau) = \int_0^{+\infty} \frac{1}{2\sqrt{\pi l}} e^{\alpha(x-z)+\beta l}[e^{-\frac{(x-z)^2}{4l}} - e^{-\frac{(x+z)^2}{4l}}]C(z,\tau)dz.$$

We now can obtain the solution of $\mathcal{A}_2$ as

$$V_2(x,l;\tau) = V_2^{(1)}(x,l;\tau) + V_2^{(2)}(x,l;\tau). \tag{3.10}$$

Applying the connectivity condition (3.8) to (3.9) and (3.10), we obtain an integral equation governing $W(\tau)$

$$\int_0^\tau W(s)\frac{\partial g_1}{\partial x}(x,\tau-s)ds|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)|_{x=0}+\int_0^{\bar{J}} W(\tau-\bar{J}+s)\frac{\partial g_2}{\partial x}(x,\bar{J}-s)ds|_{x=0}, \quad (3.11)$$

where

$$\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)|_{x=0} = \int_0^{+\infty}\frac{zC(z,\tau)}{2\sqrt{\pi\bar{J}^3}}e^{-\alpha z+\beta\bar{J}-\frac{z^2}{4\bar{J}}}dz.$$

Now, taking a simple coordinate transform, $\xi = \tau-\bar{J}+s$, in the last integral on the right-hand side of equation (3.11) leads to

$$\int_0^\tau W(s)\frac{\partial g_1}{\partial x}(x,\tau-s)ds|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)|_{x=0}+\int_{\tau-\bar{J}}^{\tau} W(\xi)\frac{\partial g_2}{\partial x}(x,\tau-\xi)d\xi|_{x=0}. \quad (3.12)$$

It can be observed that the left-hand side of (3.12) contains the information of $W(s)$ from the expiry ($\tau = 0$) to the current time to expiry, τ, while its right-hand side integral involves the value of $W(\xi)$, $\xi \in [\tau-\bar{J},\tau]$, which coincides with the projection of the "slide" (a plane is of 45° angle to both of the plane $t = 0$, and $J = 0$) passing through $(\bar{S},\tau,0)$ on the plane $J = 0$. As in [15], we also name such a projection a "window". It should be noted that $W(\tau) = 0, \forall\tau \in [-\bar{J},0]$ because $V_1(S,t) = 0$, for all $t \geq T-\bar{J},\ S \leq \bar{S}$ (as already explained in Section 2).

We now solve the integral equation (3.12) for $\tau \in [0,\bar{J}]$ to obtain the solution for $W_1(\tau)$, the value of W in the first window. Since $W(\xi) = 0, \forall\ \xi \in [-\bar{J},0]$, we can rewrite (3.12) as follows:

$$\int_0^\tau W_1(s)(\frac{\partial g_1}{\partial x}-\frac{\partial g_2}{\partial x})(x,\tau-s)ds|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)|_{x=0}. \quad (3.13)$$

Clearly, the left hand-side of the last equation is a convolution integral involving the unknown function W_1. Taking the Laplace transform of equation (3.13) with respect to τ, we obtain

$$\mathcal{L}[W_1(\tau)]\mathcal{L}\big[(\frac{\partial g_1}{\partial x}-\frac{\partial g_2}{\partial x})(x,\tau)\big]|_{x=0}=\mathcal{L}\big[\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\big]|_{x=0},$$

where $\mathcal{L}\big[(\frac{\partial g_1}{\partial x}-\frac{\partial g_2}{\partial x})(x,\tau)\big]|_{x=0}=2\sqrt{p-\beta}$ with p being the Laplace parameter as in [15]. Thus,

$$\mathcal{L}[W_1(\tau)]=\frac{1}{2\sqrt{p-\beta}}\mathcal{L}\big[\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tau)\big]|_{x=0}. \tag{3.14}$$

By taking the inverse Laplace transform on both sides of (3.14), we can obtain an expression for $W_1(\tau)$ as follows:

$$W_1(\tau)=\int_0^{\tau}\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};s)|_{x=0}\frac{e^{\beta(\tau-s)}ds}{2\sqrt{\pi(\tau-s)}}=\int_0^{+\infty}\frac{ze^{-\alpha z+\beta\bar{J}-\frac{z^2}{4\bar{J}}}}{4\pi\bar{J}^{3/2}}\int_0^{\tau}\frac{C(z,s)e^{\beta(\tau-s)}}{\sqrt{\tau-s}}dsdz. \tag{3.15}$$

Similar to the case in [15], for a state point (S,τ,J), one can evaluate W forwards, window by window, until the value at the required time τ is found. In fact, assuming W_n is known for $n\geq 1$, we can then calculate the option price V_1 or V_2 in the nth window from the formula (3.9) or (3.10), respectively. However, the determination of W_{n+1}, assuming W_n is known for $n\geq 1$, is slightly different from that of W_1. The 2-D coupled PDE systems governing the option price in the $(n+1)$th window can be expressed as

$$\mathcal{A}_3\begin{cases}\dfrac{\partial V_1}{\partial t}+\mathbb{L}V_1=0,\\[2ex] V_1(S,T-(n+1)\bar{J})=\bar{S}f_n(\ln\dfrac{S}{\bar{S}}),\\[2ex] V_1(0,t)=0,\\[2ex] V_1(\bar{S},t)=W(t),\end{cases}\qquad \mathcal{A}_4\begin{cases}\dfrac{\partial V_2}{\partial l'}+\mathbb{L}V_2=0,\\[2ex] V_2(S,\bar{J};t)=C(S,t+\bar{J}),\\[2ex] V_2(S,l';t)\backsim S\ as\ S\rightarrow+\infty,\\[2ex] V_2(\bar{S},l';t)=W(t+l'),\end{cases} \tag{3.16}$$

$$\text{connectivity condition}: \frac{\partial V_1}{\partial S}(\bar{S}, t) = \frac{\partial V_2}{\partial S}(\bar{S}, 0; t), \tag{3.17}$$

where

$$f_n(x) = \sum_{i=1}^{n} \int_{(i-1)\bar{J}}^{i\bar{J}} W_i(s) g_1(x, n\bar{J} - s) ds,$$

$\mathcal{A}_3$ is defined on $t \in [T-(n+2)\bar{J}, T-(n+1)\bar{J}],\ S \in [0, \bar{S}]$; $\mathcal{A}_4$ is defined on $l' \in [0, \bar{J}],\ S \in [\bar{S}, \infty)$, and $t \in [T-(n+2)\bar{J}, T-(n+1)\bar{J}]$ is only a parameter as far as the PDE system in $\mathcal{A}_4$ is concerned. It should be noted that the system (3.16-3.17) is very similar to that of (3.4-3.5), except that the initial condition in the former now becomes inhomogeneous as $\bar{S} f_n(\ln \frac{S}{\bar{S}}) > 0, \forall S < \bar{S}$. To non-dimensionalize the system (3.16-3.17), we use the same dimensionless variables introduced in (3.6), except that τ and $W'(\tau)$ are replaced by $\tilde{\tau} = (T-(n+1)\bar{J}-t)\sigma^2/2 = \tau - n\bar{J}'$ and $U(\tilde{\tau})$, respectively. Dropping all primes from now on, we derive the following coupled dimensionless PDE systems:

$$\mathcal{B}_3 \begin{cases} \dfrac{\partial V_1}{\partial \tilde{\tau}} = \mathbb{L} V_1, \\ V_1(x, 0) = f_n(x), \\ \lim\limits_{x \to -\infty} V_1(x, \tilde{\tau}) = 0, \\ V_1(0, \tilde{\tau}) = U(\tilde{\tau}), \end{cases} \qquad \mathcal{B}_4 \begin{cases} \dfrac{\partial V_2}{\partial l} = \mathbb{L} V_2, \\ V_2(x, 0; \tilde{\tau}) = C(x, \tilde{\tau}), \\ V_2(x, l; \tilde{\tau}) \backsim e^x \ as\ x \to +\infty, \\ V_2(0, l; \tilde{\tau}) = U(\tilde{\tau} - \bar{J} + l), \end{cases} \tag{3.18}$$

$$\text{connectivity condition}: \frac{\partial V_1}{\partial x}(0, \tilde{\tau}) = \frac{\partial V_2}{\partial x}(0, \bar{J}; \tilde{\tau}), \tag{3.19}$$

where

$$f_n(x) = \sum_{i=1}^{n} \int_{(i-1)\bar{J}}^{i\bar{J}} W_i(s) g_1(x, n\bar{J} - s) ds,$$

$\mathcal{B}_3$ is defined on $\tilde{\tau} \in [0, \bar{J}],\ x \in (-\infty, 0]$, $\mathcal{B}_4$ is defined on $l \in [0, \bar{J}],\ x \in [0, \infty)$, with $\tilde{\tau} \in [0, \bar{J}]$.

The inhomogeneous initial condition of $\mathcal{B}_3$ makes its solution procedure more complicated than that of $\mathcal{A}_1$. The solution for $\mathcal{B}_3$ can be found by splitting the linear problem into two sub-problems: one with homogeneous boundary conditions but a non-zero initial condition, and another with a zero initial condition but inhomogeneous boundary condition at $x = 0$. The first can be transferred to a standard Heat problem on a semi-infinite domain, which has standard solution [7], while the solution of the second problem, can be obtained by applying the Laplace transform technique as we did to solve $\mathcal{A}_1$. Without going through the lengthy derivation process, the solution of $\mathcal{B}_3$ is given below

$$V_1(x, \tilde{\tau}) = G(x, \tilde{\tau}) + \int_0^{\tilde{\tau}} U(s) g_1(x, \tilde{\tau} - s) ds, \tag{3.20}$$

where

$$G(x, \tilde{\tau}) = \int_{-\infty}^0 \frac{1}{2\sqrt{\pi\tilde{\tau}}} e^{\alpha(x-z)+\beta\tilde{\tau}} [e^{-\frac{(x-z)^2}{4\tilde{\tau}}} - e^{-\frac{(x+z)^2}{4\tilde{\tau}}}] f_n(z) dz.$$

Consequently, the corresponding integral equation governing $U(\tilde{\tau})$ is

$$\frac{\partial G}{\partial x}(x, \tilde{\tau})|_{x=0} + \int_0^{\tilde{\tau}} U(s) \frac{\partial g_1}{\partial x}(x, \tilde{\tau}-s) ds|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})|_{x=0} + \int_0^{\bar{J}} U(\tilde{\tau}-\bar{J}+s) \frac{\partial g_2}{\partial x}(x, \bar{J}-s) ds|_{x=0}. \tag{3.21}$$

Now, taking a simple coordinate transform, $\xi = \tilde{\tau} - \bar{J} + s$, in the integral on the right-hand side of the above equation leads to

$$\frac{\partial G}{\partial x}(x, \tilde{\tau})|_{x=0} + \int_0^{\tilde{\tau}} U(s) \frac{\partial g_1}{\partial x}(x, \tilde{\tau}-s) ds|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})|_{x=0} + \int_{\tilde{\tau}-\bar{J}}^{\tilde{\tau}} U(\xi) \frac{\partial g_2}{\partial x}(x, \tilde{\tau}-\xi) d\xi|_{x=0}. \tag{3.22}$$

Since $U(\xi) \equiv U_0(\xi) = W_n(\xi + n\bar{J}), \forall\ \xi \in [-\bar{J}, 0]$, (3.22) can be written as:

$$\int_0^{\tilde{\tau}} U(s) (\frac{\partial g_1}{\partial x} - \frac{\partial g_2}{\partial x})(x, \tilde{\tau}-s) ds|_{x=0} = \frac{\partial V_2^{(2)}}{\partial x}(x, \bar{J}; \tilde{\tau})|_{x=0} + \int_{\tilde{\tau}-\bar{J}}^0 U_0(\xi) \frac{\partial g_2}{\partial x}(x, \tilde{\tau}-\xi) d\xi|_{x=0} - \frac{\partial G}{\partial x}(x, \tilde{\tau})|_{x=0}. \tag{3.23}$$

Taking the Laplace transform on both sides of (3.23) with respect to $\tilde{\tau}$, we obtain:

$$
\begin{aligned}
\mathcal{L}[U(\tilde{\tau})]\mathcal{L}\big[(\frac{\partial g_1}{\partial x}-\frac{\partial g_2}{\partial x})(x,\tilde{\tau})\big]|_{x=0} \;&=\; \mathcal{L}\big[\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tilde{\tau})\big]|_{x=0}-\mathcal{L}\big[\frac{\partial G}{\partial x}(x,\tilde{\tau})\big]\big|_{x=0} \\
&\quad +\mathcal{L}\big[\int_{\tilde{\tau}-\bar{J}}^{0}U_0(\xi)\frac{\partial g_2}{\partial x}(x,\tilde{\tau}-\xi)d\xi\big]|_{x=0}.
\end{aligned}
$$

Therefore,

$$
\mathcal{L}[U(\tilde{\tau})]=\frac{1}{2\sqrt{p-\beta}}\left(\mathcal{L}\big[\frac{\partial V_2^{(2)}}{\partial x}(x,\bar{J};\tilde{\tau})\big]+\mathcal{L}\big[\int_{\tilde{\tau}-\bar{J}}^{0}U_0(\xi)\frac{\partial g_2}{\partial x}(x,\tilde{\tau}-\xi)d\xi\big]-\mathcal{L}\big[\frac{\partial G}{\partial x}(x,\tilde{\tau})\big]\right)\Big|_{x=0}. \tag{3.24}
$$

By taking the inverse Laplace transform on both sides of (3.24), we can obtain the solution of (3.21) as follows:

$$
\begin{aligned}
U(\tilde{\tau}) \;&=\; \int_{-\infty}^{0}\frac{e^{-\alpha z+\beta\tilde{\tau}}}{2\sqrt{\pi\tilde{\tau}}}e^{-\frac{z^2}{4\tilde{\tau}}}f_n(z)dz-\frac{e^{\beta\bar{J}}}{2\pi\sqrt{\bar{J}}}\int_0^{\tilde{\tau}}\frac{e^{\beta(\tilde{\tau}-s)}}{\sqrt{\tilde{\tau}-s}}U_0(s-\bar{J})ds \\
&\quad +\frac{U_0(0)}{2}e^{\beta\tilde{\tau}}+\int_0^{+\infty}\frac{z}{4\pi\bar{J}^{3/2}}e^{-\frac{z^2}{4\bar{J}}+\beta\bar{J}-\alpha z}\int_0^{\tilde{\tau}}\frac{C(z,s)}{\sqrt{\tilde{\tau}-s}}e^{\beta(\tilde{\tau}-s)}dsdz \\
&\quad -\frac{1}{\pi}\int_0^{\tilde{\tau}}\frac{e^{\beta(\tilde{\tau}-s)}}{\sqrt{\tilde{\tau}-s}}\int_{\sqrt{s}}^{\sqrt{\bar{J}}}e^{\beta t^2}\big[(-\beta)U_0(s-t^2)+U_0^{'}(s-t^2)\big]dtds,
\end{aligned}
$$

where $U_0(\tilde{\tau})=W_n(\tilde{\tau}+n\bar{J}),\forall\;\tilde{\tau}\in[-\bar{J},0]$.

Note that the inverse Laplace of the first term on the right hand side of (3.24) is the same as that in the calculation of W_1, while the inverse Laplace of the last two terms on the right hand side of (3.24) were also carried out analytically, the detailed calculation can be seen in Appendix A and Appendix B in [15].

Consequently, the analytical formula for $W_{n+1}(\tau), \tau \in [n\bar{J}, (n+1)\bar{J}]$, $n \geq 1$, is

$$\begin{aligned} W_{n+1}(\tau) \;\; = & \int_{-\infty}^{0} \frac{e^{-\alpha z+\beta(\tau-n\bar{J})}}{2\sqrt{\pi(\tau-n\bar{J})}} e^{-\frac{z^2}{4(\tau-n\bar{J})}} f_n(z)dz - \frac{e^{\beta\bar{J}}}{2\pi\sqrt{\bar{J}}} \int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} W_n(s-\bar{J})ds \\ & + \frac{W_n(n\bar{J})}{2} e^{\beta(\tau-n\bar{J})} + \int_0^{+\infty} \frac{z}{4\pi\bar{J}^{3/2}} e^{-\frac{z^2}{4\bar{J}}+\beta\bar{J}-\alpha z} \int_{n\bar{J}}^{\tau} \frac{C(z,s)}{\sqrt{\tau-s}} e^{\beta(\tau-s)} dsdz \\ & - \frac{1}{\pi} \int_{n\bar{J}}^{\tau} \frac{e^{\beta(\tau-s)}}{\sqrt{\tau-s}} \int_{\sqrt{s-n\bar{J}}}^{\sqrt{\bar{J}}} e^{\beta t^2} \left[(-\beta)W_n(s-t^2) + W_n'(s-t^2)\right] dtds. \end{aligned}$$

Thus, we have obtained an analytical solution for Parisian up-and-in calls. This solution can be used for the valuation of American-style and European-style Parisian up-and-in calls, once C is substituted by C_A and C_E in the above formulae of V_1, V_2, and W, respectively, as both vanilla European option and vanilla American option have been thoroughly studied in the literature [1–5, 8–14, 16].

4 Numerical example and discussion

In this section, we provide an example of pricing an American-style Parisian up-and-in call. This example will illustrate the implementation of our analytical solution as well as reveal some interesting features of a Parisian up-and-in call.

It should be noted that the calculation procedure for an American-style Parisian up-and-in call option is similar to that for a European-style Parisian up-and-out call as presented in [15], except that we have replaced the values of the vanilla European option by the numerical values of its American counterpart, which are obtained by using the highly efficient integral equation method ([2, 9]). Once the value of the embedded vanilla American option is determined, the integrals in our analytical formula are computed by using quadrature rules (Gauss-Laguerre, Gauss-Legendre, Gauss-Jacobi rules) in a very similar way as that in [15].

Figure 1 compares the values of an American-style Parisian up-and-in call for various J

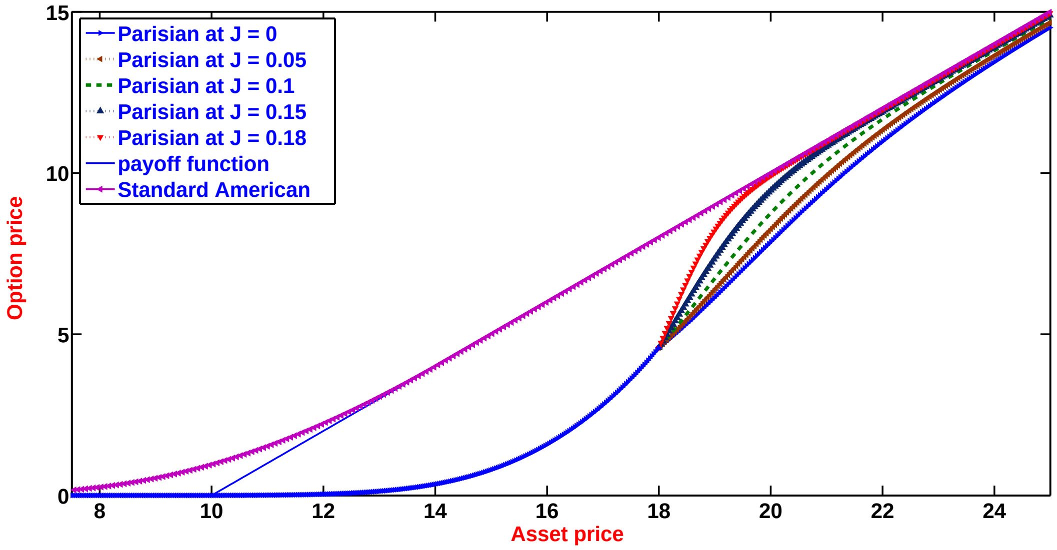

Figure 1: Price of an American-style up-and-in call with parameters: $E = 10, T - t = 0.8$, $\bar{S} = 13, \bar{J} = 0.2, \sigma = 0.3, r = 0.05, D = 0.1$

.

values with the value of its embedded vanilla American call. The parameters used in our calculations are $E = 10, T - t = 0.8, \bar{S} = 13, \bar{J} = 0.2, \sigma = 0.3, r = 0.05, D = 0.1$. As can be seen clearly from the figure that the value of the Parisian option is always less than that of its embedded vanilla option. This makes sense financially as a holder of the Parisian up-and-in call has to wait until the knock-in feature is activated, to obtain the same exercise right as the holder of the embedded vanilla option. This waiting period, with the risk that the "knock-in" may never occur, would definitely devalues the Parisian up-and-in call, in comparison with its embedded vanilla counterpart.

Figure 1 also reveals some interesting properties of a Parisian up-and-in call with respect to changes in S and J. One can observe that the Parisian call price is an increasing function of asset price. This can be financially explained that the knock-in feature is more likely to be activated when the asset price increases. Similarly, the closer J approaches $\bar{J}$, the more likely the Parisian up-and-in call becomes its embedded vanilla option. Therefore, the Parisian option price increases when J get closer to $\bar{J}$.

5 Conclusion

In this paper, we derived a simple analytical formula for Parisian up-and-in calls by using the technique proposed in Zhu and Chen [15]. Unlike "knock-out" cases, the valuation of American-style Parisian up-and-in calls is very similar to that of its European counterpart and both can be proceeded with the same solution procedure. As a result, we have obtained a pricing formula that can be used to evaluate both American-style and European-style Parisian up-and-in calls. We also provide an example to illustrate the implementation of our analytical solution as well as to reveal some interesting features of a Parisian up-and-in call.

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