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# A simple proof of Euler's continued fraction of e^{1/M}

Joseph Tonien
*University of Wollongong*, dong@uow.edu.au

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**Abstract**

A continued fraction is an expression of the form

$f_0$+ $g_0$

$f_1$+$g_1$

$f_2$+$g_2$

and we will denote it by the notation $[f_0, (g_0, f_1), (g_1, f_2), (g_2, f_3), \dots ]$. If the numerators $g_i$ are all equal to 1 then we will use a shorter notation $[f_0, f_1, f_2, f_3, \dots ]$. A *simple continued fraction* is a continued fraction with all the $g_i$ coefficients equal to 1 and with all the $f_i$ coefficients positive integers except perhaps $f_0$.

The finite continued fraction $[f_0, (g_0, f_1), (g_1, f_2), \dots, (g_{k-1}, f_k)]$ is called the *k*th convergent of the infinite continued fraction $[f_0, (g_0, f_1), (g_1, f_2), \dots]$. We define

$[f_0, (g_0, f_1), (g_1, f_2), (g_2, f_3), \dots] = \lim [f_0, (g_0, f_1), (g_1, f_2), \dots, (g_{k-1}, f_k)]$

if this limit exists and in this case we say that the infinite continued fraction *converges*.



**Keywords**

proof, 1, e, fraction, continued, euler, m, simple

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# A simple proof of Euler's continued fraction of $e^{1/M}$

JOSEPH TONIEN

*Introduction*

A continued fraction is an expression of the form

$$f_0 + \cfrac{g_0}{f_1 + \cfrac{g_1}{f_2 + \cfrac{g_2}{\cdots}}}$$

and we will denote it by the notation $[f_0, (g_0, f_1), (g_1, f_2), (g_2, f_3), \ldots]$. If the numerators $g_i$ are all equal to 1 then we will use a shorter notation $[f_0, f_1, f_2, f_3, \ldots]$. A *simple continued fraction* is a continued fraction with all the $g_i$ coefficients equal to 1 and with all the $f_i$ coefficients positive integers except perhaps $f_0$.

The finite continued fraction $[f_0, (g_0, f_1), (g_1, f_2), \ldots, (g_{k-1}, f_k)]$ is called the $k$ th *convergent* of the infinite continued fraction $[f_0, (g_0, f_1), (g_1, f_2), \ldots]$. We define

$$[f_0, (g_0, f_1), (g_1, f_2), (g_2, f_3), \ldots] = \lim_{k \to \infty} [f_0, (g_0, f_1), (g_1, f_2), \ldots, (g_{k-1}, f_k)]$$

if this limit exists and in this case we say that the infinite continued fraction *converges*.

In a foundational publication on the theory of continued fractions, *De fractionibus continuis dissertatio* [1], Euler used the Ricatti differential equation to derive the following interesting continued fraction for any positive real number $M$:

$$e^{1/M} = 1 + \cfrac{1}{M - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3M - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5M - 1 + \cfrac{1}{\cdots}}}}}}}}. \tag{1}$$

When $M = 1$, we have the following simple continued fraction expansion of $e$

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, 8, \ldots] \tag{2}$$

The fact that a rational number must have a finite simple continued fraction expansion implies that $e$ is irrational. Lagrange's theorem asserts that a real number has a periodic simple continued fraction if, and only if, it is a quadratic irrational. Since (2) is not periodic, $e$ must not be algebraic of degree 2.

Using integration of the form $\int e^{-rx}x^n(1 - x)^n\,dx$, Hermite [2] gave the first proof that $e$ is transcendental. As a by-product, Hermite also derived the identity (2). Based on Hermite's work, Olds [3] gave an expository proof of the continued fraction of $e$. Cohn [4] streamlined Olds' proof into a short presentation. Osler [5] extended Cohn's proof to the general case of $e^{1/M}$. All of these proofs rely heavily on the integration technique.

In this paper, we will present a simple proof of the continued fraction of $e^{1/M}$ – the identity (1) – which only involves the manipulation of recurrence equations. Our proof contains two steps. In the first step, we show that

$$e^{1/M} = 1 + \cfrac{1}{M - \frac{1}{2} + \cfrac{\frac{1}{4}}{3M + \cfrac{\frac{1}{4}}{5M + \cfrac{\frac{1}{4}}{7M + \cfrac{\frac{1}{4}}{\dots}}}}}. \qquad (3)$$

And in the second step of the proof, we transform the identity (3) into the form (1).

The interested reader is referred to [6, 7] for other proofs of related continued fractions which also use manipulation of recurrence relations instead of integration.

*An interesting recurrence sequence*

Let us look at the following sequence.

*Lemma* 1: For a positive real number $M$, let

$$S_0 = \sum_{i=1}^{\infty} \frac{1}{i!\,M^i}$$

$$S_1 = \sum_{i=1}^{\infty} \frac{i - 1}{(i + 1)!\,M^i}$$

$$\vdots$$

$$S_k = \sum_{i=1}^{\infty} \frac{(i - 1)(i - 2)\dots(i - k)}{(i + k)!\,M^i}$$

then

$$S_{n+2} + (4n + 6)MS_{n+1} - S_n = 0. \qquad (4)$$

*Proof*: $S_0 = e^{1/M} - 1$, and by ratio test, we can see that each of the series $S_k$ converges to a positive number. We have

$$\begin{aligned}
S_n - (4n+6)MS_{n+1} &= \sum_{i=1}^{\infty} \frac{(i-1)(i-2)\dots(i-n)}{(i+n)!M^i} - (4n+6)\sum_{i=1}^{\infty} \frac{(i-1)(i-2)\dots(i-n-1)}{(i+n+1)!M^{i-1}} \\
&= \sum_{i=1}^{\infty} \frac{(i-1)(i-2)\dots(i-n)}{(i+n)!M^i} - (4n+6)\sum_{i=1}^{\infty} \frac{i(i-1)\dots(i-n)}{(i+n+2)!M^i} \\
&= \sum_{i=1}^{\infty} \frac{(i-1)(i-2)\dots(i-n)[(i+n+1)(i+n+2)-(4n+6)i]}{(i+n+2)!M^i} \\
&= \sum_{i=1}^{\infty} \frac{(i-1)(i-2)\dots(i-n)(i-n-1)(i-n-2)}{(i+n+2)!M^i} \\
&= S_{n+2}.
\end{aligned}$$

Using the recurrence relation (4), we can consistently define $S_{-1}$ as follows:

$$\begin{aligned}
S_{-1} &= S_1 + 2MS_0 \\
&= \sum_{i=1}^{\infty} \frac{i-1}{(i+1)!\,M^i} + 2M\sum_{i=1}^{\infty} \frac{1}{i!\,M^i} \\
&= 2 + \sum_{i=1}^{\infty} \frac{i-1}{(i+1)!\,M^i} + 2\sum_{i=2}^{\infty} \frac{1}{i!\,M^{i-1}} \\
&= 2 + \sum_{i=1}^{\infty} \frac{i-1}{(i+1)!\,M^i} + 2\sum_{i=1}^{\infty} \frac{1}{(i+1)!\,M^i} \\
&= 2 + \sum_{i=1}^{\infty} \frac{i+1}{(i+1)!\,M^i} = 2 + \sum_{i=1}^{\infty} \frac{1}{i!\,M^i} \\
&= e^{1/M} + 1.
\end{aligned}$$

We now use the recurrence relation

$$S_{n+2} + (4n+6)MS_{n+1} - S_n = 0, \text{ for all } n \geqslant -1,$$

to establish a continued fraction.

*Lemma* 2: For any $n \geqslant 0$,

$$\left[M, \left(\frac{1}{4}, 3M\right), \left(\frac{1}{4}, 5M\right), \dots, \left(\frac{1}{4}, (2n+1)M\right), \left(\frac{1}{4}, \frac{S_n}{2S_{n+1}}\right)\right] = \frac{1}{e^{1/M}-1} + \frac{1}{2}.$$

*Proof*: Using the sequence $\{S_n\}$ of Lemma 1, we have

$$S_{n+2} + (4n + 6)MS_{n+1} - S_n = 0$$
$$\Rightarrow \frac{S_n}{2S_{n+1}} = (2n + 3)M + \frac{S_{n+2}}{2S_{n+1}}$$
$$\Rightarrow \frac{S_n}{2S_{n+1}} = (2n + 3)M + \frac{\frac{1}{4}}{\frac{S_{n+1}}{2S_{n+2}}}.$$

So for any $n \geqslant 0$,

$$\frac{1}{e^{1/M} - 1} + \frac{1}{2} = \frac{e^{1/M} + 1}{2(e^{1/M} - 1)} = \frac{S_{-1}}{2S_0}$$
$$= M + \frac{\frac{1}{4}}{\frac{S_0}{2S_1}}$$
$$= M + \cfrac{\frac{1}{4}}{3M + \cfrac{\frac{1}{4}}{\frac{S_1}{2S_2}}}$$
$$= M + \cfrac{\frac{1}{4}}{3M + \cfrac{\frac{1}{4}}{5M + \cfrac{\frac{1}{4}}{\frac{S_2}{2S_3}}}}$$
$$= \dots$$
$$= M + \cfrac{\frac{1}{4}}{3M + \cfrac{\frac{1}{4}}{5M + \cfrac{\frac{1}{4}}{\dots (2n+1)M + \cfrac{\frac{1}{4}}{\frac{S_n}{2S_{n+1}}}}}}.$$

Lemma 2 almost gives us the continued fraction (3) for $e^{1/M}$. All we need to do is to show that the infinite continued fraction $[M, (\frac{1}{4}, 3M), (\frac{1}{4}, 5M), \dots]$ converges to $\dfrac{1}{e^{1/M} - 1} + \dfrac{1}{2}$. To do that we will review some basic facts about continued fractions.

*Euler-Wallis recurrence formulas*

The following theorem due to Lord Brouncker, the first President of the Royal Society, is called *the fundamental theorem of continued fractions*. It gives us recursive formulas to calculate the numerator and the denominator of the convergents. Wallis and Euler made extensive use of these formulas and now they are called the Euler-Wallis formulas.

*Theorem* 1: For any $n \geqslant 0$, the $n$th convergent can be determined as

$$[f_0, (g_0, f_1), (g_1, f_2), \dots, (g_{n-1}, f_n)] = \frac{p_n}{q_n}$$

where the sequences $\{p_n\}_{n \geqslant -2}$ and $\{q_n\}_{n \geqslant -1}$ are specified as follows

$$p_{-2} = 0, p_{-1} = 1, p_n = f_n p_{n-1} + g_{n-1} p_{n-2}, \text{ for all } n \geqslant 0,$$

$$q_{-1} = 0, q_0 = 1, q_n = f_n q_{n-1} + g_{n-1} q_{n-2}, \text{ for all } n \geqslant 1.$$

The theorem can be easily proved by induction as

$$[f_0, (g_0, f_1), (g_1, f_2), \dots, (g_{n-1}, f_n), (g_n, f_{n+1})]$$
$$= \left[f_0, (g_0, f_1), (g_1, f_2), \dots, \left(g_{n-1}, f_n + \frac{g_n}{f_{n+1}}\right)\right]$$
$$= \frac{\left(f_n + \frac{g_n}{f_{n+1}}\right) p_{n-1} + g_{n-1} p_{n-2}}{\left(f_n + \frac{g_n}{f_{n+1}}\right) q_{n-1} + g_{n-1} q_{n-2}} = \frac{(f_{n+1} f_n + g_n) p_{n-1} + f_{n+1} g_{n-1} p_{n-2}}{(f_{n+1} f_n + g_n) q_{n-1} + f_{n+1} g_{n-1} q_{n-2}}$$
$$= \frac{f_{n+1}(f_n p_{n-1} + g_{n-1} p_{n-2}) + g_n p_{n-1}}{f_{n+1}(f_n q_{n-1} + g_{n-1} q_{n-2}) + g_n q_{n-1}} = \frac{f_{n+1} p_n + g_n p_{n-1}}{f_{n+1} q_n + g_n q_{n-1}} = \frac{p_{n+1}}{q_{n+1}}.$$

Using Euler-Wallis recurrence formulas, one can prove many identities. The following identity is due to Euler.

$$\frac{p_n}{q_n} = f_0 + \sum_{k=1}^{n} (-1)^{k+1} \frac{\prod_{j=0}^{k-1} g_j}{q_{k-1} q_k}, \text{ for all } n \geqslant 0. \tag{5}$$

To prove (5), we first observe that

$$p_n q_{n-1} - q_n p_{n-1} = (f_n p_{n-1} + g_{n-1} p_{n-2}) q_{n-1} - (f_n q_{n-1} + g_{n-1} q_{n-2}) p_{n-1}$$
$$= -g_{n-1}(p_{n-1} q_{n-2} - q_{n-1} p_{n-2}).$$

It follows that

$$p_n q_{n-1} - q_n p_{n-1} = (-1)^n g_{n-1} g_{n-2} \cdots g_0 (p_0 q_{-1} - q_0 p_{-1})$$
$$= (-1)^{n+1} g_{n-1} g_{n-2} \cdots g_0$$

so

$$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = (-1)^{n+1} \frac{g_{n-1} g_{n-2} \cdots g_0}{q_{n-1} q_n}, \text{ for all } n \geqslant 1.$$

Taking the sum then we have (5). Using (5), we have a simple necessary condition for a *positive* continued fraction to converge.

*Theorem* 2: Let $\varepsilon$ be a positive number. If $f_n$, $g_n$ are positive numbers and

$$\frac{f_{n+1} f_n}{g_n} > \varepsilon$$

then the infinite continued fraction $[f_0, (g_0, f_1), (g_1, f_2), \dots]$ converges.

*Proof*: Since $f_n$, $g_n$ are positive, $q_n$ is also positive. Writing (5) as

$$\frac{p_n}{q_n} = f_0 + \sum_{k=1}^{n} (-1)^{k+1} a_k$$

where

$$a_k = \frac{\prod_{j=0}^{k-1} g_j}{q_{k-1} q_k},$$

we have

$$\frac{a_k}{a_{k+1}} = \frac{q_{k+1}}{q_{k-1} g_k} = \frac{f_{k+1} q_k + g_k q_{k-1}}{q_{k-1} g_k} = 1 + \frac{f_{k+1} q_k}{q_{k-1} g_k}$$

$$= 1 + \frac{f_{k+1} (f_k q_{k-1} + g_{k-1} q_{k-2})}{q_{k-1} g_k} > 1 + \frac{f_{k+1} f_k}{g_k} > 1 + \varepsilon.$$

Thus, by Leibniz's alternating series test, the series

$$[f_0, (g_0, f_1), (g_1, f_2), \dots ] = f_0 + \sum_{k=1}^{\infty} (-1)^{k+1} a_k$$

converges.

*Corollary* 1: Let $\varepsilon$ be a positive number. If for large enough $n, f_n, g_n > 0$ and $\frac{f_{n+1} f_n}{g_n} > \varepsilon$ then the infinite continued fraction $[f_0, (g_0, f_1), (g_1, f_2), \dots ]$ converges.

By Theorem 2, we know that the infinite continued fraction

$$\left[M, \left(\frac{1}{4}, 3M\right), \left(\frac{1}{4}, 5M\right), \dots \right]$$

converges, but does Lemma 2 say that this continued fraction converges to

$$\frac{1}{e^{1/M} - 1} + \frac{1}{2}?$$

Here is an example. Since $x_0 = 1 - \sqrt{2}$ is a root of the quadratic equation $x^2 - 2x - 1 = 0$, we have $x_0 = 2 + \frac{1}{x_0}$. This gives us the following continued fraction of arbitrary length

$$1 - \sqrt{2} = 2 + \cfrac{1}{2 + \cfrac{1}{\dots 2 + \cfrac{1}{2 + (-1 - \sqrt{2})}}}.$$

Does this mean the infinite continued fraction $[2, (1, 2), (1, 2), \dots ]$ converges to $1 - \sqrt{2}$? No, in fact, this continued fraction converges to $1 + \sqrt{2}$.

The difference between Lemma 2 and the above $1 - \sqrt{2}$ example is that in Lemma 2, $\frac{S_n}{2S_{n+1}}$ is positive, whereas in the $1 - \sqrt{2}$ example, $-1 - \sqrt{2}$ is negative.

The following theorem is known as Markov's test for *positive* continued fractions.

*Theorem* 3: Assume that $f_n$, $g_n$ are positive numbers and the following infinite continued fraction converges:

$$[f_0, (g_0, f_1), (g_1, f_2), \dots] = \ell.$$

Construct a sequence $\{z_n\}$ as follows:

$$z_0 = f_0 + \cfrac{g_0}{f_1 + \cfrac{g_1}{\dots f_{n-1} + \cfrac{g_{n-1}}{f_n + z_n}}}.$$

If the terms $z_n$ are positive then $z_0 = \ell$.

*Proof*: [8]

Let $\tau_0(x) = f_0 + x, \tau_k(x) = g_{k-1}/(f_k + x)$ then

$$z_0 = T_n(z_n) = \tau_0 \circ \tau_1 \circ \dots \circ \tau_n(z_n).$$

Every function $\tau_k$ is continuous and monotonic on $[0, +\infty)$. Hence the same is true for their composition $T_n$. Picking two limit values $x = 0$ and $x = +\infty$, we find that $z_0$ must be in the interval with the end-points at

$$T_n(0) = \frac{p_n}{q_n}, \qquad T_n(+\infty) = \frac{p_{n-1}}{q_{n-1}}.$$

Since the continued fraction is assumed to converge to $\ell$, so $z_0 = \ell$.

So now, by Markov's test, it follows from Lemma 2 that $[M, (\frac{1}{4}, 3M), (\frac{1}{4}, 5M), \dots]$ converges to $\frac{1}{e^{1/M} - 1} + \frac{1}{2}$ and thus we obtain (3):

$$e^{1/M} = 1 + \cfrac{1}{M - \frac{1}{2} + \cfrac{\frac{1}{4}}{3M + \cfrac{\frac{1}{4}}{5M + \cfrac{\frac{1}{4}}{7M + \cfrac{\frac{1}{4}}{\dots}}}}}.$$

*The transformation*

We will use the following algebraic identity to transfrom (3) into (1)

$$(2k+1)M - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{x}}} = (2k+1)M - \frac{1}{2} + \frac{\frac{1}{4}}{\frac{1}{2} + x}. \quad (6)$$

*Theorem* 4: For any positive real number $M$,

$$e^{1/M} = [1, M - 1, 1, 1, 3M - 1, 1, 1, 5M - 1, 1, 1, \dots].$$

*Proof*: The coefficients of the continued fraction

$$[1, M - 1, 1, 1, 3M - 1, 1, 1, 5M - 1, 1, 1, \dots]$$

are eventually positive, so by Corollary 1, it converges.

We apply the identity (6) repeatedly as follows:

$$1 + \cfrac{1}{M - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3M - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5M - 1 + \cfrac{1}{\dots}}}}}}}}$$

$$= 1 + \cfrac{1}{M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\frac{1}{2} + 3M - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5M - 1 + \cfrac{1}{\dots}}}}}}$$

$$= 1 + \cfrac{1}{M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\frac{1}{2} + 3M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\frac{1}{2} + 5M - 1 + \cfrac{1}{\dots}}}}$$

$$= \dots$$

$$= 1 + \cfrac{1}{M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\frac{1}{2} + 3M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\frac{1}{2} + 5M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\frac{1}{2} + 7M - \frac{1}{2} + \cfrac{\frac{1}{4}}{\dots}}}}}$$

$$= 1 + \cfrac{1}{M - \frac{1}{2} + \cfrac{\frac{1}{4}}{3M + \cfrac{\frac{1}{4}}{5M + \cfrac{\frac{1}{4}}{7M + \cfrac{\frac{1}{4}}{\cdots}}}}} = e^{1/M}.$$

We have finally proved the continued fraction expansion formula for $e^{1/M}$. Our proof is self-contained and does not employ any integration.

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JOSEPH TONIEN

*School of Computing and Information Technology,*
*University of Wollongong, Australia*

e-mail: *joseph_tonien@uow.edu.au*