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Modeling of viscoelastic turbulent flow in channel and pipe

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This paper investigates turbulent flows with or without polymer additives in open channels and pipes. Equations of mean velocity, root mean square of velocity fluctuations, and energy spectrum are derived, in which the shear stress deficit model is used and the non-Newtonian properties are represented by the viscoelasticity α_* . The obtained results show that, with α_* increment, (1) the streamwise velocity fluctuations is increased, (2) the wall-normal velocity fluctuation is attenuated, (3) the Reynolds stress is reduced, and (4) there is a redistribution of energy from high frequencies to the low frequencies for the streamwise component, but dimensionless distribution over all frequencies almost remains the same as that in Newtonian fluid flows. Good agreement between the derived equations and experimental data in small drag-reduction regime is achieved, which indicates that the present model is workable for Newtonian/non-Newtonian fluid turbulent flows. © 2008 American Institute of Physics. [DOI: 10.1063/1.2920275]

I. INTRODUCTION

It is well known that the addition of a minute amount of polymer to a turbulent Newtonian fluid would result in the variation of turbulent structures, which has been known for long time. The underlying mechanism that causes this drag reduction (DR) has been intensively studied over the last 50 years, and the original assumptions made by Lumley¹ and Metzner² have been verified by the numerical models, especially the direct numerical simulation (DNS).^{3–10}

Beginning with the study of Wells and Spangler,¹¹ it has been found that the polymer must be effective in the near-wall region for DR to occur. Petrie *et al.*¹² confirmed Wells and Spangler's conclusion by carrying out experiments in a flat-plate boundary layer. However, researchers wonder how the polymers alter the turbulent structures and cause drag to be reduced.

With advances in instrumentation and visualization techniques, the coherent turbulent structures in drag-reducing flows have been measured and reported; the observations of turbulent structures in DR flows could be very useful for researchers to understand the mechanism of DR. Without intrusive probe, the turbulent structures could be measured by laser-Doppler velocimeter (LDV), particle-image velocimetry (PIV), and visualization techniques.

Early LDV measurement in polymer drag-reducing flows was reported by Rudd,¹³ then followed by Reischman and Tiederman,¹⁴ Berner and Scrivener,¹⁵ Berman,¹⁶ etc. The velocity fluctuations and energy spectrum in streamwise direction were observed and reported by them; one of the important discoveries was that the root mean square of streamwise velocity in drag-reducing flows is slightly higher than that in Newtonian fluid flows, and the energy is redistributed from high frequencies to low frequencies. Two-component

LDV measurements were conducted by many researchers.^{17–21} All these experiments confirmed that the root mean square of the velocity fluctuations in the streamwise direction increases while the rms of the fluctuations in the wall-normal direction decreases with DR, and the Reynolds shear stress decreases in the drag-reducing flows. The sum of Reynolds shear stress and viscous shear stresses is lower than the total shear stress without the presence of polymer agents. Wei and Willmarth²¹ found that the energy of the normal velocity component is dramatically suppressed over all frequency, while there is a redistribution of energy from high frequencies to low frequencies.

Flow visualization techniques have been used to study the influence of polymers on the streak spacing, bursting frequency, and Reynolds shear stress. The changes in the burst events in drag-reducing flows are particularly interesting because the variations of velocity field in the near-wall region during a burst could reveal the mechanism of DR and could also exhibit the basic relationship of turbulent structures. Donohue *et al.*²² examined the effects of polymers on turbulent structures by using the visualization technique; they reported that the streaking spacing and bursting rates in drag-reducing flow are different from that in the Newtonian flow, i.e., the average nondimensional spacing between streaks linearly increases with increasing DR and the viscous sublayer was more stable when polymer solutions were present. A suppression of the burst process and an increment of streak spacing were also reported by Berman¹⁶ and Tiederman *et al.*²³ However, Luchik and Tiederman¹⁹ found that the method for deducing the time between bursts was not accurate in these experiments because they did not mark and counted all of these events. Thus, the flow visualization revealed some important phenomena, but a lack of measure of the corresponding turbulent velocity field limits the interpretation.

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However, the PIV system would overcome this shortcoming and provide a quantitative measure of the effect of polymer additives on the near-wall turbulent structures; one is able to determine what effects of the polymer turbulent interactions are important. Warholic *et al.*²⁴ applied PIV to a channel flow with polymer additives, and they found that the role of the wall in creating turbulence greatly diminishes at large drag reductions (LDRs). White *et al.*²⁵ used PIV in a drag-reducing flat-plate boundary layer flow and found that a significant modification of the near-wall structure of turbulence with a coarsening of the low-speed velocity streaks and a reduction in the number and strength of near-wall vertical structures.

Therefore, it can be seen that since Toms²⁶ reported turbulent DR by polymer additives, there have been many studies on this phenomenon. The experimental studies provide valuable insights into polymer DR, and the numerical models reveal the principal underlying mechanism of DR. The phenomenon of DR has been used in technological applications from fire engines to oil pipes. However, semitheoretical equations expressing the turbulence characteristics are unavailable in the literature. Ordinary engineers generally prefer simple formulas, instead of numerical models, such that they are able to estimate the pipeline/channel capacity; thus, it is worthwhile to develop some simplified models.

For Newtonian turbulent flows, Dou²⁷ has made some theoretical analyses of turbulent flows by using the stochastic theory.^{28,29} The equations of mean velocity distribution has been applied to the viscoelastic flows; it is found that the equations derived are in good agreement with experimental data measured by Virk,^{30–32} Seyer and Metzner,³³ James and Acosta,³⁴ Reischman and Tiederman,¹⁴ Rudd,¹³ etc.

However, the equations of fluctuating velocity intensity and the energy spectrum in viscoelastic fluid flow have not been developed. Thus, the primary objective of this study is to determine the rms of velocity fluctuations and the energy spectrum in drag-reducing flows, i.e., the emphasis will be placed on structures of turbulence; a primary scope of this paper is confined to the turbulent intensity, Reynolds shear stress, and energy spectrum affected by polymers.

II. REYNOLDS SHEAR STRESS, TURBULENT VELOCITY INTENSITY, AND RHEOLOGICAL PROPERTY OF POLYMER ADDITIVES

For a steady, uniform, and two-dimensional (2D) flow, the velocity fluctuations can be deduced from the stochastic theory of turbulence as follows [see Eqs. (A12)–(A14) in the Appendix]:

$$u' = u'' - \left(l'_{(1)2} + \frac{1}{2} r'_2 \right) \frac{d\bar{u}}{dy}, \quad (2.1)$$

$$v' = v'' - \frac{1}{2} r'_1 \frac{d\bar{u}}{dy}, \quad (2.2)$$

where u and v are instantaneous velocities in the streamwise and wall-normal directions, respectively; u' and v' are velocity fluctuations; $\bar{u}$ is the mean velocity; u'' and v'' are the basic fluctuating velocities that are independent of each

other; $l'_{(1)2}$ is a length similar to Prandtl's mixing length; r'_1 and r'_2 are the lengths from a discussed point to center of an eddy in horizontal and vertical directions, respectively; subscripts "1" and "2" denote the streamwise and wall-normal directions, respectively.

The Reynolds stress can be obtained from Eqs. (2.1) and (2.2) as follows:

$$\begin{aligned} \overline{u'v'} = \overline{u''v''} - \left(\overline{l'_{(1)2}v''} + \frac{1}{2} \overline{r'_2v''} \right) \frac{d\bar{u}}{dy} - \frac{\overline{u''r'_1}}{2} \frac{d\bar{u}}{dy} \\ + \left(\frac{\overline{l'_{(1)2}r'_1}}{2} + \frac{\overline{r'_1r'_2}}{4} \right) \left(\frac{d\bar{u}}{dy} \right)^2. \end{aligned} \quad (2.3)$$

According to the definition, the basic velocity fluctuations u'' and v'' are two independent random variables, i.e., $\overline{u''v''} = 0$, and u'' and v'' are also independent of r' ; thus, $\overline{r'_2v''} = 0$, $\overline{u''r'_1} = 0$. From Eqs. (A20a)–(A20c) in the Appendix, one can determine these terms, $\overline{l'_{(1)2}v''} = U^2T/2$, $\overline{l'_{(1)2}r'_1} = mL^2/2$, and $\overline{r'_1r'_2} = -m^2L^2/2$, in which U , T , and L represent the eddy's characteristic velocity, time, and length, respectively; m is introduced to consider the damping effect of wall on the eddy size. Therefore, the Reynolds shear stress can be expressed as

$$\overline{u'v'} = -\frac{U^2T}{2} \frac{d\bar{u}}{dy} + \left(\frac{mL^2}{4} - \frac{m^2L^2}{8} \right) \left(\frac{d\bar{u}}{dy} \right)^2. \quad (2.4)$$

Similarly, the turbulent velocity intensities can be obtained in the following forms:

$$\begin{aligned} \overline{u'^2} = \overline{u''^2} - 2 \left(\overline{u''l'_{(1)2}} + \frac{\overline{u''r'_2}}{2} \right) \frac{d\bar{u}}{dy} \\ + \left(\overline{l'_{(1)2}^2} + \overline{r'_2l'_{(1)2}} + \frac{\overline{r'_2^2}}{4} \right) \left(\frac{d\bar{u}}{dy} \right)^2, \end{aligned} \quad (2.5)$$

$$\overline{v'^2} = \overline{v''^2} - \overline{v''r'_1} \frac{d\bar{u}}{dy} + \frac{\overline{r'_1^2}}{4} \left(\frac{d\bar{u}}{dy} \right)^2. \quad (2.6)$$

As mentioned $\overline{u''r'_2} = \overline{v''r'_1} = 0$; other unknowns in Eqs. (2.5) and (2.6) can be determined from Eqs. (A20a)–(A20c) in the Appendix, i.e., $\overline{u''^2} = U^2/2$, $\overline{v''^2} = U^2/2$, $\overline{u''l'_{(1)2}} = 0$, $\overline{l'_{(1)2}^2} = 2m^2L^2$, $\overline{r'_2l'_{(1)2}} = 0$, and $\overline{r'_2^2} = \overline{r'_1^2} = m^2L^2$. Therefore, the mean square of velocity fluctuations in streamwise and wall-normal directions can be expressed as

$$\overline{u'^2} = \frac{U^2}{2} + \frac{9}{4} m^2 L^2 \left(\frac{d\bar{u}}{dy} \right)^2, \quad (2.7)$$

$$\overline{v'^2} = \frac{U^2}{2} + \frac{1}{4} m^2 L^2 \left(\frac{d\bar{u}}{dy} \right)^2. \quad (2.8)$$

For two-dimensional turbulent shear flow, the eddy's characteristics are determined by its location, i.e., vertical distance y , thickness of viscous sublayer δ , local shear stress τ , and boundary shear stress τ_* ; the following criteria of similarity were introduced by Dou:²⁸

$$m = \frac{y}{\delta}, \quad U = u_* \sqrt{\frac{y}{\delta + y}}, \quad L = \kappa \delta \sqrt{1 - \frac{y}{h}}, \quad (2.9)$$

$$T = \kappa \left(1 - \frac{y}{h}\right) \frac{\delta + y}{u_*},$$

in which κ is the Karman constant=0.4, u_* is the shear velocity ($\tau_* = \rho u_*^2$), and h is the channel depth or pipe radius.

Besides, the shear stress τ in viscoelastic fluid flows has to be determined. As mentioned, of particular interest of polymer drag-reducing flow is the existence of “stress deficit,” i.e., the total shear stress in DR flow is greater than the sum of viscous shear stress ($=\nu d\bar{u}/dy$) and the measured Reynolds shear stress ($=-\overline{u'v'}$). Gyr and Tsinober³⁵ defined the deficit in the following form:

$$G(y) = \frac{\tau}{\rho} - \left[\nu \frac{d\bar{u}}{dy} + (-\overline{u'v'}) \right], \quad (2.10)$$

where $G(y)$ is the stress deficit, ν is the kinematic viscosity of the fluid, ρ is the density of the fluid, and τ is the total shear stress $=\rho u_*^2(1-y/h)$.

For turbulent flows of a Newtonian fluid, $G(y)$ must be equal to zero. However, for flows with polymer additives, experimental researchers found that $G(y)$ is essentially non-negligible and is mostly positive due to the elastic effect of the polymer. Gyr and Tsinober³⁵ expressed it as follows:

$$G(y) = \nu_{\text{eff}} \frac{d\bar{u}}{dy}, \quad (2.11)$$

where ν_{eff} is the effective viscosity. If the polymer concentration $C=0$, then $G(y)=0$ means that the flow is Newtonian fluid flows and no DR occurs.

The shear deficit indicates that the viscoelasticity ν_{eff} should be the most important property of a dilute polymer solution. Yang and Dou³⁶ suggested that the effective viscosity could be related to a characteristic velocity ($=u_*$) and a characteristic length ($=h$), i.e.,

$$\nu_{\text{eff}} = \alpha_* u_* h, \quad (2.12)$$

where α_* is the apparent viscoelasticity depending on the type of polymer and its concentration.

The existence of ν_{eff} can be found from various models, such as the Maxwell model, the Oldroyd-B model, and the FENE-P (finitely extensible nonlinear elastic model proposed by Peterlin) model. For example, Benzi *et al.*³⁷ derived the following momentum equation for viscoelastic flows from the FENE-P model:

$$\frac{\tau}{\rho} = -\overline{u'v'} + \nu \frac{d\bar{u}}{dy} + \nu_p \langle R_{yy} \rangle \frac{d\bar{u}}{dy} = u_*^2 \left(1 - \frac{y}{h}\right), \quad (2.13)$$

where $\langle R_{yy} \rangle$ is the ensemble average dimensionless extension tensor of the dyadic product of the end-to-end vector distance of the polymer chains in the y direction; ν_p is a viscosity parameter which is related to concentration of polymer. Then, by comparing Eq. (2.13) to Eq. (2.10), one has

$$\nu_{\text{eff}} = \nu_p \langle R_{yy} \rangle. \quad (2.14)$$

Noticing that in viscoelastic flows $\langle R_{yy} \rangle$ and ν_p are always positive, thus, ν_{eff} should be greater than zero; this is probably why Gyr and Tsinober³⁵ observed that ν_{eff} is always positive.

By substituting Eqs. (2.11) and (2.12) into Eq. (2.10), one obtains

$$\frac{\tau}{\rho} = (\nu + \nu_{\text{eff}}) \frac{d\bar{u}}{dy} - \overline{u'v'} = \nu D_* \frac{d\bar{u}}{dy} - \overline{u'v'} = u_*^2 \left(1 - \frac{y}{h}\right), \quad (2.15)$$

where D_* is the DR parameter and

$$D_* = 1 + \alpha_* \frac{u_* h}{\nu}. \quad (2.16)$$

Obviously, for Newtonian fluid flows, Eq. (2.16) gives $D_*=1$ due to $\alpha_*=0$; however, for viscoelastic fluid flows, one has $D_* > 1$ for $\alpha_* > 0$.

Equation (2.15) states that the total shear stress in viscoelastic fluid flows remains unchanged, i.e., $\tau = \rho u_*^2(1-y/h)$ or the presence of polymer does not change the distribution of total shear stress but only modifies the distribution of Reynolds shear stress and viscous shear due to the additional shear stress caused by the polymer that is expressed by $\nu_{\text{eff}} d\bar{u}/dy$. This conformity can be extended to any complicated flows, i.e., its relative distribution of total shear stress in a simple flow (e.g., laminar flow) must remain unchanged no matter how flow properties vary. For example, the relative distribution of shear stress in laminar boundary layer flow that can be theoretically determined must be the same as that in turbulent flow. Instead of directly solving Eq. (2.15), Benzi *et al.*³⁷ introduced the energy balance equation and other assumptions including $y \ll h$ in Eq. (2.15); then, they claimed that their obtained velocity equation is valid for the whole flow region. Different from the approach of Benzi *et al.*, this study makes an attempt on the direct solution of Eq. (2.15) as Yang and Dou³⁶ did.

By substituting Eq. (2.9) into Eq. (2.4), one obtains the dimensionless Reynolds shear stress,

$$-\frac{\overline{u'v'}}{u_*^2} = \frac{\kappa}{2} \left(1 - \frac{y}{h}\right) \left(y^+ \frac{du^+}{dy^+}\right) \left[1 - \frac{\kappa}{4} (2\delta^+ - y^+) \left(\frac{du^+}{dy^+}\right)\right], \quad (2.17)$$

in which $u^+ = \bar{u}/u_*$, $y^+ = u_* y/\nu$, and $\delta^+ = u_* \delta/\nu$. It is well known that for Newtonian fluid flow, $u_* \delta/\nu = 11.6$.

For turbulent flows with polymer additives, Lumley¹ found that the only difference between Newtonian flows and drag-reducing flows is that polymer molecules are expanded in the flow outside the viscous sublayer due to possible stretching of the polymer molecule (if the strain rate in the turbulent flow is large); this causes an increase in the effective viscosity, which in turn damps dissipative eddies. This effectively leads to a thickening of the viscous sublayer leading to a decrease in the velocity gradient at the wall. Consequently, the Reynolds shear stress at the wall decreases, thus leading to a reduction in drag.³⁸ Warholic *et al.*³⁹ also reported that based on their experimental data, “the log-layer is

displaced upward and thickness of viscous wall region increases with increasing drag-reduction." By examining the velocity in the near-wall region, Yang and Dou³⁶ found that the thickening of the viscous sublayer, i.e., δ^+ in Eq. (2.17), can be expressed as follows:

$$\delta^+ = \frac{u_* \delta}{\nu} = 11.6 D_*^3. \quad (2.18)$$

Therefore, the root mean square of velocity fluctuations in Eqs. (2.7) and (2.8) can be rewritten in the following dimensionless forms:

$$\frac{\overline{u'^2}}{u_*^2} = \frac{1}{2} \frac{y^+}{\delta^+ + y^+} + \frac{9}{4} \kappa^2 \left(1 - \frac{y}{h}\right) \left(y^+ \frac{du^+}{dy^+}\right)^2, \quad (2.19)$$

$$\frac{\overline{v'^2}}{u_*^2} = \frac{1}{2} \frac{y^+}{\delta^+ + y^+} + \frac{1}{4} \kappa^2 \left(1 - \frac{y}{h}\right) \left(y^+ \frac{du^+}{dy^+}\right)^2. \quad (2.20)$$

By inserting Eq. (2.17) into Eq. (2.15), one obtains

$$\begin{aligned} \left(1 - \frac{y}{h}\right) &= D_* \frac{du^+}{dy^+} + \frac{\kappa}{2} \left(1 - \frac{y}{h}\right) \left(y^+ \frac{du^+}{dy^+}\right) \\ &\times \left[1 - \frac{\kappa}{4} (2\delta^+ - y^+) \left(\frac{du^+}{dy^+}\right)\right]. \end{aligned} \quad (2.21)$$

Detailed calculation shows that $(1 - y/h)$ on both sides of Eq. (2.21) can be dropped for simplification and the result remains almost unchanged;³⁶ thus, the velocity gradient shown in Eqs. (2.17), (2.19), and (2.20) can be determined by solving Eq. (2.21) as follows:

$$y^+ \frac{du^+}{dy^+} = \frac{y^+}{D_* + 0.2\kappa y^+} + \frac{\kappa^2}{8} \frac{2\delta^+ y^{+2} - y^{+3}}{(D_* + 0.2\kappa y^+)^3}. \quad (2.22)$$

Integration of Eq. (2.22) yields

$$\begin{aligned} u^+ &= \frac{1}{\kappa} \ln \left(1 + \frac{\kappa y^+}{2D_*}\right) + \frac{1}{2} \left(\frac{\delta^+}{D_*} + \frac{1}{\kappa}\right) \left(\frac{\kappa y^+}{2D_* + \kappa y^+}\right)^2 \\ &+ \frac{1}{\kappa} \frac{\kappa y^+}{2D_* + \kappa y^+}. \end{aligned} \quad (2.23)$$

Hence, the turbulent intensities can be expressed as

$$\begin{aligned} \frac{\sqrt{\overline{u'^2}}}{u_*} &= \left\{ \frac{1}{2} \frac{y^+}{11.6 D_*^3 + y^+} + 0.36 \left(1 - \frac{y}{h}\right)^{m_0} \right. \\ &\times \left. \left[\frac{y^+}{D_* + 0.2y^+} + \frac{0.464 D_*^3 y^{+2} - 0.02 y^{+3}}{D_*^{n_0} (D_* + 0.2y^+)^3} \right]^2 \right\}^{0.5}, \end{aligned} \quad (2.24)$$

$$\begin{aligned} \frac{\sqrt{\overline{v'^2}}}{u_*} &= \left\{ \frac{1}{2} \frac{y^+}{11.6 D_*^3 + y^+} + 0.04 \left(1 - \frac{y}{h}\right) \right. \\ &\times \left. \left[\frac{y^+}{D_* + 0.2y^+} + \frac{0.464 D_*^3 y^{+2} - 0.02 y^{+3}}{(D_* + 0.2y^+)^3} \right]^2 \right\}^{0.5}. \end{aligned} \quad (2.25)$$

Equations (2.23)–(2.25) are valid for both Newtonian fluid flows ($D_* = 1$) and viscoelastic fluid flows ($D_* > 1$). The direct derivation gives $m_0 = 1$ and $n_0 = 0$. The theoretical results show that the velocity u^+ and $\sqrt{\overline{u'^2}}/u_*$ increases with the increment of D_* , but $\sqrt{\overline{v'^2}}/u_*$ will decrease with the increment of D_* . This conclusion is consistent with the experimental observations in the small drag-reduction (SDR) regime.^{20,21} However, experimental results by Warholic *et al.*³⁹ and Ptasinski *et al.*⁴⁰ indicate that the turbulent velocity decreases with the increment of D_* in the LDR regime. To express the velocity fluctuations in LDR, the empirical values of m_0 and n_0 are introduced in Eq. (2.24), and it is found that $m_0 = n_0 = 2$ could approximate the turbulent velocity in LDR.

It is well known that Prandtl's mixing-length theorem gives unreasonably zero turbulent velocity fluctuations at $y = h$ because he assumed that the velocity fluctuations were proportional to the mean velocity gradient. Equation (2.25) avoids this and, at $y = h$, it reduces to

$$\frac{\sqrt{\overline{v'^2}}}{u_*} = \left(\frac{1}{2} \frac{h^+}{11.6 D_*^3 + h^+} \right)^{0.5}. \quad (2.26)$$

It is interesting to note that from Eq. (2.26), the rms of turbulent velocity decreases with the DR parameter D_* ; this means that the turbulent structures far away from a wall are also affected by the near-wall drag-reducing phenomenon. This decrease was observed by Harder and Tiederman,²⁰ Wei and Willmarth,²¹ etc.

In the region near a solid wall where $\kappa y^+ \ll 2D_*$, one has

$$\begin{aligned} \ln \left(1 + \frac{\kappa y^+}{2D_*}\right) &\approx \frac{\kappa y^+}{2D_*}, \quad \frac{\kappa y^+}{2D_* + \kappa y^+} \approx \frac{\kappa y^+}{2D_*}, \\ \left(\frac{\kappa y^+}{2D_* + \kappa y^+}\right)^2 &\approx 0. \end{aligned}$$

Then, Eq. (2.23) becomes

$$\frac{\bar{u}}{u_*} = \frac{u_* y}{\nu D_*}. \quad (2.27)$$

Equation (2.27) states that the velocity distribution in the region very near a wall is linear, whereas in the turbulent core where $\kappa y^+/(2D_*) \gg 1$, Eq. (2.23) has the following relation:

$$\frac{1}{\kappa} \ln \left(1 + \frac{\kappa y^+}{2D_*}\right) \approx 2.5 \ln y^+ - 2.5 \ln(5D_*),$$

$$\frac{1}{2} \left(\frac{\delta^+}{D_*} + \frac{1}{\kappa} \right) \left(\frac{\kappa y^+}{2D_* + \kappa y^+} \right)^2 \approx 5.8D_*^2 + 1.25,$$

$$\frac{1}{\kappa 2D_* + \kappa y^+} \approx 2.5, \quad 5.8 + 1.25 + 2.5 - 2.5 \ln(5) = 5.52.$$

Thus, in the turbulent core, Eq. (2.23) can be rewritten as

$$\frac{\bar{u}}{u_*} = 2.5 \ln \left(\frac{u_* y}{\nu} \right) + 5.52 + \Delta B, \quad (2.28a)$$

$$\Delta B = 5.8(D_*^2 - 1) - 2.5 \ln D_*. \quad (2.28b)$$

Equation (2.28a) states that far away from the wall, the velocity is proportional to the logarithmic distance y^+ ; its slope remains the same as that in Newtonian fluid flow, but its intercept depends on the DR parameter D_* . Obviously, if $D_*=1$, Eq. (2.28b) shows $\Delta B=0$ and Eq. (2.28a) becomes the classical log law. Thus, Eqs. (2.28a) and (2.28b) are consistent with Virk's observation.³⁰ He first discovered that in turbulent core of viscoelastic fluid flow, the log law is shifted by an amount ΔB with no change of slope. The constant 5.52 in Eq. (2.28a) was first observed by Nikuradse in his experiment.

To verify the above equations, one must determine the unknown parameter D_* by comparing the measured velocity profiles with Eq. (2.23). In this study, the experimental data collected by Warholic *et al.*,^{24,39} Harder and Tiederman,²⁰ and Ptasiński *et al.*^{40,41} will be used for the verification shown in Fig. 1.

Warholic *et al.*,³⁹ using a LDV system, comprehensively measured the velocity profiles, the root mean square of the streamwise velocity fluctuation, and spectral density function in a rectangular channel that has a cross section of $5.08 \times 61.0 \text{ cm}^2$. Percol 727 was injected into the flowing water. The measured mean velocity profiles are replotted in Fig. 1(a) (polymer concentration C from 0 to 50 ppm), in which %DR was defined as the ratio of the wall shear stress for a polymer solution to that obtained for water flowing at the same volumetric flow rate. The estimated DR parameter D_* is also included in Fig. 1(a).

Warholic *et al.*²⁴ used a PIV system to experimentally investigate the effect of drag-reducing polymers on the structure of turbulence in a channel flow, under conditions of 43% and 55% DR. A solution of a copolymer of polyacrylamide and sodium acrylate (Percol 727) was injected through wall slots at the entrance to the channel, the polymer concentrations were 1.24 and 50 ppm, and the estimated D_* were 1.7 and 1.95, respectively.

Harder and Tiederman²⁰ measured the turbulent structure of turbulence by using LDV in a smooth channel; polyacrylamide Separan AP273 was used as a polymer additive and the polymer concentrations varied from 3 to 5 ppm. The estimated D_* is 1.8. It can be seen from Fig. 1(a) that if $D_*=1$, Eq. (2.23) agrees well with the measured velocity in Newtonian flow for $u_* y / \nu \geq 80$, but for the buffer region, the deviation of measured velocities from the prediction is no-

FIG. 1. Dimensionless streamwise velocity u^+ vs the dimensionless distance y^+ for DR parameter D_* with best fit. In Fig. 1(a), Virk's asymptote and L'vov's theory are included for comparison; in Fig. 1(b), DNS results are also included.

ticeable. For viscoelastic fluid flow, the inaccuracies in the buffer region become more obvious as viscoelasticity increases.

For Newtonian fluid flow, this discrepancy may be caused by the assumed constants in the theorem, i.e., $\kappa = 0.4$ and $\delta^+ = 11.6$, resulting in the slope of 2.5 and intercept of 5.52 in Eq. (2.28a). Experimenters often find that the classical log law does not fit their measured data very well; constants in Eq. (2.28a) have to be adjusted to fit their measured data. Different values have been suggested, for example, $\kappa = 0.436$, intercept = 6.13 was used by Zagarola and Smith,⁴² 0.41 and 6.0 by Luchik and Tiederman,¹⁹ 0.415 and 5.5 by Warholic *et al.*,³⁹ 0.40 and 5.0 by Ptasiński *et al.*,⁴¹ 0.408 and 4.69 by Reischman and Tiederman,¹⁴ 0.41 and 5.0 by Fontaine *et al.*,⁴³ etc. Ideally, the values of κ and δ^+ in this model should be adjusted to yield the best agreement with experimental data as the experimenters did, but to maintain its consistency and simplicity, this study only uses the constants of $\kappa = 0.4$ and $\delta^+ = 11.6$. Thus, it is understandable that the systematic discrepancies exist, as shown in Fig. 1(a).

To demonstrate that the discrepancy does not always increase with DR, the data of Ptasiński *et al.*⁴⁰ are plotted in

Fig. 1(b); it can be seen that the agreement between the predicted and measured velocities is acceptable even up to %DR=70.

In Fig. 1, Virk's ultimate asymptote,³⁰ the theory of L'vov *et al.*,⁴⁴ and DNS's prediction by Ptasiniski *et al.*⁴¹ are included for comparison. Virk believed that the asymptote is the feature of turbulence and should be independent of polymer characteristics. Ptasiniski *et al.*⁴¹ developed the DNS model to simulate the viscoelastic fluid flow near the maximum drag reduction. By comparing Eq. (2.23) with these results, one may conclude that the agreement between the present model and experimental data is acceptable.

It is interesting to note that the velocity considerably higher than Virk's asymptote could be achieved if D_* is large enough; in fact, both Eq. (2.23) ($D_*=2.7$) and the experimental data of Ptasiniski *et al.*⁴⁰ [see Fig. 1(b)] significantly excess his asymptote. Figure 1 shows that there are two regimes in polymer DR: One is the SDR regime where $1 < D_* \leq 2.3$, and the other is the LDR regime where $D_* > 2.3$. From Fig. 1, one may find that Eq. (2.23) can be used to estimate the velocity profiles in both regimes.

The measurements of the root mean square of streamwise velocity fluctuations are presented in Fig. 2, in which the values of D_* were determined from Fig. 1 based on the measured mean velocity profiles. In Fig. 2(a), the agreement between Eq. (2.24) and the measured fluctuating velocity in Newtonian fluid flow is investigated, and it includes the data from Warholic *et al.*,^{24,39} Harder and Tiederman,²⁰ and Hishida *et al.*⁴⁵ It can be seen that like the mean velocity distributions discussed above, different experimenters obtained different profiles of fluctuating velocity, but all measured profiles can be roughly represented by Eq. (2.24), suggesting that Eq. (2.24) and values of $\kappa=0.4$ and $\delta^+=11.6$ could be used in the assessment of fluctuating velocity. Thereafter, the measured profiles of fluctuating velocity in the SDR regime of viscoelastic fluid flow by Warholic *et al.*³⁹ are plotted in Fig. 2(b), where Eq. (2.24) is also included for comparison. It can be seen that both the experimental data and Eq. (2.24) give that the peaks in $\sqrt{u'^2}/u_*$ increase and are displaced rightward with increasing D_* . Equation (2.24) shows that the peaks occur at $y^+=15$; this is consistent with Harder and Tiederman's observations.²⁰ Figure 2(b) shows that Eq. (2.24) can be used to estimate the streamwise velocity fluctuations in SDR.

However, it should be stressed that $m_0=1$ and $n_0=0$ used in Figs. 2(a) and 2(b) cannot express the fluctuating velocity in the LDR regime because the measurements show that $\sqrt{u'^2}/u_*$ decreases as D_* increases. For %DR=69 in the experiment of Warholic *et al.*,³⁹ the magnitudes of $\sqrt{u'^2}/u_*$ are smaller than the values of the Newtonian flow over the whole cross section, but totally different conclusion was obtained by Ptasiniski *et al.*⁴⁰ from their experiment in which %DR=70 is very similar to that used by Warholic *et al.* (%DR=69). The data of Ptasiniski *et al.*⁴⁰ [see Fig. 2(c)] show that in the LDR regime, the measured $\sqrt{u'^2}/u_*$ is higher than the values in Newtonian flows. Therefore, experimenters have not reached a consensus for the turbulent velocity in the LDR regime, suggesting that more experi-

FIG. 2. Distribution of measured rms of streamwise velocity fluctuations: (a) In Newtonian fluid flow ($D_*=1$, $m_0=1$, $n_0=0$), (b) in the SDR regime ($m_0=1$, $n_0=0$), and (c) in the LDR regime ($m_0=2$, $n_0=2$).

mental and theoretical investigations should be conducted in the future. Figure 2(c) shows that $m_0=n_0=2$ can express the data points of Ptasiniski *et al.* well.

The measured Reynolds shear stress by Warholic *et al.*³⁹ is presented in Fig. 3; the data in the SDR regime are shown in Fig. 3(a), and the LDR regime in Fig. 3(b). It can be seen that both calculated and measured Reynolds shear stresses decrease with increasing the DR parameter D_* ; the Reynolds

FIG. 3. Comparison of Eq. (2.17) with measured Reynolds shear stress; (a) SDR regime and (b) LDR regime.

shear stress is about zero at the wall and sharply increases to its peaks at about $y^+ = 60-100$ and then gradually decreases to zero at $y/h = 1$. Again, the agreement in the SDR regime is acceptable, but large discrepancies exist in the LDR regime. According to the observation of Warholic *et al.*, the Reynolds shear stress in %DR=69 was found to be zero over most of the cross section, but the experiment of Ptasinski *et al.*⁴⁰ in %DR=70 did not support the assertion of Warholic *et al.* of zero Reynolds shear stress in the LDR regime.

III. ENERGY SPECTRUM

The velocity fluctuations in turbulence can be assumed to be represented by a Fourier integral, i.e.,

$$u'(t) = 2 \int_0^\infty [\alpha_1(\omega) \cos(\omega t) + \beta_1(\omega) \sin(\omega t)] d\omega, \quad (3.1)$$

where α_1 and β_1 are coefficients and ω is the angular frequency.

In quasisteady flows, there exists a constant average value of u'^2 , which can be represented by the energy between frequency ω and $\omega + d\omega$, i.e.,

$$u'^2 = \varphi(\omega) d\omega, \quad (3.2)$$

in which $\varphi(\omega)$ is the energy spectrum. Thus,

$$\overline{u'^2} = \int_0^\infty \varphi(\omega) d\omega, \quad (3.3)$$

When the Eulerian time correlation of the velocity fluctuation at a fixed point is discussed, it is reasonable to expect a direct relation between the energy spectrum and the autocorrelation function, that is,

$$R(\tau_1) = \overline{u'(t)u'(t - \tau_1)}, \quad (3.4)$$

where τ_1 is the time.

It is obvious that Eq. (3.4) gives $R(0) = \overline{u'^2}$; thus, there is a relation between $R(\tau_1)$ and the energy spectrum, that is, the largest eddies cause fluctuations with low frequencies, whereas the smallest eddies induce fluctuations with high frequencies. If the turbulence contains only large eddies, then the distribution of energy spectrum mainly exists in the region of lower frequencies; if there are only small eddies, it mainly occurs in the region of high frequency. The relation between the correlation function and the spectrum function was first put into mathematical form by Hinze⁴⁶ as follows:

$$R(\tau_1) = \int_0^\infty \varphi(\omega) \cos \omega \tau_1 d\omega, \quad (3.5a)$$

$$\varphi(\omega) = \frac{2}{\pi} \int_0^\infty R(\tau_1) \cos \omega \tau_1 d\tau_1. \quad (3.5b)$$

By inserting the $R(\tau_1)$ in the streamwise direction [see Eq. (A31) in the Appendix] and in the wall-normal direction [see Eq. (A32) in the Appendix] into Eq. (3.5b), one obtains

$$\varphi_{1(\omega)} = \sqrt{\frac{2}{3\pi\tilde{\omega}}} \left[B_1 - \frac{B_2}{3} \left(1 - \frac{1}{3} \frac{\omega^2}{\tilde{\omega}^2} \right) \right] \exp\left(-\frac{1}{6} \frac{\omega^2}{\tilde{\omega}^2} \right), \quad (3.6)$$

$$\varphi_{2(\omega)} = \sqrt{\frac{2}{3\pi\tilde{\omega}}} \left[C_1 - \frac{C_2}{3} \left(1 - \frac{1}{3} \frac{\omega^2}{\tilde{\omega}^2} \right) \right] \exp\left(-\frac{1}{6} \frac{\omega^2}{\tilde{\omega}^2} \right), \quad (3.7)$$

where $\varphi_{1(\omega)}$ is the function of streamwise spectrum density, $\varphi_{2(\omega)}$ is the function of wall-normal spectrum density, and $\tilde{\omega}$ is the characteristic frequency representing the local flow conditions. One can obtain the following B_1 and B_2 by substituting Eq. (2.9) into Eqs. (A33a) and (A33b) in the Appendix,

$$B_1 = \overline{u'^2}, \quad (3.8a)$$

$$\begin{aligned}
\frac{B_2}{u_*^2} = & \frac{1}{4} \frac{y^+}{11.6D_*^3 + y^+} - 0.2 \sqrt{1 - \frac{y}{h}} \left[\sqrt{1 - \frac{y}{h}} \right. \\
& + 2 \sqrt{\frac{y^+}{11.6D_*^3 + y^+}} \left(1 + \frac{1.45D_*^3}{y^+} \right) \left(y^+ \frac{du^+}{dy^+} \right) \\
& + 0.16 \left(1 - \frac{y}{h} \right) \left[1 + \sqrt{\left(1 - \frac{y}{h} \right) \left(1 + \frac{11.6D_*^3}{y^+} \right)} \right. \\
& \left. \left. + \frac{2.9D_*^3}{y^+} \right] \left(y^+ \frac{du^+}{dy^+} \right)^2. \quad (3.8b)
\end{aligned}$$

Similarly, parameters C_1 and C_2 in Eq. (3.7) can be determined from Eqs. (A34a) and (A34b) in the Appendix, i.e.,

$$C_1 = \bar{v}'^2, \quad (3.9a)$$

$$\begin{aligned}
\frac{C_2}{u_*^2} = & \frac{1}{4} \frac{y^+}{11.6D_*^3 + y^+} - \frac{0.58D_*^3}{y^+} \left[\left(1 - \frac{y}{h} \right) \frac{y^+}{11.6D_*^3 + y^+} \right]^{0.5} \\
& \times \left(y^+ \frac{du^+}{dy^+} \right). \quad (3.9b)
\end{aligned}$$

The unknown characteristic parameter $\tilde{\omega}$ in Eqs. (3.6) and (3.7) can be expressed as follows:

$$\tilde{\omega}^2 = \frac{2\omega_*^2 + \sqrt{\omega_* \omega^3}}{3}, \quad (3.10)$$

where $\omega_* = \bar{u}/h$. The wave number α is the ratio of frequency ω to the convection velocity, which can be taken as the local mean velocity,³⁹

$$\alpha = \frac{\omega}{\bar{u}} \quad \text{or} \quad \omega = \alpha \bar{u}. \quad (3.11)$$

Then, Eq. (3.10) can be rewritten as

$$\tilde{\omega}^2 = \frac{\omega_*^2}{3} \left[2 + \sqrt{\left(\frac{\omega}{\omega_*} \right)^3} \right] = \frac{\omega_*^2}{3} [2 + \sqrt{(\alpha h)^3}], \quad (3.12)$$

where αh is the dimensionless wave number. Thus, one can obtain the following expression by introducing the dimensionless wave number:

$$\begin{aligned}
\overline{u'^2} &= \int_0^\infty \varphi_1(\omega) d\omega \\
&= \int_0^\infty \varphi_1(\alpha h) d(\alpha h) \\
&= \int_0^\infty \frac{h}{\bar{u}} \varphi_1(\alpha h) d\omega. \quad (3.13)
\end{aligned}$$

Thus,

$$\varphi_1(\omega) = \frac{h}{\bar{u}} \varphi_1(\alpha h). \quad (3.14a)$$

Similarly,

$$\varphi_2(\omega) = \frac{h}{\bar{u}} \varphi_2(\alpha h). \quad (3.14b)$$

Since $\omega/\omega_* = \alpha h$, finally, we have

$$\begin{aligned}
\frac{\varphi_1(\alpha h)}{\bar{u}'^2} &= \frac{\sqrt{2/\pi}}{\sqrt{2 + (\alpha h)^{1.5}}} \left[1 - \frac{b_*}{3} \left(1 - \frac{\alpha^2 h^2}{2 + (\alpha h)^{3/2}} \right) \right] \\
&\times \exp \left[-\frac{1}{2} \frac{(\alpha h)^2}{2 + (\alpha h)^{3/2}} \right], \quad (3.15)
\end{aligned}$$

$$\begin{aligned}
\frac{\varphi_2(\alpha h)}{\bar{v}'^2} &= \frac{\sqrt{2/\pi}}{\sqrt{2 + (\alpha h)^{1.5}}} \left[1 - \frac{c_*}{3} \left(1 - \frac{\alpha^2 h^2}{2 + (\alpha h)^{3/2}} \right) \right] \\
&\times \exp \left[-\frac{1}{2} \frac{(\alpha h)^2}{2 + (\alpha h)^{3/2}} \right], \quad (3.16)
\end{aligned}$$

where $b_* = B_2/\bar{u}'^2$ and $c_* = C_2/\bar{v}'^2$.

Equations (3.15) and (3.16) show that the influence of Reynolds number, drag-reducing parameter D_* , and the relative depth y/h on the spectrum is included in the parameters of b_* and c_* . The calculated results exhibit that the values of b_* and c_* do not significantly change the spectrum distribution; one of the examples is shown in Fig. 4(a).

The predictability of Eq. (3.15) is first investigated by using the data in Newtonian fluid flows and the results are shown in Fig. 4(a) that contains the data of Lawn⁴⁷ and Warholic *et al.*³⁹ In Fig. 4(a), the lines represent Eq. (3.15), and b_* is calculated by using Eq. (3.8b) using $D_* = 1$ and y^+ is determined from the location of measuring point, y/h . $b_* = 0.03$ and 0.42 are obtained for the data.^{39,47} It can be seen that the large variation of b_* leads to a slight redistribution of the spectrum.

The energy spectrum in viscoelastic fluid flow is shown in Fig. 4(b), in which the experimental data of Warholic *et al.*³⁹ and Wei and Willmarth²¹ are included for comparison. The calculated b_* ranges from 0.42 to 0.537 ; it can be seen that the parameter b_* has a little influence on the theoretical distribution of dimensionless energy spectrum, which is consistent with the observation, as shown in Fig. 4. The widely cited $-\frac{5}{3}$ th power law is also included in Fig. 4; it can be seen that both the power law and Eq. (3.15) provide good agreement with the measured data for $20 < \alpha h < 100$, but the power law fails in the higher or lower frequency ranges.

Equation (3.15) indicates that the energy spectrum $\varphi_1(\alpha h)$ in DR flows is larger than that in Newtonian fluid flows because the mean square of streamwise velocity fluctuations increase with increasing parameter D_* . Virk *et al.*³² observed these trends in a pipe and the values of $\varphi_1(\alpha h)$ at the pipe center and near-wall region were measured and the results are presented in Fig. 5. It can be seen that the difference between the Newtonian fluid flow and the viscoelastic fluid flows is caused by the values of $\sqrt{u'^2}$.

In Fig. 5, Eq. (3.15) does not match with the experimental result well at the intermediate wave number, so in the future, more research works have to be carried out to modify the performance of Eq. (3.15) in this regime.

FIG. 4. Comparison of dimensionless measured streamwise spectrum available in the literature with Eq. (3.15): (a) Newtonian fluid flow and (b) viscoelastic fluid flow.

IV. CONCLUSIONS

Turbulence characteristics of viscoelastic fluid flows are investigated by using the concept of apparent viscoelasticity. The formulas for the root-mean-square velocity fluctuations, Reynolds shear stress, and energy spectra in addition to the mean velocity profile have been derived and validated by using experimental data available in the literature. Based on the comparisons, the following conclusions can be achieved.

The stress deficit in DR flows can be evaluated by the product of the “effective viscosity” and the velocity gradient. The effective viscosity can be approximated by $\alpha_* u_* h$; this eddylike viscosity can well represent the non-Newtonian properties.

The thickness of viscous sublayer increases with increasing DR parameter D_* that depends on the effective viscosity; the thickened viscous sublayer plays a significant role for the modifications of turbulence structures in drag-reducing flow, i.e., the mean velocity and streamwise velocity fluctuations are increased. The Reynolds stress is reduced or the correlating velocities in different directions are weakened; there is a redistribution of energy from high frequencies to the low frequencies for the streamwise component, but dimensionless distribution over all frequencies almost remains the same as that in Newtonian fluid flows.

The derived equation of mean velocity can approximately express the measured velocity in SDR and LDR regimes, but the derived equations for velocity fluctuation and Reynolds shear stress are valid only in the SDR regime when compared to experimental data, indicating that more theoretical research works are needed. On the other hand, experimenters have not reached a consensus on the turbulence structures in the LDR regime, suggesting that more experimental investigations should be done in the future.

This study shows that the difference between Newtonian fluid flow and viscoelastic fluid flow is caused by the viscoelasticity α_* , which depends on polymer type and concentration. Thus, a systematical investigation is needed to establish the relation in the future between α_* and polymer characteristics, i.e., relaxation time, intrinsic viscosity, concentration, etc.

APPENDIX: STOCHASTIC THEORY OF TURBULENCE

Turbulent flow is known to be full of eddies whose sizes and axis orientations are stochastically distributed, and so is the change of the relative motion of eddies within turbulent flow. Due to the rotation of eddies, the instantaneous velocity at the center of the eddy changes much more slowly than that at other points and the center’s velocity remains unchanged basically in a certain distance. As a result, although the eddy is continually deformed in the process of motion, deformation of eddy center can be assumed to be negligible in a short period. Based on these assumptions, the relation between the instantaneous velocities at different points can be developed as follows.

The vorticity can be written in the tensor form as follows:

$$\omega_j = \frac{1}{2} \left(\frac{\partial u_i}{\partial x_k} - \frac{\partial u_k}{\partial x_i} \right), \quad \omega_k = \frac{1}{2} \left(\frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} \right), \quad (\text{A1})$$

where i, j , and k denote 1, 2, and 3 in the x, y , and z directions, respectively, and ω_j and ω_k are the angular velocities. In Cartesian coordinates, x_1, x_2 , and x_3 , the location of point P (see Fig. 6) can be expressed by radius vector $\mathbf{x}$. It is supposed that the distance from P to the center of the eddy at time t is $\mathbf{r}'$ and the center of the eddy at time t_{0i} is located at $\mathbf{x} - \mathbf{r}' - \mathbf{l}'$, in which $\mathbf{l}'$ is the distance along which the velocity at the eddy center remains unchanged, i.e.,

$$u_i(\mathbf{x} - \mathbf{r}' - \mathbf{l}', t_{0i}) = u_i(\mathbf{x} - \mathbf{r}', t) \quad (\text{A2})$$

Equation (A2) states that the velocity of the eddy center remains unchanged until it has traveled a distance $\mathbf{l}'$; this is

FIG. 5. Comparison of energy spectrum between Eq. (3.15) and the experimental data of Virk *et al.* (Ref. 32).

very similar to the mixing-length theorem developed by Prandtl, who, by simulating momentum exchange on a macroscale in a way equivalent to that of the molecular motion of a gas, proposed that if a small body of water randomly moves in a flow field, its momentum does not change until the water element has moved a distance (l) perpendicular to the flow direction and has mixed with the local fluid.

At time t_{0i} , the velocity at P can be expressed by the Taylor series and it has the following form if only the first two terms are considered:

$$u_i(\mathbf{x}, t_{0i}) = u_i(\mathbf{x} - \mathbf{r}' - \mathbf{l}'_{(i)}, t_{0i}) + (l'_{(i)m} + r'_m) \frac{\partial u_i(\mathbf{x} - \mathbf{r}' - \mathbf{l}'_{(i)}, t_{0i})}{\partial x_m}. \quad (\text{A3})$$

By inserting Eq. (A2) into Eq. (A3), one has

FIG. 6. Scheme of vortex motion.

$$u_i(\mathbf{x}, t_{0i}) = u_i(\mathbf{x} - \mathbf{r}', t) + (l'_{(i)m} + r'_m) \frac{\partial u_i(\mathbf{x} - \mathbf{r}', t)}{\partial x_m}. \quad (\text{A4})$$

In Eq. (A4), $\partial u_i(\mathbf{x} - \mathbf{r}', t) / \partial x_m \approx \partial u_i(\mathbf{x}, t) / \partial x_m$; thus, Eq. (A4) can be rewritten as

$$u_i(\mathbf{x}, t_{0i}) = u_i(\mathbf{x} - \mathbf{r}', t) + (l'_{(i)m} + r'_m) \frac{\partial u_i(\mathbf{x}, t)}{\partial x_m}. \quad (\text{A5})$$

On the other hand, due to the rotation of the eddy, using Eq. (A1) at time t , the velocity at the point P is

$$u_i(\mathbf{x}, t) = u_i(\mathbf{x} - \mathbf{r}', t) + \omega_j r'_k - \omega_k r'_i. \quad (\text{A6})$$

By inserting Eq. (A6) into Eq. (A5), one has

$$u_i(\mathbf{x}, t) = u_i(\mathbf{x}, t_{0i}) - (l'_{(i)m} + r'_m) \frac{\partial u_i(\mathbf{x}, t)}{\partial x_m} + \omega_j r'_k - \omega_k r'_i. \quad (\text{A7})$$

Denoting that $u_i = \bar{u}_i + u'_i$, $\omega_i = \bar{\omega}_i + \omega'_i$, and noting that $\bar{u}_i(\mathbf{x}, t) = \bar{u}_i(\mathbf{x}, t_{0i})$, one can rewrite Eq. (A7) as follows:

$$\begin{aligned} u'_i &= u'_{io} - (l'_{(i)m} + r'_m) \frac{\partial u'_i}{\partial x_m} + \omega'_j r'_k - \omega'_k r'_i \\ &\quad - (l'_{(i)m} + r'_m) \frac{\partial \bar{u}_i}{\partial x_m} + \bar{\omega}_j r'_k - \bar{\omega}_k r'_j, \end{aligned} \quad (\text{A8})$$

where $u'_i = u'_i(\mathbf{x}, t)$, $u'_{io} = u'_i(\mathbf{x}, t_{0i})$ and $\bar{u}_i = \bar{u}_i(\mathbf{x}, t)$.

Equation (A8) can be rewritten as

$$u'_i = u''_i - (l'_{(i)m} + r'_m) \frac{\partial \bar{u}_i}{\partial x_m} + \bar{\omega}_j r'_k - \bar{\omega}_k r'_j, \quad (\text{A9})$$

where

$$u_i'' = u_{io}' - (l'_{(i)m} + r'_m) \frac{\partial u_i'}{\partial x_m} + \omega_j' r'_k - \omega_k' r'_i. \quad (\text{A10})$$

The term of $\bar{\omega}_j' r'_k - \bar{\omega}_k' r'_j$ in Eq. (A9) can be expressed as follows:

$$\bar{\omega}_j' r'_k - \bar{\omega}_k' r'_j = \frac{1}{2} r'_m \frac{\partial \bar{u}_i}{\partial x_m} - \frac{1}{2} r'_m \frac{\partial \bar{u}_m}{\partial x_i}. \quad (\text{A11})$$

Therefore, from Eq. (A9), the velocity fluctuation can be expressed as

$$u_i' = u_i'' - \left(l'_{(i)m} + \frac{r'_m}{2} \right) \frac{\partial \bar{u}_i}{\partial x_m} - \frac{1}{2} r'_m \frac{\partial \bar{u}_m}{\partial x_i}, \quad (\text{A12})$$

in which u' is the velocity fluctuation, u'' is the basic fluctuating velocity that represents the turbulence in the flow and is independent of mean velocity gradient, r' is the distance to the center of eddy, and l' is similar to Prandtl's mixing length representing the distance along which the change of instantaneous velocity can be neglected.

For 2D turbulent flow, these relations are valid: $\bar{u}_1 = \bar{u}(y)$, $\bar{u}_2 = \bar{v} = 0$, $\bar{u}_3 = \bar{w} = 0$, and $\partial/\partial x_1 = \partial/\partial x_3 = 0$; thus, the velocity fluctuation u' and v' can be determined from Eq. (A12),

$$u' = u'' - \left(l'_{(1)2} + \frac{1}{2} r'_2 \right) \frac{\partial \bar{u}}{\partial y}, \quad (\text{A13})$$

$$v' = v'' - \frac{1}{2} r'_1 \frac{\partial \bar{u}}{\partial y}. \quad (\text{A14})$$

The basic fluctuating velocity u'' and the eddy radius r'' are two independent random variables. If the radius vectors of these random variables in a statistic space are denoted by nt and nT , respectively, then their probability density function can be assumed to follow Gaussian statistics, i.e.,

$$p(t, T) = \left(\frac{1}{\sqrt{\pi}} \right)^6 \exp[-n^2(t^2 + T^2)], \quad (\text{A15})$$

where $n^2(t^2 + T^2) = n_1^2(t_1^2 + T_1^2) + n_2^2(t_2^2 + T_2^2) + n_3^2(t_3^2 + T_3^2)$.

If U , L , and T represent the characteristic velocity, length, and time, respectively, the following relations can be written as

$$u_i'' = U n_i t_i, \quad r_i'' = L n_i t_i. \quad (\text{A16})$$

As r' expresses the distance from the discussed point to the center of the eddy, it should be dependent on the eddy radius r'' and the deviation distance from the eddy center S' ; thus,

$$r_i' = \frac{r_i''}{2} + S_i' = \left[\frac{(1+i)}{2} n_i T_i + m(-n_j^2 T_j^2 + n_k^2 T_k^2) \right] L, \quad (\text{A17})$$

where m is a dimensionless parameter to consider dumping

effect of wall and $i = \sqrt{-1}$. The “mixing length” l' should be dependent on both the diameter of eddy, $2r''$, and the basic fluctuating velocity u'' and have the following form:

$$l'_{(ij)} = m(2r_i'') + (1+i)u_j'' T = 2mL n_i T_i + (1+i)UT n_j T_j. \quad (\text{A18})$$

By the use of the probability density function, the statistical mean values of fluctuating quantities and their correlation moment can be determined. For example,

$$\overline{u_i'' l'_{(jk)}} = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} u_i'' l'_{(jk)} p(t, T) dn_1 t_1, \dots, dn_2 T_2 dn_3 T_3. \quad (\text{A19})$$

The double correlation moments are shown as follows:

$$\overline{u_i'' u_j''} = \frac{1}{2} U^2 \delta_{ij}, \quad \overline{r_i' r_j'} = \frac{1}{2} m^2 L^2 (3 \delta_{ij} - 1), \quad (\text{A20a})$$

$$\overline{l'_{(ij)} l'_{(kl)}} = 2m^2 L^2 \delta_{ik}, \quad \overline{u_i'' r_j'} = 0, \quad (\text{A20b})$$

$$\overline{u_i'' l'_{(jk)}} = \frac{1}{2} U^2 T \delta_{jk}, \quad \overline{r_i' l'_{(jk)}} = \frac{1}{2} m^2 L^2 \delta_{ij}, \quad (\text{A20c})$$

in which δ_{ij} is Kronker's sign, $\delta_{ij} = 1$ when $i = j$, and $\delta_{ij} = 0$ when $i \neq j$. As the correlation moments represent the actual structure of turbulent flow, only the part of real values in the above formulas is taking into account.

For a steady turbulent flow, the correlation moment of fluctuating velocities in the same location but different time can be expressed as

$$R_{ij}(\tau) = \overline{u_i'(t) u_j'(t + \tau)} = \overline{u_i' \left(t - \frac{\tau}{2} \right) u_j' \left(t + \frac{\tau}{2} \right)}. \quad (\text{A21})$$

In the case of $i = j$, we have the autocorrelation function

$$R_{ii}(\tau) = \overline{u_i' \left(t - \frac{\tau}{2} \right) u_i' \left(t + \frac{\tau}{2} \right)}. \quad (\text{A22})$$

In order for the time average to be equal to the assembly average for double correlations, it has to express the random variables by the following formulas:

$$u_i'' = U \tilde{\omega}_i t_i \left(\frac{1}{\sqrt{\pi}} \right)^3 \exp \left[-\frac{1}{2} \tilde{\omega}_i^2 (t_i^2 + T_i^2) \right], \quad (\text{A23})$$

$$r_i'' = L \tilde{\omega}_i T_i \left(\frac{1}{\sqrt{\pi}} \right)^3 \exp \left[-\frac{1}{2} \tilde{\omega}_i^2 (t_i^2 + T_i^2) \right], \quad (\text{A24})$$

$$r_i' = L \left[\frac{1}{2} (1+i) \tilde{\omega}_i T_i + m(-\tilde{\omega}_j^2 T_j^2 + \tilde{\omega}_k^2 T_k^2) \right] \left(\frac{1}{\sqrt{\pi}} \right)^3 \times \exp \left[-\frac{1}{2} \tilde{\omega}_i^2 (t_i^2 + T_i^2) \right], \quad (\text{A25})$$

$$l'_{(i)j} = [2mL\tilde{\omega}_i T_i + (1+i)UT\tilde{\omega}_j t_j] \left(\frac{1}{\sqrt{\pi}} \right)^3 \times \exp \left[-\frac{1}{2} \tilde{\omega}_i^2 (t_i^2 + T_i^2) \right], \quad (\text{A26})$$

where $\tilde{\omega}$ is the characteristic angular frequency.

For the simple case of 2D flow, such as flows in circular pipes and very wide open channels, we have $\bar{u}_1 = \bar{u}$, $\bar{u}_2 = \bar{u}_3 = 0$, and $\partial \bar{u}_1 / \partial x_1 = \partial \bar{u}_1 / \partial x_3 = 0$; one can obtain from Eq. (A12) the fluctuating velocities at moment $t + \tau/2$ and $t - \tau/2$ as follows, respectively:

$$u'_1 \left(t - \frac{\tau}{2} \right) = u''_1 \left(t - \frac{\tau}{2} \right) - \left[l'_{(1)2} \left(t - \frac{\tau}{2} \right) + \frac{1}{2} r'_2 \left(t - \frac{\tau}{2} \right) \right] \frac{d\bar{u}_1}{dx_2}, \quad (\text{A27})$$

$$u'_1 \left(t + \frac{\tau}{2} \right) = u''_1 \left(t + \frac{\tau}{2} \right) - \left[l'_{(1)2} \left(t + \frac{\tau}{2} \right) + \frac{1}{2} r'_2 \left(t + \frac{\tau}{2} \right) \right] \frac{d\bar{u}_1}{dx_2}. \quad (\text{A28})$$

Thus, the autocorrelation function can be derived as follows:

$$\begin{aligned} \overline{u'_1 \left(t - \frac{\tau}{2} \right) u'_1 \left(t + \frac{\tau}{2} \right)} &= \overline{u''_1 \left(t - \frac{\tau}{2} \right) u''_1 \left(t + \frac{\tau}{2} \right)} - \left[\overline{u''_1 \left(t + \frac{\tau}{2} \right) l'_{(1)2} \left(t - \frac{\tau}{2} \right)} + \frac{1}{2} \overline{u''_1 \left(t + \frac{\tau}{2} \right) r'_2 \left(t - \frac{\tau}{2} \right)} + \overline{u''_1 \left(t - \frac{\tau}{2} \right) l'_{(1)2} \left(t + \frac{\tau}{2} \right)} \right. \\ &\quad \left. + \frac{1}{2} \overline{u''_1 \left(t - \frac{\tau}{2} \right) r'_2 \left(t + \frac{\tau}{2} \right)} \right] \frac{d\bar{u}_1}{dx_2} + \left[\overline{l'_{(1)2} \left(t - \frac{\tau}{2} \right) l'_{(1)2} \left(t + \frac{\tau}{2} \right)} + \frac{1}{2} \overline{l'_{(1)2} \left(t + \frac{\tau}{2} \right) r'_2 \left(t - \frac{\tau}{2} \right)} \right. \\ &\quad \left. + \frac{1}{2} \overline{l'_{(1)2} \left(t - \frac{\tau}{2} \right) r'_2 \left(t + \frac{\tau}{2} \right)} + \frac{1}{4} \overline{r'_2 \left(t - \frac{\tau}{2} \right) r'_2 \left(t + \frac{\tau}{2} \right)} \right] \left(\frac{d\bar{u}_1}{dx_2} \right)^2. \end{aligned} \quad (\text{A29})$$

In a similar way, the autocorrelation function in the wall-normal direction can be written as

$$\begin{aligned} \overline{u'_2 \left(t - \frac{\tau}{2} \right) u'_2 \left(t + \frac{\tau}{2} \right)} &= \overline{u''_2 \left(t - \frac{\tau}{2} \right) u''_2 \left(t + \frac{\tau}{2} \right)} - \left[\frac{1}{2} \overline{u''_2 \left(t - \frac{\tau}{2} \right) r'_1 \left(t + \frac{\tau}{2} \right)} + \frac{1}{2} \overline{u''_2 \left(t + \frac{\tau}{2} \right) r'_1 \left(t - \frac{\tau}{2} \right)} \right] \frac{d\bar{u}_1}{dx_2} \\ &\quad + \left[\overline{l'_{(1)2} \left(t - \frac{\tau}{2} \right) l'_{(1)2} \left(t + \frac{\tau}{2} \right)} + \frac{1}{4} \overline{r'_1 \left(t - \frac{\tau}{2} \right) r'_1 \left(t + \frac{\tau}{2} \right)} \right] \left(\frac{d\bar{u}_1}{dx_2} \right)^2. \end{aligned} \quad (\text{A30})$$

Each term in above can be determined by using Eqs. (A23)–(A27) and the final forms are

$$\begin{aligned} R_{11}(\tau) &= \overline{u'_1(t) u'_1(t+\tau)} \\ &= \overline{u'_1 \left(t - \frac{\tau}{2} \right) u'_1 \left(t + \frac{\tau}{2} \right)} \\ &= (B_1 - B_2 \tilde{\omega}^2 \tau^2) \exp \left(-\frac{3}{2} \tilde{\omega}^2 \tau^2 \right), \end{aligned} \quad (\text{A31})$$

$$\begin{aligned} R_{22}(\tau) &= \overline{u'_2(t) u'_2(t+\tau)} \\ &= \overline{u'_2 \left(t - \frac{\tau}{2} \right) u'_2 \left(t + \frac{\tau}{2} \right)} \\ &= (C_1 - C_2 \tilde{\omega}^2 \tau^2) \exp \left(-\frac{3}{2} \tilde{\omega}^2 \tau^2 \right), \end{aligned} \quad (\text{A32})$$

where

$$B_1 = \frac{U^2}{2} + \frac{9}{4} m^2 L^2 \left(\frac{d\bar{u}_1}{dx_2} \right)^2, \quad (\text{A33a})$$

$$\begin{aligned} B_2 &= \frac{U^2}{4} - \left(mLU + \frac{U^2 T}{2} + \frac{UL}{8} \right) \frac{d\bar{u}_1}{dx_2} \\ &\quad + \left(m^2 L^2 + mLU + \frac{mL^2}{4} \right) \left(\frac{d\bar{u}_1}{dx_2} \right)^2, \end{aligned} \quad (\text{A33b})$$

$$C_1 = \frac{U^2}{2} + \frac{1}{4} m^2 L^2 \left(\frac{d\bar{u}_1}{dx_2} \right)^2, \quad (\text{A34a})$$

$$C_2 = \frac{U^2}{4} - \frac{UL}{8} \frac{d\bar{u}_1}{dx_2}. \quad (\text{A34b})$$

For 2D turbulent flows where $u_1 = u$ and $u_2 = v$, when Eqs. (A33a) and (A34a) are compared to Eqs. (2.7) and (2.8) in the main text, one has $B_1 = \overline{u'^2}$ and $C_1 = \overline{v'^2}$ and $R_{11}(\tau) = \overline{u'(t) u'(t+\tau)}$ and $R_{22}(\tau) = \overline{v'(t) v'(t+\tau)}$.

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