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Keywords
stock, call, margin, maturity, finite, loans

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Finite Maturity Margin Call Stock Loans

Xiaoping Lu [*]   Endah R.M. Putri [†]

Abstract

In this paper, we formulate margin call stock loans in finite maturity as American down-and-out calls with rebate and time-dependent strike. The option problem is solved semi-analytically based on the approach in [17]. An explicit equation for optimal exit price and a pricing formula for loan value are obtained in Laplace space. Final results are obtained by numerical inversion. Examples are provided to show the dependency of the optimal exit price and margin call stock loan value on various parameters.

Keywords
margin call stock loan, American down-and-out call with rebate, Laplace transform method

1 Introduction

A stock loan is a financial contract that allows the borrower to obtain a loan with stocks as collateral. A borrower may retrieve the stock at any time before or at maturity by repaying the loan and accumulated interest at a predetermined interest rate. At anytime if the stock price increases, the borrower is able to pay back the loan, retrieve the stock and get the unlimited upside potential profit. For a non-recourse stock loan, if the stock price decreases below the accumulated loan, the borrower can walk away by surrendering the stock. In this case the borrower only loses the service fee paid at the beginning of the contract. A margin call stock loan contract provides more security for the lender than its non-recourse counterpart. When the stock price falls at or below the accumulated loan value, the lender issues a margin call. Once the margin call is issued, the borrower must pay back a pre-determined percentage of the loan, then the contract continues as a non-recourse loan with a reduced loan amount. Only one margin call is allowed in the life of a standard margin call stock loan contract.

A margin call stock loan is more advantageous to both sides than its non-recourse counterpart, as it not only provides protection for the lender, but also lowers the service fee for the borrower. Thus, developing an accurate and efficient pricing method for margin call stock loans is necessary, as accuracy and efficiency are the most crucial elements for financial success. However, the margin

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call feature adds more complication to the pricing problem on top of the early exit boundary, which mathematically makes the pricing problem a moving boundary problem. There is not much work documented in the literature for margin call stock loans except Ekstrm's work [7] on a perpetual margin call stock loan. In [7] a perpetual margin call stock loan is shown to resemble a perpetual American down and out call option with rebate, a possibly negative interest rate and a time-dependent strike price. Explicit formulas for the value and optimal exit time of a perpetual margin call stock loan were obtained in their study. However, the assumption of infinite maturity is not practical as stock loans usually have finite maturity of only a few years. In this work, we will formulate and solve the pricing problem of a finite maturity margin call stock loan as the corresponding American down-and-out option with rebate, negative interest rate and time dependent strike price.

Merton [14] in his fundamental paper in 1973, established a closed-form solution for valuing a European-style option with the presence of a down-and-out barrier. Subsequently, barrier options in American style were investigated by other researchers. Similar to vanilla American options, analytic solutions are available only for limited cases, such as, a closed-form solution of a perpetual American up-and-out put [9], an integral representation of the knock-in American option pricing formula [6]. Thus, both numerical and analytical approximation methods have been developed for the valuation of finite maturity American barrier options.

The main numerical methods in the literature are the tree method [2, 5, 16] and the finite difference method [3, 15, 19]. However, although easy to implement pure numerical methods can prove to be time consuming and restrictive in applications. As a result analytical approximation methods are much preferred.

Gao et al. [8] proposed a decomposition technique in which the value of up-and-in or up-and-out American barrier options are decomposed into the corresponding European value and an early exercise premium. Chang et al. [4] presented a modified quadratic approximation method based on the method in Barone-Adesi-Whaley [1] for American options. Lu and Rhodes [13] presented a valuation of American down-and-out call options based on the analytical approximation method proposed by Zhu [17], which uses Laplace transform with pseudo-steady state approximation of the moving boundary. Lauko and Sevcovic [11] pointed out that the method in [17] is particularly suitable for long term American option valuation in a comparative study of numerical and analytical approximation methods of the early exercise boundary of American puts. Since a stock loan usually has a longer life time than ordinary options, the approximation method is also proved suitable in a stock loan evaluation [12]. Therefore, the approach in [13] and [17] will form the basis for the solution procedure in this study.

This paper has four main sections. Section 1 provides an introduction, Section 2 discusses margin call stock loans as the corresponding American barrier options with rebate. Numerical results are presented in Section 3, and conclusion is given in Section 4.

2 Margin call stock loans as American barrier options

The mechanism of a margin call stock loan contract can be described as follows. At time 0, a client borrows amount q at a predetermined interest rate γ with one share of stock valued at S_0 as collateral. At any time t ($t \geq 0$), the borrower may pay back the amount $qe^{\gamma t}$ (loan principal plus accumulated interest) to regain the collateral. However, if the stock price drops to or below the accrued loan amount, the contract is suspended, the lender issues a margin call and forces the borrower to pay back a predetermined percentage Δ of the loan. After the call and payback, the contract then continues as a non-recourse loan with a reduced loan amount. It is assumed that only one margin call is allowed during the life of the loan.

As mentioned above, a stock loan contract with a margin call has two phases: a margin call phase and a non-recourse stock loan phase. The margin call phase resembles an American down-and-out call with rebate and the non-recourse phase resembles a vanilla American call, both with negative interest rate and time dependent strike. We will concentrate on the margin call phase in this work, the evaluation of non-recourse stock loans is presented by the authors in [12].

2.1 Formulation of a standard margin call stock loan

Through out this section we assume that the risk neutral stock price process is described by the following stochastic differential equation (Geometric Brownian Motion)

$$dS = (r - \delta)\, S\, dt + \sigma S\, dW_t$$

where dW_t is a Wiener process defined for $t \in [0, \infty)$, $r \geq 0$ is the risk-free interest rate, $\delta \geq 0$ the continuous dividend yield, and σ the volatility of the stock.

Let $V(S, t; q)$ be the value of a margin call stock loan with finite maturity T. By employing Ito's lemma and arbitrage-free opportunity principle, we obtain the following differential equation system under the Black-Scholes frame work:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta) S \dfrac{\partial V}{\partial S} - rV = 0, \ S_{t_q} \leq S \leq S_f(t), \ 0 \leq t \leq T \\ V(S, T) = max(S - qe^{\gamma T}, 0) \\ V(S_f(t), t) = S_f(t) - qe^{\gamma t} \\ \dfrac{\partial V}{\partial S}(S_f(t), t) = 1 \\ V(S_{t_q}, t) = R(t) \end{cases} \tag{2.1}$$

where $S_f(t)$ stands for the optimal exit price, at or above which it is optimal for the borrower to exit the loan contract. When the stock price falls on or below the accumulated loan value at $t = t_q$ ($S \leq S_{t_q} = qe^{\gamma t_q}$), the lender issues a margin call; the borrower is forced to pay back a fraction Δ of the loan and left with a non-recourse loan of amount $(1 - \Delta)qe^{\gamma t_q}$. The terminal value of the margin call phase equals the difference of the initial value of the non-recourse phase and the payback amount, that is, $R(t) = V_{nr}(S_{t_q}, 0; (1-\Delta)qe^{\gamma t_q}) - \Delta q e^{\gamma t_q}$.

The equation system (A.2) resembles one for American down-and-out call options with time dependent rebate and strike, and its solution gives rise to the values of a standard margin call stock loan. Applying the following change of variables

$$t = T - \frac{2\tau}{\sigma^2},\ S = Xqe^{\gamma t},\ V(S,t) = \tilde{V}(X,\tau)\, qe^{\gamma t}$$

we obtain a simpler differential equation system with initial and boundary conditions:

$$\begin{cases} -\dfrac{\partial \tilde{V}}{\partial \tau} + X^2 \dfrac{\partial^2 \tilde{V}}{\partial X^2} + (\alpha - \beta) X \dfrac{\partial \tilde{V}}{\partial X} - \alpha \tilde{V} = 0,\ 1 \leq X \leq X_f,\ 0 \leq \tau \leq 2T/\sigma^2 \\ \tilde{V}(X,0) = max(X-1,0) \\ \tilde{V}(X_f(\tau),\tau) = X_f(\tau) - 1 \\ \dfrac{\partial \tilde{V}}{\partial X}(X_f(\tau),\tau) = 1 \\ \tilde{V}(1,\tau) = (1-\Delta)\tilde{V}_{nr}(\frac{1}{1-\Delta},\tau) - \Delta \end{cases} \tag{2.2}$$

where $\alpha = \frac{2(r-\gamma)}{\sigma^2}$, $\beta = \frac{2\delta}{\sigma^2}$, and $\tilde{V}_{nr}(\frac{1}{1-\Delta},\tau)$ is the dimensionless initial value of the non-recourse phase.

To make the system more manageable, we define a new function

$$U = \tilde{V} + 1 - X,\ 1 \leq X \leq X_f,$$

then the PDE system (2.2) becomes,

$$\begin{cases} -\dfrac{\partial U}{\partial \tau} + X^2 \dfrac{\partial^2 U_1}{\partial X^2} + (\alpha - \beta) X \dfrac{\partial U}{\partial X} - \alpha U = \beta X - \alpha \\ U(X,0) = 0 \\ U(X_f,\tau) = 0 \\ \dfrac{\partial U}{\partial X}(X_f,\tau) = 0 \\ U(1,\tau) = (1-\Delta)U_{nr}(\frac{1}{1-\Delta},\tau) \end{cases} \tag{2.3}$$

The next step is to transform the Equation system (2.3) into Laplace space. Following the approximation method for the moving boundary in [17], we derive the following ordinary differential equation system

$$\begin{cases} X^2 \dfrac{d^2\bar{U}(X,p)}{dX^2} + (\alpha-\beta)X\dfrac{d\bar{U}(X,p)}{dX} - (\alpha+p)\bar{U}(X,p) = \dfrac{\beta X-\alpha}{p} \\ \bar{U}(p\bar{X}_f,p) = 0 \\ \dfrac{d\bar{U}}{dX}(p\bar{X}_f,p) = 0 \\ \bar{U}(1,p) = (1-\Delta)\bar{U}_{nr}(\dfrac{1}{1-\Delta},p) \end{cases} \tag{2.4}$$

where variables with a bar on top represent the corresponding Laplace transforms.

Assuming $\bar{X}_f(p)$ is known, Equation (2.4) can be solved to give the following solution:

$$\bar{U}(X,p) = C_1X^{k1} + C_2X^{k2} + \frac{p(\alpha-\beta X)+\alpha\beta(1-X)}{p(p+\alpha)(p+\beta)} \tag{2.5}$$

where $C_1 = -\frac{(p\bar{X}_f)^{-k_1}}{k_1-k_2}\left(\frac{(k_2-1)\beta}{p+\beta}\bar{X}_f - \frac{\alpha k_2}{p(p+\alpha)}\right)$ and $C_2 = \frac{(p\bar{X}_f)^{-k_2}}{k_1-k_2}\left(\frac{(k_1-1)\beta}{p+\beta}\bar{X}_f - \frac{\alpha k_1}{p(p+\alpha)}\right)$.

Applying the boundary condition at the barrier $X=1$, we obtain an explicit equation for the optimal exit price, $\bar{X}_f(p)$, in Laplace space:

$$\begin{aligned} &\bar{X}_f^{-k_1}\frac{k_2\alpha}{p^{1+k_1}(p+\alpha)} + \bar{X}_f^{1-k_1}\frac{(1-k_2)\beta}{p^{k_1}(p+\beta)} - \bar{X}_f^{-k_2}\frac{k_1\alpha}{p^{1+k_2}(p+\alpha)} - \bar{X}_f^{1-k_2}\frac{(1-k_1)\beta}{p^{k_2}(p+\beta)} \\ &= (k_1-k_2)\left\{\bar{R}(p) + \frac{\beta-\alpha}{(p+\alpha)(p+\beta)}\right\} \end{aligned} \tag{2.6}$$

where $k_{1,2} = \frac{1+\beta-\alpha}{2} \pm \sqrt{(\frac{1+\beta-\alpha}{2})^2 + (p+\alpha)}$ and $\bar{R}(p) = (1-\Delta)\bar{U}_{nr}(\frac{1}{1-\Delta},p)$, the derivation of which is detailed in Section (2.2).

Once the optimal exit price $\bar{X}_f(p)$ is obtained from Equation (2.6) the calculation of the stock loan value $\bar{U}(X,p)$ is straight-forward using Equation (2.5). Numerical Laplace inversion is then applied to obtain the values in original time space. Stehfest method is chosen following the discussion in [18].

2.2 Derivation of rebate $\bar{R}(p)$

During the life time, T, of a margin call stock loan contract, if the stock price S falls and touches the loan value at $t=t_q$ ($S_{t_q} = qe^{\gamma t_q}$), the contract is temporarily halted. The borrower must pay a pre-determined fraction Δ of the accumulated loan in order for the contract to continue. At the time of the call, the margin call phase of the stock loan is terminated. After the payment, the contract resumes as a non-recourse contract until maturity.

The connection between the two phases is time t_q which is the end of a margin call stock loan phase and the beginning of a new non-recourse phase. At time t_q, we reset the clock for the non-recourse phase by introducing $t' = t-t_q$ and $T' = T-t_q$. The non-recourse phase begins with the initial stock value S_{t_q} as collateral and a new loan size $q' = (1-\Delta)qe^{\gamma t_q}$ after the payback.

Let the value of the non-recourse phase be $V_{nr}(S, t'; q')$, then its initial value is $V_{nr}(S_{t_q}, 0; (1-\Delta)qe^{\gamma t_q})$. Because the borrower pays back a fraction of the loan, $\Delta qe^{\gamma t_q}$, the value of the borrower's holding at time t_q is reduced by this amount. That is, the net value of the borrower holds at time t_q is $V_{nr}(S_{t_q}, 0; (1-\Delta)qe^{\gamma t_q}) - \Delta qe^{\gamma t_q}$, which can be considered as a rebate (terminal value of the margin call phase) compensating the borrower due to the margin call. Therefore, the value of the margin call stock loan at t_q is

$$V(S_{t_q}, t_q; q) = V_{nr}(S_{t_q}, 0; (1-\Delta)qe^{\gamma t_q}) - \Delta qe^{\gamma t_q}. \tag{2.7}$$

After the variable change in Section 2.1, Equation (2.7) becomes

$$\tilde{V}(1, \tau_q) = (1-\Delta)\tilde{V}_{nr}(X'_{t_q}, \tau'_q) - \Delta$$

where $\tilde{V}_{nr} = V_{nr}/q' = V_{nr}/(1-\Delta)qe^{\gamma t_q}$, and $X'_{t_q} = S_{t_q}/q' = 1/(1-\Delta)$. Noting $\tau'_q = \tau_q$ at t_q, dropping the q, we obtain a dimensionless form of Equation (2.7):

$$\tilde{V}(1, \tau) = (1-\Delta)\tilde{V}_{nr}(\frac{1}{1-\Delta}, \tau) - \Delta \tag{2.8}$$

Since $\tilde{V}(1,\tau) = U(1,\tau)$ (Section 2.1), and $\tilde{V}_{nr}(\frac{1}{1-\Delta}, \tau) = U_{nr}(\frac{1}{1-\Delta}, \tau) + \frac{1}{1-\Delta} - 1$ (Appendix A), we obtain

$$U(1, \tau) = (1-\Delta)U_{nr}(\frac{1}{1-\Delta}, \tau) \tag{2.9}$$

After Laplace transform, Equation (2.9) becomes

$$\bar{U}(1, p) = (1-\Delta)\bar{U}_{nr}(\frac{1}{1-\Delta}, p), \tag{2.10}$$

that is, $\bar{R}(p) = (1-\Delta)\bar{U}_{nr}(\frac{1}{1-\Delta}, p)$, where $\bar{U}_{nr}$ can be found from Equation (A.3).

Equation (2.10) is valid for $\frac{1}{1-\Delta} < X_f^{nr}$, where X_f^{nr} is the optimal exit price for the non-recourse phase at the time of the margin call. For $\frac{1}{1-\Delta} \geq X_f^{nr}$, it is possible to exit the non-recourse phase optimally at t_q so $\tilde{V}_{nr}(\frac{1}{1-\Delta}, \tau) = \frac{1}{1-\Delta} - 1 = \frac{\Delta}{1-\Delta}$, which means rebate $R = 0$ for the margin call phase according to Equation (2.8).

It appears that there are two competing forces that are driving the rebate as Δ increases from Equation (2.7) or Equation (2.8). However, the increase of payback ($\Delta qe^{\gamma t}$) outweighs the slight increase in the initial value of the non-recourse phase. This is easier to be seen from Equation (2.8): the $1-\Delta$ factor 'cancels' the increase in the value of $\tilde{V}_{nr}$ due to the increase of Δ, and if Δ increases further that $\frac{1}{1-\Delta} \geq X_f^{nr}$, the rebate will be reduced to zero for the margin call phase. Thus, rebate R is a decreasing function of Δ. Financially it makes sense that the more payback is required the less value is left for the same collateral.

It is obvious that $\Delta = 0$ (no payment) corresponds to the case of a non-recourse loan.

2.3 Calculation of optimal exit price $\bar{X}_f(p)$

Equation (2.6) is highly non-linear, a numerical procedure is needed to calculate the values of $\bar{X}_f(p)$ for a given value of p. In order to settle on a suitable numerical technique we first have to determine the defining characteristics of Equation (2.6).

Let $y = p\bar{X}_f(p)$, Equation (2.6) becomes

$$F(y) = \frac{\alpha\left(k_2 y^{-k_1} - k_1 y^{-k_2}\right)}{p(p+\alpha)} + \frac{\beta\left[(1-k_2)y^{1-k_1} - (1-k_1)y^{1-k_2}\right]}{p(p+\beta)} - (k_1 - k_2)\left\{\bar{R}(p) + \frac{\beta-\alpha}{(p+\alpha)(p+\beta)}\right\} = 0.$$

$F'(y) = 0$ yields two points: $y = 1$ or $y = \frac{\alpha k_1 k_2 (p+\beta)}{\beta(1-k_1)(1-k_2)(p+\alpha)} < 0$. Therefore, there are only two possible local extreme points for $F(y)$. For our practical problem $y > 0$, thus, we can focus on the behavior of $F(y)$ for $y > 0$ only.

$F(1) = F_{min}$, as $F''(1) = (k_1 - k_2)\left\{\frac{k_1 k_2 \alpha}{p+\alpha} - \frac{(1-k_1)(1-k_2)\beta}{p+\beta}\right\} > 0$. Note that the following information is used in the discussion: $k_1 > 1, k_2 < 0, \alpha < 0, \beta > 0, \bar{R}(p) \geq 0$, and $p + \alpha > 0$ (p is large and positive for finite maturity). Due to the complexity of $F(y)$, we only prove

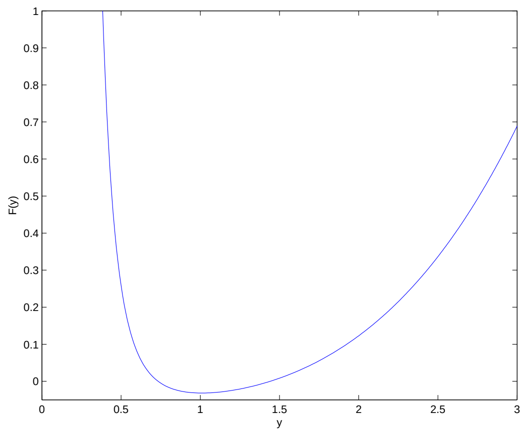


Figure 2.1: Root Testing

numerically that for $y > 0$, F decreases monotonically to its minimum at $y = 1$ and then increases monotonically without bound for $y > 1$ as shown in Figure 2.1. $F(1) = -(k_1 - k_2)\bar{R}(p) < 0$, thus, we can deduce that for $y > 0$ there exist two zeroes of $F(y)$, $y_1 \in (0, 1)$ and $y_2 \in (1, \infty)$. Further when $\bar{R}(p) = 0$, the two roots become one at $y = 1$, which corresponds to $\bar{X}_f = 1/p$. Following the discussion in [13], a financially permissible $\bar{X}_f$ has a lower bound of $1/p$ (barrier level) in Laplace space. It follows that there is a unique zero of Equation (2.6) satisfying $\bar{X}_f(p) \geq 1/p$.

After this realization a standard numerical technique could be implemented in order to solve Equation (2.6) for $\bar{X}_f(p)$. The bisection method is chosen for its simplicity, robustness, and proven convergence rate. Our computation time (Intel(R) Core(TM) i3-2120CPU@3.30 GHz 3.29 GHz, 3.21 GB of RAM) for one optimal exit price is about 0.03 second and 0.05 second for one stock loan value, respectively.

3 Numerical results and discussions

3.1 Method validation

Before a meaningful discussion is carried out, we validate our method by the following processes:

We first compute the optimal exit price of a perpetual margin call stock loan using our formulation by letting maturity $T \to \infty$, and compare our result with that obtained by the analytic solution in [7]. The following parameters are used in the calculations: $r = 0.06, \gamma = 0.1, \delta = 0, \sigma = 0.15$ and $\Delta = 5\%$. The result from our calculation, $x_{f\infty} = 1.3103$, matches to 4 decimal places with the exact value computed from the perpetual margin call formula.

We then let the payback $\Delta = 0$ (no payment) in the finite maturity margin call stock loan to degenerate the margin call loan to a standard non-recourse loan, and compare our result with that of a standard non-recourse loan in [12]. As shown in Figure 3.1, our calculated dimensionless optimal exit price for $\Delta = 0$ agrees perfectly with that of the standard non-recourse loan.

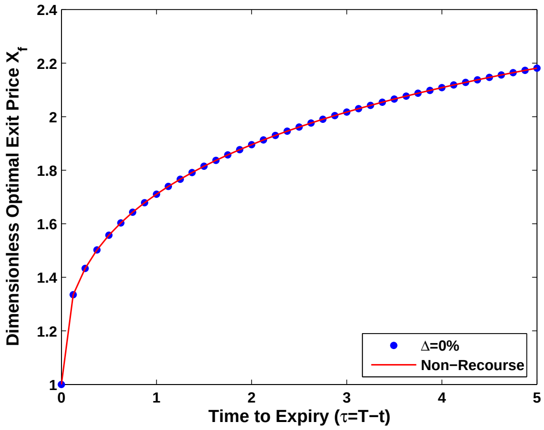


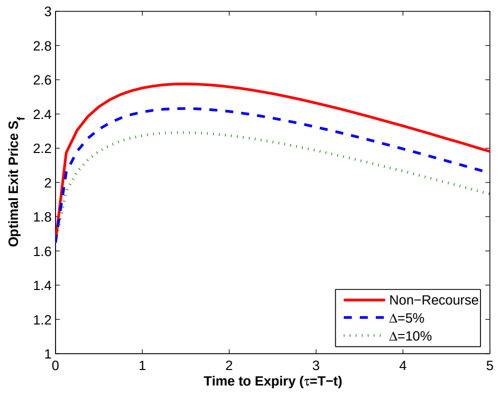
Figure 3.1: Dimensionless optimal exit price for $q = 1, r = 0.06, \gamma = 0.1, \delta = 0.03, \sigma = 0.4, T = 5$

Due to the limitation of space, we did not provide a comparison of our method with another numerical scheme, such as, the Finite Difference method. However, our numerical procedure is similar to that for the pricing of American down-and-out calls in [13], which compared well with the Adaptive Mesh method, Finite Difference method and the Least-Square Monte Carlo method. Since as far as numerical calculation is concerned the difference between our problem and that in [13] is just the rebate, we feel confident about our own method after comparing our results with those of the perpetual margin call and the finite maturity non-recourse loan.

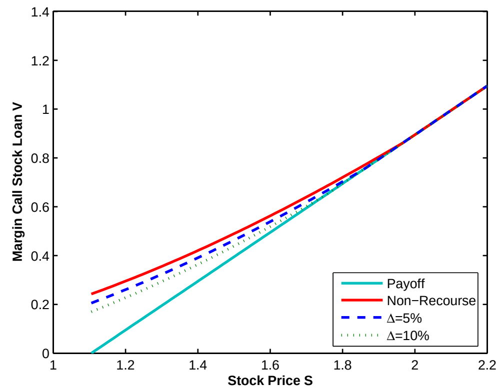
3.2 Margin call stock loans

In this section, we will present some quantitative analysis of margin call stock loans. Unless otherwise stated the following parameters are used in the calculations: initial loan value $q = \$1$, risk free interest rate $r = 0.06$, loan interest rate $\gamma = 0.1$, continuous dividend $\delta = 0.03$, volatility $\sigma = 0.4$, payback $\Delta = 10\%$ and maturity $T = 5$ years.

We first exam the impact of margin call on stock loans by looking at the variation of the optimal exit price and stock loan value for different values of Δ with other parameters fixed. Kwok [10] stated that the optimal exercise price of an American down-and-out call option with rebate is an increasing function of the rebate. As a margin call stock loan resembles an American down-and-out call option with rebate, its optimal exit price also increases with an increasing rebate. An increase in Δ leads to a decrease in the optimal exit price as shown in Figure 3.2(a). This is because rebate R is a decreasing function of Δ (more payback means lower rebate). However, even though the dimensionless optimal price for a stock loan behaves just like that of an American option, the dimensional optimal exit price is not monotonic with respect to time. Financially, the reason why the actual optimal exit price is first increasing then decreasing in time to maturity is that, since the strike price grows in time, there is a natural trend for the optimal exit price to grow; on the other hand, an American down-and-out call with rebate loses its value closer to expiry so the optimal exit price will be lower. The curvature is a result of these two competing forces.



(a)

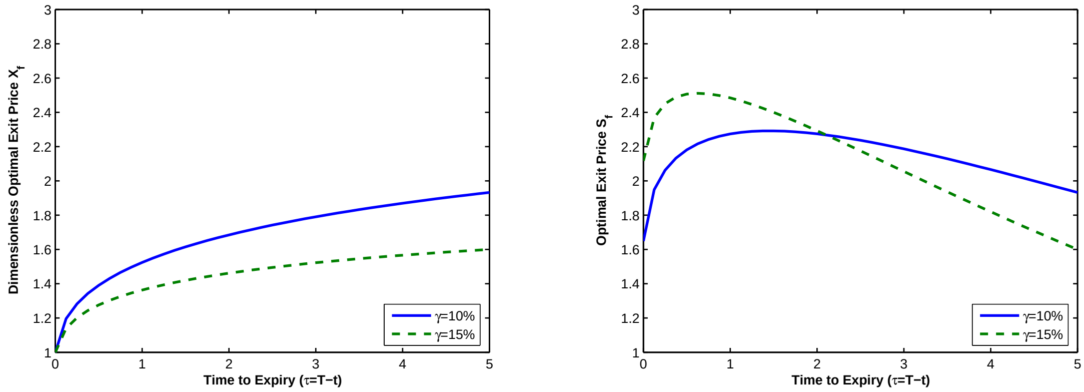


(b)

Figure 3.2: Variation of optimal exit price and stock loan value with Δ for fixed δ, γ, r and σ

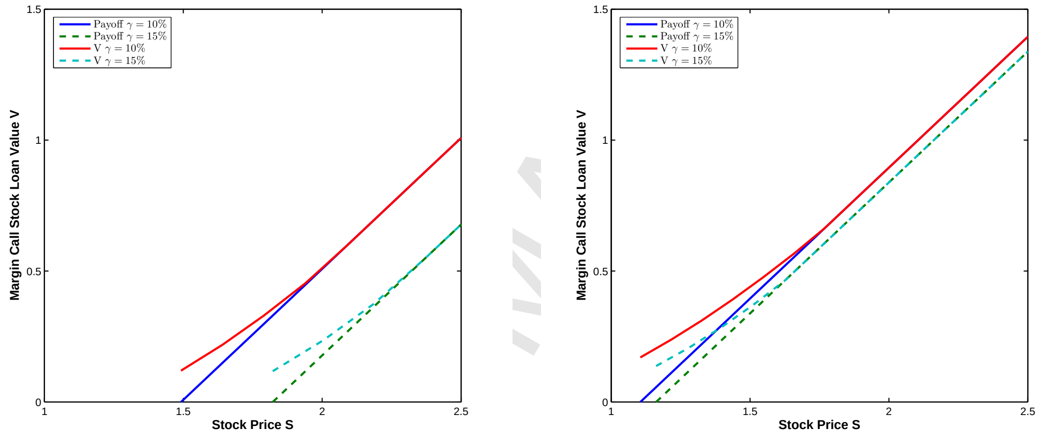
The value of a margin call stock loan as shown in Figure 3.2(b) decreases as Δ increases. It makes financial sense, as a margin call is supposed to provide more security for the lender; higher payback provides lower margin call value to the borrower. A margin call stock loan values less than a non-recourse loan so it should be cheaper to enter a margin call contract [12].

Figures 3.3 (a) and 3.3(b) show the variation of the optimal exit prices for different loan interest rate γ with all other parameters fixed. Although the dimensionless optimal price is monotonic in γ for $\tau \in [0, T]$, the dimensional price is not so that the dimensional curves cross each other in the interval. This is due to fact that the optimal exit price of a stock loan is non-monotonic with respect to time; with all other parameters fixed, the rate of the increase or decrease of the optimal price is dominated by the γ factor in the strike price $qe^{\gamma t}$.



(a) (b)

Figure 3.3: Optimal exit prices for cases $\gamma = 0.1, 0.15$ for fixed δ, r, σ and Δ

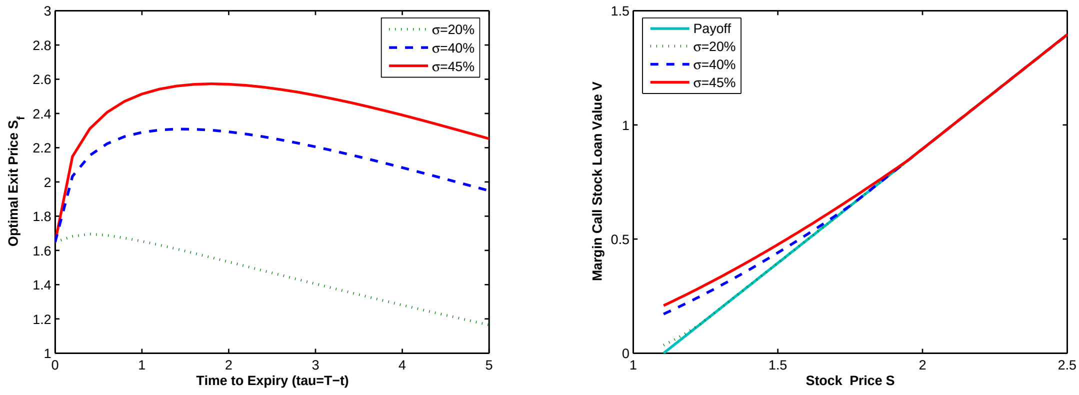


(a) (b)

Figure 3.4: Margin call stock loan value at: (a) $\tau = 1$, (b) $\tau = 4$ for fixed δ, r, σ and Δ

Figures 3.4 (a) and 3.4(b) depict the variation of a margin call stock loan value with the loan interest rate γ at different times to expiry. It is interesting to note that although the optimal exit price is not monotonic in γ, the value of the margin call stock loan changes monotonically with respect to γ. The stock loan values are dominated by the loan interest rate γ when other parameters are fixed, that is, the lower loan interest rate corresponds to the higher loan value, and vice versa. This make financial sense, as the higher γ charged by the lender, the lower value of the loan to the borrower.

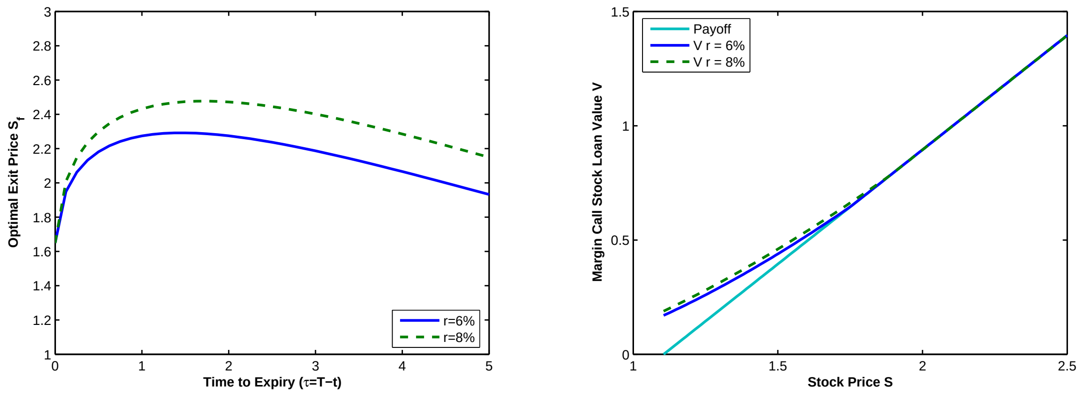
In Figures 3.5 (a) and 3.5(b) we provide a comparison of the optimal exit price and value of the stock loan for different volatility rates for fixed Δ, r, δ and γ. As it can be seen both S_f and V increase with σ. This makes sense as the optimal exit price and stock loan value increase with rebate. The rebate itself increases with the initial value of the non-recourse phase when other parameters are fixed (see Equation (2.7)). Because the non-recourse phase behaves like an



(a) (b)

Figure 3.5: Variation of optimal exit price and stock loan value with σ at fixed δ, γ, r and Δ

American call which has positive Vega, its value is an increasing function of volatility. Therefore, a higher σ corresponds to a higher initial value of the non-recourse phase, thus, more rebate. Although the usual down-and-out option may have negative Vega near the barrier, whereas the margin call stock loan behaves like an option with a rebate, which will compensate the holder for the loss of the option if the stock price does reach the barrier. Thus, the overall effect of a higher σ is to increase the value of a margin call stock loan. Our results are in agreement with the analytically proven result in [7] that the value of a perpetual margin call stock loan increases with σ for any $r < \gamma$, which is true for a stock loan that makes financial sense.



(a) (b)

Figure 3.6: Variation of optimal exit price and stock loan value with r at fixed δ, γ, σ and Δ

The impact of the risk-free interest rate r on a margin call stock loan is depicted in Figures 3.6 (a) and 3.6(b). We could see that the optimal exit price and value of the loan are higher for lower r, which corresponds to the same trend for American down-and-out calls with rebate.

4 Conclusion

The formulation and solution procedure in this paper can be readily extended to the case of a margin call with a buffer. The generalization to a stock loan with multiple margin calls should not be too difficult either. It will involve a system with multiple margin call phases with the last one being a non-recourse phase; each margin call phase will be like an American down-and-out call with rebate, which is the difference between the initial value of the next phase and the payback for the current phase. However, numerically it would add much more complexity to the solution procedure. In practice, it is not common to have multiple margin calls, but the problem could be interesting mathematically.

References

[1] Barone-Adesi, G. and Whaley, R. E. 1987. Efficient analytic approximation of American option values. *Journal of Finance 42*, 301–320.

[2] Boyle, P. P. and Lau, S. H. 1994. Bumping up against the barrier with the binomial method. *Journal of Derivatives 1,* Summer, 6–14.

[3] Boyle, P. P. and Tian, Y. S. 1998. An explicit finite difference approach to the pricing of barrier options. *Applied Mathematical Finance 5,* 1, 17–43.

[4] Chang, G., Kang, J., Kim, H.-S., and Kim, I. J. 2007. An efficient approximation method for American exotic options. *Journal of Futures Markets 27,* 1, 29–59.

[5] Cheuk, T. H. F. and Vorst, T. C. F. 1996. Complex barrier options. *Journal of Derivatives 4*, 822.

[6] Dai, M. and Kwok, Y.-K. 2004. Knock-in American options. *Journal of Futures Markets 24,* 2, 179–192.

[7] Ekstrm, E. and Wanntorp, H. 2008. Margin call stock loans. `http://www2.math.uu.se/~ekstrom/stockloans.pdf`.

[8] Gao, B., Huang, J.-Z., and Subrahmanyam, M. 2000. The valuation of American barrier options using the decomposition technique. *Journal of Economic Dynamics and Control 24,* 1112, 1783 – 1827.

[9] Karatzas, I. and Wang, H. 2000. A barrier option of American type. *Applied Mathematics and Optimization 42*, 259–279.

[10] Kwok, Y.-K. 2008. *Mathematical Models of Financial Derivatives*, 2 ed. Springer.

[11] Lauko, M. and Sevcovic, D. 2010. Comparison of numerical and analytical approximations of the early exercise boundary of American put options. *The ANZIAM Journal 51,* 04, 430–448.

[12] Lu, X. and Putri, E. R. M. 2015. Semi-analytic valuation of stock loans with finite maturity. *Communications in Nonlinear Science and Numerical Simulation 27,* 13, 206 – 215.

[13] Lu, X. and Rhodes, P. 2010. American down and out call option valuation. Working paper.

[14] Merton, R. C. Theory of rational option pricing. *Bell Journal of Economics and Management Scienece* 4.

[15] Ndogmo, J. C. and Ntwiga, D. B. 2007. High-order accurate implicit methods for the pricing of barrier options. Working paper.

[16] Ritchken, P. 1995. On pricing barrier options. *Journal of Derivatives 3,* Winter, 19–28.

[17] Zhu, S.-P. 2006. A new analytical approximation formula for the optimal exercise boundary of American put options. *International Journal of Theoretical and Applied Finance 7,* 9, 1141 – 1177.

[18] Zhu, S.-P. and Zhang, J. 2012. Using Laplace transform to price American puts. *Dynamics of Continuous, Discrete and Impulsive Systems-B: Applications and Algorithms 19,* 4, 447–469.

[19] Zvan, R., Vetzal, K. R., and Forsyth, P. A. 1997. PDE methods for pricing barrier options. *Journal of Economic Dynamics and Control 24*, 200–0.

A Solution of a non-recourse stock loan

We recall the PDE system of the non-recourse stock loan in[12]:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + (r-\delta)S\dfrac{\partial V}{\partial S} - rV = 0, \ 0 \le S \le S_f \\ V(0,t) = 0 \\ V(S,T) = \mathtt{max}(S - qe^{\gamma T}, 0) \\ V(S_f(t),t) = S_f(t) - qe^{\gamma t} \\ \dfrac{\partial V}{\partial S}(S_f(t),t) = 1 \end{cases} \tag{A.1}$$

Applying similar variable transformation as those in Section 2.1 to the PDE system (A.1), we obtain a PDE in dimensionless variables τ, X and $\tilde{V}(X,\tau)$.

$$\begin{cases} -\dfrac{\partial \tilde{V}}{\partial \tau} + X^2 \dfrac{\partial^2 \tilde{V}}{\partial X^2} + (\alpha - \beta)X\dfrac{\partial \tilde{V}}{\partial X} - \alpha\tilde{V} = 0 \\ \tilde{V}(0,\tau) = 0 \\ \tilde{V}(X,0) = max(X-1,0) \\ \tilde{V}(X_f(\tau),\tau) = X_f(\tau) - 1 \\ \dfrac{\partial \tilde{V}}{\partial X}(X_f(\tau),\tau) = 1 \end{cases} \tag{A.2}$$

Define the following function $U = \begin{cases} \tilde{V} + 1 - X, & 1 < X \le X_f; \\ \tilde{V}, & 0 \le X \le 1. \end{cases}$

The PDE system (A.2) for the non-recourse stock loan is transformed into an ordinary differential equation (ODE) system in Laplace space, and its solution of $\bar{U}$ is given by:

$$\bar{U} = \begin{cases} \dfrac{k_2(p\bar{X}_f)^{k_2-k_1}}{k_2-k_1} W X^{k_1} + \dfrac{\beta(p\bar{X}_f)^{1-k_2}}{k_2 p(p+\beta)} X^{k_2} + \dfrac{p(\alpha - \beta p\bar{X}_f) + \alpha\beta(1-p\bar{X}_f)}{p(p+\alpha)(p+\beta)}, & \text{if } 1 < X \le X_f \\ \left\{(1-(p\bar{X}_f)^{k_2-k_1})\dfrac{k_2}{k_2-k_1}W + \dfrac{(p\bar{X}_f)^{1-k_2}\beta + p}{k_1 p(p+\beta)}\right\} X^{k_1}, & \text{if } X \le 1 \end{cases} \tag{A.3}$$

where k_1 and k_2 are the solutions of the equation $\frac{1}{2}\sigma^2 k^2 + (r - \delta - \frac{1}{2}\sigma^2)k - (p+r) = 0$, and $W = \frac{\beta-\alpha}{(p+\alpha)(p+\beta)} + \frac{(p\bar{X}_f)^{1-k_2}\beta + p}{k_1 p(p+\beta)} - \frac{(p\bar{X}_f)^{1-k_2}\beta}{k_2 p(p+\beta)}$.

In order to compute the value of $\bar{U}(X,p)$, we need to numerically solve for the optimal exit price $\bar{X}_f(p)$ from the following equation first.

$$\bar{X}_f^{k_2}\left\{\frac{k_1(\beta-\alpha)+(p+\alpha)}{k_1(p+\alpha)(p+\beta)}\right\} + \bar{X}_f\left\{\frac{\beta(1-k_1)}{k_1 p^{k2}(p+\beta)}\right\} = -\frac{\alpha}{p^{1+k_2}(p+\alpha)} \tag{A.4}$$