An alternative form used to calibrate the Heston option pricing model

Xin-Jiang He
University of Wollongong, xh016@uowmail.edu.au

Song-Ping Zhu
University of Wollongong, spz@uow.edu.au
An alternative form used to calibrate the Heston option pricing model

Abstract
This paper presents an alternative form of the Heston model that preserves an essential advantage of the Heston model, its analytic tractability, by imposing the necessary and sufficient conditions for the existence of a solution in affine form, while it is in a different form so that it offers certain advantages in parameter determination. To demonstrate this, we conducted some empirical studies, exploring if this new form does have certain advantages over the original version under certain market conditions.

Disciplines
Engineering | Science and Technology Studies

Publication Details
An alternative form used to calibrate the Heston option pricing model

Xin-Jiang HE *   Song-Ping ZHU †

Abstract

This paper presents an alternative form of the Heston model that preserves an essential advantage of the Heston model, its analytic tractability, by imposing the necessary and sufficient conditions for the existence of a solution in affine form, while it is in a different form so that it offers certain advantages in parameter determination. To demonstrate this, we conducted some empirical studies, exploring if this new form does have certain advantages over the original version under certain market conditions.

AMS(MOS) subject classification.

Keywords. Alternative form, Heston model, affine solution, Empirical studies.

1 Introduction

There is a long history in the development of the option valuing problem, which is basic and essential in risk management today. Bachelier [2] seems to be the first person to

---

*School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia
†Corresponding author. School of Mathematics and Applied Statistics, University of Wollongong NSW 2522, Australia.
use Brownian motion to model stock price and value stock options, which was provided in his PhD thesis in 1900. However, he assumed that the stock price follow a Brownian motion with normal distribution, which would lead to negative stock and option prices, as pointed out by Merton [32]. After more than 70 years’ research and development, Black & Scholes [7] finally made a landmark contribution by presenting a simple and closed-form pricing formula after some simple assumptions were made to capture the essence of the problem while preserving analytical tractability. However, the implied volatility from the real market data tends to exhibit a curve of the shape of a “smile” or “smirk”, referred to as the volatility smile in the literature (e.g., [17]), which is at odds with one of those basic assumptions, i.e. the constant volatility, in Black-Scholes model. As a result, this has stimulated widely-spread research interest in proposing various forms of non-constant volatility processes in option pricing models to avoid the apparent “paradox”.

Among many attempts to modify the “constant volatility” assumption in the Black-Scholes model, the two most natural ones are either to choose a deterministic function of underlying and time as the volatility, called “local volatility model” or make volatility another random variable described by a stochastic process, called “stochastic volatility model”. While the former was proposed by Dupire [18] and Derman & Kani [15], with the deterministic volatility function being determined from the well known Dupire formula, the latter is much more popular now among market traders and academic researchers since many empirical studies suggest the “smile dynamics” are poorly captured by the local volatility model (e.g., Hagan et. al [23]).

Models in the category of “stochastic volatility” were first systematically studied by Johnson & Shanno[26], Scott [34], and Wiggins [40] with numerical methods. Specifically, Monte Carlo simulation was adopted by Johnson & Shanno and Scott, while Wiggins proposed that the finite difference method be adopted in solving the corresponding PDEs for pricing financial derivatives, such as options. Unfortunately, neither of them is satisfactory

\[1\] Recently, there are hybrid models that combine both [38].
due to the lack of closed-form solution, which could make it quite time-consuming when operating in real markets. Furthermore, although Hull & White [25] proposed a simple form of stochastic volatility process and adopted a power series approximation method, one of its main drawbacks is the zero correlation assumption. This is not reasonable since it violates the so-called “leverage effects” that the underlying price and the volatility should be negatively correlated [4]. Another well-known model is presented by Stein & Stein [37] several years later, who assumed that the volatility follow an Ornstein-Uhlenbeck process [21] and derived a closed-form pricing formula. However, except the assumption of no correlation between the underlying price and volatility, this model could not prevent the volatility from going negative, which was certainly not appropriate. Finally, a great progress was made by Heston [24] in 1993, who proposed the correlated stochastic volatility model and derived an analytic solution based on the inverse Fourier transform. Two aspects can account for the success of the Heston model; one is that the volatility process itself satisfies a wide range of basic properties, such as the obvious non-negative property and the mean-reverting property being consistent with the results of empirical studies [5], and another is that there exists a closed-form formula when pricing options, which can bring a number of advantages. In particular, with closed-form solutions, computational accuracy could certainly be guaranteed while there would exist systematical errors when numerical solutions must be resorted to for models that no closed-form solutions are associated with. Most importantly, having closed-form solution can spare us considerable amount of time and effort in parameter estimation, a vitally important process for any mathematical model to function properly as model parameters always need to be extracted from real market data during model calibration.

In this paper, we propose an alternative form of the Heston model based on a proof for the necessary and sufficient conditions to obtain an affine solution, and thus the new form also captures the essential ingredients of the original Heston version. In particular, our formula preserves the analytical tractability and has substantially reduced the computational
effort in terms of parameter determination. To make this new form more attractive, the Feller condition and the non-explosion condition are also imposed on the parameter space to guarantee that the volatility will not drop below and reach infinity respectively. The mean-reverting property is also preserved since it is consistent with real market data.

Analytically, this form is actually equivalent to the original Heston version and there would be no difference of using this form or the original Heston model, if one could determine all model parameters analytically. However, the calibration of a model in reality is so complicated that model parameters always need to be determined numerically with an optimization algorithm. In the latter case, one would never be able to obtain the “optimized” set of parameters, but probably would have to settle near it. It is for this reason that we shall show, through some empirical evidence, that the newly proposed form may yield better results than the original one in some cases. Let’s use a simple example to illustrate this point. Imagine that we want to minimize an objective function \( g(y) = (\frac{y}{3})^2 - 2(\frac{y}{3}) + 1 \) numerically with an optimization algorithm being adopted to find the optimal solution \( y^* \). In doing so, any numerical algorithm needs to impose a stopping criterion so that the search of the optimal solution would cease, once an approximation of the optimal solution is close to the true one within a pre-given tolerance level \( \epsilon \). One of the common choices is that when the changing amount, i.e. \( |\frac{g(y_{n+1}) - g(y_n)}{g(y_n)}| \), is lower than a chosen \( \epsilon \) in the searching process, the algorithm will stop and return the optimal \( y_n \). If this is the case for this example, the optimization would stop when \( |g'(y)| = |\frac{2}{9}y - \frac{2}{3}| = \epsilon \), and thus \( y^* = 3 + \frac{9}{2}\epsilon \) can be easily obtained as the optimal solution. On the other hand, if we make a transformation of \( x = \frac{y}{3} \) first to the undetermined parameter, it is not difficult to work out that the returned point then becomes \( x = 1 + \frac{1}{2}\epsilon \), which implies that the optimal solution will be \( y^* = 3 + \frac{3}{2}\epsilon \). Clearly, the obtained optimal solution with a simple transformation has made a difference; the latter is closer to the true optimal solution. Based on this simple concept, we propose a different form of the Heston model and demonstrate the possible advantages of adopting this form in some cases through an empirical study.
The rest of the paper is organized as follows. In Section 2, following a brief introduction of general underlying dynamics and the proved conditions for the existence of an affine solution, a closed-form pricing formula for European call options is presented. In Section 3, some necessary parameter restrictions such as Feller condition, non-explosion condition and the mean-reverting property are imposed. In Section 4, the results of some preliminary empirical studies for a comparison of the performances between our form and the original Heston version are discussed, followed by some concluding remarks given in the last section.

2 A new form of the Heston model

In this section we firstly introduce a general stochastic volatility model and then provide the necessary and sufficient conditions for the existence of an affine solution to the governing PDE (partial differential equation) of the option price.

Let \( S_t \), \( t \geq 0 \) denote the underlying price and \( \{v_t, t \geq 0\} \) represent the dynamic of the volatility. Then the general stochastic volatility model under the risk-neutral measure is characterized as

\[
\frac{dS}{S} = r dt + v dW_t, \\
dv = \lambda(v) dt + \sigma v dB_t,
\]

where \( W_t \) and \( B_t \) are two standard Brownian motions with correlation \( \rho \). It is obvious that \( \alpha \neq 0 \), otherwise the underlying asset is not related to the stochastic volatility.

Here, this particular model can be regarded as a general one since it includes a number of different stochastic volatility models and three examples are listed below to further illustrate this point. First of all, if \( \alpha = \beta = \frac{1}{2} \) and \( \lambda(v) = k(\theta - v) \), our model will surely degenerate to the Heston model. Secondly, if \( \beta \) is set to be zero and the values of other two terms, i.e. \( \alpha \) and \( \lambda(v) \), remain unchanged as the first case, the model is then the
Stein-Stein model. In addition, when $\alpha$ and $\beta$ take the value of $\frac{1}{2}$ and $\frac{3}{2}$ respectively, and $\lambda(v) = kv(\theta - v)$, this model degenerates to another well-known model, the so-called $\frac{3}{2}$ model [30].

Now let $U(S, v, t)$ denote the European call option price written on the underlying asset $S_t$, then according to the martingale pricing theory, which requires that $e^{-rt}U_t$ be a martingale, we can obtain the following PDE

$$\frac{1}{2}v^{2\alpha}S^2\frac{\partial^2 U}{\partial S^2} + \rho \sigma v^{\alpha+\beta}S \frac{\partial^2 U}{\partial v \partial S} + \frac{1}{2} \sigma^2 v^{2\beta} \frac{\partial^2 U}{\partial v^2} + rS \frac{\partial U}{\partial S} + \lambda \frac{\partial U}{\partial v} - ru + \frac{\partial U}{\partial t} = 0,$$

(2.2)

with boundary conditions

$$U(S, v, T) = \max(S - K, 0),$$

$$U(0, v, t) = 0,$$

$$\lim_{S \to \infty} \frac{U(S, v, t)}{S} = 1,$$

for the price of a European call option. Then, based on the form of Black-Scholes formula, we assume that the solution to PDE (2.2) takes the form of

$$U = SP_1(S, v, t) - Ke^{r(T-t)}P_2(S, v, t),$$

(2.3)

with $K$ as the strike price. As a result, by substituting Equation (2.3) into (2.2) and applying the transform $x = \ln S$ we can finally arrive at

$$\frac{1}{2}v^{2\alpha}P_2 \frac{\partial^2 P_j}{\partial x^2} + \rho \sigma v^{\alpha+\beta} \frac{\partial^2 P_j}{\partial x \partial v} + \frac{1}{2} \sigma^2 v^{2\beta} \frac{\partial^2 P_j}{\partial v^2} + (r + a_jv) \frac{\partial P_j}{\partial x} + [\lambda(v) + b_jv^{\alpha+\beta}] \frac{\partial P_j}{\partial v} + \frac{\partial P_j}{\partial t} = 0,$$

(2.4)
where \( u_j = \frac{(-1)^j}{2} \) and \( b_j = \rho \sigma (2 - j) \) for \( j = 1, 2 \). As a result, the terminal condition for \( P_j \) becomes

\[
P_j(x, v, T) = I_{\{x > \ln K\}}.
\]

To solve PDE (2.4), we only need to find the characteristic function of \( x_T \) conditional on \( x_t \) and \( v_t \) denoted by \( f_j(x, v, t; \phi) \) satisfying the same PDE as \( P_j(x, v, t; \ln [K]) \) for \( j = 1, 2 \) respectively. Actually, according to the results in [24] with an affine structure solution in the closed-form pricing formula, we also try to seek a solution of \( f_j(x, v, t; \phi) \) in a particular affine form [24] with respect the \( v^\theta \), which is stated in the following theorem.

**Theorem 1** Let \( f_j(x, v, t; \phi) \) be the solution to (2.4) with terminal condition \( f_j(x, v, T; \phi) = e^{i\phi x} \), then \( f_j(x, v, t; \phi) \) takes the affine form

\[
f(x, v, \tau; \phi) = e^{C(\tau; \phi) + D(\tau; \phi)v^\theta + i\phi x}, \tag{2.5}
\]

with arbitrary \( \theta \) and if and only if \( \alpha + \beta = 1 \), \( \lambda(v) = \lambda_1 v + \lambda_2 v^{1 - 2\alpha} \), \( \theta = 2\alpha \). Here, \( \tau = T - t \), and the two functions \( C(\tau; \phi) \) and \( D(\tau; \phi) \) are set to be independent of \( x \) and \( v \).

**Proof.** By substituting (2.5) into (2.4) and after some simplifications, we can reach

\[
\frac{\partial C}{\partial t} + iv\phi + \left[ -\frac{1}{2} \phi^2 + iu_j \phi \right] v^{2\alpha} + \frac{\partial D}{\partial t} v^\theta + (i\phi \rho \sigma + b_j)\theta D v^{\alpha + \theta + \theta - 1} \nonumber
\]

\[
+ \frac{1}{2} \sigma^2 \theta (\theta - 1) D v^{2\beta + \theta - 2} + \frac{1}{2} \sigma^2 \theta^2 v^{2\beta + 2\theta - 2} + \theta D \lambda(v) v^{-\theta - 1} = 0. \tag{2.6}
\]

As mentioned before that \( \alpha \neq 0 \), we can obviously obtain \( 2\alpha \neq 0 \). If we assume that \( 2\alpha \neq \theta \), then by setting \( \tau = 0 \) we can obtain

\[
\frac{1}{2} \phi^2 v_0^{2\alpha} + (r + u_j v_0^{2\alpha})i\phi + \left. \frac{\partial C}{\partial t} \right|_{\tau=0} + \left. \frac{\partial D}{\partial t} \right|_{\tau=0} v_0^\theta = 0,
\]
which holds for any $v_0$. Therefore the coefficients of $v^{2\alpha}$ should be zero, i.e.

$$\frac{1}{2} \phi^2 + ia_j \phi = 0,$$

which is not always true. As a result, our assumption was incorrect and we should conclude with

$$2\alpha = \theta.$$  \hfill (2.7)

Now we assume that $2\beta + 2\theta - 2 \neq \{0, \theta\}$, and then we can deduce that the coefficients of $v^{2\beta+2\theta-2}$ should satisfy

$$\frac{1}{2} \sigma^2 \theta^2 D^2 + kD = 0,$$

where $k$ is an arbitrary complex constant. To seek non-trivial solution, we must obtain

$$D = -\frac{k}{\frac{1}{2} \sigma^2 \theta^2},$$

which means that $D$ is not related to $\tau$. Thus it contradicts to the fact that $D(0) = 0$. As a result, our assumption is again incorrect and we have

$$2\beta + 2\theta - 2 = \{0, \theta\}.$$  \hfill (2.8)

To complete the proof, two cases need to be considered here.

(1) $2\beta + 2\theta - 2 = 0$.

Combining Equation (2.7) and the condition $2\beta + 2\theta - 2 = 0$, Equation (2.6) can be further simplified as

$$(-\frac{1}{2} \phi^2 + ia_j \phi + \frac{\partial D}{\partial t})v^{2\alpha} + 2\alpha D(i\phi \rho \sigma + b_j)v^\alpha + \alpha(2\alpha - 1) \sigma^2 D v^{-2\alpha}$$

$$+ (2\alpha^2 \sigma^2 D^2 + ir \phi + \frac{\partial C}{\partial t}) + 2\alpha \lambda(v) D v^{2\alpha - 1} = 0,$$
which should hold for any $v$. As a result, the coefficient of the $v^\alpha$ should be equal to zero, i.e.

$$ 2\alpha D(i\phi \rho \sigma + b_j + \lambda^0) = 0. \quad (2.9) $$

Here $\lambda^0$ should be the coefficient of the element $v^{1-\alpha}$ in $\lambda(v)$, which is an arbitrary real number, and should take the value to make the above Equation (2.9) hold. Therefore,

$$ \lambda^0 = -b_j - i\phi \rho \sigma. $$

This means that $\lambda^0$ is not a real number, which is at odds with the fact that $\lambda^0$ is supposed to be a real number stated above. Hence, we have no such solution in this situation.

(2) $2\beta + 2\theta - 2 = \theta$.

Now we have

$$ \alpha + \beta = 1, \quad (2.10) $$

and thus Equation (2.6) becomes

\[
\left[ \frac{\partial D}{\partial t} \right] + 2\alpha^2 \sigma^2 D^2 + 2\alpha(i\phi \rho \sigma + b_j)D + (ia_j \phi - \frac{1}{2} \phi^2)]v^{2\alpha} \\
+ \left[ \frac{\partial C}{\partial t} \right] + \alpha(2\alpha - 1)\sigma^2 D + ir \phi]v^0 + 2\alpha D\lambda(v)v^{2\alpha-1} = 0, \quad (2.11)
\]

which should hold for any $v$. Therefore, $\lambda(v)$ must satisfy

$$ \lambda(v) = \lambda_1 v + \lambda_2 v^{1-2\alpha}. \quad (2.12) $$

Otherwise, with another term $f(v)$ being added in $\lambda(v)$ in (2.12), it would certainly lead to a conclusion that $f(v) \equiv 0$ due to the arbitrariness of $v$.

According to condition (2.7), (2.10) and (2.12), we finally obtain the desired result. This has completed the proof.
With Theorem 1 being verified, it is clear that to make the general model (2.1) to possess such an affine structure solution, the pricing dynamics is specified as

\[
\frac{dS}{S} = r dt + v^\alpha dW_t, \\
dv = (\lambda_1 v + \lambda_2 v^{1-2\alpha}) dt + \sigma v^{1-\alpha} dB_t.
\] (2.13)

In this case, our closed-form pricing formula is presented as follows, which is the same as Equation (2.3), and we have left the proof in the Appendix since the derivation process is similar to that of the Heston model.

\[ U = SP_1(S, v, t) - Ke^{-r(T-t)} P_2(S, v, t), \] (2.14)

where

\[
P_j = \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} RE\left[ \frac{e^{-i\phi ln K}}{i\phi} \right] f_j d\phi, \\
f_j(x, v, \tau; \phi) = e^{C(\tau; \phi) + D(\tau; \phi) v^{2\alpha} + i\phi x}, \\
C(\tau; \phi) = r i \phi \tau + \frac{\alpha(2\alpha - 1)\sigma^2}{4\alpha^2\sigma^2} p(\tau; \phi), \\
D(\tau; \phi) = \frac{d - 2\alpha(b_j + \lambda_1 + i\phi \rho \sigma)}{4\alpha^2\sigma^2} \cdot \frac{1 - e^{d\tau}}{1 - ge^{d\tau}}, \\
p(\tau; \phi) = [d - 2\alpha(b_j + \lambda_1 + i\phi \rho \sigma)] \tau - 2ln\left[ \frac{1 - ge^{d\tau}}{1 - g} \right], \\
d = 2\alpha \sqrt{(b_j + \lambda_1 + i\phi \rho \sigma)^2 - \sigma^2(2u_j i\phi - \phi^2)}, \\
g = \frac{2\alpha(b_j + \lambda_1 + i\phi \rho \sigma) - d}{2\alpha(b_j + \lambda_1 + i\phi \rho \sigma) + d}, \\
u_j = \frac{(-1)^j}{2}, \quad b_j = \rho \sigma (2 - j),
\]

for \( j = 1, 2. \)

Although the closed-form solution is obtained under certain settings, we still need to check whether our model is suitable to be applied in real markets. In fact, the volatility...
process is expected to be mean-reverting, and it should also be bounded and never fall below zero. All of these features will be discussed in the next section.

3 Parameter restrictions

Like some other closed-form solutions, there are usually some restrictions that need to be imposed in the parameter space [16, 24]. To ensure that the volatility will never become negative nor reach infinity, the Feller condition [1, 19] and the non-explosion condition [16, 28] are imposed respectively. Moreover, according to some empirical studies [5], volatility displays the mean-reverting trend and thus this would further give some limitations on the parameter space of our model in order to show this property. As a result, we will put forward and verify several propositions below to set limitations for parameters in our model to meet these requirements.

Proposition 3.1 If we impose \( \alpha \in (-\infty, \frac{1}{2}) \) or \( \lambda_2 \geq \sigma^2 (1 - \alpha) \) and \( \alpha \in \left[ \frac{1}{2}, 1 \right) \), volatility will always stay non-negative.

Proof. Let us first denote \( \beta(v, t) = \lambda_1 v + \lambda_2 v^{1-2\alpha} \) and \( c(v, t) = \sigma v^{1-\alpha} \) in our model. According to the Feller condition, we have the assumption that \( \beta(0, t) \geq 0 \) and \( c(0, t) = 0 \), which imply

\[
\alpha < 1,
\]

and \( \lambda_2 \) should be non-negative when \( \alpha \in \left[ \frac{1}{2}, 1 \right) \). To ensure that volatility remains non-negative, the Feller condition specified in the following should be satisfied

\[
\lim_{v \to 0} \beta(v, t) - \frac{1}{2} \frac{\partial c^2}{\partial v} \geq 0,
\]
the LHS (left hand side) of which can be calculated as

$$\lim_{v \to 0} \left[ \beta(v) - \frac{1}{2} \frac{\partial^2 c}{\partial v^2} \right] = \lim_{v \to 0} \{ \lambda_1 v + \lambda_2 v^{1-2\alpha} - (1 - \alpha) \sigma^2 v^{1-2\alpha} \},$$

$$= \lim_{v \to 0} [\lambda_2 - \sigma^2(1 - \alpha)] v^{1-2\alpha}. \quad (3.3)$$

Thus we can immediately get the condition we need according to (3.2) and (3.3). This has completed the proof.

**Proposition 3.2** If we impose \( \alpha > -\frac{1}{2} \), or \( \alpha \in (-1, -\frac{1}{2}] \) and \( \lambda_2 \leq \sigma^2 \), volatility will obey the non-explosion condition, which means that the volatility will be bounded.

**Proof.** It is obvious that \( v \) never reaches infinity is equivalent to \( \frac{1}{v} \) will never take negative values. By applying the Itô lemma and setting \( u = \frac{1}{v} \) we have

\[
\begin{align*}
du & = \frac{d}{v}, \\
& = -v^{-2} dv + v^{-3} (dv)^2, \\
& = (-\lambda_1 v^{-1} - \lambda_2 v^{-1-2\alpha} + \sigma^2 v^{-1-2\alpha}) dt - \sigma v^{-1-\alpha} dB_t, \\
& = [-\lambda_1 u + (\sigma^2 - \lambda_2) u^{1+2\alpha}] dt - \sigma u^{1+\alpha} dB_t.
\end{align*}
\]

Now we set \( \beta(u, t) = -\lambda_1 u + (\sigma^2 - \lambda_2) u^{1+2\alpha} \) and \( c(u, t) = -\sigma u^{1+\alpha} \). According to the assumption in the Feller condition, we should have \( \beta(0, t) \geq 0 \) and \( c(0, t) = 0 \), which imply

\[
\alpha > -1, \quad (3.4)
\]

and

\[
\begin{cases}
\lambda_2 \in (-\infty, +\infty), & \alpha > -\frac{1}{2}, \\
\lambda_2 \in (-\infty, \sigma^2), & \alpha \leq -\frac{1}{2},
\end{cases}
\]
respectively. Now again imposing the Feller condition, the following should be satisfied

\[
\lim_{u \to 0} \beta(u, t) - \frac{1}{2} \frac{\partial^2}{\partial u} = \lim_{u \to 0} (-\alpha \sigma^2 - \lambda_2)u^{1+2\alpha} \geq 0. \tag{3.5}
\]

As a result, condition (3.5) yields

\[
\begin{cases}
\lambda_2 \in (-\infty, +\infty), & \alpha > -\frac{1}{2}, \\
\lambda_2 \in (-\infty, -\alpha \sigma^2), & \alpha \leq -\frac{1}{2}.
\end{cases}
\]

Combining all the conditions above, we will finally arrive at what we try to prove.

**Proposition 3.3** If we impose \(\lambda_1 > 0, \lambda_2 < 0\) and \(\alpha < 0\), or \(\lambda_1 < 0, \lambda_2 > 0\) and \(\alpha > 0\), the volatility process is mean-reverting.

**Proof.** Recall that the stochastic volatility follows

\[
dv = (\lambda_1 v + \lambda_2 v^{1-2\alpha})dt + \sigma v^{1-\alpha}dB_t. \tag{3.6}
\]

To set it mean-reverting, we should consider the ODE below

\[
dv = (\lambda_1 v + \lambda_2 v^{1-2\alpha})dt, \tag{3.7}
\]

which can be simplified as

\[
\frac{\frac{1}{2\alpha} dv^{2\alpha}}{\lambda_1 v^{2\alpha} + \lambda_2} = dt. \tag{3.8}
\]

Therefore, ODE (3.8) can be easily solved as

\[
v^{2\alpha} = C e^{2\alpha \lambda_1 t} - \frac{\lambda_2}{\lambda_1}. \tag{3.9}
\]

The following two cases are considered to find the needed restriction on parameters for the volatility process to become mean-reverting.

13
(1) $\alpha < 0$.

In this scenario, when we set $\lambda_1 < 0$,

$$\lim_{t \to +\infty} v^{2\alpha}_t = \lim_{t \to +\infty} Ce^{2\alpha \lambda_1 t} - \frac{\lambda_2}{\lambda_1} = \infty,$$

which is not suitable. When we set $\lambda_1 > 0$,

$$\lim_{t \to +\infty} v^{2\alpha}_t = \lim_{t \to +\infty} Ce^{2\alpha \lambda_1 t} - \frac{\lambda_2}{\lambda_1} = -\frac{\lambda_2}{\lambda_1}.$$

To keep $v$ non-negative, obviously $\lambda_2 < 0$.

(2) $\alpha > 0$.

Following the similar law of the first situation, to keep the mean-reverting property in this case, we can easily obtain

$$\lambda_1 < 0, \quad \lambda_2 > 0. \quad (3.10)$$

This has completed the proof.

In conclusion, to meet the requirement of the Feller condition, non-explosion condition and the mean-reverting property for the volatility, $\alpha$ should be limited in $(-1, 1)$ and other parameters should be restricted as

$$\left\{ \begin{array}{ll}
\lambda_1 > 0, \lambda_2 < 0, & \alpha \in (-1, 0), \\
\lambda_1 < 0, \lambda_2 > 0, & \alpha \in (0, \frac{1}{2}), \\
\lambda_1 < 0, \lambda_2 \geq \sigma^2 (1 - \alpha), & \alpha \in \left[ \frac{1}{2}, 1 \right].
\end{array} \right.$$  

Although some people may argue that if we make the transformation of $\tilde{v}_t = v^{2\alpha}_t$, $\tilde{\sigma} = 2\alpha \sigma$ and $k = -2\alpha \lambda_1$ together with $\theta = \frac{\frac{1}{2}(\alpha(1 - 2\alpha)\sigma^2 - 2\alpha \lambda_2)}{2\alpha \lambda_1}$, our form will
degenerate to the following original Heston version

\[
\begin{align*}
\frac{dS}{S} &= rdt + \sqrt{\bar{v}}dW_t, \\
\bar{v} &= k(\theta - \bar{v})dt + \sigma\sqrt{\bar{v}}dB_t,
\end{align*}
\]

our work is still meaningful since as stated before, our form could yield better results than the original Heston version under certain conditions. To further demonstrate this point, the behavior of our form in real markets will be compared with the original Heston version through a carefully designed empirical study in the next section.

4 Empirical studies

In this section, we shall present and discuss the results of an empirical study, aimed to benchmark the performance between our form and that of the Heston version in terms of the closeness between the calculated option prices with model parameters extracted from the “historical data” and market prices, in order to show whether it is meaningful to propose another form of the Heston model.

4.1 Data description

Our empirical study was conducted on the data of S&P 500 European call options for two separate periods with each period containing two-year data. Actually, one period was deliberately chosen during the financial crisis (between Jan 2007 and Dec 2008), while another was selected for a post-crisis period from Sept 2011 to Aug 2013, the market during which can be viewed as under a normal condition. To simplify the calculations without losing key information for the purpose of this study, we took the average value of bid and ask prices as the option price.

In order to eliminate sample noise in the estimation of parameters, appropriate filters
were applied to the raw data. First, following Bakshi et al [4] and Christoffersen et al [11], only Wednesday options data is used in the stage of parameter estimation since Wednesday is least likely to be a holiday in a week and also less likely to be affected by day-of-the-week effect. Moreover, since global optimization problems are quite time consuming, choosing one day a week helped us to effectively reduce the size of the data set so that a longer time series can be included in the process of parameter determination. Second, options with time to maturity less than 30 days were discarded since they usually possess less time value and less information about the future dynamics of the firm [29]. Options with more than 120 days to expiry were also excluded because there would be a high premium if they are traded, which makes them unpopular. Third, very deep in-the-money and very deep out-of-money options were discarded since they are not active in the market and may have liquidity-related biases [36]. More specifically, if moneyness is defined as the percentage difference between the S&P 500 Index value and the corresponding strike price, i.e. $Moneyness = \frac{S - K}{K}$, then options with the absolute value of moneyness over 10% were excluded. Finally, options with prices less than $1/8$ were all removed since these prices are rather volatile [14] and such abnormal volatility may result in unusual option prices. As a summary, the numbers of observations for original data and filtered data are reported in Table 1. It needs to be emphasized that by applying these filters, all the important information is still preserved, while higher efficiency and accuracy can be obtained in the process of parameter estimation, according to various empirical studies conducted before.

<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Original</td>
<td>112144</td>
<td>158732</td>
<td>294825</td>
<td>382637</td>
</tr>
<tr>
<td>Filtered(Whole week)</td>
<td>22095</td>
<td>23935</td>
<td>35452</td>
<td>48713</td>
</tr>
<tr>
<td>Filtered(Only Wednesday)</td>
<td>4684</td>
<td>5202</td>
<td>7235</td>
<td>9403</td>
</tr>
</tbody>
</table>

An important parameter that needs to be determined first is the risk-free interest rate. As time to maturity of options used is less than 120 days, it is reasonable to choose the
three-month U.S. Treasury Bill Rate (T-Bill rate) released daily as a proxy of the risk-free rate [6, 36].

The total set of data is divided into two consecutive subsets, with the 1st one being used for parameter determination, while the 2nd one is used for comparison. Data in the first period of each example is referred to as in-sample observations which are used for parameter determination, whereas data in the latter period serves for the purpose of out-of-sample comparison. In the current study, both periods are of the duration of one year. To illustrate it more clearly, three steps are listed as follows with the first period (2007.1-2008.12) as an example.

- First of all, filtered Wednesday data during the period of Jan 2007 to Dec 2007 was used as the input to estimate model parameters.

- Second, with a genetic algorithm, parameters in our form and the original Heston version were determined respectively and in-sample errors were calculated based on the obtained “optional” parameters for this period.

- Finally, by applying the filtered Wednesday and whole week data from Jan 2008 to Dec 2008, out-of-sample errors were calculated so that the performance of the two forms could be compared.

4.2 Parameter estimation

Model comparison empirically involving market data always begins with parameter estimation, which itself is a difficult problem. One of the most commonly adopted approaches is to find the parameter set that minimizes the distance between model and market prices. In our empirical study, a genetic algorithm was adopted to search for a solution of global minimization from the S&P 500 Index and options, in order to compare the pricing performance of the two forms.
To find the “optimal” parameter set that best fits the chosen market data, what we need to do is to minimize the distance between market and model prices. Therefore, one of the most important steps is to choose an appropriate objective function (loss function) [9]. Following Christoffersen & Jacobs [10] and Lim & Zhi [31], we adopt the dollar mean-squared errors defined as

\[
MSE = \frac{1}{N} \sum_{i=1}^{N} (C_{\text{Market}} - C_{\text{Model}})^2,
\]

where \(C_{\text{Market}}\) denotes the market price of an option contract, \(C_{\text{Model}}\) represents the corresponding calculated price with the pricing formula by using a particular set of parameters, and \(N\) is the total number of observations selected for parameter estimation. It should be noted that the objective function (4.1) is not necessarily convex and there may exist several local minima. In this case, if a local optimization approach, such as the non-linear least squared method, is employed, we would not be sure whether the solution is a local minimum or a global one. Furthermore, if a local one is reached, it is still hard for us to attain the global solution and thus there is no point of using any local minimization technique. As a result, a global optimization is preferred, in which some stochastic factors are generally introduced in their search process. This means that it will not stop searching when it finds a potential solution.

In fact, the method for optimization we adopted is a genetic algorithm [12], which is one of the most popular types of evolutionary algorithm [27] and based on the idea of natural selection. One of the most important qualities in the algorithm is that random changes are made to the potential solution to check whether there would be an improvement, instead of following the known information to determine the next step. This very special feature has made genetic algorithms very reliable. In fact, there are a number of financial applications of genetic algorithms. For example, Sefiane & Benbouziane [35] adopted a genetic algorithm in optimal portfolio selection while Gimeno & Nave conducted estimation
of the term structure of interest rates with genetic algorithms. Furthermore, it has also been applied in the area of option pricing [13, 22]. It is even pointed out by Bajpai & Kumar [3] that genetic algorithms are one of the best global optimization methods and can provide high quality solutions since they are intrinsically parallel and can explore the solution space in multiple directions each time.

In our study, the adopted genetic algorithm is Matlab built-in function `ga`, which has made it easy for us to implement. In Tables 2 and 3, estimation results obtained with data in a financial crisis and a normal market are exhibited, respectively.

**Table 2: Estimation results with option data ranging from 2007.1 to 2007.12**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$v_0$</th>
<th>$\alpha$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our form</td>
<td>0.6861</td>
<td>-0.7082</td>
<td>0.0981</td>
<td>0.9008</td>
<td>-16.4877</td>
<td>0.1882</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heston</td>
<td>0.6684</td>
<td>-0.6223</td>
<td>0.0188</td>
<td></td>
<td></td>
<td></td>
<td>7.3249</td>
<td>0.0305</td>
</tr>
</tbody>
</table>

**Table 3: Estimation results with option data ranging from 2011.9 to 2012.8**

<table>
<thead>
<tr>
<th>Parameters</th>
<th>$\sigma$</th>
<th>$\rho$</th>
<th>$v_0$</th>
<th>$\alpha$</th>
<th>$\lambda_1$</th>
<th>$\lambda_2$</th>
<th>$k$</th>
<th>$\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Our form</td>
<td>0.7224</td>
<td>-0.7093</td>
<td>0.0781</td>
<td>0.6911</td>
<td>-7.6438</td>
<td>0.2637</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Heston</td>
<td>0.8775</td>
<td>-0.6916</td>
<td>0.0298</td>
<td></td>
<td></td>
<td></td>
<td>7.5539</td>
<td>0.0510</td>
</tr>
</tbody>
</table>

The estimated parameters in Table 2 and Table 3 all appear to be in very reasonable range. For example, $\theta$ in the Heston model denotes the long-term mean of the volatility process and the extracted values are 0.0305 for the period with a financial crisis and 0.0510 for the normal period, respectively. On the other hand, the long-term mean for our form can be calculated from $-\frac{\lambda_2}{\lambda_1}$ and thus the results are 0.0114 and 0.0345 for the first and second example, respectively. These results are rather close to the one reported in [4] at 0.04. It is also interesting to notice that the long-term mean of the volatility in financial crisis is lower than that in the normal market for both forms.

Moreover, our estimated values of the so-called volatility of volatility range from 0.6684 to 0.8775, which are a little larger than those in Bakshi [4] and Eraker [20] using joint data of underlying returns and option prices. However, our results are quite similar to the results obtained in more recent studies [33]. Also, it should be noticed that 15 different sets
of parameters were obtained by Christoffersen et al. [8], who repeatedly used yearly option data as the input of the estimation between 1990 and 2004. According to their results, the value for the volatility of volatility can be as low as 0.3796 and as high as 0.8516. So, values we obtained for volatility of volatility are quite reasonable.

In addition, our estimation of the "leverage effect" $\rho$ for the two models in two different cases is quite similar, ranging from -0.7093 to -0.6223. This result is rather satisfactory since most of the empirical studies show that the value for $\rho$ was negative [4, 8, 41, 42]. In particular, if we again compare with the results in [8], it provided a quite wide range from -0.8519 to -0.5061, in which range our result is included.

With the reasonable parameters extracted from the filtered data, we are now ready to compare the performance of our form and the original Heston model with these reported parameters, which is presented in the next subsection.

### 4.3 Empirical comparison

Once model parameters have been estimated, it is natural for us to empirically compare performance of these two forms. It is obvious that we regard the performance of a model better if it results in lower pricing differences between the calculated option prices with model and the corresponding market prices. Specifically, root mean-squared error (RMSE) is adopted as a measure, which is the square root of the objective function (MSE), to reflect the pricing difference for both in-sample and out-of-sample comparison.

As for the in-sample comparison, it is clear that for both examples, the performance of the two forms is really similar. In fact, the Heston version performs slightly better than our form from the perspective of in-sample comparison. Specifically, the RMSE of the Heston version is 8.2767, while that for our form is 8.2954 in the period of financial crisis (2007.1-2007.12) with the average call option price being $51.6222. As for the normal market (2011.9-2012.8), the average call option price is lower at $46.5101. In this case,
although the absolute value of the RMSE for the two models increases, the gap between them is narrowed down, with 9.2802 and 9.2819 for the Heston version and our form respectively.

However, when we turn to the out-of-sample errors, it is quite a different story. In fact, Table 4 and 5 display the out-of-sample errors for the two periods (2008.1-2008.12, 2012.9-2013.8) respectively. It is clear that both forms generally perform better in the normal market than in the financial crisis since the RMSE between Jan 2008 and Dec 2008 is approximately twice of that from Sept 2012 to Aug 2013. Specifically, our form is only slightly better than the Heston version in the financial crisis since by replacing the Heston version with our form, the maximum improvement in the RMSE is merely 1.33% with only Wednesday data and whole week data. In contrast, when we turn to the results in the normal market, what we can see first is that using only Wednesday data and employing all the filtered data for the whole year exhibit almost same results, which show that our form is much more attractive than the Heston version in this case. In particular, the RMSE of our form is smaller than that of the Heston version in both of the two cases, with the former being 83% of the latter.

Table 4: Out-of-sample errors for the first example

<table>
<thead>
<tr>
<th>Data type</th>
<th>Model</th>
<th>Average price</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wednesday</td>
<td>Ours</td>
<td>53.0310</td>
<td>20.7824</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>53.0310</td>
<td>21.0616</td>
</tr>
<tr>
<td></td>
<td>Rate</td>
<td></td>
<td>98.67%</td>
</tr>
<tr>
<td>All</td>
<td>Ours</td>
<td>53.7271</td>
<td>21.3709</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>53.7271</td>
<td>21.6545</td>
</tr>
<tr>
<td></td>
<td>Rate</td>
<td></td>
<td>98.69%</td>
</tr>
</tbody>
</table>

On the other hand, the valuation errors for the option data of the whole year considered by moneyness are shown in Table 6 and 7. It should be noticed that the column of “No.” refers to the observation numbers and “Difference” in these tables stands for the relative
difference defined as

\[
\text{difference} = \frac{|\text{Our form error} - \text{Heston version error}|}{\text{Heston version error}}.
\]  

In addition, an option is regarded as “at the money” (A) if \(0.97 \leq S/K \leq 1.03\). The position of “in the money” (I) and “out of money” (O) refer to the case of \(S/K > 1.03\) and \(S/K < 0.97\), respectively.

Table 6 exhibits the performance of the two forms in financial crisis. Although our form generally performs better than the Heston model in this case, which is mentioned above, it turns out that the Heston version is a better choice when predicting out-of-money option prices since there would be 8.1% less errors. In contrast, when at-the-money and in-the-money options are taken into consideration, it is not difficult to find that there can be a more precise prediction with our form and its improvement can be 7.8% and 4.7% respectively.

Table 5: Out-of-sample errors for the second example

<table>
<thead>
<tr>
<th>Data type</th>
<th>Model</th>
<th>Average price</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wednesday</td>
<td>Ours</td>
<td>44.4747</td>
<td>10.8238</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>44.4747</td>
<td>12.8878</td>
</tr>
<tr>
<td></td>
<td>Rate</td>
<td>83.98%</td>
<td></td>
</tr>
<tr>
<td>All</td>
<td>Ours</td>
<td>44.3649</td>
<td>10.5969</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>44.3649</td>
<td>12.8659</td>
</tr>
<tr>
<td></td>
<td>Rate</td>
<td>82.36%</td>
<td></td>
</tr>
</tbody>
</table>

Table 6: Out-of-sample errors of the first example according to moneyness

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Model</th>
<th>Average price</th>
<th>No.</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.90 &lt; S/K &lt; 0.97(O)</td>
<td>Ours</td>
<td>19.4995</td>
<td>9688</td>
<td>21.5817</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>19.4995</td>
<td>9688</td>
<td>19.9681</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>8.1%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>0.97 ≤ S/K ≤ 1.03(A)</td>
<td>Ours</td>
<td>54.0624</td>
<td>7572</td>
<td>22.5165</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>54.0624</td>
<td>7572</td>
<td>24.4305</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>7.8%</td>
<td></td>
<td></td>
</tr>
<tr>
<td>1.03 &lt; S/K &lt; 1.10(I)</td>
<td>Ours</td>
<td>103.0018</td>
<td>6675</td>
<td>19.6566</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>103.0018</td>
<td>6675</td>
<td>20.6272</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td>4.7%</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
A different phenomenon is shown by Table 7 that our forms greatly outperforms the Heston version for options either “in the money”, “at the money” or “out of money”. To be more specific, although “at-the-money” options exhibit the largest value of RMSE for both models, the maximum absolute difference of RMSE between our form and the Heston version is shown by “in-the-money” options to be 3.86. On the other hand, the relative difference of errors between our form and the Heston version for “out-of-money” options is the lowest at 6.7%, which is only approximately one half of that for “at-the-money” options. The largest improvement appears in the category of “in-the-money” options, almost reaching 30% when we replace the Heston version with our form, which is rather significant.

<table>
<thead>
<tr>
<th>Moneyness</th>
<th>Model</th>
<th>Average price</th>
<th>No.</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Ours</td>
<td>4.0590</td>
<td>17472</td>
<td>8.5902</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>4.0590</td>
<td>17472</td>
<td>9.2077</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td></td>
<td></td>
<td>6.7%</td>
</tr>
<tr>
<td>0.90 &lt; S/K &lt; 0.97(O)</td>
<td>Ours</td>
<td>33.0472</td>
<td>15783</td>
<td>13.6092</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>33.0472</td>
<td>15783</td>
<td>15.9021</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td></td>
<td></td>
<td>14.4%</td>
</tr>
<tr>
<td>1.03 &lt; S/K &lt; 1.10(I)</td>
<td>Ours</td>
<td>100.8722</td>
<td>15618</td>
<td>9.0335</td>
</tr>
<tr>
<td></td>
<td>Heston</td>
<td>100.8722</td>
<td>15618</td>
<td>12.8935</td>
</tr>
<tr>
<td></td>
<td>Difference</td>
<td></td>
<td></td>
<td>29.9%</td>
</tr>
</tbody>
</table>

Therefore, based on these observations, we can certainly conclude that our form has certain advantages over the original Heston version in the normal market for S&P 500 with a genetic algorithm, which implies that it may provide more accurate results than the original Heston version for some cases.

5 Conclusion

In this paper, the necessary and sufficient conditions to ensure the existence of a solution in an affine form are proved and imposed to establish an alternative form of the Heston
model. This alternative form could offer certain advantages in parameter determination. After deriving the semi-closed pricing formula, our form is empirically compared with the Heston version by incorporating data of S&P 500 returns and options, with the Feller condition and another non-explosion condition correctly imposed in the parameter space to prevent the volatility from taking negative values or reaching infinity. Results show that our form generally outperforms the original Heston version for the case tested so far and it could be used as an alternative to the Heston version for some markets.

References


Appendix

Here we give the details in deriving the analytic pricing formula. Setting the coefficients of $v^{2a}$ to be zero since Equation (2.11) should hold for any $v$, we can obtain two ordinary differential equations (ODE) as follows.

\[ \frac{\partial D}{\partial \tau} = 2\alpha^2 \sigma^2 D^2 + 2\alpha (b_j + \lambda_1 + i\phi \rho \sigma) D + u_j i\phi - \frac{1}{2} \phi^2, \]
\[ \frac{\partial C}{\partial \tau} = \alpha [(2\alpha - 1) \sigma^2 + 2\lambda_2] D + ri\phi, \]

where $D(0; \phi) = 0$ and $C(0; \phi) = 0$. Let $A = 2\alpha^2 \sigma^2$, $B = 2\alpha (b_j + \lambda_1 + i\phi \rho \sigma)$ and $M = u_j i\phi - \frac{1}{2} \phi^2$. So the first ODE can be simplified as

\[ \frac{\partial D}{\partial \tau} = AD^2 + BD + M, \] (A-1)

which is exactly the Riccati Equation. Now by applying the transformation $D = \frac{y'}{-Ay}$, the following is obtained

\[ y'' - By' + AMy = 0, \] (A-2)

with initial condition $y'(0) = 0$. Equation (A-2) is obviously a second-order linear ODE, which has a general solution with the form

\[ y = C_1 e^{d+\tau} + C_2 e^{d-\tau}, \] (A-3)
where $d^+$ and $d^-$ are two roots of $d^2 - Bd + AM = 0$. Applying the initial condition of $D$ yields

$$\frac{C_1}{C_2} = \frac{d^-}{d^+};$$

and as a result, $D$ can be derived as

$$D = \frac{-1}{A} \cdot \frac{C_1 d^+ e^{d^+ \tau} + C_2 d^- e^{d^- \tau}}{C_1 e^{d^+ \tau} + C_2 e^{d^- \tau}},$$

$$= \frac{-1}{A} \cdot \frac{-d^- e^{d^+ \tau} + d^- e^{d^- \tau}}{-d^+ e^{d^+ \tau} + e^{d^- \tau}},$$

$$= \frac{d - 2\alpha(b_j + \lambda_1 + i\phi \rho \sigma)}{4\alpha^2 \sigma^2} \cdot \frac{1 - e^{d^+ \tau}}{1 - e^{d^- \tau}}.$$

Therefore, once the expression of $D(\tau; \phi)$ is obtained, it is straightforward to derive $C(\tau; \phi)$ as

$$C = \int_0^\tau \alpha[(2\alpha - 1)\sigma^2 + 2\lambda_2]D(t; \phi) + ri\phi dt,$$

$$= ri\phi \tau + \alpha(2\alpha - 1)\sigma^2 \frac{p(\tau; \phi)}{4\alpha^2 \sigma^2}.$$

Now we have derived $C(\tau; \phi)$ and $D(\tau; \phi)$ so that $f_j$ is known to us by now, which can be used to get $P_j$ according to the relationship between probability distribution function and the characteristic function of a random variable [39]. Thus the option pricing formula is obtained.