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The primitive ideals of some étale groupoid C^* -algebras

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Abstract

We consider the Deaconu-Renault groupoid of an action of a finitely generated free abelian monoid by local homeomorphisms of a locally compact Hausdorff space. We catalogue the primitive ideals of the associated groupoid C^* -algebra. For a special class of actions we describe the Jacobson topology.

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THE PRIMITIVE IDEALS OF SOME ÉTALE GROUPOID C^* -ALGEBRAS

AIDAN SIMS AND DANA P. WILLIAMS

ABSTRACT. Consider the Deaconu–Renault groupoid of an action of a finitely generated free abelian monoid by local homeomorphisms of a locally compact Hausdorff space. We catalogue the primitive ideals of the associated groupoid C^* -algebra. For a special class of actions we describe the Jacobson topology.

1. INTRODUCTION

Describing the primitive-ideal space of a C^* -algebra is typically quite difficult, but for crossed products of $C_0(X)$ by abelian groups G , a very satisfactory description is available: for each point $x \in X$ and for each character χ of G there is an irreducible representation of the crossed product on $L^2(G \cdot x)$. The map which sends (x, χ) to the kernel of this representation is a continuous open map from $X \times \widehat{G}$ to the primitive-ideal space of $C_0(X) \rtimes G$, and it carries (x, χ) and (y, ρ) to the same ideal precisely when $\overline{G \cdot x} = \overline{G \cdot y}$ and χ and ρ restrict to the same character of the stability subgroup $G_x = \{g : g \cdot x = x\}$ [29, Theorem 8.39].

Regarding $C_0(X) \rtimes G$ as a groupoid C^* -algebra leads to a natural question: what can be said about the primitive-ideal spaces of C^* -algebras of Deaconu–Renault groupoids of semigroup actions by local homeomorphisms? Examples of groupoids of this sort arise from the \mathbf{N} -actions by the shift map on the infinite-path spaces of row-finite directed graphs E with no sources. The primitive-ideal spaces of the associated graph C^* -algebras were described by Hong and Szymański [10] building on Huef and Raeburn’s description of the primitive-ideal space of a Cuntz–Krieger algebra [11]. The description given in [10] is in terms of the graph rather than its groupoid. Recasting their results in groupoid terms yields a map from $E^\infty \times \mathbf{T}$ to the primitive-ideal space of $C^*(E)$ along more or less the same lines as described above for group actions. But this map is not necessarily open, and the equivalence relation it induces on $E^\infty \times \mathbf{T}$ is complicated by the fact that orbits with the same closure need not have the same isotropy in \mathbf{Z}^k .

The complications become greater still when \mathbf{N} is replaced with \mathbf{N}^k , and the resulting class of C^* -algebras is substantial. For example, it contains the C^* -algebras of graphs [14] and k -graphs [13] and their topological generalisations [30, 31]. However, the results of [4] for higher-rank graph algebras suggest that a satisfactory description of the primitive-ideal spaces of Deaconu–Renault groupoids of \mathbf{N}^k actions

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might be achievable. Here we take a substantial first step by producing a complete catalogue of the primitive ideals of the C^* -algebra $C^*(G_T)$ of the Deaconu–Renault groupoid associated to an action T of \mathbf{N}^k by local homeomorphisms of a locally compact Hausdorff space X . Specifically, there is a surjection $(x, z) \mapsto I_{x,z}$ from $X \times \mathbf{T}^k$ to $\text{Prim}(C^*(G_T))$. Moreover, $I_{x,z}$ and $I_{x',z'}$ coincide if and only if the orbits of x and x' under T have the same closure and z and z' determine the same character of the interior of the isotropy of the reduction of G_T to this orbit closure. For a very special class of actions T we are also able to describe the topology of the primitive-ideal space of $C^*(G_T)$, but in general we can say little about it. Indeed, graph-algebra examples show that any general description will require subtle adjustments to the “obvious” quotient topology.

The paper is organised as follows. In Section 2 we establish our conventions for groupoids, and prove that if G is an étale Hausdorff groupoid and the interior $\text{Iso}(G)^\circ$ of its isotropy subgroupoid is closed as well as open, then the natural quotient $G/\text{Iso}(G)^\circ$ is also a Hausdorff étale groupoid and there is a natural homomorphism of $C^*(G)$ onto $C^*(G/\text{Iso}(G)^\circ)$.

In Section 3 we consider the Deaconu–Renault groupoids G_T associated to actions T of \mathbf{N}^k by local homeomorphisms of locally compact spaces X . We state our main theorem about the primitive ideals of $C^*(G_T)$, and begin its proof. We first show that G_T is always amenable. We then consider the situation where \mathbf{N}^k acts irreducibly on X . We show that there is then an open \mathbf{N}^k -invariant subset $Y \subset X$ on which the isotropy in $\mathbf{N}^k \times \mathbf{N}^k$ is maximal. For this set Y , $\text{Iso}(G_T|_Y)^\circ$ is closed. We finish Section 3 by showing that restriction gives a bijection between irreducible representations of $C^*(G_T)$ that are faithful on $C_0(X)$ and irreducible representations of $C^*(G_T|_Y)$ that are faithful on $C_0(Y)$. Our arguments in this section are special to \mathbf{N}^k , and make use of techniques developed in [4].

In Section 4 we show that if the subspace Y from the preceding paragraph is all of X , then $C^*(G_T)$ is an induced algebra—associated to the canonical action of \mathbf{T}^k on $C^*(G_T)$ —with fibres $C^*(G_T/\text{Iso}(G_T)^\circ)$. We use this description to give a complete characterisation of $\text{Prim}(C^*(G_T))$ as a topological space under the rather strong hypothesis that the reduction of $G_T/\text{Iso}(G_T)^\circ$ to any closed G_T -invariant subset of Y is topologically principal. In Section 5 we complete the proof of our main theorem. The fundamental idea is that for every irreducible representation ρ of $C^*(G_T)$ there is a set $Y = Y_\rho$ as above and an element $z = z_\rho \in \mathbf{T}^k$ for which ρ factors through an irreducible representation of $C^*(G_T|_Y)$ that is faithful on $C_0(Y)$ and which in turn factors through evaluation (in the induced algebra) at z .

Standing assumptions. Throughout this paper, all topological spaces (including topological groupoids) are second countable, and all groupoids are Hausdorff. By a homomorphism between C^* -algebras, we mean a $*$ -homomorphism, and by an ideal of a C^* -algebra we mean a closed, 2-sided ideal. We take the convention that \mathbf{N} is a monoid under addition, so it includes 0.

2. PRELIMINARIES

Let G be a locally compact second-countable Hausdorff groupoid with a Haar system. For subsets $A, B \subset G$, we write

$$AB := \{\alpha\beta \in G : (\alpha, \beta) \in (A \times B) \cap G^{(2)}\}.$$

We use the standard groupoid conventions that $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$, and $G_x^x = G^x \cap G_x$ for $x \in G^{(0)}$. If $K \subset G^{(0)}$, then the restriction of G to K is the subgroupoid $G|_K = \{\gamma \in G : r(\gamma), s(\gamma) \in K\}$. We will be particularly interested in the *isotropy subgroupoid*

$$\text{Iso}(G) = \{\gamma \in G : r(\gamma) = s(\gamma)\} = \bigcup_{x \in G^{(0)}} G_x^x.$$

This $\text{Iso}(G)$ is closed in G and is a group bundle over $G^{(0)}$.

A groupoid G is *topologically principal* if the units with trivial isotropy are dense in $G^{(0)}$. That is, $\overline{\{x \in G^{(0)} : G_x^x = \{x\}\}} = G^{(0)}$. It is worth pointing out that the condition we are here calling topologically principal has gone under a variety of names in the literature and that those names have not been used consistently (see [3, Remark 2.3]).

Recall that $G^{(0)}$ is a left G -space: $\gamma \cdot s(\gamma) = r(\gamma)$. If $x \in G^{(0)}$, then $G \cdot x = r(G_x)$ is called the *orbit* of x and is denoted by $[x]$. A subset A of $G^{(0)}$ is called *invariant* if $G \cdot A \subset A$. The quotient space $G \backslash G^{(0)}$ (with the quotient topology) is called the orbit space. The quasi-orbit space $\mathcal{Q}(G)$ of a groupoid G is the quotient of $G \backslash G^{(0)}$ in which orbits are identified if they have the same closure. Alternatively it is the T_0 -ization of orbit space $G \backslash G^{(0)}$ (see [29, Definition 6.9]). In particular, the quasi-orbit space has the quotient topology coming from the quotient map $q : G^{(0)} \rightarrow \mathcal{Q}(G)$.

An ideal $I \triangleleft C_0(G^{(0)})$ is called *invariant* if the corresponding closed set

$$C_I := \{x \in G^{(0)} : f(x) = 0 \text{ for all } f \in I\}$$

is invariant. If M is a representation of $C_0(G^{(0)})$ with kernel I , then C_I is called the *support* of M . We say C_I is G -irreducible if it is not the union of two proper closed invariant sets. For example, orbit closures, $\overline{[x]}$, are always G -irreducible.

Lemma 2.1. *Let G be a second-countable locally compact groupoid. A closed invariant subset C of $G^{(0)}$ is G -irreducible if and only if there exists $x \in G^{(0)}$ such that $C = \overline{[x]}$.*

Proof. It suffices to see that every closed G -invariant set is an orbit closure. This is a straightforward consequence of the lemma preceding [9, Corollary 19] and the observation that the orbit space $G \backslash G^{(0)}$ is the continuous open image of G and hence totally Baire. \square

Remark 2.2. We say that $C_0(G^{(0)})$ is G -simple if it has no nonzero proper invariant ideals. So $C_0(G^{(0)})$ is G -simple exactly when $G^{(0)}$ has a dense orbit. This is much weaker than the notion of minimality, which requires that *every* orbit is dense.

We also want to refer to a couple of old chestnuts. Recall that there is a nondegenerate homomorphism

$$V : C_0(G^{(0)}) \rightarrow M(C^*(G))$$

such that for $f \in C_c(G)$ and $\varphi \in C_0(G^{(0)})$, we have $(V(\varphi)f)(\gamma) = \varphi(r(\gamma))f(\gamma)$. In particular, if L is a nondegenerate representation of $C^*(G)$, then we obtain an associated representation M of $C_0(G^{(0)})$ by extension: $M(\varphi) = \bar{L}(V(\varphi))$. The next result is standard. A proof in the case where G is principal can be found in [5, Lemma 3.4 and Proposition 3.2], and the proof goes through in general *mutatis mutandis*.

Proposition 2.3. *Let G be a second-countable locally compact groupoid with a Haar system. Let L be a nondegenerate representation of $C^*(G)$ with associated representation M of $C_0(G^{(0)})$ as above. Then $\ker M$ is invariant. If L is irreducible, then the support of M is G -irreducible.*

Proposition 2.4. *Let G be a second-countable locally compact groupoid with a Haar system. Let L be a nondegenerate representation of $C^*(G)$ with associated representation M of $C_0(G^{(0)})$. If F is the support of M , then L factors through $C^*(G|_F)$. In particular, if L is irreducible, then L factors through $C^*(G|_{\overline{\{x\}}})$ for some $x \in G^{(0)}$.*

Proof. Since F is a closed invariant set, $U := G^{(0)} \setminus F$ is open and invariant. We have a short exact sequence

$$0 \longrightarrow C^*(G|_U) \xrightarrow{L} C^*(G) \xrightarrow{R} C^*(G|_F) \longrightarrow 0$$

of C^* -algebras with respect to the natural maps coming from extension (by 0) and restriction of functions in $C_c(G)$ [17, Lemma 2.10]. Since M has support F , the kernel of L contains the ideal corresponding to $C^*(G|_U)$, so L factors through $C^*(G|_F)$.

The last assertion follows from Proposition 2.3 and Lemma 2.1. \square

When the range and source maps in a groupoid G are open maps (in particular, when G is étale), the multiplication map is also open: Fix open $A, B \subseteq G$ and composable $(\alpha, \beta) \in A \times B$, and suppose that $\gamma_i \rightarrow \alpha\beta$. Since r is open, the $r(\gamma_i)$ eventually lie in $r(A)$; say $r(\gamma_i) = r(\alpha_i)$ with $\alpha_i \in A$. Now $\alpha_i^{-1}\gamma_i \rightarrow \beta$, and since B is open, the $\alpha_i^{-1}\gamma_i$ eventually belong to B , so that $\gamma_i = \alpha_i(\alpha_i^{-1}\gamma_i)$ eventually belongs to AB ; so AB is open.

For the remainder of this note, we specialize to the situation where G is étale. Since G is Hausdorff, this means that $G^{(0)}$ is clopen in G and that $r : G \rightarrow G^{(0)}$ is a local homeomorphism. Hence counting measures form a continuous Haar system for G . The I -norm on $C_c(G)$ is defined by

$$\|f\|_I = \sup_{x \in G^{(0)}} \max \left\{ \sum_{\gamma \in G_x} |f(\gamma)|, \sum_{\gamma \in G^x} |f(\gamma)| \right\}.$$

The groupoid C^* -algebra $C^*(G)$ is the completion of $C_c(G)$ in the norm $\|a\| = \sup\{\pi(a) : \pi \text{ is an } I\text{-norm bounded } *\text{-representation}\}$. For $x \in G^{(0)}$ there is a representation $L^x : C^*(G) \rightarrow B(\ell^2(G_x))$ given by $L^x(f)\delta_\gamma = \sum_{s(\alpha)=r(\gamma)} f(\alpha)\delta_{\alpha\gamma}$. This is called the (left-)regular representation associated to x . The reduced groupoid C^* -algebra $C_r^*(G)$ is the image of $C^*(G)$ under $\bigoplus_{x \in G^{(0)}} L^x$.

A *bisection* in a groupoid G , also known as a G -set, is a set $U \subset G$ such that r, s restrict to homeomorphisms on U . An important feature of étale groupoids is that they have plenty of open bisections: Proposition 3.5 of [8] together with local compactness implies that the topology on an étale groupoid has a basis of precompact open bisections.

If G is étale, then the homomorphism $V : C_0(G^{(0)}) \rightarrow MC^*(G)$ takes values in $C^*(G)$ and extends the inclusion $C_c(G^{(0)}) \hookrightarrow C_c(G)$ given by extension of functions (by 0). We regard $C_0(G^{(0)})$ as a $*$ -subalgebra of $C^*(G)$. If L is a representation of $C^*(G)$, then the associated representation M of $C_0(G^{(0)})$ is just the restriction of L to $C_0(G^{(0)})$. Thus $\ker M = \ker L \cap C_0(G^{(0)})$.

We write $\text{Iso}(G)^\circ$ for the interior of $\text{Iso}(G)$ in G . Since G is étale, $G^{(0)} \subset \text{Iso}(G)^\circ$ and $\text{Iso}(G)^\circ$ is an open étale subgroupoid of G .

Proposition 2.5. *Suppose that G is a second-countable locally compact Hausdorff étale groupoid such that $\text{Iso}(G)^\circ$ is closed in G .*

- (a) *The subgroupoid $\text{Iso}(G)^\circ$ acts freely and properly on the right of G , and the orbit space $G/\text{Iso}(G)^\circ$ is locally compact and Hausdorff.*
- (b) *For each $\gamma \in G$, the map $\alpha \mapsto \gamma\alpha\gamma^{-1}$ is a bijection from $\text{Iso}(G)^\circ_{s(\gamma)}$ onto $\text{Iso}(G)^\circ_{r(\gamma)}$.*
- (c) *For each $x \in G^{(0)}$, the set $\text{Iso}(G)^\circ_x$ is a normal subgroup of G_x^x .*
- (d) *The set $G/\text{Iso}(G)^\circ$ is a locally compact Hausdorff étale groupoid with respect to the operations $[\gamma]^{-1} = [\gamma^{-1}]$ for $\gamma \in G$, and $[\gamma][\eta] = [\gamma\eta]$ for $(\gamma, \eta) \in G^{(2)}$. The corresponding range and source maps are given by $r'([\gamma]) = r(\gamma)$ and $s'([\gamma]) = s(\gamma)$.*
- (e) *The groupoid $G/\text{Iso}(G)^\circ$ is topologically principal.*
- (f) *If G is amenable, then so is $G/\text{Iso}(G)^\circ$.*

Proof. (a) Since $\text{Iso}(G)^\circ$ is closed in G , it acts freely and properly on the right of G . Hence the orbit space is locally compact and Hausdorff by [19, Corollary 2.3].

(b) Conjugation by γ is a multiplicative bijection of $\text{Iso}(G)^\circ_{s(\gamma)}$ onto $\text{Iso}(G)^\circ_{r(\gamma)}$. So it suffices to show that

$$(2.1) \quad \gamma \text{Iso}(G)^\circ \gamma^{-1} \subset \text{Iso}(G)^\circ \quad \text{for all } \gamma \in G.$$

Take $\alpha \in \text{Iso}(G)^\circ$ such that $s(\gamma) = r(\alpha)$ and let U be an open neighborhood of α in $\text{Iso}(G)^\circ$. Let V be an open neighborhood of γ . Since G is étale, we can assume that U and V are bisections with $s(V) = r(U)$. Since the product of open subsets of G is open, VUV^{-1} is an open neighborhood of $\gamma\alpha\gamma^{-1}$. Since U and V are bisections and U consists of isotropy, VUV^{-1} is contained in $\text{Iso}(G)$. Hence $\gamma\alpha\gamma^{-1} \in \text{Iso}(G)^\circ$.

(c) Follows from (b) applied with $\gamma \in \text{Iso}(G)_x$.

(d) The maps r' and s' are clearly well defined. Suppose that $(\gamma, \eta) \in G^{(2)}$ and that $\gamma' = \gamma\alpha$ and $\eta' = \eta\beta$ with $\alpha, \beta \in \text{Iso}(G)^\circ$. Then $\gamma'\eta' = \gamma\eta(\eta^{-1}\alpha\eta\beta)$. But $\eta^{-1}\alpha\eta\beta \in \text{Iso}(G)^\circ$ by (b). Hence $[\gamma'\eta'] = [\gamma\eta]$. This shows that multiplication is well-defined. A similar argument shows that inversion is well-defined. Since the quotient map is open [18, Lemma 2.1], it is not hard to see that these operations are continuous. For example, suppose that $[\gamma_i] \rightarrow [\gamma]$ and $[\eta_i] \rightarrow [\eta]$ with $(\gamma_i, \eta_i) \in G^{(2)}$. It suffices to see that every subnet of $[\gamma_i\eta_i]$ has a subnet converging to $[\gamma\eta]$. But after passing to a subnet, relabeling, and passing to another subnet and relabeling, we can assume that there are $\alpha_i, \beta_i \in \text{Iso}(G)^\circ$ such that $\gamma_i\alpha_i \rightarrow \gamma$ and $\eta_i\beta_i \rightarrow \eta$ in G (see [29, Proposition 1.15]). But then $\gamma_i\alpha_i\eta_i\beta_i \rightarrow \gamma\eta$, and so $[\gamma_i\eta_i] \rightarrow [\gamma\eta]$.

We still need to see that $G/\text{Iso}(G)^\circ$ is étale. Its unit space is the image of $G^{(0)}$ which is open since the quotient map is open. So it suffices to show that r' is a local homeomorphism. Given $[\gamma] \in G/\text{Iso}(G)^\circ$, choose a compact neighborhood K of γ in G such that $r|_K$ is a homeomorphism. Let $q : G \rightarrow G/\text{Iso}(G)^\circ$ be the quotient map. Then $q(K)$ is a compact neighborhood of $[\gamma]$ and r' is a continuous bijection, and hence a homeomorphism, of $q(K)$ onto its image.

(e) Take $b \in G/\text{Iso}(G)^\circ$ such that $r'(b) = s'(b)$ but $b \neq r'(b)$. (That is, $b \in \text{Iso}(G/\text{Iso}(G)^\circ) \setminus q(G^{(0)})$, but the notation is a bit overwhelming.) It follows that $b = q(\gamma)$ for some $\gamma \in \text{Iso}(G) \setminus \text{Iso}(G)^\circ$. Let U be an open neighborhood of b . Then $q^{-1}(U)$ is an open neighborhood of γ , so meets $G \setminus \text{Iso}(G)$. Take $\delta \in q^{-1}(U) \setminus \text{Iso}(G)$;

so $s(\delta) \neq r(\delta)$. Then $q(\delta) \in U$ and $r'(q(\delta)) \neq s'(q(\delta))$. In particular, $q(\delta)$ does not belong to the interior of the isotropy of the groupoid $G/\text{Iso}(G)^\circ$. Thus the interior of the isotropy of $G/\text{Iso}(G)^\circ$ is just $q(G^{(0)})$. Now (e) follows from [3, Lemma 3.1].

(f) To see that $G/\text{Iso}(G)^\circ$ is amenable, we need to see that r' is an amenable map (see [1, Definition 2.2.8]). If G itself is amenable, then $r = r' \circ q$ is amenable. Thus r' is amenable by [1, Proposition 2.2.4]. \square

Our analysis of primitive ideals in C^* -algebras of Deaconu–Renault groupoids G will hinge on realising $C^*(G)$ as an induced algebra with fibres $C^*(G/\text{Iso}(G)^\circ)$. The first step towards this is to construct a homomorphism $C^*(G) \rightarrow C^*(G/\text{Iso}(G)^\circ)$, which can be done in much greater generality.

Proposition 2.6. *Let G be a locally compact Hausdorff étale groupoid such that $\text{Iso}(G)^\circ$ is closed in G . There is a C^* -homomorphism $\kappa : C^*(G) \rightarrow C^*(G/\text{Iso}(G)^\circ)$ such that*

$$\kappa(f)(b) = \sum_{q(\gamma)=b} f(\gamma) \quad \text{for } f \in C_c(G) \text{ and } b \in G/\text{Iso}(G)^\circ.$$

Proof. Lemma 2.9(b) of [16] implies that κ defines a surjection of $C_c(G)$ onto $C_c(G/\text{Iso}(G)^\circ)$. It clearly preserves involution, and

$$\begin{aligned} \kappa(f) * \kappa(g)(b) &= \sum_{s'(a)=r'(b)} \kappa(f)(a^{-1})\kappa(g)(ab) = \sum_{s'(a)=r'(b)} \sum_{\substack{q(\gamma)=a \\ q(\delta)=b}} f(\gamma^{-1})g(\gamma^{-1}\delta) \\ &= \sum_{q(\delta)=b} \sum_{s(\gamma)=r(\delta)} f(\gamma^{-1})g(\gamma\delta) = \sum_{q(\delta)=b} f * g(\delta) = \kappa(f * g)(b). \end{aligned}$$

It is not hard to see that κ is continuous in the inductive-limit topology (see [20, Corollary 2.17]). Since the $\|\cdot\|_I$ -norm dominates the C^* -norm, the inductive-limit topology is stronger than the C^* -norm topology. Hence κ extends to a C^* -homomorphism from $C^*(G)$ to $C^*(G/\text{Iso}(G)^\circ)$ as claimed. \square

Remark 2.7. It is fairly unusual for $\text{Iso}(G)^\circ$ to be closed in a general étale groupoid G (but see Proposition 3.10 and [15, Proposition 2.1]). For example, let X denote the union of the real and imaginary axes in \mathbf{C} , and let $T : X \rightarrow X$ be the homeomorphism $z \mapsto \bar{z}$. Regarding T as the generator of an action of \mathbf{N} by local homeomorphisms, we form the associated groupoid

$$G_T = \{(t, m, t) : t \in \mathbf{R}, m \in \mathbf{Z}\} \cup \{(z, 2m, z), (z, 2m+1, \bar{z}) : z \in i\mathbf{R}, m \in \mathbf{Z}\}.$$

Then

$$\text{Iso}(G)^\circ = \{(z, 2m, z) : z \in X, m \in \mathbf{Z}\} \cup \{(t, 2m+1, t) : t \in \mathbf{R} \setminus \{0\}, m \in \mathbf{Z}\}$$

is not closed: for example, $(0, 1, 0) \in \overline{\text{Iso}(G)^\circ} \setminus \text{Iso}(G)^\circ$.

However, we do not have an example of an étale groupoid G which acts minimally on its unit space and in which $\text{Iso}(G)^\circ$ is not closed; and [15, Proposition 2.1] implies that no such example exists amongst the Deaconu–Renault groupoids of \mathbf{N}^k actions that we consider for the remainder of the paper.

3. DEACONU–RENAULT GROUPOIDS

Given k commuting local homeomorphisms of a locally compact Hausdorff space X , we obtain an action of \mathbf{N}^k on X written $n \mapsto T^n$ (we do not assume that the T^n are surjective—cf., [7]). The corresponding *Deaconu–Renault Groupoid* is the set

$$(3.1) \quad G_T := \bigcup_{m, n \in \mathbf{N}^k} \{(x, m - n, y) \in X \times \mathbf{Z}^k \times X : T^m x = T^n y\}$$

with unit space $G_T^{(0)} = \{(x, 0, x) : x \in X\}$ identified with X , range and source maps $r(x, n, y) = x$ and $s(x, n, y) = y$, and operations $(x, n, y)(y, m, z) = (x, n + m, z)$ and $(x, n, y)^{-1} = (y, -n, x)$. For open sets $U, V \subseteq X$ and for $m, n \in \mathbf{N}^k$, we define

$$(3.2) \quad Z(U, m, n, V) := \{(x, m - n, y) : x \in U, y \in V \text{ and } T^m x = T^n y\}.$$

Lemma 3.1. *Let X be a locally compact Hausdorff space and let T be an action of \mathbf{N}^k on X by local homeomorphisms. The sets (3.2) are a basis for a locally compact Hausdorff topology on G_T . The sets $Z(U, m, n, V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms and $T^m(U) = T^n(V)$ are a basis for the same topology. Under this topology and operations defined above, G_T is a locally compact Hausdorff étale groupoid.*

Proof. When X is compact and the T^m are surjective, this result follows immediately from [7, Propositions 3.1 and 3.2]. Their proof is easily modified to show that the $Z(U, m, n, V)$ form a basis for a topology on G_T when X is assumed only to be locally compact and the T^n are not assumed to be surjective. It is not hard to see that the groupoid operations are continuous in this topology.

Since the T^m are all local homeomorphisms, each $Z(U, m, n, V)$ is a union of sets $Z(U', m, n, V')$ such that $T^m|_{U'}$ and $T^n|_{V'}$ are local homeomorphisms. Given U, V , we have

$$Z(U, m, n, V) = Z(U \cap (T^m)^{-1}(T^m U \cap T^n V), m, n, V \cap (T^n)^{-1}(T^m U \cap T^n V)).$$

So the sets $Z(U, m, n, V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms with $T^m U = T^n V$ form a basis for the same topology as claimed.

To see that this topology is locally compact, let K_1 and K_2 be compact subsets of X . Then just as in [7, Proposition 3.2], the map $(x, y) \mapsto (x, p - q, y)$ is continuous from the compact set $\{(x, y) \in K_1 \times K_2 : T^p x = T^q y\}$ onto $Z(K_1, p, q, K_2)$. Hence the latter is compact in G_T . It now follows easily that G_T is locally compact. It is étale because the source map restricts to a homeomorphism on any set of the form described in the preceding paragraph. \square

We now state our main theorem, which gives a complete listing of the primitive ideals of $C^*(G_T)$; but we need to establish a little notation first. Recall that for $x \in X$, the orbit $r((G_T)_x)$ is denoted $[x]$. So

$$[x] = \{y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbf{N}^k\}.$$

We write

$$H(x) := \bigcup_{\substack{\emptyset \neq U \subset \overline{[x]} \\ U \text{ relatively open}}} \{m - n : m, n \in \mathbf{N}^k \text{ and } T^m y = T^n y \text{ for all } y \in U\}.$$

We write $H(x)^\perp := \{z \in \mathbf{T}^k : z^g = 1 \text{ for all } g \in H(x)\}$. We shall see later that $H(x)$ is a subgroup of \mathbf{Z}^k , so this usage of $H(x)^\perp$ is consistent with the usual notation for the annihilator in \mathbf{T}^k of a subgroup of \mathbf{Z}^k . Our main theorem is the following.

Theorem 3.2. *Suppose that T is an action of \mathbf{N}^k on a locally compact Hausdorff space X by local homeomorphisms. For each $x \in X$ and $z \in \mathbf{T}^k$, there is an irreducible representation $\pi_{x,z}$ of $C^*(G_T)$ on $\ell^2([x])$ such that*

$$(3.3) \quad \pi_{x,z}(f)\delta_y = \sum_{(u,g,y) \in G_T} z^g f(u,g,y)\delta_u \quad \text{for all } f \in C_c(G_T).$$

The relation on $X \times \mathbf{T}^k$ given by

$$(x,z) \sim (y,w) \text{ if and only if } \overline{[x]} = \overline{[y]} \text{ and } \bar{z}w \in H(x)^\perp$$

is an equivalence relation, and $\ker(\pi_{x,z}) = \ker(\pi_{y,w})$ if and only if $(x,z) \sim (y,w)$. The map $(x,z) \mapsto \ker \pi_{x,z}$ induces a bijection from $(X \times \mathbf{T}^k)/\sim$ to $\text{Prim}(C^*(G_T))$.

Remark 3.3. A warning is in order. Theorem 3.2 lists the primitive ideals of $C^*(G_T)$, but it says nothing about the Jacobson topology. Example 3.4 below shows that neither the map $(x,z) \mapsto \ker \pi_{x,z}$ nor the induced map from $\mathcal{Q}(G_T) \times \mathbf{T}^k$ to $\text{Prim}(C^*(G_T))$ is open in general.

Example 3.4. Consider the directed graph E with two vertices v and w and three edges e, f, g where e is a loop at v , g is a loop at w and f points from w to v . We use the conventions of [10], so the infinite paths in E are e^∞ , g^∞ and $\{g^n f e^\infty : n = 0, 1, 2, \dots\}$. There are two orbits: $[g^\infty]$ and $[e^\infty]$. The latter is dense (because $\lim_{n \rightarrow \infty} g^n f e^\infty = g^\infty$), while the former is a singleton and is closed. As shown in [14], $C^*(E)$ is isomorphic to $C^*(G_T)$ where T is the shift operator on the infinite path space E^∞ . Hence we can apply [10] to conclude that each $\ker \pi_{e^\infty, z} \subset \ker \pi_{g^\infty, w}$, and if $I_{x,z} := \ker \pi_{x,z}$ for $x \in E^\infty$ and $z \in \mathbf{T}$, we have $\overline{\{I_{g^\infty, z}\}} = \{I_{g^\infty, z}\} \cup \{I_{e^\infty, w} : w \in \mathbf{T}\}$. So, for example, the set $E^\infty \times \{w \in \mathbf{T} : \text{Re}(w) > 0\}$ is open in $E^\infty \times \mathbf{T}$, but its image is not open in $\text{Prim}(C^*(E))$; and likewise the set $\mathcal{Q}(E) \times \{w : \text{Re}(w) > 0\}$ is open in $\mathcal{Q}(G_E) \times \mathbf{T}$, but its image is not open in $\text{Prim}(C^*(E))$.

The proof of Theorem 3.2 occupies this and the next two sections, culminating in Section 5. Our first order of business is to show that G_T is always amenable.

Lemma 3.5. *Let G_T be the locally compact Hausdorff étale groupoid arising from an action of T of \mathbf{N}^k on X by local homeomorphisms as above. Let $c : G_T \rightarrow \mathbf{Z}^k$ be the cocycle $c(x, k, y) = k$. Then both $c^{-1}(0)$ and G_T are amenable.*

Proof. For each $n \in \mathbf{N}^k$, let $F_n := \{(x, 0, y) : T^n x = T^n y\}$. Then each F_n is a closed subgroupoid containing $G^{(0)}$, and

$$c^{-1}(0) = \bigcup_{n \in \mathbf{N}^k} F_n.$$

In fact, each F_n is also open in G : for $(x, 0, y) \in F_n$ and any neighborhoods U of x and V of y , we have $(x, 0, y) \in Z(U, n, n, V) \subset F_n$.

Since N^k acts by local homeomorphisms, for $x \in X$ the set $\{y \in X : T^n y = T^n x\}$ is discrete and therefore countable. So the Mackey–Glimm–Ramsay Dichotomy [22, Theorem 2.1] implies the orbit space is standard. It then follows from [1,

Example 2.1.4(2)] that F_n is a properly amenable Borel groupoid, and hence Borel amenable as in [25, Definition 2.1]. Since F_n is open in G_T , it has a continuous Haar system (by restriction). Hence it is amenable by [25, Corollary 2.15]. It then follows from [1, Proposition 5.3.37] that $c^{-1}(0)$ is measurewise amenable. Since $c^{-1}(0)$ is open in G_T , it too is étale. Hence $c^{-1}(0)$ is amenable due to [1, Theorem 3.3.7].

The amenability of G_T now follows from [28, Proposition 9.3]. \square

Our next task is to understand the interior of the isotropy in G_T . By definition of the topology on G_T this is the union of all the sets $Z(U, m, n, U)$ such that $U \subset X$ is open and $T^m x = T^n x$ for all $x \in U$. Our approach is based on that of [4, Section 4].

Lemma 3.6. *Let T be an action of \mathbf{N}^k on X by local homeomorphisms. For each nonempty open set $U \subset X$, let*

$$(3.4) \quad \Sigma_U := \{(m, n) \in \mathbf{N}^k \times \mathbf{N}^k : T^m x = T^n x \text{ for all } x \in U\}.$$

Then

- (a) Σ_U is a submonoid of $\mathbf{N}^k \times \mathbf{N}^k$.
- (b) Σ_U is an equivalence relation on \mathbf{N}^k .
- (c) If $U \subset V$, then $\Sigma_V \subset \Sigma_U$.
- (d) For $p \in \mathbf{N}^k$ and U open and nonempty, we have $\Sigma_U \subset \Sigma_{T^p U}$.

Proof. Clearly $(0, 0) \in \Sigma_U$. Suppose that $(m, n), (p, q) \in \Sigma_U$. For $x \in U$ we have

$$(3.5) \quad T^{m+p} x = T^m T^p x = T^m T^q x = T^q T^m x = T^q T^n x = T^{n+q} x.$$

This proves (a). Statements (b) and (c) are immediate, and (d) follows from the special case of (3.5) where $p = q$. \square

Since our aim is to identify the primitive ideals of $C^*(G_T)$, and since Lemma 2.1 shows that every irreducible representation of $C^*(G_T)$ factors through the restriction of G_T to some \mathbf{N}^k -irreducible subset, we will often assume that X itself (viewed as $G^{(0)}$) is \mathbf{N}^k -irreducible. In this case, we will say that T acts irreducibly. Lemma 2.1 then implies that X has a dense orbit: $X = \overline{[x]}$ for some $x \in X$.

Lemma 3.7. *Let T be an \mathbf{N}^k -irreducible action on X by local homeomorphisms. For all open subsets $U, V \subseteq X$, there exists a nonempty open set W such that $\Sigma_U \cup \Sigma_V \subset \Sigma_W$*

Proof. Fix x with $\overline{[x]} = X$. Choose $y \in U$ and $z \in V$ such that $T^r y = T^s z$ and $T^p z = T^l x$. Then $T^{r+l} y = T^{p+s} z$, so $m = r + l$ and $n = s + p$ satisfy $T^m U \cap T^n V \neq \emptyset$. Since T^m and T^n are local homeomorphisms, and therefore open maps, $W := T^m U \cap T^n V$ is open. Parts (c) and (d) of Lemma 3.6 show that $\Sigma_U \subset \Sigma_{T^m U} \subset \Sigma_W$ and $\Sigma_V \subset \Sigma_{T^n V} \subset \Sigma_W$. \square

Given X and T as in Lemma 3.7, let

$$(3.6) \quad \Sigma := \bigcup_{\substack{\emptyset \neq U \subset X \\ U \text{ open}}} \Sigma_U.$$

We give $\mathbf{N}^k \times \mathbf{N}^k$ the usual partial order as a subset of \mathbf{N}^{2k} :

$$((n_i)_{i=1}^k, (n'_i)_{i=1}^k) \leq ((m_i)_{i=1}^k, (m'_i)_{i=1}^k) \quad \text{if } n_i \leq m_i \text{ and } n'_i \leq m'_i \text{ for all } i.$$

We let Σ^{\min} denote the collection of minimal elements of $\Sigma \setminus \{(0, 0)\}$ with respect to this order.

Lemma 3.8. *Let T be an irreducible action of \mathbf{N}^k by local homeomorphisms on a locally compact space X , and let Σ and Σ^{\min} be as above. Then Σ is a submonoid of $\mathbf{N}^k \times \mathbf{N}^k$ and an equivalence relation on \mathbf{N}^k . We have $\Sigma = (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k)$. Furthermore, Σ^{\min} is finite and generates Σ as a monoid.*

Proof. We have $(0, 0) \in \Sigma_X \subset \Sigma$. If $(m, n), (p, q) \in \Sigma$, then there are nonempty open sets U and V such that $(m, n) \in \Sigma_U$ and $(p, q) \in \Sigma_V$. Lemma 3.7 yields an open set W with $(m, n), (p, q) \in \Sigma_W$. Now $(m + p, n + q) \in \Sigma_W \subset \Sigma$ by Lemma 3.6(a), so Σ is a monoid.

To see that Σ is an equivalence relation, observe that it is reflexive and symmetric because each Σ_U is. Consider $(m, n), (n, p) \in \Sigma$; say $(m, n) \in \Sigma_U$ and $(n, p) \in \Sigma_V$. By Lemma 3.7, there is open set W with $(m, n), (n, p) \in \Sigma_W$. Hence $(m, p) \in \Sigma_W \subset \Sigma$ by Lemma 3.6(b).

The containment $\Sigma \subset (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k)$ is trivial because $(0, 0) \in \Sigma$ and $\Sigma \subset \mathbf{N}^k \times \mathbf{N}^k$. For the reverse containment, suppose that $(m, n), (p, q) \in \Sigma$ and $m - p, n - q \in \mathbf{N}^k$. By Lemma 3.7 we may choose an open W such that $(m, n), (p, q) \in \Sigma_W$. Fix $x \in T^{p+q}W$, say $x = T^{p+q}y$. Lemma 3.6 implies first that $(q, p) \in \Sigma_W$, and then that $(m + q, n + p) \in \Sigma_W$. Hence

$$T^{m-p}x = T^{m-p}(T^{p+q}y) = T^{m+q}y = T^{n+p}y = T^{n-q}(T^{q+p}y) = T^{n-q}x.$$

So $(m - p, n - q) \in \Sigma_{T^{p+q}W} \subset \Sigma$.

Now we argue as in [4, Proposition 4.4].¹ Dickson's Lemma [26, Theorem 5.1] implies that Σ^{\min} is finite. We must show that each $(m, n) \in \Sigma$ is a finite sum of elements of Σ^{\min} . We argue by induction on $|(m, n)| := \sum_{i=1}^k m_i + n_i$. If $|(m, n)| = 0$, the assertion is trivial. Now take $(m, n) \in \Sigma \setminus \{0\}$, and suppose that each $(p, q) \in \Sigma$ such that $|(p, q)| < |(m, n)|$ can be written as a finite sum of elements of Σ^{\min} . Since $(m, n) \neq 0$, by definition of Σ^{\min} there exists $(a, b) \in \Sigma^{\min}$ such that $(a, b) \leq (m, n)$. The preceding paragraph shows that $(p, q) = (m, n) - (a, b) \in (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k) = \Sigma$. The induction hypothesis implies that (p, q) is a finite sum of elements of Σ^{\min} , and then so is $(m, n) = (p, q) + (a, b)$. \square

We let

$$(3.7) \quad \begin{aligned} H(T) &:= \{m - n : (m, n) \in \Sigma\} \quad \text{and} \\ Y^{\max} &:= \bigcup \{Y \subset X : Y \text{ is open and } \Sigma_Y = \Sigma\}. \end{aligned}$$

Lemma 3.9. *Let T be an irreducible action of \mathbf{N}^k by local homeomorphisms of a locally compact Hausdorff space X . With Σ as in (3.6), we have*

$$(3.8) \quad \Sigma = \{(m, n) \in \mathbf{N}^k \times \mathbf{N}^k : m - n \in H(T)\}.$$

The set Y^{\max} is nonempty and open, and is the maximal open set in X such that $\Sigma_{Y^{\max}} = \Sigma$. We have $T^m Y^{\max} \subset Y^{\max}$ for all $m \in \mathbf{N}^k$.

Proof. By definition, $\Sigma \subset \{(m, n) : m - n \in H(T)\}$. For the reverse inclusion, suppose that $m - n = p - q$ with $(p, q) \in \Sigma$. Let $g = m - p \in \mathbf{Z}^k$. Fix $a, b \in \mathbf{N}^k$ such that $g = a - b$. Then both $(p + a, q + a)$ and (b, b) belong to Σ . Hence Lemma 3.8 implies that

$$(m, n) = (p + a, q + a) - (b, b) \in (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k) = \Sigma.$$

¹Though in [4, Proposition 4.4], the crucial use, in the induction, of the fact that $\Sigma = (\Sigma - \Sigma) \cap (\mathbf{N}^k \times \mathbf{N}^k)$ is not made explicit.

This establishes (3.8).

Now $|\Sigma^{\min}| - 1$ applications of Lemma 3.7 give a nonempty open set Y such that $\Sigma^{\min} \subset \Sigma_Y$. Since Σ_Y is monoid by Lemma 3.6, we have $\Sigma_Y = \Sigma$ by Lemma 3.8.

It now follows that Y^{\max} is open and nonempty. It is clearly maximal. Each $T^m Y^{\max} \subset Y^{\max}$ by Lemma 3.6(d) and the definition of Y^{\max} . \square

Proposition 3.10. *Let T be an irreducible action of \mathbf{N}^k by local homeomorphisms of a locally compact Hausdorff space X , and let G_T be the associated Deaconu–Renault groupoid (as in (3.1)). The set $H(T)$ of (3.7) is a subgroup of \mathbf{Z}^k . Let Σ be as in (3.6), and let $Y \subset X$ be an open set such that $\Sigma_Y = \Sigma$ and $T^p Y \subset Y$ for all $p \in \mathbf{N}^k$. Then $\text{Iso}(G_T|_Y)^\circ = \{(x, g, x) : x \in Y \text{ and } g \in H(T)\}$, and $\text{Iso}(G_T|_Y)^\circ$ is closed in $G_T|_Y$.*

Proof. Since Σ is an equivalence relation, 0 belongs to $H(T)$, and $g \in H(T)$ implies $-g \in H(T)$. Suppose that $m, n \in H(T)$, say $m = p_1 - q_1$ and $n = p_2 - q_2$ with $(p_i, q_i) \in \Sigma$. Lemma 3.8 implies that $(p_1 + p_2, q_1 + q_2) \in \Sigma$, and therefore that $m + n = p_1 + p_2 - q_1 - q_2$ belongs to $H(T)$. So $H(T)$ is a subgroup of \mathbf{Z}^k .

Take $x \in Y$ and $g \in H(T)$. By Lemma 3.9, there exists $(p, q) \in \Sigma$ such that $g = p - q$. Choose an open neighbourhood U of x in Y on which T^p and T^q are homeomorphisms. By choice of Y we have $T^p y = T^q y$ for all $y \in U$, and hence $\{(y, g, y) : y \in U\} = Z(U, p, q, U)$ is an open neighbourhood of (x, g, x) contained in $\{(y, g, y) : y \in Y, g \in H(T)\}$. So $\{(y, g, y) : y \in Y, g \in H(T)\} \subset \text{Iso}(G_T)^\circ$. For the reverse inclusion, suppose that $(z, h, z) \in \text{Iso}(G_T)^\circ$. By Lemma 3.1, there exist $m, n \in \mathbf{N}^k$ and open sets $U, V \subset Y$ such that $(z, h, z) \in Z(U, m, n, V) \subset \text{Iso}(G_T)$ with $T^m U = T^n V$. So $T^m x = T^n x$ for all $x \in U$, and then $(m, n) \in \Sigma_U \subset \Sigma$. Thus $h \in H(T)$ as required.

We now have

$$\begin{aligned} \text{Iso}(G_T|_Y)^\circ &= \{(x, g, x) : x \in Y, g \in H(T)\} \\ &= G_T \setminus \left(\bigcup_{m-n \notin H(T)} Z(U, m, n, V) \cup \bigcup_{U \cap V = \emptyset} Z(U, m, n, V) \right), \end{aligned}$$

and so $\text{Iso}(G_T|_Y)^\circ$ is closed. \square

Remark 3.11. We have an opportunity to fill a gap in the literature. The penultimate paragraph of the proof of [4, Theorem 5.3], appeals to [4, Corollary 2.8]. But unfortunately, the authors forgot to verify the hypothesis of [4, Corollary 2.8] that Γ should be aperiodic. We rectify this using our results above. Using the definition of aperiodicity of Γ [4, page 2575] and of the groupoid G_Γ of Γ [4, page 2573] as in the proof of [4, Corollary 2.8], we see that Γ is aperiodic if and only if G_Γ is topologically principal. In the situation of [4, Theorem 5.3], the groupoid $G_{H\Lambda T}$ discussed there is the restriction of the Deaconu–Renault groupoid $G_{\Lambda T}$ to $Y = H\Lambda^\infty$ which has the properties required of Y in Proposition 3.10 (see [4, Theorem 4.2(2)]), and so Proposition 3.10 shows that $\text{Iso}(G_{H\Lambda T})^\circ$ is closed. It is easy to check that G_Γ is isomorphic to $G_{H\Lambda T} / \text{Iso}(G_{H\Lambda T})^\circ$. So Proposition 2.6(e) shows that G_Γ is topologically principal and hence that Γ is aperiodic as required.

Corollary 3.12. *Let T be an irreducible action of \mathbf{N}^k by local homeomorphisms on a locally compact Hausdorff space X . Let Σ and $H(T)$ be as in (3.6) and (3.7). Suppose that Y is an open subset of X such that $T^p Y \subset Y$ for all p and such that $\Sigma_Y = \Sigma$.*

- (a) Regard $C_c(G_T|_Y)$ as a subalgebra of $C_c(G_T)$. The identity map extends to a monomorphism $\iota : C^*(G_T|_Y) \rightarrow C^*(G_T)$, and $\iota(C^*(G_T|_Y))$ is a hereditary subalgebra of $C^*(G_T)$.
- (b) The map $\pi \mapsto \pi \circ \iota$ is a bijection from the collection of irreducible representations of $C^*(G_T)$ that are injective on $C_0(X)$ to the space of irreducible representations of $C^*(G_T|_Y)$ that are injective on $C_0(Y)$. Moreover, the map $\ker \pi \mapsto \ker(\pi \circ \iota)$ is a homeomorphism from $\{I \in \text{Prim } C^*(G_T) : I \cap C_0(X) = \{0\}\}$ onto $\{J \in \text{Prim } C^*(G_T|_Y) : J \cap C_0(Y) = \{0\}\}$.

Proof. The inclusion $C_c(G_T|_Y) \hookrightarrow C_c(G_T)$ is a $*$ -homomorphism and continuous in the inductive-limit topology. Hence we get a homomorphism ι . Fix $x \in Y$. Let L^x be the regular representation of $C^*(G_T)$ on $\ell^2((G_T)_x)$. Then $L^x \circ \iota$ leaves the subspace $\ell^2\{(y, g, x) \in G_T : y \in Y\}$ invariant. Hence $L^x \circ \iota$ is equivalent to $L_Y^x \oplus 0$ where L_Y^x is the corresponding regular representation of $C^*(G_T|_Y)$. Since G_T and $G_T|_Y$ are both amenable by Lemma 3.5, ι is isometric and hence a monomorphism.

Let $\{f_i\}$ be an approximate identity for $C_0(Y)$. For $f \in C_c(G_T)$ we have $f_i f f_i \in C_c(G_T|_Y)$. Thus $\iota(C^*(G_T|_Y))$ is the closure of $\bigcup_i f_i C^*(G_T) f_i$. It follows easily that the image of ι is a hereditary subalgebra of $C^*(G_T)$ as claimed.

If π is an irreducible representation of $C^*(G_T)$ that is injective on $C_0(X)$, then it does not vanish on the ideal I_Y in $C^*(G_T)$ generated by $C_0(Y)$. Clearly, $\iota(C^*(G_T|_Y))$ is Morita equivalent to I_Y , and restriction of representations implements Rieffel induction from I_Y to $\iota(C^*(G_T|_Y))$. Since Rieffel induction between Morita equivalent C^* -algebras takes irreducibles to irreducibles ([21, Corollary 3.32]) and since $\pi|_{I_Y}$ is irreducible ([2, Theorem 1.3.4]), $\pi \circ \iota$ is irreducible and clearly injective on $C_0(Y)$. If ρ is an irreducible representation of $C^*(G_T|_Y)$, then it extends to an irreducible representation of I_Y . Since I_Y is an ideal, this representation extends to a (necessarily irreducible) representation π of $C^*(G_T)$ such that $\rho = \pi \circ \iota$. The kernel of $\pi|_{C_0(X)}$ is proper and has \mathbf{N}^k -invariant support. Since T acts irreducibly, $C_0(X)$ is G_T -simple, and so $\ker(\pi|_{C_0(X)}) = \{0\}$ and we obtain the required bijection.

The remaining assertion follows from this bijection and the Rieffel correspondence (see [21, Corollary 3.33(a)]). \square

4. THE PRIMITIVE IDEALS OF THE C^* -ALGEBRA OF AN IRREDUCIBLE DEACONU–RENAULT GROUPOID

In this section we specialize to the situation where T is an irreducible action of \mathbf{N}^k on a locally compact Hausdorff space Y with the property that, in the notation of (3.6), $\Sigma_Y = \Sigma$. We then have $\Sigma = \Sigma_U$ for all nonempty open subsets U of Y by Lemma 3.6. Lemma 3.9 says that $m - n \in H(T)$ implies $T^m x = T^n x$ for all $x \in Y$; and Proposition 3.10 gives

$$\text{Iso}(G_T)^\circ = \{(x, g, x) : x \in Y \text{ and } g \in H(T)\}.$$

We show that under these hypotheses, the primitive ideals of $C^*(G_T)$ with trivial intersection with $C_0(Y)$ are indexed by characters of $H(T)$. More precisely, we show that the irreducible representations of $C^*(G_T)$ that are faithful on $C_0(Y)$ are indexed by pairs (π, χ) where π is an irreducible representation of $C^*(G_T/\text{Iso}(G_T)^\circ)$ and χ is a character of $H(T)$. Our approach is to exhibit $C^*(G_T)$ as an induced algebra. Recall from Proposition 3.10 that $\text{Iso}(G_T)^\circ$ is closed in G_T , so Proposition 2.6 gives a homomorphism $\kappa : C^*(G_T) \rightarrow C^*(G_T/\text{Iso}(G_T)^\circ)$.

Lemma 4.1. *Suppose that T is an irreducible action of \mathbf{N}^k on a locally compact space Y such that $\Sigma_Y = \Sigma$. There is an action α of \mathbf{T}^k on $C^*(G_T)$ such that $\alpha_z(f)(x, g, y) = z^g f(x, g, y)$ for $f \in C_c(G_T)$. Let $\kappa : C^*(G_T) \rightarrow C^*(G_T/\text{Iso}(G_T)^\circ)$ be the homomorphism of Proposition 2.6. There is an action $\tilde{\alpha}$ of $H(T)^\perp$ on $C^*(G_T/\text{Iso}(G_T)^\circ)$ such that $\tilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$ for all $z \in H(T)^\perp \subset \mathbf{T}^k$.*

If $\bar{z}w \notin H(T)^\perp$, then $(\ker(\kappa \circ \alpha_z) + \ker(\kappa \circ \alpha_w)) \cap C_0(Y) \neq \{0\}$. We have $\ker(\kappa \circ \alpha_z) = \ker(\kappa \circ \alpha_w)$ if and only if $\bar{z}w \in H(T)^\perp$.

Proof. Let $c : G_T \rightarrow \mathbf{Z}^k$ be the canonical cocycle $c(x, g, y) = g$. The formula $\alpha_z(f)(\gamma) = z^{c(\gamma)} f(\gamma)$ defines a $*$ -homomorphism $\alpha_z : C_c(G_T) \rightarrow C_c(G_T)$. This α_z is trivially I -norm preserving, so extends to $\alpha_z : C^*(G_T) \rightarrow C^*(G_T)$. Since $\alpha_{\bar{z}}$ is an inverse for α_z , we have $\alpha_z \in \text{Aut}(C^*(G_T))$. The map $z \mapsto \alpha_z$ is a homomorphism because α_{zw} and $\alpha_z \circ \alpha_w$ agree on each $C_c(c^{-1}(g))$. To see that $z \mapsto \alpha_z$ is strongly continuous, first note that if $f \in C_c(G_T)$ is supported on $c^{-1}(g)$, then each $\alpha_z(f) = z^g f$, so $z \mapsto \alpha_z(f)$ is continuous. Since each $f \in C_c(G_T)$ is a finite linear combination $f = \sum_{\text{supp}(f) \cap c^{-1}(g) \neq \emptyset} f|_{c^{-1}(g)}$ of such functions, $z \mapsto \alpha_z(f)$ is continuous for each $f \in C_c(G_T)$. Now an $\varepsilon/3$ argument shows that $z \mapsto \alpha_z$ is strongly continuous.

Let $q : \mathbf{Z}^k \rightarrow \mathbf{Z}^k/H(T)$ be the quotient map. We have,

$$\text{Iso}(G_T)^\circ = \{(x, g, x) : x \in Y \text{ and } g \in H(T)\}.$$

Identify $G_T/\text{Iso}(G_T)^\circ$ with $\{(x, q(g), y) : (x, g, y) \in G_T\} \subset Y \times (\mathbf{Z}^k/H(T)) \times Y$.

Proposition 2.5 implies that the quotient map from G_T onto $G_T/\text{Iso}(G_T)^\circ$ is continuous and open, so the sets

$$\underline{Z}(U, q(m), q(n), V) = \{(x, q(m-n), y) : x \in U, y \in V \text{ and } T^m x = T^n y\}$$

are a basis for the topology on $G_T/\text{Iso}(G_T)^\circ$ (this makes sense because $T^m x = T^n y$ if and only if $T^{m+a} x = T^{n+b} y$ whenever $a - b \in H(T)$). Arguing as in the first paragraph, we get an action $\tilde{\alpha}$ of $H(T)^\perp$ on $C^*(G_T/\text{Iso}(G_T)^\circ)$ such that $\tilde{\alpha}_z(f)(x, q(g), y) = z^g f(x, q(g), y)$ for $f \in C_c(G_T/\text{Iso}(G_T)^\circ)$. For $f \in C_c(G_T)$, it is easy to check that $\tilde{\alpha}_z \circ \kappa(f) = \kappa \circ \alpha_z(f)$ for $z \in H(T)^\perp$. This identity then extends by continuity to all of $C^*(G_T)$.

Suppose that $\bar{z}w \notin H(T)^\perp$. Choose $n \in H(T)$ such that $z^n \neq w^n$. Fix a nonzero function $f \in C_c(Y)$ and define $f_n \in C_c(\{(x, n, x) : x \in Y\}) \subset C_c(G_T)$ by $f_n(x, n, x) = f(x, 0, x)$ for all $x \in Y$. Then $w^n f - f_n \in \ker(\kappa \circ \alpha_w)$ and $z^n f - f_n \in \ker(\kappa \circ \alpha_z)$. Hence $(z^n - w^n)f \in (\ker(\kappa \circ \alpha_z) + \ker(\kappa \circ \alpha_w)) \cap C_0(Y) \setminus \{0\}$ by choice of n . This proves the second-last statement of the lemma.

Since each of $\kappa \circ \alpha_z$ and $\kappa \circ \alpha_w$ is injective on $C_0(Y)$, this also proves the (contrapositive of the) implication \implies in the final statement of the lemma. For the reverse implication, suppose that $\bar{z}w \in H(T)^\perp$. Then

$$\ker(\kappa \circ \alpha_w) = \ker(\kappa \circ \alpha_{\bar{z}w} \circ \alpha_z) = \ker(\tilde{\alpha}_{\bar{z}w} \circ \kappa \circ \alpha_z) = \ker(\kappa \circ \alpha_z). \quad \square$$

The final assertion of Lemma 4.1 ensures that we can form the induced algebra $\text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha})$, namely

$$\{s \in C(\mathbf{T}^k, C^*(G_T/\text{Iso}(G_T)^\circ)) : s(wz) = \tilde{\alpha}_z(s(w)) \text{ for all } w \in \mathbf{T}^k \text{ and } z \in H(T)^\perp\}.$$

Induced algebras have a well-understood structure. Some of their elementary properties (in particular, the ones that we rely upon) are discussed in [21, §6.3].

Before proving the next result, we recall some basic results from abelian harmonic analysis. We write $C_c(H(T))$ for the set of finitely supported functions on $H(T)$. If $\varphi \in C_c(H(T))$, then its Fourier transform $\hat{\varphi} \in C(\mathbf{T}^k)$ is given by

$$\hat{\varphi}(z) = \sum_{n \in H(T)} \varphi(n)z^n,$$

and is constant on $H(T)^\perp$ cosets. Taking a few liberties with notation and terminology, we regard $\hat{\varphi}$ as an element of $C(\mathbf{T}^k/H(T)^\perp)$. The general theory implies that $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is a (uniformly) dense subalgebra of $C(\mathbf{T}^k/H(T)^\perp)$.

Lemma 4.2. *Let T be an irreducible action of \mathbf{N}^k on a locally compact space Y by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. If $(x, g, y) \in G_T$, then $(x, g + n, y) \in G_T$ for all $n \in H(T)$.*

Proof. Let $(x, g, y) = (x, p - q, y)$ with $T^p x = T^q y$. Fix $n \in H(T)$. Then $n = n_+ - n_-$ with $(n_+, n_-) \in \Sigma = \Sigma_Y$. Hence $T^{n_+} z = T^{n_-} z$ for all $z \in Y$, giving

$$T^{p+n_+} x = T^{n_+} T^p x = T^{n_+} T^q y = T^{n_-} T^q y = T^{q+n_-} y.$$

Hence $(x, g + n, y) = (x, (p + n_+) - (q + n_-), y) \in G_T$. \square

Because of Lemma 4.2, we can define a left action of $C_c(H(T))$ on $C_c(G_T)$ by

$$(4.1) \quad \varphi \cdot f(x, g, y) := \sum_{n \in H(T)} \varphi(n) f(x, g - n, y).$$

Lemma 4.3. *Let T be an irreducible action of \mathbf{N}^k on a locally compact space Y by local homeomorphisms such that $\Sigma_Y = \Sigma$, and let $\kappa : C^*(G_T) \rightarrow C^*(G_T/\text{Iso}(G_T)^\circ)$ be as in Proposition 2.6. Then*

$$\kappa(\alpha_z(\varphi \cdot f)) = \hat{\varphi}(z) \kappa(\alpha_z(f))$$

for all $f \in C_c(G_T)$, all $z \in \mathbf{T}^k$, and all $\varphi \in C_c(H(T))$.

Proof. We compute:

$$\begin{aligned} \kappa(\alpha_z(\varphi \cdot f))(x, q(g), y) &= \sum_{m \in H(T)} \alpha_z(\varphi \cdot f)(x, g + m, y) \\ &= \sum_{m \in H(T)} z^{g+m} \varphi \cdot f(x, g + m, y) \\ &= \sum_{m \in H(T)} \sum_{n \in H(T)} z^{g+m} \varphi(n) f(x, g + m - n, y). \end{aligned}$$

Since both sums are finite and we can interchange the order of summations at will, we may continue the calculation:

$$\begin{aligned} &= \sum_{m \in H(T)} \sum_{n \in H(T)} z^{g+m+n} \varphi(n) f(x, g + m, y) \\ &= \sum_{m \in H(T)} z^{g+m} \hat{\varphi}(z) f(x, g + m, y) \\ &= \hat{\varphi}(z) \kappa(\alpha_z(f))(x, q(g), y). \end{aligned} \quad \square$$

Proposition 4.4. *Let T be an irreducible action of \mathbf{N}^k on a locally compact space Y by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. Let*

$$\alpha : \mathbf{T}^k \rightarrow \text{Aut } C^*(G_T) \quad \text{and} \quad \tilde{\alpha} : H(T)^\perp \rightarrow \text{Aut } C^*(G_T/\text{Iso}(G_T)^\circ)$$

be as in Lemma 4.1, and let $\kappa : C^*(G_T) \rightarrow C^*(G_T/\text{Iso}(G_T)^\circ)$ be as in Proposition 2.6. There is an isomorphism $\Phi : C^*(G_T) \rightarrow \text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha})$ such that $\Phi(a)(z) = \kappa(\alpha_z(a))$ for $a \in C^*(G_T)$ and all $z \in \mathbf{T}$.

Proof. For $a \in C^*(G_T)$, the map $z \mapsto \kappa(\alpha_z(a))$ is continuous by continuity of α . Take $f \in C_c(G_T)$, $w \in \mathbf{T}^k$ and $z \in H(T)^\perp$. Lemma 4.1 gives $\tilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$. Hence

$$\Phi(f)(wz) = \kappa(\alpha_{wz}(f)) = \kappa(\alpha_z(\alpha_w(f))) = \tilde{\alpha}_z \kappa(\alpha_w(f)) = \tilde{\alpha}_z(\Phi(f)(w)).$$

Thus Φ takes values in $\text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha})$. It is not hard to check that Φ is a homomorphism.

To see that Φ is injective we use an averaging argument. Let \mathbf{T}^k act on the left of $\text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha})$ by left translation: $\text{lt}_z(c)(w) = c(\bar{z}w)$. We have $\Phi \circ \alpha_z = \text{lt}_{\bar{z}} \circ \Phi$. So the standard argument involving the faithful conditional expectations obtained from averaging over \mathbf{T}^k actions (see, for example, [27, Lemma 3.13]) shows that it is sufficient to check that Φ restricts to an injection on $C^*(G_T)^\alpha$.

If $f \in C_c(G_T)$, then arguing as in [29, Lemma 1.108], we have $\int_{\mathbf{T}^k} \alpha_z(f) dz \in C_c(G_T)$ and for $\gamma \in G_T$,

$$\begin{aligned} \left(\int_{\mathbf{T}^k} \alpha_z(f) dz \right)(\gamma) &= \int_{\mathbf{T}^k} \alpha_z(f)(\gamma) dz = \left(\int_{\mathbf{T}^k} z^{c(\gamma)} dz \right) f(\gamma) \\ &= \begin{cases} f(\gamma) & \text{if } \gamma \in c^{-1}(0) \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that $C^*(G_T)^\alpha = \overline{C_c(c^{-1}(0))} \subset C^*(G_T)$. Thus the inclusion map induces a monomorphism $\rho : C^*(c^{-1}(0)) \rightarrow C^*(G_T)$ whose image is exactly $C^*(G_T)^\alpha$. To see that $\Phi|_{C^*(G_T)^\alpha}$ is injective, it suffices to show that $\Phi \circ \rho$ is injective. Since $c^{-1}(0)$ is amenable by Lemma 3.5 and principal by construction, [6, Theorem 4.4] implies that we need only show that $(\Phi \circ \rho)|_{C_0(Y)}$ is injective. As ρ restricts to the canonical inclusion $C_0(Y) \hookrightarrow C^*(G_T)^\alpha$, it is enough to verify that Φ is injective on $C_0(Y)$. The homomorphism $\kappa \circ \alpha_z$ restricts to the identity map of $C_0(Y) \subset C^*(G_T)$ onto $C_0(Y) \subset C^*(G_T/\text{Iso}(G_T)^\circ)$. So if $f \in C_0(Y)$, $z \in \mathbf{T}^k$ and $b \in G_T/\text{Iso}(G_T)^\circ$, then

$$\Phi(f)(z)(b) = \kappa(\alpha_z(f))(b) = \begin{cases} f(x) & \text{if } b = (x, 0, x) \in (G_T/\text{Iso}(G_T)^\circ)^{(0)} \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $\Phi(f) = 0$, then $f = 0$. This completes the proof that Φ is injective.

We still have to show that Φ is surjective. Lemma 4.3 implies that if $\varphi \in C_c(H(T))$ and $f \in C_c(G_T)$, then $\hat{\varphi} \cdot \Phi(f) = \Phi(\varphi \cdot f)$ for the obvious action of $C(\mathbf{T}^k/H(T))$ on the induced algebra. Since $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is dense in $C(\mathbf{T}^k/H(T))$ it follows that the range of Φ is a $C(\mathbf{T}^k/H(T))$ -submodule. So it suffices to show that the range of $\kappa \circ \alpha_z$ contains $C_c(G_T/\text{Iso}(G_T)^\circ)$.

For this, fix $g \in \mathbf{Z}^k$ and $f \in \tilde{c}^{-1}(q(g))$; it suffices to show that f is in the range of $\pi \circ \alpha_z$. Define $h \in C_c(G_T)$ by

$$h(\gamma) = \begin{cases} \bar{z}^g f(\tilde{q}(\gamma)) & \text{if } c(\gamma) = g \\ 0 & \text{otherwise.} \end{cases}$$

Then h is continuous because each $c^{-1}(g)$ is clopen in G_T ; and $\kappa(\alpha_z(h)) = f$. \square

We now aim to apply [21, Proposition 6.6], which describes the primitive-ideal space of an induced algebra, to describe the topology of $\text{Prim}(C^*(G_T))$ for a special class of \mathbf{N}^k -actions T . To achieve this we first describe, in Lemma 4.6, the Jacobson topology on $\text{Prim}(C^*(G))$ when G is an amenable étale Hausdorff groupoid whose reduction to any closed invariant set is topologically principal. This topology is also described by [24, Corollary 4.9], but the statement given there is not quite the one we need.

Lemma 4.5. *Let G be a second-countable locally compact Hausdorff étale groupoid, and fix $x \in G^{(0)}$. There is an irreducible representation $\omega_{[x]} : C^*(G) \rightarrow \mathcal{B}(\ell^2([x]))$ satisfying $\omega_{[x]}(f)\delta_y = \sum_{s(\gamma)=y} f(\gamma)\delta_{r(\gamma)}$ for all $f \in C_c(G)$. If G is topologically principal and amenable and if $[x]$ is dense in $G^{(0)}$, then $\omega_{[x]}$ is faithful, and hence $C^*(G)$ is primitive.*

Proof. Let E_x denote the 1-dimensional representation of the group G_x^x . Then $\omega_{[x]} := \text{Ind}_{\{x\}}^{G_x^x} E_x$ is a representation satisfying the desired formula.² Hence $\omega_{[x]}$ is irreducible by [12, Theorem 5].

Now suppose that G is amenable and topologically principal with $[x]$ dense in $G^{(0)}$. Then clearly $\omega_{[x]}$ is faithful on $C_0(G^{(0)})$. So [6, Theorem 4.4] says that it is faithful on $C^*(G)$, whence $C^*(G)$ is primitive. \square

Recall that the quasi-orbit space $\mathcal{Q}(G) = \{ \overline{[x]} : x \in G^{(0)} \}$ carries the quotient topology for the map $q : G^{(0)} \rightarrow \mathcal{Q}(G)$ that identifies u with v exactly when $[u]$ and $[v]$ have the same closure in $G^{(0)}$. In particular, if $S \subset \mathcal{Q}(G)$, then $\overline{S} = \{ q(x) : x \in q^{-1}(S) \}$.

Lemma 4.6. *Let G be an amenable, étale Hausdorff groupoid and suppose that $G|_X$ is topologically principal for every closed invariant subset X of the unit space³ For $x \in G^{(0)}$, let ω_x be the irreducible representation of Lemma 4.5. The map $x \mapsto \ker \omega_x$ from $G^{(0)}$ to $\text{Prim}(C^*(G))$ descends to a homeomorphism of the quasi-orbit space $\mathcal{Q}(G)$ onto $\text{Prim}(C^*(G))$.*

Proof. For $x \in G^{(0)}$, we have $\ker \omega_x \cap C_0(G^{(0)}) = C_0(G^{(0)} \setminus \overline{[x]})$. Since $G|_X$ is topologically principal for every closed invariant subset $X \subset G^{(0)}$, [24, Corollary 4.9]⁴ therefore implies that $\ker \omega_x = \ker \omega_y$ if and only if $\overline{[x]} = \overline{[y]}$. Hence $x \mapsto \ker \omega_x$ descends to a well-defined injection $\overline{[x]} \mapsto \ker \omega_x$. To see that it is surjective, observe that if π is an irreducible representation of $C^*(G)$, then Proposition 2.4 implies that $\ker \pi \cap C_0(G^{(0)}) = C_0(G^{(0)} \setminus \overline{[x]})$ for some $x \in G^{(0)}$. That is,

²This is also the representation described in [3, Proposition 5.2].

³Although the term has been used inconsistently, in [24] for example, one says the G -action on $G^{(0)}$ is essentially free.

⁴Specifically, [24, Corollary 4.9] applied to the groupoid dynamical system (G, Σ, \mathcal{A}) where Σ is the bundle of trivial groups over $G^{(0)}$ and \mathcal{A} is the trivial bundle $G^{(0)} \times \mathbf{C}$ of 1-dimensional C^* -algebras—see also [3, Corollary 5.9].

$\ker \pi \cap C_0(G^{(0)}) = \ker \omega_x \cap C_0(G^{(0)})$, and then [24, Corollary 4.9] again shows that $\ker \pi = \ker \omega_x$.

To show that $\overline{[x]} \mapsto \ker \omega_x$ is a homeomorphism, it suffices to take a set $S \subset \mathcal{Q}(G)$ and an element $x \in G^{(0)}$ and show that $\overline{[x]} \in \overline{S}$ if and only if $\ker \omega_x \in \overline{\{\ker \omega_y : q(y) \in S\}}$; for then $S \subset \mathcal{Q}(G)$ is closed if and only if its image $\{\ker \omega_y : q(y) \in S\}$ is closed in $\text{Prim}(C^*(G))$.

Fix $S \subset \mathcal{Q}(G)$ and $x \in G^{(0)}$. We have

$$\overline{\{\ker \omega_y : q(y) \in S\}} = \{\ker \omega_z : \bigcap_{q(y) \in S} \ker \omega_y \subset \ker \omega_z\}.$$

Using [24, Corollary 4.9] again, we deduce that $\ker \omega_x \in \overline{\{\ker \omega_y : q(y) \in S\}}$ if and only if $(\bigcap_{q(y) \in S} \ker \omega_y) \cap C_0(G^{(0)}) \subset \ker \omega_x \cap C_0(G^{(0)})$. We have

$$\begin{aligned} \left(\bigcap_{q(y) \in S} \ker \omega_y \right) \cap C_0(G^{(0)}) &= \{f \in C_0(G^{(0)}) : f|_{q^{-1}(S)} = 0\} \\ &= \{f \in C_0(G^{(0)}) : f|_{\overline{q^{-1}(S)}} = 0\}. \end{aligned}$$

On the other hand,

$$\ker \omega_x \cap C_0(G^{(0)}) = \{f \in C_0(G^{(0)}) : f|_{\overline{[x]}} = 0\}.$$

Hence $\ker \omega_x \in \overline{\{\ker \omega_y : y \in \bigcup S\}}$ if and only if $\overline{[x]} \subset \overline{q^{-1}(S)}$, and this is equivalent to $q(x) \in \overline{S} = \{q(x) : x \in \overline{q^{-1}(S)}\}$ since $\overline{q^{-1}(S)}$ is closed and invariant. \square

For the next result, recall that the quotient map $q : G \rightarrow G/\text{Iso}(G)^\circ$ restricts to a homeomorphism of unit spaces. Since q also preserves the range and source maps, it carries G -orbits bijectively to the corresponding $(G/\text{Iso}(G)^\circ)$ -orbits, and therefore carries orbit closures in G to the corresponding orbit closures in $G/\text{Iso}(G)^\circ$. Hence the identification $G^{(0)} = (G/\text{Iso}(G)^\circ)^{(0)}$ induces a homeomorphism $\mathcal{Q}(G) \cong \mathcal{Q}(G/\text{Iso}(G)^\circ)$.

Theorem 4.7. *Let T be an irreducible action of \mathbf{N}^k on a locally compact space Y by local homeomorphisms such that, in the notation of (3.6), $\Sigma_Y = \Sigma$. Suppose that for every $y \in Y$, the set*

$$\Sigma_{\overline{[y]}} := \{(m, n) \in \mathbf{N}^k \times \mathbf{N}^k : T^m x = T^n x \text{ for all } x \in \overline{[y]}\}$$

satisfies $\Sigma_{\overline{[y]}} = \Sigma$. Let $\alpha : \mathbf{T}^k \rightarrow \text{Aut } C^(G_T)$ be as in Lemma 4.1, and let $\kappa : C^*(G_T) \rightarrow C^*(G_T/\text{Iso}(G_T)^\circ)$ be as in Proposition 2.6. For $y \in (G_T/\text{Iso}(G_T)^\circ)^{(0)}$, let ω_x be the irreducible representation of $C^*(G_T)$ described in Lemma 4.5. Then the map $(y, z) \mapsto \ker(\omega_y \circ \alpha_z)$ from $Y \times \mathbf{T}^k$ to $\text{Prim}(C^*(G_T))$ descends to a homeomorphism $\mathcal{Q}(G_T) \times H(T)^\wedge \cong \text{Prim}(C^*(G_T))$.*

Proof. Let $\Phi : C^*(G_T) \rightarrow \text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha})$ be the isomorphism of Proposition 4.4. For each $y \in Y$, let $\tilde{\omega}_y$ be the irreducible representation of $C^*(G_T/\text{Iso}(G_T)^\circ)$ obtained from Lemma 4.5. Observe that $\tilde{\omega}_y \circ \kappa = \omega_y$. We have

$$\Phi(\ker(\omega_y \circ \alpha_z)) = \{s \in \text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha}) : f(z) \in \ker \tilde{\omega}_y\}.$$

Write ε_z for the homomorphism of the induced algebra onto $C^*(G_T/\text{Iso}(G_T)^\circ)$ given by evaluation at z . It now suffices to show that

$$(4.2) \quad (y, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$$

induces a homeomorphism of $\mathcal{Q}(G_T) \times H(T)^\wedge$ onto the primitive ideal space of the induced algebra.

Proposition 3.10 combined with the hypothesis that each $\Sigma_{\overline{[y]}} = \Sigma$ ensures that $\text{Iso}(G)^\circ|_{\overline{[y]}} = \text{Iso}(G|_{\overline{[y]}})^\circ$ for each y . Hence Proposition 2.5 ensures that the reduction of $G_T/\text{Iso}(G_T)^\circ$ to any orbit closure, and hence to any closed invariant set, is topologically principal. Now Lemma 4.6 implies that $\ker(\tilde{\omega}_y \circ \varepsilon_z) = \ker(\tilde{\omega}_x \circ \varepsilon_z)$ if and only if $\overline{[y]} = \overline{[x]}$. So the map (4.2) descends to a map $(\overline{[y]}, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$. Composing this with the homeomorphism of Lemma 4.6 shows that (4.2) induces a well-defined map

$$M : (\ker \tilde{\omega}_y, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$$

from $\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ) \times \mathbf{T}^k)$ to $\text{Prim}(\text{Ind}_{H(T)^\perp}^{\mathbf{T}^k}(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha}))$. An application of Proposition 6.16 of [21]—or, rather, of the obvious primitive-ideal version of that result—shows that M induces a homeomorphism of the quotient of $(\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ)) \times \mathbf{T}^k)$ by the diagonal action of $H(T)^\perp$ onto the primitive ideal space of the induced algebra. Since the action of $H(T)^\perp$ on \mathbf{T}^k is by translation and has quotient $H(T)^\wedge$, it now suffices to show that the action of $H(T)^\perp$ on $\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ))$ is trivial. Since $\tilde{\alpha}_z$ fixes $C_0(G^{(0)}) \subset C^*(G_T/\text{Iso}(G_T)^\circ)$ pointwise, for any ideal I of $C^*(G_T/\text{Iso}(G_T)^\circ)$, we have $\tilde{\alpha}_z(I) \cap C_0(G^{(0)}) = I \cap C_0(G^{(0)})$, and then [24, Corollary 4.9] implies that $\tilde{\alpha}_z(I) = I$. So $H(T)^\perp$ acts trivially on $\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ))$. \square

5. THE PRIMITIVE IDEALS OF THE C^* -ALGEBRA OF A DEACONU–RENAULT GROUPOID

In this section, our aim is to catalogue the primitive ideals of $C^*(G_T)$. We need to refine our notation from Section 4 to accommodate actions which are not necessarily irreducible.

Notation. Let T be an action of \mathbf{N}^k on a locally compact space X by local homeomorphisms. Recall that for $x \in X$,

$$[x] = \{y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbf{N}^k\}.$$

For $x \in X$ and $U \subset \overline{[x]}$ relatively open, let

$$\Sigma(x)_U := \{(m, n) \in \mathbf{N}^k \times \mathbf{N}^k : T^m y = T^n y \text{ for all } y \in U\},$$

and define

$$\Sigma(x) := \bigcup_U \Sigma(x)_U.$$

Lemma 3.9 implies that

$$Y(x) := \bigcup \{Y \subset \overline{[x]} : Y \text{ is relatively open and } \Sigma(x)_Y = \Sigma(x)\}$$

is nonempty and is the maximal relatively open subset of $\overline{[x]}$ such that $\Sigma(x)_{Y(x)} = \Sigma(x)$. Proposition 3.10 implies that

$$H(x) := H(T|_{\overline{[x]}}) = \{m - n : (m, n) \in \Sigma(x)\}$$

is a subgroup of \mathbf{Z}^k . To lighten notation, set $\mathcal{I}(x) := \text{Iso}(G_T|_{Y(x)})^\circ$. Proposition 3.10 says that

$$\mathcal{I}(x) = \{(y, g, y) : y \in Y(x) \text{ and } g \in H(x)\},$$

and is a closed subset of $G_T|_{Y(x)}$.

Lemma 5.1. *Let T be an action of \mathbf{N}^k on a locally compact Hausdorff space X by local homeomorphisms. For $x, y \in X$, we have $Y(x) = Y(y)$ if and only if $\overline{[x]} = \overline{[y]}$.*

Proof. The “if” direction is trivial. Suppose that $Y(x) = Y(y)$. By symmetry, it suffices to show that $y \in \overline{[x]}$. Since $Y(x) = Y(y)$ is open in $\overline{[y]}$, we have $Y(x) \cap [y] \neq \emptyset$. Since $Y(x) \subset [x]$, and $[x]$ is G_T -invariant, we deduce that $y \in \overline{[x]}$. \square

The key to the proof of our main theorem is the following result, which works at the level of irreducible representations.

Theorem 5.2. *Let T be an action of \mathbf{N}^k on a locally compact Hausdorff space X by local homeomorphisms. Take $x \in X$ and $z \in \mathbf{T}^k$. Suppose that ρ is a faithful irreducible representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$. Let $\iota : C^*(G_T|_{Y(x)}) \rightarrow C^*(G_T)$ be the inclusion of Corollary 3.12. Let*

$$\Phi : C^*(G_T|_{Y(x)}) \rightarrow \text{Ind}_{H(x)^\perp}^{\mathbf{T}^k}(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \tilde{\alpha})$$

be the isomorphism of Proposition 4.4, and let

$$\varepsilon_z : \text{Ind}_{H(x)^\perp}^{\mathbf{T}^k}(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \tilde{\alpha}) \rightarrow C^*(G_T|_{Y(x)}/\mathcal{I}(x))$$

denote evaluation at z . Let $R_x : C^*(G_T) \rightarrow C^*(G_T|_{\overline{[x]}})$ be the homomorphism induced by restriction of compactly supported functions. There is a unique irreducible representation $\pi_{x,z,\rho}$ of $C^*(G_T)$ such that

- (a) $\pi_{x,z,\rho}$ factors through R_x , and
- (b) the representation $\pi_{x,z,\rho}^0$ of $C^*(G_T|_{\overline{[x]}})$ such that $\pi_{x,z,\rho} = \pi_{x,z,\rho}^0 \circ R_x$ satisfies $\pi_{x,z,\rho}^0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$.

Every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$ for some x, z, ρ .

Proof. The representation $\rho \circ \varepsilon_z \circ \Phi$ is an irreducible representation of $C^*(G_T|_{Y(x)})$, and is injective on $C_0(Y(x))$ because both Φ and ε_z restrict to injections on $C_0(Y(x))$. Corollary 3.12(b) applied to $Y(x) \subset \overline{[x]}$ yields a unique representation $\pi_{x,z,\rho}^0$ of $C^*(G_T|_{\overline{[x]}})$ such that $\pi_{x,z,\rho}^0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. The set $\overline{[x]}$ is a closed invariant set in X . As in Proposition 2.4, restriction of functions induces a homomorphism $R_x : C^*(G_T) \rightarrow C^*(G_T|_{\overline{[x]}})$. Now $\pi_{x,z,\rho} := \pi_{x,z,\rho}^0 \circ R_x$ satisfies (a) and (b).

For uniqueness, take a representation φ of $C^*(G_T)$ satisfying (a) and (b). Then φ vanishes on the ideal generated by $C_0(X \setminus \overline{[x]})$ which is precisely the kernel of R_x by Proposition 2.4. So $\varphi = \varphi_0 \circ R_x$ for some irreducible representation φ_0 of $C^*(G_T|_{\overline{[x]}})$ satisfying $\varphi_0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. We saw in the preceding paragraph that $\pi_{x,z,\rho}^0$ is the unique such representation, so $\varphi_0 = \pi_{x,z,\rho}^0$ and hence $\varphi = \pi_{x,z,\rho}$.

To see that every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$, fix an irreducible representation φ of $C^*(G_T)$. Since it is irreducible, Proposition 2.4 implies that $\varphi = \varphi^0 \circ R_x$ for some $x \in X$ and some irreducible representation φ^0 of $C^*(G_T|_{\overline{[x]}})$ that is faithful on $C_0(\overline{[x]})$. Since Φ is an isomorphism, Corollary 3.12(b) implies that φ^0 is uniquely determined by $\varphi^0 \circ \iota \circ \Phi^{-1}$, which is an irreducible representation of $\text{Ind}_{H(x)^\perp}^{\mathbf{T}^k}(C^*(G_T|_{Y(x)}/\mathcal{I}(x)), \tilde{\alpha})$ that is faithful on $C_0(Y(x))$. By [21, Proposition 6.16], there exists z such that $\ker(\varepsilon_z) \subset \ker \varphi^0 \circ \iota \circ \Phi^{-1}$, and then $\varphi^0 \circ \iota \circ \Phi^{-1}$ descends to an irreducible representation ρ of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$. That is $\rho \circ \varepsilon_z = \varphi^0 \circ \iota \circ \Phi^{-1}$. Post-composing with Φ on both sides of this equation shows that $\varphi^0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. So we now need only prove that ρ is faithful.

Since φ^0 is faithful on $C_0(\overline{[x]})$, the composition $\varphi^0 \circ \iota \circ \Phi^{-1}$ is faithful on $C_0(Y(x))$, and hence ρ is faithful on $C_0(Y(x)) = C_0((G_T|_{Y(x)})/\mathcal{I}(x))^{(0)}$. Proposition 2.5(e) implies that $G_T|_{Y(x)}/\mathcal{I}(x)$ is topologically principal, and Proposition 2.5(f) combined with Lemma 3.5 implies that $G_T|_{Y(x)}/\mathcal{I}(x)$ is amenable. So [6, Theorem 4.4] implies that ρ is faithful as claimed. \square

Proof of Theorem 3.2. Fix $x \in G_T^{(0)}$ and $z \in \mathbf{T}^k$. Let $\alpha_z \in \text{Aut}(C^*(G_T))$ be the automorphism of Lemma 4.1, and let $\omega_{[x]}$ be the irreducible representation of Lemma 4.5. Then $\pi_{x,z} := \omega_{[x]} \circ \alpha_z$ is an irreducible representation satisfying (3.3). Furthermore $\pi_{x,z}|_{C_0(G^{(0)})}$ has support $\overline{[x]}$.

It is clear that the relation \sim is an equivalence relation. To see that $\ker \pi_{x,z} = \ker \pi_{y,w}$ if and only if $\overline{[x]} = \overline{[y]}$ and $\bar{z}w \in H(x)^\perp$, first suppose that $\overline{[x]} \neq \overline{[y]}$. Then $\ker \pi_{x,z} \cap C_0(X) \neq \ker \pi_{y,w} \cap C_0(X)$.

Second, suppose that $\overline{[x]} = \overline{[y]}$ but $\bar{z}w \notin H(x)$. Then $\pi_{x,z}$ and $\pi_{y,w}$ descend to representations $\pi_{x,z}^0$ and $\pi_{y,w}^0$ of $C^*(G_T|_{\overline{[x]}})$. Corollary 3.12(b) implies that their kernels are equal if and only if the kernels of $\pi_{x,z}^0 \circ \iota$ and $\pi_{y,w}^0 \circ \iota$ are equal. Lemma 5.1 shows that $Y(x) = Y(y)$, and for $f \in C_c(G_T|_{Y(x)}) = C_c(G_T|_{Y(y)})$, we have

$$\pi_{x,z}^0 \circ \iota(f)\delta_y = \sum_{(u,g,y) \in G_T|_{Y(x)}} z^g f(u, g, y)\delta_u.$$

Lemma 4.2 shows that for $n \in H(x)$,

$$\sum_{(u,g,y) \in G_T|_{Y(x)}} z^g f(u, g, y)\delta_u = \sum_{(u,g+n,y) \in G_T|_{Y(x)}} z^g f(u, g, y)\delta_u.$$

As in Lemma 4.3, for $\varphi \in C_c(H(x))$ and $f \in C_c(G_T|_{Y(x)})$, we have $\pi_{x,z} \circ \iota(\varphi \cdot f) = \hat{\varphi}(z)(\pi_{x,z} \circ \iota)(f)$ and $\pi_{y,w} \circ \iota(\varphi \cdot f) = \hat{\varphi}(w)(\pi_{y,w} \circ \iota)(f)$. Choose φ such that $\hat{\varphi}(w) = 0$ and $\hat{\varphi}(z) \neq 0$, and choose $f \in C_c(Y(x))$ such that $f(x) = 1$. Then $\pi_{y,w} \circ \iota(\varphi \cdot f) = 0$ whereas $\pi_{x,z} \circ \iota(\varphi \cdot f)\delta_x = \hat{\varphi}(z)\delta_x \neq 0$. So the kernels are not equal.

Third, suppose that $\overline{[x]} = \overline{[y]}$ and $\bar{z}w \in H(x)^\perp$. Again Lemma 5.1 shows that $Y(x) = Y(y)$. Let $\pi_{[x]}$ and $\pi_{[y]}$ be the faithful irreducible representations of $C^*(G_T|_{Y(x)}/\mathcal{I}(x)) = C^*(G_T|_{Y(y)}/\mathcal{I}(y))$ described by Lemma 4.5. It is routine to check that $\pi_{x,z}^0 \circ \iota = \omega_{[x]} \circ \varepsilon_z \circ \Phi$ and $\pi_{y,w}^0 \circ \iota = \omega_{[y]} \circ \varepsilon_w \circ \Phi$. We have

$$\omega_{[x]} \circ \tilde{\alpha}_{\bar{z}w} \circ \varepsilon_z \circ \Phi = \omega_{[x]} \circ \varepsilon_w \circ \Phi.$$

Since $\tilde{\alpha}_{\bar{z}w}$ is an automorphism, we deduce that $\ker(\omega_{[x]} \circ \varepsilon_z \circ \Phi) = \ker(\omega_{[y]} \circ \varepsilon_w \circ \Phi)$. Thus $\ker(\pi_{x,z}^0 \circ \iota) = \ker(\pi_{y,w}^0 \circ \iota)$. Now Corollary 3.12(b) implies that $\pi_{x,z}^0$ and $\pi_{y,w}^0$ have the same kernel. Since $\overline{[x]} = \overline{[y]}$, we have $R_x = R_y$, and so

$$\ker \pi_{x,z} = R_x^{-1}(\ker \pi_{x,z}^0) = R_y^{-1}(\ker \pi_{y,w}^0) = \ker \pi_{y,w}.$$

It remains to show that $(x, z) \mapsto \ker \pi_{x,z}$ is surjective. Fix a primitive ideal $I \triangleleft C^*(G_T)$. Theorem 5.2 gives $I = \ker \pi_{x,z,\rho}$ for some x, z, ρ . Choose $y \in [x] \cap Y(x)$, and let $\tilde{\omega}_{[y]}$ be the faithful irreducible representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$ of Lemma 4.5. Since ρ is faithful on $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$, we have $\ker(\omega_{[y]} \circ \varepsilon_z \circ \Phi) = \ker(\rho \circ \varepsilon_z \circ \Phi)$. So Theorem 5.2 gives $\ker \pi_{x,z,\omega_{[y]}} = \ker(\pi_{x,z,\rho})$. As in the second step above, one checks on basis elements that $\pi_{x,z} = \pi_{x,z,\omega_{[y]}}$, completing the proof. \square

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