The primitive ideals of some etale groupoid $C^*$-algebras

Aidan Sims
University of Wollongong, asims@uow.edu.au

Dana P. Williams
Dartmouth College, danaw@uow.edu.au

Follow this and additional works at: https://ro.uow.edu.au/eispapers

Part of the Engineering Commons, and the Science and Technology Studies Commons
The primitive ideals of some etale groupoid C*-algebras

Abstract
We consider the Deaconu-Renault groupoid of an action of a finitely generated free abelian monoid by local homeomorphisms of a locally compact Hausdorff space. We catalogue the primitive ideals of the associated groupoid C*-algebra. For a special class of actions we describe the Jacobson topology.

Disciplines
Engineering | Science and Technology Studies

Publication Details

This journal article is available at Research Online: https://ro.uow.edu.au/eispapers/5300
THE PRIMITIVE IDEALS OF SOME ÉTALE GROUPOID 
$C^*$-ALGEBRAS

AIDAN SIMS AND DANA P. WILLIAMS

Abstract. Consider the Deaconu–Renault groupoid of an action of a finitely
generated free abelian monoid by local homeomorphisms of a locally compact
Hausdorff space. We catalogue the primitive ideals of the associated groupoid
$C^*$-algebra. For a special class of actions we describe the Jacobson topology.

1. Introduction

Describing the primitive-ideal space of a $C^*$-algebra is typically quite difficult,
but for crossed products of $C_0(X)$ by abelian groups $G$, a very satisfactory description
is available: for each point $x \in X$ and for each character $\chi$ of $G$ there is an
irreducible representation of the crossed product on $L^2(G \cdot x)$. The map which sends
$(x, \chi)$ to the kernel of this representation is a continuous open map from $X \times \hat{G}$ to
the primitive-ideal space of $C_0(X) \rtimes G$, and it carries $(x, \chi)$ and $(y, \rho)$ to the same
ideal precisely when $G \cdot x = G \cdot y$ and $\chi$ and $\rho$ restrict to the same character of the
stability subgroup $G_x = \{ g : g \cdot x = x \}$ [29, Theorem 8.39].

Regarding $C_0(X) \rtimes G$ as a groupoid $C^*$-algebra leads to a natural question: what
can be said about the primitive-ideal spaces of $C^*$-algebras of Deaconu–Renault
groupoids of semigroup actions by local homeomorphisms? Examples of groupoids
of this sort arise from the $\mathbb{N}$-actions by the shift map on the infinite-path spaces
of row-finite directed graphs $E$ with no sources. The primitive-ideal spaces of the
associated graph $C^*$-algebras were described by Hong and Szymański [10] building
on Huef and Raeburn’s description of the primitive-ideal space of a Cuntz–Krieger
algebra [11]. The description given in [10] is in terms of the graph rather than its
groupoid. Recasting their results in groupoid terms yields a map from $E^\infty \times T$ to
the primitive-ideal space of $C^*(E)$ along more or less the same lines as described
above for group actions. But this map is not necessarily open, and the equivalence
relation it induces on $E^\infty \times T$ is complicated by the fact that orbits with the same
closure need not have the same isotropy in $\mathbb{Z}^k$.

The complications become greater still when $\mathbb{N}$ is replaced with $\mathbb{N}^k$, and the result-
ing class of $C^*$-algebras is substantial. For example, it contains the $C^*$-algebras
of graphs [14] and $k$-graphs [13] and their topological generalisations [30,31]. However,
the results of [4] for higher-rank graph algebras suggest that a satisfactory de-
scription of the primitive-ideal spaces of Deaconu–Renault groupoids of $\mathbb{N}^k$ actions
might be achievable. Here we take a substantial first step by producing a complete
catalogue of the primitive ideals of the $C^*$-algebra $C^*(G_T)$ of the Deaconu–Renault
groupoid associated to an action $T$ of $N^k$ by local homeomorphisms of a locally
compact Hausdorff space $X$. Specifically, there is a surjection $(x, z) \mapsto I_{x,z}$ from
$X \times T^k$ to $\text{Prim}(C^*(G_T))$. Moreover, $I_{x,z}$ and $I_{x',z'}$ coincide if and only if the
orbits of $x$ and $x'$ under $T$ have the same closure and $z$ and $z'$ determine the same
character of the interior of the isotropy of the reduction of $G_T$ to this orbit closure.
For a very special class of actions $T$ we are also able to describe the topology of
the primitive-ideal space of $C^*(G_T)$, but in general we can say little about it. In
indeed, graph-algebra examples show that any general description will require subtle
adjustments to the "obvious" quotient topology.

The paper is organised as follows. In Section 2 we establish our conventions
for groupoids, and prove that if $G$ is an étale Hausdorff groupoid and the interior
Iso$(G)^o$ of its isotropy subgroupoid is closed as well as open, then the natural
quotient $G/\text{Iso}(G)^o$ is also a Hausdorff étale groupoid and there is a natural ho-
momorphism of $C^*(G)$ onto $C^*(G/\text{Iso}(G)^o)$.

In Section 3 we consider the Deaconu–Renault groupoids $G_T$ associated to ac-
tions $T$ of $N^k$ by local homeomorphisms of locally compact spaces $X$. We state
our main theorem about the primitive ideals of $C^*(G_T)$, and begin its proof. We
first show that $G_T$ is always amenable. We then consider the situation where $N^k$
acts irreducibly on $X$. We show that there is then an open $N^k$-invariant subset
$Y \subset X$ on which the isotropy in $N^k \times N^k$ is maximal. For this set $Y$, $\text{Iso}(G_T|_Y)^o$
is closed. We finish Section 3 by showing that restriction gives a bijection between
irreducible representations of $C^*(G_T)$ that are faithful on $C_0(X)$ and irreducible
representations of $C^*(G_T|_Y)$ that are faithful on $C_0(Y)$. Our arguments in this
section are special to $N^k$, and make use of techniques developed in 4.

In Section 4 we show that if the subspace $Y$ from the preceding paragraph is
all of $X$, then $C^*(G_T)$ is an induced algebra—associated to the canonical action of
$T^k$ on $C^*(G_T)$—with fibres $C^*(G_T/\text{Iso}(G_T)^o)$. We use this description to give a
complete characterisation of $\text{Prim}(C^*(G_T))$ as a topological space under the rather
strong hypothesis that the reduction of $G_T/\text{Iso}(G_T)^o$ to any closed $G_T$-invariant
subset of $Y$ is topologically principal. In Section 5 we complete the proof of our
main theorem. The fundamental idea is that for every irreducible representation $\rho$
of $C^*(G_T)$ there is a set $Y = Y_\rho$ as above and an element $z = z_\rho \in T^k$ for which $\rho$
factors through an irreducible representation of $C^*(G_T|_Y)$ that is faithful on $C_0(Y)$
and which in turn factors through evaluation (in the induced algebra) at $z$.

Standing assumptions. Throughout this paper, all topological spaces (including
topological groupoids) are second countable, and all groupoids are Hausdorff. By
a homomorphism between $C^*$-algebras, we mean a $*$-homomorphism, and by an
ideal of a $C^*$-algebra we mean a closed, 2-sided ideal. We take the convention that
$N$ is a monoid under addition, so it includes 0.

2. Preliminaries

Let $G$ be a locally compact second-countable Hausdorff groupoid with a Haar
system. For subsets $A, B \subset G$, we write
$$AB := \{\alpha \beta \in G : (\alpha, \beta) \in (A \times B) \cap G^{(2)}\}.$$
We use the standard groupoid conventions that \( G^x = r^{-1}(x) \), \( G_x = s^{-1}(x) \), and 
\( G^x = G^x \cap G_x \) for \( x \in G^{(0)} \). If \( K \subset G^{(0)} \), then the restriction of \( G \) to \( K \) is the subgroupoid \( G|_K = \{ \gamma \in G : r(\gamma), s(\gamma) \in K \} \). We will be particularly interested in the isotropy subgroupoid
\[
\text{Iso}(G) = \{ \gamma \in G : r(\gamma) = s(\gamma) \} = \bigcup_{x \in G^{(0)}} G^x_x.
\]
This \( \text{Iso}(G) \) is closed in \( G \) and is a group bundle over \( G^{(0)} \).

A groupoid \( G \) is topologically principal if the units with trivial isotropy are dense in \( G^{(0)} \). That is, \( \{ x \in G^{(0)} : G^x = \{ x \} \} = G^{(0)} \). It is worth pointing out that the condition we are here calling topologically principal has gone under a variety of names in the literature and that those names have not been used consistently (see [3, Remark 2.3]).

Recall that \( G^{(0)} \) is a left \( G \)-space: \( \gamma : s(\gamma) = r(\gamma) \). If \( x \in G^{(0)} \), then \( G \cdot x = r(G_x) \) is called the orbit of \( x \) and is denoted by \([x]\). A subset \( A \) of \( G^{(0)} \) is called invariant if \( G \cdot A \subset A \). The quotient space \( G \setminus G^{(0)} \) (with the quotient topology) is called the orbit space. The quasi-orbit space \( Q(G) \) of a groupoid \( G \) is the quotient of \( G \setminus G^{(0)} \) in which orbits are identified if they have the same closure. Alternatively it is the \( T_0 \)-ization of orbit space \( G \setminus G^{(0)} \) (see [29, Definition 6.9]). In particular, the quasi-orbit space has the quotient topology coming from the quotient map
\[
q : G^{(0)} \to Q(G).
\]
An ideal \( I \triangleleft C_0(G^{(0)}) \) is called invariant if the corresponding closed set
\[
C_I := \{ x \in G^{(0)} : f(x) = 0 \text{ for all } f \in I \}
\]
is invariant. If \( M \) is a representation of \( C_0(G^{(0)}) \) with kernel \( I \), then \( C_I \) is called the support of \( M \). We say \( C_I \) is \( G \)-irreducible if it is not the union of two proper closed invariant sets. For example, orbit closures, \([x]\), are always \( G \)-irreducible.

**Lemma 2.1.** Let \( G \) be a second-countable locally compact groupoid. A closed invariant subset \( C \) of \( G^{(0)} \) is \( G \)-irreducible if and only if there exists \( x \in G^{(0)} \) such that \( C = [x] \).

**Proof.** It suffices to see that every closed \( G \)-invariant set is an orbit closure. This is a straightforward consequence of the lemma preceding [2, Corollary 19] and the observation that the orbit space \( G \setminus G^{(0)} \) is the continuous open image of \( G \) and hence totally Baire. \( \square \)

**Remark 2.2.** We say that \( C_0(G^{(0)}) \) is \( G \)-simple if it has no nonzero proper invariant ideals. So \( C_0(G^{(0)}) \) is \( G \)-simple exactly when \( G^{(0)} \) has a dense orbit. This is much weaker than the notion of minimality, which requires that every orbit is dense.

We also want to refer to a couple of old chestnuts. Recall that there is a nondegenerate homomorphism
\[
V : C_0(G^{(0)}) \to M(C^*(G))
\]
such that for \( f \in C_c(G) \) and \( \varphi \in C_0(G^{(0)}) \), we have \( (V(\varphi)f)(\gamma) = \varphi(r(\gamma))f(\gamma) \). In particular, if \( L \) is a nondegenerate representation of \( C^*(G) \), then we obtain an associated representation \( M \) of \( C_0(G^{(0)}) \) by extension: \( M(\varphi) = L(V(\varphi)) \). The next result is standard. A proof in the case where \( G \) is principal can be found in [5, Lemma 3.4 and Proposition 3.2], and the proof goes through in general mutatis mutandis.
Proposition 2.3. Let $G$ be a second-countable locally compact groupoid with a Haar system. Let $L$ be a nondegenerate representation of $C^*(G)$ with associated representation $M$ of $C_0(G^{(0)})$ as above. Then $M$ is invariant. If $L$ is irreducible, then the support of $M$ is $G$-irreducible.

Proposition 2.4. Let $G$ be a second-countable locally compact groupoid with a Haar system. Let $L$ be a nondegenerate representation of $C^*(G)$ with associated representation $M$ of $C_0(G^{(0)})$. If $F$ is the support of $M$, then $L$ factors through $C^*(G|_F)$. In particular, if $L$ is irreducible, then $L$ factors through $C^*(G|_F)$ for some $x \in G^{(0)}$.

Proof. Since $F$ is a closed invariant set, $U := G^{(0)} \setminus F$ is open and invariant. We have a short exact sequence

$$0 \longrightarrow C^*(G|_U) \xrightarrow{\iota} C^*(G) \xrightarrow{R} C^*(G|_F) \longrightarrow 0$$

of $C^*$-algebras with respect to the natural maps coming from extension (by 0) and restriction of functions in $C_c(G)$ [17, Lemma 2.10]. Since $M$ has support $F$, the kernel of $L$ contains the ideal corresponding to $C^*(G|_U)$, so $L$ factors through $C^*(G|_F)$.

The last assertion follows from Proposition 2.3 and Lemma 2.1. □

When the range and source maps in a groupoid $G$ are open maps (in particular, when $G$ is étale), the multiplication map is also open: Fix open $A, B \subseteq G$ and composable $(\alpha, \beta) \in A \times B$, and suppose that $\gamma_i \to \alpha \beta$. Since $r$ is open, the $r(\gamma_i)$ eventually lie in $r(A)$; say $r(\gamma_i) = r(\alpha_i)$ with $\alpha_i \in A$. Now $\alpha_i^{-1} \gamma_i \to \beta$, and since $B$ is open, the $\alpha_i^{-1} \gamma_i$ eventually belong to $B$, so that $\gamma_i = \alpha_i(\alpha_i^{-1} \gamma_i)$ eventually belongs to $AB$; so $AB$ is open.

For the remainder of this note, we specialize to the situation where $G$ is étale. Since $G$ is Hausdorff, this means that $G^{(0)}$ is clopen in $G$ and that $r : G \to G^{(0)}$ is a local homeomorphism. Hence counting measures form a continuous Haar system for $G$. The $I$-norm on $C_c(G)$ is defined by

$$\|f\|_I = \sup_{x \in G^{(0)}} \max \left\{ \sum_{\gamma \in G_x} |f(\gamma)|, \sum_{\gamma \in G^s} |f(\gamma)| \right\}.$$ 

The groupoid $C^*$-algebra $C^*(G)$ is the completion of $C_c(G)$ in the norm $\|a\| = \sup\{ \pi(a) : \pi$ is an $I$-norm bounded $*$-representation $\}$. For $x \in G^{(0)}$ there is a representation $L^x : C^*(G) \to B(l^2(G_x))$ given by $L^x(f)\delta_\gamma = \sum_{s(\alpha) = r(\gamma)} f(\alpha)\alpha_\gamma$. This is called the (left-)regular representation associated to $x$. The reduced groupoid $C^*$-algebra $C^*_r(G)$ is the image of $C^*(G)$ under $\bigoplus_{x \in G^{(0)}} L^x$.

A bisection in a groupoid $G$, also known as a $G$-set, is a set $U \subset G$ such that $r, s$ restrict to homeomorphisms on $U$. An important feature of étale groupoids is that they have plenty of open bisections: Proposition 3.5 of [8] together with local compactness implies that the topology on an étale groupoid has a basis of precompact open bisections.

If $G$ is étale, then the homomorphism $V : C_0(G^{(0)}) \to MC^*(G)$ takes values in $C^*(G)$ and extends the inclusion $C_c(G^{(0)}) \hookrightarrow C_c(G)$ given by extension of functions (by 0). We regard $C_0(G^{(0)})$ as a $*$-subalgebra of $C^*(G)$. If $L$ is a representation of $C^*(G)$, then the associated representation $M$ of $C_0(G^{(0)})$ is just the restriction of $L$ to $C_0(G^{(0)})$. Thus $\ker M = \ker L \cap C_0(G^{(0)})$. 


We write $\text{Iso}(G)^{\circ}$ for the interior of $\text{Iso}(G)$ in $G$. Since $G$ is étale, $G^{(0)} \subset \text{Iso}(G)^{\circ}$ and $\text{Iso}(G)^{\circ}$ is an open étale subgroupoid of $G$.

**Proposition 2.5.** Suppose that $G$ is a second-countable locally compact Hausdorff étale groupoid such that $\text{Iso}(G)^{\circ}$ is closed in $G$.

(a) The subgroupoid $\text{Iso}(G)^{\circ}$ acts freely and properly on the right of $G$, and the orbit space $G/\text{Iso}(G)^{\circ}$ is locally compact and Hausdorff.

(b) For each $\gamma \in G$, the map $\alpha \mapsto \gamma \alpha \gamma^{-1}$ is a bijection from $\text{Iso}(G)^{\circ}_{\nu(\gamma)}$ onto $\text{Iso}(G)^{\circ}_{\rho(\gamma)}$.

(c) For each $x \in G^{(0)}$, the set $\text{Iso}(G)^{\circ}_{x}$ is a normal subgroup of $G^{\times}$.

(d) The set $G/\text{Iso}(G)^{\circ}$ is a locally compact Hausdorff étale groupoid with respect to the operations $[\gamma]^{-1} = [\gamma^{-1}]$ for $\gamma \in G$, and $[\gamma][\eta] = [\gamma \eta]$ for $(\gamma, \eta) \in G^{(2)}$. The corresponding range and source maps are given by $r'(\gamma) = r(\gamma)$ and $s'(\gamma) = s(\gamma)$.

(e) The groupoid $G/\text{Iso}(G)^{\circ}$ is topologically principal.

(f) If $G$ is amenable, then so is $G/\text{Iso}(G)^{\circ}$.

**Proof.** (a) Since $\text{Iso}(G)^{\circ}$ is closed in $G$, it acts freely and properly on the right of $G$. Hence the orbit space is locally compact and Hausdorff by [19] Corollary 2.3.

(b) Conjugation by $\gamma$ is a multiplicative bijection of $\text{Iso}(G)^{\circ}_{\nu(\gamma)}$ onto $\text{Iso}(G)^{\circ}_{\rho(\gamma)}$. So it suffices to show that

$$\gamma \text{Iso}(G)^{\circ}_{\nu(\gamma)} \gamma^{-1} \subset \text{Iso}(G)^{\circ} \quad \text{for all } \gamma \in G.$$ 

Take $\alpha \in \text{Iso}(G)^{\circ}$ such that $s(\gamma) = r(\alpha)$ and let $U$ be an open neighborhood of $\alpha$ in $\text{Iso}(G)^{\circ}$. Let $V$ be an open neighborhood of $\gamma$. Since $G$ is étale, we can assume that $U$ and $V$ are bisections with $s(V) = r(U)$. Since the product of open subsets of $G$ is open, $VUV^{-1}$ is an open neighborhood of $\gamma \alpha \gamma^{-1}$. Since $U$ and $V$ are bisections and $U$ consists of isotropy, $VUV^{-1}$ is contained in $\text{Iso}(G)$. Hence $\gamma \alpha \gamma^{-1} \in \text{Iso}(G)^{\circ}$.

(c) Follows from (a) applied with $\gamma \in \text{Iso}(G)^{\circ}$.

(d) The maps $r'$ and $s'$ are clearly well defined. Suppose that $(\gamma, \eta) \in G^{(2)}$ and that $\gamma' = \gamma \alpha$ and $\eta' = \eta \beta$ with $\alpha, \beta \in \text{Iso}(G)^{\circ}$. Then $\gamma' \eta' = \gamma \eta (\eta^{-1} \alpha \beta)$. But $\eta^{-1} \alpha \beta \in \text{Iso}(G)^{\circ}$ by (b). Hence $[\gamma' \eta'] = [\gamma \eta]$. This shows that multiplication is well-defined. A similar argument shows that inversion is well-defined. Since the quotient map is open [18] Lemma 2.1, it is not hard to see that these operations are continuous. For example, suppose that $[\gamma_i] \to [\gamma]$ and $[\eta_i] \to [\eta]$ with $(\gamma_i, \eta_i) \in G^{(2)}$. It suffices to see that every subnet of $[\gamma_i \eta_i]$ has a subnet converging to $[\gamma \eta]$. But after passing to a subnet, relabeling, and passing to another subnet and relabeling, we can assume that there are $\alpha_i, \beta_i \in \text{Iso}(G)^{\circ}$ such that $\gamma_i \alpha_i \to \gamma$ and $\eta_i \beta_i \to \eta$ in $G$ (see [29] Proposition 1.15). But then $\gamma_i \alpha_i \eta_i \beta_i \to \gamma \eta$, and so $[\gamma_i \eta_i] \to [\gamma \eta]$.

We still need to see that $G/\text{Iso}(G)^{\circ}$ is étale. Its unit space is the image of $G^{(0)}$ which is open since the quotient map is open. So it suffices to show that $r'$ is a local homeomorphism. Given $[\gamma] \in G/\text{Iso}(G)^{\circ}$, choose a compact neighborhood $K$ of $\gamma$ in $G$ such that $r|_K$ is a homeomorphism. Let $q : G \to G/\text{Iso}(G)^{\circ}$ be the quotient map. Then $q(K)$ is a compact neighborhood of $[\gamma]$ and $r'$ is a continuous bijection, and hence a homeomorphism, of $q(K)$ onto its image.

(e) Take $b \in G/\text{Iso}(G)^{\circ}$ such that $r'(b) = s'(b)$ but $b \neq r'(b)$. (That is, $b \in \text{Iso}(G)/\text{Iso}(G)^{\circ} \setminus q(G^{(0)})$, but the notation is a bit overwhelming.) It follows that $b = q(\gamma)$ for some $\gamma \in \text{Iso}(G) \setminus \text{Iso}(G)^{\circ}$. Let $U$ be an open neighborhood of $b$. Then $q^{-1}(U)$ is an open neighborhood of $\gamma$, so meets $G\setminus \text{Iso}(G)$. Take $\delta \in q^{-1}(U) \setminus \text{Iso}(G)$;
so \( s(\delta) \neq r(\delta) \). Then \( q(\delta) \in U \) and \( r'(q(\delta)) \neq s'(q(\delta)) \). In particular, \( q(\delta) \) does not belong to the interior of the isotropy of the groupoid \( G/\text{Iso}(G)^\circ \). Thus the interior of the isotropy of \( G/\text{Iso}(G)^\circ \) is just \( q(G^{(0)}) \). Now \([3]\) follows from \([3]\) Lemma 3.1.

To see that \( G/\text{Iso}(G)^\circ \) is amenable, we need to see that \( r' \) is an amenable map (see \([1]\) Definition 2.2.8). If \( G \) itself is amenable, then \( r = r' \circ q \) is amenable. Thus \( r' \) is amenable by \([1]\) Proposition 2.2.4].

Our analysis of primitive ideals in \( C^*\)-algebras of Deaconu–Renault groupoids \( G \) will hinge on realising \( C^*(G) \) as an induced algebra with fibres \( C^*(G/\text{Iso}(G)^\circ) \). The first step towards this is to construct a homomorphism \( C^*(G) \to C^*(G/\text{Iso}(G)^\circ) \), which can be done in much greater generality.

**Proposition 2.6.** Let \( G \) be a locally compact Hausdorff étale groupoid such that \( \text{Iso}(G)^\circ \) is closed in \( G \). There is a \( C^*\)-homomorphism \( \kappa : C^*(G) \to C^*(G/\text{Iso}(G)^\circ) \) such that

\[
\kappa(f)(b) = \sum_{q(\gamma) = b} f(\gamma) \quad \text{for } f \in C_c(G) \text{ and } b \in G/\text{Iso}(G)^\circ.
\]

**Proof.** Lemma 2.9(b) of \([16]\) implies that \( \kappa \) defines a surjection of \( C_c(G) \) onto \( C_c(G/\text{Iso}(G)^\circ) \). It clearly preserves involution, and

\[
\kappa(f) \ast \kappa(g)(b) = \sum_{s'(a) = r'(b)} \kappa(f)(a^{-1})\kappa(g)(ab) = \sum_{s'(a) = r'(b)} \sum_{q(\gamma) = a} f(\gamma^{-1})g(\gamma^{-1}\delta) = \sum_{q(\delta) = b} f(\gamma^{-1})g(\gamma\delta) = \sum_{q(\delta) = b} f \ast g(\delta) = \kappa(f \ast g)(b).
\]

It is not hard to see that \( \kappa \) is continuous in the inductive-limit topology (see \([20]\) Corollary 2.17]). Since the \( \|\cdot\|_1 \)-norm dominates the \( C^*\)-norm, the inductive-limit topology is stronger than the \( C^*\)-norm topology. Hence \( \kappa \) extends to a \( C^*\)-homomorphism from \( C^*(G) \) to \( C^*(G/\text{Iso}(G)^\circ) \) as claimed.

**Remark 2.7.** It is fairly unusual for \( \text{Iso}(G)^\circ \) to be closed in a general étale groupoid \( G \) (but see Proposition \([3,10]\) and \([15]\) Proposition 2.1]). For example, let \( X \) denote the union of the real and imaginary axes in \( \mathbb{C} \), and let \( T : X \to X \) be the homeomorphism \( z \mapsto z \). Regarding \( T \) as the generator of an action of \( \mathbb{N} \) by local homeomorphisms, we form the associated groupoid

\[
G_T = \{(t,m,t) : t \in \mathbb{R}, m \in \mathbb{Z}\} \cup \{(z,2m,z), (z,2m+1,z) : z \in i\mathbb{R}, m \in \mathbb{Z}\}.
\]

Then

\[
\text{Iso}(G)^\circ = \{(z,2m,z) : z \in X, m \in \mathbb{Z}\} \cup \{(t,2m+1,t) : t \in \mathbb{R} \setminus \{0\}, m \in \mathbb{Z}\}
\]

is not closed: for example, \((0,1,0) \in \overline{\text{Iso}(G)^\circ} \setminus \text{Iso}(G)^\circ \).

However, we do not have an example of an étale groupoid \( G \) which acts minimally on its unit space and in which \( \text{Iso}(G)^\circ \) is not closed; and \([15]\) Proposition 2.1] implies that no such example exists amongst the Deaconu–Renault groupoids of \( \mathbb{N}^k \) actions that we consider for the remainder of the paper.
3. Deaconu–Renault Groupoids

Given $k$ commuting local homeomorphisms of a locally compact Hausdorff space $X$, we obtain an action of $\mathbb{N}^k$ on $X$ written $n \mapsto T^n$ (we do not assume that the $T^n$ are surjective—cf., [7]). The corresponding Deaconu–Renault Groupoid is the set

$$G_T := \bigcup_{m, n \in \mathbb{N}^k} \{(x, m - n, y) \in X \times \mathbb{Z}^k \times X : T^m x = T^n y\}$$

with unit space $G_T^{(0)} = \{(x, 0, x) : x \in X\}$ identified with $X$, range and source maps $r(x, n, y) = x$ and $s(x, n, y) = y$, and operations $(x, n, y)(y, m, z) = (x, n + m, z)$ and $(x, n, y)^{-1} = (y, -n, x)$. For open sets $U, V \subseteq X$ and for $m, n \in \mathbb{N}^k$, we define

$$Z(U, m, n, V) := \{(x, m - n, y) : x \in U, y \in V \text{ and } T^m x = T^n y\}.$$

**Lemma 3.1.** Let $X$ be a locally compact Hausdorff space and let $T$ be an action of $\mathbb{N}^k$ on $X$ by local homeomorphisms. The sets (3.2) are a basis for a locally compact Hausdorff topology on $G_T$. The sets $Z(U, m, n, V)$ such that $T^m|_U$ and $T^n|_V$ are homeomorphisms and $T^m(U) = T^n(V)$ are a basis for the same topology. Under this topology and operations defined above, $G_T$ is a locally compact Hausdorff étale groupoid.

**Proof.** When $X$ is compact and the $T^n$ are surjective, this result follows immediately from [7] Propositions 3.1 and 3.2]. Their proof is easily modified to show that the $Z(U, m, n, V)$ form a basis for a topology on $G_T$ when $X$ is assumed only to be locally compact and the $T^n$ are not assumed to be surjective. It is not hard to see that the groupoid operations are continuous in this topology.

Since the $T^n$ are all local homeomorphisms, each $Z(U, m, n, V)$ is a union of sets $Z(U', m, n, V')$ such that $T^m|_{U'}$ and $T^n|_{V'}$ are local homeomorphisms. Given $U, V$, we have

$$Z(U, m, n, V) = Z(U \cap (T^m)^{-1}(T^m U \cap T^n V), m, n, V \cap (T^n)^{-1}(T^m U \cap T^n V)).$$

So the sets $Z(U, m, n, V)$ such that $T^m|_{U'}$ and $T^n|_{V'}$ are homeomorphisms with $T^m U = T^n V$ form a basis for the same topology as claimed.

To see that this topology is locally compact, let $K_1$ and $K_2$ be compact subsets of $X$. Then just as in [7] Proposition 3.2], the map $(x, y) \mapsto (x, \{p - q, y\}$ is continuous from the compact set $\{(x, y) \in K_1 \times K_2 : T^p x = T^q y\}$ onto $Z(K_1, p, q, K_2)$. Hence the latter is compact in $G_T$. It now follows easily that $G_T$ is locally compact. It is étale because the source map restricts to a homeomorphism on any set of the form described in the preceding paragraph. \qed

We now state our main theorem, which gives a complete listing of the primitive ideals of $C^*(G_T)$; but we need to establish a little notation first. Recall that for $x \in X$, the orbit $r((G_T)_x)$ is denoted $[x]$. So

$$[x] = \{y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbb{N}^k\}.$$

We write

$$H(x) := \bigcup_{\emptyset \neq U \subseteq [x]} \{m - n : m, n \in \mathbb{N}^k \text{ and } T^m x = T^n y \text{ for all } y \in U\}.$$
We write $H(x)^\perp := \{ z \in T^k : z^g = 1 \text{ for all } g \in H(x) \}$. We shall see later that $H(x)$ is a subgroup of $Z^k$, so this usage of $H(x)^\perp$ is consistent with the usual notation for the annihilator in $T^k$ of a subgroup of $Z^k$. Our main theorem is the following.

**Theorem 3.3.** Suppose that $T$ is an action of $N^k$ on a locally compact Hausdorff space $X$ by local homeomorphisms. For each $x \in X$ and $z \in T^k$, there is an irreducible representation $\pi_{x,z}$ of $C^*(G_T)$ on $L^2([x])$ such that

$$(3.3) \quad \pi_{x,z}(f)\delta_y = \sum_{(u,g,y)\in G_T} z^g f(u,g,y)\delta_u \quad \text{for all } f \in C_c(G_T).$$

The relation on $X \times T^k$ given by

$$(x,z) \sim (y,w) \quad \text{if and only if } [x] = [y] \text{ and } zw \in H(x)^\perp$$

is an equivalence relation, and $\ker(\pi_{x,z}) = \ker(\pi_{y,w})$ if and only if $(x,z) \sim (y,w)$. The map $(x,z) \mapsto \ker \pi_{x,z}$ induces a bijection from $(X \times T^k)/\sim$ to $\text{Prim}(C^*(G_T))$.

**Remark 3.3.** A warning is in order. Theorem 3.2 lists the primitive ideals of $C^*(G_T)$, but it says nothing about the Jacobson topology. Example 3.4 below shows that neither the map $(x,z) \mapsto \ker \pi_{x,z}$ nor the induced map from $Q(G_T) \times T^k$ to $\text{Prim}(C^*(G_T))$ is open in general.

**Example 3.4.** Consider the directed graph $E$ with two vertices $v$ and $w$ and three edges $e, f, g$ where $e$ is a loop at $v$, $g$ is a loop at $w$ and $f$ points from $w$ to $v$. We use the conventions of [10], so the infinite paths in $E$ are $e^\infty, g^\infty$ and $\{ g^n f e^\infty : n = 0, 1, 2, \ldots \}$. There are two orbits: $[g^\infty]$ and $[e^\infty]$. The latter is dense (because $\lim_{n \to \infty} g^n f e^\infty = g^\infty$), while the former is a singleton and is closed. As shown in [14], $C^*(E)$ is isomorphic to $C^*(G_T)$ where $T$ is the shift operator on the infinite path space $E^\infty$. Hence we can apply [10] to conclude that each $\ker \pi_{x,z} \subset \ker \pi_{y,w}$, and if $I_{x,z} := \ker \pi_{x,z}$ for $x \in E^\infty$ and $z \in T$, we have $\{ I_{g^\infty,z} \} = \{ I_{g^\infty,w} \} \cup \{ I_{e^\infty,w} : w \in T \}$. So, for example, the set $E^\infty \times \{ w \in T : \text{Re}(w) > 0 \}$ is open in $E^\infty \times T$, but its image is not open in $\text{Prim}(C^*(E))$; and likewise the set $Q(E) \times \{ w : \text{Re}(w) > 0 \}$ is open in $Q(E) \times T$, but its image is not open in $\text{Prim}(C^*(E))$.

The proof of Theorem 3.2 occupies this and the next two sections, culminating in Section 5. Our first order of business is to show that $G_T$ is always amenable.

**Lemma 3.5.** Let $G_T$ be the locally compact Hausdorff étale groupoid arising from an action of $T$ of $N^k$ on $X$ by local homeomorphisms as above. Let $c : G_T \to Z^k$ be the cocycle $c(x,k,y) = k$. Then both $c^{-1}(0)$ and $G_T$ are amenable.

**Proof.** For each $n \in N^k$, let $F_n := \{ (x,0,y) : T^n x = T^n y \}$. Then each $F_n$ is a closed subgroupoid containing $G^{(0)}$, and

$$c^{-1}(0) = \bigcup_{n \in N^k} F_n.$$

In fact, each $F_n$ is also open in $G$: for $(x,0,y) \in F_n$ and any neighborhoods $U$ of $x$ and $V$ of $y$, we have $(x,0,y) \in Z(U,n,n,V) \subset F_n$.

Since $N^k$ acts by local homeomorphisms, for $x \in X$ the set $\{ y \in X : T^n y = T^n x \}$ is discrete and therefore countable. So the Mackey–Glimm–Ramsay Dichotomy [22, Theorem 2.1] implies the orbit space is standard. It then follows from [1]...
Example 2.1.4(2)] that $F_n$ is a properly amenable Borel groupoid, and hence Borel amenable as in [25] Definition 2.1. Since $F_n$ is open in $G_T$, it has a continuous Haar system (by restriction). Hence it is amenable by [25] Corollary 2.15. It then follows from [1] Proposition 5.3.7] that $c^{-1}(0)$ is measurewise amenable. Since $c^{-1}(0)$ is open in $G_T$, it too is étale. Hence $c^{-1}(0)$ is amenable due to [1] Theorem 3.3.7.

The amenability of $G_T$ now follows from [28] Proposition 9.3].

Our next task is to understand the interior of the isotropy in $G_T$. By definition of the topology on $G_T$ this is the union of all the sets $Z(U, m, n, U)$ such that $U \subset X$ is open and $T^m x = T^n x$ for all $x \in U$. Our approach is based on that of [4] Section 4.

**Lemma 3.6.** Let $T$ be an action of $\mathbb{N}^k$ on $X$ by local homeomorphisms. For each nonempty open set $U \subset X$, let

$$
(3.4) \quad \Sigma_U := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : T^m x = T^n x \text{ for all } x \in U\}.
$$

Then

(a) $\Sigma_U$ is a submonoid of $\mathbb{N}^k \times \mathbb{N}^k$.

(b) $\Sigma_U$ is an equivalence relation on $\mathbb{N}^k$.

(c) If $U \subset V$, then $\Sigma_U \subset \Sigma_V$.

(d) For $p \in \mathbb{N}^k$ and $U$ open and nonempty, we have $\Sigma_U \subset \Sigma_{T^p U}$.

**Proof.** Clearly $(0,0) \in \Sigma_U$. Suppose that $(m,n),(p,q) \in \Sigma_U$. For $x \in U$ we have

$$
(3.5) \quad T^{m+p} x = T^m T^p x = T^n T^q x = T^n x = T^{n+q} x.
$$

This proves (a). Statements (b) and (c) are immediate, and (d) follows from the special case of (3.3) where $p = q$. \hfill $\square$

Since our aim is to identify the primitive ideals of $C^*(G_T)$, and since Lemma 2.1 shows that every irreducible representation of $C^*(G_T)$ factors through the restriction of $G_T$ to some $\mathbb{N}^k$-irreducible subset, we will often assume that $X$ itself (viewed as $G(0)$) is $\mathbb{N}^k$-irreducible. In this case, we will say that $T$ acts irreducibly.

**Lemma 3.7.** Let $T$ be an $\mathbb{N}^k$-irreducible action on $X$ by local homeomorphisms. For all open subsets $U, V \subseteq X$, there exists a nonempty open set $W$ such that $\Sigma_U \cup \Sigma_V \subset \Sigma_W$.

**Proof.** Fix $x$ with $[x] = X$. Choose $y \in U$ and $z \in V$ such that $T^r y = T^s z$ and $T^l z = T^t x$. Then $T^{r+l} y = T^{p+q} z$, so $m = r + l$ and $n = s + p$ satisfy $T^m U \cap T^n V \neq \emptyset$. Since $T^m$ and $T^n$ are local homeomorphisms, and therefore open maps, $W := T^m U \cap T^n V$ is open. Parts (c) and (d) of Lemma 3.6 show that $\Sigma_U \subset \Sigma_{T^m U} \subset \Sigma_W$ and $\Sigma_V \subset \Sigma_{T^n V} \subset \Sigma_W$. \hfill $\square$

Given $X$ and $T$ as in Lemma 3.7, let

$$
(3.6) \quad \Sigma := \bigcup_{\emptyset \neq U \subset X \atop U \text{ open}} \Sigma_U.
$$

We give $\mathbb{N}^k \times \mathbb{N}^k$ the usual partial order as a subset of $\mathbb{N}^{2k}$:

$$
\left( (m_i)_{i=1}^k \right) \leq \left( (m'_i)_{i=1}^k \right) \quad \text{if } n_i \leq m_i \text{ and } n'_i \leq m'_i \text{ for all } i.
$$

We let $\Sigma_{\text{min}}$ denote the collection of minimal elements of $\Sigma \setminus \{(0,0)\}$ with respect to this order.
Lemma 3.8. Let $T$ be an irreducible action of $\mathbb{N}^k$ by local homeomorphisms on a locally compact space $X$, and let $\Sigma$ and $\Sigma^{\text{min}}$ be as above. Then $\Sigma$ is a submonoid of $\mathbb{N}^k \times \mathbb{N}^k$ and an equivalence relation on $\mathbb{N}^k$. We have $\Sigma = (\Sigma - \Sigma) \cap (\mathbb{N}^k \times \mathbb{N}^k)$. Furthermore, $\Sigma^{\text{min}}$ is finite and generates $\Sigma$ as a monoid.

Proof. We have $(0,0) \in \Sigma_X \subset \Sigma$. If $(m,n), (p,q) \in \Sigma$, then there are nonempty open sets $U$ and $V$ such that $(m,n) \in \Sigma_U$ and $(p,q) \in \Sigma_V$. Lemma 3.7 yields an open set $W$ with $(m,n), (p,q) \in \Sigma_W$. Now $(m+p, n+q) \in \Sigma_W \subset \Sigma$ by Lemma 3.6, so $\Sigma$ is a monoid.

To see that $\Sigma$ is an equivalence relation, observe that it is reflexive and symmetric because each $\Sigma_U$ is. Consider $(m,n), (n,p) \in \Sigma$; say $(m,n) \in \Sigma_U$ and $(n,p) \in \Sigma_V$. By Lemma 3.7, there is open set $W$ with $(m,n), (n,p) \in \Sigma_W$. Hence $(m,p) \in \Sigma_W \subset \Sigma$ by Lemma 3.6.

The containment $\Sigma \subset (\Sigma - \Sigma) \cap (\mathbb{N}^k \times \mathbb{N}^k)$ is trivial because $(0,0) \in \Sigma$ and $\Sigma \subset \mathbb{N}^k \times \mathbb{N}^k$. For the reverse containment, suppose that $(m,n), (p,q) \in \Sigma$ and $m+p, n-q \in \mathbb{N}^k$. By Lemma 3.7 we may choose an open $W$ such that $(m,n), (p,q) \in \Sigma_W$. Fix $x \in T^{m+p}W$, say $x = T^{m+q}y$. Lemma 3.6 implies first that $(q,p) \in \Sigma_W$, and then that $(m+q,n+p) \in \Sigma_W$. Hence

$$T^{m+q}x = T^{m+q}y = T^{n+q} = T^{n+q} = T^{n+q} = T^{n+q}x.$$ So $(m+p, n-q) \in \Sigma_{T^{m+q}W} \subset \Sigma$.

Now we argue as in [4] Proposition 4.4. Dickson’s Lemma [26, Theorem 5.1] implies that $\Sigma^{\text{min}}$ is finite. We must show that each $(m,n) \in \Sigma$ is a finite sum of elements of $\Sigma^{\text{min}}$. We argue by induction on $|\{m,n\} := \sum_{i=1}^k m_i + n_i$. If $|\{m,n\} = 0$, the assertion is trivial. Now take $(m,n) \in \Sigma \setminus \{0\}$, and suppose that each $(p,q) \in \Sigma$ such that $|(p,q)| < |(m,n)|$ can be written as a finite sum of elements of $\Sigma^{\text{min}}$. Since $(m,n) \neq 0$, by definition of $\Sigma^{\text{min}}$ there exists $(a,b) \in \Sigma^{\text{min}}$ such that $(a,b) \leq (m,n)$. The preceding paragraph shows that $(p,q) = (m,n) - (a,b) \in (\Sigma - \Sigma) \cap (\mathbb{N}^k \times \mathbb{N}^k) \subset \Sigma$. The induction hypothesis implies that $(p,q)$ is a finite sum of elements of $\Sigma^{\text{min}}$, and then so is $(m,n) = (p,q) + (a,b)$. \hfill \square

We let

$$H(T) := \{m-n : (m,n) \in \Sigma\} \quad \text{and}$$

$$Y^{\max} := \bigcup \{Y \subset X : Y \text{ is open and } \Sigma_Y = \Sigma\}.$$ \hspace{1cm} (3.7)

Lemma 3.9. Let $T$ be an irreducible action of $\mathbb{N}^k$ by local homeomorphisms of a locally compact Hausdorff space $X$. With $\Sigma$ as in (3.6), we have

$$\Sigma = \{(m,n) \in \mathbb{N}^k \times \mathbb{N}^k : m-n \in H(T)\}. \hspace{1cm} (3.8)$$

The set $Y^{\max}$ is nonempty and open, and is the maximal open set in $X$ such that $\Sigma_{Y^{\max}} = \Sigma$. We have $T^m Y^{\max} \subset Y^{\max}$ for all $m \in \mathbb{N}^k$.

Proof. By definition, $\Sigma \subset \{(m,n) : m-n \in H(T)\}$. For the reverse inclusion, suppose that $m-n = p-q$ with $(p,q) \in \Sigma$. Let $g = m-p \in \mathbb{Z}^k$. Fix $a,b \in \mathbb{N}^k$ such that $g = a-b$. Then both $(p+a,q+a)$ and $(b,b)$ belong to $\Sigma$. Hence Lemma 3.8 implies that

$$(m,n) = (p+g, q+g) = (p+a, q+a) - (b,b) \in (\Sigma - \Sigma) \cap (\mathbb{N}^k \times \mathbb{N}^k) = \Sigma.$$ 

\footnote{Though in [4] Proposition 4.4, the crucial use, in the induction, of the fact that $\Sigma = (\Sigma - \Sigma) \cap (\mathbb{N}^k \times \mathbb{N}^k)$ is not made explicit.}
This establishes (3.8).

Now $|\Sigma_Y| - 1$ applications of Lemma 3.7 give a nonempty open set $Y$ such that $\Sigma_Y \subseteq \Sigma_Y$. Since $\Sigma_Y$ is monoid by Lemma 3.6, we have $\Sigma_Y = \Sigma$ by Lemma 3.8.

It now follows that $Y^{\max}$ is open and nonempty. It is clearly maximal. Each $T^mY^{\max} \subseteq Y^{\max}$ by Lemma 3.6(a) and the definition of $Y^{\max}$.

Proposition 3.10. Let $T$ be an irreducible action of $N^k$ by local homeomorphisms of a locally compact Hausdorff space $X$, and let $G_T$ be the associated Deaconu–Renault groupoid (as in (3.1)). The set $H(T)$ of (3.7) is a subgroup of $Z^k$. Let $\Sigma$ be as in (3.6), and let $Y \subset X$ be an open set such that $\Sigma_Y = \Sigma$ and $T^pY \subset Y$ for all $p \in N^k$. Then $\text{Iso}(G_T|_Y)^\circ = \{(x, g, x) : x \in Y \text{ and } g \in H(T)\}$, and $\text{Iso}(G_T|_Y)^\circ$ is closed in $G_T|_Y$.

Proof. Since $\Sigma$ is an equivalence relation, 0 belongs to $H(T)$, and $g \in H(T)$ implies $-g \in H(T)$. Suppose that $m, n \in H(T)$, say $m = p_1 - q_1$ and $n = p_2 - q_2$ with $(p_1, q_1) \in \Sigma$. Lemma 3.8 implies that $(p_1 + p_2, q_1 + q_2) \in \Sigma$, and therefore that $m + n = p_1 + p_2 - q_1 - q_2$ belongs to $H(T)$. So $H(T)$ is a subgroup of $Z^k$.

Take $x \in Y$ and $g \in H(T)$. By Lemma 3.9 there exists $(p, q) \in \Sigma$ such that $g = p - q$. Choose an open neighbourhood $U$ of $x$ in $Y$ on which $T^p$ and $T^q$ are homeomorphisms. By choice of $Y$ we have $T^pY = T^qY$ for all $y \in Y$, and hence $\{(y, g, y) : y \in U\} = Z(U, p, q, U)$ is an open neighbourhood of $(x, g, x)$ contained in $\{(y, g, y) : y \in Y, g \in H(T)\}$. So $\{(y, g, y) : y \in Y, g \in H(T)\} \subseteq \text{Iso}(G_T|_Y)^\circ$. For the reverse inclusion, suppose that $(z, h, z) \in \text{Iso}(G_T)^\circ$. By Lemma 3.1 there exist $m, n \in N^k$ and open sets $U, V \subset Y$ such that $(z, h, z) \in Z(U, m, n, V) \subset \text{Iso}(G_T)$ with $T^mU = T^nV$. So $T^m(x) = T^n(x)$ for all $x \in U$, and then $(m, n) \in \Sigma_U \subset \Sigma$. Thus $h \in H(T)$ as required.

We now have

$$\text{Iso}(G_T|_Y)^\circ = \{(x, g, x) : x \in Y, g \in H(T)\} = G_T \setminus \left( \bigcup_{m-n \not\in H(T)} Z(U, m, n, V) \cup \bigcup_{U \cap V = \emptyset} Z(U, m, n, V) \right),$$

and so $\text{Iso}(G_T|_Y)^\circ$ is closed.

Remark 3.11. We have an opportunity to fill a gap in the literature. The penultimate paragraph of the proof of [4, Theorem 5.3], appeals to [4, Corollary 2.8]. But unfortunately, the authors forgot to verify the hypothesis of [4, Corollary 2.8] that $\Gamma$ should be aperiodic. We rectify this using our results above. Using the definition of aperiodicity of $\Gamma$ [4, page 2575] and of the groupoid $G_T$ of $\Gamma$ [4, page 2573] as in the proof of [4, Corollary 2.8], we see that $\Gamma$ is aperiodic if and only if $G_T$ is topologically principal. In the situation of [4, Theorem 5.3], the groupoid $G_{\text{HAT}}$ discussed there is the restriction of the Deaconu–Renault groupoid $G_{\text{HAT}}$ to $Y = HA^\infty$ which has the properties required of $Y$ in Proposition 3.10 (see [4, Theorem 4.2(2)]), and so Proposition 3.10 shows that $\text{Iso}(G_{\text{HAT}})^\circ$ is closed. It is easy to check that $G_T$ is isomorphic to $G_{\text{HAT}}/\text{Iso}(G_{\text{HAT}})^\circ$. So Proposition 2.6(c) shows that $G_T$ is topologically principal and hence that $\Gamma$ is aperiodic as required.

Corollary 3.12. Let $T$ be an irreducible action of $N^k$ by local homeomorphisms on a locally compact Hausdorff space $X$. Let $\Sigma$ and $H(T)$ be as in (3.6) and (3.7). Suppose that $Y$ is an open subset of $X$ such that $T^pY \subset Y$ for all $p$ and such that $\Sigma_Y = \Sigma$. Then...
(a) Regard $C_c(G_T|Y)$ as a subalgebra of $C_c(G_T)$. The identity map extends to a monomorphism $\iota: C^*(G_T|Y) \to C^*(G_T)$, and $\iota(C^*(G_T|Y))$ is a hereditary subalgebra of $C^*(G_T)$.

(b) The map $\pi \mapsto \pi \circ \iota$ is a bijection from the collection of irreducible representations of $C^*(G_T)$ that are injective on $C_0(X)$ to the space of irreducible representations of $C^*(G_T|Y)$ that are injective on $C_0(Y)$. Moreover, the map $\ker \pi \mapsto \ker(\pi \circ \iota)$ is a homeomorphism from $\{I \in \text{Prim} C^*(G_T) : I \cap C_0(X) = \{0\}\}$ onto $\{J \in \text{Prim} C^*(G_T|Y) : J \cap C_0(Y) = \{0\}\}$.

Proof. The inclusion $C_c(G_T|Y) \hookrightarrow C_c(G_T)$ is a $*$-homomorphism and continuous in the inductive-limit topology. Hence we get a homomorphism $\iota$. Fix $x \in Y$. Let $L^x$ be the regular representation of $C^*(G_T)$ on $l^2((G_T)_x)$. Then $L^x \circ \iota$ leaves the subspace $l^2\{(y, g, x) \in G_T : y \in Y\}$ invariant. Hence $L^x \circ \iota$ is equivalent to $L^x_Y \oplus 0$ where $L^x_Y$ is the corresponding regular representation of $C^*(G_T|Y)$. Since $G_T$ and $G_T|Y$ are both amenable by Lemma 3.5, $\iota$ is isometric and hence a monomorphism.

Let $\{f_i\}$ be an approximate identity for $C_0(Y)$. For $f \in C_c(G_T)$ we have $f_i f_i^* \to C_c(G_T|Y)$. Thus $\iota(C^*(G_T|Y))$ is the closure of $\bigcup_i f_i C^*(G_T)f_i$. It follows easily that the image of $\iota$ is a hereditary subalgebra of $C^*(G_T)$ as claimed.

If $\pi$ is an irreducible representation of $C^*(G_T)$ that is injective on $C_0(X)$, then it does not vanish on the ideal $I_Y$ in $C^*(G_T)$ generated by $C_0(Y)$. Clearly, $\iota(C^*(G_T|Y))$ is Morita equivalent to $I_Y$, and restriction of representations implements Rieffel induction from $I_Y$ to $\iota(C^*(G_T|Y))$. Since Rieffel induction between Morita equivalent $C^*$-algebras takes irreducibles to irreducibles (Corollary 3.32) and since $\pi|I_Y$ is irreducible (Theorem 1.3.4), $\pi \circ \iota$ is irreducible and clearly injective on $C_0(Y)$. If $\rho$ is an irreducible representation of $C^*(G_T|Y)$, then it extends to an irreducible representation of $I_Y$. Since $I_Y$ is an ideal, this representation extends to a (necessarily irreducible) representation $\pi$ of $C^*(G_T)$ such that $\rho = \pi \circ \iota$. The kernel of $\pi|C_0(X)$ is proper and has $N^k$-invariant support. Since $T$ acts irreducibly, $C_0(X)$ is $G_T$-simple, and so $\ker(\pi|C_0(X)) = \{0\}$ and we obtain the required bijection.

The remaining assertion follows from this bijection and the Rieffel correspondence (see [21 Corollary 3.33(a)]).

4. The primitive ideals of the $C^*$-algebra of an irreducible Deaconu–Renault groupoid

In this section we specialize to the situation where $T$ is an irreducible action of $N^k$ on a locally compact Hausdorff space $Y$ with the property that, in the notation of Lemma 3.6, $\Sigma_Y = \Sigma$. We then have $\Sigma = \Sigma_U$ for all nonempty open subsets $U$ of $Y$ by Lemma 3.9 says that $m - n \in H(T)$ implies $T^m x = T^n x$ for all $x \in Y$; and Proposition 3.10 gives

$$\text{Iso}(G_T)^\circ = \{(x, g, x) : x \in Y \text{ and } g \in H(T)\}.$$  

We show that under these hypotheses, the primitive ideals of $C^*(G_T)$ with trivial intersection with $C_0(Y)$ are indexed by characters of $H(T)$. More precisely, we show that the irreducible representations of $C^*(G_T)$ that are faithful on $C_0(Y)$ are indexed by pairs $(\pi, \chi)$ where $\pi$ is an irreducible representation of $C^*(G_T/\text{Iso}(G_T)^\circ)$ and $\chi$ is a character of $H(T)$. Our approach is to exhibit $C^*(G_T)$ as an induced algebra. Recall from Proposition 3.10 that $\text{Iso}(G_T)^\circ$ is closed in $G_T$, so Proposition 2.6 gives a homomorphism $\kappa: C^*(G_T) \to C^*(G_T/\text{Iso}(G_T)^\circ)$. 
Lemma 4.1. Suppose that $T$ is an irreducible action of $\mathbb{N}^{k}$ on a locally compact space $Y$ such that $\Sigma_{Y} = \Sigma$. There is an action $\alpha$ of $T^{k}$ on $C^{*}(G_{T})$ such that $\alpha_{z}(f)(x, g, y) = z^a f(x, g, y)$ for $f \in C_{c}(G_{T})$. Let $\kappa : C^{*}(G_{T}) \to C^{*}(G_{T}/\text{Iso}(G_{T})^{\circ})$ be the homomorphism of Proposition 2.6. There is an action $\bar{\alpha}$ of $H(T)^{\perp}$ on $C^{*}(G_{T}/\text{Iso}(G_{T})^{\circ})$ such that $\bar{\alpha}_{z} \circ \kappa = \kappa \circ \alpha_{z}$ for all $z \in H(T)^{\perp} \subset T^{k}$.

If $\bar{z}w \notin H(T)^{\perp}$, then $(\ker(\kappa \circ \alpha_{z}) + \ker(\kappa \circ \alpha_{w})) \cap C_{0}(Y) \neq \{0\}$. We have $\ker(\kappa \circ \alpha_{z}) = \ker(\kappa \circ \alpha_{w})$ if and only if $\bar{z}w \in H(T)^{\perp}$.

Proof. Let $c : G_{T} \to \mathbb{Z}^{k}$ be the canonical cocycle $c(x, g, y) = g$. The formula $\alpha_{z}(f)(\gamma) = z^{c(\gamma)}f(\gamma)$ defines a $\ast$-homomorphism $\alpha_{z} : C_{c}(G_{T}) \to C_{c}(G_{T})$. This $\alpha_{z}$ is trivially $L$-norm preserving, so extends to $\alpha_{z} : C^{*}(G_{T}) \to C^{*}(G_{T})$. Since $\alpha_{z}$ is an inverse for $\alpha_{z}$, we have $\alpha_{z} \in \text{Aut}(C^{*}(G_{T}))$. The map $z \mapsto \alpha_{z}$ is a homomorphism because $\alpha_{z}w$ and $\alpha_{z} \circ \alpha_{w}$ agree on each $C_{c}(c^{-1}(g))$. To see that $z \mapsto \alpha_{z}$ is strongly continuous, first note that if $f \in C_{c}(G_{T})$ is supported on $c^{-1}(g)$, then each $\alpha_{z}(f) = z^{a}f$, so $z \mapsto \alpha_{z}(f)$ is continuous. Since each $f \in C_{c}(G_{T})$ is a finite linear combination $f = \sum_{\supp(f) \cap c^{-1}(g) \neq \emptyset} f|_{c^{-1}(g)}$ of such functions, $z \mapsto \alpha_{z}(f)$ is continuous for each $f \in C_{c}(G_{T})$. Now an $\epsilon/3$ argument shows that $z \mapsto \alpha_{z}$ is strongly continuous.

Let $q : \mathbb{Z}^{k} \to \mathbb{Z}^{k}/H(T)$ be the quotient map. We have

$$\text{Iso}(G_{T})^{\circ} = \{(x, g, y) : x \in Y \text{ and } g \in H(T)\}.$$  

Identify $G_{T}/\text{Iso}(G_{T})^{\circ}$ with $\{(x, q(g), y) : (x, g, y) \in G_{T}\} \subset Y \times (\mathbb{Z}^{k}/H(T)) \times Y$.

Proposition 2.4 implies that the quotient map from $G_{T}$ onto $G_{T}/\text{Iso}(G_{T})^{\circ}$ is continuous and open, so the sets

$$\mathbb{Z}(U, q(m), q(n), V) = \{(x, q(m-n), y) : x \in U, y \in V \text{ and } T^{m}x = T^{n}y\}$$

are a basis for the topology on $G_{T}/\text{Iso}(G_{T})^{\circ}$ (this makes sense because $T^{m}x = T^{n}y$ if and only if $T^{m+a}x = T^{n+b}y$ whenever $a - b \in H(T)$). Arguing as in the first paragraph, we get an action $\tilde{\alpha}$ of $H(T)^{\perp}$ on $C^{*}(G_{T}/\text{Iso}(G_{T})^{\circ})$ such that $\tilde{\alpha}_{z}(f)(x, q(g), y) = z^{a}f(x, q(g), y)$ for $f \in C_{c}(G_{T}/\text{Iso}(G_{T})^{\circ})$. For $f \in C_{c}(G_{T})$, it is easy to check that $\tilde{\alpha}_{z} \circ \kappa(f) = \kappa \circ \alpha_{z}(f)$ for $z \in H(T)^{\perp}$. This identity then extends by continuity to all of $C^{*}(G_{T})$.

Suppose that $\bar{z}w \notin H(T)^{\perp}$. Choose $n \in H(T)$ such that $z^{n} \neq w^{n}$. Fix a nonzero function $f \in C_{c}(Y)$ and define $f_{n} \in C_{c}(\{(x, n, x) : x \in Y\} \subset C_{c}(G_{T})$ by $f_{n}(x, n, x) = f(x, 0, x)$ for all $x \in Y$. Then $w^{n}f - f_{n} \in \ker(\kappa \circ \alpha_{w})$ and $z^{n}f - f_{n} \in \ker(\kappa \circ \alpha_{z})$. Hence $(z^{n} - w^{n})f \in (\ker(\kappa \circ \alpha_{z}) + \ker(\kappa \circ \alpha_{w})) \cap C_{0}(Y) \setminus \{0\}$ by choice of $n$. This proves the second-last statement of the lemma. Since each of $\kappa \circ \alpha_{z}$ and $\kappa \circ \alpha_{w}$ is injective on $C_{0}(Y)$, this also proves the (contrapositive of the) implication $\implies$ in the final statement of the lemma. For the reverse implication, suppose that $\bar{z}w \in H(T)^{\perp}$. Then

$$\ker(\kappa \circ \alpha_{w}) = \ker(\kappa \circ \alpha_{z}w \circ \alpha_{z}) = \ker(\tilde{\alpha}_{z}w \circ \kappa \circ \alpha_{z}) = \ker(\kappa \circ \alpha_{z}). \quad \Box$$

The last assertion of Lemma 4.1 ensures that we can form the induced algebra $\text{Ind}_{H(T)^{\perp}}^{T^{k}}(\alpha)$, namely

$$\{s \in C(T^{k}, C^{*}(G_{T}/\text{Iso}(G_{T})^{\circ})) : s(wz) = \tilde{\alpha}_{z}(s(w)) \text{ for all } w \in T^{k} \text{ and } z \in H(T)^{\perp}\}.$$  

Induced algebras have a well-understood structure. Some of their elementary properties (in particular, the ones that we rely upon) are discussed in [21, §6.3].
Before proving the next result, we recall some basic results from abelian harmonic analysis. We write $C_c(H(T))$ for the set of finitely supported functions on $H(T)$. If $\varphi \in C_c(H(T))$, then its Fourier transform $\hat{\varphi} \in C(\mathbb{T}^k)$ is given by

$$\hat{\varphi}(z) = \sum_{n \in H(T)} \varphi(n) z^n,$$

and is constant on $H(T)^\perp$ cosets. Taking a few liberties with notation and terminology, we regard $\hat{\varphi}$ as an element of $C(\mathbb{T}^k/H(T)^\perp)$. The general theory implies that $\{\hat{\varphi} : \varphi \in C_c(H(T))\}$ is a (uniformly) dense subalgebra of $C(\mathbb{T}^k/H(T)^\perp)$.

**Lemma 4.2.** Let $T$ be an irreducible action of $\mathbb{N}^k$ on a locally compact space $Y$ by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. If $(x,g,y) \in G_T$, then $(x,g+n,y) \in G_T$ for all $n \in H(T)$.

**Proof.** Let $(x,g,y) = (x,p-q,y)$ with $T^p x = T^q y$. Fix $n \in H(T)$. Then $n = n_+ - n_-$ with $(n_+, n_-) \in \Sigma = \Sigma_Y$. Hence $T^{n+z} = T^{n-z}$ for all $z \in Y$, giving

$$T^{p+n_+} x = T^{n_+} T^p x = T^{n_+} T^q y = T^{n_-} T^q y = T^{q+n_-} y.$$

Hence $(x,g+n,y) = (x,(p+n_+)-(q+n_-),y) \in G_T$. \qed

Because of Lemma 4.2, we can define a left action of $C_c(H(T))$ on $C_c(G_T)$ by

$$\varphi \cdot f(x,g,y) := \sum_{n \in H(T)} \varphi(n) f(x,g-n,y).$$

**Lemma 4.3.** Let $T$ be an irreducible action of $\mathbb{N}^k$ on a locally compact space $Y$ by local homeomorphisms such that $\Sigma_Y = \Sigma$, and let $\kappa : C^\ast(G_T) \to C^\ast(G_T/\text{Iso}(G_T)^\circ)$ be as in Proposition 2.6. Then

$$\kappa(\alpha_z(\varphi \cdot f)) = \hat{\varphi}(z) \kappa(\alpha_z(f))$$

for all $f \in C_c(G_T)$, all $z \in \mathbb{T}^k$, and all $\varphi \in C_c(H(T))$.

**Proof.** We compute:

$$\kappa(\alpha_z(\varphi \cdot f))(x,q(g),y) = \sum_{m \in H(T)} \alpha_z(\varphi \cdot f)(x,g+m,y)$$

$$= \sum_{m \in H(T)} z^{g+m} \varphi \cdot f(x,g+m,y)$$

$$= \sum_{m \in H(T)} \sum_{n \in H(T)} z^{g+m+n} \varphi(n) f(x,g+m-n,y).$$

Since both sums are finite and we can interchange the order of summations at will, we may continue the calculation:

$$= \sum_{m \in H(T)} \sum_{n \in H(T)} z^{g+m+n} \varphi(n) f(x,g+m,y)$$

$$= \sum_{m \in H(T)} z^{g+m} \hat{\varphi}(z) f(x,g+m,y)$$

$$= \hat{\varphi}(z) \kappa(\alpha_z(f))(x,q(g),y).$$ \qed
Proposition 4.4. Let $T$ be an irreducible action of $\mathbb{N}^k$ on a locally compact space $Y$ by local homeomorphisms, and suppose that $\Sigma_Y = \Sigma$. Let

$$\alpha : \mathbf{T}^k \to \text{Aut} C^*(G_T) \quad \text{and} \quad \tilde{\alpha} : H(T)^\perp \to \text{Aut} C^*(G_T / \text{Iso}(G_T)^0)$$

be as in Lemma 4.1 and let $\kappa : C^*(G_T) \to C^*(G_T / \text{Iso}(G_T)^0)$ be as in Proposition 2.6. There is an isomorphism $\Phi : C^*(G_T) \to \text{Ind}_{H(T)^\perp} \bigl(C^*(G_T / \text{Iso}(G_T)^0), \tilde{\alpha}\bigr)$ such that $\Phi(a)(z) = \kappa(\alpha_z(a))$ for $a \in C^*(G_T)$ and all $z \in T$.

Proof. For $a \in C^*(G_T)$, the map $z \mapsto \kappa(\alpha_z(a))$ is continuous by continuity of $\alpha$. Take $f \in C_c(G_T)$, $w \in \mathbf{T}^k$ and $z \in H(T)^\perp$. Lemma 4.1 gives $\tilde{\alpha}_z \circ \kappa = \kappa \circ \alpha_z$. Hence

$$\Phi(f)(wz) = \kappa(\alpha_{wz}(f)) = \kappa(\alpha_z(\alpha_w(f))) = \tilde{\alpha}_z \kappa(\alpha_w(f)) = \tilde{\alpha}_z(\Phi(f)(w)).$$

Thus $\Phi$ takes values in $\text{Ind}_{H(T)^\perp} \bigl(C^*(G_T / \text{Iso}(G_T)^0), \tilde{\alpha}\bigr)$. It is not hard to check that $\Phi$ is a homomorphism.

To see that $\Phi$ is injective we use an averaging argument. Let $\mathbf{T}^k$ act on the left of $\text{Ind}_{H(T)^\perp} \bigl(C^*(G_T / \text{Iso}(G_T)^0), \tilde{\alpha}\bigr)$ by left translation: $\text{lt}_z(c)(w) = c(zw)$. We have $\Phi \circ \alpha_z = \text{lt}_z \circ \Phi$. So the standard argument involving the faithful conditional expectations obtained from averaging over $\mathbf{T}^k$ actions (see, for example, [27, Lemma 3.13]) shows that it is sufficient to check that $\Phi$ restricts to an injection on $C^*(G_T)^0$.

If $f \in C_c(G_T)$, then arguing as in [29, Lemma 1.108], we have $\int_{\mathbf{T}^k} \alpha_z(f) \, dz \in C_c(G_T)$ and for $\gamma \in G_T$,

$$\left( \int_{\mathbf{T}^k} \alpha_z(f) \, dz \right)(\gamma) = \int_{\mathbf{T}^k} \alpha_z(f)(\gamma) \, dz = \left( \int_{\mathbf{T}^k} z^{c(\gamma)} \, dz \right) f(\gamma) = \begin{cases} f(\gamma) & \text{if } \gamma \in c^{-1}(0) \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $C^*(G_T)^0 = \overline{C_c(c^{-1}(0))} \subset C^*(G_T)$. Thus the inclusion map induces a monomorphism $\rho : C^*(c^{-1}(0)) \to C^*(G_T)$ whose image is exactly $C^*(G_T)^0$. To see that $\Phi|_{C^*(G_T)^0}$ is injective, it suffices to show that $\Phi \circ \rho$ is injective. Since $c^{-1}(0)$ is amenable by Lemma 3.5 and principal by construction, [6, Theorem 4.4] implies that we need only show that $(\Phi \circ \rho)|_{C_0(Y)}$ is injective. As $\rho$ restricts to the canonical inclusion $C_0(Y) \hookrightarrow C^*(G_T)^0$, it is enough to verify that $\Phi$ is injective on $C_0(Y)$.

The homomorphism $\kappa \circ \alpha_z$ restricts to the identity map of $C_0(Y) \subset C^*(G_T)$ onto $C_0(Y) \subset C^*(G_T / \text{Iso}(G_T)^0)$. So if $f \in C_0(Y)$, $z \in \mathbf{T}^k$ and $b \in G_T / \text{Iso}(G_T)^0$, then

$$\Phi(f)(z)(b) = \kappa(\alpha_z(f))(b) = \begin{cases} f(x) & \text{if } b = (x, 0, x) \in (G_T / \text{Iso}(G_T)^0)^0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus if $\Phi(f) = 0$, then $f = 0$. This completes the proof that $\Phi$ is injective.

We still have to show that $\Phi$ is surjective. Lemma 4.3 implies that if $\varphi \in C_c(H(T))$ and $f \in C_c(G_T)$, then $\hat{\varphi} \cdot \Phi(f) = \Phi(\varphi \cdot f)$ for the obvious action of $C(\mathbf{T}^k / H(T))$ on the induced algebra. Since $\{ \hat{\varphi} : \varphi \in C_c(H(T)) \}$ is dense in $C(\mathbf{T}^k / H(T))$ it follows that the range of $\Phi$ is a $C(\mathbf{T}^k / H(T))$-submodule. So it suffices to show that the range of $\kappa \circ \alpha_z$ contains $C_c(G_T / \text{Iso}(G_T)^0)$.
For this, fix $g \in \mathbb{Z}^k$ and $f \in \hat{c}^{-1}(q(g))$; it suffices to show that $f$ is in the range of $\pi \circ \alpha_z$. Define $h \in C_c(G_T)$ by

$$h(\gamma) = \begin{cases} \overline{\mathcal{P}}^g f(\tilde{q}(\gamma)) & \text{if } c(\gamma) = g \\ 0 & \text{otherwise}. \end{cases}$$

Then $h$ is continuous because each $c^{-1}(g)$ is clopen in $G_T$; and $\kappa(\alpha_z(h)) = f$. □

We now aim to apply [21, Proposition 6.6], which describes the primitive-ideal space of an induced algebra, to describe the topology of $\text{Prim}(C^*(G_T))$ for a special class of $\mathbb{N}^k$-actions $T$. To achieve this we first describe, in Lemma 4.6, the Jacobson topology on $\text{Prim}(C^*(G))$ when $G$ is an amenable étale Hausdorff groupoid whose reduction to any closed invariant set is topologically principal. This topology is also described by [24, Corollary 4.9], but the statement given there is not quite the one we need.

**Lemma 4.5.** Let $G$ be a second-countable locally compact Hausdorff étale groupoid, and fix $x \in G^{(0)}$. There is an irreducible representation $\omega_x : C^*(G) \to \mathcal{B}(\ell^2([x]))$ satisfying $\omega_x(f)\delta_y = \sum_{x(\gamma) = y} f(\gamma)\delta_{\gamma}$ for all $f \in C_c(G)$. If $G$ is topologically principal and amenable and if $[x]$ is dense in $G^{(0)}$, then $\omega_x$ is faithful, and hence $C^*(G)$ is primitive.

**Proof.** Let $E_x$ denote the 1-dimensional representation of the group $G^{(0)}$. Then $\omega_x := \text{Ind}_{G^{(0)}}^{G} E_x$ is a representation satisfying the desired formula. Hence $\omega_x$ is irreducible by [12, Theorem 5].

Now suppose that $G$ is amenable and topologically principal with $[x]$ dense in $G^{(0)}$. Then clearly $\omega_x$ is faithful on $C_0(G^{(0)})$. So [6, Theorem 4.4] says that it is faithful on $C^*(G)$, whence $C^*(G)$ is primitive. □

Recall that the quasi-orbit space $Q(G) = \{[x] : x \in G^{(0)}\}$ carries the quotient topology for the map $q : G^{(0)} \to Q(G)$ that identifies $u$ with $v$ exactly when $[u]$ and $[v]$ have the same closure in $G^{(0)}$. In particular, if $S \subset Q(G)$, then $S = \{q(x) : x \in q^{-1}(S)\}$.

**Lemma 4.6.** Let $G$ be an amenable, étale Hausdorff groupoid and suppose that $G|_X$ is topologically principal for every closed invariant subset $X$ of the unit space. For $x \in G^{(0)}$, let $\omega_x$ be the irreducible representation of $G^{(0)}$ and $\omega_x$ from $G^{(0)}$ to $\text{Prim}(C^*(G))$ descends to a homeomorphism of the quasi-orbit space $Q(G)$ onto $\text{Prim}(C^*(G))$.

**Proof.** For $x \in G^{(0)}$, we have $\ker \omega_x \cap C_0(G^{(0)}) = C_0(G^{(0)} \setminus \overline{\{x\}})$. Since $G|_X$ is topologically principal for every closed invariant subset $X \subset G^{(0)}$, [24, Corollary 4.9] therefore implies that $\ker \omega_x = \ker \omega_y$ if and only if $\overline{\{x\}} = \overline{\{y\}}$. Hence $x \mapsto \ker \omega_x$ descends to a well-defined injection $\overline{\{x\}} \mapsto \ker \omega_x$. To see that it is surjective, observe that if $\pi$ is an irreducible representation of $C^*(G)$, then Proposition 2.4 implies that $\ker \pi \cap C_0(G^{(0)}) = C_0(G^{(0)} \setminus \overline{\{x\}})$ for some $x \in G^{(0)}$. That is,

\footnote{This is also the representation described in [5, Proposition 5.2].}

\footnote{Although the term has been used inconsistently, in [24] for example, one says the $G$-action on $G^{(0)}$ is essentially free.}

\footnote{Specifically, [24, Corollary 4.9] applied to the groupoid dynamical system $(G, \Sigma, s')$ where $\Sigma$ is the bundle of trivial groups over $G^{(0)}$ and $s'$ is the trivial bundle $G^{(0)} \times C$ of 1-dimensional $C^*$-algebras—see also [3, Corollary 5.9].}
\[ \ker \pi \cap C_0(G^{(0)}) = \ker \omega_x \cap C_0(G^{(0)}), \] and then Corollary 4.9 again shows that \( \ker \pi = \ker \omega_x \).

To show that \( \overline{x} \mapsto \ker \omega_x \) is a homeomorphism, it suffices to take a set \( S \subset Q(G) \) and an element \( x \in G^{(0)} \) and show that \( \overline{x} \in S \) if and only if \( \ker \omega_x \in \{ \ker \omega_y : q(y) \in S \} \); for then \( S \subset Q(G) \) is closed if and only if its image \( \{ \ker \omega_y : q(y) \in S \} \) is closed in \( \text{Prim}(C^*(G)) \).

Fix \( S \subset Q(G) \) and \( x \in G^{(0)} \). We have
\[
\{ \ker \omega_y : q(y) \in S \} = \{ \ker \omega_z : \bigcap_{q(y) \in S} \ker \omega_y \subset \ker \omega_z \}.
\]

Using Corollary 4.9 again, we deduce that \( \ker \omega_x \subset \{ \ker \omega_y : q(y) \in S \} \) if and only if \( \bigcap_{q(y) \in S} \ker \omega_y \subset \ker \omega_x \), and therefore carries orbit closures in \( G/\omega \) bijectively to the corresponding \( \text{Prim}(C^*(G)) \)-orbits, and hence the identification \( \ker \omega_x \subset \{ \ker \omega_y : q(y) \in S \} \) is equivalent to \( q(x) \in S \in \{ q(x) : x \in q^{-1}(S) \} \), since \( q^{-1}(S) \) is closed and invariant.

For the next result, recall that the quotient map \( q : G \to G/\text{Iso}(G)^\circ \) restricts to a homeomorphism of unit spaces. Since \( q \) also preserves the range and source maps, it carries \( G \)-orbits bijectively to the corresponding \( (G/\text{Iso}(G)^\circ) \)-orbits, and therefore carries orbit closures in \( G/\text{Iso}(G)^\circ \) to the corresponding orbit closures in \( G/\text{Iso}(G)^\circ \). Hence the identification \( G^{(0)} = (G/\text{Iso}(G)^\circ)^{(0)} \) induces a homeomorphism \( Q(G) \cong Q(G/\text{Iso}(G)^\circ) \).

**Theorem 4.7.** Let \( T \) be an irreducible action of \( \mathbb{N}^k \) on a locally compact space \( Y \) by local homeomorphisms such that, in the notation of Proposition 3.6, \( \Sigma_Y = \Sigma \). Suppose that for every \( y \in Y \), the set
\[
\Sigma_{[y]} := \{ (m, n) \in \mathbb{N}^k \times \mathbb{N}^k : T^m x = T^n x \text{ for all } x \in [y] \}
\]
satisfies \( \Sigma_{[y]} = \Sigma \). Let \( \alpha : \mathbb{T}^k \to \text{Aut} C^*(G_T) \) be as in Lemma 4.1 and let \( \kappa : C^*(G_T) \to C^*(G_T/\text{Iso}(G_T)^\circ) \) be as in Proposition 2.6. For \( y \in (G_T/\text{Iso}(G_T)^\circ)^{(0)} \), let \( \omega_x \) be the irreducible representation of \( C^*(G_T) \) described in Lemma 4.6. Then the map \( (y, z) \mapsto \ker(\omega_y \circ \alpha_z) \) from \( Y \times \mathbb{T}^k \) to \( \text{Prim}(C^*(G_T)) \) descends to a homeomorphism \( Q(G_T) \times H(T)^\wedge \cong \text{Prim}(C^*(G_T)) \).

**Proof.** Let \( \Phi : C^*(G_T) \to \text{Ind}_{H(T)^\wedge}^H(C^*(G_T/\text{Iso}(G_T)^\circ), \hat{\alpha}) \) be the isomorphism of Proposition 4.4. For each \( y \in Y \), let \( \hat{\omega}_y \) be the irreducible representation of \( C^*(G_T/\text{Iso}(G_T)^\circ) \) obtained from Lemma 4.5. Observe that \( \hat{\omega}_y \circ \kappa = \omega_y \). We have
\[
\Phi(\ker(\omega_y \circ \alpha_z)) = \{ s \in \text{Ind}_{H(T)^\wedge}^H(C^*(G_T/\text{Iso}(G_T)^\circ), \hat{\alpha}) : f(z) \in \ker \hat{\omega}_y \}.
\]

Write \( \varepsilon_z \) for the homomorphism of the induced algebra onto \( C^*(G_T/\text{Iso}(G_T)^\circ) \) given by evaluation at \( z \). It now suffices to show that
\[
(y, z) \mapsto \ker(\hat{\omega}_y \circ \varepsilon_z).
\]
induces a homeomorphism of $Q(G_T) \times H(T)^\wedge$ onto the primitive ideal space of the induced algebra.

Proposition 3.10 combined with the hypothesis that each $\Sigma_{[y]} = \Sigma$ ensures that $\text{Iso}(G)^\circ_{[y]} = \text{Iso}(G)^\circ_{[y]}$ for each $y$. Hence Proposition 2.5 ensures that the reduction of $G_T/\text{Iso}(G_T)^\circ$ to any orbit closure, and hence to any closed invariant set, is topologically principal. Now Lemma 4.6 implies that $\ker(\tilde{\omega}_y \circ \varepsilon_z) = \ker(\tilde{\omega}_z \circ \varepsilon_z)$ if and only if $[y] = [x]$. So the map (4.2) descends to a map $([y], z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$.

Composing this with the homeomorphism of Lemma 4.6 shows that (4.2) induces a well-defined map

$$M : (\ker \tilde{\omega}_y, z) \mapsto \ker(\tilde{\omega}_y \circ \varepsilon_z)$$

from $\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ)) \times T^k$ to $\text{Prim}(\text{Ind}_{H(T)}^T(C^*(G_T/\text{Iso}(G_T)^\circ), \tilde{\alpha}))$. An application of Proposition 6.16 of [21]—or, rather, of the obvious primitive-ideal version of that result—shows that $M$ induces a homeomorphism of the quotient of $(\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ)) \times T^k)$ by the diagonal action of $H(T)^+$ onto the primitive ideal space of the induced algebra. Since the action of $H(T)^+$ on $T^k$ is by translation and has quotient $H(T)^\wedge$, it now suffices to show that the action of $H(T)^+$ on $\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ))$ is trivial. Since $\tilde{\alpha}_z$ fixes $C_0(G(0)) \subset C^*(G_T/\text{Iso}(G_T)^\circ)$ pointwise, for any ideal $I$ of $C^*(G_T/\text{Iso}(G_T)^\circ)$, we have $\tilde{\alpha}_z(I) \cap C_0(G(0)) = I \cap C_0(G(0))$, and then [24 Corollary 4.9] implies that $\tilde{\alpha}_z(I) = I$. So $H(T)^+$ acts trivially on $\text{Prim}(C^*(G_T/\text{Iso}(G_T)^\circ))$. □

5. THE PRIMITIVE IDEALS OF THE $C^*$-ALGEBRA OF A DEACONU–RENAULT GROUPOID

In this section, our aim is to catalogue the primitive ideals of $C^*(G_T)$. We need to refine our notation from Section 4 to accommodate actions which are not necessarily irreducible.

Notation. Let $T$ be an action of $\mathbb{N}^k$ on a locally compact space $X$ by local homeomorphisms. Recall that for $x \in X$,

$$[x] = \{ y \in X : T^m x = T^n y \text{ for some } m, n \in \mathbb{N}^k \}.$$  

For $x \in X$ and $U \subset [x]$ relatively open, let

$$\Sigma(x)_U := \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : T^m y = T^n y \text{ for all } y \in U \},$$

and define

$$\Sigma(x) := \bigcup_U \Sigma(x)_U.$$  

Lemma 3.9 implies that

$$Y(x) := \bigcup \{ Y \subset [x] : Y \text{ is relatively open and } \Sigma(x)_Y = \Sigma(x) \}$$

is nonempty and is the maximal relatively open subset of $[x]$ such that $\Sigma(x)_Y = \Sigma(x)$. Proposition 3.10 implies that

$$H(x) := H(T{[x]}) = \{ m - n : (m, n) \in \Sigma(x) \}$$

is a subgroup of $\mathbb{Z}^k$. To lighten notation, set $\mathcal{I}(x) := \text{Iso}(G_T|_Y(x))^\circ$. Proposition 3.10 says that

$$\mathcal{I}(x) = \{(y, g, y) : y \in Y(x) \text{ and } g \in H(x) \},$$

and is a closed subset of $G_T|_Y(x)$.
Lemma 5.1. Let $T$ be an action of $N^k$ on a locally compact Hausdorff space $X$ by local homeomorphisms. For $x, y \in X$, we have $Y(x) = Y(y)$ if and only if $[x] = [y]$.

Proof. The “if” direction is trivial. Suppose that $Y(x) = Y(y)$. By symmetry, it suffices to show that $y \in [x]$. Since $Y(x) = Y(y)$ is open in $[y]$, we have $Y(x) \cap [y] \neq \emptyset$. Since $Y(x) \subset [x]$, and $[x]$ is $G_T$-invariant, we deduce that $y \in [x]$. □

The key to the proof of our main theorem is the following result, which works at the level of irreducible representations.

Theorem 5.2. Let $T$ be an action of $N^k$ on a locally compact Hausdorff space $X$ by local homeomorphisms. Take $x \in X$ and $z \in T^k$. Suppose that $\rho$ is a faithful irreducible representation of $C^*(G_T \mid Y(x) / \mathcal{I}(x))$. Let $\iota : C^*(G_T \mid Y(x)) \to C^*(G_T)$ be the inclusion of Corollary 3.12. Let

$$\Phi : C^*(G_T \mid Y(x)) \to \text{Ind}^T_{H(Z)}(C^*(G_T \mid Y(x) / \mathcal{I}(x)), \hat{\alpha})$$

be the isomorphism of Proposition 4.3, and let

$$\varepsilon_z : \text{Ind}^T_{H(Z)}(C^*(G_T \mid Y(x) / \mathcal{I}(x)), \hat{\alpha}) \to C^*(G_T \mid Y(x) / \mathcal{I}(x))$$

denote evaluation at $z$. Let $R_z : C^*(G_T) \to C^*(G_T \mid z^T)$ be the homomorphism induced by restriction of compactly supported functions. There is a unique irreducible representation $\pi_{x,z,\rho}$ of $C^*(G_T)$ such that

(a) $\pi_{x,z,\rho}$ factors through $R_z$, and

(b) the representation $\pi^0_{x,z,\rho}$ of $C^*(G_T \mid z^T)$ such that $\pi_{x,z,\rho} = \pi^0_{x,z,\rho} \circ R_z$ satisfies

$$\pi^0_{x,z,\rho} \circ \iota = \rho \circ \varepsilon_z \circ \Phi.$$

Every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,z,\rho}$ for some $x, z, \rho$.

Proof. The representation $\rho \circ \varepsilon_z \circ \Phi$ is an irreducible representation of $C^*(G_T \mid Y(x))$, and is injective on $C_0(Y(x))$ because both $\Phi$ and $\varepsilon_z$ restrict to injections on $C_0(Y(x))$. Corollary 3.12(b) applied to $Y(x) \subset [x]$ yields a unique representation $\pi^0_{x,z,\rho}$ of $C^*(G_T \mid z^T)$ such that $\pi^0_{x,z,\rho} \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. The set $[x]$ is a closed invariant set in $X$. As in Proposition 2.4, restriction of functions induces a homomorphism $R_x : C^*(G_T) \to C^*(G_T \mid [x])$. Now $\pi_{x,x,\rho} := \pi^0_{x,x,\rho} \circ R_x$ satisfies (a) and (b).

For uniqueness, take a representation $\varphi$ of $C^*(G_T)$ satisfying (a) and (b). Then $\varphi$ vanishes on the ideal generated by $C_0(X \setminus [x])$ which is precisely the kernel of $R_x$ by Proposition 2.4. So $\varphi = \varphi_0 \circ R_x$ for some irreducible representation $\varphi_0$ of $C^*(G_T \mid [x])$ satisfying $\varphi_0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. We saw in the preceding paragraph that $\pi^0_{x,x,\rho}$ is the unique such representation, so $\varphi_0 = \pi^0_{x,x,\rho}$ and hence $\varphi = \pi_{x,x,\rho}$.

To see that every irreducible representation of $C^*(G_T)$ has the form $\pi_{x,x,\rho}$, fix an irreducible representation $\varphi$ of $C^*(G_T)$. Since it is irreducible, Proposition 2.4 implies that $\varphi = \varphi_0 \circ R_x$ for some $x \in X$ and some irreducible representation $\varphi_0$ of $C^*(G_T \mid [x])$ that is faithful on $C_0([x])$. Since $\Phi$ is an isomorphism, Corollary 3.12(b) implies that $\varphi_0$ is uniquely determined by $\varphi_0 \circ \iota \circ \Phi^{-1}$, which is an irreducible representation of $\text{Ind}^T_{H(Z)}(C^*(G_T \mid Y(x) / \mathcal{I}(x)), \hat{\alpha})$ that is faithful on $C_0(Y(x))$. By [21 Proposition 6.16], there exists $z$ such that $\ker(\varepsilon_z) \subset \ker \varphi_0 \circ \iota \circ \Phi^{-1}$, and then $\varphi_0 \circ \iota \circ \Phi^{-1}$ descends to an irreducible representation $\rho$ of $C^*(G_T \mid Y(x) / \mathcal{I}(x))$. That is $\rho \circ \varepsilon_z = \varphi_0 \circ \iota \circ \Phi^{-1}$. Post-composing with $\Phi$ on both sides of this equation shows that $\varphi_0 \circ \iota = \rho \circ \varepsilon_z \circ \Phi$. So we now need only prove that $\rho$ is faithful.
Since $\varphi^0$ is faithful on $C_0([x])$, the composition $\varphi^0\circ\Phi^{-1}$ is faithful on $C_0(Y(x))$, and hence $\rho$ is faithful on $C_0(Y(x)) = C_0(G_T|_{Y(x)}/\mathcal{I}(x))$. Proposition 2.5 implies that $G_T|_{Y(x)}/\mathcal{I}(x)$ is topologically principal, and Proposition 2.5 combined with Lemma 3.3 implies that $G_T|_{Y(x)}/\mathcal{I}(x)$ is amenable. So Theorem 4.4 implies that $\rho$ is faithful as claimed.

**Proof of Theorem 3.3.** Fix $x \in G_T^{(0)}$ and $z \in T^k$. Let $\alpha_z \in \text{Aut}(C^*(G_T))$ be the automorphism of Lemma 4.1 and let $\omega_{[x]}$ be the irreducible representation of Lemma 4.5. Then $\pi_{x,z} := \omega_{[x]} \circ \alpha_z$ is an irreducible representation satisfying (3.3). Furthermore $\pi_{x,z}|_{C_0(G^{(0)})}$ has support $[x]$.

It is clear that the relation $\sim$ is an equivalence relation. To see that $\ker\pi_{x,z} = \ker\pi_{y,w}$ if and only if $[x] = [y]$ and $\exists w \in H(x)^\perp$, first suppose that $[x] \neq [w]$. Then $\ker\pi_{x,z} \cap C_0(X) \neq \ker\pi_{y,w} \cap C_0(X)$.

Second, suppose that $[x] = [y]$ but $\exists w \notin H(x)$. Then $\pi_{x,z}$ and $\pi_{y,w}$ descend to representations $\pi^0_{x,z}$ and $\pi^0_{y,w}$ of $C^*(G_T|_{[y]})$. Corollary 3.12(b) implies that their kernels are equal if and only if the kernels of $\pi^0_{x,z} \circ \epsilon$ and $\pi^0_{y,w} \circ \epsilon$ are equal. Lemma 5.1 shows that $Y(x) = Y(y)$, and for $f \in C_c(G_T|_{Y(x)}) = C_c(G_T|_{Y(y)})$, we have

$$\pi^0_{x,z} \circ \epsilon(f)\delta_y = \sum_{(u,g,y) \in G_T|_{Y(x)}} z^g f(u,g,y)\delta_u.$$ 

Lemma 4.2 shows that for $n \in H(x)$,

$$\sum_{(u,g,y) \in G_T|_{Y(x)}} z^n f(u,g,y)\delta_u = \sum_{(u,g+n,y) \in G_T|_{Y(x)}} z^n f(u,g,y)\delta_u.$$ 

As in Lemma 4.3, for $\varphi \in C_c(H(x))$ and $f \in C_c(G_T|_{Y(x)})$, we have $\pi_{x,z} \circ \epsilon(\varphi \cdot f) = \hat{\varphi}(z)(\pi_{x,z} \circ \epsilon)(f)$ and $\pi_{y,w} \circ \epsilon(\varphi \cdot f) = \hat{\varphi}(w)(\pi_{y,w} \circ \epsilon)(f)$. Choose $\varphi$ such that $\hat{\varphi}(w) = 0$ and $\hat{\varphi}(z) \neq 0$, and choose $f \in C_c(Y(x))$ such that $f(x) = 1$. Then $\pi_{y,w} \circ \epsilon(\varphi \cdot f) = 0$ whereas $\pi_{x,z}(\varphi \cdot f)\delta_y = \hat{\varphi}(z)\delta_x \neq 0$. So the kernels are not equal.

Third, suppose that $[x] = [y]$ and $\exists w \in H(x)^\perp$. Again Lemma 5.1 shows that $Y(x) = Y(y)$. Let $\pi_{x,z}$ and $\pi_{y,w}$ be the faithful irreducible representations of $C^*(G_T|_{Y(x)})$ described by Lemma 4.5. It is routine to check that $\pi^0_{x,z} \circ \epsilon = \omega_{[x]} \circ \epsilon_z \circ \Phi$ and $\pi^0_{y,w} \circ \epsilon = \omega_{[y]} \circ \epsilon_w \circ \Phi$. We have

$$\omega_{[x]} \circ \delta \epsilon_w \circ \epsilon_z \circ \Phi = \omega_{[y]} \circ \epsilon_w \circ \Phi.$$ 

Since $\epsilon_w$ is an automorphism, we deduce that $\ker(\omega_{[x]} \circ \epsilon_z \circ \Phi) = \ker(\omega_{[y]} \circ \epsilon_w \circ \Phi)$. Thus $\ker(\pi^0_{x,z} \circ \epsilon) = \ker(\pi^0_{y,w} \circ \epsilon)$. Now Corollary 3.12(b) implies that $\pi^0_{x,z}$ and $\pi^0_{y,w}$ have the same kernel. Since $[x] = [y]$, we have $R_x = R_y$, and so

$$\ker\pi_{x,z} = R^{-1}_x(\ker\pi^0_{x,z}) = R^{-1}_y(\ker\pi^0_{y,w}) = \ker\pi_{y,w}.$$ 

It remains to show that $(x, z) \mapsto \ker\pi_{x,z}$ is surjective. Fix a primitive ideal $I \triangleleft C^*(G_T)$. Theorem 5.2 gives $I = \ker\pi_{x,z}$, for some $x, z, \rho$. Choose $y \in [x] \cap Y(x)$, and let $\omega_{[y]}$ be the faithful irreducible representation of $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$ of Lemma 4.5. Since $\rho$ is faithful on $C^*(G_T|_{Y(x)}/\mathcal{I}(x))$, we have $\ker(\omega_{[y]} \circ \epsilon_z \circ \Phi) = \ker(\rho \circ \epsilon_z \circ \Phi)$. So Theorem 5.2 gives $\ker\pi_{x,z} \circ \omega_{[y]} = \ker(\pi_{x,z} \circ \rho)$. As in the second step above, one checks on basis elements that $\pi_{x,z} = \pi_{x,z} \circ \omega_{[y]}$, completing the proof. 

References


(A. Sims) School of Mathematics and Applied Statistics, University of Wollongong, NSW 2522, Australia
E-mail address: asims@uow.edu.au

(D. Williams) Department of Mathematics, Dartmouth College, Hanover, NH 03755-3551
E-mail address: dana.williams@Dartmouth.edu