A Schauder estimate for stochastic PDEs

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A Schauder estimate for stochastic PDEs

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Abstract. Considering stochastic partial differential equations of parabolic type with random coefficients in vector-valued Hölder spaces, we establish a sharp Schauder theory. The existence and uniqueness of solutions to the Cauchy problem is obtained.

1. Introduction

We consider the second-order stochastic partial differential equations (SPDEs) of the Itô type

\[ du = (a^{ij} u_{x^i x^j} + b^i u_{x^i} + cu + f) dt + (\sigma^{ik} u_{x^i} + \nu^k u + g^k) dw^k_t, \]

in \( \mathbb{R}^n \times (0, \infty) \), where \( w^k \) are countable independent standard Wiener processes defined on a filtered complete probability space \( (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}}, \mathbb{P}) \) for \( k = 1, 2, \cdots \). The matrix \( a = (a^{ij}) \) is symmetric, and the uniform parabolic condition is assumed throughout the paper, namely there is a constant \( \lambda > 0 \) such that

\[ 2a^{ij} - \sigma^{ik} \sigma^{jk} \geq \lambda \delta_{ij} \text{ on } \mathbb{R}^n \times (0, \infty) \times \Omega, \]

where \( \delta_{ij} \) is the Kronecker delta. The random fields \( u, a^{ij}, b^i, f \) are all real-valued, while \( \sigma^i, \nu \) and \( g \) take values in \( \ell^2 \). One of the most important examples of (1.1) is the Zakai equation arising in the nonlinear filtering problem [15].

The regularity of solutions of (1.1) in Sobolev spaces has already been investigated by many researchers. Various aspects of \( L^2 \)-theory were studied since 1970s, see [11, 9, 13, 1] and references therein. Later on, a complete \( L^p \)-theory was established by Krylov in 1990s, see [7, 8]. By using Sobolev’s embedding, one then has the regularity in Hölder spaces, which is however not sharp. As an open problem mentioned in [8], one desires a sharp \( C^{2+\alpha} \)-theory in the sense that not only that for \( f, g \) belonging to a proper space \( \mathcal{F} \), the solution belongs to some kind of stochastic \( C^{2+\alpha} \)-spaces, but also that every element of this stochastic space can be obtained as a solution for certain \( f, g \) belonging to the same \( \mathcal{F} \).

Key words and phrases. Stochastic partial differential equation, Schauder estimate, Hölder space.
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The purpose of this paper is to establish a Schauder theory of Equation (1.1), which is sharp in the above sense. In order to state our main results, we first introduce a notion of quasi-classical solutions.

**Definition 1.1.** A random field $u$ is called a **quasi-classical** solution of (1.1) if

(1) For each $t \in (0, \infty)$, $u(\cdot, t)$ is a twice strongly differentiable function from $\mathbb{R}^n$ to $L^\gamma := L^\gamma(\Omega; \mathbb{R})$ for some $\gamma \geq 2$; and

(2) For each $x \in \mathbb{R}^n$, the process $u(x, \cdot)$ satisfies (1.1) in the Itô integral form with respect to the time variable.

If furthermore, $u(\cdot, t, \omega) \in C^2(\mathbb{R}^n)$ for any $(t, \omega) \in (0, \infty) \times \Omega$, then $u$ is a classical solution of (1.1).

Analogously to classical Hölder spaces, we can define the $L^\gamma$-valued Hölder spaces $C^m_{x,t}(Q_T; L^\gamma)$ and $C^{m+\alpha, \alpha/2}_{x,t}(Q_T; L^\gamma)$, where $T > 0$, $Q_T = \mathbb{R}^n \times (0, T)$, and $L^\gamma := L^\gamma(\Omega; \mathbb{R})$ is a Banach space equipped with the norm $\|\cdot\|_{L^\gamma} := (E|\cdot|^\gamma)^{1/\gamma}$. More specifically, we define $C^m_{x,t}(Q_T; L^\gamma)$ to be the set of all $L^\gamma$-valued strongly continuous functions $u$ such that

$$
|u|_{m+\alpha, Q_T} := \sup_{(x,t) \in Q_T} \|D^\alpha u(x,t)\|_{L^\gamma} + \sup_{t, x \neq y, |\beta| = m} \frac{\|D^\beta u(x,t) - D^\beta u(y,t)\|_{L^\gamma}}{|x-y|^\alpha} < \infty.
$$

Using the parabolic module $|X|_p := |x| + \sqrt{|t|}$ for $X = (x,t) \in \mathbb{R}^n \times \mathbb{R}$, we define $C^{m+\alpha, \alpha/2}_{x,t}(Q_T; L^\gamma)$ to be the set of all $u \in C^m_{x,t}(Q_T; L^\gamma)$ such that

$$
|u|_{(m+\alpha/2), Q_T} := |u|_{m, Q_T} + \sup_{|\beta| = m, X \neq Y} \frac{\|D^\beta u(X) - D^\beta u(Y)\|_{L^\gamma}}{|X-Y|^\alpha} < \infty.
$$

Similarly, we can define the norms (1.3) and (1.4) over a domain $Q = \mathcal{O} \times I$, for any domains $\mathcal{O} \subset \mathbb{R}^n$ and $I \subset \mathbb{R}$.

Our main result is the following

**Theorem 1.1.** Assume that the classical $C^\alpha$-norms of $a^{ij}, b^i, c, \sigma_i^j, \nu, \nu_x$ are all dominated by a constant $K$ uniformly in $(t, \omega) \in (0, T) \times \Omega$, and the condition (1.2) is satisfied. If $f \in C^\alpha(\mathbb{R}^n \times \mathbb{R}; Q_T)$, $g \in C^{1+\alpha}_{x,t}(Q_T; L^\gamma)$ for some $\gamma > 2$, then Equation (1.1) with a zero initial condition admits a unique quasi-classical solution $u$ in $C^{2+\delta, \delta/2}_{x,t}(Q_T; L^\gamma)$.

We remark that the problem with nonzero initial value can be easily reduced to our case by a simple transform. We also remark that by an anisotropic Kolmogorov continuity theorem (see [2]), if $\alpha \gamma > n + 2$, the above obtained quasi-classical solution $u$ has a $C^{2+\delta, \delta/2}$ modification for $0 < \delta < \alpha - (n + 2)/\gamma$ as a classical solution of (1.1).

In order to prove the solvability in Theorem 1.1, by means of the standard method of continuity, it suffices to establish the following a priori estimate.

**Theorem 1.2.** Under the hypotheses of Theorem 1.1, letting $u \in C^{2,0}_{\text{loc}}(Q_T; L^\gamma)$ be a quasi-classical solution of (1.1) and $u(\cdot, 0) = 0$, there is a positive constant $C$ depending only on $n, \lambda, \gamma, \alpha$ and $K$ such that

$$
|u|_{2+\alpha, Q_T} \leq C e^{C T} (|f|_{\alpha, Q_T} + |g|_{1+\alpha, Q_T}).
$$

The Hölder regularity in spaces $C^{m+\alpha}_{x,t}(Q_T; L^\gamma)$ for Equation (1.1) was previously investigated by Rozovsky [12], and later was improved by Mikulevicius [10]. However, both works addressed only the equations with nonrandom coefficients and with no derivatives of the unknown function in the stochastic term, namely $a^{ij}$ is deterministic and $\sigma^{ik} \equiv 0$. Moreover, both previous works did
not obtain the time-continuity of second-order derivatives of \( u \), comparing to our estimate (1.5) and Theorem 1.1.

The Schauder estimate we obtained in Theorem 1.2 is sharp in the sense that mentioned in [8], and is for the general form (1.1) with natural assumptions, where all coefficients are random. The approach to \( C^{2+\alpha} \)-theory in [10] was based on several delicate estimates for the heat kernel. Our method is completely different and more straightforward by combining certain integral estimates and a perturbation argument of Wang [14]. A sketch of proof of Theorem 1.2 is given in Section 2. Full details in addition to applications and further remarks are contained in our separate paper [3].

2. Schauder estimates

In this section we give an outline of the proof of our main estimate (1.5). For simplicity we will first deal with a simplified model equation, and then extend to the general ones.

Consider the model equation

\[
(2.1) \quad du = (a^{ij} u_{ij} + f) dt + (\sigma^{ik} u_i + g^k) dw^k,
\]

where \( a^{ij}, \sigma^{ik} \) are predictable processes, independent of \( x \), satisfying the condition (1.2). We shall consider the model equation in the entire space \( \mathbb{R}^n \times \mathbb{R} \). Suppose that \( f(t, \cdot) \) and \( g_z(t, \cdot) \) are Dini continuous with respect to \( x \) uniformly in \( t \), namely

\[
\int_0^1 \frac{\varpi(r)}{r} \, dr < \infty,
\]

where

\[
\varpi(r) = \sup_{t \in \mathbb{R}, |x-y| \leq r} (\|f(t,x) - f(t,y)\|_{L^2} + \|g_z(t,x) - g_z(t,y)\|_{L^2}).
\]

For any \( r > 0 \), we denote

\[
(2.2) \quad B_r(x) = \{ y \in \mathbb{R}^n : |y-x| < r \}, \quad Q_r(x,t) = B_r(x) \times (t-r^2, t),
\]

and further define \( B_r = B_r(0) \) and \( Q_r = Q_r(0,0) \).

**Lemma 2.1.** Let \( u \in C^{2,0}_{x,t}(Q; L^2) \) be a quasi-classical solution of (2.1). Then there is a positive constant \( C \), depending only on \( n, \lambda, \gamma \) such that for any \( X, Y \in Q_{1/4} \),

\[
(2.3) \quad \|u_{XX}(X) - u_{XX}(Y)\|_{L^2} \leq C \left[ \delta M_1 + \int_0^\delta \frac{\varpi(r)}{r} \, dr + \delta \int_\delta^1 \frac{\varpi(r)}{r^2} \, dr \right],
\]

where \( \delta = |X-Y| \) and \( M_1 = |u|_{0,Q_1} + |f|_{0,Q_1} + |g|_{1,Q_1} \).

An important consequence of Lemma 2.1 is the fundamental Schauder estimate that the solution \( u \in C^{2,0}_{x,t,\alpha,\alpha/2}(Q_{1/4}; L^2) \) when \( f \in C^\alpha_x(Q; L^2) \) and \( g \in C^{1+\alpha}_x(Q; L^2) \) for some \( \alpha \in (0,1) \).

**Outline of proof.** Without loss of generality, we may assume \( X = 0 \). Let \( \rho = 1/2 \), and denote

\[
Q^\kappa = Q_{\rho^\kappa} = Q_{\rho^\kappa}(0,0), \quad \kappa = 0, 1, 2, \cdots.
\]

Construct a sequence of Cauchy problems

\[
du^\kappa = [a^{ij} u^\kappa_{ij} + f(0,t)] \, dt + [\sigma^{ik} u^\kappa_i + g^k(0,t)] \, dw^k \quad \text{in } Q^\kappa,
\]

\[
u^\kappa = u \quad \text{on } \partial Q^\kappa.
\]

**Claim 1.** For each \( \kappa \), there is a unique generalised solution \( u^\kappa \) such that \( u^\kappa(\cdot, t) \in L^\gamma(\Omega; C^m(B_{\rho^\kappa})) \) for any \( m \geq 0 \) and \( \varepsilon \in (0, \rho^\kappa) \). Moreover, for any \( r < \rho^\kappa \) there is a constant \( C = C(n, \gamma) \) such that

\[
(2.4) \quad \|u\|_{L^\gamma(\Omega; L^2(\partial Q_{1/2}))} \leq C \left( r^2 \|f\|_{L^\gamma(\Omega; L^2(Q_{1/2}))} + r \|g\|_{L^\gamma(\Omega; L^2(Q_{1/2}))} \right).
\]
Proof. In fact, for \( \gamma = 2 \), the unique solvability and interior smoothness of \( u^k \) follows from [5, Theorem 2.1]. For \( \gamma \geq 2 \), higher order \( L_2^\gamma \)-integrability (2.4) can be achieved by a truncation technique.

**Claim 2.** There is a constant \( C = C(n, \lambda, \gamma) \) such that

\[
|D^m(u^k - u^{k+1})|_{0, L^{2k+x}} \leq C\rho^{2-m} \varpi(\rho^k), \quad m = 1, 2, \ldots.
\]

**Proof.** Note that \((u^k - u^{k+1})\) satisfies a homogeneous equation. By a delicate computation, we have

\[
|D^m(u^k - u^{k+1})|_{0, L^{2k+x}} \leq C\rho^{-m}\left\| \int_{Q^{k+1}} (u^k - u^{k+1})^2 \, dX \right\|^{1/2}_{L_2^{k+1}} := I_{k,m}.
\]

On the other hand, \((u^k - u)\) satisfies a zero initial condition. By Claim 1,

\[
J_k := \left\| \int_{Q^k} (u^k - u)^2 \, dX \right\|^{1/2}_{L_2^k} \leq C\rho^2 \varpi(\rho^k).
\]

Thus, Claim 2 is proved, since

\[
I_{k,m} \leq C\rho^{-m}(J_k + J_{k+1}) \leq C\rho^{2-m} \varpi(\rho^k).
\]

It is worth remarking that instead of using the maximum principle to estimate the term \(|D^m(u^k - u^{k+1})|_{0, L^{2k+x}}\) as in [14], we obtain the inequality (2.5) by subtle integral estimates.

**Claim 3.** \( \{u_{xx}^k(0)\} \) converges in \( L_\infty^\gamma \) (here \( 0 \in \mathbb{R}^{n+1} \)), and the limit is \( u_{xx}(0) \).

**Proof.** By Claim 2 and the assumption of Dini continuity,

\[
\sum_{\kappa \geq 1} |(u^k - u^{k+1})|_{0, L^{\infty + 2k}} \leq C \sum_{\kappa \geq 1} \varpi(\rho^k) \leq C \int_0^1 \frac{\varpi(r)}{r} \, dr < \infty,
\]

which implies that \( u_{xx}^k(0) \) converges in \( L_\infty^\gamma \). Since \( \gamma \geq 2 \), it suffices to show that

\[
\lim_{\kappa \to \infty} \|u_{xx}^k(0) - u_{xx}(0)\|_{L_\infty^\gamma} = 0,
\]

which can also be achieved straightforward by our integral estimates.

Now for any \( Y = (y, s) \in \mathcal{Q}_{1/4} \) we can select an \( \kappa \) such that \( |Y|_p \in [\rho^{k+2}, \rho^{k+1}] \). By decomposition, one has

\[
\|u_{xx}(Y) - u_{xx}(0)\|_{L_\infty^\gamma} \leq \|u_{xx}^k(Y) - u_{xx}^k(0)\|_{L_\infty^\gamma} + \|u_{xx}^k(0) - u_{xx}(0)\|_{L_\infty^\gamma} + \|u_{xx}^k(Y) - u_{xx}(Y)\|_{L_\infty^\gamma} =: I_1 + I_2 + I_3.
\]

**Claim 4.** \( I_1 \leq C\delta M_1 + C\delta \int_0^1 \frac{\varpi(r)}{r} \, dr \), where \( \delta := |Y|_p \) and \( M_1 \) was given in (2.3).

**Proof.** The proof is by induction. When \( \kappa = 0 \), note that \( u_{xx}^0 \) satisfies the following homogeneous equation:

\[
\frac{d}{dt} u_{xx}^0 = a^{ij} D_{ij} u_{xx}^0 \, dt + \sigma^{ik} D_k u_{xx}^0 \, dt \in \mathcal{Q}_{3/4}.
\]

From interior estimates, we have

\[
\|u_{xx}^0(X) - u_{xx}^0(Y)\|_{L_\infty^\gamma} \leq CM_1 |X - Y|_p, \quad \forall X, Y \in \mathcal{Q}_{1/4}.
\]

When \( \kappa \geq 1 \), denote \( h^I = u^I - u^{I-1} \), for \( I = 1, 2, \ldots, \kappa \), then \( h^I \) satisfies

\[
\frac{d}{dt} h^I = a^{ij} h^I_{ij} \, dt + \sigma^{ik} h^I_k \, dt \in \mathcal{Q}^I.
\]
From Claim 2, we have for $-\rho^{2(k+1)} \leq t \leq 0$ and $|x| \leq \rho^{k+1}$,
\begin{equation}
(2.9) \quad \|h_{xx}^k(x, t) - h_{xx}^k(0, 0)\|_{L^2} \leq C\rho^{k-1}\varpi(\rho^{-1}).
\end{equation}
Using (2.8) and (2.9), we can obtain the estimate
\begin{align*}
I_1 & \leq \|u_{xx}^k(Y) - u_{xx}^{k-1}(0)\|_{L^2} + \|h_{xx}^k(Y) - h_{xx}^k(0)\|_{L^2} \\
& \leq \|u_{xx}^0(Y) - u_{xx}^{0}(0)\|_{L^2} + \sum_{i=1}^{\kappa} \|h_{xx}^i(Y) - h_{xx}^{i}(0)\|_{L^2} \\
& \leq C\delta M_1 + C\delta \int_\delta^1 \frac{\varpi(r)}{r^2} \, dr.
\end{align*}
Claim 4 is proved. \hfill \square

**Claim 5.** $I_i \leq C \int_0^\delta \frac{\varpi(\tau)}{\tau} \, d\tau$, for $i = 2, 3$.

**Proof.** The estimate of $I_2$ is a refinement of convergence in Claim 3. In fact, by Claim 2 we have the precise estimate
\begin{equation}
(2.10) \quad I_2 = \|u_{xx}^k(t) - u_{xx}(0)\|_{L^2} \leq \sum_{j \geq 2} \|u^j - u^{j+1}\|_{L^2} \leq C \int_0^\delta \frac{\varpi(r)}{r} \, dr,
\end{equation}
where $C = C(n, \lambda, \gamma)$. We can obtain a similar estimate for $I_3$ by shifting the centre of domains. \hfill \square

To sum up, Lemma 2.1 is proved.

Having proved Lemma 2.1, we are in a position to derive the global estimate of solutions of (1.1) and complete the proof of Theorem 1.2.

**Outline of proof of Theorem 1.2.** The proof is by an argument of frozen coefficients. Denote $Q_{r, \tau} = B_r \times (0, \tau)$, and let
\begin{equation}
M_{x,r}(u) = \sup_{0 \leq t \leq \tau} \left( \int_{B_r(x)} \mathbb{E}|u(t, y)|^{\gamma} \, dy \right)^{1/\gamma} \quad M_{x,r}(u) = \sup_{x \in \mathbb{R}^n} M_{x,r}(u).
\end{equation}
By multiplying cut-off functions and applying Lemma 2.1 we can get
\begin{equation}
|u_{xx}|_{(\alpha, \alpha/2) ; Q_{r/2}} \leq C \left( M_{0,r}^\rho(u) + |f|_{\alpha; Q_r} + |g|_{1+\alpha; Q_r} \right),
\end{equation}
for some sufficiently small $\rho > 0$. The derivation of (2.11) involves a rather delicate computation, which makes use of interpolation inequalities in Hölder spaces (see [4, Lemma 6.35] or [6, Theorem 3.2.1]). Since the centre of domains can shift to any point $x \in \mathbb{R}^n$, we obtain
\begin{equation}
|u|_{(2+\alpha, \alpha/2) ; Q_r} \leq C \left( M_{\rho}^\rho(u) + |f|_{\alpha; Q_r} + |g|_{1+\alpha; Q_r} \right),
\end{equation}
where $C = C(n, \lambda, \gamma, \alpha)$.

To estimate $M_{\rho}^\rho(u)$, applying Itô's formula, and using Hölder and Sobolev-Gagliardo-Nirenberg inequalities, we can get
\begin{equation}
M_{\rho}^\rho(u) \leq C_1 \tau(M_{\rho}^\rho(u) + |u_{xx}|_{0; Q_r} + |f|_{0; Q_r} + |g|_{0; Q_r}),
\end{equation}
where $C_1 = C_1(n, \lambda, \gamma)$. Letting $\tau = (2CC_1 + C_1)^{-1}$, by virtue of (2.12) we obtain
\begin{equation}
|u|_{(2+\alpha, \alpha/2) ; Q_r} \leq C_0 \left( |f|_{\alpha; Q_r} + |g|_{1+\alpha; Q_r} \right),
\end{equation}
where $C_0 = C_0(n, \lambda, \gamma, \alpha)$.

Finally, the proof of (1.5) and Theorem 1.2 is completed by induction. \hfill \square
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