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Abstract

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Keywords

curved, slightly, microplates, behavior, deformable, dynamical, transfer, energy, internal, shear

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Internal Energy Transfer in Dynamical Behavior of Slightly Curved Shear Deformable Microplates

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Keywords: internal energy transfer, size-dependent dynamical behavior, shear deformable microplate, stability

1 Introduction

Due to new technological advances in microfabrication, microscale continuous elements such as microbeams and microplates are widely used in many advanced micromachine components and engineering devices [1–18]. Microplates are present, for example, in biosensors, biomechanical organs, microactuators, and microswitches. These widespread applications have motivated a substantial amount of research on the dynamical behavior of this class of microscale continuous elements.

Experimental investigations [19–21] showed that microscale continuous elements display size-dependent deformation behavior, which cannot be predicted by classical continuum theories. This necessitated the development of new continuum theories such as the strain gradient [22–25] and modified couple stress [26–28] theories.

In general, due to improper manufacturing process, it is very likely to make slightly curved microplates, even if it is not intended to do so. Moreover, in some applications, such as in microswitches, microshutters, and bandpass filters, the microplate is manufactured in a curved shape. This geometric imperfection is taken into account in this analysis by an initial curvature in the out-of-plane direction; it is explored that how this initial curvature changes the dynamical behavior of the microplate.

The literature regarding the dynamical behavior of *initially perfectly flat* microplates is quite large, for the cases where *linear* systems are considered; however, it is not large where *nonlinear* systems are considered. Reviewing some of the recent papers

which analyzed the dynamical behavior of *perfectly flat* microplates, starting with the *linear* analyses, Hashemi and Samaei [29] investigated the linear buckling of microplates subject to an in-plane excitation load, based on the nonlocal Mindlin plate theory. Sharma and Sharma [30] analyzed the transverse linear dynamics of a generalized thermoelastic circular plate subject to different temperature. He et al. [31] developed a size-dependent four-variable functionally graded microplates employing the modified couple stress theory and Hamilton's principle; they obtained the closed form solutions for linear bending, buckling, and free dynamics of a simply supported microplate. Wang et al. [32] derived a size-dependent microplate model by means of the Kirchhoff plate theory based on a strain gradient elasticity theory. Ramezani [33] employed a first-order shear deformation theory in order to model the motion characteristics of a microplate via use of a strain gradient elasticity theory. Tahani et al. [34] employed the modified couple stress theory in order to investigate the size-dependent free dynamic characteristics of rectangular microplates predeformed by an electric field. Roque et al. [35] examined the bending of a shear deformable microplate on the basis of the modified couple stress theory. The application of *nonlinear* theories to the dynamical behavior of microplates led to few published papers in the literature; for instance, Asghari [36] obtained the size-dependent equations of motion of microplates on the basis of the modified couple stress theory. Thai and Choi [37] developed a size-dependent model of functionally graded Kirchhoff Mindlin plates via use of the modified couple stress theory. In both of the valuable studies in Refs. [36,37], only the equations of motion were derived, and no solutions for the nonlinear dynamics were obtained. Ansari et al. [38] examined the nonlinear bending and postbuckling of functionally graded Mindlin rectangular plates based on the modified couple stress theory. There have been several studies on the dynamical behavior of initially curved

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microbeams (MEMS microarches), for example, by Das and Batra [39], who examined the pull-in and snap-through instabilities in MEMS shallow arches under dynamic loading; Ouakad and Younis [40,41] and Ouakad [42], who analyzed the dynamic behavior of an arch resonator and its application as a band-pass filter; Mohammad and Ouakad [43], who performed a bifurcation analysis of electrostatically actuated MEMS arches; and Medina et al. [44–46], who performed theoretical and experimental investigations on the static behavior of electrostatically actuated initially curved microbeams. However, the literature regarding the dynamical behavior of initially curved microplates is limited [15].

An important issue which has never been studied in the dynamical behavior of microplates is the *internal energy transfer and internal resonances*. This paper examines the internal energy transfer and modal interactions in the dynamical behavior of slightly curved shear deformable microplates, with special consideration to the effect of the initial curvature on the internal energy transfer mechanisms, and employing the third-order shear deformation theory. Taking into account small-size effects, via use of the modified couple stress theory, the elastic potential energy of the continuous system is obtained; the kinetic energy as well as the work due to the damping and external excitation forces is also obtained as continuous functions of the displacement field. The contribution of the initial curvature, modeled as a slight initial displacement in the out-of-plane direction, in the potential energy is included in the modeling. The Lagrange equations along with an assumed-mode technique are employed to obtain a reduced-order model. A linear analysis is performed to verify the possibility of the occurrence of internal energy transfer and modal interactions. The pseudo-arclength continuation technique is then employed to solve the reduced-order model of the system; stability analysis is conducted using the Floquet theory. For the systems tuned to internal energy transfer, the results are illustrated in the form of frequency-responses and force-responses. As we shall see, an interesting nonlinear dynamical behavior, with multiple solution branches due to internal energy transfer mechanisms, is displayed by the system. Differences and similarities between the steady-state coupled dynamical response of the initially perfectly flat and initially curved microplates are highlighted via comparing the corresponding frequency-response and force-response curves; this illustrates that how even small imperfections may change the system dynamics and the internal energy transfer substantially.

2 Problem Statement and Displacement Field

Schematic representation of the system is shown in Fig. 1; the system consists of a microplate with in-plane dimensions a and b in the x and y directions, respectively, thickness h , mass density ρ , Poisson's ratio ν , length-scale parameter l , and Young's modulus E . A Cartesian coordinate system ($O; x, y, z$), with the origin O at one corner, defines the displacement field; x and y axes define the midplane coordinates while z -axis defines the out-of-plane coordinate from the midplane, from the static equilibrium state. The displacement components of each point of the midplane of the microplate in the x , y , and z directions are denoted by

Fig. 1 Schematic representation of an initially slightly curved shear deformable microplate

$u = u(x, y, t)$, $v = v(x, y, t)$, and $w = w(x, y, t)$, respectively, t is time. $\phi_1 = \phi_1(x, y, t)$ and $\phi_2 = \phi_2(x, y, t)$ also denote the rotations of the transverse normal at $z=0$ about the y and x axes, respectively. A distributed harmonically varying force per unit area, $f_1 \cos(\omega t)$, is exerted to the microplate in the z direction, where f_1 and ω are in N/m^2 and rad/s , respectively. An initial curvature in the positive out-of-plane direction in the form of $w_0(x, y) = A_0 \sin(\pi x/a) \sin(\pi y/b)$ is considered. The components of the displacement vector ($\mathbf{u}$) of a generic point (in the x , y , and z directions, respectively) of a geometrically imperfect third-order shear deformable microplate are given by

$$u_1(x, y, z, t) = u + z\phi_1 - \frac{4}{3h^2}z^3\left(\phi_1 + \frac{\partial w}{\partial x}\right) \quad (1)$$

$$u_2(x, y, z, t) = v + z\phi_2 - \frac{4}{3h^2}z^3\left(\phi_2 + \frac{\partial w}{\partial y}\right) \quad (2)$$

$$u_3(x, y, z, t) = w + w_0 \quad (3)$$

3 Expressions for Kinetic Energy, External Work, and Damping

The kinetic energy of a geometrically imperfect shear deformable microplate can be modeled as

$$\begin{aligned} T = & \frac{1}{2}\rho h \int_0^a \int_0^b \left\{ \left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 + \frac{17h^2}{315} \right. \\ & \times \left[\left(\frac{\partial \phi_1}{\partial t} \right)^2 + \left(\frac{\partial \phi_2}{\partial t} \right)^2 \right] - \frac{8h^2}{315} \left(\frac{\partial \phi_1}{\partial t} \frac{\partial^2 w}{\partial x \partial t} + \frac{\partial \phi_2}{\partial t} \frac{\partial^2 w}{\partial y \partial t} \right) \\ & \left. + \frac{h^2}{252} \left[\left(\frac{\partial^2 w}{\partial x \partial t} \right)^2 + \left(\frac{\partial^2 w}{\partial y \partial t} \right)^2 \right] \right\} dx dy \end{aligned} \quad (4)$$

The work of the external distributed harmonic excitation load per unit area, $f_1 \cos(\omega t)$, acting on the system in the positive z direction, can be formulated as

$$W = \int_0^a \int_0^b (f_1 \cos(\omega t) w) dx dy \quad (5)$$

A nonconservative external viscous damping with the coefficient of c is introduced to the model via using the Rayleigh dissipation function as

$$\begin{aligned} F = & \frac{1}{2}c \int_0^a \int_0^b \left[\left(\frac{\partial u}{\partial t} \right)^2 + \left(\frac{\partial v}{\partial t} \right)^2 + \left(\frac{\partial w}{\partial t} \right)^2 \right. \\ & \left. + \frac{17}{315}h^2 \left(\left(\frac{\partial \phi_1}{\partial t} \right)^2 + \left(\frac{\partial \phi_2}{\partial t} \right)^2 \right) \right] dx dy \end{aligned} \quad (6)$$

4 Continuous Model for Elastic Potential Energy Via Modified Couple Stress Theory

In this section, a continuous model for the elastic strain energy stored in the microplate is proposed based on the modified couple stress theory.

The strain energy of the microplate, on the basis of the modified couple stress theory [47], can be formulated as

$$U = \frac{1}{2} \int_V (\boldsymbol{\sigma} : \boldsymbol{\varepsilon} + \mathbf{m} : \boldsymbol{\chi}) dv \quad (7)$$

in which $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ denote the stress and strain tensors, respectively; $\mathbf{m}$ and $\boldsymbol{\chi}$ denote the deviatoric part of the symmetric couple stress

and the symmetric curvature tensors, respectively. The expanded form of Eq. (7) is given by

$$U = \frac{1}{2} \int_0^a \int_0^b \left\{ \int_{-h/2}^{h/2} [(\sigma_{xx}\epsilon_{xx} + \sigma_{yy}\epsilon_{yy} + \tau_{xy}\gamma_{xy} + \tau_{xz}\gamma_{xz} + \tau_{yz}\gamma_{yz}) + (m_{xx}\chi_{xx} + m_{yy}\chi_{yy} + m_{zz}\chi_{zz} + 2m_{xy}\chi_{xy} + 2m_{xz}\chi_{xz} + 2m_{yz}\chi_{yz})] dz \right\} dx dy \quad (8)$$

The symmetric curvature tensor χ is related to the rotation vector θ by

$$\chi = \frac{1}{2} \begin{bmatrix} 2\frac{\partial\theta_x}{\partial x} & \left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) & \left(\frac{\partial\theta_x}{\partial z} + \frac{\partial\theta_z}{\partial x}\right) \\ \left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) & 2\frac{\partial\theta_y}{\partial y} & \left(\frac{\partial\theta_y}{\partial z} + \frac{\partial\theta_z}{\partial y}\right) \\ \left(\frac{\partial\theta_x}{\partial z} + \frac{\partial\theta_z}{\partial x}\right) & \left(\frac{\partial\theta_y}{\partial z} + \frac{\partial\theta_z}{\partial y}\right) & 2\frac{\partial\theta_z}{\partial z} \end{bmatrix} \quad (9)$$

where the rotation vector θ is a function of the displacement field $\mathbf{u}$ as

$$\theta = \frac{1}{2} \nabla \times \mathbf{u} = \left\{ \frac{1}{2} \left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z} \right), \frac{1}{2} \left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x} \right), \frac{1}{2} \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y} \right) \right\} \quad (10)$$

Inserting Eqs. (1)–(3) into Eq. (10) and the resultant equations into Eq. (9) gives the components of the symmetric curvature tensor as

$$\begin{Bmatrix} \chi_{xx} \\ \chi_{yy} \\ \chi_{zz} \\ \chi_{xy} \end{Bmatrix} = \begin{Bmatrix} \chi_{xx,0} \\ \chi_{yy,0} \\ \chi_{zz,0} \\ \chi_{xy,0} \end{Bmatrix} + z^2 \begin{Bmatrix} \kappa_x^{(2)} \\ \kappa_y^{(2)} \\ \kappa_z^{(2)} \\ \kappa_{xy}^{(2)} \end{Bmatrix} \quad (11)$$

$$\begin{Bmatrix} \chi_{xz} \\ \chi_{yz} \end{Bmatrix} = \begin{Bmatrix} \chi_{xz,0} \\ \chi_{yz,0} \end{Bmatrix} + z \begin{Bmatrix} \kappa_{xz}^{(1)} \\ \kappa_{yz}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} \kappa_{xz}^{(3)} \\ \kappa_{yz}^{(3)} \end{Bmatrix} \quad (12)$$

$$\begin{Bmatrix} \chi_{xx,0} \\ \chi_{yy,0} \\ \chi_{zz,0} \\ \chi_{xy,0} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{2} \frac{\partial\phi_2}{\partial x} + \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} \\ \frac{1}{2} \frac{\partial\phi_1}{\partial y} - \frac{1}{2} \frac{\partial^2 w}{\partial x \partial y} \\ \frac{1}{2} \left(\frac{\partial\phi_2}{\partial x} - \frac{\partial\phi_1}{\partial y} \right) \\ \frac{1}{4} \left(\frac{\partial\phi_1}{\partial x} - \frac{\partial^2 w}{\partial x^2} - \frac{\partial\phi_2}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \end{Bmatrix} \quad (13)$$

$$\begin{Bmatrix} \kappa_x^{(2)} \\ \kappa_y^{(2)} \\ \kappa_z^{(2)} \\ \kappa_{xy}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \frac{2}{h^2} \left(\frac{\partial\phi_2}{\partial x} + \frac{\partial^2 w}{\partial x \partial y} \right) \\ -\frac{2}{h^2} \left(\frac{\partial\phi_1}{\partial y} + \frac{\partial^2 w}{\partial x \partial y} \right) \\ -\frac{2}{h^2} \left(\frac{\partial\phi_2}{\partial x} - \frac{\partial\phi_1}{\partial y} \right) \\ -\frac{1}{h^2} \left(\frac{\partial\phi_1}{\partial x} + \frac{\partial^2 w}{\partial x^2} - \frac{\partial\phi_2}{\partial y} - \frac{\partial^2 w}{\partial y^2} \right) \end{Bmatrix} \quad (14)$$

$$\begin{Bmatrix} \chi_{xz,0} \\ \chi_{yz,0} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{4} \left(\frac{\partial^2 v}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} \right) \\ \frac{1}{4} \left(\frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 u}{\partial y^2} \right) \end{Bmatrix} \quad (15)$$

$$\begin{Bmatrix} \kappa_{xz}^{(1)} \\ \kappa_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} -\frac{1}{4} \left(\frac{\partial^2 \phi_1}{\partial x \partial y} - \frac{\partial^2 \phi_2}{\partial x^2} \right) + \frac{2}{h^2} \left(\phi_2 + \frac{\partial w}{\partial y} \right) \\ -\frac{1}{4} \left(\frac{\partial^2 \phi_1}{\partial y^2} - \frac{\partial^2 \phi_2}{\partial x \partial y} \right) - \frac{2}{h^2} \left(\phi_1 + \frac{\partial w}{\partial x} \right) \end{Bmatrix} \quad (16)$$

$$\begin{Bmatrix} \kappa_{xz}^{(3)} \\ \kappa_{yz}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \frac{1}{3h^2} \left(\frac{\partial^2 \phi_1}{\partial x \partial y} - \frac{\partial^2 \phi_2}{\partial x^2} \right) \\ \frac{1}{3h^2} \left(\frac{\partial^2 \phi_1}{\partial y^2} - \frac{\partial^2 \phi_2}{\partial x \partial y} \right) \end{Bmatrix} \quad (17)$$

The strain-displacement relations for the third-order shear deformable microplate with the initial curvature can be expressed as

$$\begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \gamma_{xy} \end{Bmatrix} = \begin{Bmatrix} \epsilon_{xx,0} \\ \epsilon_{yy,0} \\ \gamma_{xy,0} \end{Bmatrix} + z \begin{Bmatrix} k_x^{(1)} \\ k_y^{(1)} \\ k_{xy}^{(1)} \end{Bmatrix} + z^3 \begin{Bmatrix} k_x^{(3)} \\ k_y^{(3)} \\ k_{xy}^{(3)} \end{Bmatrix} \quad (18)$$

$$\begin{Bmatrix} \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \gamma_{xz,0} \\ \gamma_{yz,0} \end{Bmatrix} + z^2 \begin{Bmatrix} k_{xz}^{(2)} \\ k_{yz}^{(2)} \end{Bmatrix} \quad (19)$$

$$\begin{Bmatrix} \epsilon_{xx,0} \\ \epsilon_{yy,0} \\ \gamma_{xy,0} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial u}{\partial x} + \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 + \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial x} \\ \frac{\partial v}{\partial y} + \frac{1}{2} \left(\frac{\partial w}{\partial y} \right)^2 + \frac{\partial w}{\partial y} \frac{\partial w_0}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \frac{\partial w}{\partial x} \frac{\partial w}{\partial y} + \frac{\partial w}{\partial x} \frac{\partial w_0}{\partial y} + \frac{\partial w_0}{\partial x} \frac{\partial w}{\partial y} \end{Bmatrix} \quad (20)$$

$$\begin{Bmatrix} k_x^{(1)} \\ k_y^{(1)} \\ k_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \frac{\partial\phi_1}{\partial x} \\ \frac{\partial\phi_2}{\partial y} \\ \frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x} \end{Bmatrix} \quad (21)$$

$$\begin{Bmatrix} k_x^{(3)} \\ k_y^{(3)} \\ k_{xy}^{(3)} \end{Bmatrix} = \begin{Bmatrix} -\frac{4}{3h^2} \left(\frac{\partial\phi_1}{\partial x} + \frac{\partial^2 w}{\partial x^2} \right) \\ -\frac{4}{3h^2} \left(\frac{\partial\phi_2}{\partial y} + \frac{\partial^2 w}{\partial y^2} \right) \\ -\frac{4}{3h^2} \left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x} + 2 \frac{\partial^2 w}{\partial x \partial y} \right) \end{Bmatrix} \quad (22)$$

$$\begin{Bmatrix} \gamma_{xz,0} \\ \gamma_{yz,0} \end{Bmatrix} = \begin{Bmatrix} \phi_1 + \frac{\partial w}{\partial x} \\ \phi_2 + \frac{\partial w}{\partial y} \end{Bmatrix} \quad (23)$$

$$\begin{Bmatrix} k_{xz}^{(2)} \\ k_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} -\frac{4}{h^2} \left(\phi_1 + \frac{\partial w}{\partial x} \right) \\ -\frac{4}{h^2} \left(\phi_2 + \frac{\partial w}{\partial y} \right) \end{Bmatrix} \quad (24)$$

The relation between the deviatoric part of the symmetric couple stress tensor and the symmetric curvature tensor is given by

$$m_{ij} = \frac{El^2}{1+\nu} \chi_{ij} \quad (25)$$

The stress-strain relations for an isotropic material, under plane-stress condition, are given by

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \tau_{xy} \\ \tau_{xz} \\ \tau_{yz} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & (1-\nu)/2 & 0 & 0 \\ 0 & 0 & 0 & (1-\nu)/2 & 0 \\ 0 & 0 & 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \gamma_{xy} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} \quad (26)$$

The expression for the elastic strain energy of the shear deformable microplate is obtained by inserting Eqs. (11)–(26) into Eq. (8), as follows:

$$\begin{aligned} U = & \frac{1}{2} \frac{Eh}{1-\nu^2} \int_0^a \int_0^b \left[\varepsilon_{xx,0}^2 + \varepsilon_{yy,0}^2 + 2\nu \varepsilon_{xx,0} \varepsilon_{yy,0} + \frac{1-\nu}{2} (\gamma_{xy,0}^2 + \gamma_{xz,0}^2 + \gamma_{yz,0}^2) \right] dx dy \\ & + \frac{1}{2} \frac{Eh^3}{12(1-\nu^2)} \int_0^a \int_0^b \left[\left(k_x^{(1)} \right)^2 + \left(k_y^{(1)} \right)^2 + 2\nu k_x^{(1)} k_y^{(1)} + \frac{1-\nu}{2} \left(\left(k_{xy}^{(1)} \right)^2 + 2\gamma_{xz,0} k_{xz}^{(2)} + 2\gamma_{yz,0} k_{yz}^{(2)} \right) \right] dx dy \\ & + \frac{1}{2} \frac{Eh^5}{80(1-\nu^2)} \int_0^a \int_0^b \left[2 \left(k_x^{(1)} k_x^{(3)} + k_y^{(1)} k_y^{(3)} + \nu k_x^{(1)} k_y^{(3)} + \nu k_x^{(3)} k_y^{(1)} \right) + \frac{1-\nu}{2} \left(\left(k_{xz}^{(2)} \right)^2 + \left(k_{yz}^{(2)} \right)^2 + 2k_{xy}^{(1)} k_{xy}^{(3)} \right) \right] dx dy \\ & + \frac{1}{2} \frac{Eh^7}{448(1-\nu^2)} \int_0^a \int_0^b \left[\left(k_x^{(3)} \right)^2 + \left(k_y^{(3)} \right)^2 + 2\nu k_x^{(3)} k_y^{(3)} + \frac{1-\nu}{2} \left(k_{xy}^{(3)} \right)^2 \right] dx dy \\ & + \frac{1}{2} \frac{El^2 h}{1+\nu} \int_0^a \int_0^b \left[\chi_{xx,0}^2 + \chi_{yy,0}^2 + \chi_{zz,0}^2 + 2\chi_{xy,0}^2 + 2\chi_{xz,0}^2 + 2\chi_{yz,0}^2 \right] dx dy \\ & + \frac{1}{2} \frac{El^2 h^3}{6(1+\nu)} \int_0^a \int_0^b \left[\chi_{xx,0} \kappa_x^{(2)} + \chi_{yy,0} \kappa_y^{(2)} + \chi_{zz,0} \kappa_z^{(2)} + 2\chi_{xy,0} \kappa_{xy}^{(2)} + \left(\kappa_{xz}^{(1)} \right)^2 + \left(\kappa_{yz}^{(1)} \right)^2 \right] dx dy \\ & + \frac{1}{2} \frac{El^2 h^5}{80(1+\nu)} \int_0^a \int_0^b \left[\left(\kappa_x^{(2)} \right)^2 + \left(\kappa_y^{(2)} \right)^2 + \left(\kappa_z^{(2)} \right)^2 + 2 \left(\kappa_{xy}^{(2)} \right)^2 + 4\kappa_{xz}^{(1)} \kappa_{xz}^{(3)} + 4\kappa_{yz}^{(1)} \kappa_{yz}^{(3)} \right] dx dy \\ & + \frac{1}{2} \frac{El^2 h^7}{224(1+\nu)} \int_0^a \int_0^b \left[\left(\kappa_{xz}^{(3)} \right)^2 + \left(\kappa_{yz}^{(3)} \right)^2 \right] dx dy \end{aligned} \quad (27)$$

5 Reduced-Order Model

The analysis is conducted for a shear deformable simply-supported microplate with immovable edges. The following approximate series expansions are assumed for the in-plane and out-of-plane displacements as well as the rotations:

$$u(x, y, t) = \sum_{m=1}^{\tilde{M}} \sum_{n=1}^{\tilde{N}} u_{m,n}(t) \sin(m\pi x/a) \sin(n\pi y/b) \quad (28)$$

$$v(x, y, t) = \sum_{m=1}^{\tilde{M}} \sum_{n=1}^{\tilde{N}} v_{m,n}(t) \sin(m\pi x/a) \sin(n\pi y/b) \quad (29)$$

$$w(x, y, t) = \sum_{m=1}^{\tilde{M}} \sum_{n=1}^{\tilde{N}} w_{m,n}(t) \sin(m\pi x/a) \sin(n\pi y/b) \quad (30)$$

$$\phi_1(x, y, t) = \sum_{m=1}^{\tilde{M}} \sum_{n=1}^{\tilde{N}} \phi_{1,m,n}(t) \cos(m\pi x/a) \sin(n\pi y/b) \quad (31)$$

$$\phi_2(x, y, t) = \sum_{m=1}^{\tilde{M}} \sum_{n=1}^{\tilde{N}} \phi_{2,m,n}(t) \sin(m\pi x/a) \cos(n\pi y/b) \quad (32)$$

in which m and n are the numbers of half-waves in the x and y directions, respectively, and $u_{m,n}(t)$, $v_{m,n}(t)$, $w_{m,n}(t)$, $\phi_{1,m,n}(t)$, and $\phi_{2,m,n}(t)$ denote the generalized coordinates. The series expansion defined in Eqs. (28)–(32) satisfy the following boundary conditions of the system:

$$u = v = w = 0, \quad \phi_2 = 0, \quad M_x = 0 \quad \text{at } x = 0, a \quad (33)$$

$$u = v = w = 0, \quad \phi_1 = 0, \quad M_y = 0 \quad \text{at } y = 0, b \quad (34)$$

where M_x and M_y represent the bending moment resultants per unit length.

The following notation is introduced for brevity:

$$\mathbf{q} = \{u_{m,n}(t), v_{m,n}(t), w_{m,n}(t), \phi_{1,m,n}(t), \phi_{2,m,n}(t)\}^T \quad (35)$$

in which the dimension of the vector $\mathbf{q}$, N , is the number of degrees of freedom of the reduced-order model.

The Lagrange equations are given by

$$\frac{d}{dt} \left(\frac{\partial T}{\partial \dot{q}_j} \right) - \frac{\partial T}{\partial q_j} + \frac{\partial U}{\partial q_j} + \frac{\partial F}{\partial \dot{q}_j} = \frac{\partial W}{\partial q_j} \quad j = 1, \dots, N \quad (36)$$

Substitution of Eqs. (28)–(32) into Eqs. (4)–(6) and (27), and substitution of the resultant expressions into Eq. (36) results in a set of N second-order nonlinear ordinary differential equations with *inertial coupling*. These equations are solved numerically via the methods explained in Sec. 6.

6 Solution Techniques

Application of the procedure explained in Sec. 5 results in N second-order nonlinear ordinary differential equations with

Fig. 2 Variations of the dimensionless natural frequencies of the out-of-plane motion of the microplate with the amplitude of the initial curvature

inertial coupling, which can be shown in a matrix equation form with *nonlinear* coefficients, as follows:

$$\mathbf{M}\ddot{\mathbf{q}} + \mathbf{C}\dot{\mathbf{q}} + \mathbf{K}_1\mathbf{q} + \mathbf{K}_2(\mathbf{q})\mathbf{q} + \mathbf{K}_3(\mathbf{q}, \mathbf{q})\mathbf{q} = \mathbf{f} \cos(\omega t) \quad (37)$$

where $\mathbf{M}$ is the *nondiagonal* mass matrix, $\mathbf{C}$ is the damping matrix, and $\mathbf{f}$ is the forcing amplitude vector; $\mathbf{K}_1$, $\mathbf{K}_2$, and $\mathbf{K}_3$ denote the linear, quadratic, and cubic stiffness matrices, respectively.

Premultiplying both sides of Eq. (37) by $\mathbf{M}^{-1}$ gives the following *inertially decoupled* matrix equation:

$$\mathbf{I}\ddot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{C}\dot{\mathbf{q}} + \mathbf{M}^{-1}\mathbf{K}_1\mathbf{q} + \mathbf{M}^{-1}\mathbf{K}_2(\mathbf{q})\mathbf{q} + \mathbf{M}^{-1}\mathbf{K}_3(\mathbf{q}, \mathbf{q})\mathbf{q} = \mathbf{M}^{-1}\mathbf{f} \cos(\omega t) \quad (38)$$

The set of nonlinear stiffnesswise coupled (and inertially decoupled) equations, given in Eq. (38), is then recast into a new double-dimensional ($2N$) set of first-order nonlinear ordinary differential equations via a change of variables through use of $y_i = \dot{q}_i$ for $i = 1, 2, \dots, N$. A high-dimensional reduced-order model is analyzed via choosing $N = 32$. More specifically, $2N = 64$ first-order ordinary differential equations are solved together to obtain the time-varying values of the generalized coordinates. A linear analysis, upon the linear terms of the reduced-order model, is conducted to verify the possibility of the occurrence of internal energy transfer and modal interactions. For the cases tuned to internal resonances, the pseudo-arclength continuation technique is employed to analyze the nonlinear dynamical behavior and internal energy transfer mechanisms of the system via constructing the frequency- and force-response curves. Stability of solution branches is determined via use of the Floquet theory.

The numerical results are obtained for a square epoxy microplate with: $l = 17.6 \mu\text{m}$ [48–50], $h = 4.0l$, $a = 300h$, $b = 300h$, $E = 1.44 \text{ GPa}$, $\rho = 1220 \text{ kg/m}^3$, and $\nu = 0.38$. Throughout the numerical simulations in this paper, the excitation frequency ω , the forcing amplitude f_1 , the linear natural frequencies for the out-of-plane motion, time t , and the damping coefficient c are made dimensionless via

$$f_1^* = \frac{f_1 b^4}{Dh}, \quad \Omega = \omega \sqrt{\frac{\rho h b^4}{D}}, \quad t^* = t \sqrt{\frac{D}{\rho h b^4}}, \quad (39)$$

$$\omega_{i,j}^* = \omega_{i,j} \sqrt{\frac{\rho h b^4}{D}}, \quad c^* = \frac{c b^4}{D} \sqrt{\frac{D}{\rho h b^4}}$$

where $D = Eh^3/12(1 - \nu^2)$. Moreover, the substitution $c^* = 2\zeta\omega_{1,1}^*$ is made in the numerical simulations. In what follows, the asterisk notations of the dimensionless parameters are dropped for brevity.

For the nonlinear analysis, the following 32 generalized coordinates are used: $u_{2,1}, u_{4,1}, u_{6,1}, u_{8,1}, u_{2,3}, u_{4,3}, u_{2,5}, v_{1,2}, v_{1,4}, v_{1,6}, v_{1,8}, v_{3,2}, v_{3,4}, v_{5,2}, w_{1,1}, w_{3,1}, w_{5,1}, w_{1,3}, w_{3,3}, w_{1,5}, \phi_{1,1,1}, \phi_{1,3,1}, \phi_{1,5,1}, \phi_{1,1,3}, \phi_{1,3,3}, \phi_{1,5,3}, \phi_{2,1,1}, \phi_{2,3,1}, \phi_{2,5,1}, \phi_{2,1,3}, \phi_{2,3,3}, \phi_{2,5,3}$; 64 first-order nonlinear ordinary differential equations with coupled terms are solved together.

7 Possibility of the Occurrence of Internal Energy Transfer and Modal Interactions

The possibility of the occurrence of internal energy transfer and modal interactions is verified by plotting the variations of different natural frequencies for the out-of-plane motion (dimensionless) of the linear system versus the amplitude of the initial curvature; this question is answered that which value of the initial curvature causes internal energy transfer and modal interactions.

Figure 2(a) shows that the first possible internal resonance occurs for $A_0/h \approx 0.36$, where $\omega_{3,3} \approx 8\omega_{1,1}$. Figure 2(b) illustrates two other possible cases with internal resonances; in particular, an internal resonance in the form of $\omega_{3,1} \approx 4\omega_{1,1}$ occurs in the

Fig. 3 Frequency-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$; $f_1 = 12.00$, $A_0/h = 0.36$, and $\omega_{1,1} = 24.8747$

vicinity of $A_0/h = 0.535$. As the amplitude of the initial curvature is increased even further, another possible internal energy transfer in the form of $\omega_{3,1} \approx 3 \omega_{1,1}$ is detected in the vicinity of $A_0/h = 1.00$. In what follows, the nonlinear dynamical behavior of the system for each of these cases is examined to study the nonlinear effects on such internal energy transfer mechanisms.

8 Nonlinear Energy Transfer for the Case With $\omega_{3,3} \approx 8\omega_{1,1}$

In this section, the frequency- and force-response curves of the system are obtained to illustrate the nonlinear energy transfer between the transverse modes of the system for the case when $A_0/h = 0.36$ (resulting in $\omega_{3,3} \approx 8 \omega_{1,1}$).

The frequency-response curves of the system, when $A_0/h = 0.36$ and $f_1 = 12.00$, are depicted in Fig. 3, for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$; it should be noted that due to symmetry, $w_{3,1} = w_{1,3}$ and $u_{2,1} = v_{1,2}$. The modal damping ratio of $\zeta = 0.008$ is used throughout the numerical simulations. The fundamental natural frequency, i.e., $\omega_{1,1}$, for this case is obtained as 24.8747. It is seen that due to the presence of a nonlinear energy transfer mechanism, extra peaks are observed in the responses of the generalized coordinates $w_{3,1}$ and $w_{3,3}$. In particular, as seen in subfigures (b) and (c), as the frequency of excitation is increased, an internal energy transfer occurs in the vicinity of $\Omega = 28.9$, causing an extra peak in the frequency-response curves of $w_{3,1}$ and $w_{3,3}$. By increasing the excitation frequency even further, another internal resonance takes place at $\Omega = 34.4$, causing a peak in the response of $w_{3,3}$. Moreover, the system displays a hardening-type nonlinear behavior with two limit point bifurcations at $\Omega = 34.8830$ (LP₁) and $\Omega = 25.8604$ (LP₂)—the system response is unstable between the two limit point bifurcations and stable elsewhere.

A comparison between the frequency-response curves of the system obtained via the modified couple stress and classical continuum theories is made by plotting the corresponding curves in Fig. 4; $\omega_{1,1} = 24.8747$ for the modified couple stress theory and $\omega_{1,1} = 22.9816$ for the classical continuum theory. It is seen that the classical continuum theory predicts the first internal resonance around $\Omega = 26.1$, compared to $\Omega = 28.9$ predicted by the modified couple stress theory. Moreover, the second internal resonance occurs in the vicinity of $\Omega = 31.2$ when the system is modeled via the classical continuum theory, as opposed to $\Omega = 34.4$ for the case of modified couple stress theory. This shows the significance of taking into account the length-scale parameter in order to examine the nonlinear resonant response of a microplate.

Shown in Fig. 5 are the force-response curves of the system for the case with $A_0/h = 0.36$ and $\Omega = 29.1033$. For the ranges of f_1 studied here, two limit point bifurcations occur at $f_1 = 72.2694$ and 7.1435. Due to the presence of internal energy transfers between transverse modes, extra peaks are observed in the response of $w_{3,1}$ and $w_{3,3}$. In particular, for the $w_{3,1}$ motion (as seen in subfigure (b)), an energy transfer occurs in the vicinity of $f_1 = 39$, creating a local maximum amplitude in the $w_{3,1}$ motion. For the case of $w_{3,3}$ motion, there are two energy transfer mechanisms around $f_1 = 39$ and $f_1 = 110$, due to the presence of an internal resonance between $\omega_{3,3}$ and $\omega_{1,1}$.

9 Nonlinear Energy Transfer for the Case With $\omega_{3,1} \approx 4 \omega_{1,1}$

This section investigates the nonlinear internal energy transfers in the resonant response of the system for the case when the amplitude of the initial curvature is set to $A_0/h = 0.535$ causing a four-to-one internal resonance in the form of $\omega_{3,1} \approx 4 \omega_{1,1}$. For this case, $\omega_{1,1} = 28.0375$.

Fig. 4 Frequency-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$, obtained via modified couple stress and classical continuum theories; $f_1 = 12.00$ and $A_0/h = 0.36$

Fig. 5 Force-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$; $\Omega = 29.1033$ and $A_0/h = 0.36$

Fig. 6 Frequency-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$; $f_1 = 11.00$, $A_0/h = 0.535$, and $\omega_{1,1} = 28.0375$

Fig. 7 Frequency-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$, obtained via modified couple stress and classical continuum theories; $f_1 = 11.00$ and $A_0/h = 0.535$

Fig. 8 The effect of forcing amplitude on the frequency-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$; $A_0/h = 0.535$ and $\omega_{1,1} = 28.0375$

Figure 6 shows the frequency-response curves of the system where $A_0/h = 0.535$ and $f_1 = 11.00$. Compared to the system of Fig. 3, it is seen that due to increased amplitude of the initial curvature, the system displays an initial softening behavior followed by a hardening one. Moreover, four limit point bifurcations occur as a result of varying the excitation frequency, at $\Omega = 26.4336$ (LP₁), $\Omega = 26.2643$ (LP₂), $\Omega = 31.6375$ (LP₃), and $\Omega = 27.2057$ (LP₄). It is seen in subfigures (b) and (c) that there is an extra peak in the response of $w_{3,1}$ and $w_{3,3}$ generalized coordinates around $\Omega = 28.5$ due to presence of an internal resonance between $\omega_{3,1}$ and $\omega_{1,1}$. It is interesting to note that for both $w_{3,1}$ and $w_{3,3}$ motions, the maximum amplitude due to occurrence of the internal resonance is more than that due to the nonlinear resonance (i.e., LP₃), showing the importance of internal energy transfers in the resonant response of microplates. Figure 7 shows a comparison between the results obtained via the modified couple stress and classical continuum theories. It is seen that the frequency-response curves are shifted to the left, since the natural frequency predicted by the classical continuum theory is smaller than that predicted by the modified couple stress theory (which takes into account small size effects). Hence, as seen in subfigure (b), the classical continuum theory predicts the internal energy transfer at a smaller excitation frequency, compared to the modified couple stress theory. Moreover, as seen in subfigure (c), two energy transfer mechanisms are detected by the classical continuum theory, while the modified couple stress theory predicts only one internal energy transfer mechanism.

Figure 8 illustrates the frequency-response of the system for several forcing amplitudes. It is seen that for small values of f_1 , the system displays only a softening-type nonlinear behavior. As the forcing amplitude is set to larger values, the system displays a secondary hardening-type behavior as well as an internal energy transfer. It is worthwhile noting that the internal resonance has

been detected only when the forcing amplitude is set to larger values.

10 Nonlinear Energy Transfer for the Case With $\omega_{3,1} \approx 3 \omega_{1,1}$

This section examines the nonlinear energy transfer mechanisms between the transverse modes when there is a three-to-one internal resonance in the form of $\omega_{3,1} \approx 3 \omega_{1,1}$, due to presence of an initial curvature with the amplitude of $A_0/h = 1.00$.

Figure 9 shows the frequency-response plots of the system with $A_0/h = 1.00$, $f_1 = 14.50$, and $\omega_{1,1} = 39.3379$. As seen in this figure, the system displays a softening-type nonlinear behavior, due to increased amplitude of the initial curvature, with two limit point bifurcations at $\Omega = 37.3226$ (LP₁) and $\Omega = 28.6435$ (LP₂). As shown in the figure, an internal energy transfer occurs in the vicinity of $\Omega = 37$, causing extra peaks in the frequency-response of the $w_{1,1}$ and $w_{3,1}$ motions. There is also another energy transfer mechanism around $\Omega = 32.7$, which only affects the $w_{3,1}$ generalized coordinate.

11 Validations

The numerical simulations employed in this study are validated by comparing the frequency-response of the $w_{1,1}$ motion of a *macroplate* (hence way simpler model) obtained by the present study and plotted versus that given in Ref. [51] in Fig. 10. The figure shows that the results match perfectly, hence proving the validity of the numerical simulations employed in the present study.

12 Conclusions

Internal energy transfer and modal interactions in the dynamical behavior of slightly curved shear deformable microplates have

Fig. 9 Frequency-response curves of the system for the generalized coordinates $w_{1,1}$, $w_{3,1}$, $w_{3,3}$, and $u_{2,1}$; $f_1 = 14.50$, $A_0/h = 1.00$, and $\omega_{1,1} = 39.3379$

been investigated numerically in this paper. On the basis of the modified couple stress theory, the third-order shear deformation theory was employed to derive the continuous and reduced-order models of the system by retaining the in-plane displacements and inertia. The initial curvature was modeled by an initial displacement in the out-of-plane direction. A model reduction was conducted via Lagrange's equations along with an assumed-mode technique. The reduced-order model was then solved numerically through use of the pseudo-arclength continuation. Stability analysis was conducted via the Floquet theory. The possibility of the occurrence of modal interactions and internal energy transfers was

verified by means of a linear analysis via obtaining different natural frequencies of the system and determining their ratio as a function of the initial curvature. The nonlinear resonant response of the system was obtained for different cases with internal energy transfer, showing that: (i) depending on the amplitude of the initial curvature, the system displays softening, hardening, or softening-hardening behaviors; (ii) due to the presence of internal energy transfers, extra peaks appear in the frequency- and force-responses; (iii) the maximum response amplitude due to an internal energy transfer is more than that due to a nonlinear resonance for some cases; (iv) the modified couple stress theory predicts the occurrence of the internal energy transfer at larger frequencies, compared to the classical continuum theory; (v) the energy transfer due to internal resonances is detected only at sufficiently large forcing amplitudes.

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Fig. 10 Frequency-response of a macroplate for the $w_{1,1}$ mode: solid line and symbols represent the results obtained by the numerical simulations employed in the present study and those given in Ref. [51], respectively

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