Structure and classification of generalised bunce-deddens algebras and their KMS States

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Structure and Classification of Generalised Bunce–Deddens Algebras and their KMS States

A thesis submitted in fulfillment of the requirements for the award of the degree

Doctor of Philosophy in Mathematics

from the

University of Wollongong, NSW

by

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B. Math Advanced (Hons)

Supervised by Professor Aidan Sims
and Doctor David Robertson

School of Mathematics and Applied Statistics

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Declaration

This thesis contains no material which has been submitted for the award of any higher degree to any other university or institution.

During the writing of this thesis I have received advice, guidance and assistance from my supervisors. Apart from their help, this thesis has been all my own work. Some of the results of this thesis are contained in [44] written jointly with my supervisors.

__________________________
James Rout
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Abstract

A Bunce-Deddens algebra is a direct limit of matrix algebras over $C(\mathbb{T})$ arising from periodic weighted shift operators. These were introduced by Bunce and Deddens in 1975 as examples of simple $C^*$-algebras.

In 2007 Kribs and Solel introduced a class of limit algebras arising from periodic weighted shift operators on the path spaces of directed graphs. These are known as the generalised Bunce–Deddens algebras. They showed that the generalised Bunce–Deddens algebras can be realised as direct limits of graph algebras.

Here we study the generalised Bunce–Deddens algebras and their Toeplitz extensions constructed by Kribs and Solel from a directed graph $E$ and an increasing sequence $\omega = (n_k)_{k=1}^{\infty}$ of natural numbers $n_k$ such that $n_k | n_{k+1}$. We describe both of these $C^*$-algebras in terms of novel universal properties. We use these descriptions to prove uniqueness theorems, characterise simplicity and calculate the KMS states for the gauge action, but we use Kribs and Solel’s original description to prove a classification theorem. If $\omega$ is strictly increasing, then our uniqueness theorem, unlike the uniqueness theorem for graph algebras, requires no aperiodicity condition. We apply Perron–Frobenius theory to characterise simplicity of generalised Bunce–Deddens algebras constructed from strongly connected finite directed graphs $E$ in terms of the period of $E$ and the sequence $\omega$. We compute the $K$-theory of the generalised Bunce–Deddens algebras in order to generalise the classification theorem of Bunce–Deddens algebras to the generalised Bunce–Deddens algebras constructed from a large class of strongly connected finite directed graphs whose vertex matrix has 1 as an eigenvalue. This answers a question asked by Kribs and Solel for a large class of generalised Bunce–Deddens algebras. We calculate the KMS states for the gauge action in the Toeplitz algebra when the underlying graph is finite. We deduce that the generalised Bunce–Deddens algebra is simple if and only if it supports exactly one KMS state, and this is equivalent to the terms in the sequence $\omega$ all being coprime with the period of the underlying graph.
Chapter 1

Introduction

In 1980 Cuntz and Krieger [8] constructed a class of $C^*$-algebras from finite \{0,1\}-matrices. For an \(n \times n\) \{0,1\}-matrix \(A\), the Cuntz–Krieger algebra \(\mathcal{O}_A\) is generated by partial isometries \(\{S_i : 1 \leq i \leq n\}\) satisfying a set of relations which are known as the Cuntz–Krieger relations. For a large class of matrices, including all irreducible matrices which are not permutation matrices, Cuntz and Krieger showed that all \(C^*\)-algebras generated by non-zero partial isometries satisfying the Cuntz–Krieger relations are isomorphic. This result is known as the Cuntz–Krieger Uniqueness theorem. They also showed that \(\mathcal{O}_A\) is simple if \(A\) is irreducible.

A directed graph consists of a set of vertices connected by directed edges. A directed graph \(E\) can be represented as a nonnegative integer matrix \(A_E\) with rows and columns indexed by the vertices of \(E\), and conversely, a nonnegative matrix \(A\) can be represented as a graph \(E_A\) with \(n\) vertices. In 1980, Enomoto and Watatani [11] sought a graph-theoretical description of Cuntz–Krieger algebras. For a class of finite directed graphs they obtained a universal \(C^*\)-algebra \(C^*(E)\) generated by a set of partial isometries indexed by the edges of the graph. These \(C^*\)-algebras are known as graph algebras. Enomoto and Watatani established an isomorphism between \(\mathcal{O}_A\) and \(C^*(E_A)\).

Many authors including Bates, Kumjian, Pask, Raeburn, Renault and Szymański, for example [40, 35, 34, 13, 15, 3, 42], have since developed the theory of infinite Cuntz–Krieger algebras and extended the graph algebra construction to row-finite directed graphs in which every vertex receives only a finite number of edges. In [34] Kumjian, Pask and Raeburn proved an analogue of the Cuntz–Krieger uniqueness theorem for the graph algebra of a row-finite graph \(E\) whose cycles each have an exit. They were also able to understand properties of the \(C^*\)-algebra such as simplicity and pure infiniteness in terms of properties of the cycles in the graph.

The Kubo–Martin–Schwinger (KMS) condition was first introduced in quantum sta-
statistical mechanics in [32] and [39] as a condition satisfied by thermodynamic Green’s functions. It was first formulated in the language of \( C^* \)-algebras in [19].

Every Cuntz–Krieger algebra \( \mathcal{O}_A \) carries a gauge action of \( \mathbb{R} \). Enomoto, Fujii and Watatani [10] proved that when \( A \) is irreducible, \((\mathcal{O}_A, \alpha)\) has a unique KMS state, which occurs at inverse temperature equal to the logarithm \( \ln \rho(A) \) of the spectral radius of \( A \). Exel and Laca [14] extended this result to Cuntz–Krieger algebras of infinite matrices and also described the KMS states of their Toeplitz extensions. Recently, many authors including an Huef, Laca, Raeburn and Sims, for example [11, 22, 23, 36, 24, 25, 20], have developed the theory of KMS states for the \( C^* \)-algebras of directed graphs and their generalisations.

In 1975 Bunce and Deddens [5] introduced a class of direct limit \( C^* \)-algebras arising from their earlier work on the \( C^* \)-algebras generated by weighted shift operators [4]. These are known as the Bunce–Deddens algebras. Bunce and Deddens showed that these \( C^* \)-algebras can be realised as the direct limits of matrix algebras over \( C(T) \) and showed that they are simple and, as with UHF algebras [18], are classified by supernatural numbers.

In [30] Kribs generalised the Bunce–Deddens algebras by introducing a family of direct limit \( C^* \)-algebras determined by noncommutative multivariable versions of weighted shift operators. He realised these algebras as the direct limits of full matrix algebras over Cuntz–Toeplitz and Cuntz algebras, and proved a classification theorem paralleling that of the UHF algebras and the Bunce–Deddens algebras.

In [31] Kribs and Solel initiated the study of a class of direct limit \( C^* \)-algebras determined by weighted shift operators on the path space of a row-finite directed graph \( E \) with no sinks or sources. They called these algebras generalised Bunce–Deddens algebras. Kribs and Solel constructed directed graphs \( E(n) \) for natural numbers \( n \geq 1 \), and realised the generalised Bunce–Deddens algebras as the direct limits of the \( C^* \)-algebras of the \( E(n) \).

The class of generalised Bunce–Deddens algebras contains the Bunce–Deddens algebras as the subclass generated by the graph consisting of a single vertex and a single loop edge, and also contains the class of algebras studied in [30] as the subclass generated by the graph consisting of a single vertex and \( k \) loop edges for \( k \geq 2 \).

Kribs and Solel presented their algebras as topological graph algebras and used the results of Katsura [26, 27, 28] to compute \( K \)-theory and to characterise simplicity. They considered the special case when the graph consists of \( j \) vertices connected by a single cycle of \( j \) edges. In this case they characterised the simplicity in terms of \( j \). They also showed that the classification theorem of the Bunce–Deddens algebras applies to this example. They asked whether it holds in general or for which class of graphs it does hold,
but were unable to provide a complete answer.

In this thesis we give a new universal description of Kribs and Solel’s generalised Bunce–Deddens algebras as universal $C^*$-algebras. We prove a Cuntz–Krieger uniqueness theorem which, unlike the Cuntz–Krieger uniqueness theorem for graph algebras, requires no aperiodicity condition. We use Perron–Frobenius theory to understand the structure of these algebras. We use this to extend Kribs and Solel’s characterisation of simplicity to the generalised Bunce–Deddens algebras constructed from finite strongly connected directed graphs. We compute the $K$-theory of generalised Bunce–Deddens algebras to prove a classification theorem for a large class of generalised Bunce–Deddens algebras constructed from finite strongly connected directed graphs. Finally, we study the KMS states on the generalised Bunce–Deddens algebras under the dynamics coming from the gauge actions on the underlying graph algebras.

1.1 Outline of the thesis

Chapter 2. In this chapter we review material about the $C^*$-algebras of directed graphs including uniqueness theorems, simplicity, $K$-theory and KMS states for the dynamics coming from the gauge action. We review the projective limit of a sequence of topological spaces, the direct limit of a sequence of $C^*$-algebras and supernatural numbers, and then discuss the Bunce–Deddens algebras and the generalised Bunce–Deddens algebras. We describe the inverse limit of a sequence of topological measure spaces and review the space of finite signed Borel measures.

Chapter 3. In this chapter we give a new universal description of the graph algebra of $E(n)$ and its Toeplitz extension. We show that there are injective homomorphisms $\mathcal{T}C^*(E(n)) \to \mathcal{T}C^*(E(mn))$ for $n, m \geq 1$. Upon restriction to the canonical abelian subalgebra in $\mathcal{T}C^*(E(mn))$, these inclusions are compatible with a natural surjection $E^{<mn} \to E^{<n}$, so $\varprojlim \mathcal{T}C^*(E(n))$ has an abelian subalgebra isomorphic to $C_0(\varprojlim E^{<n})$. We show that the generalised Bunce–Deddens algebra $\varprojlim C^*(E(n_k))$ of a directed graph $E$ corresponding to a sequence $\omega = (n_k)_{k=1}^\infty$ of positive integers such that $n_k | n_{k+1}$ for all $k \geq 1$ is generated by a copy of $C^*(E)$ and a copy of $C_0(\varprojlim E^{<n_k})$. We give an analogous description of the Toeplitz extension $\varprojlim \mathcal{T}C^*(E(n_k))$. We label these universal descriptions by $C^*(E, \omega)$ and their Toeplitz extensions by $\mathcal{T}C^*(E, \omega)$.

Chapter 4. In this chapter we prove uniqueness theorems for $C^*(E, \omega)$ and $\mathcal{T}(E, \omega)$. The uniqueness theorem for $\mathcal{T}(E, \omega)$ (Proposition 4.1.1) is analogous to that for the Toeplitz extension of a graph algebra, and we prove it using that technology. Interestingly, our Cuntz–Krieger uniqueness theorem (Theorem 4.2.1) for $C^*(E, \omega)$ requires no
APERIODICITY HYPOTHESIS, EMPHASISING KRIBS AND SOLEL’S VIEW OF THESE ALGEBRAS AS GENERALISED Bunce–Deddens Algebras.

Chapter 5 In this chapter we establish some results using Perron–Frobenius theory which will be useful for characterising simplicity, calculating the $K$-theory of $C^*(E,\omega)$, and understanding the structure of KMS factor states. We begin by defining an equivalence relation on $E(n)$. This equivalence relation leads to a very satisfactory characterisation of ideal-structure for $C^*(E,\omega)$ for finite, strongly connected $E$: $C^*(E,\omega)$ decomposes as a direct sum of simple subalgebras indexed by the finite group of integers modulo $\lim_{k\to\infty}\gcd(P_E,n_k)$, the greatest common divisor of the period $P_E$ of the graph $E$ and the sequence $\omega$. It follows that $C^*(E,\omega)$ is simple if and only if $\lim_{k\to\infty}\gcd(P_E,n_k) = 1$ (Corollary 5.2.2). We also use the equivalence relation to calculate cokernels of vertex matrices.

Chapter 6 Kribs and Solel construct a topological graph $E(\infty)$ and use the results of Katsura to study some properties of their direct-limit algebras. They show that $C^*(E,\omega)$ is isomorphic to the topological-graph $C^*$-algebra $C^*(E(\infty))$, allowing them to use Katsura’s structure theory. In this chapter we provide a slightly different description of $E(\infty)$ that we feel clarifies the construction somewhat, and study its structure in greater depth than appears in [31]. We give an alternative proof for Theorem 4.2.1 by applying Katsura’s uniqueness theorem for topological graph $C^*$-algebras together with Kribs and Solel’s observation that their topological graph $E(\infty)$ contains no loops. We use Katsura’s characterisation of simplicity to give an alternative proof for Corollary 5.2.2.

Chapter 7 In this chapter we generalise the classification theorem for Bunce–Deddens algebras ([5, Theorem 4]) to the generalised Bunce-Deddens algebras constructed from a large class of strongly connected finite directed graphs whose vertex matrix has eigenvalue 1. This theorem says that generalised Bunce–Deddens algebras $C^*(E,\omega)$ and $C^*(E,\omega')$ constructed from a large class of strongly connected finite directed graphs $E$ whose vertex matrices $A_E'$ have 1 as an eigenvalue are isomorphic if and only if the supernatural numbers determined by $\omega$ and $\omega'$ are equal. This answers the question asked by Kribs and Solel in [31, Remark 7.7] for a large class of finite strongly connected graph $E$ such that $K_0(C^*(E))$ has a nontrivial torsion-free component. The presentation of $C^*(E,\omega)$ given in Chapter 3 allows for an elementary proof of the “if” implication. We prove the “only if” implication by computing $K_0(C^*(E,\omega))$ and studying its torsion-free component; the assumption that 1 is an eigenvalue of $E$ ensures that the torsion-free component is non-trivial.

Chapter 8 In this chapter we study the KMS states for the dynamics on $\mathcal{T}(E,\omega)$ coming from the gauge action. We concentrate on finite strongly connected graphs $E$.
allowing us to use the Perron–Frobenius theory of Chapter 5. We follow the program of [14, 37]. The vertex matrices for the graphs $E(n_k)$ induce a linear operator on the space of signed Borel measures on $\lim_{\leftarrow} E^{<n_k}$ (Theorem 8.2.1). We find that the KMS condition for states on $\mathcal{T}(E, \omega)$ can be characterised as a subinvariance condition for this operator. We construct KMS$_\beta$ states for all $\beta > \ln \rho(A_E)$ (Proposition 8.3.3), and show that there is an affine isomorphism between the KMS$_\beta$-simplex of $\mathcal{T}(E, \omega)$ and the simplex of probability measures on $\lim_{\leftarrow} E^{<n_k}$ (Corollary 8.3.5). Finally, we investigate which KMS states factor through $C^*(E, \omega)$. In contrast with [10, 22], strong connectedness of $E$ is not sufficient to ensure that $C^*(E, \omega)$ admits a unique KMS state. Following the approach of [25] we show that there are exactly $\lim_{k \to \infty} \gcd(P_E, n_k)$ extremal KMS states for $\mathcal{T}(E, \omega)$ at the critical temperature $\ln \rho(A_E)$ and that these factor through KMS states for $C^*(E, \omega)$. We use the results of [10, 23] to show that there cannot be any KMS states for $C^*(E, \omega)$ at any other temperatures. We deduce that $\phi$ is the only KMS state of $C^*(E, \omega)$ if and only if $\gcd(P_E, \omega) = 1$, and hence if and only if $C^*(E)$ is simple; we further show that this is equivalent to $\phi$ being a factor state.

1.2 Connections to the literature

The material in Chapters 3, 5, 6 and 8 is summarised in the paper [44] written jointly with my supervisors. This work constituted the first part of my PhD research. The approach to the uniqueness theorem presented in Chapter 4 and the approach to simplicity presented in Chapter 5 appeared in an earlier draft of [44], but was replaced with the approach in Chapter 6 for the published version. I have included both as I feel that the hands-on approach in Chapters 4 and 5 give a useful alternative point of view and have independent value. The material in Chapter 7, which answers the question posed by Kribs and Solel in [31, Remark 7.7] for a very large class of graphs $E$, is new to this thesis.
Chapter 2

Background

2.1 Graph algebras

A directed graph \( E = (E^0, E^1, r, s) \) is a combinatorial object consisting of two countable sets \( E^0, E^1 \) and two functions \( r, s : E^0 \to E^1 \). The elements of \( E^0 \) are called the vertices and the elements of \( E^1 \) are called the edges. The map \( s \) is called the source map and \( r \) is called the range map. For each edge \( e \in E^1 \), \( s(e) \in E^0 \) is called the source of \( e \) and \( r(e) \in E^0 \) is called the range of \( e \).

We use the convention for directed graphs appearing in Raeburn’s book [41]. So if \( E = (E^0, E^1, r, s) \) is a directed graph, then a path in \( E \) is a word \( \mu = e_1 \ldots e_n \) in \( E^1 \) such that \( s(e_i) = r(e_{i+1}) \) for all \( i \), and we write \( r(\mu) = r(e_1), s(\mu) = s(e_n) \), and \( |\mu| = n \). We write \( E^* \) for the collection of all finite paths in \( E \) (including the vertices, which we regard as paths of length 0). As usual, we denote \( E^n := \{ \mu \in E^* : |\mu| = n \} \); we also write \( E^{<n} := \{ \mu \in E^* : |\mu| < n \} \). We borrow the convention from the higher-rank graph literature in which we write, for example \( vE^* \) for \( \{ \mu \in E^* : r(\mu) = v \} \), and \( vE^1w \) for \( \{ e \in E^1 : r(e) = v \text{ and } s(e) = w \} \).

We say that \( E \) is finite if the sets \( E^0 \) and \( E^1 \) are both finite. We say that \( E \) is row-finite if \( vE^1 \) is finite for all \( v \in E^0 \), and that \( E \) has no sources if each \( vE^1 \) is nonempty.

**Definition 2.1.1.** Let \( E \) be a row-finite directed graph with no sources. A Toeplitz–Cuntz–Krieger E-family in a \( C^* \)-algebra \( A \) is a pair \( (S, P) \), where \( S = \{ S_e : e \in E^1 \} \subseteq A \) is a collection of partial isometries and \( P = \{ P_v : v \in E^0 \} \subseteq A \) is a set of mutually orthogonal projections such that \( S_e^* S_e = P_{s(e)} \) for all \( e \in E^1 \) and

\[
p_v \geq \sum_{e \in vE^1} S_e S_e^* \quad (2.1.1)
\]
for all \( v \in E^0 \). If equality holds in (2.1.1) for all \( v \in E^0 \), then \((S, P)\) is a Cuntz–Krieger \( E \)-family.

It follows from (2.1.1) that the projections \( \{S_e S_e^* : e \in E^1\} \) are mutually orthogonal: algebraically, \( S_e S_f = 0 \) whenever \( e \neq f \) (see, for example, [22, Corollary 1.2]).

For each path \( \mu \in E^\ast \), we define
\[
S_\mu := \begin{cases} 
P_\mu & \text{if } \mu \in E^0 \\ S_{\mu_1} S_{\mu_2} \ldots S_{\mu_{|\mu|}} & \text{otherwise.}
\end{cases}
\]

The following lemma summarises results from [41, Chapter 1].

**Lemma 2.1.2.** Let \( E \) be a row-finite directed graph and let \((S, P)\) be a Cuntz–Krieger \( E \)-family. Then

1. For each \( \mu \in E^\ast \), the element \( S_\mu \) is a partial isometry with \( S_\mu^* S_\mu = P_{s(\mu)} \).

2. For each \( v \in E^0 \) and each \( n \in \mathbb{N} \), we have \( P_v = \sum_{\mu \in vE^\ast_n} S_\mu S_\mu^* \).

3. For each \( \mu \in E^\ast \) and \( v \in E^0 \), we have \( P_v S_\mu = \delta_{v, r(\mu)} S_\mu \) and \( S_\mu P_v = \delta_{s(\mu), v} S_\mu \).

4. Let \((S_e, P_v)\) be a Cuntz–Krieger \( E \)-family, and \( \mu, \nu \in E^\ast \). Then
\[
S_\mu^* S_\nu = \begin{cases} 
S_{\mu'}^* & \text{if } \mu = \nu \mu' \text{ for some } \mu' \in E^\ast \\
S_{\nu'} & \text{if } \nu = \mu \nu' \text{ for some } \nu' \in E^\ast \\
0 & \text{otherwise.}
\end{cases}
\]

5. For \( \mu, \nu \in E^\ast \), we have \( S_\mu S_\nu^* = 0 \) if \( s(\mu) \neq s(\nu) \).

**Proposition 2.1.3** ([41, Corollary 1.15]). Let \( E \) be a row-finite directed graph and let \((S, P)\) be a Cuntz–Krieger \( E \)-family. For \( \mu, \nu, \sigma, \tau \in E^\ast \), we have
\[
(S_\mu S_\nu^*)(S_\sigma S_\tau^*) = \begin{cases} 
S_{\mu \sigma'} S_{\tau}^* & \text{if } \sigma = \nu \sigma' \\
S_{\mu \tau \nu'} S_{\tau}^* & \text{if } \nu = \sigma \nu' \\
0 & \text{otherwise.}
\end{cases}
\]

**Corollary 2.1.4** ([41, Corollary 1.16]). Let \( E \) be a row-finite directed graph and let \((S, P)\) be a Cuntz–Krieger \( E \)-family. Then \( C^* (S, P) := C^* (S \cup P) \) satisfies
\[
C^* (S, P) = \overline{\text{span}} \{ S_\mu S_\nu^* : \mu, \nu \in E^\ast, s(\mu) = s(\nu) \}.
\]
The following theorem summarises Theorem 1.2 and Remark 1.3 of [34] or Proposition 1.21 and Corollary 1.22 of [41]. It says that for each row-finite directed graph there is a universal $C^*$-algebra which is unique up to isomorphism.

**Theorem 2.1.5.** Let $E$ be a row-finite directed graph with no sources. There exists a $C^*$-algebra $C^*(E)$ generated by a Cuntz–Krieger $E$-family $(s, p)$ which is universal in the sense that given any other Cuntz–Krieger $E$-family $(S, P)$, there is a homomorphism $\pi_{S, P}: C^*(E) \to C^*(S, P)$ satisfying $\pi_{S, P}(p_v) = P_v$ and $\pi_{S, P}(s_v) = S_v$ for all $v \in E^0$ and $e \in E^1$. Moreover, the pair $(C^*(E), (s, p))$ is unique up to isomorphism in the sense that if $(B, (S, P))$ is another pair with the same properties, then there is an isomorphism of $C^*(E)$ into $B$ which carries each $p_v$ to $P_v$ and each $s_v$ to $S_v$.

The $C^*$-algebra $C^*(E)$ is called the Cuntz–Krieger algebra of $E$ or the graph algebra of $E$, and we refer to the Cuntz–Krieger $E$-family $(p, s)$ as the universal Cuntz–Krieger $E$-family.

The Toeplitz algebra $T^*C^*(E)$ is the universal $C^*$-algebra generated by a Toeplitz–Cuntz–Krieger family ([16, Proposition 1.3]).

Graph algebras have several generalisations including the higher rank graph algebras of Kumjian and Pask [33] and the topological graph algebras of Katsura [26]; we will make use of the latter in Chapter 6.

A **topological graph** $F$ consists of locally compact Hausdorff spaces $F^0$ and $F^1$ and maps $r, s: F^1 \to F^0$ such that $r$ is continuous and $s$ is a homeomorphism. Katsura [26] associates to each topological graph $F$ a $C^*$-algebra that we denote $C^*(F)$. This $C^*(F)$ is generated by a homomorphism $t_0^F: C_0(F^0) \to C^*(F)$ and a linear map $t_1^F: C_c(F^1) \to C^*(F)$ satisfying relations reminiscent of the Cuntz–Krieger relations for graph algebras (for a description that avoids the machinery of Hilbert modules, see [38]). The pair $(t_0^F, t_1^F)$ is called a Cuntz–Krieger $F$-pair. When $F^0$ and $F^1$ are discrete and countable, $C^*(F)$ coincides with the usual graph $C^*$-algebra.

### 2.2 Uniqueness and simplicity of graph algebras

Given a directed graph $E$, we denote by $E^* \times E^*$ the collection of all pairs $(\mu, \nu) \in E^* \times E^*$ such that $s(\mu) = s(\nu)$.

An **action** of a group $G$ on a $C^*$-algebra $A$ is a group homomorphism $\alpha: G \to \text{Aut} A$. The image of $g \in G$ under $\alpha$ is usually denoted by $\alpha_g$. The action is said to be **strongly continuous** if for each $a \in A$ the map $g \mapsto \alpha_g(a)$ is continuous from $G$ to $A$. 
Proposition 2.2.1 ([11 Proposition 2.1]). Let $E$ be a row-finite directed graph. There is a strongly continuous action $\gamma$ of $\mathbb{T}$ on $C^*(E)$, called the gauge action, such that $\gamma_z(s_\mu s_\nu^*) = z^{||\mu||-||\nu||} s_\mu s_\nu^*$ for all $(\mu, \nu) \in E^* \ast E^*$. In particular, $\gamma_z(s_e) = z s_e$ and $\gamma_z(p_v) = p_v$ for all $z \in \mathbb{T}, e \in E^1$ and $v \in E^0$.

The following theorem is known as the gauge-invariant uniqueness theorem. The first version of this theorem was formulated by an Huef and Raeburn for Cuntz–Krieger algebras in [21]. It has become a fundamental tool of graph $C^*$-algebra theory.

Theorem 2.2.2 ([3], Theorem 2.1). Let $E$ be a row-finite directed graph with no sources, and let $(S,P)$ be a Cuntz–Krieger $E$-family in a $C^*$-algebra $B$. Suppose that each $P_v$ is nonzero, and that there is a strongly continuous action $\beta : \mathbb{T} \to \text{Aut} B$ satisfying $\beta_z(t_e) = z t_e$ for all $e \in E^1$. Then $\pi_{S,P}$ is injective.

Definition 2.2.3. A cycle in $E$ is a path $\lambda \in E^* \setminus E^0$ such that $r(\lambda) = s(\lambda)$. An entrance to a cycle $\lambda = \lambda_1 \ldots \lambda_n \in E^n$, $n \geq 1$, is an edge $e \in E^1$ such that $r(e) = r(\lambda_i)$ but $e \neq \lambda_i$ for some $i$.

The following uniqueness theorem is a generalisation of Cuntz and Krieger’s [8, Theorem 2.13], and is known as the Cuntz–Krieger uniqueness theorem.

Theorem 2.2.4 ([11], Theorem 2.4). Let $E$ be a row-finite directed graph with no sources, and suppose that every cycle in $E$ has an entrance. Let $(S,P)$ be a Cuntz–Krieger $E$-family. Suppose that each $P_v$ is nonzero. Then $\pi_{S,P}$ is injective.

A uniqueness theorem for the Toeplitz algebra $TC^*(E)$ is established in [16, Theorem 4.1]. It states that for a Toeplitz–Cuntz–Krieger $E$-family $(T,Q)$ the representation $\pi_{T,Q}$ is faithful if and only if every $Q_v$ is nonzero, and $Q_v > \sum_{e \in vE^1} T_e T_e^*$ for every vertex $v$ such that $E^1 v$ is finite.

A $C^*$-algebra is simple if it has no nontrivial closed two-sided ideals. Simplicity for graph algebras can be determined from the graph $E$. For $v, w \in E^0$ we write $v \geq w$ if there exists a path $\alpha \in vE^* w$. A graph is cofinal if every for every $v \in E^0$ and every infinite path $\lambda$ there exists $n \geq 1$ such that $v \geq r(\lambda_n)$. A loop is an edge $e \in E^1$ such that $r(e) = s(e)$.

Theorem 2.2.5 ([3, Proposition 5.1]). Let $E$ be a row-finite directed graph with no sources. Then $C^*(E)$ is simple if and only if $E$ is cofinal and every loop has an entrance.
2.3 Perron-Frobenius theory

Let $X$ be a finite set. We denote by $M_X(\mathbb{R})$ the collection of $|X|\times|X|$ matrices indexed by elements of $X$ and having entries in $\mathbb{R}$. Formally, $a \in M_X(\mathbb{R})$ is a function $a : X \times X \to \mathbb{R}$, and these form a finite dimensional algebra with pointwise addition and scalar multiplication, and algebraic multiplication given by $(ab)(x,y) = \sum_{z\in X} a(x,z)b(z,y)$. A matrix $A \in M_X(\mathbb{R})$ is said to be nonnegative if $A(x,y) \geq 0$ for all $x,y \in X$, and irreducible if, for each $x \in X$, there exists $n \in \mathbb{N}$ such that $A^n(x,y) \neq 0$. For a nonnegative, irreducible matrix $A \in M_X(\mathbb{R})$, the Perron–Frobenius theorem (see, for example, [48, Theorem 1.5]) implies that the spectral radius $\rho$ (example, [48, Theorem 1.5]) implies that the spectral radius $\rho(A) := \max\{|\lambda| : \lambda \in \mathbb{R} : \lambda \text { is an eigenvalue of } A\}$ is an eigenvalue of $A$ for which there exists an eigenvector $r = (r_x)_{x \in X}$ with $r_x > 0$ for all $x \in X$. The eigenvector $r$ of $A$ such that $\sum_{x \in X} r_x = 1$ is the unimodular Perron–Frobenius eigenvector of $A$. The unimodular Perron–Frobenius eigenvector can be regarded as a probability measure on $X$ (see Chapter 8).

The vertex matrix of a directed graph $E$ is the $|E^0| \times |E^0|$ integer matrix $A_E$ with entries $A_E(v,w) = |vE^1w|$. Let $\mathbb{Z}^{E^0} := \bigoplus_{v \in E^0} \mathbb{Z}$ be the free abelian group over $E^0$ with basis $\{\delta_v : v \in E^0\}$. It is often useful to view $A_E$ as a map $A_E : \mathbb{Z}^{E^0} \to \mathbb{Z}^{E^0}$ defined by $A_E|_{E^0} = \sum_{w \in E^0} A_E(w,v)\delta_w$ for $v \in E^0$.

A directed graph is strongly connected if for every pair of vertices $v,w \in E^0$, there exists $\lambda \in E^*\setminus E^0$ such that $r(\lambda) = v$ and $s(\lambda) = w$. The vertex matrix $A_E$ is irreducible if and only if the graph $E$ is strongly connected.

The period $\mathcal{P}_E$ of a strongly connected directed graph $E$ is given by $\mathcal{P}_E = \gcd\{|\mu| : \mu \in E^*, r(\mu) = s(\mu)\}$ (see for example [36, Section 6] with $k = 1$). The group $\mathcal{P}_E \mathbb{Z}$ is then equal to the subgroup generated by $\{|\mu| : \mu \in vE^*v\}$ for any vertex $v$ of $E$, and so is equal to $\{|\mu| - |\nu| : |\mu|, |\nu| \in vE^*v\}$ for any $v$.

2.4 $K$-theory of graph algebras

The $K$-theory of a $C^*$-algebra $A$ is a pair of abelian groups $K_0(A)$ and $K_1(A)$. The following outlines the construction of the $K$-theory of unital $C^*$-algebras. This is based on [45, Chapter 3]. See [45, Chapter 4] for the nonunital case.

If $A$ is unital, then the group $K_0(A)$ is formed from equivalence classes of projections in matrix algebras over $A$. For each $n \geq 1$, let $\text{Proj } M_n(A)$ be the set of projections in $M_n(A)$, the $C^*$-algebra of $n \times n$ matrices over $A$. For $a \in M_n(A)$ and $M_m(A)$, define

$$a \oplus b = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \in M_{m+n}(A).$$
Identifying $p \in \Proj M_n(A)$ with $p \oplus 0$ gives an embedding $\Proj M_n(A)$ in $\Proj M_{n+1}(A)$. Define $\Proj_{\infty}(A) := \bigcup_{n=1}^{\infty} \Proj(M_n(A))$. There is an equivalence relation $\sim_0$ on $\Proj_{\infty}(A)$ defined for $p \in \Proj M_n(A)$ and $q \in \Proj M_m(A)$ by $p \sim_0 q$ if there exists $u \in M_{m,n}(A)$ such that $p = u^*u$ and $q = uu^*$. The set $D(A) := \Proj_{\infty}(A)/\sim_0$ of equivalence classes $\{[p] : p \in \Proj_{\infty}(A)\}$ is an abelian semigroup, with $[p] + [q] = [p \oplus q]$. Then $K_0(A)$ is the Grothendieck group of $D(A)$. That is, $K_0(A)$ is the group of all formal differences

$$K_0(A) = \{[p] - [q] : p, q \in \Proj_{\infty}(A)\}$$

with

$$([p] - [q]) + ([r] - [s]) = ([p] + [r]) - ([q] + [s]).$$

For the details, see [45, pages 21–42].

The group $K_1(A)$ is formed from equivalence classes of unitary elements in $M_n(A)$. Let $UM_n(A)$ denote the unitary elements of $M_n(A)$. Identifying $u \in UM_n(A)$ with $u \oplus 1$ gives an embedding $UM_n(A)$ in $UM_{n+1}(A)$. We let $U_{\infty}(A) := \bigcup_{n=1}^{\infty} U(M_n(A))$. There is an equivalence relation $\sim_1$ on $U_{\infty}(A)$ defined for $u \in UM_n(A)$ and $v \in UM_m(A)$ by $u \sim_1 v$ if there is a natural number $k \geq \max\{m, n\}$ and a continuous path $t \mapsto u_t : [0, 1] \to UM_k(A)$ such that $u_0 = u \oplus 1_{k-n}$ and $u_1 = v \oplus 1_{k-m}$.

The group $K_1(A)$ is the set of equivalence classes $\{[u]_1 : u \in U_{\infty}(A)\}$ with addition given by $[u]_1 + [v]_1 = [u \oplus v]_1$; the identity is the class containing the identity matrices $1_n$. The group $K_1(A)$ is an abelian group (see [15, Section 8]).

**Remark 2.4.1.** A homomorphism $\phi : A \to B$ between unital $C^*$-algebras induces homomorphisms $\phi_n : M_n(A) \to M_n(B)$ satisfying $\phi_n((a_{ij})) = (\phi(a_{ij}))$. These homomorphisms map projections to projections and unitaries to unitaries, and hence induce homomorphisms $K_0(\phi) : K_0(A) \to K_0(B)$ and $K_1(\phi) : K_1(A) \to K_1(B)$ satisfying

$$K_0(\phi)([p] - [q]) = [\phi_n(p)] - [\phi_n(q)] \text{ and } K_1(\phi)([u]_1) = [\phi_n(u)]_1.$$ 

This process is functorial: the identity homomorphism induces the identity map on $K$-groups, and $K_i(\phi \circ \psi) = K_i(\phi) \circ K_i(\psi)$ for $i = 1, 2$.

The $K$-theory of graph algebras is described in [11, Chapter 7]. The main result is the following.

**Theorem 2.4.2.** (11, Theorem 7.1). Let $E$ be a row-finite graph with no sources, and let $A_E$ be the vertex matrix of $E$. Then $K_1(C^*(E))$ is isomorphic to the kernel of $1 - A_E^* : \mathbb{Z}^E \to \mathbb{Z}^E$, and $K_0(C^*(E))$ is isomorphic to the cokernel.
The vertex projections \( \{ p_v : v \in E^0 \} \) define classes \([p_v] \) in \( D(C^*(E)) \), and the Cuntz–Krieger relations imply that \([p_v] = \sum_{e \in E^1} s_es_e^* \sum_{e \in E^1} s_es_e^* = \sum_{e \in E^1} [p_{n(e)}] \) in \( K_0(C^*(E)) \). The above theorem says that \( K_0(C^*(E)) \) is generated by \( \{ [p_v] : v \in E^0 \} \) subject only to these relations. Looking into the proof of [41, Theorem 7.1] shows that the isomorphism \( K_0(C^*(E)) \rightarrow \text{coker}(1-A_E^t) \) is given by \( \sum_{v \in E^0} a_v [p_v] \rightarrow a + \text{Im}(1-A_E^t) \) for \( a \in \mathbb{Z}^{E^0} \).

### 2.5 KMS states on the \( C^* \)-algebras of finite graphs

We use the definition of KMS states given in [2] Definition 5.3.1. Let \((A, \mathbb{R}, \alpha)\) be a \( C^* \)-dynamical system. An element \( a \in A \) is analytic for \( \alpha \) if \( t \rightarrow \alpha_t(a) \) extends to an entire function \( z \mapsto \alpha_z(a) \) on \( \mathbb{C} \) (the \( \alpha_z \) will typically not be homomorphisms for \( z \not\in \mathbb{R} \)).

Let \( A_\alpha \) denote the collection of analytic elements of \( A \). A state \( \phi \) of \( A \) is said to be a KMS state for \( \alpha \) at inverse temperature \( \beta \in \mathbb{R} \setminus \{0\} \) if

\[
\phi(ab) = \phi(b \alpha_{i\beta}(a)) \quad \text{for all } a, b \in A_\alpha.
\]

It suffices to verify this KMS condition on any \( \alpha \)-invariant set of analytic elements spanning a dense subspace of \( A \). Proposition 5.3.3 of [2] says that if \( \phi \) is KMS for \( \alpha \) and \( \beta \neq 0 \), then \( \phi \) is \( \alpha \)-invariant. If \( \beta = 0 \), then the KMS condition reduces to requiring that \( \phi \) is a trace, and we then impose \( \alpha \)-invariance as an additional requirement.

The following result characterises the KMS states for the gauge actions on Toeplitz algebras of finite directed graphs.

**Proposition 2.5.1** ([22 Proposition 2.1]). Let \( E \) be a finite directed graph, and let \( A_E \) be the vertex matrix. Let \( \gamma : \mathbb{T} \rightarrow \text{Aut} \mathcal{T}C^*(E) \) be the gauge action, and define \( \alpha : \mathbb{R} \rightarrow \text{Aut} \mathcal{T}C^*(E) \) by \( \alpha_t = \gamma_{e^{it}} \). Let \( \beta \in \mathbb{R} \).

1. A state \( \phi \) of \( \mathcal{T}C^*(E) \) is a KMS state of \( (\mathcal{T}C^*(E), \alpha) \) if and only if

\[
\phi(s_\mu s_\nu^*) = \delta_{\mu, \nu} e^{-\beta|\mu|} \phi(p_{s(\mu)}) \quad \text{for all } \mu, \nu \in E^*.
\]

2. Suppose that \( \phi \) is a KMS state of \( (\mathcal{T}C^*(E), \alpha) \), and define \( m^\phi = (m^\phi_v) \in [0, \infty)^{E^0} \) by \( m^\phi_v = \phi(p_v) \). Then \( m^\phi \) is a probability measure on \( E^0 \) satisfying the subinvariance relation \( A_\beta m^\phi \leq e^\beta m^\phi \).

3. A KMS state \( \phi \) of \( (\mathcal{T}C^*(E), \alpha) \) factors through \( C^*(E) \) if and only if \( (A_\beta m^\phi)_v = e^\beta m^\phi_v \) whenever \( v \) is not a source.
The following shows how KMS\(\beta\) states are constructed from probability measures for \(\beta > \ln \rho(A_E)\).

**Theorem 2.5.2** ([22, Theorem 3.1]). Let \(E\) be a finite directed graph. Let \(\gamma : \mathbb{T} \to \text{Aut} \mathcal{T}C^*(E)\) be the gauge action, and define \(\alpha : \mathbb{R} \to \text{Aut} \mathcal{T}C^*(E)\) by \(\alpha_t = \gamma_{\epsilon_t}\). Assume that \(\beta > \ln \rho(A_E)\).

1. For \(v \in E^0\), the series \(\sum_{\mu \in E^* v} e^{-\beta |\mu|}\) either converges or is finite, with sum \(y_v \geq 1\). Set \(y := (y_v) \in [1, \infty)^{E^0}\), and consider \(\epsilon \in [0, \infty)^{E^0}\). Then \(m := (I - e^{-\beta A_E})^{-1}\epsilon\) is a probability measure on \(E^0\) if and only if \(\epsilon \cdot y = 1\).

2. Suppose \(\epsilon \in [0, \infty)^{E^0}\) satisfies \(\epsilon \cdot y = 1\), and set \(m := (I - e^{-\beta A_E})^{-1}\epsilon\). Then there is a KMS\(\beta\) state \(\phi\) of \((\mathcal{T}C^*(E), \alpha)\) satisfying
\[
\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} e^{-\beta |\mu|} m s(\mu),
\]

3. The map \(\epsilon \mapsto \phi\) is an affine isomorphism of
\[
\Sigma_{\beta} := \{\epsilon \in [0, \infty)^{E^0} : \epsilon \cdot y = 1\}
\]
onto the simplex of KMS\(\beta\) states of \((\mathcal{T}C^*(E), \alpha)\). The inverse of this isomorphism takes a KMS\(\beta\) state \(\phi\) to \(\alpha m\).

The critical inverse temperature is \(\beta = \ln \rho(A_E)\). Corollary 4.2 of [22] shows that for any finite graph, there exists at least one KMS state at the critical temperature. To obtain uniqueness, \(E\) is assumed to be strongly connected.

**Theorem 2.5.3** ([22, Theorem 4.3]). Let \(E\) be a strongly connected finite directed graph. Let \(\gamma : \mathbb{T} \to \text{Aut} \mathcal{T}C^*(E)\) be the gauge action, and define \(\alpha : \mathbb{R} \to \text{Aut} \mathcal{T}C^*(E)\) by \(\alpha_t = \gamma_{\epsilon_t}\). Let \(x\) be the unimodular Perron-Frobenius eigenvector of the vertex matrix \(A_E\).

1. The system \((\mathcal{T}C^*(E), \alpha)\) has a unique KMS\(_{\ln \rho(A_E)}\) state \(\phi\). This state satisfies
\[
\phi(s_\mu s_\nu^*) = \delta_{\mu,\nu} \rho(A_E)^{-|\mu|} x s(\mu),
\]
and factors through a KMS\(_{\ln \rho(A_E)}\) state \(\tilde{\phi}\) of \((C^*(E), \alpha)\).

2. The state \(\tilde{\phi}\) is the only KMS state of \((C^*(E), \alpha)\).

3. If \(\beta < \ln \rho(A_E)\), then \((\mathcal{T}C^*(E), \alpha)\) has no KMS\(\beta\) states.
2.6 Projective limits of topological spaces

The following results for projective (or inverse) limits of topological spaces are standard (see [43, Chapter 1] for details).

Definition 2.6.1. A directed partially ordered set or directed poset $I$ is a set with a binary relation $\leq$ that is reflexive, antisymmetric and transitive, such that for all $i, j \in I$, there exists $k \in I$ such that $i, j \leq k$.

Definition 2.6.2. A projective system of topological spaces over $I$ consists of a collection $\{X_i : i \in I\}$ of topological spaces indexed by a directed poset $I$, and a collection of continuous maps $\varphi_{i,j} : X_i \to X_j$, defined whenever $i \geq j$, such that the diagrams of the form

$$
\begin{array}{ccc}
X_i & \xrightarrow{\varphi_{i,k}} & X_k \\
\downarrow{\varphi_{i,j}} & & \uparrow{\varphi_{j,k}} \\
X_j
\end{array}
$$

commute whenever $i, j, k \in I$ and $i \geq j \geq k$. We assume that $\varphi_{i,i}$ is the identity map $\text{id}_{X_i}$ on $X_i$. We shall denote such a system by $(X_i, \varphi_{i,j}, I)$, or by $(X_i, \varphi_{i,j})$ if the index set $I$ is clearly understood.

Definition 2.6.3. Let $(X_i, \varphi_{i,j}, I)$ be a projective system of topological spaces. Let $X = \prod_{i \in I} X_i$ be the product space. The subset

$$\lim_{\leftarrow} X_i = \{x = (x_i)_{i \in I} \in \prod_{i \in I} X_i : \varphi_{i,j}(x_i) = x_j \text{ whenever } i \geq j\}$$

of $X$ is called the projective limit of the projective system.

For each $j \in I$, the map $p_j : \prod_{i \in I} X_i \to X_j$ is called the projection map and its restriction to $\lim_{\leftarrow} X_i$ is called the canonical map and is sometimes denoted by $\varphi_{\infty,j} : \lim_{\leftarrow} X_i \to X_j$. The map $\varphi_{\infty,j}$ is continuous because each $p_j$ is. By the definition of $\lim_{\leftarrow} X_i$, it follows that $\varphi_{\infty,j} = \varphi_{j,k} \circ \varphi_{\infty,k}$ whenever $k \geq j$. The sets $\{\varphi_{\infty,j}^{-1}(U_j) : U_j \text{ open in } X_j\}$ form a base of the topology of $\lim_{\leftarrow} X_i$.

Proposition 2.6.4 ([43, Proposition 1.1.1]). Let $(X_i, \varphi_{i,j}, I)$ be a projective system of topological spaces. Suppose there is a topological space $Y$ and maps $\psi_j : Y \to X_j$ such that $\varphi_{j,k} \circ \psi_j = \psi_k$ whenever $k \leq j$. Then there is a unique continuous map $\psi : Y \to \lim_{\leftarrow} X_i$ such that $\varphi_{\infty,j} \circ \psi = \psi_j$ for all $j \in I$. 
**Proposition 2.6.5** ([43, Lemma 1.1.2]). Let \((X_i, \varphi_{i,j}, I)\) be a projective system of nonempty Hausdorff spaces. Then \(\varprojlim X_i\) is a closed subset of \(\prod_{i \in I} X_i\). If the \(X_i\) are compact, so is \(\varprojlim X_i\).

### 2.7 Direct limits of \(C^\ast\)-algebras

Viewing \(C^\ast\)-algebras as algebras of functions on “noncommutative topological spaces”, a direct limit of \(C^\ast\)-algebras is analogous to a projective limit of topological spaces. The following results are standard (see [43, Chapter 6] for details).

**Definition 2.7.1.** Let \((A_n)_{n=1}^\infty\) be a sequence of \(C^\ast\)-algebras and let \((\varphi_n)_{n=1}^\infty\) be a sequence of homomorphisms \(\varphi_n : A_n \to A_{n+1}\) (called connecting maps). A direct limit for the sequence \((A_n, \varphi_n)\) is a pair \((\varinjlim A_n, (\varphi_{n,\infty})_{n=1}^\infty)\), where \(\varinjlim A_n\) is a \(C^\ast\)-algebra and, for each \(n \geq 1\), \(\varphi_{n,\infty} : A_n \to \varinjlim A_n\) is a homomorphism, and where the following two conditions hold.

1. The following diagram commutes for each \(n \geq 1\).

\[
\begin{array}{ccc}
A_n & \xrightarrow{\varphi_n} & A_{n+1} \\
\downarrow{\varphi_{n,\infty}} & & \Downarrow{\varphi_{n+1,\infty}} \\
\varinjlim A_n & & \\
\end{array}
\]

2. For any \(C^\ast\)-algebra \(B\), and any sequence of homomorphisms \(\psi_n : A_n \to B\) such that \(\psi_{n,\infty} = \psi_{n+1,\infty} \circ \varphi_n\), for all \(n \geq 1\), there is a unique homomorphism \(\psi : \varinjlim A_n \to B\) such that for each \(n \geq 1\) the following diagram commutes.

\[
\begin{array}{ccc}
A_n & \xrightarrow{\varphi_{n,\infty}} & \varinjlim A_n \\
\downarrow{\psi_{n,\infty}} & & \Downarrow{\psi} \\
B & & \\
\end{array}
\]

For \(n, m \geq 1\), we define \(\varphi_{n,m} := \varphi_{n+m-1} \circ \cdots \circ \varphi_n : A_n \to A_{n+m}\).

**Proposition 2.7.2** ([43, Proposition 6.2.4]). Every sequence of \(C^\ast\)-algebras \((A_n)_{n=1}^\infty\) with connecting maps \((\varphi_n)_{n=1}^\infty\) has a direct limit \((\varinjlim A_n, (\varphi_{n,\infty})_{n=1}^\infty)\). Moreover, the following hold.
1. \( \lim_{n \to \infty} A_n = \bigcup_{n=1}^{\infty} \varphi_n(A_n) \).

2. \( \|\varphi_n(a)\| = \lim_{m \to \infty} \|\varphi_{n+m}(a)\| \) for all \( n \geq 1 \) and \( a \in A_n \).

3. \( \ker \varphi_n = \{ a \in A_n : \lim_{m \to \infty} \|\varphi_{n+m}\| = 0 \} \).

4. Let \( (B_n, (\psi_n)_{n=1}^{\infty}) \) and \( \psi : \lim_{n \to \infty} A_n \to B \) as in Definition 2.7.1. Then

(a) \( \ker \varphi_n \subseteq \ker \psi_n \) for all \( n \geq 1 \).

(b) \( \psi \) is injective if and only if \( \ker \psi_n \subseteq \ker \varphi_n \) for all \( n \geq 1 \).

(c) \( \psi \) is surjective if and only if \( B = \bigcup_{n=1}^{\infty} \psi_n(A_n) \).

Example 2.7.3. Let \( (X_n, \phi_{n+1,n})_{n=1}^{\infty} \) be an inverse system of locally compact topological spaces indexed by \( n \geq 1 \). The homomorphisms \( \phi_{n+1,n} : X_{n+1} \to X_n \) induce homomorphisms \( \phi_{n+1,n}^* : C_0(X_n) \to C_0(X_{n+1}) \) satisfying \( (\phi_{n+1,n}^*(f))(x) = f(\phi_{n+1,n}(x)) \) for \( f \in C(X_n) \) and \( x \in X_{n+1} \). Moreover, \( (C_0(X_n), \psi_{n+1,n})_{n=1}^{\infty} \) is a direct sequence of commutative \( C^* \)-algebras, and

\[
C_0(\lim_{n \to \infty} (X_n, \varphi_{n+1,n})) \cong \lim_{n \to \infty}(C_0(X_n), \varphi_{n+1,n}).
\]

The direct limit of a sequence of Abelian groups and group homomorphisms is defined similarly and also exists (see [45] Proposition 6.2.5). By [45] Theorem 6.3.2 there is an isomorphism \( \theta_0 : K_0(\lim_{n \to \infty}(A_n, \varphi_n)) \to \lim_{n \to \infty}(K_0(A_n), K_0(\varphi_n)). \) Similarly, by [45] Proposition 8.2.7 there is an isomorphism \( \theta_1 : K_1(C^*(\lim_{n \to \infty}(A_n, \varphi_n))) \to \lim_{n \to \infty}(K_1(A_n), K_1(\varphi_n)). \) If the \( A_n \) are unital, then \( \theta_0([\varphi_{n,\infty}(p)]_0) = K_0(\varphi_{n,\infty})([p]_0) \) for any projection \( p \) over \( A_n \) and \( \theta_1([\varphi_{n,\infty}(u)]_1) = K_1(\varphi_{n,\infty})([u]_1) \) for any unitary \( u \) over \( A_n \).

2.8 Supernatural numbers

Supernatural numbers (or Steinitz numbers) are an invariant for some operator algebras (see for example [45] Section 7.4). They also appear in field theory and group theory.

A supernatural number is a sequence \( m = (m_j)_{j=1}^{\infty} \) in \( \mathbb{N} \cup \{ \infty \} \). If we let \( \{p_1, p_2, \ldots \} \) be the set of primes listed in increasing order, we can view \( m \) as an infinite prime factorisation \( m = \prod_{j=1}^{\infty} p_j^{m_j} \). We say \( m \) is infinite if \( \prod_{j=1}^{\infty} p_j^{m_j} = \infty \), or equivalently if \( \sum_{j=1}^{\infty} m_j = \infty \). When all but finitely many exponents \( m_j \) are zero, \( m \) is a natural number.

If \( m = \prod_{j=1}^{\infty} p_j^{m_j} \) and \( n = \prod_{j=1}^{\infty} p_j^{n_j} \) are natural or supernatural numbers, then we define the product of \( m \) and \( n \) by \( mn = (m_j + n_j)_{j=1}^{\infty} \). We say \( n \) divides \( m \), and write \( n|m \), if
$n_j \leq m_j$ for all $j$. Each supernatural number $m$ defines an additive subgroup

$$Q(m) = \left\{ \frac{p}{q} : p \in \mathbb{Z}, q \in \mathbb{N}, q|m \right\} \subseteq \mathbb{Q}.$$  

A multiplicative sequence is a sequence $\omega = (n_k)_{k=1}^\infty$ of natural numbers with $n_k|n_{k+1}$ for all $k \geq 1$. We say that a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$ divides a multiplicative sequence $\omega' = (m_j)_{j=1}^\infty$, and write $\omega|\omega'$, if for each $k \in \mathbb{N}$ there exists $j(k) \in \mathbb{N}$ such that $n_k|m_{j(k)}$. Define an equivalence relation $\sim$ on $\{(n_k)_{k=1}^\infty : n_k|n_{k+1} \text{ for all } k\}$ by $\omega \sim \omega'$ if $\omega|\omega'$ and $\omega|\omega'$. Write $[\omega]$ for the equivalence class of $\omega$ under $\sim$.

To each multiplicative sequence $\omega = (n_k)_{k=1}^\infty$, we can assign a supernatural number $n^\omega$ by $n_j^\omega := \sup\{r : p_j^r|n_k \text{ for some } k \in \mathbb{N}\}$. Conversely, to each supernatural number $m = (m_j)_{j=1}^\infty$, we can associate a multiplicative sequence $\omega^m = (n_k^m)_{k=1}^\infty$ by $n_k^m := \Pi_{j=1}^k p_j^{\min(k,m_j)}$. It is routine to check that $\omega \mapsto n^\omega$ induces a bijection between $\{(n_k)_{k=1}^\infty : n_k|n_{k+1} \text{ for all } k\}/\sim$ and $\{(m_j)_{j=1}^\infty : m_j \in \mathbb{N}\cup\{\infty\} \text{ for all } j\}$.

### 2.9 The Bunce–Deddens algebras

In [5] Bunce and Deddens studied a class of simple $C^*$-algebras which arose from their study of $C^*$-algebras generated by weighted shift operators [4]. These $C^*$-algebras were amongst the earliest examples of simple non-type $I$ $C^*$-algebras, and have been much studied ever since Bunce and Deddens’ paper.

Let $H$ be the Hilbert space $\ell^2(\mathbb{N})$ with canonical orthonormal basis $\{e_i : i \geq 1\}$. Let $\mathcal{B}(H)$ be the set of bounded operators on $H$, and let $\mathcal{K}$ be the ideal of all compact operators on $H$. For $T \in \mathcal{B}(H)$ we denote by $C^*(T)$ the smallest $C^*$-subalgebra of $\mathcal{B}(H)$ containing $T$ and the identity operator $1$.

A bounded operator $T \in \mathcal{B}(H)$ is a weighted shift operator if there is a bounded sequence of complex scalars $(a_n)_{n=1}^\infty$ such that $Te_n = a_{n+1}e_{n+1}$ for all $n \geq 1$. A weighted shift is said to be $p$-periodic if its weight sequence satisfies $a_n = a_{n+p}$ for all $n \geq 1$. Let $A(p)$ be the $C^*$-algebra generated by all $p$-periodic weighted shift operators.

Fix a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. Note that every $n_k$-periodic weighted shift is also $n_{k+1}$-periodic. Thus there is a natural inclusion $\iota_k$ of $A(n_k)$ into $A(n_{k+1})$ for every $k$. The Bunce–Deddens–Toeplitz $C^*$-algebra $A_\omega$ is defined to be the direct limit of the $A(n_k)$ under the inclusions $\iota_k$. In [5] Bunce and Deddens observed that $\mathcal{K}$ is contained in $A(n_k)$ as an ideal, and defined $B(n_k) := A(n_k)/\mathcal{K}$. The inclusions $\iota_k : A(n_k) \to A(n_{k+1})$ descend to inclusions $\tilde{\iota}_k : B(n_k) \to B(n_{k+1})$. The Bunce–Deddens algebra $B_\omega$ is defined to be the direct limit of the $B(n_k)$ under the inclusions $\tilde{\iota}_k$. 


There are isomorphisms $B(n_k) \to M_{n_k}(C(T))$ for each $k$, so $B_\omega$ can also be viewed as the direct limit of the matrix algebras $M_{n_k}(C(T))$ (see [9] Section V.3 for details).

Bunce and Deddens showed that $B_\omega$ is simple ([5, Theorem 2]). They also proved a classification theorem. As with UHF algebras (see [18, Theorem 1.12]), Bunce–Deddens algebras are classified by supernatural numbers.

**Theorem 2.9.1** ([5, Theorem 4]). Fix two multiplicative sequences $\omega := \{n_k\}_{k=1}^\infty$ and $\omega' := \{m_j\}_{j=1}^\infty$. The algebras $B_\omega$ and $B_{\omega'}$ are isomorphic if and only if $[\omega] = [\omega']$.

## 2.10 A generalisation of the Bunce–Deddens algebras

Let $E$ be a row-finite directed graph with no sources or sinks. In Section 2.1 we gave a universal presentation of $\mathcal{T}C^*(E)$ and $C^*(E)$. We now give a concrete construction of these algebras. Let $H_E = \ell^2(E^*)$ be the Hilbert space with orthonormal basis $\{\xi_\mu : \mu \in E^*\}$ indexed by the finite paths in $E$. Fix $\nu \in E^*$ and define an operator $L_\nu$ on $H_E$ by $L_\nu \xi_\mu = \delta_{s(\nu), \tau(\mu)} \xi_{\nu\mu}$. Then $L_\nu$ is a partial isometry. For $\nu \in E^0$, $L_\nu$ is a projection and we write $P_\nu := L_\nu$. The pair $(\nu, P)$ is a Toeplitz–Cuntz–Krieger $E$-family. The Toeplitz algebra of $E$ is isomorphic to the $C^*$-algebra generated by $\{L_e : e \in E^1\}$ ([16, Theorem 4.1]).

For $\nu \in E^*$, define an operator $R_\nu$ on $H_E$ by $R_\nu \xi_\mu = \delta_{\nu, s(\nu)} \lambda_{\nu\mu} \xi_\mu$ for $\mu \in E^*$. Let $\mathcal{K}_E$ be the ideal of all compact operators on $H_E$. Define $\mathcal{K}_E := \bigoplus_{\nu \in E^0} R_\nu \mathcal{K} R_\nu$. Kribs and Solel observe that $\mathcal{K}_E$ is a $C^*$-subalgebra of $\mathcal{T}C^*(E)$ ([31, Proposition 2.1]) and that $\mathcal{T}C^*(E)/\mathcal{K}_E \cong C^*(E)$ ([31, Theorem 2.2]).

Let $e \in E^1$. A bounded operator $T_e$ on $H_E$ is a weighted shift if there are scalars $\{\lambda_\mu : \mu \in E^* \setminus E^0\}$ such that $T_e \xi_\mu = \delta_{s(e), \tau(\mu)} \lambda_{e\mu} \xi_\mu$. For $\mu \in E^*$ and $n \in \mathbb{N}$, we write $[\mu]_n$ for the unique element of $E^{\leq n}$ such that $\mu = [\mu]_n \mu'$ for some $\mu' \in E^*$ with $|\mu'| \in n\mathbb{N}$; we think of $[\mu]_n$ as the residue of $\mu$ modulo $n$. A weighted shift $T_e$ is $p$-periodic if $\lambda_\mu = \lambda_{[\mu]_p}$ and hence $T_e \xi_\mu = \lambda_{e\mu} \xi_\mu = \lambda_{e[\mu]_p} \xi_\mu$ for all $\mu \in s(e)E^*$. Let $A(p)$ be the $C^*$-algebra generated by all $p$-periodic weighted shifts $T_e$, $e \in E^1$, on $H_E$. Let $\omega = \{n_k\}_{k=1}^\infty$ be a multiplicative sequence. Since every $n_k$-periodic weighted shift $T_e$ is also $n_{k+1}$-periodic, there is a natural inclusion of $A(n_k)$ into $A(n_{k+1})$, and we may consider the norm-closed limit algebra $A_E(\omega) := \bigcup_{k \geq 1} A(n_k)$.

Since $\mathcal{T}C^*(E)$ is generated by the unweighted shifts $\{L_e : e \in E^1\}$, we have that $\mathcal{K}_E$ is contained as an ideal in $\mathcal{T}C^*(E) \subseteq A_E(p)$ for each $p \in \mathbb{N}$. Let $B(p) := A(p)/\mathcal{K}_E$. Again, we have inclusions $B(n_k) \to B(n_{k+1})$, so we may also consider the norm-closed limit algebra $B_E(\omega) := \bigcup_{k \geq 1} B(n_k)$. Kribs and Solel referred to $B_E(\omega)$ as a generalised Bunce–Deddens algebra.
Kribs and Solel showed that $B(n)$ can be written as a graph algebra ([31 Theorem 4.2]). Their construction is as follows. Let $E = (E^0, E^1, r, s)$ be a row-finite directed graph with no sources, and fix $n \geq 1$. Define sets

$$E(n)^0 := E^{<n} \quad \text{and} \quad E(n)^1 := \{(e, \mu) : e \in E^1, \mu \in s(e)E^{<n}\},$$

and maps

$$s_n(e, \mu) := \mu \quad \text{and} \quad r_n(e, \mu) = \begin{cases} e\mu & \text{if } |\mu| < n-1 \\ r(e) & \text{if } |\mu| = n-1. \end{cases}$$

Then $E(n) = (E(n)^0, E(n)^1, r_n, s_n)$ is a row-finite directed graph with no sources. This construction has recently been used to calculate the nuclear dimension of graph algebras and Kirchberg algebras [46, 47].

By [31, Theorem 4.2] $A(n) \cong TC^*(E(n))$ and $B(n) \cong C^*(E(n))$. So, for a multiplicative sequence $\omega = (n_k)_{k=1}^{\infty}$, $A_\omega \cong \varprojlim TC^*(E(n_k))$ and $B_\omega \cong \varprojlim C^*(E(n_k))$.

**Example 2.10.1.** Let $C$ be the single cycle with one vertex and let $n \geq 1$. Then $C(n)$ is the single cycle $C_n$ with $n$ edges. The diagrams for $C$ and $C(4)$ follow.

$$
\begin{array}{c}
\qquad (e, v) \qquad (e, e)
\\
\quad v \\
\quad (e, e)
\end{array}
$$

The graph algebra of $C_n$ is isomorphic to the matrix algebra $M_n(C(T))$ (see [12 Theorem 2.2], [21 Remark 2.5]). So, if $\omega = (n_k)_{k=1}^{\infty}$ is a multiplicative sequence, then $B_{C}(\omega) \cong C^*(C_{n_k}) \cong \varprojlim M_{n_k}(C(T))$, yielding the classical Bunce–Deddens algebra $B_{\omega}$.

In [31, Section 6] Kribs and Solel presented $B_\omega$ as a topological graph algebra and used the results of Katsura (see [26, 27, 28]) to discuss the K-theory of $B_\omega$ ([31 Section 8]) and to give a characterisation of simplicity in terms of the topological graph ([31 Section 9]). In [31, Section 7] they considered the generalised Bunce–Deddens algebras constructed from the single cycle $C_j$ with $j$ vertices for some $j \geq 1$. For a given multiplicative sequence $\omega$ they proved a classification result for $B_{C_j}(\omega)$ along the lines of [5 Theorem 4]. They also provided a characterisation of simplicity for $B_{\omega}(C_j)$ in terms of $j$ and $\omega$. In this thesis we extend these results about $B_{\omega}(C_j)$ to a much broader class of generalised Bunce–Deddens algebras (see Corollary 5.2.2 and Theorem 7.0.1).
2.11 Projective limits of measure spaces

Projective (or inverse) limits of measure spaces have been investigated in probability theory (see for example [6]).

Definition 2.11.1. Let $I$ be a directed set. A family $(X_i, M_i, m_i)_{i \in I}$ of measure spaces is called an projective system of measure spaces if, for $j \leq i$, there exist maps $\varphi_{i,j} : X_i \to X_j$ such that

1. $(X_i, \varphi_{i,j})$ is a projective system,
2. $\varphi_{i,j}^{-1}(M_j) \subseteq M_i$,
3. $m_j(E) = m_i(\varphi_{i,j}^{-1}(E))$ for $E \in M_j$.

Let $(X_i, M_i, m_i)$ be a projective system of measure spaces and let $X_\infty = \varprojlim(X_i, \varphi_{i,j})$. Fix $i \in I$. Since $M_i$ is a $\sigma$-ring of subsets of $X_i$, we have that $M_i^* := \varphi_{\infty,i}^{-1}(M_i)$ is a $\sigma$-ring of subsets of $X_\infty$. For each $E \in M_i$, let $E^* := \varphi_{\infty,i}^{-1}(E)$, and define $m_i^*(E^*) := m_i(E)$. Since $m_i$ is a measure on $(X_i, M_i)$, we have that $m_i^*$ is a measure on $(X_\infty, M_i^*)$. So the triple $(X_\infty, M_i^*, m_i^*)$ is a measure space.

We check that $M_j^* \subseteq M_i^*$ for $j \leq i$:

$$M_j^* = \varphi_{\infty,j}^{-1}(M_j) = \varphi_{\infty,i}^{-1}(\varphi_{i,j}^{-1}(M_j)) \subseteq \varphi_{\infty,i}^{-1}(M_i) = M_i^*.$$ 

We check that $m_j^*(E^*) = m_i^*(E^*)$ for $j \leq i$ and $E^* \in M_j^* \subseteq M_i^*$:

$$m_j^*(E^*) = m_j(E) = m_i(\varphi_{i,j}^{-1}(E)) = m_i^*(\varphi_{\infty,i}^{-1}(\varphi_{i,j}^{-1}(E))) = m_i^*(\varphi_{\infty,j}^{-1}(E)) = m_i^*(E^*).$$

Define $M := \bigcup_{i \in I} M_i^*$. Then $M$ is a ring of sets. Define $m$ by $m(E) = m_i^*(E)$ for $E \in M_i^*$. Then $m$ is a finitely additive set function but, in general, is not $\sigma$-additive, and so has no extension to $S(M)$, the $\sigma$-ring generated by $M$.

Definition 2.11.2. If $m$ has a $\sigma$-additive extension (also called $m$) to $S(M)$ then we call $(X_\infty, S(M), m)$ the projective limit of the family $(X_i, M_i, m_i)$.

Definition 2.11.3. A measure space $(X, N, \mu)$ is called a topological measure space if

1. $X$ is a topological space,
2. for every $E \in N$ such that $\mu(E) < \infty$, and every $\epsilon > 0$, there exists a closed compact set $C \subseteq E$ such that $C \in N$, and $\mu(E \setminus C) < \epsilon$. 

Theorem 2.11.4 ([6, Theorem 2.2]). Let \((X_i, M_i, m_i)\) be a projective system of topological measure spaces, where the \(X_i\) are, moreover, compact Hausdorff spaces. Then \(X_\infty\) is compact Hausdorff, and the projective limit measure space \((X_\infty, S(M), m)\) is a topological measure space.

2.12 The space of finite signed Borel measures

If \(M\) is a \(\sigma\)-algebra of subsets of a set \(X\), then a real-valued function \(m\) defined on \(M\) is said to be a finite signed measure if \(m(\emptyset) = 0\) and \(m\) is countably additive in the sense that if \(\{E_n : n \in \mathbb{N}\}\) is a disjoint collection of sets in \(X\), then \(m\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} m(E_n)\).

Suppose that \(X\) is a compact Hausdorff space. We denote by \(\mathcal{M}(X)\) the space of all finite signed Borel measures on \(X\), by \(\mathcal{M}^+(X)\) the subset of \(\mathcal{M}(X)\) consisting of positive Borel measures, and by \(\mathcal{M}^+_1(X)\) the subset of \(\mathcal{M}^+(X)\) consisting of probability measures on \(X\).

Let \(m \in \mathcal{M}(X)\). By the Hahn decomposition theorem [1, Theorem 8.2] there are sets \(P, N \subseteq X\) such that \(X = P \cup N\) and \(P \cap N = \emptyset\), and such that \(m(E \cap P) \geq 0\) and \(m(E \cap N) < 0\) for all Borel \(E \subseteq X\).

Let \(m^+\) and \(m^-\) be given by \(m^+(E) = m(E \cap P)\) and \(m^-(E) = -m(E \cap N)\) for Borel \(E\). Then \(m^+, m^- \in \mathcal{M}^+(X)\). The Jordan decomposition theorem [1, Theorem 8.5] says that \(m = m^+ - m^-\) and that if \(m', m'' \in \mathcal{M}^+(X)\) satisfy \(m = m' - m''\), then \(m'(E) \geq m^+(E)\) and \(m''(E) \geq m^-(E)\) for all Borel \(E \subseteq X\).

The space \(\mathcal{M}(X)\) of finite signed measures is a real Banach space under the norm \(\|m\| = m^+(X) + m^-(X)\).
Chapter 3

Generalised Bunce–Deddens algebras

In this chapter, we give an alternative presentation of Kribs and Solel’s $C^*$-algebras $\mathcal{T}C^*(E(n))$ and $C^*(E(n))$ for $n \geq 1$, and of their direct-limit algebras $\varinjlim \mathcal{T}C^*(E(n_k))$ and $\varinjlim C^*(E(n_k))$ for a given multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. We show that $\mathcal{T}C^*(E(n))$ is the universal $C^*$-algebra generated by a Toeplitz–Cuntz–Krieger $E$-family and mutually orthogonal projections indexed by $E^{<n}$. This presentation has the advantage that the connecting maps $\mathcal{T}C^*(E(n)) \to \mathcal{T}C^*(E(nm))$ have a particularly simple form: they preserve the generating Toeplitz–Cuntz–Krieger $E$-family, and resolve the projection associated to each $\mu \in E^{<n}$ into a sum of projections associated to paths of the form $\mu \tau \in E^{<nm}$. This leads to a very natural presentation of $\varinjlim \mathcal{T}C^*(E(n_k))$ in terms of a Toeplitz–Cuntz–Krieger $E$-family and a representation of the algebra of continuous functions on a natural projective limit of the $E^{<n}$. We show that all of this descends naturally to the $C^*(E(n))$ and $\varinjlim C^*(E(n_k))$.

A condensed account of the results in this chapter appears in joint work with my supervisors [44, Section 3]. Since the primary purpose of this material is both to describe an alternative approach to Kribs and Solel’s construction and also to lay the foundations for our analysis of simplicity and of KMS states later, we present the material in this chapter in fairly fine detail.

3.1 The $C^*$-algebras $\mathcal{T}(E, n)$ and $C^*(E, n)$

Definition 3.1.1. Let $E$ be a row-finite directed graph with no sources, and fix $n \in \mathbb{N}$. A Toeplitz $n$-representation of $E$ in a $C^*$-algebra $A$ is a triple $(T, Q, \Theta)$ where

1. $(T, Q)$ is a Toeplitz–Cuntz–Krieger $E$-family in $A$,
2. $\Theta = \{\Theta_\mu : \mu \in E^{<n}\}$ is a collection of mutually orthogonal projections,
3. \( Q_v = \sum_{\mu \in v^{E < n}} \Theta_\mu \) for all \( v \in E^0 \), and

4. 

\[
T^*_e \Theta_\mu = \begin{cases} 
\Theta_\mu' T^*_e & \text{if } \mu = e\mu' \\
\sum_{e\nu \in E^n} \Theta_\nu T^*_e & \text{if } \mu = r(e) \\
0 & \text{otherwise.}
\end{cases}
\tag{3.1.1}
\]

If \((T, Q)\) is a Cuntz–Krieger \( E \)-family, we call \((T, Q, \Theta)\) a Cuntz–Krieger \( n \)-representation of \( E \).

We show that Kribs and Solel’s \( \mathcal{T}C^*(E(n)) \) is universal for Toeplitz \( n \)-representations of \( E \) and that \( C^*(E(n)) \) is universal for Cuntz–Krieger \( n \)-representations. We first describe a convenient family of spanning elements. We will need the following notation: given a directed graph \( E \), \( n > 0 \) and \( \mu \in E^* \), we write \( \tau_n(\mu) \) for the unique element of \( E^{< n} \) such that \( \mu = \mu' \tau_n(\mu) \) with \( |\mu'| \in n\mathbb{N} \); so \( |\tau_n(\mu)| \equiv |\mu| \pmod{n} \).

Remark 3.1.2. The definition of the paths \( \tau_n(\mu) \) is related to that of the paths \([\mu]_n\) discussed earlier: \( |\tau_n(\mu)| = |[\mu]_n| \), and \( \mu = [\mu]_n \mu' = \mu'' \tau_n(\mu) \) for some \( \mu', \mu'' \in E^* \) such that \( |\mu'| = |\mu''| \in n\mathbb{N} \).

The next two results give us a multiplication formula for elements of Toeplitz \( n \)-representations and Cuntz–Krieger \( n \)-representations. These results will be useful when we introduce a new presentation of the generalised Bunce–Deddens algebras, and will be particularly useful in our calculation of KMS states in Chapter 8.

Lemma 3.1.3. Let \( E \) be a row-finite directed graph with no sources, and take \( n \in \mathbb{N} \). Let \((T, Q, \Theta)\) be a Toeplitz \( n \)-representation of \( E \), and fix \( \mu \in E^* \) and \( \alpha \in E^{<n} \).

1. If \( |\mu| \in n\mathbb{N} \), then \( T^*_e \Theta_{r(\mu)} = \Theta_{s(\mu)} T^*_\mu \).

2. 

\[
T^*_\mu \Theta_\alpha = \begin{cases} 
\Theta_\alpha' T^*_\mu & \text{if } \alpha = \mu\alpha' \\
\Theta_s(\mu) T^*_\mu & \text{if } \mu = \alpha\mu' \text{ and } |\mu'| \in n\mathbb{N} \\
\sum_{|\tau_n(\mu)\lambda| = n} \Theta_\lambda T^*_\mu & \text{if } \mu = \alpha\mu' \text{ and } |\mu'| \notin n\mathbb{N} \\
0 & \text{otherwise.}
\end{cases}
\]

Proof. [1] Suppose \( |\mu| = kn \). We have

\[
T^*_e \Theta_{r(\mu)} = T^*_{\mu_{kn}} \cdots T^*_{\mu_2} T^*_{\mu_1} \Theta_{r(\mu_1)} = T^*_{\mu_{kn}} \cdots T^*_{\mu_2} \left( \sum_{\mu_1 \lambda \in E^n} \Theta_\lambda \right) T^*_{\mu_1}
\]
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by the second case in Definition 3.1.1(1). Repeated applications of the first case in the same relation then give

\[ T_{\mu_3}^* \cdots T_{\mu_n}^* \left( \sum_{\mu_1, \mu_2, \lambda \in E^\alpha} \Theta_\lambda \right) T_{\mu_1}^* = \cdots = T_{\mu_2}^* \cdots T_{\mu_n+1}^* \Theta_{s(\mu_n)} T_{\mu_1}^* \cdots T_{\mu_n}^*. \]  

(3.1.2)

Now induction on \( k \) yields the desired relation.

Next suppose that \( \alpha = \mu \alpha' \). Then repeated applications of Definition 3.1.1(4) give

\[ T_{\mu}^* \Theta_{\alpha} = T_{\mu_{|\mu|}}^* \cdots T_{\mu_1}^* \Theta_{\alpha} = T_{\mu_{|\mu|}}^* \cdots T_{\mu_2}^* \Theta_{\alpha_2} \cdots T_{\mu_1}^* = \cdots = \Theta_{\alpha'} T_{\mu}^*. \]

Now suppose that \( \mu = \alpha \mu' \). Write \( \mu' = \mu'' \tau_n(\mu') \). Then \(|\mu''| \in n \mathbb{N} \), so we calculate, using part (1) at the fourth equality,

\[ T_{\mu}^* \Theta_{\alpha} = T_{\mu_{|\mu|}}^* T_{\mu_1}^* \Theta_{\alpha} = T_{\mu_{|\mu|}}^* T_{\mu_2}^* \Theta_{s(\mu)} T_{\mu_1}^* = T_{\mu_{|\mu|}}^* \Theta_{s(\mu)} T_{\mu_1}^*. \]

If \(|\mu'| \in n \mathbb{N} \), then \( \alpha \mu'' = \mu \) and \( \tau_n(\mu') = s(\mu) \), so the preceding displayed equation gives \( T_{\mu}^* \Theta_{\alpha} = \Theta_{s(\mu)} T_{\mu}^* \). Otherwise, we repeat the first \(|\mu''| \) steps of the calculation (3.1.2) to obtain

\[ T_{\mu}^* \Theta_{\alpha} = \sum_{|\tau_n(\mu')\lambda| = n} \Theta_\lambda T_{\mu}^*. \]

Finally, if \( \mu \neq \alpha \mu' \) and \( \alpha \neq \mu \alpha' \), then we can write \( \mu = \lambda e \mu' \) and \( \alpha = \lambda f \alpha' \) for distinct \( e, f \in E^1 \). Using the first case in part (2), we obtain

\[ T_{\mu}^* \Theta_{\alpha} = T_{\mu_{|\mu|}}^* T_{\mu}^* \Theta_{f \alpha'} T_{\lambda}^*, \]

which is zero by Definition 3.1.1(4).

Lemma 3.1.4. Let \( E \) be a row-finite directed graph with no sources, take \( n \in \mathbb{N} \) and suppose that \( (T, Q, \Theta) \) is a Toeplitz \( n \)-representation of \( E \). For \( \alpha, \beta, \gamma, \delta \in E^* \) and \( \mu, \nu \in E^{<n} \),

\[ (T_{\alpha} \Theta_{\mu} T_{\beta}^*)(T_{\gamma} \Theta_{\nu} T_{\delta}^*) = \begin{cases} T_{\alpha} \Theta_{\mu} T_{\delta}^* & \text{if } \beta = \gamma \beta' \text{ and } \nu = \beta' \mu \\ T_{\alpha} \Theta_{\mu} T_{\delta}^* & \text{if } \beta = \gamma \nu \rho \text{ with } |\rho \mu| \in n \mathbb{N} \\ T_{\alpha} \Theta_{\mu} T_{\delta}^* & \text{if } \gamma = \beta \gamma' \text{ and } \mu = \gamma' \nu \\ T_{\alpha} \Theta_{\mu} T_{\delta}^* & \text{if } \gamma = \beta \mu \rho \text{ with } |\rho \nu| \in n \mathbb{N} \\ 0 & \text{otherwise.} \end{cases} \]

Proof. We consider the case where \(|\beta| \geq |\gamma|\); the case where \(|\gamma| > |\beta|\) will then follow by
taking adjoints. By [41, Corollary 1.14(b)], we have
\[(T_\alpha \Theta_\mu T^*_\beta)(T_\gamma \Theta_\nu T^*_\delta) = \begin{cases} T_\alpha \Theta_\mu T^*_\delta \Theta_\nu T^*_\gamma & \text{if } \beta = \gamma \beta' \\ 0 & \text{otherwise.} \end{cases}\]

Suppose that \(\beta = \gamma \beta'\). By Lemma 3.1.3(2) we have
\[(T_\alpha \Theta_\mu T^*_\beta)(T_\gamma \Theta_\nu T^*_\delta) = \begin{cases} T_\alpha \Theta_\mu \Theta_\nu T^*_\beta & \text{if } \nu = \beta' \nu' \\ T_\alpha \Theta_\mu \Theta_{s(\beta')} T^*_\delta & \text{if } \beta' = \nu \rho \text{ with } |\rho| \in n \mathbb{N} \\ T_\alpha \Theta_\mu \sum_{\tau_n(\rho) \lambda \in E^n} \Theta_\lambda T^*_\delta & \text{if } \beta' = \nu \rho \text{ with } |\rho| \notin n \mathbb{N} \\ 0 & \text{otherwise.} \end{cases}\]

Since \(\tau_n(\rho) \mu \in E^n\) if and only if \(|\rho \mu| \in n \mathbb{N}\), the result follows.

**Theorem 3.1.5.** Let \(E\) be a row-finite directed graph with no sources and let \(n \geq 1\). Let \((t_{(e,\mu)}, q_\mu)\) be the universal Toeplitz–Cuntz–Krieger \(E(n)\)-family in \(TC^*(E(n))\). Then the elements
\[t_{n,e} := \sum_{\mu \in s(e) E^n} t_{(e,\mu)}, \quad q_{n,v} := \sum_{\mu \in v E^n} q_\mu, \quad \text{and} \quad \theta_{n,\mu} := q_\mu\]
constitute a Toeplitz \(n\)-representation of \(E\) and generate \(TC^*(E(n))\). For every Toeplitz \(n\)-representation \((T,Q,\Theta)\) of \(E\) in a \(C^*\)-algebra \(B\), there is a \(*\)-homomorphism \(\pi_{T,Q,\Theta} : TC^*(E(n)) \to B\) such that \(\pi_{T,Q,\Theta}(t_{n,e}) = T_e, \pi_{T,Q,\Theta}(q_{n,v}) = Q_v\) and \(\pi_{T,Q,\Theta}(\theta_{n,\mu}) = \Theta_\mu\).

If \((T,Q)\) is a Cuntz–Krieger \(E\)-family, then \(\pi_{T,Q,\Theta}\) factors through a homomorphism \(\tilde{\pi}_{T,Q,\Theta} : C^*(E(n)) \to B\).

**Proof.** We begin by showing that \((t_n,q_n)\) defines a Toeplitz–Cuntz–Krieger \(E\)-family. Let \(v, w \in E^0\) with \(v \neq w\). Since the \(q_\mu\) are mutually orthogonal, we have
\[q_{n,v} q_{n,w} = \left( \sum_{\mu \in v E^n} q_\mu \right) \left( \sum_{\nu \in w E^n} q_\nu \right) = \sum_{\mu \in v E^n} \sum_{\nu \in w E^n} q_\mu q_\nu = 0.\]
Let $e \in E^1$. Then

$$t_{n,e}^* t_{n,e} = \left( \sum_{\mu \in s(e)E^{<n}} t_{(e,\mu)} \right)^* \left( \sum_{\nu \in s(e)E^{<n}} t_{(e,\nu)} \right)$$

$$= \sum_{\mu,\nu \in s(e)E^{<n}} t_{(e,\mu)}^* t_{(e,\nu)} = \sum_{\mu \in s(e)E^{<n}} q_{\mu} = q_{n,s(e)}.$$

Let $v \in E^0$. We calculate

$$q_{n,v} \left( \sum_{e \in vE^1} t_{n,e}^* t_{n,e} \right) = \left( \sum_{\mu \in vE^{<n}} q_{\mu} \right) \left( \sum_{e \in vE^1} \sum_{\tau,\nu \in s(e)E^{<n}} t_{(e,\tau)} t_{(e,\nu)}^* \right)$$

$$= \sum_{\mu \in vE^{<n}} \sum_{e \in vE^1} \sum_{\tau,\nu \in s(e)E^{<n}} q_{\mu} t_{(e,\tau)} t_{(e,\nu)}^*$$

$$= \sum_{e \in vE^1} \sum_{\tau,\nu \in s(e)E^{<n}} t_{(e,\tau)} t_{(e,\nu)}^*$$

$$= \sum_{e \in vE^1} t_{n,e}^* t_{n,e}.$$

So $q_{n,v} \geq \sum_{e \in vE^1} t_{n,e}^* t_{n,e}.$

We now show that $(t_n, q_n, \theta_n)$ is a Toeplitz $n$-representation. Let $e \in E^1$ and $\mu \in E^{<n}$. First suppose that $\mu = e\mu'$. Since $r_n(e, \mu') = \mu$ and $s(e, \mu') = \mu'$, we have

$$t_{n,e}^* \theta_{n,\mu} = \left( \sum_{\nu \in s(e)E^{<n}} t_{(e,\nu)}^* \right) q_{\mu} = t_{(e,\mu')}^* = q_{\mu'} \left( \sum_{\nu \in s(e)E^{<n}} t_{(e,\nu)}^* \right) = \theta_{n,\mu'}$$

Now suppose that $\mu = r(e)$. Then

$$t_{n,e}^* \theta_{n,\mu} = \left( \sum_{\nu \in s(e)E^{<n}} t_{(e,\nu)}^* \right) q_{\mu} = \sum_{e \in vE^n} t_{(e,\nu)}^*,$$

since $r_n(e, \nu) = \mu$ forces $|e\nu| = n$. So $(t_n, q_n, \theta_n)$ is a Toeplitz $n$-representation of $E$.

Observe that $t_{(e,\mu)} = t_{n,e}^* \theta_{n,\mu}$ for each $e \in E^1$ and $\mu \in s(e)E^{<n}$ and $q_{\mu} = \theta_{n,\mu}$ for each $\mu \in E^{<n}$. It follows that the $t_{n,e}$, the $q_{n,v}$ and the $\theta_{n,\mu}$ generate $T C^* (E(n))$.

Fix a Toeplitz $n$-representation $(T, Q, \Theta)$. Define a pair $(\tilde{T}, \tilde{Q})$ by $\tilde{T}_{(e,\mu)} := T_e \Theta_{\mu}$ and $\tilde{Q}_{\mu} := \Theta_{\mu}$. We claim that this pair is a Toeplitz–Cuntz–Krieger $E(n)$-family. The $\tilde{Q}_{\mu}$ are mutually orthogonal projections since the $\Theta_{\mu}$ are. Let $e \in E^1$ and $\mu \in s(e)E^{<n}$. We have

$$\tilde{T}_{(e,\mu)}^* \tilde{T}_{(e,\mu)} = \Theta_{\mu} T_e^* T_e \Theta_{\mu} = \Theta_{\mu} Q_{s(e)} \Theta_{\mu} = \Theta_{\mu} = \tilde{Q}_{\mu}.$$
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Let \( \mu \in E^{<n} \). We calculate, using Definition 3.1.4 at the third equality,

\[
\tilde{Q}_\mu \left( \sum_{(e,\nu) = \mu E(n)}^* T_{e,\nu}^* \right) = \sum_{(e,\nu) = \mu E(n)} \tilde{Q}_\mu T_{e,\nu}^* T_{e,\nu}^* = \sum_{(e,\nu) = \mu E(n)} \Theta_\mu T_e \Theta_\nu T_e^* 
\]

\[
= \sum_{(e,\nu) = \mu E(n)}^* T_e \Theta_\nu \Theta_e T_e^* = \sum_{(e,\nu) = \mu E(n)}^* \tilde{T}_{(e,\nu)} \tilde{T}_{(e,\nu)}^* . 
\]

So \( \tilde{Q}_\mu \geq \sum_{\tau \in E^{<n}} \tilde{T}_{(e,\nu)} \tilde{T}_{(e,\nu)}^* \). Therefore the pair \((\tilde{T}, \tilde{Q})\) is a Toeplitz–Cuntz–Krieger \( E(n) \)-family, and so induces the desired homomorphism \( \pi_{T,Q,\Theta} \).

Now, suppose \((T, Q, \Theta)\) is a Cuntz–Krieger \( n \)-representation, and fix \( v \in E^0 \). We have,

\[
\sum_{\mu \in v E^{<n}} \sum_{(e,\nu) = \mu E(n)}^* \tilde{T}_{(e,\nu)} \tilde{T}_{(e,\nu)}^* = \sum_{e \in v E^1} \sum_{(e,\nu) = \mu E(n)}^* \tilde{T}_{e,\nu} \tilde{T}_{e,\nu}^* 
\]

\[
= \sum_{e \in v E^1} T_e \left( \sum_{\nu \in (e) E^{<n}} \theta_\nu \right) T_e^* = \sum_{e \in v E^1} T_e T_e^* 
\]

\[
= Q_v = \sum_{\mu \in v E^{<n}} \Theta_\mu = \sum_{\mu \in v E^{<n}} \tilde{Q}_\mu . 
\]

So \((\tilde{T}_{(e,\nu)}, \tilde{Q}_\mu)\) is a Cuntz–Krieger \( E(n) \)-family. Hence \( \pi_{T,Q,\Theta} \) factors through \( \tilde{\pi}_{T,Q,\Theta} : C^*(E(n)) \to B \).

\[ \square \]

Remark 3.1.6. We denote by \( T(E, n) \) the universal \( C^* \)-algebra generated by a Toeplitz \( n \)-representation, and by \( C^*(E, n) \) the universal \( C^* \)-algebra generated by a Cuntz–Krieger \( n \)-representation. By Theorem 3.1.5 there is an isomorphism \( \pi_n : T(E, n) \to TC^*(E(n)) \) satisfying

\[
\pi_n(t_{n,e}) = \sum_{\mu \in v(e) E^{<n}} t_{(e,\mu)}, \quad \pi_n(q_{n,v}) = \sum_{\mu \in v E^{<n}} q_\mu, \quad \text{and} \quad \pi_n(\theta_{n,\mu}) = q_\mu, 
\]

with inverse satisfying \( \pi_n^{-1}(t_{(e,\mu)}) = t_{n,e} \theta_{n,\mu} \) and \( \pi_n^{-1}(q_\mu) = \theta_{n,\mu} \). This isomorphism descends to an isomorphism \( \tilde{\pi}_n : C^*(E, n) \to C^*(E(n)) \).

We now describe the injective homomorphisms \( TC^*(E(n)) \to TC^*(E(mn)) \) and \( C^*(E(n)) \to C^*(E(mn)) \) of Kribs and Solel [31, Section 5]. The details are not given there so we include them here.

Lemma 3.1.7. Let \( E \) be a row-finite directed graph and \( n, m \in \mathbb{N} \). There is an injective homomorphism \( j_{n,mn} : TC^*(E(n)) \to TC^*(E(mn)) \) satisfying

\[
j_{n,mn}(t_{n,(e,\nu)}) = \sum_{\nu \in E^{<mn}, |\nu|_n = \mu} t_{mn,(e,\nu)} \quad \text{and} \quad j_{n,mn}(q_{n,\mu}) = \sum_{\nu \in E^{<mn}, |\nu|_n = \mu} q_{mn,\nu}. 
\]
Moreover \( j_{n,mn} \) descends to an injective homomorphism \( \tilde{j}_{n,mn} : C^*(E(n)) \to C^*(E(mn)) \).

**Proof.** Define \( T_{(e,\mu)} := \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} t_{mn,(e,\nu)} \) for \( e \in E^1 \) and \( \mu \in s(e)E^{<n} \), and \( Q_{\mu} := \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} q_{mn,\nu} \) for \( \mu \in E^{<n} \).

We check that \((T, Q)\) is a Toeplitz–Cuntz–Krieger \( E(n)\)-family. Let \( e \in E^1 \) and \( \mu \in s(e)E^{<n} \). We have

\[
T^*_{(e,\mu)}T_{(e,\mu)} = \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} \sum_{\lambda \in E^{<mn}, [\lambda]_n = \mu} t^*_{mn,(e,\lambda)} t_{mn,(e,\nu)}
= \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} q_{mn,\nu} = Q_{\mu}.
\]

Now let \( \mu \in E^{<n} \). Then

\[
Q_{\mu} \left( \sum_{(e,\nu) \in \mu E(E)} T_{(e,\nu)}T^*_{(e,\nu)} \right)
= \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} q_{mn,\tau} \left( \sum_{(e,\nu) \in \mu E(E)} \sum_{\lambda \in E^{<mn}, [\lambda]_n = \nu} \sum_{\lambda' \in E^{<mn}, [\lambda']_n = \nu} t_{mn,(e,\lambda)} t^*_{mn,(e,\lambda')} \right)
= \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} \sum_{\tau' \in (E(E))^1} \sum_{\lambda \in E^{<mn}, [\lambda]_n = \nu} q_{mn,\tau} t_{mn,(e,\lambda)} t^*_{mn,(e,\lambda')}
= \sum_{(e,\nu) \in \mu E(E)^1} \sum_{\lambda \in E^{<mn}, [\lambda]_n = \nu} t_{mn,(e,\lambda)} t^*_{mn,(e,\lambda)},
\]

since \([r_{mn}(e,\lambda)]_n = [e\lambda]_n = [e\nu]_n = \mu\). Therefore \( Q_{\mu} \geq \sum_{(e,\nu) \in \mu E(E)^1} T_{(e,\nu)}T^*_{(e,\nu)} \). So \((T, Q)\) is a Toeplitz–Cuntz–Krieger \( E(n)\)-family and the universal property of \( TC^*(E(n)) \) gives a homomorphism \( j_{n,mn} \) satisfying the desired formulas.

We check that \( j_{n,mn} \) is injective. For \( \mu \in E^{<n} \), we have

\[
Q_{\mu} - \sum_{(e,\tau) \in \mu E(E)^1} T_{(e,\tau)}T^*_{(e,\tau)} = \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} \left( q_{mn,\nu} - \sum_{(e,\lambda), (e,\lambda') \in \nu E(E)^1} t_{mn,(e,\lambda)} t^*_{mn,(e,\lambda')} \right)
= \sum_{\nu \in E^{<mn}, [\nu]_n = \mu} \left( q_{mn,\nu} - \sum_{(e,\lambda) \in \nu E(E)^1} t_{mn,(e,\lambda)} t^*_{mn,(e,\lambda)} \right).
\]

Theorem 4.1 of [16] implies that each term on the right hand side of the preceding displayed equation is nonzero. Therefore the terms on the left hand side are also nonzero and hence [16] Theorem 4.1 implies that \( j_{n,mn} \) is injective.

For the final statement, observe that \( j_{n,mn} \) preserves the Cuntz–Krieger relation, so it descends to a homomorphism \( \tilde{j}_{n,mn} : C^*(E(n)) \to C^*(E(mn)) \). Since \( C^*(E(mn)) \) carries a gauge action by [11] Proposition 2.1 it follows from the gauge invariant uniqueness theorem [3] Theorem 2.1 for \( C^*(E(n)) \) that \( \tilde{j}_{n,mn} \) is injective. □
For a multiplicative sequence $\omega = (n_k)_{k=1}^{\infty}$, the homomorphisms of the previous proposition give us direct limits $\varprojlim \mathcal{T}C^*(E(n_k))$ and $\varinjlim C^*(E(n_k))$. For $k < l$, we write $j_{n_k,n_l} : \mathcal{T}C^*(E(n_k)) \to \mathcal{T}C^*(E(n_l))$ for the connecting maps, and we write $j_{n_k,\infty} : \mathcal{T}C^*(E(n_k)) \to \varinjlim \mathcal{T}C^*(E(n_k))$ for the canonical inclusion.

Next, we describe the homomorphisms $j_{n,mn}$ and $\tilde{j}_{n,mn}$ in terms of the universal properties described in Theorem 3.1.5.

**Proposition 3.1.8.** Let $E$ be a row-finite directed graph with no sources. Take integers $m,n \geq 1$. There are injective homomorphisms $i_{n,mn} : \mathcal{T}(E,n) \to \mathcal{T}(E,mn)$ satisfying

$$i_{n,mn}(t_{n,e}) = t_{mn,e}, \quad i_{n,mn}(q_{n,v}) = q_{mn,v}, \quad \text{and} \quad i_{n,mn}(\theta_{n,\mu}) = \sum_{\nu \in E < mn, [\nu]_n = \mu} \theta_{mn,\nu},$$

and $\tilde{i}_{n,mn} : C^*(E,n) \to C^*(E,mn)$ satisfying the same formulas for the generators of $C^*(E,n)$ and $C^*(E,mn)$.

**Proof.** Define $i_{n,mn} := \pi_{mn}^{-1} \circ j_{n,mn} \circ \pi_n$. This defines an injective homomorphism since $\pi_{mn}$ and $\pi_n$ are bijective and $j_{n,mn}$ is injective. We check that $i_{n,mn}$ satisfies the desired formulas. For $e \in E^1$, we have

$$i_{n,mn}(t_{n,e}) = \pi_{mn}^{-1}(j_{n,mn}(\pi_n(t_{n,e}))) = \pi_{mn}^{-1}(j_{n,mn}\left(\sum_{(e,\mu) \in E(n)^1} t_{n,(e,\mu)}\right)) = \pi_{mn}^{-1}\left(\sum_{(e,\nu) \in E(mn)^1} t_{mn,(e,\nu)}\right) = t_{mn,e}.$$

For $v \in E^0$, we have

$$i_{n,mn}(q_{n,v}) = \pi_{mn}^{-1}(j_{n,mn}(\pi_n(q_{n,v}))) = \pi_{mn}^{-1}(j_{n,mn}\left(\sum_{\mu \in v E < mn} q_{n,\mu}\right)) = \pi_{mn}^{-1}\left(\sum_{\nu \in v E < mn} q_{mn,\nu}\right) = q_{mn,v}.$$

For $\mu \in E^{<n}$, we have

$$i_{n,mn}(\theta_{n,\mu}) = \pi_{mn}^{-1}(j_{n,mn}(\pi_n(\theta_{n,\mu}))) = \pi_{mn}^{-1}(j_{n,mn}(q_{n,\mu})) = \pi_{mn}^{-1}\left(\sum_{\nu \in E < mn, [\nu]_n = \mu} q_{mn,\nu}\right) = \sum_{\nu \in E < mn, [\nu]_n = \mu} \theta_{mn,\nu}.$$

Similarly, define $\tilde{i}_{n,mn} := \pi_{mn}^{-1} \circ \tilde{j}_{n,mn} \circ \pi_n$. This defines an injective homomorphism since $\pi_{mn}$ and $\pi_n$ are bijective and $\tilde{j}_{n,mn}$ is injective. Calculations as above show that
\[ \hat{i}_{n,mn} \] satisfies the desired formulas.

For a multiplicative sequence \((n_k)_{k=1}^\infty\), we use the homomorphisms of the preceding proposition to form the direct limits \(\lim \mathcal{T}(E, n_k)\) and \(\lim C^*(E, n_k)\). We write \(i_{n_k,mn} : \mathcal{T}(E, n_k) \to \mathcal{T}(E, n_l)\) for the connecting maps with \(k < l\), and we write \(i_{n_k,\infty} : \mathcal{T}(E, n_k) \to \lim \mathcal{T}(E, n_k)\) for the canonical inclusion. We also use these same symbols to denote the corresponding maps in the direct system associated to the \(C^*(E, n_k)\); the meaning should be clear from context.

**Corollary 3.1.9.** Let \(\omega = (n_k)_{k=1}^\infty\) be a multiplicative sequence. There are isomorphisms \(\varphi : \lim \mathcal{T}C^*(E(n_k)) \to \lim \mathcal{T}(E, n_k)\) satisfying \(\varphi \circ i_{n_k,\infty} = j_{n_k,\infty} \circ \pi_{n_k}\) and \(\hat{\varphi} : \lim C^*(E(n_k)) \cong \lim C^*(E, n_k)\) satisfying \(\hat{\varphi} \circ i_{n_k,\infty} = j_{n_k,\infty} \circ \hat{\pi}_{n_k}\) for all \(k \geq 1\).

**Proof.** Fix \(k \geq 1\). We have

\[
(j_{n_{k+1},\infty} \circ \pi_{n_{k+1}}) \circ i_{n_k,n_{k+1}} = j_{n_{k+1},\infty} \circ j_{n_k,n_{k+1}} \circ \pi_{n_k} = j_{n_k,\infty} \circ \pi_{n_k}.
\]

So the universal property of \(\lim \mathcal{T}(E, n_k)\) gives a homomorphism \(\varphi : \lim \mathcal{T}(E, n_k) \to \lim \mathcal{T}C^*(E(n_k))\) such that \(\varphi \circ i_{n_k,\infty} = j_{n_k,\infty} \circ \pi_{n_k}\). Similarly, the universal property of \(\lim \mathcal{T}C^*(E(n_k))\) gives a homomorphism \(\psi : \lim \mathcal{T}C^*(E(n_k)) \to \mathcal{T}(E, n_k)\) such that \(\psi \circ j_{n_k,\infty} = i_{n_k,\infty} \circ \pi_{n_k}^{-1}\). We have

\[
(\varphi \circ \psi) \circ j_{n_k,\infty} = \varphi \circ i_{n_k,\infty} \circ \pi_{n_k}^{-1} = j_{n_k,\infty} \circ \pi_{n_k} \circ \pi_{n_k}^{-1} = j_{n_k,\infty}
\]

and similarly

\[
(\psi \circ \varphi) \circ i_{n_k,\infty} = \psi \circ j_{n_k,\infty} \circ \pi_{n_k} = i_{n_k,\infty} \circ \pi_{n_k} \circ \pi_{n_k}^{-1} = i_{n_k,\infty}.
\]

Therefore \(\varphi \circ \psi\) is the identity map on each \(j_{n_k,\infty}(\mathcal{T}C^*(E(n_k)))\) and \(\psi \circ \varphi\) is the identity on each \(i_{n_k,\infty}(\mathcal{T}(E, n_k))\), so continuity shows that \(\varphi\) and \(\psi\) are mutually inverse.

The same argument shows that there are mutually inverse maps \(\hat{\varphi} : \lim C^*(E, n_k) \to \lim C^*(E(n_k))\) and \(\hat{\psi} : \lim C^*(E(n_k)) \to \lim C^*(E, n_k)\).

### 3.2 The \(C^*\)-algebras \(\mathcal{T}(E, \omega)\) and \(C^*(E, \omega)\)

In this section we describe \(\lim \mathcal{T}(E, n_k)\) and \(\lim C^*(E, n_k)\) by universal properties. We begin by analysing the underlying projective limit \(\lim E^{<n_k}\). We will show that \(\lim \mathcal{T}(E, n_k)\) is generated by a copy of \(\mathcal{T}C^*(E)\) and a copy of \(C_0(\lim E^{<n_k})\) and that \(\lim C^*(E)\) is generated by a copy of \(C^*(E)\) and a copy of \(C_0(\lim E^{<n_k})\).
Fix a directed graph $E$. For $m, n \geq 1$ such that $m \mid n$, we define $p_{n,m} : E^{<n} \to E^{<m}$ by $p_{n,m}(\nu) = [\nu]_m$. Let $(n_k)_{k=1}^\infty$ be a multiplicative sequence. Then $(E^{<n_k}, p_{n_k, n_{k-1}})$ is a projective system. The projective limit $\lim\limits_{\leftarrow} E^{<n_k}$ can be realised as the topological subspace

$$\left\{ (\mu_k)_{k=1}^\infty \in \prod_{k=1}^\infty E^{<n_k} : \mu_k = [\mu_{k+1}]_{n_k} \text{ for all } k \geq 1 \right\}$$

of the infinite product $\prod_{k=1}^\infty E^{<n_k}$ of the discrete spaces $E^{<n_k}$. If $E$ is a finite graph, then $\lim\limits_{\leftarrow} E^{<n_k}$ is a compact space by [43, Lemma 1.1.2].

Now, fix $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$. We write $Z(\mu,k)$ for the cylinder set $\{ (\nu_i)_{i=1}^\infty \in \lim\limits_{\leftarrow} E^{<n_k} : \nu_k = \mu \}$; these are the canonical clopen basis sets for the projective limit space $\lim\limits_{\leftarrow} E^{<n_k}$. We write $\chi_{Z(\mu,k)}$ for the characteristic function of $Z(\mu,k) \subseteq \lim\limits_{\leftarrow} E^{<n_k}$.

**Definition 3.2.1.** Let $E$ be a row-finite directed graph with no sources, and suppose that $\omega = (n_k)_{k=1}^\infty$ is a multiplicative sequence. A Toeplitz $\omega$-representation of $E$ is a triple $(T, Q, \psi)$ where

1. $(T, Q)$ is a Toeplitz–Cuntz–Krieger $E$-family in a $C^*$-algebra $B$,
2. $\psi : C_0(\lim\limits_{\leftarrow} E^{<n_k}) \to B$ is a homomorphism,
3. $Q_v = \sum_{\mu \in v} \psi(\chi_{Z(\mu,1)})$ for each $v \in E^0$, and
4. $T^*_e \psi(\chi_{Z(\mu,k)}) = \begin{cases} \psi(\chi_{Z(\mu',k)}) T^*_e & \text{if } \mu = e \mu' \\ \sum_{\nu \in E^{n_k}} \psi(\chi_{Z(\nu,k)}) T^*_e & \text{if } \mu = r(e) \\ 0 & \text{otherwise} \end{cases}$

for all $e \in E^1$, $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$.

If the pair $(T, Q)$ is a Cuntz–Krieger $E$-family, then we call $(T, Q, \psi)$ a Cuntz–Krieger $\omega$-representation, or just an $\omega$-representation of $E$.

We show that the universal $C^*$-algebra generated by an $\omega$-representation coincides with Kribs and Solel’s algebra $\lim\limits_{\leftarrow} C^*(E(n_k))$. We first need a multiplication formula analogous to that of Lemma 3.1.4. To lighten notation a bit, given a homomorphism $\psi : C_0(\lim\limits_{\leftarrow} E^{<n_k}) \to B$, we will write $\psi(\mu,k)$ for the image $\psi(\chi_{Z(\mu,k)})$ of $\chi_{Z(\mu,k)}$ under $\psi$, which is a projection in $B$.

**Lemma 3.2.2.** Let $E$ be a row-finite directed graph with no sources, and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. Let $(T, Q, \psi)$ be a Toeplitz $\omega$-representation of $E$. For
\[ (T_\alpha \psi_{(\mu,k)} T_\beta^*)(T_\gamma \psi_{(v,k)} T_\delta^*) = \begin{cases} 
\alpha, \beta, \gamma, \delta \in E^*, k \geq 1 \text{ and } \mu, \nu \in E^{<nk}, we have \]
\[
T_\alpha \psi_{(\mu,k)} T_\delta^* & \text{ if } \beta = \gamma \beta' \text{ and } \nu = \beta' \mu \\
T_\alpha \psi_{(\mu,k)} T_{\delta \nu \rho} & \text{ if } \beta = \gamma \nu \rho \text{ with } |\rho \mu| \in nN \\
T_{\alpha \gamma} \psi_{(\nu,k)} T_\delta^* & \text{ if } \gamma = \gamma \beta' \text{ and } \mu = \gamma \nu \\
T_{\mu \nu \rho} \psi_{(\nu,k)} T_\delta^* & \text{ if } \gamma = \beta \mu \rho \text{ with } |\rho \nu| \in nN \\
0 & \text{ otherwise.} 
\end{cases}
\]

In particular, \( C^*(T, Q, \psi) = \text{span}\{T_\alpha \psi_{(\mu,k)} T_\beta^* : k \geq 1, \mu \in E^{<nk}, \alpha, \beta \in E^{*r}(\mu)\} \).

Proof. For each \( k \in \mathbb{N} \), \( (T, Q, \psi_{(\cdot, k)}) \) is a Toeplitz \( n_k \)-representation of \( E \). So the first statement follows from Lemma 3.1.4. For the second statement, first observe that \( \{T_\alpha \psi_{(\mu,k)} T_\beta^* : k \geq 1, \mu \in E^{<nk}, \alpha, \beta \in E^{*r}(\mu)\} \) contains each \( T_\alpha = \sum_{\mu \in s(\alpha) E^{<nk}} T_\alpha \psi_{(\mu,1)} T_{s(\alpha)}^* \), each \( Q_v = \sum_{\mu \in v E^{<nk}} T_v \psi_{(\mu,1)} T_v^* \) and each \( \psi_{(\mu,k)} = T_{r(\mu)} \psi_{(\mu,k)} T_{r(\mu)}^* \). It is clearly closed under adjoints. So it suffices to show that it is closed under multiplication. To see this, we consider a product \( (T_\alpha \psi_{(\mu,k)} T_\beta^*)(T_\gamma \psi_{(v,l)} T_\delta^*) \). Suppose that \( k \geq l \) (the case where \( k < l \) will follow by taking adjoints). Then \( Z(\nu, l) = \bigcup_{\lambda \in E^{<nk}, |\lambda| \nu = \nu} Z(\lambda, k) \), and so we have

\[
T_\alpha \psi_{(\mu,k)} T_\beta^* T_\gamma \psi_{(v,l)} T_\delta^* = \sum_{\lambda \in E^{<nk}, |\lambda| \nu = \nu} T_\alpha \psi_{(\mu,k)} T_\beta^* T_\gamma \psi_{(\lambda,k)} T_\delta^*,
\]

and this belongs to \( \text{span}\{T_\alpha \psi_{(\mu,k)} T_\beta^* : k \geq 1, \mu \in E^{<nk}, \alpha, \beta \in E^{*r}(\mu)\} \) by the first statement.

\[ \square \]

Theorem 3.2.3. Let \( E \) be a row-finite directed graph with no sources, and let \( \omega = (n_k)_{k=1}^{\infty} \) be a multiplicative sequence. Let \( (t_{n_k}, q_{n_k}, \theta_{n_k}) \) be the universal Toeplitz \( n_k \)-representation. There is a Toeplitz \( \omega \)-representation \( (t, q, \pi) \) of \( E \) in \( \lim T(E, n_k) \) such that

\[
t_e = i_{n_1, \infty}(t_{n_1,e}), \quad q_v = i_{n_1, \infty}(q_{n_1,v}), \quad \text{and} \quad \pi_{(\mu,k)} = i_{n_k, \infty}(\theta_{n_k,\mu})
\]

for all \( e \in E^1 \), all \( v \in E^0 \), and all \( k \in \mathbb{N} \) and \( \mu \in E^{<nk} \). This Toeplitz \( \omega \)-representation is universal in the sense that if \( (T, Q, \psi) \) is a Toeplitz \( \omega \)-representation of \( E \) in a C*-algebra \( B \), then there is a homomorphism \( \varphi_{T,Q,\psi} : \lim T(E, n_k) \to B \) such that

\[
\varphi_{T,Q,\psi}(t_e) = T_e, \quad \varphi_{T,Q,\psi}(q_v) = Q_v, \quad \text{and} \quad \varphi_{T,Q,\psi} \circ \pi = \psi.
\]

Proof. The pair \( (t_{n_1}, q_{n_1}) \) is a Toeplitz–Cuntz–Krieger \( E \)-family. Since \( i_{n_1, \infty} \) is a homomorphism, it follows that \( t_e := i_{n_1, \infty}(t_{n_1,e}) \) and \( q_v = i_{n_1, \infty}(q_{n_1,v}) \) is a Toeplitz–Cuntz–Krieger
We check that \( \psi \): \( \Theta \) universal property of \( T \). So
\[
\pi_k(i_{\mu,\mu}) := i_{n_k,\infty}(\theta_{n_k,\mu})
\]
gives a homomorphism \( \pi_k : \text{span}\{\chi_{Z(\mu,k)} : \mu \in E^{<n_k}\} \to \lim_{\to} T(E,n_k) \). So the universal property of \( C_0(\lim E^{<n_k}) \cong \lim C_0(E^{<n_k}) \) yields a homomorphism \( \pi : C_0(\lim E^{<n_k}) \to \lim_{\to} T(E,n_k) \) satisfying \( \pi(\mu,k) = i_{n_k,\infty}(\theta_{n_k,\mu}) \).

We check that \( (t,q,\pi) \) is a Toeplitz \( \omega \)-representation. Let \( v \in E^0 \). We have \( q_v = i_{n_1,\infty}(q_{n_1,v}) = i_{n_1,\infty}\left(\sum_{\mu\in E^{<n_1}} \theta_{n_1,\mu}\right) = \sum_{\mu\in E^{<n_1}} \pi(\mu,n_1) \). Now, let \( e \in E^1 \) and \( \mu \in E^{<n_k} \).

Then
\[
t^*_e \pi(\mu,k) = i_{n_k,\infty}(t^*_{n_k,e}) \pi(\mu,k)
= i_{n_k,\infty}(i_{n_k,n_k}(t^*_{n_k,e})\theta_{n_k,\mu})
= i_{n_k,\infty}(t^*_{n_k,e}\theta_{n_k,\mu})
= \begin{cases} 
i_{n_k,\infty}(\theta_{n_k,\mu'}t^*_{n_k,e}) & \text{if } \mu = e\mu' \\
i_{n_k,\infty}\left(\sum_{e\lambda\in E^{n_k}} \theta_{n_k,\lambda}t^*_{n_k,e}\right) & \text{if } \mu = r(e) \\
0 & \text{otherwise} \end{cases}
= \begin{cases} 
\pi(\mu',k)t^*_e & \text{if } \mu = e\mu' \\
\sum_{e\lambda\in E^{n_k}} \pi(\lambda,k)t^*_e & \text{if } \mu = r(e) \\
0 & \text{otherwise} \end{cases}
\]

So \( (t,q,\pi) \) is a Toeplitz \( \omega \)-representation of \( E \) in \( \lim_{\to} T(E,n_k) \).

Let \( (T,Q,\psi) \) be another \( \omega \)-representation of \( E \) in \( B \), and fix \( k \in \mathbb{N} \). For \( \mu \in E^{<n_k} \), let \( \Theta_{\mu} := \psi(\mu,k) \). Then for each \( k \in \mathbb{N} \), \( (T,Q,\Theta) \) is a Toeplitz \( n_k \)-representation of \( E \). The universal property of \( T(E,n_k) \) gives a homomorphism \( \varphi_{n_k,\infty} : T(E,n_k) \to B \) satisfying
\[
\varphi_{n_k,\infty}(t_e) = T_e, \quad \varphi_{n_k,\infty}(q_v) = Q_v, \quad \text{and} \quad \varphi_{n_k,\infty}(\theta_{n_k,\mu}) = \psi(\mu,k).
\]

We check that \( \varphi_{n_k+1,\infty} \circ i_{n_k,n_{k+1}} = \varphi_{n_k,\infty} \). We have
\[
\varphi_{n_k+1,\infty}(i_{n_k,n_{k+1}}(t_{n_k,e})) = \varphi_{n_k+1,\infty}(t_{n_{k+1},e}) = T_e = \varphi_{n_k,\infty}(t_{n_k,e}).
\]
and similarly $\varphi_{n_{k+1}} \circ i_{n_k,n_{k+1}}(q_{n_k,v}) = Q_v = \varphi_{n_k}(q_{n_k,v})$. For $\mu \in E^{< n_k}$,

$$\varphi_{n_{k+1},\infty}(i_{n_k,n_{k+1}}(\theta_{n_k,\mu})) = \varphi_{n_k,\infty}\left( \sum_{\lambda \in E^{< n_{k+1}}, |\lambda|_{n_k} = \mu} \theta_{n_{k+1},\lambda} \right) = \sum_{\lambda \in E^{< n_{k+1}}, |\lambda|_{n_k} = \mu} \varphi_{n_k,\infty}(\theta_{n_{k+1},\lambda}) = \sum_{\lambda \in E^{< n_{k+1}}, |\lambda|_{n_k} = \mu} \psi(\lambda, k+1) = \psi(\sum_{\lambda \in E^{< n_{k+1}}, |\lambda|_{n_k} = \mu} \chi\lambda) = \psi(\chi(\mu, k)) = \varphi_{n_k,\infty}(\theta_{n_k,\mu}).$$

The universal property of $\varprojlim T(E, n_k)$ now gives a homomorphism $\varphi_{T,Q,\psi}$ making the diagram

$$\begin{array}{ccc}
T(E, n_k) & \xrightarrow{i_{n_k,n_{k+1}}} & T(E, n_{k+1}) \\
\varphi_{n_k,\infty} \downarrow & & \varphi_{n_k,\infty} \downarrow \\
\lim \varprojlim T(E, n_k) & \xrightarrow{i_{n_k,\infty}} & \varphi_{T,Q,\psi} \\
\downarrow & & \downarrow \\
B & & T(E, n_{k+1})
\end{array}$$

commute, and this homomorphism has the desired properties. \qed

Given $E$ and $\omega$ as in Theorem 3.2.3, we write $T(E, \omega)$ for the universal $C^*$-algebra generated by a Toeplitz $\omega$-representation of $E$. Since the universal $C^*$-algebra for a given set of generators and relations is unique up to canonical isomorphism, we can and will identify $T(E, \omega)$ with $\varprojlim T(E(n_k))$ via the homomorphism of Theorem 3.2.3. The following theorem follows from the same argument as Theorem 3.2.3 where we use the universal properties of $C^*(E, n_k)$ and $\varprojlim C^*(E, n_k)$ in place of those of $T(E, n_k)$ and $\varprojlim T(E, n_k)$.

**Theorem 3.2.4.** Let $E$ be a row-finite directed graph with no sources, and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. Let $(s_{n_k}, p_{n_k}, \rho_{n_k})$ be the universal Cuntz–Krieger $n_k$-representation. There is an $\omega$-representation $(s, p, \rho)$ of $E$ in $\varprojlim C^*(E, n_k)$ such that

$$s_e = i_{n_1,\infty}(s_{n_1,e}), \quad p_v = i_{n_1,\infty}(p_{n_1,v}), \quad \text{and} \quad \rho_{(\mu,k)} = i_{n_k,\infty}(\epsilon_{n_k,\mu})$$

for all $e \in E^1$, all $v \in E^0$, and all $k \in \mathbb{N}$ and $\mu \in E^{< n_k}$. This $\omega$-representation is universal in the sense that if $(S, P, \psi)$ is an $\omega$-representation of $E$ in a $C^*$-algebra $B$, then there is
a homomorphism \( \varphi_{S,P,\psi} : \lim C^*(E, n_k) \to B \) such that

\[
\varphi_{S,P,\psi}(s_e) = S_e, \quad \varphi_{S,P,\psi}(p_e) = P_e, \quad \text{and} \quad \varphi_{S,P,\psi} \circ \rho = \psi.
\]

We write \( C^*(E, \omega) \) for the universal \( C^* \)-algebra generated by an \( \omega \)-representation of \( E \), and we identify it with \( \lim C^*(E(n)) \) via the homomorphism of Theorem 3.2.4.
Chapter 4

Uniqueness theorems

In this chapter we prove uniqueness theorems for $\mathcal{T}(E, \omega)$ and $C^*(E, \omega)$. The uniqueness theorem for $C^*(E, \omega)$ requires no aperiodicity condition provided that $n_k \to \infty$. This is interesting since the Cuntz–Krieger uniqueness theorem for graph algebras requires cycles to have exits to ensure uniqueness.

In [44, Theorem 5.2] the uniqueness theorem for $C^*(E, \omega)$ is proved using Katsura’s uniqueness theorems for $C^*$-algebras associated to topological graphs together with Kribs and Solel’s realisation of $C^*(E, \omega)$ as a topological-graph $C^*$-algebra [31, Theorem 6.3]. We think that the direct argument provided in this chapter gives complementary insight. We will use our uniqueness theorems in Chapter 5 to improve upon Kribs and Solel’s simplicity results (see [31, Section 9] and Chapter 6).

4.1 A uniqueness theorem for $\mathcal{T}(E, \omega)$

Our first uniqueness theorem is for $\mathcal{T}(E, \omega)$, and follows relatively easily from Fowler and Raeburn’s uniqueness theorem [16, Theorem 4.1] for Toeplitz algebras of Hilbert bimodules.

**Proposition 4.1.1.** Let $E$ be a row-finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. Let $(T, Q, \psi)$ be an $\omega$-representation of $E$ in a $C^*$-algebra $A$. The induced homomorphism $\pi_{T, Q, \psi} : \mathcal{T}(E, \omega) \to A$ is injective if and only if

$$\left( Q_{r(\mu)} - \sum_{e \in r(\mu)E^1} T_e T_e^* \right)\psi(\mu, k) \neq 0 \quad (4.1.1)$$

for all $k \in \mathbb{N}$ and $\mu \in E^{< n_k}$.

**Proof.** Fix $k \in \mathbb{N}$ and let $(t_{n_k}, q_{n_k})$ be the universal Toeplitz–Cuntz–Krieger $E(n_k)$-family.
in $\mathcal{T}C^*(E(n_k))$. Let $\pi_k : \mathcal{T}C^*(E(n_k)) \to \mathcal{T}(E(n_k))$ be as in Remark 3.1.6 of [16] shows that $\varphi_{T,Q,\psi} \circ i_{n_k,\infty} \circ \pi_{n_k}^{-1} : \mathcal{T}C^*(E(n_k)) \to A$ is injective if and only if

$$0 \neq (\varphi_{T,Q,\psi} \circ i_{n_k,\infty} \circ \pi_{n_k}^{-1})(q_{n_k,\mu} - \sum_{(e,\nu) \in \mu E(n_k)}^* t_{n_k,\pi_{n_k}(e,\nu)}^*)$$

$$= (\varphi_{T,Q,\psi} \circ i_{n_k,\infty})(\theta_{n_k,\mu} - \sum_{(e,\nu) \in \mu E(n_k)} t_{n_k,\pi_{n_k}(e,\nu)}^* \pi_{n_k}(e,\nu))$$

$$= \varphi_{T,Q,\psi}(\pi_{(\mu,k)} - \sum_{(e,\nu) \in \mu E(n_k)}^* T_e \psi_{(\nu,k)} T_e^* \pi_{(\mu,k)}$$

$$= \psi_{(\mu,k)} - \sum_{e \in r(\mu) E^1} T_e T_e^* \psi_{(\mu,k)}$$

$$= Q_{r(\mu)} \psi_{(\mu,k)} - \sum_{e \in r(\mu) E^1} T_e T_e^* \psi_{(\mu,k)}$$

for all $\mu \in E^{<n_k}$. Since $i_{n_k,\infty}$ is injective and $\pi_{n_k}$ is bijective for each $k \in \mathbb{N}$, we have that $\psi_{T,Q,\psi}$ is injective if and only if equation 4.1.1 holds for all $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$. 

4.2 A Cuntz–Krieger uniqueness theorem for $C^*(E,\omega)$

We now state our main uniqueness result, which characterises the injective homomorphisms of $C^*(E,\omega)$.

**Theorem 4.2.1.** Let $E$ be a row-finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. Suppose that $n_k \to \infty$ as $k \to \infty$. Suppose that $(S,P,\psi)$ is an $\omega$-representation of $E$. Then $\varphi_{S,P,\psi}$ is injective if and only if $\psi_{(\mu,k)} \neq 0$ for all $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$.

To prove Theorem 4.2.1, we need a series of preliminary results. We first show that there is a gauge action $\gamma$ of $\mathbb{T}$ on $C^*(E,\omega)$. We then consider the fixed-point algebra

$$C^*(E,\omega)^\gamma := \{a \in C^*(E,\omega) : \gamma_z(a) = a \text{ for all } z \in \mathbb{T}\}$$

and show that $\varphi_{S,P,\psi}$ is isometric on $C^*(E,\omega)^\gamma$.

**Proposition 4.2.2.** Let $E$ be a row-finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. There is a strongly continuous action $\gamma$ of $\mathbb{T}$ on $C^*(E,\omega)$ such that $\gamma_z(s_e) = z s_e$, $\gamma_z(p_e) = p_e$, and $\gamma_z \circ \rho = \rho$, where $(s,p,\rho)$ is the universal $\omega$-representation generating $C^*(E,\omega)$. 

Proof. Each $C^*(E(n_k))$ carries a gauge action $\gamma^{n_k}$ (see [41, Proposition 2.1]). Define $\beta^{n_k} := \gamma^{n_k} \circ \tilde{\pi}_{n_k}$ for each $z \in \mathbb{T}$. Then $\beta$ is a strongly continuous action of $\mathbb{T}$ on $C^*(E, n_k)$.

Let $(s_{n_k}, p_{n_k}, z_{n_k})$ be the universal $n_k$-representation of $C^*(E, n_k)$. Then

\[
\beta^m_{n_k}(s_{n_k, e}) = \gamma^{n_k}_{z} \left( \sum_{\mu \leq s(e) \in E^{< n_k}} s_{n_k, (e, \mu)} \right) = \sum_{\mu \leq s(e) \in E^{< n_k}} z s_{n_k, (e, \mu)} = z s_{n_k, e},
\]

for $e \in E^1$. Similarly $\beta^m_{n_k}(p_{n_k, v}) = p_{n_k, v}$ for $v \in E^0$ and $\beta_{n_k, \mu} = \varepsilon_{n_k, \mu}$ for $\mu \in E^{< n_k}$.

Fix $k \in \mathbb{N}$ and $z \in \mathbb{T}$, and let $e \in E^1$. Then

\[
\beta_{n_k}^{n_k+1}(i_{n_k, n_{k+1}}(s_{n_k, e})) = \beta_{n_k+1}^{n_k+1}(s_{n_{k+1}, e}) = z s_{n_{k+1}, e} = i_{n_k, n_{k+1}}(\beta_{n_k}^{n_k}(s_{n_k, e})).
\]

Similarly $\beta_{n_k}^{n_k+1}(i_{n_k, n_{k+1}}(p_{n_k, v})) = i_{n_k, n_{k+1}}(\beta_{n_k}^{n_k}(p_{n_k, v}))$ for $v \in E^0$ and $\beta_{n_k}^{n_k+1}(i_{n_k, n_{k+1}}(\varepsilon_{n_k, \mu})) = i_{n_k, n_{k+1}}(\beta_{n_k}^{n_k}(\varepsilon_{n_k, \mu}))$. So $\beta_{n_k}^{n_k+1} \circ i_{n_k, n_{k+1}} = i_{n_k, n_{k+1}} \circ \beta_{n_k}^{n_k}$ for all $k \in \mathbb{N}$ and $z \in \mathbb{T}$.

Since

\[
(i_{n_k+1, \infty} \circ \beta_{n_k}^{n_k+1}) \circ i_{n_k, n_{k+1}} = i_{n_k+1, \infty} \circ i_{n_k, n_{k+1}} \circ \beta_{n_k}^{n_k} = i_{n_k, \infty} \circ \beta_{n_k}^{n_k}
\]

for each $k \in \mathbb{N}$ and $z \in \mathbb{T}$, the universal property of $\lim \gamma_{n_k}(E, \omega)$ gives a homomorphism $\gamma_{z}$ satisfying the desired formulas.

Using the action of Proposition 4.2.2 we obtain a faithful conditional expectation $\Phi : C^*(E, \omega) \rightarrow C^*(E, \omega)\gamma$ given by $\Phi(a) = \int_{\mathbb{T}} \gamma_z(a) \, dz$. For details, see [41, Proposition 3.2].

Lemma 4.2.3. Let $E$ be a row-finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. With the notation just discussed, we have

\[
\Phi(s_{\mu, \rho}(f)s_{\nu}^*) = \delta_{|\mu|, |\nu|} s_{\mu, \rho}(f)s_{\nu}^* \quad (4.2.1)
\]

for all $\mu, \nu \in E^*$ and $f \in C_0(\lim \mathbb{T})$, where

\[
C^*(E, \omega)\gamma = \overline{\text{span}} \{ s_{\mu, \rho}(\alpha, k)s_{\nu}^* : k \in \mathbb{N}, \alpha \in E^{< n_k}, \mu, \nu \in E^* r(\alpha) \text{ and } |\mu| = |\nu| \}. \quad (4.2.2)
\]

Proof. For (4.2.1), we calculate

\[
\Phi(s_{\mu, \rho}(f)s_{\nu}^*) = \int_{\mathbb{T}} \gamma_z(s_{\mu, \rho}(f)s_{\nu}^*) \, dz = \int_{\mathbb{T}} z^{1-|\nu|} s_{\mu, \rho}(f)s_{\nu}^* \, dz = \delta_{|\mu|, |\nu|} s_{\mu, \rho}(f)s_{\nu}^*,
\]

as required.

The inclusions $\supseteq$ in (4.2.2) is immediate from the definition of $\gamma$. For the reverse inclusion, observe that since $\Phi$ is continuous and linear, the final statement of Lemma 3.2.2
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gives

\[ \Phi(C^*(E, \omega)) = \overline{\text{span}} \{ \Phi(s_\mu \rho(\alpha, k) s_\nu^*) : k \in \mathbb{N}, \alpha \in E^{<n_k}, \mu, \nu \in E^* r(\alpha) \} \].

So the containment \( \subseteq \) in (4.2.2) follows from (4.2.1). \( \square \)

We now show that \( C^*(E, \omega) \) is an AF algebra. We will use this to characterise the homomorphisms of \( C^*(E, \omega) \) that are injective on \( C^*(E, \omega) \).

Given a countable set \( X \), we write \( K_X \) for the unique \( C^* \)-algebra generated by nonzero elements \( \{ \theta_{x,y} : x, y \in X \} \) such that \( \theta_{x,y}^* = \theta_{y,x} \) and \( \theta_{x,y} \theta_{w,z} = \delta_{y,w} \theta_{x,z} \). This \( K_X \) is canonically isomorphic to \( K(\ell^2(X)) \), so is AF.

Lemma 4.2.4. Let \( E \) be a row-finite directed graph with no sources, and take a multiplicative sequence \( \omega = (n_k)_{k=1}^\infty \). For \( k \geq 1 \) and \( p \geq 0 \), define

\[ F_{k,p} := \overline{\text{span}} \{ s_\mu \rho(\alpha, k) s_\nu^* : \alpha \in E^{<n_k}, \mu, \nu \in E^* r(\alpha) \} \subseteq C^*(E, \omega). \]

For \( \alpha \in E^{<n_k} \), let

\[ F_{k,p}(\alpha) := \overline{\text{span}} \{ s_\mu \rho(\alpha, k) s_\nu^* : \mu, \nu \in E^* r(\alpha) \} \subseteq F_{k,p}. \]

For each \( k, p, \alpha \), there is an isomorphism \( F_{k,p}(\alpha) \cong K_{E^* r(\alpha)} \) that carries \( s_\mu \rho(\alpha, k) s_\nu^* \) to \( \theta_{\mu, \nu} \).

We have \( F_{k,p} = \bigoplus_{\alpha \in E^{<k}} F_{k,p}(\alpha) \), and \( F_{k,p} \subseteq F_{l,q} \) whenever \( k \leq l \) and \( p \leq q \).

Proof. To obtain the desired isomorphism \( F_{k,p}(\alpha) \cong K_{E^* r(\alpha)} \), it suffices to show that the elements \( \Theta_{\mu, \nu} := s_\mu \rho(\alpha, k) s_\nu^* \) where \( \mu, \nu \in E^* r(\alpha) \) satisfy \( \Theta_{\mu, \nu}^* = \Theta_{\nu, \mu} \), and \( \Theta_{\mu, \nu} \Theta_{\eta, \zeta} = \delta_{\nu, \eta} \Theta_{\mu, \zeta} \) and are nonzero. The first relation is trivial, and the second follows immediately from the Cuntz–Krieger relation \( s_\nu^* s_\eta = \delta_{\nu, \eta} p_\nu \) and that \( p_\nu = p_r(\alpha) \geq \rho(\alpha, k) \). For distinct \( \alpha, \beta \in E^{<k} \) and spanning elements

\[ s_\mu \rho(\alpha, k) s_\nu^* \in F_{k,p}(\alpha) \quad \text{and} \quad s_\eta \rho(\beta, k) s_\xi^* \in F_{k,p}(\beta), \]

we have \( s_\mu \rho(\alpha, k) s_\nu^* s_\eta \rho(\beta, k) s_\xi^* = s_\nu^* s_\mu \rho(\alpha, k) s_\rho(\beta, k) s_\xi^* = 0 \) if \( \alpha \neq \beta \), so the \( F_{k,p}(\alpha) \) are mutually orthogonal for fixed \( k, p \) giving \( F_{k,p} = \bigoplus_{\alpha \in E^{<k}} F_{k,p}(\alpha) \).

For the last assertion, fix \( k \leq l \) and \( p \leq q \), and take a spanning element \( s_\mu \rho(\alpha, k) s_\nu^* \in
Using the Cuntz–Krieger relation and Lemma 3.1.3 we have

\[ s_\mu \rho(\alpha,k) s_\nu^* = \sum_{\lambda \in s(\mu)} s_\mu s_\lambda^* \rho(\alpha,k) s_\nu^* \]

\[ = \begin{cases} 
  s_{\mu \alpha'} \rho(\alpha'',k) s_{\mu \alpha''}^* & \text{if } |\alpha| \geq q - p, \text{ and } \alpha = \alpha' \alpha'' \\
  \sum_{\lambda' \in s(\alpha) E^{q-p}} s_{\mu \alpha \lambda'} \rho(\lambda',k) s_{\nu \alpha \lambda'}^* & \text{if } q - p - |\alpha| \in n k \mathbb{N} \setminus \{0\} \\
  \sum_{\lambda' \in s(\alpha) E^{q-p}} - |\alpha| \sum_{\tau(n(\lambda')) \eta \in E^n} s_{\mu \alpha \lambda'} \rho(\eta,k) s_{\nu \alpha \lambda'}^* & \text{otherwise.} 
\end{cases} \]

Hence \( s_\mu \rho(\alpha,k) s_\nu^* \in F_{q,k} \), giving \( F_{p,k} \subseteq F_{q,k} \). Now fix a spanning element \( s_\eta \rho(\alpha,k) s_\zeta^* \in F_{q,k} \).

We have \( \rho(\alpha,k) = \sum_{\beta \in E^{n_1}, |\beta| = \alpha} \rho(\beta,l) \), and so

\[ s_\eta \rho(\alpha,k) s_\zeta^* = \sum_{\beta \in E^{n_1}, |\beta| = \alpha} s_\eta \rho(\beta,l) s_\zeta^* \in F_{q,l}. \]

It follows from the preceding Lemma that \( C^*(E,\omega)^\gamma \) is AF—we have presented an explicit decomposition as the closure of an increasing union over the directed set \( \mathbb{N} \times \mathbb{N} \) of direct sums of algebras of compact operators. In particular, we obtain the desired characterisation of the homomorphisms that are injective in this subalgebra of \( C^*(E,\omega) \).

**Lemma 4.2.5.** Let \( E \) be a row-finite directed graph with no sources, and take a multiplicative sequence \( \omega = (n_k)_{k=1}^\infty \). Suppose that \( (S,P,\psi) \) is an \( \omega \)-representation of \( E \) such that each \( \psi(\alpha,k) \) is nonzero. Then \( \varphi_{S,P,\psi}|_{C^*(E,\omega)^\gamma} \) is injective.

**Proof.** For each of the spanning elements \( S_\mu \psi(\alpha,k) S_\nu^* \) of \( \varphi_{S,P,\psi}(F_{p,k}) \), we have \( 0 \neq \psi(\alpha,k) = S_\mu^* S_\mu \psi(\alpha,k) S_\nu^* S_\nu \), and hence \( S_\mu \psi(\alpha,k) S_\nu^* \neq 0 \). Since each \( F_{p,k}(\alpha) \cong K_{E^r(\alpha)} \) is simple, it follows that \( \varphi_{S,P,\psi} \) is injective on each \( F_{p,k}(\alpha) \). It is therefore also injective, and hence isometric, on each \( F_{p,k} = \bigoplus_\alpha F_{p,k}(\alpha) \). It follows that \( \varphi_{S,P,\psi} \) is isometric on \( \bigcup_{p,k} F_{p,k} \), which is dense in \( C^*(E,\omega) \), giving the result.

**Proof of Theorem 4.2.1** It suffices to prove that

\[ \| \varphi_{S,P,\psi}(\Phi(a)) \| \leq \| \varphi_{S,P,\psi}(a) \| \quad \text{for all } a \in C^*(E,\omega). \]  

(4.2.3)

Indeed, suppose that (4.2.3) holds. Then the following standard argument (see, for ex-
ample, \[7, 34\] amongst many others) completes the proof:

\[
\varphi_{S,P,\psi}(a) = 0 \implies \varphi_{S,P,\psi}(a^*a) = 0 \\
\implies \varphi_{S,P,\psi}(\Phi(a^*a)) = 0 \quad \text{by (4.2.3)} \\
\implies \Phi(a^*a) = 0 \quad \text{by Lemma 4.2.5} \\
\implies a^*a = 0 \quad \text{because } \Phi \text{ is a faithful expectation} \\
\implies a = 0,
\]

so \(\varphi_{S,P,\psi}\) is injective.

So it suffices to establish (4.2.3). By continuity, it suffices to prove it for a finite linear combination \(a = \sum_{i=1}^{m} z_i s_{\mu_i} \rho(\alpha_i, k_i) s_{\nu_i}^*\). Following an argument that goes back to [7], we seek a projection \(Q\) such that

\[
\|Q \varphi_{S,P,\psi}(\Phi(a))Q\| = \|\varphi_{S,P,\psi}(\Phi(a))\|,
\]

(4.2.4)

and

\[
QS_{\mu_i} \psi_{(\alpha_i, k_i)} S_{\nu_i}^* Q = 0 \text{ whenever } |\mu_i| \neq |\nu_i|.
\]

(4.2.5)

Let \(N := \max\{|\mu_i|, |\nu_i| : i \leq m\}\). By the Cuntz–Krieger relation and Lemma 3.1.3(2) we can rewrite each

\[
S_{\mu_i} \psi_{(\alpha_i, k_i)} S_{\nu_i}^* = \sum_{\lambda \in s(\mu_i) E^{N-|\nu_i|}} S_{\mu_i} \lambda S_{\lambda}^* \psi_{(\alpha_i, k_i)} S_{\nu_i}^* = S_{\mu_i} \alpha_i \psi_{(\alpha_i', k_i)} S_{\nu_i}^*,
\]

where \(\alpha_i = \alpha_i' \alpha_i''\) and \(|\alpha_i'| = N - |\mu_i|\). So we may further assume without loss of generality that there exists \(p \in \mathbb{N}\) such that each \(|\mu_i| = p\) and each \(|\nu_i| \leq 2p\). Since the \(n_k \to \infty\), we can choose \(k \geq \max_i k_i\) such that \(n_k > 2p\), and we can then rewrite each \(\psi_{(\alpha_i, k_i)} = \sum_{\beta \in E^{<n_k}, |\beta| n_k = \alpha} \psi_{(\beta, k)}\). So we may assume without loss of generality that each \(|\mu_i| = p\), that each \(|\nu_i| \leq 2p\), that each \(k_i = k\) and that \(n_k \geq 2p\).

Equation 4.2.1 gives

\[
\Phi(a) = \sum_{|\nu_i| = p} z_i s_{\mu_i} \rho(\alpha_i, k_i) s_{\nu_i}^*.
\]

(4.2.6)

Since \(\mathcal{F}_{k,p} = \bigoplus_{\alpha \in E^{<n_k}} \mathcal{F}_{k,p}(\alpha)\), there exists \(\beta \in E^{<n_k}\) such that

\[
\|\Phi(a)\| = \left\| \sum_{|\nu_i| = p, \alpha_i = \beta} z_i s_{\mu_i} \rho(\beta, k_i) s_{\nu_i}^* \right\|.
\]

(4.2.7)
Let $I := \{i \leq m : |\nu_i| = p \text{ and } \alpha_i = \beta\}$, and let $G := \{\mu_i, \nu_i : i \in I\}$. Define

$$Q := \sum_{\lambda \in G} S_{\lambda\beta} \psi_{s(\beta),k} S_{\lambda\beta}^*.$$

We claim that $Q$ is a projection satisfying (4.2.4) and (4.2.5). Since $G \subset E^p$, the terms $S_{\lambda\beta} \psi_{s(\beta),k} S_{\lambda\beta}^*$ are mutually orthogonal projections, so $Q$ is a projection.

For (4.2.4), fix $i \leq m$ such that $|\nu_i| = p$. Using Lemma 3.1.3(2) at the third equality with $|\mu| = 0$, and that $S_{\beta} S_{\beta}^* \geq \psi_{(\beta,k)}$ at the final equality, we calculate:

$$Q S_{\mu_i} \psi_{(\alpha_i,k)} S_{\nu_i}^* Q = \sum_{\lambda, \tau \in G} S_{\lambda\beta} \psi_{s(\beta),k} S_{\lambda\beta}^* S_{\mu_i} \psi_{(\alpha_i,k)} S_{\nu_i}^* S_{\tau\beta} \psi_{s(\beta),k} S_{\tau\beta}^*$$

$$= S_{\mu_i \beta} \psi_{s(\beta),k} S_{\beta}^* S_{\mu_i \beta} \psi_{s(\beta),k} S_{\nu_i \beta}$$

$$= S_{\mu_i \beta} S_{\beta}^* \psi_{s(\beta),k} \psi_{(\beta,k)} S_{\beta} S_{\nu_i}^*$$

$$= \delta_{\alpha_i, \beta} S_{\mu_i \beta} S_{\beta}^* \psi_{s(\beta),k} S_{\beta} S_{\nu_i}^*$$

$$= \delta_{\alpha_i, \beta} S_{\mu_i \beta} \psi_{(\beta,k)} S_{\nu_i}^*.$$

By (4.2.6), $\Phi(a) \in C^*(E, \omega)^\gamma$. Now, Lemma 4.2.5 and (4.2.7) give $\|\psi_{s,P,\psi}(\Phi(a))\| = \| \sum_{|\nu_i| = p} z_i S_{\mu_i \beta} \psi_{s(\beta),k} S_{\nu_i}^* \|$. Hence, by the above calculation, we have

$$\|Q \varphi_{s,P,\psi}(\Phi(a))Q\| = \left\| Q \left( \sum_{|\nu_i| = p} z_i S_{\mu_i \beta} \psi_{s(\beta),k} S_{\nu_i}^* \right) Q \right\|$$

$$= \left\| \left( \sum_{|\nu_i| = p} z_i S_{\mu_i \beta} \psi_{s(\beta),k} S_{\nu_i}^* \right) \right\|$$

$$= \|\varphi_{s,P,\psi}(\Phi(a))\|.$$

To establish (4.2.5), take $i \leq m$ such that $i \notin I$, so that either $|\nu_i| \neq p$ or $\alpha_i \neq \beta$, and


\[ QS_{\mu_i} \psi(\alpha_i, k) S^*_{\nu_i} Q = \sum_{\lambda, \tau \in G} S_{\lambda \beta} \psi(s(\beta), k) S^*_{\lambda \beta} S_{\mu_i} \psi(\alpha_i, k) S^*_{\nu_i} S_{\tau} \psi(s(\beta), k) S^*_{\tau} \]

\[ = \sum_{\lambda, \tau \in G} S_{\lambda \beta} S^*_{\beta} \psi(s(\beta), k) S^*_{\mu_i} S_{\nu_i} S_{\tau} \psi(s(\beta), k) S^*_{\beta} S^*_\tau \]

\[ = \sum_{\tau \in G} S_{\mu_i} S^*_{\beta} \psi(s(\beta), k) S^*_{\nu_i} S_{\tau} \psi(s(\beta), k) S^*_{\beta} S^*_\tau \]

\[ = \delta_{\alpha_i, \beta} \sum_{\tau \in G} S_{\mu_i} S^*_{\beta} S^*_{\nu_i} S_{\tau} \psi(s(\beta), k) S^*_{\beta} S^*_\tau \]

We must show that this is zero. This is automatic if \( \alpha_i \neq \beta \), so we suppose that \( \alpha_i = \beta \), and hence \( |\nu_i| \neq p \). Using Lemma 3.1.3, we see that \( \psi(s(\beta), k) \in \text{span}\{ S^*_{\nu_i} \psi(s(\eta), k) : |\eta| \equiv |\beta| + |\nu_i| \ (\text{mod } n_k) \} \) and that each \( S_{\tau} \psi(s(\beta), k) \in \text{span}\{ \psi(s(\zeta), k) S_{\tau} : |\zeta| \equiv |\beta| + p \ (\text{mod } n_k) \} \). Since \( n_k > 2p \), we have \( |\nu_i| \neq p \ (\text{mod } n_k) \). So \( \psi(s(\eta), k) \psi(s(\zeta), k) = 0 \) whenever \( |\eta| \equiv |\beta| + |\nu_i| \ (\text{mod } n_k) \) and \( |\zeta| \equiv |\beta| + p \ (\text{mod } n_k) \), and we deduce that \( QS_{\mu_i} \psi(\alpha_i, k) S^*_{\nu_i} Q = 0 \). This establishes that \( Q \) satisfies (4.2.5).

We can now finish off:

\[ \| \varphi_{S, P, \psi}(\Phi(a)) \| = \| Q \varphi_{S, P, \psi}(\Phi(a)) Q \| \quad \text{by (4.2.4)} \]

\[ = \| Q \left( \sum_{|\nu_i| = p} z_i S_{\mu_i} \psi(\alpha_i, k) S^*_{\nu_i} \right) Q \| \]

\[ = \| Q \left( \sum_{i=1}^m z_i S_{\mu_i} \psi(\alpha_i, k) S^*_{\nu_i} \right) Q \| \quad \text{by (4.2.5)} \]

\[ \leq \| \sum_{i=1}^m z_i S_{\mu_i} \psi(\alpha_i, k) S^*_{\nu_i} \| = \| \varphi_{S, P, \psi}(a) \|, \]

and so \( \varphi_{S, P, \psi} \) is injective as claimed.
Chapter 5

Perron–Frobenius theory

In this chapter we describe a sequence of equivalence relations on $E^0$ which will allow us to understand the connectivity of the graphs $E(n)$, and thence characterise the simplicity of the generalised Bunce–Deddens algebras $C^*(E, \omega)$. We also use these equivalence relations to calculate the cokernels of powers of the vertex matrix of $E$ which will be useful when we compute the $K$-theory of $C^*(E, \omega)$ in Chapter 7.

The equivalence relations described in this chapter appears in joint work with my supervisors [44, Section 4], as does the direct-sum decomposition and simplicity result of Section 5.2. Related ideas are described for $k$-graphs in [36, Section 6]. There they define the group of periods of a $k$-graph and describe an equivalence relation on the set of vertices. The results we need here are 1-graph versions of these results and are really a graph-based version of well-known ideas from Perron–Frobenius theory for irreducible matrices; we must do a little extra work to relate these ideas from Perron–Frobenius theory for $E$ to the graphs $E(n)$.

The calculation of cokernels of Section 5.3 is new to this thesis and independent of [44]. We will use this for the $K$-theory and classification results in Chapter 7.

Recall that the period $P_E$ of a strongly connected directed graph $E$ is given by $P_E = \gcd\{|\mu| : \mu \in E^*, r(\mu) = s(\mu)\}$.

5.1 An equivalence relation on $E(n)$

Here we establish a sequence of equivalence relations on $E^0$ which will be useful for characterising the simplicity of $C^*(E, \omega)$, computing the $K$-theory of $C^*(E, \omega)$ and studying the structure of factor KMS states.

Lemma 5.1.1. Let $E$ be a strongly connected finite graph, and fix $n \geq 1$. There is a surjective map $C_n : E^0 \times E^0 \to \mathbb{Z}/\gcd(P_E, n)\mathbb{Z}$ such that $C_n(r(\lambda), s(\lambda)) = |\lambda| + \gcd(P_E, n)\mathbb{Z}$.
for all $\lambda \in E^*$. There is also an equivalence relation $\sim_n$ on $E^0$ such that $v \sim_n w$ if and only if $C_n(v, w) = [0]$, and $\sim_n$ has exactly $\gcd(P_E, n)$ (nonempty) equivalence classes.

Proof. Fix $v, w \in E^0$ and $\mu, \nu \in vE^*w$. Since $E$ is strongly connected, there is a path $\lambda \in wE^*v$, and then $\mu \lambda, \nu \lambda \in vE^*v$. Hence $|\mu| - |\nu| = |\mu \lambda| - |\nu \lambda| \in P_E \mathbb{Z} \subseteq \gcd(P_E, n) \mathbb{Z}$. So there is a well-defined function $C_n : \{ (v, w) \in E^0 \times E^0 : vE^*w \neq \emptyset \} \to \mathbb{Z}/\gcd(P_E, n) \mathbb{Z}$ such that it is symmetric, suppose that $\sim_n$ is an equivalence relation. We clearly have $C_n(v, w) = [0]$ if and only if $\lambda \mu \in \gcd(P_E, n) \mathbb{Z}$, and so $C_n(w, v) = [0]$ as well. For transitivity, suppose that $C_n(u, v) = [0]$ and $C_n(v, w) = [0]$. Then there exist $\mu \in uE^*v$ and $\nu \in vE^*w$ with $|\mu|, |\nu| \in \gcd(P_E, n) \mathbb{Z}$. So $|\mu \nu| = |\mu| + |\nu| \in \gcd(P_E, n) \mathbb{Z}$, and hence $C_n(u, w) = [0]$ too. For the final assertion, fix $v_0 \in E^0$. For $v, w \in E^0$, choose a path $e_1 e_2 \ldots e_{\gcd(P_E, n)} \in E^{\gcd(P_E, n)}$. Then $C_n(v, w) = \lambda \mu + \gcd(P_E, n) = |\lambda| + \gcd(P_E, n) + |\mu| + \gcd(P_E, n) = C_n(v, v_0) + C_n(v_0, w)$. Choose $\nu \in v_0E^*v$. Then $C_n(v_0, v_0) = |\mu \nu| + \gcd(P_E, n)$, so $|\mu| + \gcd(P_E, n) = -|\nu| + \gcd(P_E, n)$, giving $C_n(v_0, v_0) = -C_n(v_0, v)$. Thus $v \sim_n w$ if and only if $C_n(v_0, w) = C_n(v_0, v)$. Since $C_n$ is a surjection onto $\{ 0, [1], \ldots, [\gcd(P_E, n) - 1] \}$, it follows that $\sim_n$ has exactly $\gcd(P_E, n)$ equivalence classes.

We write $\approx_E$ for the smallest equivalence relation on $E^0$ such that $r(e) \approx_E s(e)$ for all $e \in E^1$. We call the equivalence classes of $\approx_E$ the connected components of $E$. For $v, w \in E^0$, $vE^*w \neq \emptyset$ implies that $v \approx_E w$.

There is a map $\{ (\mu, \nu) : \mu \in E^*, \nu \in s(\mu)E^{<n} \} \to E(n)^*$ given by

$$(\mu, \nu) \mapsto (\mu_1, [\mu_2 \ldots \mu_{|\mu|} \nu]_n)(\mu_2, [\mu_3 \ldots \mu_{|\mu|} \nu]_n)\ldots(\mu_{|\mu|}, \nu),$$

with inverse given by $(e_1, \nu_1)(e_2, \nu_2)\ldots(e_m, \nu_m) \mapsto (e_1 e_2 \ldots e_m, \nu_m)$ for $(e_i, \nu_i) \in E(n)^1$.

We use this bijection to identify $E(n)^*$ with $\{ (\mu, \nu) : \mu \in E^*, \nu \in s(\mu)E^{<n} \}$, and we then have $s_n(\mu, \nu) = \nu$, and $r_n(\mu, \nu) = [\mu \nu]_n$. This implies, in particular, that the lengths of the paths $r_n(\mu, \nu)$ and $s_n(\mu, \nu)$ in $E^{<n}$ differ by $|\mu|$ modulo $n$. Thus, for
\(v, w \in E^0 \subseteq E^\leq_n\), we have

\[vE(n)^*w \neq \emptyset \quad \text{if and only if} \quad vE^{jn}w \neq \emptyset \text{ for some } j \in \mathbb{N}. \quad (5.1.1)\]

If \(\mu \in E^l\) for some \(l \in \mathbb{N}\), then we identify \((\mu, s(\mu))\) with \((\mu_1, [\mu_2 \ldots \mu_\ell]_n)\ldots(\mu_t, s(\mu)) \in E(n)^l\).

The following results will be useful for characterising simplicity.

**Lemma 5.1.2.** Let \(E\) be a strongly connected finite directed graph, and fix \(n \geq 1\). Let \(\Lambda \in E^0/\sim_n\), and let \(\mu \in E^*\). Then \(s([\mu]_n) \in \Lambda\) if and only if \(s(\mu) \in \Lambda\). Moreover, if \(\alpha \in E^\leq_n\) and \(\lambda \in E^*r(\alpha)\), then \(r_n(\lambda, \alpha) \in \Lambda\) if and only if \(s(\alpha) \in \Lambda\).

**Proof.** Writing \(\mu = [\mu]_n\mu'\) for some \(\mu' \in E^*\) with \([\mu'] \in n\mathbb{N}\), we have \(C_n(s([\mu]_n), s(\mu)) = C_n(r(\mu'), s(\mu')) = |\mu'| + \gcd(P_E, n)\mathbb{Z} = [0]\). The second statement follows from the same argument since \(r_n(\lambda, \alpha) = [\lambda\alpha]_n\).

**Proposition 5.1.3.** Let \(E\) be a strongly connected finite directed graph, and let \(n \geq 1\). For \(\Lambda \in E^0/\sim_n\), let \(E(n)^\Lambda = \{\mu \in E^\leq_n : s(\mu) \in \Lambda\}\). Then the sets \(\{E(n)^\Lambda : \Lambda \in E^0/\sim_n\}\) are precisely the connected components of \(E(n)\). These connected components are all strongly connected: if \(\mu, \nu \in E(n)^\Lambda\), then \(E(n)^*\nu \neq \emptyset\). In particular, \(E(n)\) is strongly connected if and only if \(\gcd(P_E, n) = 1\).

**Proof.** Since \(\mu E(n)^*\nu \neq \emptyset\) implies \(\mu \approx_{E(n)} \nu\), it suffices to show that if \(s(\mu) \sim_n s(\nu)\) then \(\mu E(n)^*\nu \neq \emptyset\), and that if \(\mu \approx_{E(n)} \nu\), then \(s(\mu) \sim_n s(\nu)\).

First suppose that \(s(\mu) \sim_n s(\nu)\). Since \(E\) has no sources and is strongly connected, it has no sinks, so we can choose \(\alpha = \alpha_1 \ldots \alpha_k \in E^*r(\nu)\) such that \([\alpha\nu] \in n\mathbb{N}\). It follows that \(C(r(\alpha), s(\nu)) = [0]\) and so \(s(\mu) \sim_n r(\alpha)\). Let \(v := s(\mu)\) and \(w := r(\alpha)\). Since \(v \sim_n w\), there is \(\lambda \in vE^*w\) such that \(|\lambda| + \gcd(P_E, n)\mathbb{Z} = C_n(v, w) = [0]\), so \(|\lambda| \in \gcd(P_E, n)\mathbb{Z}\). Choose \(k\) such that \(kP_E \equiv \gcd(P_E, n)\) (mod \(n\)). Since \(E\) is strongly connected, we have \(P_E\mathbb{Z} = \{[\eta] - [\zeta] : \eta, \zeta \in wE^*w\}\). So there are cycles \(\eta, \zeta \in wE^*w\) such that \(|\eta| - |\zeta| = P_E^\ast\). In particular, \(|\eta\zeta^{-1} + [\eta] - [\zeta] + |\zeta^n| - P_E + n|\zeta| = P_E\) (mod \(n\)). Hence \(\beta := (\eta\zeta^{-1})^k \in wE^*w\) satisfies \(|\beta| \equiv kP_E\) (mod \(n\)) \(\equiv \gcd(P_E, n)\) (mod \(n\)). Choose \(q \in \mathbb{N}\) such that \(qn \geq |\lambda|\). Since \(|\lambda|\) is divisible by \(\gcd(P_E, n)\), the number \(l := \frac{qn - |\lambda|}{\gcd(P_E, n)}\) is an integer. Now \(|\lambda\beta^j| \in vE^{jn}w\) for some \(j\). So \((5.1.1)\) gives a path \(\tilde{\lambda} \in vE(n)^*w\). Now \((\mu, v)\tilde{\lambda}(\alpha, \nu) \in \mu E(n)\nu\) as required.

Now suppose that \(\mu \approx_{E(n)} \nu\). Since \((\mu, s(\mu)) \in \mu E(n)^*s(\mu)\) and likewise for \(\nu\), and since \(\approx_{E(n)}\) is an equivalence relation, we have \(s(\mu) \approx_{E(n)} s(\nu)\). So it suffices to show that \(v \approx_{E(n)} w\) implies \(v \sim_n w\) for \(v, w \in E^0\). By definition of \(\approx_{E(n)}\) it then suffices, by induction, to show that if \(vE(n)^*w \neq \emptyset\), say \((\lambda, w) \in vE(n)^*w\), then \(v \sim_n w\). By \((5.1.1)\)
we have $\lambda \in vE^jw$ for some $j$. In particular, $C(v, w) = |\lambda| + \gcd(P_E, n)\mathbb{Z} = 0 + \gcd(P_E, n)\mathbb{Z}$ and so $v \sim_n w$. 

5.2 Simplicity of $C^*(E, \omega)$

We use Proposition 5.1.3 together with Theorem 4.2.1 to extend [31, Corollary 8.7] to finite strongly connected graphs provided that the terms $n_k$ in $\omega$ diverge to infinity. (If the $n_k$ are bounded then they are eventually constant, and then $C^*(E, \omega) \cong C^*(E(N))$; and so simplicity of $C^*(E, \omega)$ is characterised by [3, Proposition 5.1].) Our result for simplicity can be obtained by using Kribs and Solel’s topological graph $E(\infty)$ and Katsura’s results about topological graphs (see [44, Corollary 5.5] and Chapter 6). We include the following direct proof based on our uniqueness theorem from Chapter 4 because we think that it provides complementary insight.

Given a multiplicative sequence $\omega = (n_k)_{k=1}^{\infty}$, and given $p \in \mathbb{N}$, the sequence $\gcd(p, n_k)$ is nondecreasing and bounded above by $p$, so it is eventually constant. We write $\gcd(p, \omega)$ for its eventual value. We let $l := \gcd(P_E, \omega)$.

The following technical lemma will be useful again in our analysis of the structure of the factor KMS states on $\mathcal{T}(E, \omega)$ in Theorem 8.4.2.

**Lemma 5.2.1.** Let $E$ be a strongly connected finite directed graph, and take a multiplicative sequence $\omega = (n_k)_{k=1}^{\infty}$. Fix $k$ such that $\gcd(P_E, n_k) = \gcd(P_E, \omega)$. For each equivalence class $\Lambda \in E_0/\sim_{n_k}$, let

$$Q_{k, \Lambda} := \sum_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} \pi(\mu, k) \in \mathcal{T}(E, \omega).$$

Then the $Q_{k, \Lambda}$ are nonzero mutually orthogonal projections, and

$$\mathcal{T}(E, \omega) = \bigoplus_{\Lambda \in E^0/\sim_{n_k}} Q_{k, \Lambda} \mathcal{T}(E, \omega) Q_{k, \Lambda}.$$

The images $P_{k, \Lambda}$ of the $Q_{k, \Lambda}$ in the quotient $C^*(E, \omega)$ are all nonzero.

**Proof.** For $\Lambda \in E^0/\sim_{n_k}$, we put

$$\Theta_{k, \Lambda} := \sum_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} \theta_{n_k, \mu} \in \mathcal{T}(E, n_k).$$

The $\Theta_{k, \Lambda}$ are mutually orthogonal by Proposition 5.1.3 and nonzero because the generators of $\mathcal{T}(E, n_k) \cong \mathcal{T}C^*(E(n_k))$ are all nonzero.
CHAPTER 5. PERRON–FROBENIUS THEORY

We claim that for $\alpha \in E^{<n_k}$ and $\mu, \nu \in E^*(\alpha)$, we have $\sum_\Lambda Q_\Lambda t_{\mu,\Lambda}t^*_\nu Q_\Lambda = t_{\mu,\Lambda}t^*_\nu$. Let $(t, q)$ be the universal Toeplitz–Cuntz–Krieger E-family in $T^*(E(n_k))$. Recall the isomorphism $\pi_{n_k}$ of Remark 3.1.6. By Lemma 5.1.2 we have $s(r_{n_k}(\mu, \alpha)), s(r_{n_k}(\nu, \alpha)) \in \Lambda$ if and only if $s(\alpha) \in \Lambda$. Thus, we have

$$\Theta_{k,\Lambda}t_{n_k,\mu}t^*_{n_k,\nu} \Theta_{k,\Lambda} = \sum_{n, \eta \in E^{<n_k}, s(\eta), s(\zeta) \in \Lambda} \pi^{-1}_{n_k}(g_{\eta}t_{(\mu, \alpha)}t^*_{(\nu, \alpha)}g_{\zeta})$$

$$= \pi^{-1}_{n_k}(t_{(\mu, \alpha)}t^*_{(\nu, \alpha)}) = t_{n_k,\mu}t^*_{n_k,\nu},$$

and since the $\Theta_{k,\Lambda}$ are mutually orthogonal, the claim follows.

We now show that each $i_{n_k, n_{k+1}}(\Theta_{k,\Lambda}) = \Theta_{k+1,\Lambda}$. Let $\zeta \in E^{<n_{k+1}}$. By Lemma 5.1.2 $s([\zeta]_{n_k}) \in \Lambda$ if and only if $s(\zeta) \in \Lambda$. We use this at the third equality in the following calculation:

$$i_{n_k, n_{k+1}}(\Theta_{k,\Lambda}) = i_{n_k, n_{k+1}}\left(\sum_{\eta \in E^{<n_k}, s(\eta) \in \Lambda} \theta_{n_k, \eta}\right)$$

$$= \sum_{\zeta \in E^{<n_{k+1}}, s([\zeta]_{n_k}) \in \Lambda} \theta_{n_{k+1}, \zeta} = \sum_{\zeta \in E^{<n_{k+1}}, s(\zeta) \in \Lambda} \theta_{n_{k+1}, \zeta} = \Theta_{k+1,\Lambda}.$$

The preceding two paragraphs show that every element of the spanning family for $T(E, \omega)$ described in the final statement of Lemma 3.2.2 belongs to $Q_{k,\Lambda}T(E, \omega)Q_{k,\Lambda}$ for some $\Lambda$, giving the desired direct-sum decomposition.

To see that the images $P_{k,\Lambda}$ of the $Q_{k,\Lambda}$ in $C^*(E, \omega)$ are nonzero, observe that for any $\Lambda$, and any $v \in \Lambda$, we have $P_{k,\Lambda} \geq \rho(v, k) = p_{n_k, v}$, which is nonzero since all the generators of $C^*(E(n_k))$ are nonzero.

**Corollary 5.2.2.** Let $E$ be a strongly connected finite directed graph, and take a multiplicative sequence $\omega = (n_k)_{k=0}^\infty$. Suppose that $n_k \to \infty$ as $k \to \infty$. Then $C^*(E, \omega)$ is simple if and only if $\gcd(P_E, \omega) = 1$.

**Proof.** First suppose that $C^*(E, \omega)$ is simple. Fix $k$ with $\gcd(P_E, n_k) = \gcd(P_E, \omega)$. Lemma 5.2.1 shows that $C^*(E, \omega)$, being a quotient of $T(E, \omega)$ is a direct sum $C^*(E, \omega) = \bigoplus_{\Lambda \in E^0 \sim n_k} P_{k,\Lambda}C^*(E, \omega)P_{k,\Lambda}$ and that each summand is nonzero. Since $C^*(E, \omega)$ is simple, there can be only one summand, and so $v \sim_{n_k} w$ for all $v, w$. Lemma 5.1.1 shows that $\sim_{n_k}$ has $\gcd(P_E, n_k)$ equivalence classes, and we deduce that $\gcd(P_E, \omega) = \gcd(P_E, n_k) = 1$.

Now suppose that $\gcd(P_E, \omega) = 1$. Suppose that $\kappa : C^*(E, \omega) \to B$ is a nonzero homomorphism, and fix $k \in \mathbb{N}$. Since $\sum_{\mu \in E^{<n_k}} \rho(\mu, k) = 1$, we have $\kappa(\rho(\mu, k)) \neq 0$ for some $\mu$. Choose $v \in E^{<n_k}$. Proposition 5.1.3 implies that $E(n_k)$ is strongly connected, so there exists $(\lambda, \mu) \in \nu E(n_k)^*\mu$. Using the isomorphism $\tilde{\pi}_{n_k} : C^*(E, n_k) \to C^*(E(n_k))$, we
see that
\[ \rho(\mu,k) = \pi_{nk}^{-1}(p_{nk,\mu}) = \pi_{nk}^{-1}(s_{nk,\lambda,\mu}^* s_{nk,\lambda,\mu}) = \pi_{nk}^{-1}(s_{nk,\lambda,\mu}^* p_{nk,\mu} s_{nk,\lambda,\mu}) \]
belongs to the ideal generated by \( p_{nk,\mu} \). Since \( \kappa(\rho(\mu,k)) \neq 0 \), it follows that each \( \kappa(\rho(\nu,k)) \neq 0 \). So \( \kappa \) is injective by Theorem 4.2.1.

\[ \square \]

5.3 Calculating cokernels

Here we use the equivalence relation \( \sim_l \) of Lemma 5.1.1, where \( l := \gcd(P_E, \omega) \), to show that \( \text{coker}(1 - A_E^l)^t \) is isomorphic to \( l \) copies of \( \text{coker}(1 - A_E^1) \). This will be very useful when we compute the \( K \)-theory of \( C^*(E, \omega) \) in Chapter 7.

**Lemma 5.3.1.** Let \( E \) be a strongly connected finite directed graph. Then \( A_E^l \delta_v + \text{Im}(1 - A_E^1) = \delta_v + \text{Im}(1 - A_E^1) \) for all \( v \in E^0 \).

**Proof.** Fix \( v \in E^0 \). We have that \( \delta_v - A_E^l \delta_v = (1 - A_E^1) \delta_v \in \text{Im}(1 - A_E^1) \), so \( A_E^l \delta_v + \text{Im}(1 - A_E^1) = \delta_v + \text{Im}(1 - A_E^1) \).

We enumerate the equivalence classes for \( \sim_l \). Fix \( v \in E^0 \), and let \( \Lambda_0 = [v] \). Now iteratively fix \( e \in E^1 \) with \( r(e) \in \Lambda_i \) and let \( \Lambda_{i+1} = [s(e)] \), where addition in the subscript is modulo \( l \). Then \( \Lambda_0, \ldots, \Lambda_{l-1} \) is an enumeration of the equivalence classes in \( E^0 / \sim_l \).

**Lemma 5.3.2.** Let \( E \) be a strongly connected finite directed graph. Let \( \omega = (n_k)_{k=1}^\infty \) be a multiplicative sequence, and let \( l := \gcd(P_E, \omega) \). There is an isomorphism

\[ \Theta : \text{coker}(1 - A_E^l)^t \to \bigoplus_{i=0}^{l-1} \mathbb{Z}^{\Lambda_i} / (1 - A_E^1)^t \mathbb{Z}^{\Lambda_i} \]

satisfying

\[ \Theta(\delta_v + \text{Im}(1 - A_E^1)^t) = (0, \ldots, 0, \delta_v + (1 - A_E^1)^t \mathbb{Z}^{\Lambda_j}, 0, \ldots, 0), \]

where \( v \in \Lambda_j \) for some \( 0 \leq j \leq l - 1 \) and \( \delta_v + (1 - A_E^1)^t \mathbb{Z}^{\Lambda_j} \) appears in the \( j \)-th position.

**Proof.** Fix \( 0 \leq j \leq l - 1 \), and \( v \in \Lambda_j \). Since \( E^0 = \bigsqcup_{i=0}^{l-1} \Lambda_i \), there is an isomorphism \( \theta : \mathbb{Z}^{E^0} \to \bigoplus_{i=0}^{l-1} \mathbb{Z}^{\Lambda_i} \) such that \( \theta(\delta_v) = (0, \ldots, 0, \delta_v, 0, \ldots, 0) \), where \( \delta_v \) is in the \( j \)-th position.

Our choice of \( \Lambda_0, \ldots, \Lambda_{l-1} \) ensures that \( (A_E^1)^t \delta_v = \sum_{w \in E^0} |v E^t w| \delta_w \in \mathbb{Z}^{\Lambda_j} \) and so \( (1 - A_E^1)^t \delta_v \in \mathbb{Z}^{\Lambda_j} \). Hence \( \theta((1 - A_E^1)^t \delta_v) = (0, \ldots, 0, (1 - A_E^1)^t \delta_v, 0, \ldots, 0) \in \bigoplus_{i=0}^{l-1} (1 - A_E^1)^t \mathbb{Z}^{\Lambda_i} \).
Therefore $\theta$ descends to an isomorphism $\Theta : \operatorname{coker}(1 - A_E^t) \to \bigoplus_{i=0}^{l-1} \mathbb{Z}^A/(1 - A_E^t)\mathbb{Z}^A_i$ satisfying the desired formula.

\begin{proof}

Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. For each $0 \leq j \leq l - 1$, there is an isomorphism $\Phi_j : \mathbb{Z}^A_j/(1 - A_E^t)\mathbb{Z}^A_j \to \mathbb{Z}^{E_0}/\operatorname{Im}(1 - A_E^t)$ satisfying

$$\Phi_j(\delta_v + (1 - A_E^t)\mathbb{Z}^A_j) = \delta_v + (1 - A_E^t)\mathbb{Z}^{E_0},$$

for some $v \in \Lambda_j$.

\begin{flushright}
$\square$
\end{flushright}

\textbf{Lemma 5.3.3.} Let $E$ be a strongly connected finite directed graph. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. For each $0 \leq j \leq l - 1$, there is an

$$\Phi_j(\delta_v + (1 - A_E^t)\mathbb{Z}^A_j) = \delta_v + (1 - A_E^t)\mathbb{Z}^{E_0},$$

for some $v \in \Lambda_j$.

\begin{proof}

Fix $0 \leq j \leq l - 1$. The formula $(1 - A_E^t)^i = (1 - A_E^t)(\sum_{i=0}^{l-1} (A_E^t)^i)$ shows that $\operatorname{Im}(1 - A_E^t)^i \subseteq \operatorname{Im}(1 - A_E^t)$. Since $(1 - A_E^t)^i \mathbb{Z}^A_j \subseteq \operatorname{Im}(1 - A_E^t)^i$, it follows that the map $\mathbb{Z}^A_j \to \mathbb{Z}^{E_0}$ given by $\delta_v \mapsto \delta_v$ for $v \in \Lambda_j$, descends to a homomorphism $\Phi_j : \mathbb{Z}^A_j/(1 - A_E^t)^i \mathbb{Z}^A_j \to \mathbb{Z}^{E_0}/\operatorname{Im}(1 - A_E^t)^i$ satisfying $\Phi_j(\delta_v + (1 - A_E^t)^i \mathbb{Z}^A_j) = \delta_v + \operatorname{Im}(1 - A_E^t)$, for $v \in \Lambda_j$.

We must show that $\Phi_j$ is an isomorphism. To see that $\Phi_j$ is surjective, fix $0 \leq k \leq l - 1$ and $v \in \Lambda_k$. Then $(A_E^{l-k})^i \delta_v \in \mathbb{Z}^A_j$ and

$$\delta_v + \operatorname{Im}(1 - A_E^t) = (A_E^{l-k})^i \delta_v + \operatorname{Im}(1 - A_E^t) = \Phi_j((A_E^{l-k})^i \delta_v + (1 - A_E^t)^i \mathbb{Z}^A_j).$$

To see that $\Phi_j$ is injective, fix $a = \sum_{v \in \Lambda_j} a_v \delta_v \in \mathbb{Z}^A_j$ such that $\Phi_j(a + (1 - A_E^t)^i \mathbb{Z}^A_j) = 0$. That is, $a \in \operatorname{Im}(1 - A_E^t)$. Say $a = (1 - A_E^t)b$ where $b = \sum_{w \in \mathcal{E}} b_w \delta_w$. Let $b_k := b|_{\Lambda_k} = \sum_{w \in \Lambda_k} b_w \delta_w$ for each $0 \leq k \leq l - 1$. Since $a \in \mathbb{Z}^A_j$, we have $0 = a|_{\Lambda_k} = ((1 - A_E^t)b)|_{\Lambda_k} = b_k - A_E^t b_{k-1}$, for all $0 \leq k \leq l - 1$, $k \neq j$, where subtraction in the subscript is modulo $l$. Therefore $b_k = (A_E^t)^{k-j}b_j$ for each $0 \leq k \leq l - 1$, $k \neq j$, where subtraction in the superscript is modulo $l$. Hence

$$(1 - A_E^t)b = (1 - A_E^t)(b_0 + \cdots + b_{l-1}) = (1 - A_E^t)\left(\sum_{k=0}^{l-1} (A_E^t)^k \right) b_j = (1 - A_E^t)^l b_j.$$

So $a = (1 - A_E^t)^l b_j \in (1 - A_E^t)^i \mathbb{Z}^A_j$.

\end{proof}

\begin{flushright}
$\square$
\end{flushright}

\textbf{Corollary 5.3.4.} Let $E$ be a strongly connected finite directed graph. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. There is an isomorphism $\rho : \operatorname{coker}(1 - A_E^t) \to \bigoplus_{i=1}^l \operatorname{coker}(1 - A_E^t)$ satisfying

$$\rho(\delta_v + \operatorname{Im}(1 - A_E^t)^i) = (0, \ldots, 0, \delta_v + \operatorname{Im}(1 - A_E^t), 0, \ldots, 0),$$

where $v \in \Lambda_j$ for some $0 \leq j \leq l - 1$ and $\delta_v + \operatorname{Im}(1 - A_E^t)$ appears in the $j$-th position.
Proof. Define \( \rho := \left( \bigoplus_{i=0}^{l-1} \Phi_i \right) \circ \Theta \). It follows from Lemma 5.3.3 and Lemma 5.3.4 that \( \rho \) is an isomorphism that satisfies the desired formula.
Chapter 6

The topological graph \( E(\infty) \)

In this chapter we use the theory of topological graph \( C^* \)-algebras to give alternative proofs of Theorem 4.2.1 and Corollary 5.2.2. This approach also appears in joint work with my supervisors in [44, Section 4]. Kribs and Solel construct a topological graph \( E(\infty) \) from a graph \( E \) and a multiplicative sequence \( \omega = (n_k)_{k=1}^\infty \). They show in [31, Theorem 6.3] that \( C^*(E, \omega) \) is isomorphic to the \( C^* \)-algebra \( C^*(E(\infty)) \) of this topological graph in the sense of Katsura [26]. We give a slightly different description of the topological graph \( E(\infty) \), and use this description to present the details of the isomorphism \( C^*(E, \omega) \cong C^*(E(\infty)) \).

For the most part, this involves filling in some of the details of the proofs of results in [31] and [27], and also recording explicit formulas for the isomorphisms established there.

Let \( E \) be a row-finite directed graph with no sources, and take a multiplicative sequence \( \omega = (n_k)_{k=1}^\infty \). Suppose that \( n_k \to \infty \) as \( k \to \infty \). We first recall Kribs and Solel’s construction of a topological graph from \((E, \omega)\). Let \( X_i = \{ \lambda \in E^*: 0 \leq |\lambda| < n_i, |\lambda| \equiv 0 \pmod{n_{i-1}} \} \), let \( X = \prod_{i=1}^\infty X_i \) and let \( Y = \{ y \in X : s(y_k) = r(y_{k+1}) \) for all \( k \}\). For each \( e \in E_1 \), let \( D_e = \{ y \in Y : r(y_1) = e(e) \} \) and \( R_e = \{ y \in Y : \text{there exists } l \leq \infty \text{ such that } y_i = r(e) \) for all \( i < l \) and \( (if \ l \neq \infty) \ y_l = e y' \) for some \( |y'| \equiv -1 \pmod{n_{i-1}} \} \).

For \( y \in D_e \), write \( i(y) := \min \{ i \geq 1 : |y_i| < n_i - n_{i-1} \} \) or \( i(y) := \infty \) if \( |y_i| = n_i - n_{i-1} \) for every \( i \). If \( i(y) < \infty \), write \( \sigma_e(y) = u \), where

\[
    u_i = \begin{cases} 
        r(e) & \text{if } i < i(y) \\
        e y_1 \ldots y_{i(y)} & \text{if } i = i(y) \\
        y_i & \text{if } i > i(y).
    \end{cases}
\]
If \( i(y) = \infty \), set \( \sigma_\epsilon(y) = \langle r(e), r(e) \rangle \).

Kribs and Solel construct a topological graph \( E(\infty) \) with \( E(\infty)^0 = Y, E(\infty)^1 = \{(e, y) \in E^1 \times Y : y \in D_e\}, s_{E(\infty)}(e, y) = y \) and \( r_{E(\infty)}(e, y) = \sigma_\epsilon(y) \). Here we give another presentation of \( E(\infty) \) which is more natural within our framework.

**Lemma 6.0.1.** Let \( E \) be a row-finite directed graph with no sources, and take a multiplicative sequence \( \omega = (n_k)_{k=1}^\infty \). Suppose that \( n_k \to \infty \) as \( k \to \infty \). Define \( F^0_{E,\omega} = \lim \nleftarrow E^{<n_k}; F^1_{E,\omega} = \{(e, x) \in E^1 \times \lim \nearrow E^{<n_k} : r(x_1) = s(e)\}, s_F(e, x) = x, \) and \( r_F(e, x) = r_{n_k}(e, x_k) = [ex_k]_{n_k} \). Then \( \phi = (\phi^0, \phi^1) : E(\infty) \to F \) defined by \( \phi^0(y) = y_1 y_2 \ldots y_n, \) for \( y \in E(\infty)^0 \) and \( \phi^1(e, y) = (e, \phi^0(y)) \) for \( (e, y) \in E(\infty)^1 \) is an isomorphism of topological graphs.

**Proof.** We abbreviate \( F^i := F^i_{E,\omega}, i = 0, 1 \). Define \( \psi = (\psi^0, \psi^1) : F \to Y \) by \( \psi^0(x_1) = x_1 \) and \( \psi^0(x_{i+1}) = x_{i+1} \) for all \( i \geq 1 \), for \( x \in F^0 \), and \( \psi^1(e, x) = (e, \psi^0(x)) \) for \( (e, x) \in F^1 \).

We show that \( \psi^0 \) is an inverse for \( \phi^0 \). Fix \( y \in Y \). We have \( \psi^0(\phi^0(y))_i = \psi^0(y_1 \ldots y_j^\infty) \). Since \( [y_1 \ldots y_{i-1}] = \sum_{j=1}^{i-1} [y_j] < n_{i-1} \) and \( [y_i] \in n_{i-1} \mathbb{N} \), we have \( [y_1 \ldots y_i] = y_1 \ldots y_{i-1} \) and hence \( \psi^0(\phi^0(y))_i = y_i \). Now, fix \( x \in F^0 \). Then \( \phi^0(\psi^0(x))_i = \psi^0(x_1 \ldots x_j) = x_1 \psi^0(x_2) \psi^0(x_3) = \ldots = x_i \). Therefore \( \psi^0 \) is an inverse for \( \phi^0 \).

The basic open sets in \( Y \) are given by \( Z_Y(w_1, \ldots, w_k) = \{y \in Y : y_i = w_i \text{ for } 1 \leq i \leq k\} \), where \( w_i \in X_i \) and \( s(w_i) = r(w_{i+1}) \).

We calculate

\[
\phi^0(Z_Y(y_1, \ldots, y_k)) = \{x \in F^0 : x_i = y_1 \ldots y_i \text{ for all } i \leq k\} = \{x \in F^0 : x_k = y_1 \ldots y_k\} = Z(y_1 \ldots y_k, k).
\]

So \( \psi^0 = (\phi^0)^{-1} \) is continuous.

Conversely, for \( \mu \in E^{<n_k} \), express \( \mu = y_1 \ldots y_k \), where \( y_1 = [\mu]_{n_1} \) and \( [\mu]_{n_i} y_{i+1} = [y]_{n_{i+1}} \). Then \( \psi^0(Z(\mu, k)) = Z_Y(y_1, \ldots, y_k) \). So \( \phi^0 \) is continuous. Therefore \( \phi^0 \) is a homeomorphism of \( E(\infty)^0 \) onto \( F^0 \). It then follows immediately that \( \phi^1 : E(\infty)^1 \to F^1 \) is also a homeomorphism.

We have \( \phi^0(s_{E(\infty)}(e, y)) = s_F(e, \phi^0(y)) = s_F(\phi^1(e, y)) \). We also have \( \phi^0(r_{E(\infty)}(e, y)) = [ey_1 \ldots y_i]_{n_i} \), so \( \phi^0(r_{E(\infty)}(e, y)) = r_F(e, \phi^0(y)) = r_F(\phi^1(e, y)) \). Therefore \( \phi \) is an isomorphism of topological graphs.

\[\square\]

### 6.1 Uniqueness

Here we give an alternative proof of Theorem 4.2.1 using Katsura’s results about topological graph \( C^* \)-algebras, and the isomorphism \( C^*(E, \omega) \cong C^*(E(\infty)) \) established by Kribs.
and Solel. The following result follows from the isomorphism $F \cong E(\infty)$, and Katsura’s arguments in [27], but a precise description of the isomorphism that we need to use is not provided there, so we give a detailed statement.

**Proposition 6.1.1.** Let $E$ be a row-finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^{\infty}$. Suppose that $n_k \to \infty$ as $k \to \infty$. There is an isomorphism $\pi : \lim_{\rightarrow} C^*(E(n_k)) \to C^*(F)$ such that

$$\pi(j_{n_k,\infty}(p_{n_k,\lambda})) = t_F^0(\chi_{\{\epsilon\}}Z(\lambda,k)),$$

and $\pi(j_{n_k,\infty}(s_{n_k,\lambda})) = t_F^1(\chi_{\{\epsilon\}}Z(\lambda,k))$,

where $(t_F^0, t_F^1)$ is the universal Cuntz–Krieger pair for $C^*(F)$.

**Proof.** For a topological graph $E$ we denote by $(t_E^0, t_E^1)$ the universal Cuntz–Krieger pair for $C^*(E)$. The argument of Katsura [27, Proposition 2.9] shows that each regular factor map $m : E \to F$ of topological graphs $E$ and $F$ induces a homomorphism $\mu_m : C^*(F) \to C^*(E)$ such that $\mu_m \circ t_F = t_E \circ m^*$, for $i = 0, 1$.

Let $j_{n_k,\infty}$ be the universal map from $C^*(E(n_k))$ into $\lim_{\rightarrow} C^*(E(n_k))$. Kribs and Solel define regular factor maps $m_{k,k+1} : E(n_{k+1}) \to E(n_k)$ by $m_{k,k+1}^0(\mu) = [\mu]_{n_k}$ for $\mu \in E^{<n_{k+1}}$ and $m_{k,k+1}^1(e,\mu) = (e, [\mu]_{n_k})$ for $e \in E^1$ and $\mu \in s(e)E^{<n_{k+1}}$. For each $k$, write $m_k : E(\infty) \to E(n_k)$ for the induced factor map. In [31, Theorem 6.3] Kribs and Solel invoke [27, Proposition 4.13] to show that there is an isomorphism $\rho : \lim_{\rightarrow} C^*(E(n_k)), j_{k,k+1}) \to C^*(E(\infty))$; it follows from the arguments of [27, Proposition 4.13] that $\rho \circ j_{k,\infty} = \mu_m$. Let $\psi : F \to E(\infty)$ be the inverse of the isomorphism of Lemma [6.0.1] Define $\pi := \mu_{\psi} \circ \rho$. Since $\psi$ is an isomorphism, so is $\mu_{\psi}$, and

$$\pi(j_{n_k,\infty}(p_{n_k,\lambda})) = \mu_{\psi}(\rho(j_{n_k,\infty}(p_{n_k,\lambda}))) = \mu_{\psi} \circ \mu_m(p_{n_k,\lambda}) = \mu_\psi(t_{E(\infty)}(\chi_{\{\epsilon\}}Z(\lambda,k))) = t_F^0(\chi_{\{\epsilon\}}Z(\lambda,k)).$$

A similar calculation gives $\pi(j_{n_k,\infty}(s_{n_k,\lambda})) = t_F^1(\chi_{\{\epsilon\}}Z(\lambda,k))$. \hfill \qed

**Alternative proof of Theorem 4.2.1.** Let $\varphi_{S,P,\psi}$ be the homomorphism induced by the universal property of $C^*(E,\omega)$, and let $\pi : \lim_{\rightarrow} C^*(E(n_k)) \to C^*(F)$ be the isomorphism of Proposition [6.1.1]. The isomorphism of Corollary [3.1.9] induces an isomorphism $\alpha : \lim_{\rightarrow} C^*(E(n_k)) \to C^*(E,\omega)$. Then $(\pi \circ \alpha^{-1})(\rho(\mu,k)) = t_F^0(\chi_{\{\epsilon\}}Z(\mu,k))$ for all $k \in \mathbb{N}, \mu \in E^{<n_k}$. Hence $\varphi_{S,P,\psi} \circ \alpha \circ \pi^{-1}$ is a homomorphism of $C^*(F)$ that carries $t_F^0(\chi_{\{\epsilon\}}Z(\mu,k))$ to $\psi(\mu,k)$. By [31, Lemma 9.1] $E(\infty)$ has no cycles, so Lemma [6.0.1] shows that $F$ has no cycles. So [26, Theorem 5.12] implies that $\varphi_{S,P,\psi} \circ \alpha \circ \pi^{-1}$ is injective if and only if each $\psi(\mu,k) \neq 0$. Since $\alpha$ and $\pi$ are isomorphisms, the result follows. \hfill \qed
6.2 Simplicity

Here we use Katsura’s characterisation of simplicity for $C^*$-algebras of topological graphs [28, Theorem 8.12] to give an alternative proof of Corollary 5.2.2.

Lemma 6.2.1. Let $E$ be a strongly connected finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. Fix $k$ with $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$. For each equivalence class $\Lambda \in E^0 / \sim_{n_k}$, let $X_\Lambda = \bigcup_{\mu \in E^{<n_k}, s(\mu) \in \Lambda} Z(\mu, k)$. The $X_\Lambda$ are mutually disjoint and cover $F^0 = \lim \downarrow E^{<n_k}$. Each $X_\Lambda$ is invariant in the sense of [28, Definition 2.1], and the $X_\Lambda$ are the minimal nonempty closed invariant subsets of $F^0$.

Proof. Take $\Lambda,\Lambda' \in E^0 / \sim_{n_k}$ with $\Lambda \neq \Lambda'$. Since $x \in X_\Lambda$ if and only if $s(x_k) \in \Lambda$, it is clear that $X_\Lambda$ and $X_{\Lambda'}$ are mutually orthogonal.

To see that each $X_\Lambda$ is invariant, take $(e, x) \in F^1$. By Lemma 5.1.2, we have $s(x_k) \in \Lambda$ if and only if $s(r_{n_k}(e, x_k)) \in \Lambda$. So $x \in X_\Lambda$ if and only if $r_F(e, x) = [ex_k]_{n_k} \in X_\Lambda$.

For the final assertion, fix $x = (x_k)_{k=1}^\infty$ and $y = (y_k)_{k=1}^\infty$ in a given $X_\Lambda$. It suffices to show that for every $k \in \mathbb{N}$, there exists $\mu_k \in E^*$ such that $s_F(\mu_k) = x$ and $r_F(\mu_k) \in Z(y_k, k)$. Fix $k$ such that $\gcd(\mathcal{P}_E, n_k) = \gcd(\mathcal{P}_E, \omega)$. Proposition 5.1.3 implies that the component $E(n_k)_{\Lambda}^0$ is strongly connected. So there exists $\lambda \in E(n_k)^*$ such that $s_{n_k}(\lambda) = x_k$ and $r_{n_k}(\lambda) = y_k$. Say $\lambda = (\lambda_1, [\lambda_2 \ldots \lambda_i x_k]_{n_k}) \ldots (\lambda_{i-1}, [\lambda_i x_k]_{n_k})(\lambda_i, x_k)$. Define $\mu_i := (\lambda_i, x) \in F^1$ and inductively let $\mu_j = (\lambda_j, r_F(\mu_{j+1})) \in F^1$ for $1 \leq j \leq i - 1$. Then $\mu = \mu_1 \ldots \mu_i \in F^1$ and $s_F(\mu) = s_F(\mu_i) = x$. By construction, $(\mu_j)_k = (\lambda_j, [\lambda_{j+1} x_k]_{n_k})$ for each $1 \leq j \leq i - 1$, so $r_F(\mu)_k = r_F(\mu_1)_k = r_{n_k}(\lambda_1, [\lambda_2 \ldots \lambda_i x_k]_{n_k}) = r_{n_k}(\lambda) = y_k$, so $r_F(\mu) \in Z(y_k, k)$.

Alternative proof of Corollary 5.2.2. We have $C^*(E, \omega) \cong \bigoplus_{\Lambda \in E^0 / \sim_{n_k}} P_{k, \Lambda} C^*(E, \omega) P_{k, \Lambda}$ by Lemma 5.2.1. We claim that the direct summands are simple. Observe that the isomorphism $C^*(E, \omega) \cong C^*(F)$ determined by Proposition 6.1.1 carries each $P_{k, \Lambda}$ to $t_1^F (\chi_{X_\Lambda})$, where the $X_\Lambda$ are the minimal invariant subsets of $F^0$ described in Lemma 6.2.1. For each $\Lambda$, let $F_\Lambda$ be the topological subgraph of $F$ given by $F_\Lambda^0 = X_\Lambda$ and $F_\Lambda^1 = r_F^{-1}(X_\Lambda)$. Since $F_\Lambda^0$ and $F_\Lambda^1$ are clopen in $F^0$ and $F^1$, there are canonical inclusions $C(F_\Lambda^0) \hookrightarrow C(F^0)$ and $C(F_\Lambda^1) \hookrightarrow C(F^1)$, and it is easy to verify that the universal property of $C^*(F_\Lambda)$ applied to these inclusions gives surjective homomorphisms $\iota_\Lambda : C^*(F_\Lambda) \rightarrow t_1^F (\chi_{X_\Lambda}) C^*(F) t_1^F (\chi_{X_\Lambda})$. Lemma 6.2.1 shows that each $F_\Lambda$ is invariant. By [31, Lemma 9.1] $E(\infty)$ has no loops, so Lemma 6.0.1 shows that $F$ also has no loops, and hence each $F_\Lambda$ has no loops. Hence [28, Theorem 8.12] shows that each $C^*(F_\Lambda)$ is simple. Hence each $P_{k, \Lambda} C^*(E, \omega) P_{k, \Lambda} \cong C^*(F_\Lambda)$ is simple. Hence $C^*(E, \omega)$ is simple if and only if there is exactly one equivalence class $\Lambda$ for $\sim_t$. So the final statement of Lemma 5.1.1 shows that $C^*(E, \omega)$ is simple if and only if $l = 1$. \qed
Chapter 7

K-theory and classification

Supernatural numbers have been used to classify UHF algebras ([18, Theorem 1.12]) and the classical Bunce–Deddens algebras ([4, Theorem 3.7] and [5, Theorem 4]). Kribs showed in [30, Theorem 5.1] that the generalised Bunce–Deddens algebras corresponding to the graph $B_N$ consisting of a single vertex with $N$ loop edges, are classified by their associated supernatural numbers in the sense that $C^*(B_N,\omega) \cong C^*(B_N,\omega')$ if and only if $[\omega] = [\omega']$. The special case $N = 1$ is Bunce and Deddens’ theorem. Kribs and Solel later showed in [31, Theorem 7.5] that the generalised Bunce–Deddens algebras corresponding to the single cycle with $j$ edges, are classified by their associated supernatural numbers; again the special case $j = 1$ is the original result of Bunce and Deddens. Kribs and Solel asked in [31, Remark 7.7] for what class of graphs $E$ a similar classification theorem could be obtained. Here we prove such a theorem can be obtained for the class of generalised Bunce–Deddens algebras corresponding to a given strongly connected finite directed graph $E$ such that 1 is an eigenvalue of the vertex matrix, and the only roots of unity that are eigenvalues are the $P_E$-th roots of unity, where $P_E$ is the period of the graph $E$. The main result of this chapter is the following theorem.

**Theorem 7.0.1.** Fix a strongly connected finite directed graph $E$. Let $P_E$ denote the period of $E$ and suppose that 1 is an eigenvalue of $A_E^1$ and that the only roots of unity that are eigenvalues of $A_E^1$ are $P_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ and $\omega' = (n_k')_{k=1}^\infty$ be multiplicative sequences. Then $C^*(E,\omega) \cong C^*(E,\omega')$ if and only if $[\omega] = [\omega']$.

**Remark 7.0.2.** We prove the forward direction of Theorem 7.0.1 (see Corollary 7.2.3) by studying the torsion-free component of $K_0(C^*(E,\omega))$; we assume that 1 is an eigenvalue of $A_E^1$ to ensure that this is nontrivial. If 1 is not an eigenvalue of $A_E^1$, then $K_0(C^*(E,\omega))$ is purely torsion and another argument (perhaps along the lines of [30, Theorem 5.1]) will be needed. We have not addressed that case in this thesis.
The Perron–Frobenius theorem (see [17, Theorem 8.2.1]) says that if 1 is an eigenvalue of $A_E$, then the $P_E$-th roots of unity are also eigenvalues of $A_E$. The hypothesis that these are the only roots of unity that are eigenvalues of $A_E$ is nontrivial. The nonnegative inverse eigenvalue problem asks which sets of $n$ complex numbers $\lambda_1, \ldots, \lambda_n$ occur as the eigenvalues of some $n \times n$ nonnegative matrix. Deep results of [29] regarding this problem show that it is possible for any collection of roots of unity to appear as eigenvalues of a nonnegative matrix. We thank Mike Boyle for pointing us to [29] and for a helpful email conversation about the spectra of nonnegative integer matrices.

Example 7.0.3. Let $E$ be the following graph.

The adjacency matrix $A_E = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1 and 3, so $E$ satisfies the hypotheses of Theorem 7.0.1.

We begin this chapter by showing that the reverse direction of Theorem 7.0.1 holds for row-finite directed graphs. Our presentation of $C^*\bigl(E,\omega\bigr)$ in Theorem 3.2.4 allows for an elementary proof. The following proposition appears in joint work with my supervisors ([44, Proposition 3.11]).

Proposition 7.0.4. Let $E$ be a row-finite directed graph. Let $\omega = (n_k)_{k=1}^\infty$ and $\omega' = (m_j)_{j=1}^\infty$ be multiplicative sequences. If $n_k \mid m_j(k)$ for all $k \geq 1$, then there is an injective homomorphism $\varphi_{\omega,\omega'} : T(E,\omega) \to T(E,\omega')$ such that

$$\varphi_{\omega,\omega'} \circ i_{n_k,\infty} = i_{m_j(k),\infty} \circ i_{n_k,m_j(k)} \quad \text{for all } k \geq 1 \text{ and any } j(k) \text{ such that } n_k \mid m_j(k). \quad (7.0.1)$$

Moreover, $\varphi_{\omega,\omega'}$ descends to a homomorphism $\tilde{\varphi}_{\omega,\omega'} : C^*(E,\omega) \to C^*(E,\omega')$. If $[\omega] = [\omega']$ then $\varphi_{\omega,\omega'} : T(E,\omega) \to T(E,\omega')$, and $\tilde{\varphi}_{\omega,\omega'} : C^*(E,\omega) \to C^*(E,\omega')$ are isomorphisms.

Proof. Fix natural numbers $j(k)$ such that $n_k \mid m_j(k)$ for all $k$. Then $i_{m_j(k),\infty} \circ i_{n_k,m_j(k)} : T(E,n_k) \to \lim T(E,m_k)$ is a homomorphism for each $k$. Since

$$i_{m_j(k+1),\infty} \circ i_{n_k+1,m_j(k+1)} \circ i_{n_k,m_j(k+1)} = i_{m_j(k+1),\infty} \circ i_{n_k,m_j(k+1)} = i_{m_j(k+1),\infty} \circ i_{m_j(j),m_j(k+1)} \circ i_{n_k,m_j(k)} = i_{m_j(k),\infty} \circ i_{n_k,m_j(k)},$$
the universal property of \( \lim \mathcal{T}(E, n_k) \) gives a homomorphism \( \varphi \) that satisfies (7.0.1). The same argument shows that \( \varphi \) descends to a homomorphism \( \tilde{\varphi} : \lim \mathcal{T}(E, n_k) \to \lim \mathcal{T}(E, m_l) \). Now suppose that \( \omega' | \omega \) as well. The preceding paragraph gives a homomorphism \( \gamma : \mathcal{T}(E, \omega') \to \mathcal{T}(E, \omega) \) such that \( \gamma \circ i_{m_j, \infty} = i_{n_{k(j)} \infty} \circ i_{m_j, n_{k(j)}} \) for all \( j \geq 1 \) and any \( k(j) \) such that \( m_j | n_{k(j)} \), and which descends to \( \tilde{\gamma} : C^*(E, \omega') \to C^*(E, \omega) \).

For each \( k \geq 1 \), we calculate

\[
\gamma \circ \varphi \circ i_{n_k, \infty} = \gamma \circ i_{m_j(k), \infty} \circ i_{n_k, m_j(k)} = i_{n_{k(j)} \infty} \circ i_{m_j(k), n_{k(j)}} \circ i_{n_k, m_j(k)} = i_{m_j(k), \infty} \circ i_{n_k, m_j(k)} = i_{n_k, \infty}.
\]

Similarly, for each \( j \geq 1 \), we have

\[
\varphi \circ \gamma \circ i_{m_j, \infty} = \varphi \circ i_{n_{k(j)} \infty} \circ i_{m_j, n_{k(j)}} = i_{m_j(k), \infty} \circ i_{n_{k(j)}, m_j(k)} \circ i_{m_j, n_{k(j)}} = i_{n_{k(j)}, \infty} \circ i_{m_j, n_{k(j)}} = i_{m_j, \infty}.
\]

Therefore \( \gamma \circ \phi \) is the identity map on each \( i_{n_k, \infty}(\mathcal{T}(E, n_k)) \) and \( \phi \circ \gamma \) is the identity on each \( i_{m_j, \infty}(\mathcal{T}(E, m_j)) \), so continuity shows that \( \phi \) and \( \gamma \) are mutually inverse. For the final assertion, let \( q : \mathcal{T}(E, n) \to C^*(E, n) \) be the canonical quotient map. Since \( \tilde{\gamma} \circ \tilde{\varphi} \circ q = \tilde{\gamma} \circ q \circ \varphi = q \circ \gamma \circ \varphi = q \circ \text{id}_{\mathcal{T}(E, \omega)} = q \), we have that \( \tilde{\gamma} \circ \tilde{\varphi} = \text{id}_{C^*(E, \omega)} \), and likewise for \( \tilde{\varphi} \circ \tilde{\gamma} \).

\[\Box\]

### 7.1 Computing \( K_0(C^*(E, \omega)) / \text{tor}(E, \omega) \)

In this section we calculate the torsion-free component of \( K_0(C^*(E, \omega)) \). We use this group to recover the supernatural number \([\omega]\). In order to state our main theorem, we need the following lemma.

**Lemma 7.1.1.** Let \( A \) be a free abelian group and let \( \omega = (n_k)_{k=1}^\infty \) be a multiplicative sequence. Define an equivalence relation \( \sim \) on \( A \times \mathbb{N} \), by \((a, j) \sim (a', j')\) if

\[
\frac{\max\{n_j, n_{j'}\}}{n_j} a = \frac{\max\{n_j, n_{j'}\}}{n_{j'}} a',
\]

and define

\[
A[\frac{1}{\omega}] := \{(a, j) : a \in A, j \geq 1\} / \sim.
\]
Then $A[\frac{1}{\omega}]$ is a torsion-free abelian group under the operation

$$[(a,j)] + [(a',j')] = \begin{cases} \left[\left(\frac{n_j}{n_j'}a + a',j'\right)\right] & \text{if } j' \geq j \\ \left[\left(a + \frac{n_j}{n_j'}a',j\right)\right] & \text{if } j \geq j'. \end{cases}$$

Moreover, rank $A[\frac{1}{\omega}] = \text{rank } A$.

**Proof.** Closure, associativity, and commutativity follow easily since $A$ is abelian. Let 0 be the identity element of $A$. Then $[(0,i)] + [(a,i)] = [(0 + a,i)] = [(a,i)]$ so $[(0,i)]$ is an identity for $A[\frac{1}{\omega}]$. Fix $a \in A$ and let $-a$ be the inverse. Then $[(a,i)] + [(-a,i)] = [(a - a,i)] = [(0,i)]$, so $[(-a,i)]$ is an inverse for $[(a,i)]$.

If $k \cdot [(a,i)] = [(0,i)]$, then $[(k \cdot a,i)] = [(0,i)]$, so $k \cdot a = 0$ forcing $a = 0$ since $A$ is free abelian. To see that rank $A[\frac{1}{\omega}] = \text{rank } A$, let $\{a_\alpha\}$ be a maximal linearly independent subset of $A$. Suppose $[(0,i)] = \sum_\alpha c_\alpha [(a_\alpha,i)]$ for $c_\alpha \in \mathbb{N}$ but all but finitely many nonzero. Then $[(0,i)] = \sum_\alpha [(c_\alpha \cdot a_\alpha,i)] = [(\sum_\alpha c_\alpha \cdot a_\alpha,i)]$, so $0 = \sum_\alpha c_\alpha \cdot a_\alpha$, and since $\{a_i\}$ is linearly independent, $c_\alpha = 0$ for all $\alpha$. Hence $\{(a_\alpha,i)\}$ is a linearly independent subgroup of $A[\frac{1}{\omega}]$. To see that it is maximal, take $c \in \mathbb{N}$ and $b \in A$. Then $\sum_\alpha c_\alpha [(a_\alpha,i)] + c[(b,i)] = [(\sum_\alpha c_\alpha \cdot a_\alpha + c \cdot b,i)] = [(0,i)]$, by the maximality of $\{a_\alpha\}$. \qed

**Remark 7.1.2.** We have $A[\frac{1}{\omega}] \cong A \otimes \mathbb{Z}[\frac{1}{\omega}]$ via the map $[a,j] \rightarrow a \otimes \frac{1}{n_j}$. We will regard the elements $[(a,j)]$ as formal fractions and write $a/n_j$ for $[(a,j)]$.

We can now state the main theorem of this section about the torsion-free component of $K_0(C^*(E,\omega))$. Recall that the torsion subgroup of an abelian group $A$ consists of the nonzero elements of $A$ which have finite order.

**Theorem 7.1.3.** Let $E$ be a strongly connected finite directed graph. Let $P_E$ denote the period of $E$, and let $l = \gcd(P_E,\omega)$. Suppose 1 is an eigenvalue of $A_E^l$ and that the only roots of unity that are eigenvalues of $A_E^l$ are the $P_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. Let $\text{tor}_E$ denote the torsion subgroup of $K_0(C^*(E))$, and $\text{tor}_{(E,\omega)}$ the torsion subgroup of $K_0(C^*(E,\omega))$. There is an isomorphism

$$\Psi : K_0(C^*(E,\omega))/\text{tor}_{(E,\omega)} \rightarrow \bigoplus_{i=1}^l \left(K_0(C^*(E))/\text{tor}_E\right)[\frac{1}{\omega}]$$

satisfying

$$\Psi([1_{C^*(E,\omega)}]_0 + \text{tor}_{E,\omega}) = ([1_{C^*(E)}]_0 + \text{tor}_E, \ldots, [1_{C^*(E)}]_0 + \text{tor}_E)/l.$$
CHAPTER 7. K-THEORY AND CLASSIFICATION

To prove Theorem 7.1.3 we begin by studying $K_0(C^*(E(n))) \cong \text{coker}(1 - A^t_{E(n)})$ for $n \geq 1$. Let $\{\delta_v : v \in E^O\}$ be the generators of $Z^{E^0}$ and let $\{\delta_{\mu,n} : \mu \in E^{<n}\}$ be the generators of $Z^{E^{<n}}$.

**Lemma 7.1.4.** Let $E$ be a row-finite directed graph with no sources and let $n \geq 1$. Then

$$A^t_{E(n)} \delta_{\mu,n} = \begin{cases} \delta_{\mu_2,...,\mu_{|\mu|},n} & \text{if } \mu \in E^{<n} \setminus E^0 \\ \sum_{\lambda \in \mu E^n} \delta_{\lambda_2,...,\lambda_n,n} & \text{if } \mu \in E^0. \end{cases} \quad (7.1.1)$$

Moreover, $\delta_{\mu,n} - \delta_{s(\mu),n} \in \text{Im}(1 - A^t_{E(n)})$ for each $\mu \in E^{<n}$.

**Proof.** Let $\mu \in E^n \setminus E^0$. We calculate

$$A^t_{E(n)} \delta_{\mu,n} = \sum_{\nu \in E^{<n}} A^t_{E(n)}(\nu,\mu) \delta_{\nu,n} = \sum_{\nu \in E^{<n}} |\mu E(n)|^1 \nu | \delta_{\nu,n}$$

$$= \sum_{\nu \in E^{<n} \nu \in E^1 r(\nu), |\nu\rangle = \mu} \delta_{\nu,n} = \delta_{\mu_2,...,\mu_{|\mu|},n}.$$

Let $\mu \in E^0$. Then

$$A^t_{E(n)} \delta_{\mu,n} = \sum_{\nu \in E^{<n}} A^t_{E(n)}(\nu,\mu) \delta_{\nu,n} = \sum_{\nu \in E^{<n}} |\mu E(n)|^1 \nu | \delta_{\nu,n} = \sum_{\lambda \in \mu E^n} \delta_{\lambda_2,...,\lambda_n,n}.$$

The final statement clearly holds when $\mu \in E^0$, so let $\mu \in E^{<n} \setminus E^0$. Repeated applications of the first case of (7.1.1) give $(A^{|\mu|} E(n)^t) \delta_{\mu,n} = \delta_{s(\mu),n}$, so $\delta_{\mu,n} - \delta_{s(\mu),n} = (1 - A^t_{E(n)}) \delta_{\mu,n} \in \text{Im}(1 - A^t_{E(n)})$. \hfill \square

**Lemma 7.1.5.** Let $E$ be a row-finite directed graph with no sources and let $n \geq 1$. There is an isomorphism $\psi_n : \text{coker}(1 - A^t_{E(n)}) \to \text{coker}(1 - A^t_{E(n)})$ satisfying $\psi_n(\delta_v + \text{Im}(1 - A^n_{E})^t) = \delta_v + \text{Im}(1 - A^t_{E(n)})$ for $v \in E^O$.

**Proof.** Define a map $\psi_n : Z^{E^0} \to \text{coker}(1 - A^t_{E(n)})$ by $\psi_n(\delta_v) = \delta_v + \text{Im}(1 - A^t_{E(n)})$. We show that $\psi_n(\text{Im}(1 - A^t_{E(n)})) \subseteq \text{Im}(1 - A^t_{E(n)})$. Let $v \in E^0$. Repeated applications of (7.1.1) give $(A^n_{E(n)})^t \delta_v,n = \sum_{\lambda \in v E^n} \delta_{s(\lambda),n} = \sum_{w \in E^0} |v E^n| w | \delta_{w,n} = \sum_{w \in E^0} (A^n_{E})^t(w, v) \delta_w = \psi_n((A^n_{E})^t \delta_v)$, so

$$\psi_n((1 - A^n_{E})^t \delta_v) = (1 - A^n_{E(n)})^t \delta_v,n + \text{Im}(1 - A^n_{E})^t = \text{Im}(1 - A^n_{E(n)})^t \subseteq \text{Im}(1 - A^t_{E(n)}).$$

Thus $\psi_n$ descends to a homomorphism $\text{coker}(1 - A^t_{E}) \to \text{coker}(1 - A^t_{E(n)})$, which we also label by $\psi_n$, satisfying $\psi_n(\delta_v + \text{Im}(1 - A^n_{E})^t) = \delta_v + \text{Im}(1 - A^t_{E(n)})$ for $v \in E^O$. 


Define a map \( \varphi_n : \mathbb{Z}^{E < n} \to \text{coker}(1 - A^n_E)^t \) by \( \varphi_n(\delta_{\mu,n}) = \delta_{s(\mu)} + \text{Im}(1 - A^n_E)^t \). We show that \( \varphi_n(\text{Im}(1 - A^n_{E(n)})) = \text{Im}(1 - A^n_E)^t \). Take \( (1 - A^n_{E(n)})\delta_{\mu,n} \in \text{Im}(1 - A^n_{E(n)}) \). If \( \mu \in E^{<n} \setminus E^0 \), then

\[
\varphi_n((1 - A^n_{E(n)})\delta_{\mu,n}) = \varphi_n(\delta_{\mu,n} - \delta_{\mu_2...\mu_{|\mu|},n}) = \delta_{s(\mu)} - \delta_{s(\mu)} + \text{Im}(1 - A^n_E)^t = \text{Im}(1 - A^n_E)^t,
\]

by the first case of (7.1.1). If \( \mu \in E^0 \), then applying the second case of (7.1.1) at the first equality, we have

\[
\varphi_n(\delta_{v,n} - A^n_{E(n)}\delta_{v,n}) = \varphi_n(\delta_{v,n} - \sum_{\lambda \in vE^n} \delta_{\lambda_2...\lambda_{v,n}})
\]

\[
= \delta_v - \sum_{\lambda \in vE^n} \delta_{s(\lambda)} + \text{Im}(1 - A^n_E)^t
\]

\[
= \delta_v - \sum_{w \in E^0} |vE^nw| \delta_w + \text{Im}(1 - A^n_E)^t
\]

\[
= \delta_v - \sum_{w \in E^0} (A^n_E)^t(w, v) \delta_w + \text{Im}(1 - A^n_E)^t
\]

\[
= (1 - A^n_E)^t \delta_v + \text{Im}(1 - A^n_E)^t
\]

\[
= \text{Im}(1 - A^n_E)^t.
\]

Thus \( \varphi_n \) descends to a homomorphism \( \text{coker}(1 - A^n_{E(n)}) \to \text{coker}(1 - A^n_E)^t \), which we also label \( \varphi_n \).

To show that \( \psi_n \) is an isomorphism, we show that \( \psi_n \) and \( \varphi_n \) are mutually inverse. Let \( \mu \in E^{<n} \). Then

\[
\psi_n(\varphi_n(\delta_{\mu,n} + \text{Im}(1 - A^n_{E(n)}))) = \varphi_n(\delta_{s(\mu)} + \text{Im}(1 - A^n_E)^t)
\]

\[
= \delta_{s(\mu),n} + \text{Im}(1 - A^n_{E(n)}) = \delta_{\mu,n} + \text{Im}(1 - A^n_{E(n)})
\]

by Lemma 7.1.4, so \( \psi_n \circ \varphi_n \) is the identity on \( \text{coker}(1 - A^n_{E(n)}) \). Now, let \( v \in E^0 \). Then

\[
\varphi_n(\psi_n(\delta_v + \text{Im}(1 - A^n_E)^t)) = \varphi_n(\delta_{v,n} + \text{Im}(1 - A^n_{E(n)})) = \delta_v + \text{Im}(1 - A^n_E)^t,
\]

so \( \varphi_n \circ \psi_n \) is the identity on \( \text{coker}(1 - A^n_E)^t \). \( \square \)

**Remark 7.1.6.** For each \( n \geq 1 \), let \( \sigma_n : \text{coker}(1 - A^n_{E(n)}) \to K_0(C^*(E(n))) \) be the isomorphism of [11] Theorem 7.16]. Looking into the proof of [11] Theorem 7.1 shows that this isomorphism is given by \( \sigma_n(\delta_{\mu,n} + \text{Im}(1 - A^n_{E(n)})) = [p_{\mu,n}]_0 \) for \( \mu \in E^{<n} \). So \( \sigma_n \circ \psi_n : \text{coker}(1 - A^n_E)^t \to K_0(C^*(E(n))) \) is an isomorphism satisfying \((\sigma_n \circ \psi_n)(\delta_v + \text{Im}(1 - A^n_E)^t) = [p_{v,n}]_0 \) for \( v \in E^0 \). By Lemma 7.1.4 we have \([p_{\mu,n}]_0 - [p_{v,n}]_0 = 0\).
\[ \sigma_n(\delta_{\mu,n} - \delta_{v,n} + \text{Im}(1 - A_{E(n)}^t)) = 0 \] for any \( \mu \in E^{<n}v \). So \( (\sigma_n \circ \psi_n)(\delta_v + \text{Im}(1 - A_{E}^n)^t) = [p_{\mu,n}]_0 \) for any \( \mu \in E^{<n}v \).

**Lemma 7.1.7.** Let \( E \) be a row-finite directed graph with no sources and let \( n, m \geq 1 \). The following diagram commutes.

\[
\begin{array}{ccc}
\mathbb{Z}^{E^0} & \xrightarrow{\sum_{i=0}^{m-1}(A_{E}^n)^t} & \mathbb{Z}^{E^0} \\
\downarrow & & \downarrow \\
\text{coker}(1 - A_{E}^n)^t & \xrightarrow{\sigma_n \circ \psi_n} & \text{coker}(1 - A_{E}^m)^t \\
\downarrow & & \downarrow \\
K_0(C^*(E(n))) & \xrightarrow{K_0(j_{n,mn})} & K_0(C^*(E(mn)))
\end{array}
\]

**Proof.** Let \( \eta_n := \sigma_n \circ \psi_n \), and fix \( v \in E^0 \). Then

\[
K_0(j_{n,mn})(\eta_n)(\delta_v + \text{Im}(1 - A_{E}^n)^t)) = K_0(j_{n,mn})([p_{v,n}]_0)
\]

\[
= \sum_{\mu \in vE^{<mn}, \mu \in nN} [p_{\mu,mn}]_0 = \sum_{i=0}^{m-1} \sum_{\mu \in vE^{in}} [p_{\mu,mn}]_0.
\]

Now, by Remark 7.1.6 we have

\[
(\eta_{mn})\left( \sum_{i=0}^{m-1}(A_{E}^m)^t \delta_v + \text{Im}(1 - A_{E}^m)^t \right) = \eta_{mn}\left( \sum_{i=0}^{m-1} \sum_{\mu \in vE^{in}} \delta_{s(\mu)} + \text{Im}(1 - A_{E}^m)^t \right)
\]

\[
= \sum_{i=0}^{m-1} \sum_{\mu \in vE^{in}} (\eta_{mn})(\delta_{s(\mu)} + \text{Im}(1 - A_{E}^m)^t)
\]

\[
= \sum_{i=0}^{m-1} \sum_{\mu \in vE^{in}} [p_{s(\mu),mn}]_0 = \sum_{i=0}^{m-1} \sum_{\mu \in vE^{in}} [p_{\mu,mn}]_0. \quad \square
\]

**Corollary 7.1.8.** Let \( E \) be a row-finite directed graph with no sources and let \( n, m \geq 1 \). There exists a homomorphism \( \phi_{n,mn} : \text{coker}(1 - A_{E}^n)^t \rightarrow \text{coker}(1 - A_{E}^m)^t \) satisfying

\[
\phi_{n,mn}(\delta_v + \text{Im}(1 - A_{E}^n)^t) = \sum_{\mu \in vE^{<mn}, \mu \in nN} \delta_{s(\mu),mn} + \text{Im}(1 - A_{E}^m)^t
\]

for \( v \in E^0 \).

**Proof.** Let \( \eta_n := \sigma_n \circ \psi_n \), and define \( \phi_{n,mn} : \text{coker}(1 - A_{E}^n)^t \rightarrow \text{coker}(1 - A_{E}^m)^t \) by
\( \phi_{n,mn} := \eta_{mn}^{-1} \circ K_0(j_{n,mn}) \circ \eta_n \). Let \( v \in E^0 \). By Remark 7.1.6 we have

\[
\phi_{n,mn}(\delta_v + \text{Im}(1 - A^n_E)^t) = (\eta_{mn}^{-1} \circ K_0(j_{n,mn}) \circ \eta_n)(\delta_v + \text{Im}(1 - A^n_E)^t)
\]

\[
= (\eta_{mn}^{-1} \circ K_0(j_{n,mn}))[p_{s(\mu),n}]_0
\]

\[
= \eta_{mn}^{-1}\left( \sum_{\mu \in E^{<mn}, |\mu| \in \mathbb{N}} [p_{\mu,mn}]_0 \right)
\]

\[
= \eta_{mn}^{-1}\left( \sum_{\mu \in E^{<mn}, |\mu| \in \mathbb{N}} [p_{s(\mu),mn}]_0 \right)
\]

\[
= \sum_{\mu \in E^{<mn}, |\mu| \in \mathbb{N}} \delta_{s(\mu),mn} + \text{Im}(1 - A^n_{mn})^t.
\]

\[\square\]

We now look at direct limits of quotients of abelian groups by their torsion subgroups. We seek to apply the following result to the sequence \((\text{coker}(1 - A^n_E)^t, \phi_{n,n+1})_{n=1}^{\infty}\).

**Lemma 7.1.9.** Let \((G_k, \phi_{k,k+1})\) be a directed system of abelian groups. Let \(\text{tor}_k := \text{tor}(G_k)\) for each \(k \geq 1\), and \(\text{tor}_\infty := \text{tor}(\lim \downarrow G_k)\). For each \(k\) there exists a homomorphism \(\tilde{\phi}_{k,k+1} : G_k/\text{tor}_k \to G_{k+1}/\text{tor}_{k+1}\) such that \(\tilde{\phi}_{k,k+1}(g + \text{tor}_k) = \phi_{k,k+1}(g) + \text{tor}_{k+1}\). Moreover, there is an isomorphism

\[
\tilde{q}_\infty : \lim \downarrow (G_k, \phi_{k,k+1})/\text{tor}_\infty \to \lim \downarrow (G_k/\text{tor}_k, \tilde{\phi}_{k,k+1})
\]

such that \(\tilde{q}_\infty(\phi_{k,\infty}(g) + \text{tor}_\infty) = \tilde{\phi}_{k,\infty}(g + \text{tor}_k)\).

**Proof.** Write \(Q_k := G_k/\text{tor}_k\). For each \(k \geq 1\), let \(q_k : G_k \to Q_k\) be the quotient map. Let \(r \in \text{tor}_k\). Then there exists \(n \geq 1\) such that \(nr = 0\), and then \(n\phi_{k,k+1}(r) = \phi_{k,k+1}(nr) = 0\). So \(\phi_{k,k+1}(\text{tor}_k) \subseteq \text{tor}_{k+1}\), and hence \(q_{k+1} \circ \phi_{k,k+1}\) descends to a homomorphism \(\tilde{\phi}_{k,k+1} : Q_k \to Q_{k+1}\) such that \(\tilde{\phi}_{k,k+1}(g + \text{tor}_k) = \phi_{k,k+1}(g) + \text{tor}_{k+1}\) for all \(g \in G_k\). So

\[
(\tilde{\phi}_{k+1,\infty} \circ q_{k+1}) \circ \phi_{k,k+1} = (\tilde{\phi}_{k+1,\infty} \circ \phi_{k,k+1}) \circ q_k = \tilde{\phi}_{k,\infty} \circ q_k
\]

for all \(k \geq 1\). Therefore the universal property of \(\lim \downarrow (G_k, \phi_k)\) gives a homomorphism \(q_\infty : \lim \downarrow (G_k, \phi_{k,k+1}) \to \lim \downarrow (Q_k, \tilde{\phi}_{k,k+1})\) satisfying \(q_\infty \circ \phi_{k,\infty} = \tilde{\phi}_{k,\infty} \circ q_k\).

We show that \(q_\infty\) descends to a homomorphism satisfying the desired formula. Let \(p \in \text{tor}_\infty\). Then there exists \(r \in G_k\) and \(n \geq 1\) such that \(0 = np = n\phi_{k,\infty}(r) = \phi_{k,\infty}(nr)\). By [45, Proposition 6.2.5(ii)] we have \(\ker \phi_{k,\infty} = \bigcup_{m \geq 0} \ker \phi_{k,k+m}\), so there exists \(m \geq 0\) such that \(0 = \phi_{k,k+m}(nr) = n\phi_{k,k+m}(r)\), giving \(\phi_{k,k+m}(r) \in \text{tor}_{k+m}\). Therefore \(q_\infty(p) = q_\infty(\phi_{k,\infty}(r)) = q_\infty(\phi_{k+m,\infty}(\phi_{k,k+m}(r))) = \tilde{\phi}_{k+m,\infty}(\phi_{k+m}(\phi_{k,k+m}(r))) = 0\). So \(q_\infty(\text{tor}_\infty) \subseteq \{0\}\), and hence \(q_\infty\) descends to a homomorphism \(q_\infty : \lim \downarrow (G_k, \phi_{k,k+1})/\text{tor}_\infty \to\)
Lemma 7.1.10. For each \( \phi \) such that \( \gcd(\phi, k) \in \mathbb{Z}^+ \), we find that \( \phi \) is a homomorphism. Therefore As in the first paragraph, we find that \( \phi \) descends to a homomorphism \( \psi : Q_k \to \lim\limits_{\rightarrow} (G_k, \phi_{k, k+1})/\text{tor} \) satisfying \( \psi_k(g + \text{tor}_k) = \phi_k(g) + \text{tor}_k \) for all \( g \in G_k \).

It remains to show that \( \psi \) is an isomorphism. We do this by finding an inverse. Therefore \( \phi \) descends to a homomorphism \( \psi_k : Q_k \to \lim\limits_{\rightarrow} (G_k, \phi_{k, k+1})/\text{tor}_k \) satisfying \( \psi_k(g + \text{tor}_k) = \phi_k(g) + \text{tor}_k \) for all \( g \in G_k \). We have

\[
\psi_{k+1}(\phi_{k+1}(g + \text{tor}_k)) = \psi_{k+1}(\phi_{k+1}(g) + \text{tor}_{k+1}) = \phi_{k+1}(\phi_{k, k+1}(g)) + \text{tor}_k
\]

So \( \psi_{k+1} \circ \phi_{k+1} = \psi_k \), and hence the universal property of \( \lim\limits_{\rightarrow} (Q_k, \phi_k) \) gives a homomorphism \( \psi : \lim\limits_{\rightarrow} (Q_k, \phi_k) \to \lim\limits_{\rightarrow} (G_k, \phi_{k, k+1})/\text{tor}_k \) satisfying \( \psi(\phi_k(g + \text{tor}_k)) = \phi_k(g) + \text{tor}_k \) for all \( g \in G_k \).

We check that \( \psi \) is an inverse for \( \psi \). Let \( g \in G_k \). Then

\[
\psi(\phi_k(g + \text{tor}_k)) = \psi(\phi_k(g) + \text{tor}_k) = \phi_k(g) + \text{tor}_k.
\]

We also have

\[
\psi(\psi(\phi_k(g) + \text{tor}_k)) = \psi(\phi_k(g)) = \phi_k(g) + \text{tor}_k.
\]

So \( \psi \) is the identity on \( \phi_k(G_k) \) and \( \psi \circ \phi \) is the identity on \( \phi_k(G_k) \) for all \( g \in G_k \).

\( \square \)

In order to apply Lemma 7.1.9 to the sequence \( (\text{coker}(1 - A_{E^l})^t, \phi_{n_k, n_{k+1}})_{k=1}^\infty \), we analyse the invertibility of the \( |E^n| \times |E^n| \) matrix \( \sum_{i=0}^{(n_k/l)-1}(A_{E^l})^t \), where \( l = \gcd(P_E, \omega) \) and \( k \) is such that \( \gcd(P_E, n_k) = l \).

Lemma 7.1.10. For each \( n \geq 1 \), let \( R_n \) be the polynomial over \( \mathbb{C} \) given by \( R_n(x) = \sum_{i=0}^{n-1} x^i \). The roots of \( R_n \) are the \( n \)-th roots of unity excluding 1.

Proof. We have \( (1 - x)R_n(x) = 1 - x^n \), so the roots of \( (1 - x)R_n \) are the \( n \)-th roots of unity. The only root of \( 1 - x \) is 1, so every \( n \)th root of unity other than 1 is itself a root of \( R_n \). Since the degree of \( R_n \) is \( n - 1 \), these are all the roots of \( R_n \). \( \square \)
\textbf{Lemma 7.1.11.} Let $E$ be a strongly connected finite directed graph, let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence, and let $l = \gcd(P_E, \omega)$. Then $P_E/l$ and $n_k/l$ are coprime for all $k$ such that $\gcd(P_E, n_k) = l$. Hence, if the only roots of unity that are eigenvalues of $A_E^l$ are the $P_E$-th roots of unity, then $0 \not\in \sigma(R_{n_k/l}(A_E^l)^t)$ for $k$ such that $\gcd(P_E, n_k) = l$.

\textit{Proof.} Suppose for contradiction that $k \geq K$ and that $P_E/l$ is not coprime to $n_k/l$. Say $p \neq 1$ satisfies $p|(P_E/l)$ and $p|(n_k/l)$. Then $pl|P_E$ and $pl|n_k$. This implies that $pl \leq l$, which is a contradiction.

For the second statement, we have

$$\sigma((A_E^l)^t) \cap \mathbb{T} = \{e^{(2\pi ij/P_E)l} : j \in \mathbb{N}\} = \{e^{2\pi ij/(P_E/l)} : j \in \mathbb{N}\},$$

by the spectral mapping theorem. By Lemma \ref{7.1.10} the roots of $R_{n_k/l}$ are the $n_k/l$-th roots of unity. Since $\gcd(P_E/l, n_k/l) = 1$, we have that $e^{2\pi ji/(n_k/l)} \not\in \sigma((A_E^l)^t)$ for any $j \geq 1$. So $0 \not\in \sigma(R_{n_k/l}(A_E^l)^t)$. \hfill $\square$

For $n \geq 1$, the torsion subgroup of $\coker(1 - A_E^n)^t$ is

$$\{a + \text{Im}(1 - A_E^n)^t : a \in \mathbb{Z}E^0, ma \in \text{Im}(1 - A_E^n)^t \text{ for some } m \geq 1\}.$$  

Define

$$T_n := \{a \in \mathbb{Z}E^0 : ma \in \text{Im}(1 - A_E^n)^t \text{ for some } m \geq 1\}.$$  

So $T_n = q_n^{-1}(\text{tor}_n)$ where $q_n : \mathbb{Z}E^0 \rightarrow \coker(1 - A_E^n)^t$ is the quotient map.

\textbf{Proposition 7.1.12.} Let $E$ be a strongly connected finite directed graph. Suppose 1 is an eigenvalue of $A_E^t$ and that the only roots of unity that are eigenvalues of $A_E^t$ are the $P_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence and let $l := \gcd(P_E, \omega)$. Then $T_{n_k} = T_l$ for all $k$ such that $\gcd(P_E, n_k) = l$.

\textit{Proof.} Fix $k$ such that $\gcd(P_E, n_k) = l$. Let $C := \sum_{i=0}^{n_k/l-1} (A_E^l)^t$. We have

$$(1 - A_E^n)^t = (1 - A_E^l)^t \left( \sum_{i=0}^{n_k/l-1} (A_E^l)^t \right) = (1 - A_E^l)^t C. \quad (7.1.2)$$

So $\text{Im}(1 - A_E^n)^t \subseteq \text{Im}(1 - A_E^l)^t$. Now take $x \in T_{n_k}$. Then there exists $m \geq 1$ such that $mx \in \text{Im}(1 - A_E^n)^t \subseteq \text{Im}(1 - A_E^l)^t$. Hence $T_{n_k} \subseteq T_l$.

For the reverse inclusion, take $x \in T_l$. Then $mx = (1 - A_E^l)^ty$, for some $m \geq 1$ and
$y \in \mathbb{Z}^{E^0}$. Equation \((7.1.2)\) gives
\[
(m \det C)x = (\det C)(1 - A_E^l)^ty = (1 - A_E^l)^tC(\det C)C^{-1}y = (1 - A^{n_k}_E)^t(\det C)C^{-1}y \in \Im(1 - A^{n_k}_E)^t.
\]

By Lemma \(7.1.11\) $\det C \neq 0$, so $T_l \subseteq T_{n_k}$. \(\Box\)

**Remark 7.1.13.** If we could compute $\det C$, we could compute $\det(1 - A^{n_k}_E)^t$. Then (when 1 is not an eigenvalue), we could calculate $|K_0(C^*(E, n_k))|$ and try to use Kribs’ argument for \(30\) Theorem 5.1 to prove Theorem \(7.0.1\) for the generalised Bunce–Deddens algebras constructed from a finite strongly connected graph whose vertex matrix does not have eigenvalue 1.

**Lemma 7.1.14.** Let $E$ be a strongly connected finite directed graph with no sources. Suppose 1 is an eigenvalue of $A_E^l$ and that the only roots of unity that are eigenvalues of $A_E^l$ are the $P_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, and let $l := \gcd(P_E, \omega)$. For each $k$ such that $\gcd(P_E, n_k) = l$, there is an isomorphism $\tau : \coker(1 - A^{n_k}_E)^t/\text{tor}_{n_k} \to \mathbb{Z}^{E^0}/T_l$ satisfying
\[
\tau(a + \Im(1 - A^{n_k}_E)^t + \text{tor}_{n_k}) = a + T_l
\]
for $a \in \mathbb{Z}^{E^0}$.

**Proof.** To see that the formula \((7.1.3)\) is well-defined, suppose $(a + \Im(1 - A^{n_k}_E)^t) + \text{tor}_{n_k} = (b + \Im(1 - A^{n_k}_E)^t) + \text{tor}_{n_k}$, where $a, b \in \mathbb{Z}^{E^0}$. Then $a + \Im(1 - A^{n_k}_E)^t = b + \Im(1 - A^{n_k}_E)^t + t$, where $t \in \text{tor}_{n_k}$, that is, $t = c + \Im(1 - A^{n_k}_E)^t$ for some $c \in T_{n_k}$. Then $a - b - c \in \Im(1 - A^{n_k}_E)^t \subseteq \Im(1 - A^{n_k}_E)^t \subseteq T_l$. By Proposition \(7.1.12\) $c \in T_l$, so $a - b \in T_l$. So there is a map $\tau$ satisfying \((7.1.3)\).

The map $\tau$ is clearly a surjective group homomorphism. To see that it is injective, suppose $a + T_l = b + T_l$ for $a, b \in \mathbb{Z}^{E^0}$. We have $a = b + c$, for some $c \in T_l$, and hence $a + \Im(1 - A^{n_k}_E)^t = b + c + \Im(1 - A^{n_k}_E)^t$. So $a + \Im(1 - A^{n_k}_E)^t = b + \Im(1 - A^{n_k}_E)^t + c + \Im(1 - A^{n_k}_E)^t$. By Proposition \(7.1.12\) $c \in T_{n_k}$. Therefore $a + \Im(1 - A^{n_k}_E)^t + \text{tor}_{n_k} = b + \Im(1 - A^{n_k}_E)^t + \text{tor}_{n_k}$. \(\Box\)

**Corollary 7.1.15.** Let $E$ be a strongly connected finite directed graph. Suppose that 1 is an eigenvalue of $A_E^l$ and that the only roots of unity that are eigenvalues of $A_E^l$ are the $P_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence, let $l := \gcd(P_E, \omega)$. For each $k$ such that $\gcd(P_E, n_k) = l$, there is an isomorphism $\theta_{n_k} : \coker(1 - A^{n_k}_E)^t/\text{tor}_{n_k} \to \coker(1 - A^l_E)^t/\text{tor}_l$ given by $\theta_{n_k}((a + \Im(1 - A^{n_k}_E)^t) + \text{tor}_{n_k}) = (a + \Im(1 - A^l_E)^t) + \text{tor}_l$ for $a \in \mathbb{Z}^{E^0}$.\(\Box\)
Proof. Fix \( k \) such that \( \gcd(P_E, n_k) = l \). The previous Lemma gives an isomorphism \( \text{coker}(1 - A_E^n)^t / \text{tor}_n \to \mathbb{Z}^{E_0} / T_l \) satisfying \( (a + \text{Im}(1 - A_E^n)^t) + \text{tor}_n \mapsto a + T_l \), where \( a \in \mathbb{Z}^{E_0} \). The result follows since \( \mathbb{Z}^{E_0} / T_l \) is isomorphic to \( \text{coker}(1 - A_E^n)^t / \text{tor} \) via \( a + T_l \mapsto a + \text{Im}(1 - A_E^n)^t + \text{tor} \). We take \( \theta_{n_k} \) to be the composition of these isomorphisms. \( \square \)

We give another description of the torsion-free abelian group \( A \left[ \frac{1}{\omega} \right] \) of Lemma \( 7.1.1 \)

Lemma 7.1.16. Let \( A \) be a free abelian group and let \( \omega = (n_k)_{k=1}^{\infty} \) be a multiplicative sequence, and let \( m_k := n_{k+1}/n_k \) for all \( k \geq 1 \). Define maps \( M_k : A \to A \) by \( M_k(a) = m_k \cdot a \), and let \( M_k, \omega \) be the natural map \( A \to \lim_{\to} \omega \). There is an isomorphism \( \phi : \lim_{\to} (A, M_k) \cong A \left[ \frac{1}{\omega} \right] \) satisfying \( \phi(M_k, \omega)(a) = a/n_k \) for each \( k \geq 1 \) and \( a \in A \).

Proof. Fix \( k \geq 1 \). Define \( j_k, \omega : A \to A \left[ \frac{1}{\omega} \right] \) by \( j_k, \omega(a) = a/n_k \) for \( a \in \mathbb{Z} \). This \( j_k, \omega \) is a homomorphism by definition of the operation on \( A \left[ \frac{1}{\omega} \right] \). We calculate \( j_{k+1, \omega}(M_k(a)) = (m_k \cdot a)/n_{k+1} = (n_{k+1}/n_k) \cdot (a/n_{k+1}) = a/n_k = j_k, \omega(a) \). So the universal property of \( \lim_{\to} (A, M_k) \) induces a homomorphism \( \phi \) satisfying the desired formula. It remains to check that \( \phi \) is an isomorphism. To see that \( \phi \) is injective, fix \( a \in A \) such that \( \phi(M_k, \omega)(a) = 0 \). Then \( a/n_k = 0 \), so \( a = 0 \). To see that \( \phi \) is surjective, fix \( a/n_k \in A \left[ \frac{1}{\omega} \right] \). Then \( \phi(M_k, \omega)(a) = a/n_k \).

Proposition 7.1.17. Let \( E \) be a strongly connected finite directed graph. Suppose that \( 1 \) is an eigenvalue of \( A_E^t \) and that the only roots of unity that are eigenvalues of \( A_E^t \) are \( \mathcal{P}_E \)-th roots of unity. Let \( \omega = (n_k)_{k=1}^{\infty} \) be a multiplicative sequence, and let \( l := \gcd(P_E, \omega) \).

Fix \( K \) such that \( \gcd(P_E, n_K) = l \), and define \( \omega' := (n'_k)_{k=1}^{\infty} \) where \( n'_k = l \) and \( n'_k = n_{K+k-1} \) for \( k \geq 2 \). For each \( k \geq 1 \), the map \( \phi_{n_k, n_{k+1}} \) descends to a map \( \hat{\phi}_{n_k, n_{k+1}} \) such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{coker}(1 - A_E^n)^t / \text{tor}_n & \xrightarrow{\hat{\phi}_{n_k, n_{k+1}}^t} & \text{coker}(1 - A_E^n)^t / \text{tor}_n \\
\downarrow{\theta_{n_k}^t} & & \downarrow{\theta_{n_{k+1}^t}} \\
\text{coker}(1 - A_E^n)^t / \text{tor} & \xrightarrow{M'_l} & \text{coker}(1 - A_E^n)^t / \text{tor}
\end{array}
\]

Proof. Fix \( k \geq 1 \). Applying the first assertion of Lemma \( 7.1.9 \) we see that \( \hat{\phi}_{n_k, n_{k+1}} \) descends to a homomorphism \( \hat{\phi}_{n_k, n_{k+1}} : \text{coker}(1 - A_E^n)^t / \text{tor}_n \to \text{coker}(1 - A_E^n)^t / \text{tor}_n \), satisfying \( \hat{\phi}_{n_k, n_{k+1}}(g + \text{tor}_n) = \phi_{n_k, n_{k+1}}(g) + \text{tor}_n \).

Define \( B_k := \sum_{i=0}^{n_{k+1}} -1 (A_E^{n_i} - 1)^t \). Note that \( B_k + m_k = \sum_{i=0}^{n_{k+1}} -1 (A_E^{n_i})^t \). We have that \( (A_E^{n_i} - 1)^t = (A_E^{n_i} - 1)((\sum_{j=0}^{n_{i}-1} (A_E^{n_j})^t) \), so \( \text{Im} B_k \subseteq \text{Im}(1 - A_E^n)^t \subseteq T_n = T_l \) by Lemma \( 7.1.12 \). Thus \( \text{Im} B_k + \text{Im}(1 - A_E^n)^t \subseteq \text{tor} \).
Fix $x \in \mathbb{Z}^{E_0}$. By the preceding paragraph, we have

$$\theta_{n_{k+1}} \left( \tilde{\phi}_{n_k,m_{k+1}} (x + \text{Im}(1 - A_{E}^{n_k})^t) + \text{tor}_{n_k} \right) = \theta_{n_{k+1}} \left( \phi_{n_k,m_{k+1}} (x + \text{Im}(1 - A_{E}^{n_k})^t) + \text{tor}_{n_k} \right)$$

$$= \sum_{i=0}^{m_k-1} (A_{E}^{m_k})^t x + \text{Im}(1 - A_{E}^i)^t + \text{tor}_i$$

$$= (B_k + m_k^i)(x) + \text{Im}(1 - A_{E}^i)^t + \text{tor}_i$$

$$= (m_k^i 1)(x) + \text{Im}(1 - A_{E}^i)^t + \text{tor}_i$$

$$= M_k (x + \text{Im}(1 - A_{E}^i)^t + \text{tor}_i)$$

$$= (M_k \circ \theta_{n_k}^i)(x + \text{Im}(1 - A_{E}^{n_k})^t + \text{tor}_{n_k}).$$

Recall the isomorphism $\rho : \text{coker}(1 - A_{E}^l)^t \rightarrow \bigoplus_{i=1}^l \text{coker}(1 - A_{E}^i)$ of Lemma 5.3.4 satisfying

$$\rho(\delta_v + \text{Im}(1 - A_{E}^i)^t) = (0, \ldots, \delta_v + \text{Im}(1 - A_{E})^i, \ldots, 0),$$

where $v \in \Lambda_j$ for some $0 \leq j \leq l - 1$, and $\delta_v + \text{Im}(1 - A_{E})^i$ appears in the $j$-th position.

**Lemma 7.1.18.** Let $E$ be a strongly connected finite directed graph. Suppose that 1 is an eigenvalue of $A_{E}^l$ and that the only roots of unity that are eigenvalues of $A_{E}^l$ are the $\mathcal{P}_E$-roots of unity. Let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence, and let $l = \gcd(\mathcal{P}_E, \omega)$. There is an isomorphism $\psi : K_0(C^*(E(l))) \rightarrow \bigoplus_{i=1}^l K_0(C^*(E))$ such that the following diagram commutes.

\[
\begin{array}{ccc}
\text{coker}(1 - A_{E}^l)^t & \xrightarrow{\rho} & \bigoplus_{i=1}^l \text{coker}(1 - A_{E}^i) \\
| \downarrow \sigma_1 \circ \psi | & & | \downarrow \bigoplus_{i=1}^l \sigma_1 |
\end{array}
\]

\[
\begin{array}{ccc}
K_0(C^*(E(l))) & \xrightarrow{\psi} & \bigoplus_{i=1}^l K_0(C^*(E))
\end{array}
\]

Moreover, $\psi \left( \sum_{\mu \in \mathcal{E} \gets l} [p_{\mu} \mu^i]_0 \right) = ([1_{C^*(E)}]_0, \ldots, [1_{C^*(E)}]_0)$.

**Proof.** We define $\psi := (\bigoplus_{i=1}^l \sigma_1) \circ \rho \circ (\sigma_1 \circ \psi)^{-1}$. Since $\rho, \sigma_1 \circ \psi$, and $\sigma_1$ are all isomorphisms, so is $\psi$.

We now show that $\psi$ satisfies the second statement. Fix $0 \leq i \leq l - 1$, and $v \in \Lambda_i$. Using Lemma 5.3.1 at the second equality, we have

$$\rho \left( \sum_{j=0}^{i-1} (A_{E}^{l-j}) \delta_v + \text{Im}(1 - A_{E}^l)^t \right) = \left( (A_{E}^{l-i}) \delta_v + \text{Im}(1 - A_{E}^l), \ldots, (A_{E}^{l-i}) \delta_v + \text{Im}(1 - A_{E}^l) \right)$$

$$= \left( \delta_v + \text{Im}(1 - A_{E}^l), \ldots, \delta_v + \text{Im}(1 - A_{E}^l) \right).$$
Proof of Theorem 7.1.3. Fix 

\[ \psi \left( \sum_{\mu \in E^{<\ell}} [p_{s(\mu),0}] \right) = \left( \bigoplus_{i=1}^{l} \sigma_{i} \right) \circ \rho \circ (\sigma_{i} \circ \psi_{i})^{-1} \left( \sum_{\mu \in E^{<\ell}} [p_{s(\mu),0}] \right) \]

\[ = \left( \bigoplus_{i=1}^{l} \sigma_{i} \right) \circ \rho \left( \sum_{\mu \in E^{<\ell}} \delta_{s(\mu)} + \text{Im}(1 - A_{E}^{t}) \right) \]

\[ = \left( \bigoplus_{i=1}^{l} \sigma_{i} \right) \sum_{\nu \in E^{0}} \sum_{j=0}^{l-1} (A_{E}^{t})^{j} \delta_{\nu} + \text{Im}(1 - A_{E}^{t}) \]

\[ = \left( \bigoplus_{i=1}^{l} \sigma_{i} \right) \sum_{\nu \in E^{0}} \delta_{\nu} + \text{Im}(1 - A_{E}) + \ldots, \sum_{\nu \in E^{0}} \delta_{\nu} + \text{Im}(1 - A_{E}^{t}) \]

\[ = ([1_{C^{*}(E)}], [1_{C^{*}(E)\omega}]), \] we have an isomorphism \( C^{*}(E, \omega) \cong C^{*}(E, \omega') \) by Theorem 7.0.4. Hence

\[ (K_{0}(C^{*}(E, \omega)), [1_{C^{*}(E, \omega)}]) \cong (K_{0}(C^{*}(E, \omega')), [1_{C^{*}(E, \omega')}]). \]

So it suffices to prove the theorem for \( \omega' \).

Let \( \text{tor}_{\omega'} := \text{tor} \left( \varinjlim \left( K_{0}(C^{*}(E(n_{k})), K_{0}(j_{n_{k},n_{k+1}^{t}})) \right) \right) \). By [15], Theorem 6.3.2 there is an isomorphism

\[ K_{0}(C^{*}(E, \omega')) \cong \varinjlim \left( K_{0}(C^{*}(E(n_{k})), K_{0}(j_{n_{k},n_{k+1}^{t}})) \right) \]

satisfying

\[ [1_{C^{*}(E, \omega')}]) \mapsto K_{0}(j_{n_{0}}, \infty) \left( \sum_{\mu \in E^{<n_{0}}_{1}} [p_{\mu, n_{0}^{t}}] \right). \]

This isomorphism descends to an isomorphism

\[ K_{0}(C^{*}(E, \omega'))/\text{tor}(E, \omega') \cong \varinjlim \left( K_{0}(C^{*}(E(n_{k})), K_{0}(j_{n_{k},n_{k+1}^{t}})) \right)/\text{tor}_{\omega'} \]

satisfying

\[ [1_{C^{*}(E, \omega')}])_{0} + \text{tor}(E, \omega') \mapsto K_{0}(j_{n_{0}}, \infty) \left( \sum_{\mu \in E^{<n_{0}}_{1}} [p_{\mu, n_{0}^{t}}] \right) + \text{tor}_{\omega'}. \]

Let \( x := \sum_{\mu \in E^{<n_{1}}_{1}} \delta_{s(\mu)} \in \mathbb{Z}^{E^{0}} \), and let \( \text{tor}_{\infty} := \text{tor} \left( \varinjlim \left( \text{coker}(1 - A_{E}^{t})^{t}, \phi_{n_{k},n_{k+1}^{t}} \right) \right) \).
The isomorphisms \((\sigma_{n_k'} \circ \psi_{n_k'})^{-1}\) discussed in Remark 7.1.6 induce an isomorphism
\[
\lim_{n_{k+1}'} (K_0(C^*(E(n_k'))), K_0(j_{n_k',n_{k+1}'})) / \text{tor}_{\omega'} \cong \lim_{n_{k+1}'} (\text{coker}(1 - A_{E_k'}^{n_k'} t), \phi_{n_k',n_{k+1}'}) / \text{tor}_{\infty}
\]
satisfying
\[
K_0(j_{n_k',\infty}) \left( \sum_{\mu \in E^{<n_k'}} [p_{\mu,n_k'}]_0 \right) + \text{tor}_{\omega'} \mapsto \phi_{n_k',\infty} \left( (\sigma_{n_k'} \circ \psi_{n_k'})^{-1} \left( \sum_{\mu \in E^{<n_k'}} [p_{\mu,n_k'}]_0 \right) \right) + \text{tor}_{\infty}
= \phi_{n_k',\infty} (x + \text{Im}(1 - A_{E_k'}^{n_k'} t)) + \text{tor}_{\infty}.
\]
By Lemma 7.1.9 there is an isomorphism
\[
\lim_{n_{k+1}'} (\text{coker}(1 - A_{E_k'}^{n_k'} t), \phi_{n_k',n_{k+1}'}) / \text{tor}_{n_k'} \cong \lim_{n_{k+1}'} (\text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{n_k'}, \tilde{\phi}_{n_k',n_{k+1}'})
\]
satisfying \(\phi_{n_k',\infty} (x + \text{Im}(1 - A_{E_k'}^{n_k'} t)) + \text{tor}_{n_k'} \mapsto \tilde{\phi}_{n_k',\infty} (x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{n_k'}).\)
By Proposition 7.1.17 there is an isomorphism
\[
\lim_{n_{k+1}'} (\text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{n_k'}, \tilde{\phi}_{n_k',n_{k+1}'}) \cong \lim_{n_{k+1}'} (\text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{t}, M_{n_k'})
\]
satisfying \(\tilde{\phi}_{n_k',\infty} (x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{n_k'}) \mapsto M_{n_k',\infty} (x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{t}).\)
By Lemma 7.1.16 there is an isomorphism
\[
\lim_{n_{k+1}'} (\text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{t}, M_{n_k'}) \cong (\text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{t}) \left[ \frac{1}{\omega'} \right]
\]
satisfying \(m_{n_k',\infty} (x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{t}) \mapsto (x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{t}) / n_k'.\)

The isomorphism \(\eta_l := \sigma_l \circ \psi_l : \text{coker}(1 - A_{E_k'}^{n_k'} t) \to K_0(C^*(E(l)))\) of Remark 7.1.6 descends to an isomorphism \(\tilde{\eta}_l : \text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{t} \to K_0(C^*(E(l))) / \text{tor}_{E(l)}\). This \(\tilde{\eta}_l\) induces an isomorphism
\[
(\text{coker}(1 - A_{E_k'}^{n_k'} t) / \text{tor}_{t}) \left[ \frac{1}{\omega'} \right] \cong (K_0(C^*(E(l))) / \text{tor}_{E(l)}) \left[ \frac{1}{\omega'} \right],
\]
satisfying
\[
(x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{t}) / n_k' \mapsto \tilde{\eta}_l (x + \text{Im}(1 - A_{E_k'}^{n_k'} t) + \text{tor}_{t}) / n_k'.
\]

The isomorphism of Lemma 7.1.18 descends to an isomorphism \(K_0(C^*(E(l))) / \text{tor}_{E(l)} \to \)
\[ \bigoplus_{i=1}^{l} K_0(C^*(E)) / \text{tor}_E, \] and this induces an isomorphism

\[ \left( K_0(C^*(E(l))) / \text{tor}_{E(l)} \right) \left[ \frac{1}{2^L} \right] \cong \left( \bigoplus_{i=1}^{l} K_0(C^*(E)) / \text{tor}_E \right) \left[ \frac{1}{2^L} \right], \]

satisfying

\[ \left( \sum_{\mu \in E<1} [p_{s(\mu)}, l_0 + \text{tor}_{E(l)}] / n_1' \right) \mapsto \tilde{\psi} \left( \sum_{\mu \in E<1} [p_{s(\mu)}, l_0 + \text{tor}_{E(l)}] / n_1' \right) = ([1_{C^*(E)}], 0 + \text{tor}_E, \ldots, [1_{C^*(E)}], 0 + \text{tor}_E) / n_1'. \]

Composing the isomorphisms of the previous seven paragraphs gives an isomorphism

\[ \Psi : K_0(C^*(E, \omega')) / \text{tor}_{(E, \omega')} \rightarrow \bigoplus_{i=1}^{l} \left( K_0(C^*(E)) / \text{tor}_E \right) \left[ \frac{1}{2^L} \right] \]

satisfying \( \Psi([1_{C^*(E, \omega') }]) = ([1_{C^*(E)}], 0 + \text{tor}_E, \ldots, [1_{C^*(E)}], 0 + \text{tor}_E) / l \), since \( n_1' = l \).

**Remark 7.1.19.** In the proof of Theorem 7.1.3 we needed to apply Lemma 7.1.18 to relate the torsion-free component of \( K_0(C^*(E(l))) \) back to the torsion-free component of \( K_0(C^*(E)) \). This uses Corollary 5.3.4 which requires Lemma 5.3.3, where it is crucial that the power of \( A_t^E \) in the term \((1 - A_t^E)^t\) matches the number of equivalence classes for the equivalence relation \( \sim_l \). We also needed to apply Corollary 7.1.15 to obtain an isomorphism between the torsion-free component of \( K_0(C^*(E(l))) \) and the torsion-free component of \( K_0(C^*(E(n_k))) \) for all \( k \) such that \( \gcd(P_E, n_k) = l \). This uses Lemma 7.1.12 which depends on Lemma 7.1.11 explaining why we require that the only roots of unity that are eigenvalues of \( A_t^E \) are the \( P_E \)-th roots of unity.

### 7.2 Classifying generalised Bunce–Deddens algebras

We use Theorem 7.1.3 to prove the forward direction of Theorem 7.0.1.

**Lemma 7.2.1.** Let \( D \subseteq \mathbb{N} \setminus \{0\} \). Suppose \( |D| = m \) for some \( 1 \leq m \leq \infty \), enumerate \( D \) in increasing order, \((d_1, d_2, \ldots, d_m)\), and define a nondecreasing sequence \( \text{lcm}(D) \) by

\[ \text{lcm}(D) := (d_1, \text{lcm}(d_1, d_2), \text{lcm}(d_1, d_2, d_3), \ldots, \text{lcm}(d_1, d_2, \ldots, d_m), \text{lcm}(d_1, d_2, \ldots, d_m), \ldots). \]

Then \( \text{lcm}(D) \) is a multiplicative sequence such that \( d_k | \text{lcm}(D) \) for all \( 1 \leq k \leq m \). Moreover, if \( \omega = (n_k)_{k=1}^\infty \) is another multiplicative sequence such that \( d_k | \omega \) for all \( 1 \leq k \leq m \),
then \( \lcm(D) \) divides \([\omega]\).

**Proof.** Clearly \( \lcm(D)_k \mid \lcm(D)_{k+1} \) for each \( k \geq 1 \). It is also clear that, for each \( 1 \leq k \leq m \), \( d_k \mid \lcm(D)_l \) for all \( l \geq k \), and so \( d_k \mid \lcm(D) \).

For the final statement, fix \( \omega \) such that \( d_k \mid \omega \) for each \( 1 \leq k \leq m \). For each \( 1 \leq k \leq m \), there exist natural numbers \( l_1, \ldots, l_k \) such that \( d_1 \mid n_{l_1}, \ldots, d_k \mid n_k \). Let \( l(k) = \max\{l_1, \ldots, l_k\} \). Then \( d_i \mid n_{l(k)} \) for each \( 1 \leq i \leq k \), so \( \lcm(d_1, \ldots, d_k) \mid n_{l(k)} \).

If \( A \) is a free abelian group, \( a \in A \) and \( n \geq 1 \), we write \( n \mid a \) if there exists \( a' \in A \) such that \( na' = a \).

**Theorem 7.2.2.** Fix a strongly connected finite directed graph \( E \), and a generalised Bunce–Deddens algebra \( C^*(E, \omega) \). Suppose that the only roots of unity that are eigenvalues of \( A^*_E \) are the \( \mathcal{P}_E \)-th roots of unity. Set

\[
D := \{ n \geq 1 : n \mid \left( [1_{C^*_E(\omega)}]_0 + \tor_{(E, \omega)} \right) \in K_0(C^*(E, \omega))/\tor_{(E, \omega)} \}
\]

and let

\[
d := \lcm\{ n \geq 1 : n \mid \left( [1_{C^*_E}]_0 + \tor_E \right) \in K_0(C^*(E))/\tor_E \}.
\]

Then \( [\omega] = [l \cdot \lcm(D)]/d \).

**Proof.** There is an isomorphism \( \theta : K_0(C^*(E))/\tor_E \to \mathbb{Z}^N \), where \( N = \text{rank} \ K_0(C^*(E)) \).

Let \( (u_1, \ldots, u_N) := \theta([1_{C^*_E}]_0 + \tor_E) \in \mathbb{Z}^N \).

We claim that \( \gcd(u_1, \ldots, u_N) = d \). Let \( e_1, \ldots, e_N \) be the generators of \( \mathbb{Z}^N \), and let \( n \geq 1 \) such that \( n \mid u_i \) for each \( 1 \leq i \leq N \). Then \( n \) divides \( \sum_{i=1}^N u_i \theta^{-1}(e_i) = \theta^{-1}(u_1, \ldots, u_N) = [1_{C^*_E}]_0 + \tor_E \). So \( n \mid d \), and hence \( \gcd(u_1, \ldots, u_N) \mid d \).

Now, fix \( n \geq 1 \) such that \( n \mid ([1_{C^*_E}]_0 + \tor_E) \). Then there exists \( a \in K_0(C^*(E)) \) such that \( na + \tor_E = [1_{C^*_E}]_0 + \tor_E \). We then have that \( n \theta(a + \tor_E) = (u_1, \ldots, u_N) \). So \( n \) is a common divisor of \( u_1, \ldots, u_N \), and hence \( n \mid \gcd(u_1, \ldots, u_N) \). So \( \gcd(u_1, \ldots, u_N) \) is a common multiple of \( \{ n \geq 1 : n \mid ([1_{C^*_E}]_0 + \tor_E) \in K_0(C^*(E))/\tor_E \} \), giving \( d \mid \gcd(u_1, \ldots, u_N) \), and so \( \gcd(u_1, \ldots, u_N) = d \).

Next we claim that for \( n \geq 1 \), we have \( n \mid \lcm(D) \) if and only if \( n \in D \). If \( n \in D \), it is clear that \( n \mid \lcm(D) \). For the other direction, suppose \( n \mid \lcm(D) \). Then there is an \( i \geq 1 \) such that \( n \mid \lcm(d_1, \ldots, d_i) \). Since \( d_1, \ldots, d_i \in D \), we have that \( \lcm(d_1, \ldots, d_i) \) divides \( [1_{C^*_E(\omega)}]_0 + \tor_{(E, \omega)} \), and so \( n \in D \).
We now show that \([\text{lcm}(D)]\) divides \([d\omega/l]\). Fix \(n \geq 1\). Then

\[
n \in D \iff n\left(\left[1_{C^*(E,\omega)}0 + \text{tor}(E,\omega)\right] \in K_0(C^*(E,\omega))/\text{tor}(E,\omega)\right)
\]

\[
\iff n\left(\left[1_{C^*(E)}0 + \text{tor}_E, \ldots, \left[1_{C^*(E)}0 + \text{tor}_E\right]\right] / l \in \bigoplus_{i=1}^l \left( K_0(C^*(E))/\text{tor}_E \right) \left[\frac{1}{\omega}\right]\right)
\]

\[
\iff n\left(\left[1_{C^*(E)}0 + \text{tor}_E\right] / l \in \left( K_0(C^*(E))/\text{tor}_E \right) \left[\frac{1}{\omega}\right]\right)
\]

\[
\iff n\left(\left[1_{C^*(E)}0 + \text{tor}_E\right] / l \in \left( K_0(C^*(E))/\text{tor}_E \right) \left[\frac{1}{\omega}\right]\right)
\]

\[
\iff n\left( (d/l) \in \mathbb{Z} \left[\frac{1}{\omega}\right] \right)
\]

\[
\iff n\left( 1 \in \mathbb{Z} \left[\frac{1}{(d\omega)/l}\right] \right)
\]

\[
\iff n\left( 1 \in \mathbb{Z} \left[\frac{1}{(d\omega)/l}\right] \right)
\]

Hence \(n|d\omega\) for all \(n \in D\), and so \([\text{lcm}(D)]\) divides \([d\omega/l]\) by Lemma 7.2.1.

To see that \([d\omega/l]\) divides \([\text{lcm}(D)]\), fix \(k \geq 1\). We have that \(n_k|1 \in \mathbb{Z} \left[\frac{1}{\omega}\right]\), so \((dn_k/l)|(d/l) \in \mathbb{Z} \left[\frac{1}{\omega}\right]\). The above string of implications gives us \((dn_k/l)\text{lcm}(D)\) for each \(k \geq 1\), so \([d\omega/l]\) divides \([\text{lcm}(D)]\), and the result follows.

We can now prove the forward direction of Theorem 7.0.1.

**Corollary 7.2.3.** Fix a strongly connected finite directed graph \(E\). Let \(\omega = \left(n_k\right)_{k=1}^\infty\) and \(\omega' = \left(n'_k\right)_{k=1}^\infty\) be multiplicative sequences. Suppose \(1\) is an eigenvalue of \(A^t_E\) and that the only roots of unity that are eigenvalues of \(A^t_E\) are the \(P_E\)-th roots of unity. If \(C^*(E,\omega) \cong C^*(E,\omega')\) then \([\omega] = [\omega']\).

**Proof.** Let \(l = \text{gcd}(P_E,\omega)\) and \(l' = \text{gcd}(P_E,\omega')\). Since \(C^*(E,\omega) \cong C^*(E,\omega')\), the number of summands in Theorem 7.1.3 must be equal, so \(l = l'\).

Let \(d\) be as in Theorem 7.2.2. Let

\[
D := \{ n \geq 1: n\left( \left[1_{C^*(E,\omega)}0 + \text{tor}(E,\omega) \right] \in K_0(C^*(E,\omega))/\text{tor}(E,\omega) \right) \}
\]

and let

\[
D' := \{ n \geq 1: n\left( \left[1_{C^*(E,\omega')}0 + \text{tor}(E,\omega') \right] \in K_0(C^*(E,\omega))/\text{tor}(E,\omega') \right) \}.
\]

Fix \(n \geq 1\). Since \(C^*(E,\omega) \cong C^*(E,\omega')\), we have that \(n\) divides \([1_{C^*(E,\omega)}0 + \text{tor}(E,\omega)\) precisely when \(n\) divides \([1_{C^*(E,\omega')}0 + \text{tor}(E,\omega')\), so \(D = D'\). By Theorem 7.2.2 we have that \([\omega] = [l \cdot \text{lcm}(D)]/d = [l \cdot \text{lcm}(D')] /d = [\omega']\).
Remark 7.2.4. Theorem 7.0.1 says that for a given graph $E$ and $[\omega] \neq [\omega']$, we have $C^*(E, \omega) \neq C^*(E, \omega')$. One might ask whether this can be extended to say that given graphs $E$ and $F$ and given $[\omega] \neq [\omega']$, we must have $C^*(E, \omega) \neq C^*(F, \omega')$. The following example demonstrates that the answer is no. Let $C_1$ be the graph consisting of a single vertex connected by a single loop and let $C_3$ be the graph with three vertices connected by a single cycle. Let $\omega = (3, 6, 12, 24, \ldots)$ and let $\omega' = (1, 2, 4, 8, 16, \ldots)$. Note that $\omega = 3 \omega'$. Since $C_1(3) = C_3$, we have that $C^*(C_1, \omega) \cong C^*(C_3, \omega')$. This illustrates why Theorem 7.0.1 is a theorem about generalised Bunce–Deddens algebras constructed from the same graph.

7.3 Computing $K_1(C^*(E, \omega))$

In this section we compute $K_1(C^*(E, \omega))$ where $E$ is a strongly connected finite graph $E$ such that the only roots of unity that are eigenvalues of $A_E^t$ are the $\mathcal{P}_E$-th roots of unity, $\omega$ is a multiplicative sequence. The main result of this section is the following.

Theorem 7.3.1. Let $E$ be a strongly connected finite graph and suppose that the only roots of unity that are eigenvalues of $A_E^t$ are the $\mathcal{P}_E$-th roots of unity. Let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence and let $l := \gcd(\mathcal{P}_E, \omega)$. Then

$$K_1(C^*(E, \omega)) = \bigoplus_{i=1}^l \ker(1 - A_E^t).$$

To prove Theorem 7.3.1 we need a series of results. We begin by studying $\ker(1 - A_E^{t|_{(n)}})$ for $n \geq 1$.

For $0 \leq k \leq n-1$ and $a = \sum_{\mu \in E^{<n}} a_{\mu} \delta_{\mu,n} \in \mathbb{Z}^{E^{<n}}$, we define $a_k := \sum_{\mu \in E^k} a_{\mu} \delta_{\mu,k} \in \mathbb{Z}^{E^k}$ and $a|_{\mathbb{Z}^{E^k}} := \sum_{\mu \in E^k} a_{\mu} \delta_{\mu,k} \in \mathbb{Z}^{E^k}$. For $b = \sum_{v \in E^0} b_v \delta_v \in \mathbb{Z}^{E^0}$, we define $t_n(b) := \sum_{v \in E^0} b_v \delta_v,n \in \mathbb{Z}^{E^{<n}}$.

Lemma 7.3.2. Let $E$ be a row-finite directed graph with no sources and let $n \geq 1$.

There is an isomorphism $\psi_n : \ker(1 - A_{E(n)}^t) \rightarrow \ker(1 - A_E^n)^t$ satisfying $\psi_n(a) = a|_{\mathbb{Z}^E}$ for $a \in \ker(1 - A_{E(n)}^t)$.

Proof. Define $\psi_n : \ker(1 - A_{E(n)}^t) \rightarrow \mathbb{Z}^{E^0}$ by $\psi_n(a) = a|_{\mathbb{Z}^E}$ for $a \in (1 - A_{E(n)}^t)$. We check that $\psi_n(\ker(1 - A_{E(n)}^t)) \subseteq \ker(1 - A_E^n)^t$. Let $a \in \ker(1 - A_{E(n)}^t)$. Then

$$(1 - A_E^n)^t(\psi_n(a)) = (1 - A_E^n)^t(a|_{\mathbb{Z}^E}) = ((1 - A_{E(n)}^t)a_0)|_{\mathbb{Z}^E} = 0.$$
So $\psi_n(a) \in \ker(1 - A_n^E)^t$, and hence $\psi_n$ descends to a homomorphism $\ker(1 - A_{E(n)}^t) \to \ker(1 - A_n^E)^t$ which we also label $\psi_n$.

Define $\varphi_n: \ker(1 - A_n^E)^t \to \mathbb{Z}^{E < n}$ by $\varphi_n(b) = \sum_{i=0}^{n-1} (A_{E(n)}^i)^t(\iota_n(b))$ for $b \in \ker(1 - A_n^E)^t$. We check $\varphi_n(\ker(1 - A_n^E)^t) \subseteq \ker(1 - A_{E(n)}^t)$. Let $b \in \ker(1 - A_n^E)^t$. Then

$$(1 - A_{E(n)}^t)(\varphi_n(b)) = (1 - A_{E(n)}^n)\left(\sum_{i=0}^{n-1} (A_{E(n)}^i)^t(\iota_n(b))\right)$$

$$= (1 - A_{E(n)}^n)^t(\iota_n(b))$$

$$= \iota_n((1 - A_{E(n)}^n)^t(b)) = 0.$$  

So $\varphi_n(b) \in \ker(1 - A_{E(n)}^t)$, and hence $\varphi_n$ descends to a homomorphism $\ker(1 - A_n^E)^t \to \ker(1 - A_{E(n)}^t)$ which we also label $\varphi_n$.

We check that $\varphi_n$ is an inverse for $\psi_n$. Let $a \in \ker(1 - A_{E(n)}^t)$. Fix $k < n$. We have

$$0 = (1 - A_{E(n)}^t)(a_k) = \begin{cases} a_k - A_{E(n)}^t(a_{k+1}) & \text{if } k \neq n - 1 \\ a|_{E^{n-1}} - A_{E(n)}^t(a_0) & \text{if } k = n - 1. \end{cases}$$

So $a_{n-1} = A_{E(n)}^t(a_0)$. Then $a_{n-2} = A_{E(n)}^t(a_{n-1}) = (A_{E(n)}^2)^t(a_0)$. Repeating this step yields $a_{n-i} = (A_{E(n)}^i)^t(a_0)$ for $i < n$. Since $a_0 = \iota_n(a|_{Z^{E^0}})$, we have $\varphi_n(\psi_n(a)) = \varphi_n(a|_{Z^{E^0}}) = \sum_{i=0}^{n-1} (A_{E(n)}^i)^t(a_0) = a$.

Now, we check that $\psi_n$ is an inverse for $\varphi_n$. Let $v \in E^0$ and $0 \leq i < n$. Repeated applications of (7.1.1) shows that $(A_{E(n)}^i)^t \delta_{v,n} \in \text{span}\{\delta_{\mu,n} : \mu \in E^{n-i}\}$. Thus

$$((A_{E(n)}^i)^t \delta_{v,n})|_{Z^{E^0}} = \begin{cases} \delta_v & \text{if } i = 0 \\ 0 & \text{otherwise}. \end{cases} \quad (7.3.1)$$

Now, let $b \in \ker(1 - A_n^E)^t$. By (7.3.1), we have

$$\psi_n(\varphi_n(b)) = \left(\sum_{i=0}^{n-1} (A_{E(n)}^i)^t(\iota_n(b))\right)|_{Z^{E^0}} = b.$$  

Suppose $E$ is a row-finite directed graph. Define the skew-product graph $E \times_1 \mathbb{Z}$ as the graph with edge set $(E \times_1 \mathbb{Z})^1 = E^1 \times \mathbb{Z}$ and vertex set $(E \times_1 \mathbb{Z})^0 = E^0 \times \mathbb{Z}$ and range and source maps defined by

$$r(e,k) = (r(e),k-1) \text{ and } s(e,k) = (s(e),k).$$
CHAPTER 7. K-THROUGH AND CLASSIFICATION

For each \( n \geq 1 \), we denote by \( s_{n,((\epsilon,\mu),k)} \) and \( p_{n,(\mu,k)} \) the generators of \( C^*(E(n) \times_1 \mathbb{Z}) \). Proposition 6.7 of \([41]\) gives a natural action \( (\beta_{E(n)})^m(s_{n,((\epsilon,\mu),k)}) = s_{n,((\epsilon,\mu),k+l)} \). By \([41]\, \text{Lemma 7.10}\) there is an isomorphism \( \phi_{E(n)} \) of \( C^*(E \times_1 \mathbb{Z}) \) onto the crossed product \( C^*(E(n)) \rtimes \mathbb{T} \) such that \( \phi_{E(n)} \circ (\beta_{E(n)})^m = \hat{\gamma}_n^m \circ \phi_{E(n)} \), where \( \hat{\gamma}_n \) is the dual of the gauge action \( \gamma_n \) of \( C^*(E(n)) \).

Lemma 7.3.3. Let \( E \) be a row-finite directed graph with no sources, and let \( n, m \geq 1 \). There is a homomorphism \( i_{n,mn} : C^*(E(n) \times_1 \mathbb{Z}) \to C^*(E(mn) \times_1 \mathbb{Z}) \) such that

\[
i_{n,mn}(s_{n,((\epsilon,\mu),1)}) = \sum_{\tau \in s(\epsilon)E^{<m},[\tau]_n = \mu} s_{mn,((\epsilon,\tau),1)} \quad \text{and} \quad i_{n,mn}(p_{n,(\mu,1)}) = \sum_{\tau \in E^{<m},[\tau]_n = \mu} p_{mn,((\mu,\tau),1)},
\]

for all \( n \geq 1 \).

Proof. Recall the injective homomorphism \( \hat{j}_{n,mn} : C^*(E(n)) \to C^*(E(mn)) \) of Lemma 3.1.7. Let \( (i_{C^*(E(n))}, i_{\mathbb{T}}) \) be the universal covariant representation of \( (C^*(E(n)), \mathbb{T}, \gamma_n) \). We show that \( \hat{j}_{n,mn} \) is \( \mathbb{T} \)-equivariant. For \( e \in E^1 \) and \( \mu \in s(\epsilon)E^{<n} \) and \( z \in \mathbb{T} \), we have

\[
\hat{j}_{n,mn}(\gamma_z^n(s_{n,((\epsilon,\mu),1)})) = \hat{j}_{n,mn}(zs_{n,((\epsilon,\mu),1)}) = \sum_{\tau \in E^{<m},[\tau]_n = \mu} zs_{mn,((\epsilon,\tau),1)} = \gamma_z^{mn}(\hat{j}_{n,mn}(s_{n,((\epsilon,\mu),1)})),
\]

and similarly for \( \mu \in E^{<n} \), \( \hat{j}_{n,mn}(\gamma_z^n(p_{n,\mu})) = (\gamma_z^{mn}(\hat{j}_{n,mn}(p_{n,\mu}))). \)

By \([49]\, \text{Corollary 2.48}\) there is a homomorphism \( \hat{j}_{n,mn} \times 1 : C^*(E(n)) \rtimes \mathbb{T} \to C^*(E(mn)) \rtimes \mathbb{T} \) satisfying

\[
(\hat{j}_{n,mn} \times 1)(i_{C^*(E(n))}(a))i_{\mathbb{T}}(z) = i_{C^*(E(mn))}(\hat{j}_{n,mn}(a))i_{\mathbb{T}}(z)
\]

for all \( a \in C^*(E(n)) \) and \( z \in \mathbb{T} \).

Define \( i_{n,mn} := \phi_{E(mn)}^{-1} \circ (\hat{j}_{n,mn} \times 1) \circ \phi_{E(n)} \). Let \(((\epsilon,\mu),1) \in E(n)^1 \times_1 \mathbb{Z} \) and let \( f_1(z) = z \) for \( z \in \mathbb{T} \). We calculate

\[
(\phi_{E(mn)}^{-1} \circ (\hat{j}_{n,mn} \times 1) \circ \phi_{E(n)})(s_{n,((\epsilon,\mu),1)}) = \phi_{E(n)}^{-1}(i_{A}(s_{n,((\epsilon,\mu),1)})) = \phi_{E(n)}^{-1}(i_{A}(s_{n,((\epsilon,\mu),1)})) \sum_{\tau \in s(\epsilon)E^{<m},[\tau]_n = \mu} s_{mn,((\epsilon,\tau),1)}i_{\mathbb{T}}(f_1))
\]

\[
= \phi_{E(n)}^{-1}(i_{A}(\sum_{\tau \in s(\epsilon)E^{<m},[\tau]_n = \mu} s_{mn,((\epsilon,\tau),1)}))i_{\mathbb{T}}(f_1))
\]

\[
= \sum_{\tau \in s(\epsilon)E^{<m},[\tau]_n = \mu} s_{mn,((\epsilon,\tau),1)}.
\]
Similarly, for $(\mu, 1) \in E(n)^0 \times_1 \mathbb{Z}$, we have

\[
(\phi_{E[\mu]}^{-1} \circ (j_{\mu,n} \times \text{id}) \circ \phi_{E(n)})(p_{n,\mu,1}) = \phi_{E(n)}^{-1}(j_{\mu,n} \times \text{id})(i_A(p_{n,\mu})i_T(\delta_{f,1}))
\]

\[
= \phi_{E(n)}^{-1}(i_A \left( \sum_{\tau \in E^{<mn}, \|\tau\| = \mu} p_{mn,\tau} \right) i_T(\delta_{f,1}))
\]

\[
= \sum_{\tau \in E^{<mn}, \|\tau\| = \mu} p_{mn,\tau}.
\]

**Proposition 7.3.4.** Let $E$ be a row-finite directed graph with no sources and let $n, m \geq 1$. There are isomorphisms $K_1(C^*(E(n))) \to \ker(1 - A_E^n)$ and $K_1(C^*(E(mn))) \to \ker(1 - A_E^{mn})$ such that the following diagram commutes

\[
\begin{array}{ccc}
K_1(C^*(E(n))) & \xrightarrow{K_1(j_{n,mn})} & K_1(C^*(E(mn))) \\
\downarrow & & \downarrow \\
\ker(1 - A_E^n) & \xrightarrow{\sigma_n} & \ker(1 - A_E^{mn})
\end{array}
\]

**Proof.** The naturality of the Pimsner–Voiculescu diagram gives the following commutative diagram (see [41, Lemma 7.12]).

\[
\begin{array}{ccc}
K_1(C^*(E(n))) & \xrightarrow{K_1(j_{n,mn})} & K_1(C^*(E(mn))) \\
\downarrow & & \downarrow \\
\ker(1 - (\beta_{E(n)})^{-1}) & \xrightarrow{\sigma_n} & \ker(1 - (\beta_{E(mn)})^{-1})
\end{array}
\]

By [41, Lemma 7.13] there is an injection $\sigma_n : \mathbb{Z}^{E^{<n}} \to K_0(C^*(E(n) \times_1 \mathbb{Z}))$ satisfying $\sigma_n(\delta_{\mu,n}) = [p_{n,\mu,1}]_0$. Define $\phi_{n,mn} : \mathbb{Z}^{E^{<n}} \to \mathbb{Z}^{E^{<mn}}$ by $\phi_{n,mn}(\delta_{\mu,n}) = \sum_{\tau \in E^{<mn}, \|\tau\| = \mu} \delta_{\tau,mn}$ for $\mu \in E^{<n}$. We claim that the following diagram commutes.

\[
\begin{array}{ccc}
K_0(C^*(E(n) \times_1 \mathbb{Z})) & \xrightarrow{K_0(j_{n,mn})} & K_0(C^*(E(mn) \times_1 \mathbb{Z})) \\
\sigma_n & & \sigma_{mn} \\
\mathbb{Z}^{E^{<n}} & & \mathbb{Z}^{E^{<mn}}
\end{array}
\]  

(7.3.2)
To prove this claim, fix $\mu \in E^{<n}$. Then
\[
(\sigma^{-1}_{mn} \circ K_0(i_n, mn) \circ \sigma_n)(\delta_{\mu,n}) = (\sigma^{-1}_{mn} \circ K_0(i_n, mn) \circ [p_{(n,\mu,1)}]_0)
= \sigma^{-1}_{mn}([i_n, mn]([p_{(n,\mu,1)}]_0)
= \sigma^{-1}_{mn} \left( \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} [p_{(mn,\tau,1)}]_0 \right)
= \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} \delta_{\tau,mn}
= \phi_{n,mn} (\delta_{\mu,n}).
\]

It follows from [41, Theorem 7.16] that $\sigma_n$ restricts to an isomorphism of $\ker(1 - A_t^0)$ onto $\ker(1 - (\beta_{E(n)})_*^{-1})$. Restricting diagram 7.3.2 to the subgroups $\ker(1 - (\beta_{E(n)})_*^{-1}) \subseteq K_0(C^*(E(n) \times 1 \mathbb{Z}))$ and $\ker(1 - A_{E(n)}^t) \subseteq \mathbb{Z}^{E^{<n}}$ yields the following commuting diagram.

To prove this claim, fix $x \in \ker(1 - A_{E(n)}^t)$. Then
\[
\psi_{mn} (\phi_{n,mn}(x)) = \psi_{mn} \left( \sum_{\mu \in E^{<n}} x_\mu \sum_{\tau \in E^{<mn}, [\tau]_n = \mu} \delta_{\tau,mn} \right) = \sum_{v \in E^0} x_v \delta_v = \psi_n (x).
\]

Combining the preceding commutative diagrams gives the desired commutative diagram.

**Proof of Theorem 7.3.1** By [45, Theorem 6.3.2], we have
\[
K_1(C^*(E,\omega)) \cong \lim_{\rightarrow} \left( K_1(C^*(E(n_k)), K_1(j_{n_k,n_{k+1}})) \right).
\]

By Proposition 7.3.4, we have
\[
\left( \lim_{\rightarrow} K_1(C^*(E(n_k))), K_1(j_{n_k,n_{k+1}}) \right) \cong \lim_{\rightarrow} \left( \ker(1 - A_{E(n)}^t), x \mapsto x \right).
\]
By Lemma 7.1.11 the matrix $\sum_{j=0}^{n_k/l-1}(A^j_k)^t$ is invertible for $k$ such that $\gcd(P_E, n_k) = \gcd(P_E, \omega)$. So

$$\ker(1 - A_{E}^{n_k})^t = \ker\left(\left(\sum_{j=0}^{n_k/l-1}(A^j_k)^t\right)(1 - A_{E}^l)^t\right) = \ker(1 - A_{E}^l)^t,$$

for $k$ such that $\gcd(P_E, n_k) = \gcd(P_E, \omega)$. Hence

$$\left( \ker(1 - A_{E}^{n_k})^t, x \mapsto x \right) \cong \ker(1 - A_{E}^l)^t.$$

Combining the previous three isomorphisms gives an isomorphism

$$K_1(C^*(E, \omega)) \cong \ker(1 - A_{E}^l)^t.$$

Now, $\ker(1 - A_{E}^l)^t \cong \mathbb{Z}^r$, where $r = \text{rank coker}(1 - A_{E}^l)^t = l \cdot \text{rank coker}(1 - A_{E}^l)$ by Corollary 5.3.4. So $\ker(1 - A_{E}^l)^t \cong \bigoplus_{i=1}^{l} \ker(1 - A_{E}^l)^t$, giving the result.
Chapter 8

KMS states

In this chapter we study the KMS states for the gauge actions on $\mathcal{T}(E, \omega)$ and $C^*(E, \omega)$. Our approach follows the program of [14, 37]. We also use the results of [22] for KMS states for graph algebras and of [36] for higher-rank graph algebras. The results in this chapter appear in joint work with my supervisors in [44, Section 6].

Throughout this chapter, if $X$ is a compact topological space, then $\mathcal{M}_1^+(X)$ denotes the collection of Borel probability measures on $X$. We write $A_E$ for the vertex matrix $A_{E}(v,w) = |vE^1w|$ of a finite graph $E$, and $\rho(A_E)$ for its spectral radius.

The following summarises our main results about KMS states on $\mathcal{T}(E, \omega)$ and $C^*(E, \omega)$.

**Theorem 8.0.1.** Let $E$ be a strongly connected finite graph, and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. Let $\alpha : \mathbb{R} \to \text{Aut} \mathcal{T}(E, \omega)$ be given by $\alpha_t = \gamma_{e^it}$.

1. For $\beta > \ln \rho(A_E)$ there is an affine isomorphism (described in Corollary 8.3.5) of $\mathcal{M}_1^+(\lim E^{<n_k})$ onto the KMS$\beta$-simplex of $\mathcal{T}(E, \omega)$.

2. There are exactly $\gcd(\mathcal{P}_E, \omega)$ extremal KMS$\ln \rho(A_E)$-states of $\mathcal{T}(E, \omega)$ (described explicitly in Theorem 8.4.1).

3. For $\beta < \ln \rho(A_E)$, there are no KMS$\beta$ states for $\mathcal{T}(E, \omega)$.

4. A KMS$\beta$ state of $\mathcal{T}(E, \omega)$ factors through $C^*(E, \omega)$ if and only if $\beta = \ln \rho(A_E)$.

8.1 A transformation on finite signed Borel measures

Let $E$ be a finite directed graph with no sources, and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. In this section we consider the Banach space $\mathcal{M}(\lim E^{<n_k})$ of finite signed measures on $\lim E^{<n_k}$, the projective limit described in Chapter 3. Recall that $\lim E^{<n_k}$
is compact when $E$ is finite. We show that the vertex matrices $A_{E(n_k)}$ induce a bounded linear transformation $A_w$ of $\mathcal{M}(\lim E^{<n_k})$. We use Perron–Frobenius theory to show that $\|A_w\| = \rho(A_E)$, and that $A_w$ always admits a positive measure $m$ such that $Am = \rho(A_E)m$. We say that $m$ is an eigenmeasure of $A_w$. We provide a condition under which $m$ is unique up to scalar multiples.

For $k \geq 1$, define a map $p^*_{n_{k+1},n_k} : \mathcal{M}(E^{<n_{k+1}}) \to \mathcal{M}(E^{<n_k})$ by $p^*_{n_{k+1},n_k}(m)(U) = m(p^{-1}_{n_{k+1},n_k}(U))$ for every Borel measurable subset $U$ of $E^{<n_k}$. Then $p^*_{n_{k+1},n_k}$ is linear and the $(\mathcal{M}(E^{<n_k}), p^*_{n_{k+1},n_k})$ form a projective sequence of Banach spaces. The following lemma describes an injective linear map from $\lim_{\leftarrow}(\mathcal{M}(E^{<n_k}), p^*_{n_{k+1},n_k})$ to the Banach space $\mathcal{M}(\lim_{\leftarrow} E^{<n_k})$.

**Lemma 8.1.1.** Let $E$ be a finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. There is a continuous injective linear map $\iota_\omega : \mathcal{M}(\lim_{\leftarrow} E^{<n_k}) \to \lim_{\leftarrow}(\mathcal{M}(E^{<n_k}), p^*_{n_{k+1},n_k})$ such that $\iota_\omega(m)_k(\{\tau\}) = m(Z((\tau,k))$ for all $k \in n, m \in \mathcal{M}(E^{<n_k})$ and $\tau \in E^{<n_k}$.

**Proof.** For each $k \geq 1$, define $p^*_{\infty,n_k} : \mathcal{M}(\lim_{\leftarrow} E^{<n_k}) \to \mathcal{M}(E^{<n_k})$ by $p^*_{\infty,n_k}(m)(\{\tau\}) = m(p^{-1}_{\infty,k}(\tau))$. Then each $p^*_{\infty,n_k}$ is linear, and we have

$$p^*_{n_{k+1},n_k}(p^*_{\infty,n_k+1}(m))(\{\tau\}) = m(p^{-1}_{\infty,n_{k+1}}(p^{-1}_{n_{k+1},n_k}(m))(\{\tau\})) = m(p^{-1}_{\infty,n_k}(\tau)) = p^*_{\infty,n_k}(m)(\{\tau\})$$

for all $k$. So the universal property of $\lim_{\leftarrow}(\mathcal{M}(E^{<n_k}), p^*_{n_{k+1},n_k})$ implies that there is a continuous map $\iota_\omega$ such that $\iota_\omega(m)_k(\tau) = m(Z(\tau,k)$ for all $k \in n, m \in \mathcal{M}(E^{<n_k})$ and $\tau \in E^{<n_k}$. The formula for $\iota_\omega$ is clearly linear.

For injectivity, let $m \in \mathcal{M}(\lim_{\leftarrow} E^{<n_k})$ such that $\iota_\omega(m) = 0$. For each $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$, we have $m(Z(\mu,k)) = \iota_\omega(m)_k(\{\mu\}) = 0$, and since the $Z(\mu,k)$ are a basis for $\lim_{\leftarrow} E^{<n_k}$, we deduce that $m = 0$. $\square$

**Remark 8.1.2.** The map $\iota_\omega$ is typically not surjective. For example, let $E$ be the directed graph with one vertex $v$ and one loop edge $e$. Define $m_0 \in \mathcal{M}(E^0)$ by $m_0(\{v\}) = 1$. Let $n_k = 2^k$ for all $k \in \mathbb{N}$, and inductively define $m_k \in \mathcal{M}(E^{<n_k})$ by

$$m_k(\{e^j\}) = 2m_{k-1}(\{e^j\}) \quad \text{and} \quad m_k(\{e^j+2^{k-1}\}) = -m_{k-1}(\{e^j\})$$

for $j \in \{0, \ldots, 2^{k-1} - 1\}$. Then $(m_k)_{k=1}^\infty \in \lim_{\leftarrow} \mathcal{M}(E^{<n_k})$, but $m_0(\{v\}) = 2^k \to \infty$. For any $m \in \mathcal{M}(\lim_{\leftarrow} E^{<n_k})$, we have $\iota_\omega(m)_k(\{v\}) = m(Z(k,\tau)) \leq m^+(Z(k,\tau))$ for all $k \in \mathbb{N}$, so the sequence $\iota_\omega(m)_k(\{v\})$ is bounded. So $(m_k)_{k=1}^\infty$ does not belong to the range of $\iota_\omega$.

In what follows, if $m \in \mathcal{M}(\lim_{\leftarrow} E^{<n_k})$, we will frequently write $m_{n_k}$ for $\iota_\omega(m)_k \in \mathcal{M}(E^{<n_k})$. We also regard the vertex matrix $A_{E(n_k)}$ as a linear transformation of the
Now consider linear transformation of a finite-dimensional vector space $\mathcal{M}(E^{<n_k}) \cong \mathbb{R}^{E^{<n_k}}$. We show how the $A_{E(n_k)}$ induce a linear transformation of $\lim\limits_\sim \mathcal{M}(E^{<n_k})$. The following lemma is [44, Lemma 6.4]. In the proof of [44, Lemma 6.4] there are two mistakes in the first paragraph. The statement is still correct and we include a revised proof here.

**Lemma 8.1.3.** Let $E$ be a finite directed graph with no sources, and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. For $k \geq 1$, let $A_{E(n_k)}$, regarded as a linear transformation of $\mathcal{M}(E^{<n_k})$. For $m \in \mathcal{M}(E^{<n_k})$, we have

\begin{equation}
(A_{n_k}m)(\{\mu\}) = \begin{cases} 
m\{\mu_2 \cdots \mu_{|\mu|}\} & \text{if } \mu \in E^{<n_k} \setminus E^0 \\
\sum_{e \in \mu \in E^{n_k}} m(\{\nu\}) & \text{if } \mu \in E^0,
\end{cases}
\tag{8.1.1}
\end{equation}

and

\begin{equation}
A_{n_k-1}(p_{n_k,n_k-1}^*(m)) = p_{n_k,n_k-1}^*(A_{n_k}(m)).
\tag{8.1.2}
\end{equation}

**Proof.** We write $\{\delta_{\lambda,k} : \lambda \in E^{<n_k}\}$ for the basis of Dirac measures on $E^{<n_k}$. We have

\[A_{n_k}(\delta_{\lambda,k}) = \sum_{\tau \in E^{<n_k}} A_{n_k}(\tau,\lambda)\delta_{\tau,k} = \sum_{\tau \in E^{<n_k}} |\tau E(n_k)^1\lambda|\delta_{\tau,k} = \sum_{e \in E^1r(\lambda)} \delta_{[e\lambda]_{n_k}}.\]

So, for $\mu \in E^{<n_k}$, we have

\[A_{n_k}(\delta_{\lambda,k})(\{\mu\}) = \sum_{e \in E^1r(\lambda)} \delta_{[e\lambda]_{n_k}}(\{\mu\}) = \begin{cases} 
\delta_{\lambda,k}(\{\mu_2 \cdots \mu_{|\mu|}\}) & \text{if } \mu \in E^{<n_k} \setminus E^0 \\
\sum_{e \in \mu \in E^{n_k}} \delta_{\lambda,k}(\{\nu\}) & \text{if } \mu \in E^0.
\end{cases}
\]

Now (8.1.1) follows from linearity.

To prove (8.1.2), first consider $\mu \in E^{<n_k-1} \setminus E^0$. We have

\[A_{n_k-1}(p_{n_k,n_k-1}^*(m))(\{\mu\}) = p_{n_k,n_k-1}^*(m)(\{\mu_2 \cdots \mu_{|\mu|}\}) = \sum_{\tau \in E^{<n_k} |\tau|_{n_k-1} = \mu_2 \cdots \mu_{|\mu|}} m(\{\tau\}) = \sum_{\eta \in E^{<n_k} |\eta|_{n_k-1} = \mu} A_{n_k}(m)(\{\eta\}) = p_{n_k,n_k-1}^*(A_{n_k}(m))(\{\mu\}).
\]

Now consider $\mu = v \in E^0$. We have

\[A_{n_k-1}(p_{n_k,n_k-1}^*(m))(\{v\}) = \sum_{e \in v^{E^{n_k-1}}} p_{n_k,n_k-1}^*(m)(\{\tau\}) = \sum_{e \in v^{E^1 \setminus E^{<n_k}}} m(\{\lambda\}) = \sum_{\lambda \in E^{<n_k-1} |\lambda|_{n_k-1} = v} A_{n_k}(m)(\{\lambda\}) = p_{n_k,n_k-1}^*(A_{n_k}(m))(\{v\}). \]
Proposition 8.1.4. Let $E$ be a finite directed graph with no sources, and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. For $k \geq 1$, let $A_{n_k} := A_{E(n_k)}$, regarded as a linear transformation of $\mathcal{M}(E^{<n_k})$. There is a linear transformation $A_\omega$ of $\lim \mathcal{M}(E^{<n_k})$ given by $A_\omega m = (A_{n_1}m_1, A_{n_2}m_2, \ldots)$. The inclusion $\iota_\omega$ of Lemma 8.1.1 satisfies

$$A_\omega(\iota_\omega(\mathcal{M}(\lim E^{<n_k}))) \subseteq \iota_\omega(\mathcal{M}(\lim E^{<n_k}))$$

Proof. Fix $(m_1, m_2, \ldots) \in \lim \mathcal{M}(E^{<n_k})$. By Lemma 8.1.3 we have $p_{n_k,n_{k-1}}^*(A_{n_k}(m_{n_k})) = A_{n_k-1}(p_{n_k,n_{k-1}}^*)(m_{n_k}) = A_{n_k-1}m_{n_{k-1}}$, so $(A_{n_1}m_1, A_{n_2}m_2, \ldots) \in \lim \mathcal{M}(E^{<n_k})$. The universal property of $\lim \mathcal{M}(E^{<n_k})$ gives a continuous map $A_\omega : \lim \mathcal{M}(E^{<n_k}) \to \lim \mathcal{M}(E^{<n_k})$ satisfying $A_\omega m = (A_{n_1}m_1, A_{n_2}m_2, \ldots)$. It is clear that $A_\omega$ is linear.

By Lemma 8.1.3 we have $p_{n_k,n_{k-1}}^*(A_{n_k}(m_{n_k})) = A_{n_k-1}(p_{n_k,n_{k-1}}^*)(m_{n_k}) = A_{n_k-1}m_{n_{k-1}}$. So by [6, Theorem 2.2], there is a positive Borel measure $M^+$ on $\lim E^{<n_k}$ such that $M^+(Z(\mu, k)) = (A_{n_k}m_{n_k})(\mu)$ for all $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$. Similarly, there is a positive Borel measure $M^-$ on $\lim E^{<n_k}$ such that $M^-(Z(\mu, k)) = (A_{n_k}m_{n_k})(\mu)$ for $\mu \in E^{<n_k}$.

For calculations later, we will want to understand the transformation $A_\omega$ in terms of the measures of cylinder sets.

Lemma 8.1.5. Let $E$ be a finite directed graph with no sources, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. For $m \in \mathcal{M}(\lim E^{<n_k})$, $k \in \mathbb{N}$ and $\mu \in E^{<n_k}$, the transformation $A_\omega$ of Proposition 8.1.4 satisfies

$$(A_\omega m)(Z(\mu, k)) = \begin{cases} m(Z(\mu_2 \ldots \mu_{|\mu|}, k)) & \text{if } \mu \in E^{<n_k} \setminus E^0 \\ \sum_{\nu \in E^{<n_k}} m(Z(\nu, k)) & \text{if } \mu \in E^0 \end{cases} = \sum_{\nu \in E^{<n_k}} |\nu|E(n_k)|\nu|m(Z(\mu, k)).$$

Proof. Since $A_\omega m(Z(\mu, k)) = A_\omega m(p_{n_k,n_{k-1}}^{-1}(\mu)) = A_{n_k}m_{n_k}(\mu)$, the result follows from Lemma 8.1.3.

We now show that $A_\omega$ admits a positive measure $m$ such that $A_\omega m = \rho(E)m$. We say that $m$ is an eigenmeasure of $A_\omega$. We also show that the norm of $A_\omega$, as an operator on the Banach space $\mathcal{M}(\lim E^{<n_k})$, is $\rho(A_E)$. Recall that the unimodular Perron-Frobenius eigenvector of an irreducible nonnegative matrix $A$ is its unique positive eigenvector with unit 1-norm.
Proposition 8.1.6. Let $E$ be a strongly connected finite directed graph, and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. Let $x^E$ be the unimodular Perron-Frobenius eigenvector of $A_E$. The transformation $A_\omega$ of Proposition 8.1.4 admits a positive eigenmeasure $m$ such that

$$m(Z(\mu, k)) = \frac{1}{n_k} \rho(A_E)^{-|\mu|} x^E_{s(\mu)} \quad \text{for all } \mu \in E^{<n_k},$$

(8.1.3)

and the corresponding eigenvalue is $\rho(A_E)$, and is equal to the operator norm of $A_\omega$ as a transformation of $\mathcal{M}(\lim E^{<n_k})$.

Proof. To see that (8.1.3) specifies an element $m \in \mathcal{M}(\lim E^{<n_k})$, define measures $m_k$ by $m_k(\{\mu\}) := \frac{1}{n_k} \rho(A_E)^{-|\mu|} x^E_{s(\mu)}$ for $\mu \in E^{<n_k}$. Let $a_k := n_k+1/n_k$ for each $k$. Using at the fifth equality that $A_j^j x^E = \rho(A_j^j) x^E$ for all $j$, we calculate

$$p^*_{nk+1,n_k}(m_{nk+1})(\{\mu\}) = \sum_{\tau \in E^{<nk}, |\tau|_{nk} = \mu} \frac{1}{n_{k+1}} \rho(A_E)^{-|\tau|} x^E_{s(\tau)}$$

$$= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \sum_{\lambda \in s(\mu) E^j n_k} a_k \rho(A_E)^{-jn_k} x^E_{s(\lambda)}$$

$$= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \rho(A_E)^{-jn_k} s(\mu, w) x^E_{w}$$

$$= \sum_{j=0}^{a_k-1} \frac{1}{n_k} \rho(A_E)^{|\mu|} \rho(A_E)^{-jn_k} s(\mu, w) x^E_{w} = m_{nk}(\{\mu\}).$$

Now [8.1.3] implies that there is a positive measure $m$ on $\lim E^{<n_k}$ satisfying (8.1.3).

To see that $m$ is an eigenmeasure for $A_\omega$ with eigenvalue $\rho(A_E)$, observe that for $\mu \in E^{<nk} \setminus E^0$, we have

$$(A_\omega m)(Z(\mu, k)) = m(Z(\mu_2 \ldots \mu_{|\mu|}, k)) = \frac{1}{n_k} \rho(A_E)^{-|\mu|+1} x^E_{s(\mu)} = \rho(A_E) m(Z(\mu, k)),$$

and for $v \in E^0$, we have

$$(A_\omega m)(Z(v, k)) = \sum_{e \in E^1 \tau \in E^{<nk-1}} \frac{1}{n_k} \rho(A_E)^{-|\tau|} x^E_{s(\tau)} = \frac{1}{n_k} \sum_{w \in E^0} \sum_{\lambda \in E^{<nk} w} \rho(A_E)^{-|\lambda|+1} x^E_{w}$$

$$= \frac{1}{n_k} (A_{nk} \rho(A_E)^{-nk+1} x^E_{w})_v = \frac{1}{n_k} \rho(A_E) x^E_{v}.$$
So $m$ is an eigenmeasure for $A_\omega$ with corresponding eigenvalue $\rho(A_E)$. It follows immediately that $\|A_\omega\| \geq \rho(A_E)$. For the reverse inequality, take $m \in \mathcal{M}(\varprojlim E^{<nk})$ and consider its Jordan decomposition $m = m^+ - m^-$. Since $A_\omega$ is linear, we have $A_\omega m^+ - A_\omega m^- = A_\omega m$, and since the $A_n$ are positive matrices, the measures $A_\omega m^\pm$ are positive measures. So the Jordan Decomposition Theorem implies that $A_\omega m^+ \geq (A_\omega m)^+$ and $A_\omega m^- \geq (A_\omega m)^-$. So

$$
\|A_\omega\| = \sup_{\|m\| = 1} \|A_\omega m\| = \sup_{\|m\| = 1} \left( (A_\omega m)^+ (\varprojlim E^{<nk}) + (A_\omega m)^- (\varprojlim E^{<nk}) \right)
$$

$$
\leq \sup_{\|m\| = 1} \left( (A_\omega m)^+ (\varprojlim E^{<nk}) + (A_\omega m)^- (\varprojlim E^{<nk}) \right)
$$

$$
= \sup_{\|m\| = 1} \left( (A_\omega m)^+ (E^0) + (A_\omega m)^- (E^0) \right)
$$

$$
\leq \sup_{\|m\| = 1} \left( \rho(A_E) m^+_1 (E^0) + \rho(A_E) m^-_1 (E^0) \right)
$$

$$
= \rho(A_E) \sup_{\|m\| = 1} \left( m^+(\varprojlim E^{<nk}) + m^- (\varprojlim E^{<nk}) \right)
$$

$$
= \rho(A_E).
$$

We now consider the eigenmeasures for $A_\omega$ when $E$ is a finite strongly connected graph. Fix $k \in \mathbb{N}$ such that $\gcd(P_E, n_k) = \gcd(P_E, \omega)$ and recall the equivalence relation $\sim_{nk}$ on $E^0$ of Lemma 5.1.1. We show that there are $\gcd(P_E, \omega)$ normalised eigenmeasures $m_\Lambda$ for $A_\omega$, where $\Lambda \in E^0/\sim_{nk}$, and that every eigenmeasure for $A_\omega$ is a convex combination of the $m_\Lambda$. In particular, if $\gcd(P_E, \omega) = 1$, then the measure $m$ of Proposition 8.1.6 is the only positive eigenmeasure of norm 1 for the transformation $A_\omega$.

**Lemma 8.1.7.** Let $E$ be a strongly connected finite directed graph, and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. Let $m$ be the measure of Proposition 8.1.6 and fix $k \in \mathbb{N}$ such that $\gcd(P_E, n_k) = \gcd(P_E, \omega)$.

1. Let $\sim_{nk}$ be the equivalence relation on $E^0$ of Lemma 5.1.1. For $\Lambda \in E^0/\sim_{nk}$, let $X_\Lambda = \bigcup_{\mu \in E^{<nk}, s(\mu) \in \Lambda} Z(\mu, k) \subseteq \varprojlim E^{<nk}$, and define $m^\Lambda \in \mathcal{M}(\varprojlim E^{<nk})$ by $m^\Lambda(U) := \frac{1}{m(X_\Lambda)} m(U \cap X_\Lambda)$. Then each $m^\Lambda$ is a normalised eigenmeasure for $A_\omega$ with eigenvalue $\rho(A_E)$.

2. For each $l \geq k$, and for each $\Lambda \in E^0/\sim_{nk}$, the block $A_{n_l}^\Lambda \in M_{E^{<n_l}}(\mathbb{Z})$ of $A_{n_l}$ is an irreducible matrix. We have $\rho(A_{n_l}^\Lambda) = \rho(A_E)$ and $m_{n_l}^\Lambda = (m^\Lambda(Z(\mu, l)))_{\mu \in E^{<n_l}}$ is the unimodular Perron–Frobenius eigenvector of $A_{n_l}^\Lambda$.

3. Every positive eigenmeasure for $A_\omega$ is a convex combination of the $m^\Lambda$. 
Proof. Proposition 8.1.6 shows that $m$ is an eigenmeasure with $A_\omega m = \rho(A_E)m$. Lemma 6.2.1 shows that each $\mathcal{M}(X_\Lambda) \subseteq \mathcal{M}(\lim E^{<\infty})$ is invariant for $A_\omega$, and it follows that $A_\omega m^\Lambda = \rho(A_E)m^\Lambda$ for each $\Lambda$.

(2) For each $l \geq k$ and each $\Lambda \in E^0/\sim_{n_1}$, the matrix $A^\Lambda_{n_l}$ is irreducible by Proposition 5.1.3. By definition of $A_\omega$, we have $A_\omega \chi_Z(\mu,l) = \sum_{\nu \in E^{<\infty}} A^\Lambda_{n_l}(\nu,\mu) \chi_Z(\nu,l)$, and so (1) shows that $A^\Lambda_{n_l}m^\Lambda_{n_l} = \rho(A_E)m^\Lambda_{n_l}$. The Perron-Frobenius theorem [48, Theorem 1.5] implies that every entry of the Perron–Frobenius eigenvector of the irreducible matrix $A_E$ is nonzero, and so (8.1.3) shows that $m^\Lambda_{n_l}$ is the unimodular Perron–Frobenius eigenvector of $A^\Lambda_{n_l}$, and so its eigenvalue $\rho(A_E)$ is equal to $\rho(A^\Lambda_{n_l})$.

(3) Suppose that $m' \in \mathcal{M}^+\left(\lim E^{<\infty}\right)$ and $z \in \mathbb{C}$ satisfy $A_\omega m' = zm'$. Then in particular $A_{n_l}(m')^\Lambda_{n_l} = (zm')^\Lambda_{n_l}$ for each $l \geq k$ and $\Lambda \in E^0/\sim_{n_1}$. Since each $A^\Lambda_{n_l}$ is irreducible, this forces $z = \rho(A_{n_k}) = \rho(A_E)$, and $(m')^\Lambda_{n_k}$ is a scalar multiple of $m^\Lambda_{n_k}$, so $m' = \sum_{\Lambda} \alpha_{\Lambda} m^\Lambda_{n_k}$. Since the supports of the $m^\Lambda_{n_k}$ are disjoint and the $m^\Lambda_{n_k}$ are positive, the $\alpha_{\Lambda}$ are equal, and their sum is 1 because $m'$ and the $m^\Lambda_{n_k}$ are normalised. This is true for all $l$, continuity implies that $m' = \sum_{\Lambda} \alpha_{\Lambda} m^\Lambda$.

Lemma 8.1.8. Let $E$ be a strongly connected finite directed graph, and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. Suppose that $s > 0$ and $m \in \mathcal{M}^+\left(\lim E^{<\infty}\right)$ satisfy $A_\omega m \leq sm$. Then $s \geq \rho(A_E)$. Moreover, $s = \rho(A_E)$ if and only if $A_\omega m = sm$.

Proof. Since $A_\omega m \leq sm$, we have $A_E m_1 \leq sm_1$, and since $A_E$ is irreducible, the subinvariance theorem [48, Theorem 1.6] implies that $s \geq \rho(A_E)$.

Suppose that $s = \rho(A_E)$. For $k$ such that $\gcd(P_E,n_k) = \gcd(P_E,\omega)$, the matrix $A^\Lambda_{n_k}$ is irreducible by Proposition 5.1.3 so the forward implication of the last assertion of [48, Theorem 1.6] implies that $A^\Lambda_{n_k}m_{n_k} = \rho(A^\Lambda_{n_k})m_{n_k}$. Since $\rho(A^\Lambda_{n_k}) = \rho(A_E)$ for all $k$ by part (2) of Lemma 8.1.7, we deduce that $A_{n_k}m_{n_k} = \rho(A_E)m_{n_k}$ for all $k$. So $A_\omega m = \rho(A_E)m$.

Now suppose that $A_\omega m = sm$. Then part (3) of Lemma 8.1.7 gives $s = \rho(A_E)$.

8.2 Characterising KMS states

In this section we characterise the KMS$_\beta$-states for the gauge action on $\mathcal{T}(E,\omega)$ in terms of their values at spanning elements $t_{\mu} \pi_{(a,k)} t_{\nu}^*$. We show that KMS states $\phi$ induce probability measures $m^\phi$ on $\lim E^{<\infty}$ which satisfy the subinvariance condition $A_\omega m^\phi \leq e^\beta m^\phi$. We also show that a KMS state factors through $C^*(E,\omega)$ if and only if this subinvariance condition is invariance. Our approach follows the general program of [37].
CHAPTER 8. KMS STATES

Theorem 8.2.1. Let $E$ be a strongly connected finite directed graph, and let $\omega = (n_k)_{k=1}^{\infty}$ be a multiplicative sequence. Let $\alpha : \mathbb{R} \to \text{Aut} \mathcal{T}(E, \omega)$ be given by $\alpha_t = \gamma_{\omega^t}$. Let $\beta \in \mathbb{R}$.

1. A state $\phi$ of $\mathcal{T}(E, \omega)$ is a KMS state for $\alpha$ if and only if
   \[
   \phi(t_\mu \pi(\tau,k)t_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} \phi(\pi(\tau,k)) \tag{8.2.1}
   \]
   for all $k \in \mathbb{N}$, all $\tau \in E^{<n_k}$ and all $\mu, \nu \in E^* r(\tau)$.

2. Suppose that $\phi$ is a KMS state for $(\mathcal{T}(E, \omega), \alpha)$, and let $m^\phi$ be the measure on $
\lim\leftarrow E^{<n_k}$ such that $m^\phi(Z(\mu,k)) = \phi(\pi(\mu,k))$ for $\mu \in E^{<n_k}$. Then $m^\phi$ is a probability measure and satisfies the subinvariance relation $A_{\omega} m^\phi \leq e^{\beta} m^\phi$.

3. A KMS state $\phi$ of $(\mathcal{T}(E, \omega), \alpha)$ factors through $C^*(E, \omega)$ if and only if $A_{\omega} m^\phi = e^{\beta} m^\phi$.

Proof. (1) Suppose that $\phi$ is KMS. If $\beta = 0$, then $\phi$ is $\alpha$-invariant by definition, and if $\beta \neq 0$, then $\phi$ is $\alpha$-invariant by [2, Proposition 5.33]. So $\phi$ is also $\gamma$-invariant, and then
   \[
   \phi(t_\mu \pi(\tau,k)t_\nu^*) = \int_{\mathbb{T}} \phi(\gamma_z(t_\mu \pi(\tau,k)t_\nu^*)) dz = \int_{\mathbb{T}} z^{|\mu|-|\nu|} dz \phi(t_\mu \pi(\tau,k)t_\nu^*),
   \]
   which is zero if $|\mu| \neq |\nu|$. If $|\mu| = |\nu|$, then the KMS condition gives
   \[
   \phi(t_\mu \pi(\tau,k)t_\nu^*) = e^{-i\beta|\mu|} \phi(t_\mu \pi(\tau,k)) = \delta_{\mu,\nu} \phi(\pi(\tau,k)).
   \]
   If $\phi$ satisfies (8.2.1), then the argument of [22, Proposition 2.1(a)] and the formula for multiplying spanning elements of $\mathcal{T}(E, \omega)$ in Lemma 3.2.2 shows that $\phi$ is KMS.

(2) We have $m^\phi \geq 0$ because $\phi$ is a state. To see that $m^\phi$ is a probability measure, just observe that $\phi$ restricts to a state of $\pi(C_0(\lim \leftarrow E^{<n}))$, and so $m^\phi$ is a probability measure by the Riesz representation theorem. To see that it satisfies the subinvariance condition,
let $\mu \in E^{<n_k}$ and calculate

$$
\sum_{e \in r(\mu) E^1} \phi(t_e^* \pi(\mu, k)) = \sum_{e \in r(\mu) E^1} e^{-\beta} \phi(t_{e^*} \pi(\mu, k) t_e)
$$

$$
= e^{-\beta} \left\{ \begin{array}{ll}
\phi(\pi(\mu_2 ... \mu_{|\mu|}, k)) & \text{if } \mu \notin E^0 \\
\sum_{ev \in r(\nu) E^{n_k}} \phi(\pi(\nu, k) t_e^* t_e) & \text{if } \mu \in E^0
\end{array} \right.
$$

$$
= e^{-\beta} \left\{ \begin{array}{ll}
m^\phi(Z(\mu_2 ... \mu_{|\mu|}, k)) & \text{if } \mu \notin E^0 \\
\sum_{ev \in r(\nu) E^{n_k}} m^\phi(Z(\nu, k)) & \text{if } \mu \in E^0
\end{array} \right.
$$

$$
= e^{-\beta} A_\omega m^\phi(Z(\mu, k)) \tag{8.2.2}
$$

by Lemma 8.1.5. Hence each

$$
e^\beta m^\phi(Z(\mu, k)) = e^\beta \phi(\pi(\mu, k)) = e^\beta \phi(p_r(\mu) \pi(\mu, k)) \geq \sum_{e \in r(\mu) E^1} e^\beta \phi(t_{e^*} \pi(\mu, k)) = A_\omega m^\phi(Z(\mu, k)).
$$

(3) Recall that $C^*(E, \omega)$ is the quotient of $\mathcal{T}(E, \omega)$ by the ideal generated by the projections $q_v - \sum_{e \in v E^1} t_e^* t_e$, where $v \in E^0$. Thus by Lemma 2.2 of [22] it suffices to check that $\phi(q_v - \sum_{e \in v E^1} t_e^* t_e) = 0$ for all $v$ if and only if $A_\omega m^\phi = e^\beta m^\phi$. For each $v \in E^0$ and $k \in \mathbb{N}$, we have

$$
q_v - \sum_{e \in v E^1} t_e^* t_e = \sum_{\mu \in v E^{<n_k}} (q_r(\mu)) - \sum_{e \in r(\mu) E^1} t_e^* t_e \pi(\mu, k).
$$

Since each term in the last sum is nonnegative, $\phi(q_v - \sum_{e \in v E^1} t_e^* t_e) = 0$ for each $v$ if and only if $\phi((q_r(\mu)) - \sum_{e \in r(\mu) E^1} t_e^* t_e \pi(\mu, k)) = 0$ for all $\mu \in E^{<n_k}$. By (8.2.2) we have

$$
\phi\left((q_r(\mu)) - \sum_{e \in r(\mu) E^1} t_e^* \pi(\mu, k)\right) = e^\beta m^\phi(Z(\mu, k)) - (A_\omega m^\phi)(Z(\mu, k)),
$$

and the result follows. 

\section{8.3 Constructing KMS states at large inverse temperatures}

In this section, for a strongly connected directed graph $E$ and a multiplicative sequence $\omega$, we construct for each measure $m$ satisfying the subinvariance relation $A_\omega m \leq \rho(A_E)m$
of Theorem 8.2.1 (2) a KMS state $\phi_m$ of $\mathcal{T}(E, \omega)$ such that the measure induced by $\phi_m$ is equal to $m$. We also show that there is an affine isomorphism of the collection of subinvariant Borel probability measures onto the KMS simplex of $(\mathcal{T}(E, \omega), \alpha)$. We begin this section by constructing a representation of $\mathcal{T}(E, \omega)$.

Let $E$ be a row-finite directed graph with no sources and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. Define

$$E^* \times_{E^0} \lim E^{<n_k} := \{(\lambda, x) : \lambda \in E^*, x \in \lim E^{<n_k}, s(\lambda) = r(x_1)\},$$

and let $\{h_{\lambda,x} : (\lambda, x) \in E^* \times_{E^0} \lim E^{<n_k}\}$ be the canonical basis for $\ell^2(E^* \times_{E^0} \lim E^{<n_k})$.

Let $\lambda \in E^*$ and let $x \in \lim E^{<n_k}$ and choose a sequence $(x^j)_{j=1}^\infty$ in $\lim E^{<n_k}$ converging to $x$. Then $(r_{n_i}(\lambda, x^j))_{j=1}^\infty$ converges to $r_{n_i}(\lambda, x_i)$ for each $i \geq 1$ and so the function defined by $x \mapsto (r_{n_i}(\lambda, x_i))_{i=1}^\infty$ is continuous from $\lim E^{<n_k}$ to $\lim E^{<n_k}$. So for each $\lambda \in E^*$, there is a map $\alpha_\lambda : C_0(\lim E^{<n_k}) \to C_0(\lim E^{<n_k})$ such that

$$\alpha_\lambda(\chi_{Z(\mu,k)})(x) := \begin{cases} \chi_{Z(\mu,k)}((r_{n_i}(\lambda, x_i))_{i=1}^\infty) & \text{if } s(\lambda) = r(x) \\ 0 & \text{otherwise.} \end{cases}$$

**Proposition 8.3.1.** Let $E$ be a row-finite directed graph with no sources, and let $\omega = (n_k)_{k=1}^\infty$ be a multiplicative sequence. There is a representation $\varsigma : \mathcal{T}(E, \omega) \to \mathcal{B}(\ell^2(E^* \times_{E^0} \lim E^{<n_k}))$ such that for $e \in E^1$ and $v \in E^0$,

$$\varsigma(t_e)h_{\lambda,x} = \delta_{r(\lambda),s(e)}h_{e,\lambda,x} \quad \text{and} \quad \varsigma(q_v)h_{\lambda,x} = \delta_{r(\lambda),v}h_{\lambda,x},$$

and such that for $\mu \in E^{<n_k}$, we have $\varsigma(\pi(\mu,k))h_{\lambda,x} = \alpha_\lambda(\chi_{Z(\mu,k)})(x)h_{\lambda,x}$.

**Proof.** We aim to invoke the universal property of $\mathcal{T}(E, \omega)$. Let $\lambda \in E^*$ and $x \in \lim E^{<n_k}$. Define $T_e$ by $T_e h_{\lambda,x} := \delta_{r(\lambda),s(e)}h_{e,\lambda,x}$ for $e \in E^1$ and define $Q_v$ by $Q_v h_{\lambda,x} := \delta_{r(\lambda),v}h_{\lambda,x}$ for $v \in E^0$. We have

$$(T_e^*T_e) h_{\lambda,x} = \delta_{r(\mu),s(e)}h_{e,\lambda,x} \delta_{r(\mu),s(e)}h_{\lambda,x} = Q_v(e)h_{\lambda,x},$$

and

$$Q_v \left( \sum_{e \in E^1} T_e T_e^* \right) h_{\lambda,x} = Q_v \left( \sum_{e \in E^1} \delta_{e,\lambda_1} T_e h_{\lambda_2 \ldots \lambda_{|\lambda|},x} \right) = Q_v(\delta_{r(\lambda),v}h_{\lambda,x}) = \delta_{r(\lambda),v}h_{\lambda,x} = \left( \sum_{e \in E^1} T_e T_e^* \right) h_{\lambda,x}.$$
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Therefore \((T, Q)\) is a Toeplitz–Cuntz–Krieger \(E\)-family in \(\mathcal{B}(\ell^2(E^* \times E^0 \limleftarrow E^{<n_k}))\).

For each \(k \geq 1\) and \(\mu \in E^{<n_k}\), define \(\Theta_{\mu,k}\) by \(\Theta_{\mu,k} h_{\lambda,x} := \alpha_{\lambda}(\chi_{Z(\mu,k)})(x) h_{\lambda,x}\). Then the \(\Theta_{\mu,k}\) are mutually orthogonal projections satisfying \(\Theta_{\mu,k} = \sum_{\nu \in E^{<n_k+1} | [\nu]_{n_k} = \mu} \Theta_{\nu,k+1}\), so they determine a homomorphism \(\tilde{\Theta} : C_0(\limleftarrow E^{<n_k}) \to \mathcal{B}(\ell^2(E^* \times E^0 \limleftarrow E^{<n_k}))\) such that
\[
\tilde{\Theta}(\chi_{Z(\mu,k)}) h_{\lambda,x} = \alpha_{\lambda}(\chi_{Z(\mu,k)})(x) h_{\lambda,x}.
\]

We show that \((T, Q, \tilde{\Theta})\) is a Toeplitz \(\omega\)-representation of \(E\). Take \(e \in E^1\) and \(\mu \in E^{<n_k}\) and suppose that \(\mu = e \mu'\). For \((\lambda, x) \in E^* \times E^0 \limleftarrow E^{<n_k}\), we have
\[
T_e^* \tilde{\Theta}(\mu, k) h_{\lambda,x} = T_e^* \alpha_{\lambda}(\chi_{Z(\mu,k)})(x) h_{\lambda,x} = \begin{cases} 
\alpha_{\lambda}(\chi_{Z(\mu,k)})(x) h_{\lambda',x} & \text{if } \lambda = e \lambda' \\
0 & \text{otherwise.}
\end{cases}
\]

Also,
\[
\tilde{\Theta}(\mu', k) T_e^* h_{\lambda,x} = \begin{cases} 
\tilde{\Theta}(\mu', k) h_{\lambda',x} & \text{if } \lambda = e \lambda' \\
0 & \text{otherwise}.
\end{cases}
\]

If \(\lambda \neq e \lambda'\) then both \(T_e^* \tilde{\Theta}(\mu, k) h_{\lambda,x}\) and \(\tilde{\Theta}(\mu', k) T_e^* h_{\lambda,x}\) are zero, so suppose that \(\lambda = e \lambda'\). Then \(\alpha_{\lambda}(\chi_{Z(\mu,k)})(x) = \chi_{Z(\mu,k)}(r_{n_i}(\lambda, x_i)_{i=1}^\infty) = 1\) if and only if \(\alpha_{\lambda'}(\chi_{Z(\mu', k)})(x) = 1\) as well; so \(T_e^* \tilde{\Theta}(\mu, k) = \tilde{\Theta}(\mu', k) T_e^*\).

Now let \(v = r(e)\), and observe that
\[
T_e^* \tilde{\Theta}(v, k) h_{\lambda,x} = \begin{cases} 
\alpha_{\lambda}(\chi_{Z(v,k)})(x) h_{\lambda',x} & \text{if } \lambda = e \lambda' \\
0 & \text{otherwise,}
\end{cases}
\]

while
\[
\sum_{\tau \in E^{n_k}} \tilde{\Theta}(\tau, k) T_e^* h_{\lambda,x} = \begin{cases} 
\sum_{\tau \in E^{n_k}} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) h_{\lambda',x} & \text{if } \lambda = e \lambda' \\
0 & \text{otherwise.}
\end{cases}
\]

Again, if \(\lambda \neq e \lambda'\), then both expressions are zero, so we suppose that \(\lambda = e \lambda'\). We have \(\alpha_{\lambda}(\chi_{Z(v,k)})(x) = 1\) if and only if \(r(\lambda) = v\) and \(|\lambda x_i| \in n_i N\) for large \(i\). Also, \(\sum_{\tau \in E^{n_k}} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) = 1\) if and only if \(|\lambda' x_i| \in n_i - 1\) (mod \(n_i\)) for large \(i\), which is equivalent to \(|\lambda' x_i| \equiv n_i - 1\) (mod \(n_i\)) for large \(i\), and so \(T_e^* \tilde{\Theta}(v, k) h_{\lambda,x} = \sum_{\tau \in E^{n_k}} \tilde{\Theta}(\tau, k) T_e^* h_{\lambda,x}\) as required.

Finally, suppose that \(\mu \neq e \mu'\) and \(\mu \neq r(e)\). We immediately see that \(T_e^* \tilde{\Theta}(\mu, k) = 0\) if \(\mu \in E^0 \setminus r(e)\). If \(\mu \notin E^0\), then \(\mu_1 \neq e\), so that \(\tilde{\Theta}(\mu, k)\) is the projection onto a subspace of \(\text{span}\{h_{\lambda,x} : (\lambda x_i)_{1} = \mu_1 \text{ for large } i}\}\), which is orthogonal to the projection \(T_e T_e^*\) onto \(\text{span}\{h_{\lambda,x} : \lambda_1 = e\}\).
We have now established that \((T,Q,ξ)\) is an \(ω\)-representation, and so the universal property of \(T(E,ω)\) gives the desired homomorphism \(ς\).

The following technical result will help in our construction of KMS states.

**Lemma 8.3.2.** Let \(E\) be a strongly connected finite directed graph, and let \(ω = (n_k)_{k=1}^∞\) be a multiplicative sequence. Take \(β > \ln ρ(A_E)\). The series \(\sum_{j=0}^∞ e^{-βj}A_j^ω\) converges in norm to an inverse for \(1 - e^{-β}A_ω\). For \(ε ∈ M^+(\lim_{← E} E^{<nk})\) and \(τ ∈ E^{<nk}\),

\[
(1 - e^{-β}A_ω)^{-1}(ε)(Z(τ, k)) = \sum_{(λ,ν)∈τE^{(nk)^∗}} e^{-β|λ|}ε(Z(ν, k)).
\]

**Proof.** Proposition 8.1.6 gives \(||A_ω|| = ρ(A_E)\). Since \(β > \ln ρ(A_E)\), we have \(||e^{-β}A_ω|| < 1\), and so \(\sum_{j=0}^∞ e^{-βj}A_j^ω\) converges in operator norm to \((1 - e^{-β}A_ω)^{-1}\).

Now take \(τ ∈ E^{<nk}\). Using Lemma 8.1.5 at the second equality, we calculate

\[
(1 - e^{-β}A_ω)^{-1}(ε)(Z(τ, k)) = \sum_{j=0}^∞ e^{-βj}(A_j^ωε)(Z(τ, k))
\]

\[
= \sum_{j=0}^∞ \sum_{ν∈E^{<nk}} e^{-βj}τE(n_k)^jν|ε(Z(ν, k))
\]

\[
= \sum_{j=0}^∞ \sum_{(λ,ν)∈τE(n_k)^j} e^{-βj}ε(Z(ν, k))
\]

\[
= \sum_{(λ,ν)∈τE(n_k)^∗} e^{-β|λ|}ε(Z(ν, k)).
\]

We can now construct a KMS state for each measure that satisfies the subinvariance relation in Theorem 8.2.1(2).

**Proposition 8.3.3.** Let \(E\) be a strongly connected finite directed graph, and let \(ω = (n_k)_{k=1}^∞\) be a multiplicative sequence. Take \(β > \ln ρ(A_E)\). Suppose that \(m ∈ M^+(\lim_{← E} E^{<nk})\) satisfies \(A_ωm ≤ e^β m\). Then there is a KMS\(β\) state \(φ_m\) of \((T(E,ω),α)\) satisfying

\[
φ_m(t_μπ(τ,k)t_ν^*) = δ_{μ,ν}e^{-β|μ|}m(Z(τ, k))
\]

for all \(τ ∈ E^{<nk}\) and all \(μ, ν ∈ E^*r(τ)\).

**Proof.** Let \(ε := (1 - e^{-β}A_ω)m\). Since \(m\) is subinvariant, \(ε\) is a positive measure on \(\lim_{← E} E^{<nk}\). Let \(ς : T(E,ω) → B(ℓ^2(E^* × E^*; \lim_{← E} E^{<nk}))\) be the representation of Proposition 8.3.1. We
aim to define $\phi_m$ by

$$\phi_m(a) = \sum_{\lambda \in E^*} e^{-\beta|\lambda|} \int_{x \in \lim E^{<n_k}} \chi_{Z(s(\lambda),1)}(x) (\varsigma(a)h_{\lambda,x} \mid h_{\lambda,x}) \, d\varepsilon(x). \quad (8.3.2)$$

We first show that for $a \in T(E,\omega)$, the function $f_a : E^* \times E^0 \lim E^{<n_k} \to \mathbb{C}$ given by

$$f_a(\lambda, x) = (\varsigma(a)h_{\lambda,x} \mid h_{\lambda,x})$$

is integrable. First consider $a = t_\mu \pi(\tau,k)t_\nu^*$. We have

$$\varsigma(t_\mu \pi(\tau,k)t_\nu^*)h_{\lambda,x} = (\varsigma(\pi(\tau,k)t_\nu^*)h_{\lambda,x} \mid \varsigma(\pi(\tau,k)t_\mu)h_{\lambda,x}) = \begin{cases} \alpha_{\lambda'}(\chi_{Z(\tau,k)})((x) & \text{if } \lambda = \nu \lambda' = \mu \lambda' \\ 0 & \text{otherwise.} \end{cases} \quad (8.3.3)$$

So $f_a$ is the characteristic function of the clopen set $\bigcup \{Z(\tau, k) : \tau \in E^{<n_k}, [\lambda \tau] \in_k = \mu\}$, and hence integrable. Consequently $f_a$ is integrable for $a \in \text{span}\{t_\mu \pi(\tau,k)t_\nu^*\}$. Now as in \cite[Lemma 10.1(b)]{25}, for $a \in T(E,\omega)$, $f_a$ is a pointwise limit of integrable functions and hence itself integrable as claimed.

Since each $Z(s(\lambda),1)$ is measurable, the functions $\chi_{Z(s(\lambda),1)}f_a$ are also integrable. Since $f_a(\lambda, x) \leq \|a\|$ for all $(\lambda, x)$, we have $\int_{\lim E^{<n_k}} \chi_{Z(s(\lambda),1)}f_a(\lambda, x) \, d\mu(x) < \|a\|$. Since $\beta > \ln \rho(A_E)$, Lemma \ref{8.3.2} implies that $\sum_{\lambda \in E^*} e^{-\beta|\lambda|}$ is convergent for each $\nu$, and so the series on the right-hand side of (8.3.2) is bounded above by the convergent series $\sum_{\nu \in E^0} \sum_{\lambda \in E^*} e^{-\beta|\lambda|}\|a\|$, and hence itself convergent. So there is a bounded linear map $\phi_m : T(E,\omega) \to \mathbb{C}$ satisfying (8.3.2).

This $\phi_m$ is positive because $f_{a_{\nu}}$ is positive-valued. We check that $\phi_m$ is a state. We use Lemma \ref{8.3.2} at the penultimate equality to calculate

$$\phi_m(1) = \sum_{\lambda \in E^*} e^{-\beta|\lambda|} \int_{x \in \lim E^{<n_k}} \chi_{Z(s(\lambda),1)}(x) \, d\varepsilon(x)$$

$$= \sum_{\lambda \in E^*} e^{-\beta|\lambda|} \varepsilon(Z(s(\lambda),1)) = \sum_{w \in E^0} m(Z(w,1)) = 1.$$ 

Since $\mu \lambda' = \nu \lambda'$ forces $\mu = \nu$, we have $\phi_m(t_\mu \pi(\tau,k)t_\nu^*) = 0$ if $\mu \neq \nu$. Moreover, each

$$\varsigma(t_\mu \pi(\tau,k)t_\nu^*)h_{\lambda,x} = (\varsigma(\pi(\tau,k)t_\nu^*)h_{\lambda,x} \mid h_{\lambda,x})$$

is positive-valued. We check that

$$(\varsigma(t_\mu \pi(\tau,k)t_\nu^*)h_{\lambda,x} \mid h_{\lambda,x})^2 = \begin{cases} \alpha_{\lambda'}(\chi_{Z(\tau,k)})(x) & \text{if } \lambda = \mu \lambda' \\ 0 & \text{otherwise.} \end{cases}$$
CHAPTER 8. KMS STATES

Hence

\[
\phi_m(t_\mu \pi(\tau,k) t^*_\mu) = \sum_{\mu \lambda' \in E_*} e^{-\beta |\mu|} \int_{x \in \lim E^{<n_k}} \alpha_{\mu'}(\chi_{Z(\tau,k)})(x) \, d\varepsilon(x) \\
= e^{\beta \mu} \sum_{\lambda' \in s(\mu) E_*} e^{-\beta |\lambda'|} \int_{x \in Z(s(\lambda'),1)} \chi_{Z(\tau,k)}(\{r_{n_i}(\lambda', x_i)\}_{i=1}^{\infty}) \, d\varepsilon(x) \\
= e^{\beta \mu} \sum_{\lambda' \in s(\mu) E_*} e^{-\beta |\lambda'|} \varepsilon(\{x : r_{n_k}(\lambda', x_k) = \tau\}) \\
= e^{\beta \mu} \sum_{(\lambda', x) \in \tau E(n_k)^*} e^{-\beta |\lambda'|} \varepsilon(Z(\nu, k)) \\
= e^{\beta \mu} m(Z(\tau, k)),
\]

which is \(8.3.1\). Putting \(\mu = r(\tau)\) gives \(\phi_m(\pi(\tau,k)) = m(Z(\tau, k))\), and so \(\phi_m\) also satisfies \(8.2.1\), and is therefore KMS by Theorem \(8.2.1\). \(\square\)

To finish off, we show how to obtain subinvariant measures of \(\lim E^{<n_k}\). This will allow us to show that there is an affine isomorphism from the collection of all Borel probability measures on \(\lim E^{<n_k}\) onto the KMS\(_{\beta}\)-simplex.

**Theorem 8.3.4.** Let \(E\) be a strongly connected finite directed graph, and let \(\omega = (n_k)_{k=1}^{\infty}\) be a multiplicative sequence. Let \(\alpha : \mathbb{R} \to \text{Aut}(\mathcal{T}(E, \omega))\) be given by \(\alpha_t = \gamma_{e^t}\). Take \(\beta > \ln \rho(A_E)\).

1. Take \(\varepsilon \in \mathcal{M}^+(\lim E^{<n_k})\). For each \(x \in \lim E^{<n_k}\), the series \(\sum_{\nu \in E^{*}(x)} e^{-\beta |\nu|}\) converges; we write \(y(x)\) for its limit. We have \((1 - e^{-\beta A_\omega})^{-1} \varepsilon \in \mathcal{M}^+_{\text{loc}}(\lim E^{<n_k})\) if and only if

\[
\int_{x \in \lim E^{<n_k}} y(x) \, d\varepsilon(x) = 1.
\]

2. Suppose that \(\varepsilon \in \mathcal{M}^+(\lim E^{<n_k})\) satisfies \(\int_{\lim E^{<n_k}} y(x) \, d\varepsilon(x) = 1\), and define \(m := (1 - e^{-\beta A_\omega})^{-1} \varepsilon\). There is a KMS\(_{\beta}\) state \(\phi_\varepsilon\) of \((\mathcal{T}(E, \omega), \alpha)\) such that

\[
\phi_\varepsilon(t_\mu \pi(\tau,k) t^*_\mu) = \delta_{\mu,\nu} e^{-\beta |\mu|} m(Z(\tau, k)).
\]

3. The map \(\varepsilon \mapsto \phi_\varepsilon\) is an affine isomorphism of

\[
\Omega_{\beta} := \{\varepsilon \in \mathcal{M}^+(\lim E^{<n_k}) : \int y(x) \, d\varepsilon(x) = 1\}
\]

onto the simplex of KMS\(_{\beta}\) states of \((\mathcal{T}(E, \omega), \alpha)\). The inverse of this isomorphism takes a KMS\(_{\beta}\) state \(\phi\) to \((1 - e^{-\beta A_\omega})m^{\phi}\).
In this section we show that \((8.4 \text{ KMS states at the critical temperature})\)

Let Corollary 8.3.5.

\((1)\) The series \(\sum_{j=0}^{\infty} (e^{-\beta} A_{\omega}^j) \xi\) converges to \(m := (1 - e^{-\beta} A_{\omega})^{-1} \xi\) because \(\beta > \ln \rho(A_E)\). This shows that \(m \geq 0\).

Fix \(k \in \mathbb{N}\). Using Lemma 8.3.2 we calculate

\[
m(\lim E^{<nk}) = \sum_{(\lambda,\nu) \in E(nk)^{\ast}} e^{-\beta|\lambda|} \xi(Z(\nu, k)) = \sum_{\nu \in E^{<nk}} \sum_{\lambda \in E^{+\nu}} e^{-\beta|\lambda|} \xi(Z(\nu, k)) = \sum_{\nu \in E^{<nk}} \int_{x \in Z(\nu, k)} y(x) \, d\xi(x) = \int_{x \in \lim E^{<nk}} y(x) \, d\xi(x).
\]

(2) We claim that \(A_{\omega} m \leq e^{\beta} m\). We calculate

\[
A_{\omega} m = A_{\omega} \left( \sum_{j=0}^{\infty} e^{-\beta} A_{\omega}^j \xi \right) \leq e^{\beta} \left( \sum_{j=0}^{\infty} e^{-\beta} A_{\omega}^j \xi \right) = e^{\beta} m.
\]

Now Proposition 8.3.3 gives a KMS\(_{\beta}\) state \(\phi_{\xi}\) satisfying (8.3.4).

(3) We claim that every KMS\(_{\beta}\) state \(\phi\) has the form \(\phi_{\xi}\). Fix a KMS\(_{\beta}\) state \(\phi\), and let \(m^\phi\) be the measure such that \(m^\phi(Z(\mu, k)) = \phi(\pi(\mu, k))\). By part (2), \(m^\phi\) is a subinvariant probability measure. Let \(\xi := (1 - e^{-\beta} A_{\omega})^{-1} m^\phi\). Then \(m^\phi = (1 - e^{-\beta} A_{\omega}) \xi\) by construction, and comparing (8.3.4) with (8.2.1) shows that \(\phi = \phi_{\xi}\).

The formula (8.3.4) also shows that the map \(F : \xi \mapsto \phi_{\xi}\) is injective and weak*-continuous from \(\Omega_{\beta}\) to the state space of \(\mathcal{T}(E, \omega)\). We have just seen that it is surjective onto the KMS\(_{\beta}\) simplex, which is compact since \(C^*(E, \omega)\) is unital. Hence \(F\) is a homeomorphism of \(\Omega_{\beta}\) onto the KMS\(_{\beta}\) simplex. The formula (8.3.2) shows that \(F\) is affine, and the formula for the inverse follows from our proof of surjectivity in the preceding paragraph.

\[\square\]

**Corollary 8.3.5.** Let \(E\) be a strongly connected finite directed graph, and let \(\omega = (n_k)_{k=1}^{\infty}\) be a multiplicative sequence. Let \(\alpha : \mathbb{R} \rightarrow \text{Aut}(\mathcal{T}(E, \omega))\) be given by \(\alpha_t = \gamma^{it}\). Take \(\beta > \ln \rho(A_E)\). Let \(y\) be as in part (1) of Theorem 8.3.4. The map \(m \mapsto \phi_{y^{-1} m}\) is an affine isomorphism of \(\mathcal{M}_1^+ \left( \lim E^{<nk} \right)\) onto the KMS\(_{\beta}\)-simplex of \((\mathcal{T}(E, \omega), \alpha)\).

**Proof.** Since \(y\) takes strictly positive values and is bounded, \(m \mapsto y^{-1} m\) is an affine isomorphism of \(\mathcal{M}_1^+ \left( \lim E^{<nk} \right)\) onto \(\Omega_{\beta}\), so the result follows from Theorem 8.3.4(3).

\[\square\]

### 8.4 KMS states at the critical temperature

In this section we show that \((\mathcal{T}(E, \omega), \alpha)\) admits exactly \(\gcd(P_E, \omega)\) extreme KMS states at the critical temperature \(\ln \rho(A_E)\); these states factor through KMS\(_{\ln \rho(A_E)}\) states of \((C^*(E, \omega), \alpha)\). We show that there are no KMS\(_{\beta}\)-states for \(C^*(E, \omega)\) for any \(\beta \neq \ln \rho(A_E)\).
We also show that \( C^*(E, \omega) \) is simple precisely when there is a unique KMS state for the gauge action.

**Theorem 8.4.1.** Let \( E \) be a strongly connected finite directed graph, and let \( \omega = (n_k)^{\infty}_{k=1} \) be a multiplicative sequence. Fix \( K \) such that \( \gcd(P_E, n_K) = \gcd(P_E, \omega) \), and let \( \sim_{n_K} \) be the equivalence relation on \( E^0 \) of Lemma 5.1.4. Let \( \alpha : \mathbb{R} \to \text{Aut}(\mathcal{T}(E, \omega)) \) be given by \( \alpha_t = \gamma_{e^t} \). Let \( x^E \) be the unimodular Perron–Frobenius eigenvector of \( A_E \).

1. For each \( \Lambda \in E^0/\sim_{n_K} \), there is a KMS\(_{\ln \rho(A_E)} \) state \( \phi^\Lambda \) for \( (\mathcal{T}(E, \omega), \alpha) \) satisfying

\[
\phi^\Lambda(t_\mu \pi(\tau, k) t_\nu^*) = \chi_\Lambda(s(\tau)) \delta_{\mu, \nu} \frac{1}{\sum_{v \in \Lambda} x_v^E} \rho(A_E)^{-|\mu| - |\nu|} x_v^E.
\]

This is the unique KMS\(_{\ln \rho(A_E)} \) state for \( (\mathcal{T}(E, \omega), \alpha) \) satisfying \( \phi^\Lambda(\pi(v, k)) = 0 \) for all \( v \in E^0 \setminus \Lambda \), and it factors through a KMS\(_{\ln \rho(A_E)} \) state \( \overrightarrow{\phi}^\Lambda \) of \( (C^*(E, \omega), \alpha) \).

2. The states \( \overrightarrow{\phi}^\Lambda \) are the extreme points of the KMS\(_{\ln \rho(A_E)} \)-simplex of \( (C^*(E, \omega), \alpha) \), and there are no KMS\(_\beta \)-states for \( (C^*(E, \omega), \alpha) \) for any \( \beta \neq \ln \rho(A_E) \).

**Proof.**

1. Fix \( \Lambda \in E^0/\sim_{n_K} \). We first prove the existence of a KMS\(_{\ln \rho(A_E)} \) state satisfying (8.4.1). For each \( k \geq K \), let \( E(n_k)_\Lambda \) be the component of \( E(n_k) \) with vertices \( E^{<n_k}_\Lambda \). Theorem 4.3(a) of [23] shows that there is a unique KMS state \( \phi^\Lambda_k \) of \( C^*(E(n_k)) \cong C^*(E, n_k) \) that vanishes on \( \varepsilon_{n_k, \mu} \) for \( \mu \in E^{<n_k}(E^0 \setminus \Lambda) \). Since each \( \phi^\Lambda_{k+1} \) must restrict to a KMS state of \( C^*(E, n_k) \), the \( \phi^\Lambda_k \) are compatible under the inclusions \( C^*(E, n_k) \hookrightarrow C^*(E, n_{k+1}) \). So continuity yields a state \( \phi^\Lambda \) on \( C^*(E, \omega) \) that agrees with each \( \phi^\Lambda_{n_k} \) on the image of \( C^*(E, n_k) \), and hence satisfies (8.4.1). It follows that \( \phi^\Lambda(\pi(v, k)) = 0 \) for all \( v \in E^0 \setminus \Lambda \). Uniqueness follows from uniqueness of the \( \phi^\Lambda_{n_k} \). Theorem 8.2.1[3] shows that \( \phi^\Lambda \) factors through \( (C^*(E, \omega), \alpha) \).

2. The \( \phi^\Lambda \) are linearly independent, and so are the extreme points of the convex set they generate. So it suffices to show that every KMS state of \( C^*(E, \omega) \) is a convex combination of the \( \phi^\Lambda \). Suppose that \( \psi \) is a KMS\(_\beta \) state of \( (C^*(E, \omega), \alpha) \). Let \( q : \mathcal{T}(E, \omega) \to C^*(E, \omega) \) be the quotient map. Theorem 8.2.1[3] implies that \( A_\omega m^{\psi_{\omega\beta}} = e^\beta m^{\psi_{\omega\beta}} \). Hence Lemma 8.1.7[3] shows that \( m^{\psi_{\omega\beta}} \) is a convex combination \( m^{\psi_{\omega\beta}} = \sum_k t_k m^\Lambda \) of the \( m^\Lambda \). It then follows from Theorem 8.2.1[3] that \( \psi \circ q = \sum_k t_k \phi^\Lambda \).

Theorem 8.2.1[3] combined with Lemma 8.1.7[3] shows that there are no KMS states for \( C^*(E, \omega) \) at any other inverse temperature. \( \square \)

**Proof of Theorem 8.0.4.** Item (1) follows from Corollary 8.3.5 and item (2) follows from Theorem 8.4.1.
For item (4), recall that Theorem 8.2.1(3) implies that a KMS$_\beta$ state $\phi$ factors through $C^*(E, \omega)$ if and only if $A_\omega m^\phi = e^{-\beta} m^\phi$. If $\phi$ factors through $C^*(E, \omega)$, then $m^\phi$ is a positive eigenmeasure for $A_\omega$ and Lemma 8.1.7 gives $\beta = \ln \rho(A_E)$. On the other hand, if $\beta = \ln \rho(A_E)$, then Theorem 8.2.1(2) gives $A_\omega m^\phi \leq \rho(A_E)m^\phi$, and then Lemma 8.1.8 forces equality.

Finally, for (3), suppose that $\phi$ is a KMS$_\beta$ state of $(\mathcal{C}(E, \omega), \alpha)$. Then Theorem 8.2.1(2) implies that $A_\omega m^\phi \leq e^{\beta} m^\phi$, and then Lemma 8.1.8 gives $e^\beta \geq \rho(A_E)$ and hence $\beta \geq \ln \rho(A_E)$.

We deduce that simplicity of $C^*(E, \omega)$ is reflected by the existence of a unique KMS state for the gauge action.

Recall that a KMS state $\phi$ is a factor state if the von Neumann algebra generated by the GNS representation $\pi$ has a trivial center. By [2, Theorem 5.3.30], extremal KMS states are factor states.

**Proposition 8.4.2.** Let $E$ be a strongly connected finite directed graph, and take a multiplicative sequence $\omega = (n_k)_{k=1}^\infty$. Let $\alpha : \mathbb{R} \to \text{Aut}(\mathcal{T}(E, \omega))$ be given by $\alpha_t = \gamma_{te^{it}}$. The following are equivalent

1. $\text{gcd}(\mathcal{P}_E, \omega) = 1$;
2. $C^*(E, \omega)$ is simple;
3. there is a unique KMS state for $(C^*(E, \omega), \alpha)$ and the state (8.4.1) factors through this state; and
4. the state (8.4.1) is a factor state.

**Proof.** Corollary 5.2.2 gives (1) $\iff$ (2), and Theorem 8.4.1 gives (1) $\implies$ (3). To establish (3) $\implies$ (4), suppose that $\phi$ factors through the unique KMS state of $(C^*(E, \omega), \alpha)$. Then it is an extreme point of the KMS simplex and hence a factor state by [2, Theorem 5.3.30](3).

For (4) $\implies$ (1) let $\phi$ be the state given by (8.4.1) and suppose that $\phi$ is a factor state for $\mathcal{T}C^*(E, \omega)$. Fix $k$ such that $\text{gcd}(\mathcal{P}_E, n_k) = \text{gcd}(\mathcal{P}, \omega)$. Recall the equivalence relation $\sim_{n_k}$ of Lemma 5.1.1 and the projections $Q_{k, \Lambda}$ of Lemma 5.2.1. We have $\phi(\pi_{(\mu, k)}) = \frac{1}{n_k} \rho(A_E)^{-|\mu|} x_{s(\mu)}^{E} \neq 0$ for all $\mu$ because the Perron–Frobenius eigenvector has strictly positive entries. So each $\phi(Q_{k, \Lambda}) \neq 0$. So the GNS representation $\pi_\phi$ is also nonzero on the $Q_{k, \Lambda}$. Lemma 5.2.1 implies that the $Q_{k, \Lambda}$ are central in $\mathcal{T}(E, \omega)$, and so the $\pi_\phi(Q_{k, \Lambda})$ are mutually orthogonal central projections in $\pi_\phi(\mathcal{T}(E, \omega))^\prime$. Since $\phi$ is a factor state, it follows that there is only one equivalence class $\Lambda$ for $\sim_{n_k}$, and so $\text{gcd}(\mathcal{P}_E, \omega) = 1$. \qed
Bibliography


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