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Abstract

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ON THE CLASSICAL SOLVABILITY OF NEAR FIELD REFLECTOR PROBLEMS

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Abstract. In this paper we prove the existence of classical solutions to near field reflector problems, both for a point light source and for a parallel light source, with planar receivers. These problems involve Monge-Ampère type equations, subject to nonlinear oblique boundary conditions. Our approach builds on earlier work in the optimal transportation case by Trudinger and Wang and makes use of a recent extension of degree theory to oblique boundary conditions by Li, Liu and Nguyen.

1. **Introduction.** The near field reflector problem, with a point light source, can be described as follows: there is a light source at the origin O, a reflecting surface Γ which is a radial graph over a domain $U \subset \mathbb{S}^n$, where $\mathbb{S}^n$ is the unit sphere in $\mathbb{R}^{n+1}$, and a bounded smooth receiver Σ to be illuminated. Let $\hat{f} \in L^1(U)$ be the illumination on U, i.e. the intensity of incident rays, and let $\hat{g} \in L^1(\Sigma)$ be a nonnegative function satisfying

$$\int_U \hat{f} = \int_\Sigma \hat{g}. \tag{1}$$

We are concerned with the existence of reflector Γ such that the light emitting from O with intensity $\hat{f}$ is reflected off to the receiver Σ and the intensity of reflected light on Σ is equal to $\hat{g}$.

We always assume that the reflection system is ideal, namely there is no loss of energy in reflection, (1). We represent the reflector Γ in the polar coordinate system as

$$\Gamma = \{X\rho(X) \,:\, X \in U\} \tag{2}$$

with a positive function ρ. Assuming that U lies in the northern hemisphere $\mathbb{S}^n_+ := \mathbb{S}^n \cap \{x_{n+1} > 0\}$, we project U to $\Omega \subset \{x_{n+1} = 0\}$ so that

$$x = (x_1, \cdots, x_n) \in \Omega$$

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if and only if

$$X = (x, x_{n+1}) \in U, \quad x_{n+1} = \sqrt{1 - |x|^2} =: \omega(x).$$

Hence, we may regard $\rho = \rho(x)$ as a function in Ω. Throughout this paper, we also assume that the closure $\overline{U} \subset \mathbb{S}^n_+$, or equivalently, the closure $\overline{\Omega} \subset B_1(0)$, where $B_1(0)$ is the unit ball in $\mathbb{R}^n$; and the receiver $\Sigma = \Omega^*$ for a bounded domain $\Omega^* \subset \{x_{n+1} = 0\}$.

By setting $u = \rho^{-1}$ and assuming $f = \hat{f}(\cdot, \omega)/\omega, g = \hat{g}(\cdot, 0) > 0$, one has a Monge-Ampère type equation (see [9] and §2.1)

$$\det [D^2 u] = \frac{f \left||Du|^2 - (u - Du \cdot x)^2\right|^{n+1}}{2^n g \circ T \left||Du|^2 + u^2 - (Du \cdot x)^2\right|} \quad \text{in } \Omega, \tag{3}$$

for elliptic solutions u, that is $D^2 u > 0$, together with a constraint

$$|Du| < u - x \cdot Du. \tag{4}$$

The corresponding boundary condition is

$$T(\Omega) = \Omega^*, \tag{5}$$

where $T = T_u$ is the reflection mapping given by

$$T_u(x) = Y(x, u, Du) = \frac{2Du}{|Du|^2 - (u - Du \cdot x)^2}. \tag{6}$$

Another special case of reflector problems is the far field case, which is related to the reflector antenna design problem [27] and has been extensively studied. It can be regarded as the limit of the above problem with $\Sigma = \{dX : X \in V\}$, $d \to \infty$, where V is a domain in $\mathbb{S}^n$ [9]. The existence and interior regularity for weak solutions were first established in [27] in dimension two, which can be extended to higher dimensions by the a priori estimates in [5]. By a duality, the far field case can be formulated as an optimal transportation problem [28]. Global regularity and the existence of classical solutions then follows from [22]. Mathematically, one may also consider the case when the reflector is a closed surface without boundary. In this case the existence of weak solutions was proved in [2], and the regularity was proved in [5] if $f, g \in C^\infty$ and f, g are pinched by two positive constants.

In the near field reflector problem, weak solutions were introduced and obtained in [9, 10], and criteria for local interior regularity for general targets were found in [9]. Along the lines of [28], in [14] the near field reflector problem was formulated as a nonlinear optimization problem, to which the regularity results in optimal transportation theory cannot be applied directly. In this paper we first establish the global C^2 estimate for solutions of (3)–(6) and then obtain the existence of classical solutions.

Our essential estimate is the following.

Theorem 1.1. *Assume that f, g are smooth and have positive upper and lower bounds. Let Ω, Ω^* be bounded C^4 domains in $\mathbb{R}^n$ and $\overline{\Omega} \Subset B_1(0)$. Suppose that Ω is uniformly convex, and Ω^* is uniformly Y^*-convex with respect to $\Omega \times \mathcal{I}$ for any bounded interval $\mathcal{I} \subset (\delta, \infty)$ for some fixed $\delta > 0$. Let $u > \delta, \in C^4(\overline{\Omega})$ be an elliptic solution of (3)–(6). Then we have the a priori estimate*

$$\sup_{\Omega} |D^2 u| \le C, \tag{7}$$

where C depends on f, g, Ω, Ω^* and $\inf_\Omega u$.

The notion of Y^*-convexity, which is adapted from [19, 20], is defined in Section 2.2. In particular if $0 \in \Omega^*$ and Ω^* is convex in the usual sense then Ω^* is uniformly Y^*-convex with respect to $\Omega \times \mathcal{I}$ for any domain $\Omega \Subset B_1(0)$ and interval $\mathcal{I} \Subset (0, \infty)$. We point out that the boundary condition (5)–(6) is related, but not equivalent, to the boundary condition of prescribing the image of the gradient mapping,

$$Du(\Omega) = \Omega^*, \tag{8}$$

where Ω^* is a domain in $\mathbb{R}^n$. The boundary problem (8) has been extensively studied; see for example [1, 3, 23, 24] and references therein.

As a consequence of Theorem 1.1, we have the following existence result for classical solutions. The proof is based on a degree theory recently developed in [11] for second order elliptic operators with nonlinear oblique boundary conditions.

Theorem 1.2. *Suppose in addition to the hypotheses in Theorem 1.1 that the balance condition (1) is satisfied. Then, there exists a solution $\rho \in C^3(\overline{\Omega})$ of the near field reflector problem satisfying $\rho \leq 1/\delta$.*

In the last part of this paper, we introduce another model of reflector problems with a parallel light source. Consider the situation when light is emitted from a bounded domain $\Omega \subset \mathbb{R}^n \hookrightarrow \mathbb{R}^{n+1}$ along direction e_{n+1}, where $\mathbb{R}^n$ is identified with $\mathbb{R}^{n+1} \times \{0\}$, and $e_{n+1} = (0, \cdots, 0, 1)$. We assume that the reflector Γ is represented as a graph over Ω and the light is reflected back to a bounded domain $\Omega^* \subset \mathbb{R}^n$ so that the prescribed intensities f, g on Ω, Ω^* are realized, respectively. This problem has many applications, for example, in the design of reflectors for lamps.

Similarly as above we show that for uniformly Y, Y^*-convex domains $\Omega, \Omega^* \subset \mathbb{R}^n$ with smooth distributions $C^{-1} \leq f, g \leq C$ supported on Ω, Ω^*, respectively, there exists a reflector Γ such that light emitted with intensity f from Ω is reflected to Ω^* and the intensity g is realized provided that

$$\int_\Omega f = \int_{\Omega^*} g. \tag{9}$$

A detailed description of this model and a formulation of corresponding theorems are contained in Section 50. Moreover the essential features in this case also occur in refractor problems for parallel beams, as introduced in the work of Gutiérrez and Tournier [6] and Oliker, Rubinstein and Wolanski [16, 17] so we also conclude classical existence in these cases by applying our general estimates and existence procedure.

This paper is organized as follows. In Section 2 we first derive equation (3) by considering general prescribed Jacobian equations, and then introduce some preliminary notations and results. In Section 3 we prove that the boundary condition (5)–(6) is oblique in the context of general prescribed Jacobian equations and estimate the obliqueness under the hypotheses of uniform Y and Y^*-convexity of Ω and Ω^*, by following the argument in [20]. Specializing to the Monge-Ampère equation (3),Theorem 1.1 then follows from [25]. In Section 4 we prove Theorem 1.2 by using the degree theory for oblique boundary value problems developed in [11]. In Section 5 we present the reflector problem with a parallel light source and state the main theorem for this problem as well as the extensions to refractor problems.

Finally we point out that the existence of an infinite number of classical solutions follows from Theorem 1.2 and moreover we can correspondingly refine our domain convexity conditions . In a sequel paper we consider the existence of solutions with value prescribed at a fixed point. For this we use the method proposed in [20] which necessitates more complicated estimates.

2. Preliminaries.

2.1. Derivation of equation.

2.1. Derivation of equation. Suppose that the ray $X \in U$ is reflected off at $X\rho(X) \in \Gamma$ in direction $\hat{Y}$ and reaches $Y \in \Sigma$. Let γ be the unit normal of Γ at $X\rho(X)$. By calculations one has

$$\gamma = \frac{(D\rho, 0) - X(\rho + D\rho\cdot x)}{\sqrt{|D\rho|^2 + \rho^2 - (D\rho\cdot x)^2}}, \tag{10}$$

then from the reflection law, the reflection direction is

$$\begin{aligned}\hat{Y} &= X - 2(X\cdot\gamma)\gamma \\ &= X\frac{|D\rho|^2 - (\rho + D\rho\cdot x)^2}{|D\rho|^2 + \rho^2 - (D\rho\cdot x)^2} + \frac{2\rho(D\rho,0)}{|D\rho|^2 + \rho^2 - (D\rho\cdot x)^2} \\ &= \frac{a}{b}X + \frac{2\rho}{b}(D\rho, 0),\end{aligned} \tag{11}$$

where $a := |D\rho|^2 - (\rho + D\rho\cdot x)^2$ and $b := |D\rho|^2 + \rho^2 - (D\rho\cdot x)^2$.

Let $d = |Y - X\rho|$ be the length of the reflected ray. Then,

$$Y = T(X) = X\rho + \hat{Y}d, \tag{12}$$

where $T : U \to \Sigma$ is the reflection mapping. Since $Y \in \{x_{n+1} = 0\}$, from (12) we have

$$0 = x_{n+1}\rho + \hat{y}_{n+1}d$$

and from (11),

$$\hat{y}_{n+1} = \frac{a}{b}x_{n+1}.$$

Therefore, $d = -\frac{b}{a}\rho$, and by (11)–(12) again, we obtain

$$Y = -\frac{2\rho^2}{a}(D\rho, 0). \tag{13}$$

Regarding T as a mapping from $\Omega \subset \mathbb{R}^n$ to $\Omega^* \subset \mathbb{R}^n$, we then have

$$y = T(x) = -\frac{2\rho^2 D\rho}{|D\rho|^2 - (\rho + D\rho\cdot x)^2}. \tag{14}$$

Let $u = \rho^{-1}$. The reflection mapping T in (14) can be written as

$$y = T(x) = \frac{2Du}{|Du|^2 - (u - Du\cdot x)^2}. \tag{15}$$

This is a special case of considering a general mapping Y from $\Omega\times\mathbb{R}\times\mathbb{R}^n$ into $\mathbb{R}^n$ [20]. Denoting points in $\Omega\times\mathbb{R}\times\mathbb{R}^n$ by (x, z, p), we see that from (15)

$$Y(x, z, p) = \frac{2p}{|p|^2 - (z - p\cdot x)^2}. \tag{16}$$

The reflector equation is a special case of a prescribed Jacobian equation

$$|\det DY(\cdot, u, Du)| = f/g\circ Y(\cdot, u, Du). \tag{17}$$

Since

$$DY = Y_p\, D^2u + Y_x + Y_z\otimes Du, \tag{18}$$

we then obtain, when $\det Y_p \neq 0$, the following Monge-Ampère type equation

$$\det\left[D^2u - A(\cdot, u, Du)\right] = B(\cdot, u, Du), \tag{19}$$

for elliptic solutions u, that is $D^2u > A(\cdot, u, Du)$, where the matrix function A and scalar function B are given by

$$A(\cdot, z, p) = -Y_p^{-1}(Y_x + Y_z \otimes p), \quad B(\cdot, z, p) = |\det Y_p|^{-1}\frac{f}{g\circ Y}. \tag{20}$$

For $Y = Y(x, z, p)$ in (16), it is easy to check that

$$Y_x + Y_z \otimes Du = 0, \tag{21}$$

that is $A = 0$, and

$$Y_p = \frac{2}{|Du|^2 - (u - Du\cdot x)^2}\left[I - \frac{2Du\otimes(Du + (u - Du\cdot x)x)}{|Du|^2 - (u - Du\cdot x)^2}\right]. \tag{22}$$

Using the formula $\det[I + \xi\otimes\eta] = 1 + \xi\cdot\eta$ for any vector $\xi, \eta \in \mathbb{R}^n$, we have

$$\det Y_p = \frac{2^n[(Du\cdot x)^2 - |Du|^2 - u^2]}{[|Du|^2 - (u - Du\cdot x)^2]^{n+1}}. \tag{23}$$

Combining (19), (20), (21) and (23), we then obtain equation (3) for elliptic solutions u.

Remark 1. Noting that U lies in the northern hemisphere, we have $x_{n+1} > 0$, $\hat{y}_{n+1} < 0$. Hence

$$a = \frac{\hat{y}_{n+1}}{x_{n+1}}b < 0, \tag{24}$$

which implies (4) and shows also that T is well defined.

By computing the Jacobian determinant of $T = T_\rho$ in (14), Karakhanyan and Wang obtained the equation for ρ in [9]

$$\det\left[-D^2\rho + \frac{2}{\rho}D\rho\otimes D\rho\right] = \frac{f}{2^n\rho^{2n}\omega g}\left|\frac{a^{n+1}}{b}\right| \quad \text{in } \Omega, \tag{25}$$

which is equivalent to equation (3).

2.2. **Domain convexity.** We introduce some domain convexity notions adapted from [19, 20]. Let us suppose that the mapping Y is defined and C^1 in an open set $\mathcal{U}_0 \subset \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n$, with $Y_p \neq 0$ in $\mathcal{U}_0$. In our reflector problem, Y is equal to (16) and $\mathcal{U}_0 = \{(x, z, p) \in \mathbb{R}^n\times\mathbb{R}\times\mathbb{R}^n : x \in B_1, |p| < z - x\cdot p\}$. Let $\mathcal{I}$ be an open interval such that the sets $\mathcal{P}_0(x, z) = \{p \in \mathbb{R}^n : (x, z, p) \in \mathcal{U}_0\}$ are non-empty for all $(x, z) \in \overline{\Omega}\times\overline{\mathcal{I}}$. For $\overline{\Omega}^* \subset Y(\mathcal{U}_0)$, we define the subsets, $\mathcal{P}(x, z) = \{p \in \mathcal{P}_0(x, z) : Y(x, z, p) \in \Omega^*\}$.

We first define the appropriate convexity notions for the target domain Ω^* as it is already used in the formulation of Theorem 1.1. Namely,

Definition 2.1. The target domain Ω^* is Y^*-convex with respect to a point $(x, z) \in \overline{\Omega}\times\overline{\mathcal{I}}$ if the set $\mathcal{P}(x, z)$ is convex in $\mathbb{R}^n$ and is uniformly Y^*-convex with respect to $\Omega\times\mathcal{I}$ if $\mathcal{P}(x, z)$ is uniformly convex for all $(x, z) \in \overline{\Omega}\times\overline{\mathcal{I}}$.

By pulling back from $\mathcal{P}(x, z)$ to Ω^* and using the local invertibility of Y with respect to p, we may express these notions in terms of boundary data for C^2 domains Ω^*, and mappings $Y \in C^2(\mathcal{U}_0)$, as done in [22]. Accordingly we have that the target domain Ω^* is Y^*-convex (uniformly Y^*-convex) with respect to $\Omega\times\mathcal{I}$, if it is connected and

$$[D_i\gamma_j^*(y) - A^*_{ij,k}(x, z, p)\gamma_k^*(y)]\tau_i\tau_j \geq 0, \quad (\delta_0^*), \tag{26}$$

for all $(x,z) \in \overline{\Omega} \times \overline{\mathcal{I}}$, $y = Y(x,z,p) \in \partial\Omega^*$, unit outer normal γ^* and unit tangent vector τ, (for some constant $\delta_0^* > 0$), where

$$[A^*_{ij,k}](x,z,p) = -Y_p^{-1} D_{pp} Y^k (Y_p^{-1})^t.$$

In optimal transportation, Y is independent of z and the Y^*-convexity condition agrees with the corresponding c^*-convexity of the target Ω^* introduced in [15, 22].

As in [22] the key role of the uniform convexity condition in global estimates is in barrier constructions arising from a further formulation in terms of defining functions. Using the distance function as in [22], we obtain that Ω^* is uniformly Y^*-convex with respect to $\Omega \times \mathcal{I}$, if it is connected and there exists a defining function $\phi^* \in C^2(\overline{\Omega^*})$ satisfying $\phi^* = 0$, $D\phi^* \neq 0$ on $\partial\Omega^*$ together with

$$D_p^2 \phi^* \circ Y(x,z,p) \geq \kappa_0^* I_{n\times n}, \tag{27}$$

for all $(x,z) \in \overline{\Omega} \times \overline{\mathcal{I}}$, $y = Y(x,z,p) \in \partial\Omega^*$, where $\kappa_0^* > 0$ is a constant. Again using the local invertibility of Y with respect to p, it follows that (27) holds more generally for $y \in \mathcal{N}^* \cap \Omega^*$ for some neighbourhood $\mathcal{N}^*$ of $\partial\Omega^*$, for a further constant $\kappa_0^* > 0$. Note that by dividing ϕ^* by an appropriate constant, we can assume $\kappa_0^* = 1$.

For optimal generality in our obliqueness estimate, we will employ the analogue of this characterisation of uniform convexity for our definitions for the initial domain Ω.

Definition 2.2. The C^2 domain Ω is Y-uniformly convex with respect to $\Omega^* \times \mathcal{I}$, if it is connected and there exists a defining function $\phi \in C^2(\overline{\Omega})$ satisfying $\phi = 0$, $D\phi \neq 0$ on $\partial\Omega$ together with,

$$D^2\phi(x) - \partial_{p_k} A(x,z,p) D_k \phi(x) \geq I_{n\times n}, \tag{28}$$

for all $x \in \mathcal{N} \cap \Omega$, $z \in \mathcal{I}$, $y = Y(x,z,p) \in \Omega^*$, for some neighbourhood $\mathcal{N}$ of $\partial\Omega$, where A is the matrix in (20) generated by the mapping Y.

Note that for our reflector problem $A \equiv 0$ and uniform Y-convexity is equivalent to the usual uniform convexity. These convexity notions can be equivalently expressed in terms of boundary data, corresponding to (26), when the mapping Y is globally invertible with respect to p for each $(x,z) \in \overline{\Omega} \times \overline{\mathcal{I}}$, which is the case for our parallel beam examples in Section 6, (and more generally when Y arises from a generating function as in [21]). It then follows that Ω is uniformly Y-convex with respect to $\Omega^* \times \mathcal{I}$, if it is connected and

$$[D_i\gamma_j(x) - A_{ij,p_k}(x,z,p)\gamma_k(x)]\tau_i\tau_j \geq \delta_0, \tag{29}$$

for all $x \in \partial\Omega, z \in \mathcal{I}, Y(x,z,p) \in \Omega^*$, unit outer normal γ and unit tangent vector τ, for some constant $\delta_0 > 0$. Conversely we note that (28) always implies (29).

2.3. **Gradient estimates.** By writing the constraint (4) in the form,

$$|D\log u| < 1 - x \cdot D\log u, \tag{30}$$

we immediately infer a bound

$$|D\log u| < 1/d_0, \tag{31}$$

where $d_0 = \mathrm{dist}(\Omega, \partial B_1)$ from which follows a Harnack inequality

$$\sup_\Omega u < C \inf_\Omega u, \tag{32}$$

with C depending on d_0, which is a special case of Lemma 4.2 in [9]. More generally, if we only assume $\Omega \subset B_1$, as in [9], then we obtain corresponding estimates under the condition

$$|x \cdot y| \leq (1 - d_0)|y|. \tag{33}$$

for all $x \in \Omega, y \in \Omega^*$, for some positive constant $d_0 > 0$, which then corresponds to Lemmas 4.2, 4.3 in [9].

Next we need to use the boundary condition (5) to strengthen (4), in order to estimate $|\det Y_p|$ in (20), thereby controlling the right hand side of (3). First writing $y = Y(x, z, p)$ in (16), we can estimate,

$$z - p \cdot x - |p| \geq \min\{z, 1/|y|\}/C, \tag{34}$$

for $(x, z, p) \in \mathcal{U}_0$, $x \in \Omega$ and a further constant C depending on d_0. Consequently setting $d^* = \sup_{y \in \Omega^*} |y|$, we obtain from (5), (6) and (32),

$$u - x \cdot Du - |Du| \geq \min\{\inf_{\Omega} u, 1/d^*\}/C. \tag{35}$$

Also we note here that the mapping Y is globally invertible with respect to p in $\mathcal{U}_0$ and we have an explicit formula for the inverse,

$$p = P(x, y, z) = \frac{-z^2 y}{\sqrt{z^2|y|^2 + 2z(x \cdot y) + 1} + z(x \cdot y) + 1}, \tag{36}$$

which gives us also an explicit expression for the gradient Du in terms of u and Tu.

2.4. **Convex targets.** For our reflector problem we clearly have that the sets $\mathcal{P}(x, z)$ are bounded, independently of the target domain Ω^*. Let us now suppose that Ω^* is convex and contains the origin. We claim that Ω^* is uniformly Y^*-convex with respect to any $\Omega \Subset B_1(0)$ and $\mathcal{I} \Subset (0, \infty)$. To see this we fix a point $p_0 = P(x, z, y_0) \in \partial\mathcal{P}(x, z)$ and a support hyperplane H_0 to Ω^* at y_0, given by $\{\alpha \cdot y = 1\}$ for some vector $\alpha \in \mathbb{R}^n$. Then for $y \in H_0$, $p = P(x, y, z)$, we have

$$\frac{2\alpha \cdot p}{|p|^2 - (z - x \cdot p)^2} = 1$$

and hence there exists a supporting enclosing ellipsoid to $\mathcal{P}(x, z)$ at p_0, with equation,

$$|p|^2 - (x \cdot p)^2 + 2(zx - \alpha) \cdot p = z^2. \tag{37}$$

If $0 \in \partial\Omega^*$, then we clearly have that Ω^* is Y^*-convex and uniformly Y^*-convex if Ω^* is uniformly convex. We remark that in general, if the origin is outside $\overline{\Omega}^*$, and Ω^* is uniformly convex then Ω^* will be uniformly Y^*-convex if z is sufficiently small, that is $\sup \mathcal{I}$ is sufficiently small which is equivalent to the reflector being sufficiently high above the target hyperplane, $\{y_{n+1} = 0\}$. This may be shown in a similar way by considering a supporting enclosing sphere to Ω^* instead of the hyperplane H_0.

3. **Obliqueness.** Recall that a boundary condition of the form

$$G(\cdot, u, Du) = 0 \quad \text{on } \partial\Omega \tag{38}$$

for a second order partial differential equation in a domain Ω is called *oblique* (or *degenerate oblique*), with respect to $u \in C^1(\bar{\Omega})$ if

$$G_p(\cdot, u, Du) \cdot \nu \geq c_0 > 0 \quad (\textit{or} \geq 0) \tag{39}$$

where c_0 is a positive constant and ν is the unit outer normal to $\partial\Omega$.

In this section we consider general mappings $Y \in C^2(\mathcal{U}_0)$ where $\mathcal{U}_0 \subset \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n$ and elliptic solutions $u \in C^2(\Omega)$ of the associated prescribed Jacobian equations (17)–(20), whose one jets $J_1[u](\Omega) = (\cdot, u, Du)(\Omega)$ lie in $\mathcal{U}_0$. We will henceforth refer to such solutions as admissible.

Let $\Omega^* \Subset Y(\mathcal{U}_0)$ be a C^2 domain in $\mathbb{R}^n$ and ϕ^* be a C^2 defining function for Ω^* satisfying $\phi^* = 0$, $|\nabla\phi^*| \neq 0$ on $\partial\Omega^*$ and $\phi^* < 0$ in Ω^*. The condition $T_u(\Omega) = Y(\cdot, u, Du)(\Omega) = \Omega^*$ implies the boundary condition,

$$G(x, u, Du) := \phi^* \circ Y(x, u, Du) = 0 \quad \text{on } \partial\Omega. \tag{40}$$

The main estimate in this section is the following

Theorem 3.1. *Let $u \in C^3(\overline{\Omega})$ be an admissible solution of (19) in Ω, with range $u(\Omega) \subset \mathcal{I}$ for some interval $\mathcal{I}$ and one jet $J_1 = J_1[u](\Omega) \Subset \mathcal{U}_0$ and $B > 0, \in C^1(\mathcal{U}_0)$. Suppose that Ω and Ω^* are C^3 smooth and respectively uniformly Y and Y^*-convex with respect to $\Omega^* \times \mathcal{I}$ and $\Omega \times \mathcal{I}$. Then the boundary condition (40) satisfies a strict obliqueness estimate (39), for some positive constant c_0, depending on $n, \Omega, \Omega^*, Y, B$ and $J_1[u]$.*

Proof. The proof essentials have already been given in [20]. For completeness and the convenience of readers, we provide the detailed proof here. The boundary condition $Y_u(\Omega) = \Omega^*$ implies that

$$\phi^*(Y_u) = 0 \text{ on } \partial\Omega, \quad \phi^*(Y_u) < 0 \text{ near } \partial\Omega. \tag{41}$$

By differentiation we have

$$\phi^*_k D_j Y^k \tau_j = 0 \quad \text{on } \partial\Omega \tag{42}$$

for any unit tangential vector τ on $\partial\Omega$, and

$$\phi^*_k D_\nu Y^k \geq 0 \quad \text{on } \partial\Omega \tag{43}$$

where ν is the outer normal to $\partial\Omega$, whence

$$\phi^*_i D_j Y^i = \chi \nu_j \tag{44}$$

for some χ. At this point we observe that $\chi > 0$ on $\partial\Omega$, since $|\nabla\phi^*| \neq 0$ on $\partial\Omega$ and $\det DY \neq 0$.

Denote $w_{ij} = D_{ij}u - A_{ij}(\cdot, u, Du)$. Since $\det Y_p \neq 0$, from (18) one has

$$D_j Y^i = \partial_{p_k} Y^i w_{kj}, \tag{45}$$

where $\partial_{p_k} Y^i$ denotes $\frac{\partial Y^i}{\partial p_k}$.

Consequently, from (44)–(45)

$$\phi^*_i \partial_{p_k} Y^i w_{kj} = \chi \nu_j. \tag{46}$$

Letting $\{w^{ij}\}$ denote the inverse matrix of $\{w_{ij}\}$, we then have

$$\phi^*_i \partial_{p_k} Y^i = \chi w^{jk} \nu_j. \tag{47}$$

Combining (40) and (47), we denote

$$\beta_k := G_{p_k} = \chi w^{jk} \nu_j, \tag{48}$$

which subsequently indicates that

$$\beta \cdot \nu = \chi w^{ij} \nu_i \nu_j > 0 \tag{49}$$

on $\partial\Omega$. However, from (46) we also see that

$$\begin{aligned}\phi_i^*\partial_{p_k}Y^i w_{jk}\phi_l^*\partial_{p_j}Y^l &= \chi\phi_l^*\partial_{p_j}Y^l\nu_j\\ &= \chi(\beta\cdot\nu).\end{aligned} \tag{50}$$

Eliminating χ from (48)–(50), we have

$$(\beta\cdot\nu)^2 = (w^{ij}\nu_i\nu_j)(w_{kl}\partial_{p_k}Y^i\partial_{p_l}Y^j\phi_i^*\phi_j^*). \tag{51}$$

Let $x_0\in\partial\Omega$ be a point where $\beta\cdot\nu|_{\partial\Omega}$ has its minimum value. We may make a rotation of coordinates so that $e_1,\cdots,e_{n-1}$ are tangential to $\partial\Omega$ at x_0 and $\nu(x_0)=e_n$. Then (42) and (43) become

$$\phi_k^*D_\alpha Y^k = 0 \qquad \text{at } x_0,\ \text{for } \alpha=1,\cdots,n-1, \tag{52}$$

$$\phi_k^*D_n Y^k \geq 0 \qquad \text{at } x_0. \tag{53}$$

We now consider a function

$$v := \beta\cdot\nu - KG, \tag{54}$$

for a sufficiently large constant K, where ν is extended C^2 smoothly inside Ω and G is given by (40) with the defining function ϕ^* chosen so that

$$D_p^2G(x,u,Du) \geq I_{n\times n} \tag{55}$$

for $Y(x,u,Du)\in\mathcal{N}^*\cap\Omega^*$ for some neighbourhood $\mathcal{N}^*$ of $\partial\Omega^*$, in accordance with (27) and the uniform Y^*-convexity of Ω^*. By appropriate modification of G as in [22], we can then assume (55) holds throughout Ω. Since $v|_{\partial\Omega}$ has a minimum at x_0, $D_\alpha v(x_0)=0$ for $\alpha=1,\cdots,n-1$, which can be written as

$$D_\alpha(\beta\cdot\nu) - K\phi_k^*D_\alpha Y^k = 0, \quad \text{for } \alpha=1,\cdots,n-1. \tag{56}$$

We *claim* at the moment that $D_n v(x_0)\leq C$ for some constant C. This can be rewritten as

$$D_n(\beta\cdot\nu) - K\phi_k^*D_nY^k \leq C. \tag{57}$$

By (40) and (48),

$$\begin{aligned}D_i(\beta\cdot\nu) =& D_i(\phi_k^*\partial_{p_j}Y^k\nu_j)\\ =&\phi_{kl}^*\frac{\partial Y^k}{\partial p_j}\frac{\partial Y^l}{\partial p_s}w_{si}\nu_j + \phi_k^*\frac{\partial Y^k}{\partial p_j}D_i\nu_j\\ &+\phi_k^*\nu_j\left(\frac{\partial^2Y^k}{\partial x_i\partial p_j}+\frac{\partial^2Y^k}{\partial z\partial p_j}u_i+\frac{\partial^2Y^k}{\partial p_s\partial p_j}u_{si}\right).\end{aligned} \tag{58}$$

From (20) we have

$$-Y^i_{p_k}A_{kj} = Y^i_j + Y^i_z p_j,$$

by differentiation with respect to p_l,

$$\frac{\partial^2Y^i}{\partial x_j\partial p_l}+\frac{\partial^2Y^i}{\partial z\partial p_l}u_j = -\frac{\partial^2Y^i}{\partial p_k\partial p_l}A_{kj} - \frac{\partial Y^i}{\partial p_k}\partial_{p_l}A_{kj} - \frac{\partial Y^i}{\partial z}\delta_j^l. \tag{59}$$

Combining (58) and (59), we obtain

$$D_i(\beta\cdot\nu) = D^2_{p_jp_s}(\phi^*\circ Y)w_{si}\nu_j + \phi_k^*\frac{\partial Y^k}{\partial p_j}(D_i\nu_j - \partial_{p_s}A_{ij}\nu_s) - \phi_k^*\nu_i\frac{\partial Y^k}{\partial z}. \tag{60}$$

Multiplying (60) by $\phi^*_r \partial_{p_i} Y^r$ and sum over i from 1 to n, we have

$$\begin{aligned}\phi^*_r \partial_{p_i} Y^r D_i(\beta\cdot\nu) =& \phi^*_k \phi^*_r \frac{\partial Y^k}{\partial p_j}\frac{\partial Y^r}{\partial p_i}(D_i \nu_j - \partial_{p_s} A_{ij}\nu_s) + \chi D^2_{p_j p_k}(\phi^*\circ Y)\nu_j\nu_k \\ &+ G_{p_i}\nu_i \phi^*_k \frac{\partial Y^k}{\partial z} \\ \geq& \delta_0 |G_p|^2 + \chi\kappa^*_0 - C G_{p_n}\end{aligned} \tag{61}$$

by (27), (29) and noticing that $\nu_i = \delta^n_i$ at x_0.

From (56), (57) and (61), we assert that

$$K w_{ij}\partial_{p_i} Y^r \partial_{p_j} Y^k \phi^*_r \phi^*_k \geq \tau_0 - C(\beta\cdot\nu)$$

for positive constants τ_0 and C. Hence if $\beta\cdot\nu \leq \tau_0/2C$, we have the lower bound

$$w_{ij}\partial_{p_i} Y^r \partial_{p_j} Y^k \phi^*_r \phi^*_k \geq \frac{\tau_0}{2K}. \tag{62}$$

To complete the estimation of $\beta\cdot\nu$, by (51) it remains to obtain a lower bound of $w^{ij}\nu_i\nu_j$ at x_0. For this we use the technique introduced in [20, 26], which avoids invoking a dual problem as in [22, 24], and which is not available in our generality. At x_0, $D_\alpha v = 0$ for $\alpha = 1, \cdots, n-1$ and the assertion $D_n v \leq C$ indicate that

$$Dv(x_0) = \tau\nu \quad \text{for some } \tau \leq C.$$

Therefore,

$$w^{ij}\nu_i D_j v \leq C w^{ij}\nu_i\nu_j. \tag{63}$$

It then suffices to have a lower bound for $w^{ij}\nu_i D_j v$ at x_0.

From (47) we have

$$w^{ij}\nu_i = \chi^{-1}\phi^*_k \partial_{p_j} Y^k. \tag{64}$$

Therefore, by (50) and (61)

$$\begin{aligned} w^{ij}\nu_i D_j v \geq& \frac{\tau_0}{\chi} + \delta^*_0 - \frac{C}{\chi}(\beta\cdot\nu) - K(\beta\cdot\nu) \\ \geq& \delta^*_0 - K(\beta\cdot\nu).\end{aligned} \tag{65}$$

Thus, if $\beta\cdot\nu < \frac{\min\{\tau_0,\delta^*_0\}}{2(K+C)\chi}$, then it follows that

$$w^{ij}\nu_i\nu_j \geq c_0, \tag{66}$$

for a constant $c_0 > 0$.

Combining (51), (62) and (66), the desired obliqueness estimate (39) is thus derived

$$G_p\cdot\nu \geq c_0 \tag{67}$$

on $\partial\Omega$ for a different positive constant c_0 depending only on domains Ω, Ω^* and the claim that $D_n v(x_0) \leq C$.

In the rest, it remains to prove this assertion. By differentiating equation (3), we obtain, for $r = 1, \cdots, n$,

$$w^{ij}\left(D_{ij}u_r - \partial_{p_k} A_{ij} D_k u_r - \partial_z A_{ij} u_r - \partial_{x_r} A_{ij}\right) = D_r \log h, \tag{68}$$

where h denotes the inhomogeneous term. Introducing the linearised operator L,

$$Lv = w^{ij}\left(D_{ij} v - \partial_{p_k} A_{ij} D_k v\right), \tag{69}$$

we need to compute Lv for v given by (54).

Setting

$$F(x,z,p) = G_p(x,z,p)\cdot\nu(x) - KG(x,z,p), \tag{70}$$

where G is defined by (40), we see that

$$v(x) = F(x,u,Du). \tag{71}$$

Then we obtain

$$\begin{aligned} Lv = w^{ij}\{&F_{p_r}D_{ij}u_r + F_{p_rp_s}D_{ir}uD_{js}u + F_{zp_r}D_juD_{ir}u + F_{zp_r}D_iuD_{jr}u \\ &+ F_{ip_r}D_{jr}u + F_{jp_r}D_{ir}u + F_zD_{ij}u + F_{zz}D_iuD_ju + F_{iz}D_ju \\ &+F_{jz}D_iu + F_{ij} - D_{p_k}A_{ij}(F_k + F_zu_k + F_{p_r}D_ku_r)\}\,. \end{aligned} \tag{72}$$

By choosing K in (70) sufficiently large, from (55) we can then ensure that

$$F_{p_ip_j}(x,u,Du)\xi_i\xi_j \le -\frac{K}{2}|\xi|^2 \tag{73}$$

near $\partial\Omega$. Substituting (73) into (72), it follows that

$$\begin{aligned} Lv &\le w^{ij}F_{p_rp_s}D_{ir}uD_{js}u + C(w^{ii}+1) + F_{p_r}D_r\log h \\ &\le -\frac{K}{4}w_{ii} + C(w^{ii}+1), \end{aligned} \tag{74}$$

where C is a constant depending on $h,\Omega,\Omega^*,\|u\|_{C^1(\overline{\Omega})}$ and K, see [22].

A suitable barrier is now provided by the uniform Y-convexity of Ω, which provides a defining function ϕ of Ω satisfying (28) in a fixed neighbourhood of $\partial\Omega$. From this, we infer by the standard barrier argument (which entails further modifying ϕ and fixing a small enough neighbourhood of $\partial\Omega$, [4]) that

$$\nu\cdot Dv(x_0) \le C, \tag{75}$$

where again C is a constant depending on $\Omega,\Omega^*,\|u\|_{C^1(\overline{\Omega})}$ and h. Since x_0 is a minimum point of v on $\partial\Omega$, we can write

$$Dv(x_0) = \tau\nu(x_0) \tag{76}$$

where $\tau \le C$. This completes the proof of the claim that $D_nv(x_0)\le C$. Therefore, the proof of the strict obliqueness estimate (39) is finished. □

Note that in Theorem 3.1, we need only assume B is defined for $x\in\overline{\Omega}$, $z\in\overline{\mathcal{I}}$ and $p\in\mathcal{P}(x,z)$ so that it applies to (20) when $f>0,\in C^1(\overline{\Omega}), g\in C^1(\overline{\Omega}^*)$. The dependence on $J_1[u]$ is determined through $\sup(|u|+|Du|)$ and $\operatorname{dist}(J_1[u],\partial\mathcal{U}_0)$. Taking account of the gradient estimate (34), we then obtain as a consequence of Theorem 3.1

Corollary 1. *Under the hypotheses of Theorem 1.1, the boundary condition (5)–(6) satisfies a strict obliqueness estimate (39) for some positive constant c_0.*

The oblique boundary value problem,(1), (2) for Monge-Ampère type equations has been studied in [13, 18, 22, 25]. In particular, the global C^2 estimate in Theorem 1.1 now follows from [25] where the case A = 0 is treated for uniformly convex domains Ω and the uniform convexity condition, (55).

Corollary 2. *Let $\rho\in C^4(\overline{\Omega})$ be an admissible solution of the reflector problem, and $\overline{\Omega}\Subset B_1(0)$. Then we have the estimate*

$$\sup_{\overline{\Omega}}|D^2\rho| \le C, \tag{77}$$

where the constant depends on f,g,Ω,Σ and $|\rho|_{1,\Omega}$.

Once the second derivatives are bounded, equations (3) and (25) are effectively uniformly elliptic. This combined with the obliqueness estimate yields global $C^{2,\alpha}$ estimates, [12]. Moreover, the higher order estimates follow from the theory of linear elliptic equations with oblique boundary conditions [4].

4. **Proof of Theorem 1.2.** In this section we prove Theorem 1.2 by the degree theory recently developed in [11] for second order elliptic operators with nonlinear oblique boundary conditions. For the standard Monge-Ampère equation with boundary condition (8), the existence via the method of continuity was given by Urbas in [23]. The situation here is more complicated because of the dependence of Y on u in the boundary condition (5).

First, we adopt the method of domain deformation in [20, 22]. By approximation we may assume the domains Ω and Ω^* are C^5 smooth. Fix a point $x_0 \in \Omega$, by a translation we may assume that $x_0 = 0$. Define the function

$$u_0(x) = \frac{1}{2}|x|^2 + p_0 \cdot x + b_0 \tag{78}$$

in a small ball $\Omega_0 = B_r(0) \subset \Omega$, where $p_0 \in \mathbb{R}^n$ is chosen such that

$$T_{u_0}(0) = \frac{2p_0}{|p_0|^2 - b_0^2} = y_0 \tag{79}$$

for some point $y_0 \in \Omega^*$. We may also assume that $y_0 = 0$, thus $p_0 = 0$ and u_0 has a simple form $u_0(x) = \frac{1}{2}|x|^2 + b_0$, where the constant $b_0 > 0$ is chosen large enough such that

$$|Du_0|^2 - (u_0 - Du_0 \cdot x)^2 < 0,$$

for all $x \in \Omega$ and $T_{u_0}(\Omega) \Subset \Omega^*$. For $r > 0$ sufficiently small, one has the image $\Omega_0^* := T_{u_0}(\Omega_0)$ is uniformly Y^*-convex with respect to $\Omega \times \mathcal{I}$, where $\mathcal{I}$ is an interval depending on b_0 and r, and the function u_0 is admissible.

From Definition 2.1 of Y^*-convexity, the sets $\mathcal{P}_0(x_0, u_0) = \{p \in \mathbb{R}^n : (x_0, u_0, p) \in \Omega_0^*\}$ and $\mathcal{P}_1(x_0, u_0) = \{p \in \mathbb{R}^n : (x_0, u_0, p) \in \Omega^*\}$ are uniformly convex. Let $\{\mathcal{P}_t\}$, $0 \le t \le 1$, be a foliation of uniformly convex sets such that $\mathcal{P}_0 = \mathcal{P}_0(x_0, u_0)$ and $\mathcal{P}_1 = \mathcal{P}_1(x_0, u_0)$. For example, one can take $\mathcal{P}_t = \{h_t < 0\}$ and

$$h_t = (1-t)h_0 + th_1, \tag{80}$$

where h_0 is the defining function of $\mathcal{P}_0(x_0, u_0)$ and h_1 is the defining function of $\mathcal{P}_1(x_0, u_0)$. By the invertibility of Y with respect to p, one has a foliation of $\{\Omega_t^*\}$ from Ω_0^* to Ω^* such that Ω_t^* is uniformly Y^*-convex with respect to (x_0, u_0) for all $0 \le t \le 1$. Note that by choosing the initial $r > 0$ sufficiently small, we can assume Ω_t^* is uniformly Y^*-convex with respect to $\Omega_0 \times \mathcal{I}$ for all $0 \le t \le 1$. Namely, we end up with a continuous increasing family of subdomains $\{\Omega_t^*\}$, $0 \le t \le 1$, satisfying:

(i) $\Omega_t^* \subset \Omega^*$, $\Omega_1^* = \Omega^*$;
(ii) $\partial\Omega_t^* \in C^5$, uniformly with respect to t;
(iii) Ω_0, Ω_t^* are uniformly convex and Y^*-convex, respectively.

Next, we define a family of homotopy problems and apply the degree theory in [11]. Note that by (16) and (17), equation (3) is equivalent to

$$\det [DT_u] = \frac{f/\omega}{g(T_u)} \quad \text{in } \Omega, \tag{81}$$

where DT_u is the Jacobian of the reflection mapping T_u in (15), f and g are intensity functions satisfying the energy conservation

$$\int_{\Omega} f/\omega = \int_{\Omega^*} g. \tag{82}$$

Given the family of domains $\{\Omega_t^*\}$, $0 \le t \le 1$, we consider the corresponding family of problems

$$\begin{aligned} F_t[u_t] &= \det[DT_{u_t}] - e^{\varepsilon(u_t-u_0)}\left\{\frac{tf/\omega}{g(T_{u_t})} + (1-t)\det[DT_{u_0}]\right\} = 0 \quad \text{in } \Omega_0, \\ G_t[u_t] &= \varphi_t^* \circ T(\cdot, u_t, Du_t) = 0 \quad \text{on } \partial\Omega_0, \end{aligned} \tag{83}$$

where φ_t^* is the corresponding defining function of Ω_t^*. From our construction and the obliqueness, u_0 is the unique solution of (83) at $t = 0$. To prove this, let u be another elliptic solution of

$$\begin{aligned} &\det[DT_u] = e^{\varepsilon(u-u_0)}\det[DT_{u_0}] \quad \text{in } \Omega_0, \\ &\varphi_0^* \circ T(\cdot, u, Du) = 0 \quad \text{on } \partial\Omega_0. \end{aligned} \tag{84}$$

Recall that $\Omega_0 = B_r(0)$ for a small $r > 0$ and $u_0(x) = \frac{1}{2}|x|^2 + b_0$ for a large $b_0 > 0$, we have $\Omega_0^* = T_{u_0}(\Omega_0) = B_{r_*}(0)$ for some $0 < r_* < Cr/b_0^2$, and we may choose $\varphi_0^*(y) = \frac{1}{2}(|y|^2 - r_*^2)$.

Step 1. Integrating (84),

$$|\Omega_0^*| = \int_{\Omega_0} \det[DT_u] = \int_{\Omega_0^*} e^{\varepsilon(u-u_0)}.$$

From §2.3,

$$\int_{\Omega_0^*} e^{\varepsilon \sup u/C} e^{-\varepsilon b_0} \le C|\Omega_0^*|,$$

which implies that

$$\sup u \le C(b_0 + \frac{C}{\varepsilon}) \le Cb_0, \tag{85}$$

since $\varepsilon > 0$ is a fixed constant. By §2.3 again,

$$\sup |Du| \le Cb_0. \tag{86}$$

Similarly, we also have

$$\inf u \ge C^{-1} b_0. \tag{87}$$

Step 2. From $T_u(B_r) = B_{r_*} \subset B_{Cr/b_0^2}$ and (5),

$$|Du| \le \frac{Cr}{b_0^2}\left||Du|^2 - (u - Du\cdot x)^2\right| \le Cr. \tag{88}$$

Therefore, we have $u \simeq b_0$ and $|Du| \simeq r$, where $b_0 > 0$ is large and $r > 0$ is small from our construction. Heuristically, $T_u \simeq -\frac{2Du}{u^2} = 2D\rho$, where $\rho = 1/u$.

Step 3. Let $\rho = 1/u$ and $\rho_0 = 1/u_0$. From (25), ρ is a solution of

$$\begin{aligned} &\det\left[-D^2\rho + \frac{2}{\rho} D\rho \otimes D\rho\right] = e^{\varepsilon(\frac{1}{\rho} - \frac{1}{\rho_0})} h(\rho_0) \quad \text{in } B_r \\ &\frac{1}{2}\left(|T_\rho|^2 - r_*^2\right) = 0 \quad \text{on } \partial B_r, \quad T_\rho = \frac{-2\rho^2 D\rho}{|D\rho|^2 - (\rho + D\rho\cdot x)^2}, \end{aligned} \tag{89}$$

where $h(\rho_0) = \det\left[-D^2\rho_0 + \frac{2}{\rho_0} D\rho_0 \otimes D\rho_0\right]$ is a fixed positive function. From our construction, ρ_0 is a solution of (89).

Let $w = \rho - \rho_0$, then w satisfies the linearized equation

$$L[w] = a^{ij}(x)D_{ij}w + b^i(x)D_iw + c(x)w = 0 \quad \text{in } B_r, \qquad (90)$$
$$B[w] = \beta_i(x)D_iw + \gamma(x)w = 0 \quad \text{on } \partial B_r,$$

where the coefficients $a^{ij}, b^i, c, \beta_i, \gamma$ are evaluated at $\hat{\rho} = \theta\rho + (1-\theta)\rho_0$ for some $\theta \in (0,1)$. Since ρ, ρ_0 are both admissible, $\{a^{ij}\}$ is bounded and positive definite. By differentiation,

$$c(x) = a^{ij}(x)\frac{\hat{\rho}_i\hat{\rho}_j}{\hat{\rho}^2} - \frac{\varepsilon}{\hat{\rho}^2}e^{\varepsilon(\frac{1}{\hat{\rho}} - \frac{1}{\rho_0})}.$$

From (85), (87) and (88), $\hat{\rho} \simeq 1/b_0$ and $|D\hat{\rho}| \simeq r/b_0^2$. Choosing $b_0 > 0$ sufficiently large and $r > 0$ sufficiently small, we have $c(x) < 0$ for all $x \in B_r$.

By the obliqueness estimate, $\beta \cdot \nu > 0$ on ∂B_r, where ν is the unit outer normal of ∂B_r. Let's estimate the coefficient γ,

$$\begin{aligned} |\gamma| &= \left| T_\rho \cdot \frac{\partial T_\rho}{\partial \rho} \right| = \left| \frac{8\rho^3|D\rho|^2}{(|D\rho|^2 - (\rho + D\rho\cdot x)^2)^2} + \frac{8\rho^4|D\rho|^2(\rho + D\rho\cdot x)}{(|D\rho|^2 - (\rho + D\rho\cdot x)^2)^3} \right| \\ &\leq C\frac{|D\rho|^2}{\rho} \leq Cr^2/b_0^3. \end{aligned} \qquad (91)$$

Therefore, for any small $\epsilon > 0$, by choosing $b_0 > 0$ sufficiently large and $r > 0$ sufficiently small we have $|\gamma| < \epsilon$. This implies that the kernel of (L, B) in (90) is trivial, namely $w = 0$. Consequently, u_0 is the unique solution of (83) at $t = 0$.

Note that the intensity functions f, g have positive lower and upper bounds. By integrating equation (83), we obtain uniform bounds for the quantities

$$C_1 \leq \int_{\Omega_0} e^{\varepsilon(u_t - u_0)} \leq C_2$$

for two controlled positive constants C_1 and C_2. From §2.3, it follows that u_t and Du_t are bounded. In fact, if $\sup u_t > M$, by §2.3 $u_t > cM$, which will contradicts the upper bound C_2 when M is sufficiently large. By §2.3 again one has the gradient bound for Du_t. Similarly, one can also obtain the lower bound for $\inf u_t > C$. Thus we have $\|u_t\|_{C^1(\overline{\Omega}_0)} \leq C_\varepsilon$. By Theorem 1.1, $\|u_t\|_{C^{4,\alpha}(\overline{\Omega}_0)} \leq C_\varepsilon$.

Note that there exists a diffeomorphism $\Phi_0 \in C^5 : \mathbb{R}^n \to \mathbb{R}^n$ such that $\Phi_0(\Omega_0) = B_1(0)$. Define

$$\tilde{F}_t[u] = F_t[u \circ \Phi_0] \quad \text{in } B_1(0), \qquad (92)$$
$$\tilde{G}_t[u] = G_t[u \circ \Phi_0] \quad \text{on } \partial B_1(0),$$

for any $u \in C^{4,\alpha}(\overline{B}_1)$. It is straightforward to check that (92) has the same ellipticity and obliqueness as (83) (which is essentially a change of variables), and $(F_t, G_t)[u_t] = 0$ for $u_t \in C^{4,\alpha}(\overline{\Omega}_0)$ if and only if $(\tilde{F}_t, \tilde{G}_t)[\tilde{u}_t] = 0$ for $\tilde{u}_t = u_t \circ \Phi_0^{-1} \in C^{4,\alpha}(\overline{B}_1)$. Moreover,

$$\|\tilde{u}_t\|_{C^{4,\alpha}(\overline{B}_1)} \leq C\|u_t\|_{C^{4,\alpha}(\overline{\Omega}_0)} \leq CC_\varepsilon, \qquad (93)$$

where C is a uniform positive constant independent of t.

Let $\mathcal{O}$ be a bounded open set in $C^{4,\alpha}(\overline{B}_1)$ such that $\tilde{F}_t$ is elliptic and $\tilde{G}_t$ is oblique on $\mathcal{O}$, and $\partial\mathcal{O} \cap (\tilde{F}_t, \tilde{G}_t)^{-1}(0) = \emptyset$ for all $t \in [0,1]$. In [11] an integer-valued degree for $(\tilde{F}_t, \tilde{G}_t)$ on $\mathcal{O}$ at 0 is defined, which satisfies the homotopy invariance property that $\deg((\tilde{F}_t,\tilde{G}_t),\mathcal{O},0)$ is independent of t. Recall that u_0 is the unique solution of $(F_0,G_0)[u]=0$.

$$\begin{aligned}\deg((\tilde{F}_1,\tilde{G}_1),\mathcal{O},0)&=\deg((\tilde{F}_0,\tilde{G}_0),\mathcal{O},0)\\&=\deg((F_0,G_0),\mathcal{O}_0,0)\neq 0,\end{aligned}\tag{94}$$

where $\mathcal{O}_0\subset\{u\in C^{4,\alpha}(\overline{\Omega}_0)\,:\,\|u\|_{C^{4,\alpha}(\overline{\Omega}_0)}\le C_\varepsilon+1\}$. This implies that there exists a solution $\tilde{u}_\varepsilon\in C^{4,\alpha}(\overline{B}_1)$ of the boundary value problem (92) at $t=1$. Hence there exists a solution $u_\varepsilon\in C^{4,\alpha}(\overline{\Omega}_0)$ of the boundary value problem

$$\begin{aligned}&\det[DT_u]=e^{\varepsilon(u-u_0)}\frac{f/\omega}{g(T_u)}\quad\text{in }\Omega_0,\\&T_u(\Omega_0)=\Omega^*\end{aligned}\tag{95}$$

for arbitrary small $\varepsilon>0$. Then we need deform Ω_0 to Ω while fixing the target Ω^*. From the hypotheses of Theorem 1.1 that Ω^* is uniformly Y^*-convex with respect to $\Omega\times\mathcal{I}$, similarly as (80) we have a continuous increasing family of subdomains $\{\Omega_t\}$, $0\le t\le 1$, satisfying:

(i') $\Omega_t\subset\Omega$, $\Omega_1=\Omega$;
(ii') $\partial\Omega_t\in C^5$, uniformly with respect to t;
(iii') Ω_t,Ω^* are uniformly convex and Y^*-convex, respectively.

By constructing a family of homotopy problems similar to (83) over each pair (Ω_t,Ω^*) and using the degree argument as above, we obtain a solution $u_\varepsilon\in C^{4,\alpha}(\overline{\Omega})$ of the boundary value problem

$$\begin{aligned}&\det[DT_u]=e^{\varepsilon(u-u_0)}\frac{f/\omega}{g(T_u)}\quad\text{in }\Omega,\\&T_u(\Omega)=\Omega^*\end{aligned}\tag{96}$$

for arbitrary small $\varepsilon>0$. To complete the existence proof we now need to let $\varepsilon\to 0$. Write equation (96) in the form of

$$g(T_u)\,\det[DT_u]=e^{\varepsilon(u-u_0)}\frac{f}{\omega}(x)\quad\text{in }\Omega.\tag{97}$$

Let $\{u_\varepsilon\}$ be the family of solutions of the problems (97). From (82)

$$\int_\Omega f/\omega=\int_{\Omega^*}g=\int_\Omega e^{\varepsilon(u_\varepsilon-u_0)}f/\omega,$$

we see that $u_\varepsilon-u_0$ must be zero somewhere in Ω. Hence, from §2.3 $\sup_\Omega u_\varepsilon$ is bounded independently of ε, so is $|Du_\varepsilon|$ and u_ε. By Theorem 1.1, $\|u_\varepsilon\|_{C^{4,\alpha}(\overline{\Omega})}$ is bounded independently of ε. Thus a subsequence of $\{u_\varepsilon\}$ converges in $C^{4,\beta}(\overline{\Omega})$ for $0<\beta<\alpha$ to a solution u solving (3)–(5), as required. Note that the C^0 bound of u depends on the initial choice of b_0 in (78), which in turn determines the constant δ in Theorems 1.1 and 1.2. Using the Harnack inequality, (32), we can also find a solution in an interval $(\delta,K\delta)$ for some constant K independent of δ for any $\delta>0$, so in particular there also exist an infinite number of solutions.

5. **Parallel light source.**

5.1. **Parallel reflector.** Instead of the point light source, in this section we consider another model of reflector problem with parallel light source. We start with the derivation of the equation fulfilled by this model as follows.

Assume that the light emits from $\Omega \subset \mathbb{R}^n$ with intensity $f \in L^1(\Omega)$ and illuminates $\Omega^* \subset \mathbb{R}^n$ with intensity $g \in L^1(\Omega^*)$, where f, g satisfy (9). Represent the reflector Γ as a graph $u|_\Omega$ for a positive function u, namely,

$$\Gamma = \{(x, u(x)) \, : \, x \in \Omega\}. \tag{98}$$

Let us trace a beam of light that moves upwards from $(x, 0) \in (\Omega, 0)$ along direction $e_{n+1} = (0, \cdots, 0, 1)$. Suppose that the ray is reflected off at a point $(x, u(x)) \in \Gamma$ in direction V and reaches point $(y, 0) \in (\Omega^*, 0)$. Denote by γ the (downwards) unit normal of Γ. Then by calculations,

$$\gamma = \frac{(Du, -1)}{\sqrt{|Du|^2 + 1}}. \tag{99}$$

From the reflection law,

$$V = e_{n+1} - 2(e_{n+1} \cdot \gamma)\gamma = \frac{(2Du, |Du|^2 - 1)}{|Du|^2 + 1}. \tag{100}$$

On the other hand, due to our hypotheses the reflected ray meets the hyperplane $\mathbb{R}^n \times \{0\}$ at $(y, 0)$. Thus

$$V = \frac{(y - x, -u)}{\sqrt{|y - x|^2 + u^2}}. \tag{101}$$

From (100)–(101), we have

$$y = x + \frac{2uDu}{1 - |Du|^2} =: T_u(x). \tag{102}$$

Therefore, we obtain the reflection mapping $T : x \in \Omega \to y \in \Omega^*$ given by formula (102). It is easy to see that T is a diffeomorphism onto its image for a smooth positive function u with $|Du| < 1$; we will assume this in the following content.

Write $T_u(x) = Y(x, u, Du)$, where $Y = Y(x, z, p)$ is a mapping from $\mathcal{U}_0 = \{(x, z, p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : z > 0, |p| < 1\}$ into $\mathbb{R}^n$ given by

$$Y(x, z, p) = x + \frac{2zp}{1 - |p|^2}. \tag{103}$$

From the conservation of energy and (17), (19) the corresponding prescribed Jacobian equation is a Monge-Ampère type equation

$$\begin{aligned} \det[D^2 u - A(\cdot, u, Du)] &= B(\cdot, u, Du), \\ T(\Omega) &= \Omega^*, \end{aligned} \tag{104}$$

where

$$A = -Y_p^{-1}(Y_x + Y_z \otimes p), \quad B = (\det Y_p)^{-1} f/g(Y). \tag{105}$$

By (102) we have the partial derivatives

$$Y_x = \left\{ \frac{\partial Y_i}{\partial x_j} \right\} = I, \quad Y_z = \partial_z Y = \frac{2Du}{1 - |Du|^2},$$

and

$$Y_p = \left\{ \frac{\partial Y_i}{\partial p_j} \right\} = \frac{2u}{1 - |Du|^2} \left(I + \frac{2Du \otimes Du}{1 - |Du|^2} \right).$$

By further calculations,

$$\det\, Y_p = \left(\frac{2u}{1-|Du|^2}\right)^n \frac{1+|Du|^2}{1-|Du|^2} = \frac{(2u)^n(1+|Du|^2)}{(1-|Du|^2)^{n+1}}, \tag{106}$$

$$Y_p^{-1} = \frac{1-|Du|^2}{2u}\left(I - \frac{2Du\otimes Du}{1+|Du|^2}\right). \tag{107}$$

Hence, combining (106)–(107) into (104)–(105) we obtain the equation

$$\det\left[D^2u + \frac{1-|Du|^2}{2u}I\right] = \frac{(1-|Du|^2)^{n+1}}{(2u)^n(1+|Du|^2)} f/g(Y). \tag{108}$$

We remark that (108) is also derived in [8]. Extension of these models to non-flat targets are also considered in [7].

The (uniform) Y and Y^*-convexity for domains Ω and Ω^* can be defined similarly as in §2.2, see Definitions 2.1, 2.2. Similarly to Theorem 1.2, we have the following classical solvability result.

Theorem 5.1. *Let Ω, Ω^* be C^4 smooth domains such that Ω and Ω^* are respectively uniformly Y and Y^*-convex, with respect to $\Omega^*\times\mathcal{I}$ and $\Omega\times\mathcal{I}$ for any bounded interval $\mathcal{I}\subset(\delta,\infty)$ for some $\delta>0$, with (28) holding for all $x\in\Omega$. Assume that f, g are C^2 smooth and have positive upper and lower bounds, and the balance condition (9) is satisfied. Then, there exists an admissible solution $u\in C^3(\overline{\Omega})$ of the reflector problem satisfying $u>\delta$.*

The proof of Theorem 5.1 follows from our previous considerations as follows.
(i) Similarly to (34) we need to use the boundary condition (104) to control the gradient $|Du|$ in order to estimate $|\det Y_p|$ in (106), thereby controlling the right hand side of (105). Let $d=\sup\{|x-y| : x\in\Omega, y\in\Omega^*\}$. For an admissible solution $u>\delta$ for some constant $\delta>0$, from (102) we have

$$d > \frac{2\delta|Du|}{(1+|Du|)(1-|Du|)} \ge \frac{\delta|Du|}{1-|Du|}, \tag{109}$$

and therefore, $|Du| < d/(d+\delta)$.
(ii) Denote the matrix A in (104)–(105) by

$$A(z,p) = \frac{|p|^2-1}{2z}\, I_{n\times n} \tag{110}$$

which is generated by the mapping Y in (103). By differentiating we have that for any $\xi,\eta\in\mathbb{R}^n$,

$$\begin{aligned}\sum_{i,j,k,l} D^2_{p_kp_l}A_{ij}(u,Du)\xi_i\xi_j\eta_k\eta_l &= \frac{1}{u}|\xi|^2|\eta|^2 \\ &\ge c_0|\xi|^2|\eta|^2,\end{aligned} \tag{111}$$

where the constant $c_0 = 1/\sup_\Omega u > 0$. It implies the matrix A is strictly regular for $z>0$ [19, 20]; in fact the matrix A satisfies the (A3) condition in optimal transportation without the orthogonality restriction. The required global C^2 estimate, corresponding to Theorem 1.1, then follows from Theorem 3.1 and [22].
(iii) As in the proof of Theorem 1.2, we start with $u_0(x)=\frac{1}{2}|x|^2+b_0$ in a small ball $\Omega_0 = B_r(0)\Subset\Omega$, where $r>0$ is small such that $|Du_0|<1$ and b_0 is large such that u_0 is admissible, and construct a family of homotopy problems along a foliation of

Ω^*. By (28) we can then deform Ω_0 to Ω. Similarly using the method of degree theory we then conclude the solvability of

$$\det[DT_u] = e^{\varepsilon(u-u_0)}\frac{f}{g(T_u)} \quad \text{in } \Omega,$$
$$T_u(\Omega) = \Omega^*.$$

The proof then proceeds exactly as before. Sending $\varepsilon \to 0$ and by the balance condition (9) we then obtain a solution u of (104) as required. Therefore, Theorem 5.1 is proved.

5.2. **Parallel refractor.** Finally we show our general estimates in §3 and existence procedure in §4 can be extended to refractor problems for parallel beams as introduced for example in the recent work of Gutiérrez and Tournier [6] and Oliker, Rubinstein and Wolanski [16, 17].

The parallel refractor problem can be described as follows: Suppose that a parallel light emits from $\Omega \subset \mathbb{R}^n \times \{0\}$ along $e_{n+1} = (0, \cdots, 0, 1)$ with positive intensity $f \in L^1(\Omega)$, and Ω^* is a hypersurface in $\mathbb{R}^{n+1}$, which is referred to as the target domain. Suppose that Ω and Ω^* are surrounded by two homogeneous and isotropic media I and II, respectively. One seeks an optical surface $\mathcal{R}$ interface between media I and II such that all rays refracted by $\mathcal{R}$ into medium II are received at the surface Ω^* and the prescribed radiation intensity received at each point $y \in \Omega^*$ is $g(y)$. Let $n_1, n_2 > 0$ be the indices of refraction of media I, II, respectively, and $\kappa = n_1/n_2$. As with our reflection problems we will assume we have a flat target lying in a hyperspace $\{y_{n+1} = h\}$ for some positive constant h and our surface R is the graph of a smooth function u over a domain $\Omega \subset \mathbb{R}^n$. By vertical translation we can assume that $h = 0$ and $u < 0$. The refractor mapping $T = T_u$ is now given by $T_u = Y(\cdot, u, Du)$, where

$$Y(x,z,p) = x + (1-\kappa^2)\frac{zp}{\kappa\sqrt{1-(\kappa^2-1)|p|^2}+1} \tag{112}$$

is defined on the set $\mathcal{U}_0 = \{(x,z,p) \in \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n : z < 0, (\kappa^2-1)|p|^2 < 1\}$.

Case (i), $\kappa < 1$: Assume that media II is denser than media I, that is, $\kappa < 1$. Rescaling $u \to u/\sqrt{(1-\kappa^2)}$, we have

$$Y(x,z,p) = x + \frac{zp}{\kappa\sqrt{1+|p|^2}+1}, \quad \mathcal{U}_0 = \{z < 0\}, \tag{113}$$

and we obtain the Monge-Ampère type equation

$$\det\left[D^2u + \frac{1+\kappa\sqrt{1+|Du|^2}}{u}(I + Du\otimes Du)\right] = \frac{(1+\kappa\sqrt{1+|Du|^2})^{n+1}}{\kappa(-u)^n}\frac{f}{g}, \tag{114}$$

together with the natural boundary condition $Y_u(\Omega) = \Omega^*$.

Again we use the boundary condition to estimate the gradient $|Du|$. Let $d = \sup|x-y|$ for $x \in \Omega$ and $y \in \Omega^*$. For an admissible solution $u < -\delta$ with some constant $\delta > \kappa d$, from (113) we have

$$\delta|Du| < d(1+\kappa\sqrt{1+|Du|^2}) < d + d\kappa(1+|Du|), \tag{115}$$

whence $|Du| < d(1+\kappa)/(\delta - \kappa d)$.

Next the matrix function

$$A(z,p) = \frac{1+\kappa\sqrt{1+p^2}}{-z}\,(I + p\otimes p)] \tag{116}$$

is strictly regular for $z<0$; (cf Example 2 in [22]) and we once again get a second derivative estimate corresponding to Theorem 1.1

Case (ii), $\kappa > 1$: Rescaling $u \to u/\sqrt{(\kappa^2-1)}$, we have

$$Y(x,z,p) = x - \frac{zp}{\kappa\sqrt{1-|p|^2}+1}, \quad \mathcal{U}_0 = \{z<0, |p|<1\}, \tag{117}$$

and the corresponding Monge-Ampère equation

$$\det\left[D^2u - \frac{1+\kappa\sqrt{1-|Du|^2}}{u}\,(I - Du\otimes Du)\right] = \frac{(1+\kappa\sqrt{1-|Du|^2})^{n+1}}{\kappa(-u)^n}\frac{f}{g}. \tag{118}$$

To bound $|Du|$ away from 1, let $d = \sup|x-y|$ for $x\in\Omega$ and $y\in\Omega^*$. For an admissible solution $u<-\delta$ with some constant $\delta>d$, similarly to (109) and (115), from (117) we have

$$\delta|Du| < d(1+\kappa\sqrt{1-|Du|^2}),$$

so that by calculation,

$$\kappa\sqrt{1-|Du|^2} > \frac{\delta}{d}|Du| - 1 > -\frac{\delta}{d}\sqrt{1-|Du|^2} + \frac{\delta}{d} - 1.$$

Hence,

$$\sqrt{1-|Du|^2} > \frac{\delta-d}{\delta+\kappa d},$$

and then we have

$$|Du| < \sqrt{1-\left(\frac{\delta-d}{\delta+\kappa d}\right)^2}. \tag{119}$$

As for the previous cases, the matrix function,

$$A(z,p) = \frac{1+\kappa\sqrt{1-p^2}}{z}\,(I + p\otimes p)] \tag{120}$$

is strictly regular for $z<0$, (cf Example 1, [22]).

Assume the energy conservation condition

$$\int_\Omega f = \int_{\Omega^*} g. \tag{121}$$

We thus conclude the following classical existence for both cases *(i)* and *(ii)*:

Theorem 5.2. *Let Ω,Ω^* be C^4 smooth domains such that Ω and Ω^* are respectively uniformly Y and Y^*-convex, with respect to $\Omega^*\times\mathcal{I}$ and $\Omega\times\mathcal{I}$ for any bounded interval $\mathcal{I}\subset(-\infty,-\delta)$ for some $\delta > d\min\{1,\kappa\}$, where $d=\sup\{|x-y| : x\in\Omega, y\in\Omega^*\}$, with (28) holding for all $x\in\Omega$. Assume that $f,g\in C^2$ are C^2 smooth and have positive upper and lower bounds, and the balance condition (121) is satisfied. Then, there exists an admissible solution $u\in C^3(\overline{\Omega})$ of the refractor problem satisfying $u<-\delta$.*

5.3. **Further remarks.** We conclude this paper with some brief remarks.
(i) The parallel beam models in this section may also be derived from supporting quadrics, which can be represented as graphs of generating functions in the sense of [21], and provide an appropriate notion of weak solution analogous to that in optimal transportation. In particular the reflection problem in Section 5.1 arises from paraboloids given by graphs:

$$x_{n+1} = \frac{1}{2v} - \frac{v}{2}|x-y|^2, \tag{122}$$

where $y \in \mathbb{R}^n$ and $v > 0$ denotes a dual variable, while the refraction problems in Section 5.2 arise from ellipsoids and hyperboloids respectively for $\kappa < 1$, $\kappa > 1$, given by graphs,

$$x_{n+1} = -\frac{1}{|k^2-1|}\big(\kappa v + \sqrt{v^2 + (k^2-1)|x-y|^2}\big), \tag{123}$$

where again $y \in \mathbb{R}^n$ and the dual variable $v > \sqrt{(1-\kappa^2)}|x-y|$ for $\kappa < 1$, > 0 for $\kappa > 1$. For more details about the geometric and physical aspects of the refraction problems see [6, 16, 17]. We remark also that our point source reflection problem in Sections 1 and 2 is similarly modelled by linear functions of the form

$$x_{n+1} = \frac{1}{v}\big(\sqrt{v+|y|^2} - x\cdot y\big), \tag{124}$$

for $y \in \mathbb{R}^n$ and $v > 0$, which correspond to the polar representations of supporting ellipsoids; see [14, 21].
(ii) Corresponding to our criterion in Section 2.4, we remark here that if the target Ω^* is just convex and $\Omega \Subset \Omega^*$, then Ω^* is uniformly Y^*-convex with respect to $\Omega \times \mathcal{I}$ for any interval $\mathcal{I}$ satisfying the hypotheses of Theorems 5.1 or 5.2. The arguments here are the same as in Section 2.4, namely by applying the mappings Y to supporting hyperplanes. For the parallel reflector we then obtain a supporting enclosing sphere to $\mathcal{P}(x,z)$ at each boundary point, while for the parallel refractor we obtain a supporting enclosing ellipsoid in the case $\kappa > 1$ and a supporting convex hyperboloid in the case $\kappa < 1$. If we only assume $\Omega \subset \Omega^*$, then Ω^* will be Y^*-convex if Ω^* is convex and uniformly Y^*-convex if Ω^* is uniformly convex, while for general Ω, the target domain Ω^* will be uniformly Y^*-convex if it is uniformly convex and δ is sufficiently large. The domain Ω will be uniformly Y-convex, satisfying also (28) in all of Ω, in the above examples if Ω is uniformly convex and δ is sufficiently large or the curvatures of $\partial\Omega$ are sufficiently large. When Ω^* is convex, (uniformly convex) and $\Omega \Subset (\subset)\Omega^*$ we only need Ω to be uniformly Y-convex to carry out the deformation argument in Section 4.

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