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Applications of compact topological graph c*-algebras to noncommutative solenoids

Mitchell Richard Hawkins
University of Wollongong

Applications of Compact Topological Graph C^*-Algebras to Noncommutative Solenoids

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Mitchell Richard Hawkins

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Certification

I, Mitchell Richard Hawkins, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Mitchell Richard Hawkins

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Abstract

We study the KMS states of the Toeplitz extension of the noncommutative solenoids introduced by Latrémolière and Packer. We demonstrate that noncommutative solenoids cannot be constructed as a direct limit of C^*-algebras arising from inverse sequences of topological graphs associated to local homeomorphisms of $\mathbb{T}$. We employ a different approach, utilizing a topological analogue of higher power graphs and Katsura's factor maps to obtain a noncommutative solenoid as a direct limit of topological graph algebras. This approach is compatible with the Toeplitz algebra of topological graphs, and enables us to define a Toeplitz extension of each noncommutative solenoid.

We expand on the results of KMS states of finite-graph Toeplitz-algebras, developing analagous results for the Toeplitz algebras of compact topological graphs. This is done with the aim of understanding the KMS states of noncommutative solenoids. The final chapter of the thesis deals with an attempt to characterise the KMS states of Toeplitz noncommutative solenoids, under a positivity assumption. We conclude with the conjecture that these KMS states are unique for $\beta > 0$.

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Contents

Chapter 1

Introduction

1.1 Introduction

A recent development in the theory of C^*-algebras was the introduction of noncommutative solenoids by Latrémolière and Packer in 2011 [38], and studied further in [37]. Of particular interest is [38, Theorem 3.7], which describes a noncommutative solenoid as a direct limit of rotation algebras, sometimes refered to as noncommutative tori [7]. The objective of this thesis is to construct a Toeplitz extension of a noncommutative solenoid, and study it's collection of KMS states.

In 1980 Cuntz and Krieger introduced a class of C^*-algebras associated to $\{0, 1\}$ matricies [9]. These are now called the *Cuntz-Krieger algebras* and were the genesis of a whole raft of research in the field of operator algebras. The first to generalise this approach to directed graphs were Enomoto and Watatani [14], and this has inspired a whole range of research, starting with [32, 33]. See the bibliography of [47] for an indication of the level of interest in the area in the late 90s and early 2000s. A particularly appealing aspect of this area is the rigid structure of graph C^*-algebras. A consequence of this rigid structure is that it restricts the number of examples that can be generated. For instance, all

simple graph C^*-algebras are either AF or purely infinite [32], and the K-theory is such that the K_0-group of a graph C^*-algebra is always a free abelian group, and the K_1-group of any graph C^*-algebra is always torsion free [47]. This is why generalisations were sought—be they higher dimensional [31] or continuous [11]. Katsura, utilizing work by Pimnser on the C^*-algebras associated to C^*-correspondences [46], defined in a very natural way, a topological analogue of a directed graph [25]. In a series of papers [25, 27, 28, 29], he developed some key structure theorems of topological graph C^*-algebras, and showed that this class of examples contains graph C^*-algebras, Kirchberg algebras, homeomorphism algebras, and every AF algebra. In particular, the noncommutative tori can be realised as topological graph C^*-algebras. This suggests that it may be possible to construct a sequence of topological graphs such that the direct limit of their C^*-algebras provides a natural realisation of a noncommutative solenoid. One advantage of this approach is that it suggests a natural Toeplitz extension of each noncommutative solenoid. This is important for us from the point of view of KMS states.

KMS-states were defined by Haag, Hugenholtz and Winnink [19] in 1967, based on earlier work by Kubo [30], and Martin and Schwinger [40]. Despite the motivation for this definition being deeply rooted in mathematical physics, the KMS condition can be defined purely in terms of a C^*-dynamical system (A, α), and a great deal of interesting information can be gathered from these KMS states, even when (A, α) is not a physical system. One such example is the number-theoretic work of Bost and Connes [4], which relates the Riemann zeta function to noncommutative geometry. There has also been a considerable amount of work done into the KMS states of the Cuntz algebras and their various generalisations by the likes of Enomoto, Fujii and Watatani [13] , Evans [15], Exel and Laca [16], Kajiwari and Watantani [23], and Olesen and Pedersen [42]. Typically, the Toeplitz exten-

sions of graph C^*-algebras and their analogues have a much richer supply of KMS states, than the graph algebras we typically deal with [17, 34]. The KMS states on Toeplitz algebras and on Cuntz-Krieger algebras are very similar, but they differ in one critical aspect: the KMS states on the Toeplitz algebra satisfy a subinvariance relation, whilst the those on the Cuntz-Krieger algebra satisfy a stronger invariance relation [34, Theorem 2.5]. This subinvariance condition gives rise to a much larger simplex of KMS states, that seem to illuminate some underlying structure present in the C^*-algebra: the extreme points of the simplex of KMS states of the Toeplitz algebra of a finite directed graph are indexed by its verticies [2, Theorem 3.1], the simplex of KMS states of the Toeplitz algebra of a self-similar group is isomorphic to the collection of traces on the group C^*-algebra [35, Theorem 5.1], and the simplex of KMS states of the Toeplitz algebra associated to a local homeomorphism of a compact space X is isomorphic to the set of probability measures on X [1, Theorem 5.1]. The invariance condition is much more rigid, however, and often places strict requirements the KMS states must satisfy to factor through to the Cuntz-Pimsner algebra [2, Theorem 4.3], [35, Proposition 7.1, Theorem 7.3], [1, Theorem 6.1]. For example, the C^*-algebra of a strongly connected finite graph has a unique KMS state. This suggests that we should seek to construct a Toeplitz extension of each noncommutative solenoid if we wish to see interesting structure via KMS theory. We construct such an extension in Chapter 4. The final chapter of this thesis deals the set of KMS states of Toeplitz noncommutative solenoids assuming a positivity condition.

1.2 Outline of Results

In Chapter 2, we introduce the necessary background concepts used throughout the thesis. Basing our exposition on [48], we introduce Hilbert modules and

C^*-correspondences (also called Hilbert bimodules) and some basic results about them. This enables us to discuss the Toeplitz and Cuntz–Pimsner algebras of C^*-correspondences. Following on from this, in Section 2.2, we recall the topological graphs and their associated C^*-algebras introduced by Katsura in [25], and illustrate the ideas involved by discussing graph C^*-algebras, and crossed products of commutative C^*-algebras by $\mathbb{Z}$ as examples. For example, we discuss the topological graph E_θ with vertex set $\mathbb{T}$, and one edge from each vertex $z \in \mathbb{T}$ to the vertex $e^{2i\pi\theta}z$. Chapter 2 concludes with a section introducing the fundamentals of KMS states of C^*-algebras, and some well-known examples that served as motivation for our investigation.

Chapter 3 is concerned with constructing projective limits of topological graphs, and then investigating the resulting direct limits of Cuntz–Pimsner algebras. We show that maps between topological graphs and homomorphisms between topological graph C^*-algebras have a contravariant functorial relationship. This is expressed most clearly in Theorem 3.1.11 and Theorem 3.2.19. Katsura does do this in [27], but this was unknown to the author at the time, and we expand upon the details provided in [27] for a number of results.

In Chapter 4, we show how to obtain the noncommutative solenoid $\mathcal{A}_\theta^{\mathscr{S}}$ as a direct limit of topological graph C^*-algebras (Definition 4.1.1). This is more complicated than it sounds. In particular, we are not able to realise $\mathcal{A}_\theta^{\mathscr{S}}$ as the C^*-algebra of a projective limit of topological graphs of the form E_{θ_j}. Instead, we first cover each E_{θ_j} graph with the n-th power graph of $E_{\theta_{j+1}}$, inducing an inclusion of C^*-algebras $C^*(E_{\theta_j}) \to C^*(E_{\theta_{j+1}}^{(n)})$, as in the previous chapter. We then describe an embedding of the Toeplitz algebra $\mathcal{T}(E^{(n)})$ into $\mathcal{T}(E)$ for an arbitrary topological graph E, and provide a sufficient condition, which is satisfied by E_{θ_j} for this to factor through to an inclusion of $C^*(E^{(n)})$ into $C^*(E)$. Composition of these maps gives us an inclusion $C^*(E_{\theta_j}) \to C^*(E_{\theta_{j+1}})$. We show that $\mathcal{A}_\theta^{\mathscr{S}}$ is the direct limit

of the $C^*(E_{\theta_j})$ under these maps. Since each of these inclusions are inherited from maps induced between Toeplitz algebras, we thereby define a Toeplitz extension of a noncommutative solenoid, which we call a Toeplitz noncommutative solenoid (Definition 4.1.12), whose KMS states we study in Chapter 6.

To do so, we first develop in Chapter 5, an analysis of KMS states of C^*-algebras of topological graphs, and their Toeplitz extensions. As is consistent with previous results about KMS states for graph algebras and their generalisations, for any given topological graph E, the Toeplitz algebra $\mathcal{T}(E)$ has a much richer supply of KMS states than $C^*(E)$ does.

We begin with Theorem 5.1.3, which provides a necessary and sufficient condition for a state of $\mathcal{T}(E)$ to be a KMS state for the gauge action. In keeping in with other results of the literature, every KMS state factors through the canonical expectation onto the core of the C^*-algebra. The KMS condition then implies that any KMS state can be described in terms of what happens on a commutative subalgebra, which allows a characterisation of KMS states in terms of Borel probability measures on compact spaces. A version of this theorem for Toeplitz C^*-algebras for arbitrary C^*-correspondences was obtained independently and in parallel to our work by Asfar, an Huef and Raeburn [1, Theorem 3.1] (see the generous remark preceeding the theorem).

We also describe a subinvariance relation for measures giving rise to KMS states on the Toeplitz algebras of topological graphs. We show that for KMS states that factor through to $C^*(E)$, the subinvariance relation becomes invariance. We conclude that the boundary of the simplex of KMS_β states of $\mathcal{T}(E)$ is homeomorphic to the vertex set for large values of β. We pay particular attention to graphs of the form E_θ as described in Chapter 2, whose C^*-algebras are the building blocks of noncommutative solenoids and their Toeplitz extensions in Chapter 4. We show that our results coincide with results concerning C^*-algebras of finite

directed graphs [2] and C^*-algebras associated to local homeomorphsisms [1]. We conclude the chapter by examining the KMS states of a projective limit E_∞ of topological graphs E_n. We show that the maps induced between $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_n))$ and $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_{n+1}))$ preserve extreme points, and hence the projective limit structure of E_∞.

Chapter 6 deals with the KMS states of Toeplitz noncommutative solenoids. We begin by constructing an action α_∞ of $\mathbb{R}$ on $\mathcal{T}_\theta^{\mathscr{S}}$ that respects the direct limit structure present in Definition 4.1.1. Interestingly, rather than being the lift to $\mathbb{R}$ of an action of $\mathbb{T}$, as in the Toeplitz algebras studied in Chapter 5, this action is nonperiodic, so is not a lift of a circle action. We investigate the KMS states for this action, which requires some intricate analysis. We show that, assuming the linking transforms of $\mathcal{M}^1(\mathbb{T})$ induced by the inclusions of the direct sequences all carry positive measures to positive measures, the boundary of the simplex of KMS states is homeomorphic to the solenoid from classical topology. This sounds like an intuitive result, however, the analysis is quite intricate, and the maps between the boundaries of $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_{\theta_n}))$ and $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_{\theta_{n+1}}))$ are not what one would expect. In particular, they are not simply self coverings of $\mathbb{T}$, that appears in the usual description of the solenoid. We proceed to show that this positivity assumption is false: we provide clear counter examples as to why this is the case. Nevertheless, we have left in the original arguements which require this assumption, in place (making clear where this assumption is required). We have done so because the details are instructive, and we believe that the tools developed may provide useful tools for future work. Our counterexamples seem to indicate that a KMS state on a noncommutative solenoid must restrict to a sequence of states arising from rotationally invariant measures on the approximating subalgebras in the direct limit. This leads us to the concluding conjecture: that for $\beta > 0$, there exists a unique KMS state on $\mathcal{T}_\theta^{\mathscr{S}}$, when θ is a sequence of irrational numbers.

Chapter 2

Background Material

2.1 Hilbert modules and Toeplitz algbras

The first two sections borrow heavily from [47]. Details have been added in some places since material on Cuntz-Pimsner algebras in [47] was intended as summary. We will assume some basic knowledge of C^*-algebras. For background, see [41] and [48].

First, we introduce Hilbert modules and develop some theory. We will use these Hilbert modules to assign C^*-algebras to topological graphs.

Definition 2.1.1 ([48, Definition 2.1]). *Let A be a C^*-algebra, and X a right A-module. An A-**valued inner product on** X is a function*

$$\langle \cdot, \cdot \rangle_A : X \times X \to A$$

such that for all $x, y, z \in X$, $a \in A$ and $\alpha \in \mathbb{C}$,

1. $\langle x, \alpha y + z \rangle = \alpha \langle x, y \rangle + \langle x, z \rangle$;

2. $\langle x, x \rangle_A \geq 0$ *with equality only if* $x = 0$;

3. $\langle x, y \rangle_A^* = \langle y, x \rangle_A$ *and*

4. $\langle x, y \cdot a\rangle_A = \langle x, y\rangle_A a$.

We call A the **coefficient algebra of** X. When the coefficient algebra of X is clear from the context, it is common to drop it from the notation. However, later we will be dealing with maps between Hilbert modules, so we will endeavour to keep the coefficient algebra explicit in the notation when necessary.

Conditions (1) and (3) imply that $(x, y) \mapsto \langle x, y\rangle_A$ is conjugate linear in the first variable. Taking adjoints in (4) gives the identity $\langle x \cdot a, y\rangle_A = a^*\langle x, y\rangle_A$.

We have a number of familiar looking norm properties — most importantly, a variation of the Cauchy-Schwarz inequality. This will allow us to define a norm on X.

To prove the following result, recall from [45, Proposition 1.3.5] and [48, Corollary 2.22], that given a C^*-algebra A, and $b, a \in A$ such that a is positive,

$$b^*ab \leq \|a\|\, b^*b. \tag{2.1}$$

Proposition 2.1.2 ([48, Lemma 2.5],[36, Proposition 1.1]). *Let A be a C^*-algebra, and X a right A-module with an A-valued inner-product. Then for $x, y \in X$ we have*

$$\langle x, y\rangle^*\langle x, y\rangle \leq \|\langle x, x\rangle\|\, \langle y, y\rangle$$

We present a modified version of the proof of [36, Proposition 1.1].

Proof. Fix $x, y \in X$. If $x = 0$ then the result is trivial, so we will suppose otherwise. Then,

$$\begin{aligned}
0 &\leq \langle x \cdot \langle x, y\rangle - \|\langle x, x\rangle\|\, y, x \cdot \langle x, y\rangle - \|\langle x, x\rangle\|\, y\rangle \\
&= \langle x, y\rangle^*\langle x, x\rangle\langle x, y\rangle - \|\langle x, x\rangle\|\, \langle x, y\rangle^*\langle x, y\rangle - \|\langle x, x\rangle\|\, \langle y, x\rangle\langle x, y\rangle \\
&\qquad + \|\langle x, x\rangle\|^2\, \langle y, y\rangle \\
&\leq \|\langle x, x\rangle\|\, \langle x, y\rangle^*\langle x, y\rangle - 2\, \|\langle x, x\rangle\|\, \langle x, y\rangle^*\langle x, y\rangle + \|\langle x, x\rangle\|^2\, \langle y, y\rangle \qquad \text{by (2.1)}
\end{aligned}$$

$$= \|\langle x,x\rangle\|^2 \langle y,y\rangle - \|\langle x,x\rangle\| \langle x,y\rangle^*\langle x,y\rangle.$$

Since $x \neq 0$, we have $\|\langle x,x\rangle\| > 0$, and so can divide through by $\|\langle x,x\rangle\|$ to obtain $\langle x,y\rangle^*\langle x,y\rangle \leq \|\langle x,x\rangle\| \langle y,y\rangle$. $\square$

Proposition 2.1.2 is reminiscient of the Cauchy-Schwarz inequality, so naturally this leads us define a norm on X.

Proposition 2.1.3 ([48, Corollary 2.7])**.** *Let X be a right inner-product A module. If for all $x \in X$, we define*

$$\|x\| := \|\langle x,x\rangle\|_A^{\frac{1}{2}}, \tag{2.2}$$

then $\|\cdot\|$ is a norm on X.

Proof. Let $x, y \in X$ and $\alpha \in \mathbb{C}$. Then

$$0 = \|x\| \Leftrightarrow \|\langle x,x\rangle\|^{\frac{1}{2}} = 0 \Leftrightarrow \langle x,x\rangle = 0 \Leftrightarrow x = 0.$$

Also,

$$\|\alpha x\| = \|\langle \alpha x, \alpha x\rangle\|^{\frac{1}{2}} = \|\overline{\alpha}\alpha\langle x,x\rangle\| = (|\alpha|^2\|\langle x,x\rangle\|)^{\frac{1}{2}} = |\alpha|\|x\|,$$

and finally

$$\begin{aligned}
\|x+y\|^2 &= \|\langle x+y, x+y\rangle\| \\
&= \|\langle x,x\rangle + \langle x,y\rangle + \langle y,x\rangle + \langle y,y\rangle\| \\
&\leq \|\langle x,x\rangle\| + \|\langle x,y\rangle\| + \|\langle y,x\rangle\| + \|\langle y,y\rangle\| \\
&\leq \|\langle x,x\rangle\| + \|\langle y,y\rangle\| \\
&= \|x\|^2 + \|y\|^2,
\end{aligned}$$

so the triangle inequality holds, and hence $\|\cdot\|$ is a norm. $\square$

Each element h of a Hilbert space $\mathcal{H}$ is uniquely determined by the linear functional $k \mapsto (k|h)$. The following proposition can be used to establish a similar property for Hilbert modules (Definition 2.1.6).

Proposition 2.1.4 ([3, Proposition II.7.1.9])**.** *Let X be a right inner-product module over A. For all $x \in X$, we have*

$$\|x\| = \sup_{\|y\|=1} \|\langle x, y\rangle\| .$$

If $x, y \in X$ are such that $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in X$, then $x = y$.

Proof. Fix $x \in X$. Then

$$\sup_{\|y\|=1} \|\langle x, y\rangle\|^2 = \sup_{\|y\|=1} \|\langle x, y\rangle^* \langle x, y\rangle\| \leq \sup_{\|y\|=1} \|\langle x, x\rangle\| \, \|\langle y, y\rangle\| = \|x\|^2 .$$

For the reverse inequality, we have

$$\sup_{\|y\|=1} \|\langle x, y\rangle\| \geq \left\| \left\langle x, \frac{x}{\|x\|} \right\rangle \right\| = \|x\| ,$$

which gives us the desired result. For the final assertion, observe that if $\langle x, z\rangle = \langle y, z\rangle$ for all $z \in X$, then

$$\|x - y\| = \sup_{\|z\|=1} \|\langle x - y, z\rangle\| = \sup_{\|z\|=1} \|\langle x, z\rangle - \langle y, z\rangle\| = 0.$$

That is, $x - y = 0$. □

Lemma 2.1.5 ([48, Corollary 2.7])**.** *Let X be a right inner-product module over a C^*-algebra A. For $x \in X$ and $a \in A$,*

$$\|x \cdot a\| \leq \|x\| \|a\|.$$

Proof. Fix $a \in A$ and $x \in X$. Then

$$\|x \cdot a\|^2 = \|\langle x \cdot a, x \cdot a\rangle\| = \|a^* \langle x, x\rangle a\| \leq \|a\|^2 \|x\|^2$$

by submultiplicativity of the C^*-norm. Hence $\|x \cdot a\| \leq \|x\| \|a\|$. □

Definition 2.1.6 ([48, Definition 2.8])**.** *Let A be a C^*-algebra, and X a right A-module, with an A-valued inner-product. If X is complete with respect to the norm* (2.2)*, we say X is a* ***right Hilbert*** A***-module****.*

A right-Hilbert A-module X is commonly denoted by X_A.

Notation 2.1.7. *We will use the convention that the inner-product on a Hilbert space is linear in the first variable. This is different to our convention for A-valued Hilbert modules. Following the convention of [48] , to avoid confusion with these conventions, we use the notation $(\cdot|\cdot)$ to denote the left-linear inner product on a Hilbert space, and $\langle\cdot,\cdot\rangle$ for right-linear inner-product over a C^*-algebra.*

Example 2.1.8 ([48, Example 2.9])**.** Consider the C^*-algebra $\mathbb{C}$. Let $\mathcal{H}$ be a Hilbert space, with inner product $(h,k) \mapsto (h|k)$ for all $h,k \in \mathcal{H}$. Define a right action of $\mathbb{C}$ on $\mathcal{H}$ by scalar multiplication, and $\langle h,k\rangle := (k|h)$ for all $h,k \in \mathcal{H}$. Then $\mathcal{H}$ is a Hilbert $\mathbb{C}$-module.

Notation 2.1.9. *Let X be a locally compact Hausdorff space. For $f : X \to \mathbb{C}$, we write*

$$\operatorname{supp}(f) = \overline{\{x \in X \mid f(x) \neq 0\}},$$

and call this set the support of f. We write

$$C(X) := \{f : X \to \mathbb{C} \mid f \textit{ is continuous}\}$$

for the collection of continuous functions on X. We denote the collection of compactly supported functions by

$$C_c(X) := \{f \in C(X) \mid \operatorname{supp}(f) \textit{ is compact}\}.$$

An element of the set

$$C_0(X) := \{f \in C(X) \mid \textit{for all } \epsilon > 0, \textit{ the set } \{x \in X : |f(x)| \geq \epsilon\} \textit{ is compact}\}$$

is said to vanish at infinity. Finally, we denote by

$$C(X)^+ := \{f \in C(X) \mid f(x) \geq 0 \textit{ for all } x \in X\},$$

the collection of nonnegative functions.

When X is discrete topological space, $C_c(X)$ is the collection of functions with finite support, and $C_0(X)$ is the collection of functions that are arbitrarily small off finite sets.

We take the following example from [18]. The convention with regards to left and right actions has been reversed however.

Example 2.1.10 ([18, Example 1.2])**.** Let $E = (E^0, E^1, r, s)$ be a directed graph. Then $C_c(E^1)$ is a right $C_0(E^0)$-module under the right-action defined by

$$(x \cdot a)(e) = x(e)a(s(e))$$

for all $x \in C_c(E^1)$, $a \in C_0(E^0)$ and $e \in E^1$. There exists a $C_0(E^0)$-valued inner product on $C_c(E^1)$ such that

$$\langle x, y\rangle(v) = \sum_{\{e \in E^1 : s(e)=v\}} \overline{x(e)}y(e)$$

for all $x, y \in C_c(E^1)$ and $x \in E^0$. (The sum is finite, because $s^{-1}(v)$ is a discrete set, and x, y have compact support). The completion $X(E)$ of $C_c(E^1)$ with respect to the norm arising from the inner-product is a right A-Hilbert module. It follows from [25] that

$$X(E) = \Big\{x \in C(X) \mid \sum_{e \in s^{-1}(v)} |x(e)|^2 < \infty \text{ and } v \mapsto \big\|x|_{s^{-1}(v)}\big\|_2 \in C_0(E^0)\Big\}.$$

Bounded linear operators on a Hilbert space automatically admit an adjoint, but this is not the case for Hilbert modules — there exist some operators on a Hilbert module with no adjoint — as we will see in Example 2.1.13. It is for this reason that we examine adjointable operators on Hilbert modules, which are automatically bounded and linear.

Definition 2.1.11 ([48, Definition 2.17])**.** *Let A be a C^*-algebra and X a right Hilbert A-module. An **adjointable operator** on X is a map $T : X \to X$ such*

that there exists $S : X \to X$ satisfying

$$\langle Tx, y\rangle_A = \langle x, Sy\rangle_A$$

for all $x, y \in X$. The set of adjoinable operators on X is denoted by $\mathcal{L}(X)$.

Lemma 2.1.12. *Let X be a right Hilbert module over A. For $T \in \mathcal{L}(X)$, there exists a unique operator $T^* : X \to X$ such that $\langle Tx, y\rangle = \langle x, T^*y\rangle$ for all $x, y \in X$. We call T^* the adjoint of T. Moreover, T is bounded and linear, with $\|T\| = \|T^*\|$.*

Proof. Fix $T \in \mathcal{L}(X)$, and suppose $\langle x, Sy\rangle = \langle Tx, y\rangle = \langle x, Ry\rangle$ for all $x, y \in X$. Fix $x, y \in X$. Then

$$\langle Rx - Sx, y\rangle = \langle Rx, y\rangle - \langle Sx, y\rangle = \langle x, Ty\rangle - \langle x, Ty\rangle = 0.$$

So, Proposition 2.1.4 implies that $R = S$, so adjoints are unique. From here on, we write T^* for the adjoint of T.

Since

$$\langle T(\lambda x + y), z\rangle = \overline{\lambda}\langle x, T^*z\rangle + \langle y, T^*z\rangle = \overline{\lambda}\langle Tx, z\rangle + \langle Ty, z\rangle = \langle \lambda Tx + Ty, z\rangle, \quad (2.3)$$

the final statement of Proposition 2.1.4 shows that T is linear.

Now,

$$\begin{aligned} \sup_{\|x\|\leq 1} \|T^*x\| &= \sup_{\|x\|,\|y\|\leq 1} \|\langle T^*x, y\rangle\| \\ &= \sup_{\|x\|,\|y\|\leq 1} \|\langle x, Ty\rangle\| \\ &= \sup_{\|x\|,\|y\|\leq 1} \|\langle Ty, x\rangle\| \\ &= \sup_{\|y\|\leq 1} \|Ty\| \\ &= \|T\|, \end{aligned}$$

so T^* is bounded, with $\|T\| = \|T^*\|$. $\square$

Example 2.1.13. We show that $\mathcal{L}(X) \neq \mathcal{B}(X)$. Consider the C^*-algebra $M_2(\mathbb{C})$, as a right Hilbert module over itself, such that for $A, B \in M_2(\mathbb{C})$, the right action and inner-product are given resectively by

$$A \cdot B = AB \text{ and } \langle A, B \rangle = A^* B.$$

Let $T : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be such that for all $a, b, c, d \in \mathbb{C}$,

$$T \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & 0 \\ 0 & 0 \end{pmatrix}.$$

Then T is linear and $\|T\| = 1$, so $T \in \mathcal{B}(M_2(\mathbb{C}))$. Suppose that T has an adjoint. Then there exist $x, y, z, w \in \mathbb{C}$ such that

$$T^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} x & y \\ z & w \end{pmatrix}.$$

Then

$$\left\langle T \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}, \tag{2.4}$$

and

$$\left\langle \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, T^* \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right\rangle = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}^* \begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ z & w \end{pmatrix}. \tag{2.5}$$

By considering the case where $a = b = 1$, we can see this is absurd. So T cannot be adjointable.

Proposition 2.1.14 ([48, Proposition 2.21])**.** *Let A be a C^*-algebra, and X a right-Hilbert A-module. Then $\mathcal{L}(X)$ is a C^*-algebra under the operator norm.*

Proof. Fix $x, y \in X$, $\lambda \in \mathbb{C}$, and $S, T \in \mathcal{L}(X)$. The algebraic operations of addition and multiplication (taken to be composition of operators) are clear, so we check that adjoints behave as we expect i.e. $(ST)^* = T^* S^*$, $(\lambda S + T)^* = \overline{\lambda} S^* + T^*$ and $S^{**} = S$. First,

$$\langle (\lambda S + T)x, y \rangle = \langle \lambda S x, y \rangle + \langle T x, y \rangle = \overline{\lambda} \langle x, S^* y \rangle + \langle x, T^* y \rangle = \langle x, (\overline{\lambda} S^* + T^*) y \rangle,$$

so $(\lambda S + T)^* = \overline{\lambda} S^* + T^*$. Further,

$$\langle (ST)x, y\rangle = \langle S \circ Tx, y\rangle = \langle Tx, S^*y\rangle = \langle x, T^* \circ S^*y\rangle = \langle x, (T^*S^*)y\rangle,$$

so $(ST)^* = T^*S^*$. Also,

$$\langle Sx, y\rangle = \langle x, S^*y\rangle = \langle S^{**}x, y\rangle,$$

so $S = S^{**}$.

All that remains is to check that $\mathcal{L}(X)$ is complete, and satisfies the C^*-identity. Since $\mathcal{L}(X) \subseteq \mathcal{B}(X)$, which is complete, to show $\mathcal{L}(X)$ is complete, it suffices to show that $\mathcal{L}(X)$ is closed. Fix a sequence $\{T_n\}_{n=1}^{\infty} \in \mathcal{L}(X)$ such that $T_n \to T \in \mathcal{B}(X)$. Then $(T_n^*)_{n=1}^{\infty}$ is a Cauchy sequence, since $T \mapsto T^*$ is isometric. Since $\mathcal{B}(X)$ is complete, T_n^* converges to some $S \in \mathcal{B}(X)$. Then for $x, y \in X$,

$$\langle Tx, y\rangle = \lim_{n\to\infty} \langle T_n x, y\rangle = \lim_{n\to\infty} \langle x, T_n^* y\rangle = \langle x, Sy\rangle.$$

Hence, T is adjointable, and so $\mathcal{L}(X)$ is closed.

For the C^*-identity, fix $T \in \mathcal{L}(X)$. Since $\mathcal{B}(X)$ is a Banach algebra under the operator norm,

$$\|T^*T\| \leq \|T^*\| \, \|T\| = \|T\|^2 .$$

We also have

$$\begin{aligned}
\|T^*T\| &= \sup_{\|x\|\leq 1} \|T^*Tx\| \\
&= \sup_{\|x\|,\|y\|\leq 1} \|\langle T^*Tx, y\rangle\| \\
&\geq \sup_{\|x\|\leq 1} \|\langle T^*Tx, x\rangle\| \\
&= \sup_{\|x\|\leq 1} \|\langle Tx, Tx\rangle\| \\
&= \sup_{\|x\|\leq 1} \|Tx\|^2
\end{aligned}$$

$$= \Big(\sup_{\|x\| \leq 1} \|Tx\| \Big)^2$$
$$= \|T\|^2,$$

giving $\|T^*T\| = \|T\|^2$. $\square$

Given $T \in \mathcal{L}(X)$, $x \in X$ and $a \in A$, we have $T(x \cdot a) = Tx \cdot a$, because

$$\langle T(x \cdot a), y \rangle = \langle x \cdot a, T^* y \rangle = a^* \langle x, T^* y \rangle = a^* \langle Tx, y \rangle = \langle Tx \cdot a, y \rangle.$$

Before investigating the next example, we require a lemma.

Lemma 2.1.15. *Let A be a C^*-algebra, and suppose $a \in A$ is self-adjoint. Then*

$$\|a^n\| = \|a\|^n$$

for all $n \in \mathbb{N}$.

Proof. Since $a = a^*$, then $\sigma(a) \subseteq \mathbb{R}$. For $f \in C(\sigma(a))$ and $n \in \mathbb{N}$,

$$\|f\|^n = \sup_{x \in \sigma(a)} |f(x)|^n = \sup_{x \in \sigma(a)} |f^n(x)| = \|f^n\|,$$

so the continuous functional calculus implies that $\|a^n\| = \|a\|^n$. $\square$

Example 2.1.16 ([48])**.** Let X be a right Hilbert A-module. For $x, y, z \in X$, define $\Theta_{x,y}(z) = x \cdot \langle y, z \rangle$. For all $x, y, z, w \in X$,

$$\langle \Theta_{x,y}(w), z \rangle = \langle x \cdot \langle y, w \rangle, z \rangle = \langle w, y \rangle \langle x, z \rangle = \langle w, y \cdot \langle x, z \rangle \rangle = \langle w, \Theta_{y,x}(z) \rangle.$$

So $\Theta_{x,y} \in \mathcal{L}(X)$ with $\Theta_{x,y}^* = \Theta_{y,x}$. The closed span

$$\mathcal{K}(X) = \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in X\}$$

of the $\Theta_{x,y}$ is a two sided-ideal in $\mathcal{L}(X)$. To see this, fix $T \in \mathcal{L}(X)$ and $x, y, z \in X$. Then

$$T \circ \Theta_{x,y}(z) = T(x \cdot \langle y, z \rangle) = T(x) \cdot \langle y, z \rangle = \Theta_{T(x),y}(z)$$

and

$$\Theta_{x,y} \circ T(z) = \Theta_{x,y}(Tz) = x \cdot \langle y, T(z)\rangle = x \cdot \langle T^*(y), z\rangle = \Theta_{x,T^*(y)}(z).$$

We claim that, $\|\Theta_{x,x}\| = \|x\|^2$. On one hand

$$\|\Theta_{x,x}\| = \sup_{\|y\|=1} \|\Theta_{x,x}(y)\| = \sup_{\|y\|=1} \|x \cdot \langle x, y\rangle\| \leq \sup_{\|y\|=1} \|x\| \, \|\langle x, y\rangle\| = \|x\|^2$$

by Proposition 2.1.5. On the other hand,

$$\|\Theta_{x,x}\| \geq \left\| \Theta_{x,x}\Big(\frac{x}{\|x\|}\Big) \right\| = \frac{1}{\|x\|} \|x \cdot \langle x, x\rangle\| = \frac{1}{\|x\|} \|x\|^3$$

by Lemma 2.1.15, since $\langle x, x\rangle$ is self-adjoint. Hence, $\|\Theta_{x,x}\| = \|x\|^2$.

The C^*-algebras we are interested in are built from Hilbert modules with some extra structure. Some of the C^*-algebras generated in this way coinicide with graph algebras the reader may be familiar with.

Definition 2.1.17 ([47, Chapter 8])**.** *A **C^*-correspondence** X **over a C^*-algebra** A is a right Hilbert A-module X, together with a homomorphism $\phi : A \to \mathcal{L}(X)$.*

Given a C^*-correspondence X over A, we think of the homomorphism $\phi : A \to \mathcal{L}(X)$ as determining a left action of A on X_A and write $a \cdot x := \phi(a)x$ for $a \in A, x \in X$.

Since ϕ is a homomorphism of C^*-algebras, we have $\phi(a^*) = \phi(a)^*$. This forces

$$\langle a \cdot x, y\rangle = \langle \phi(a)x, y\rangle = \langle x, \phi(a)^* y\rangle = \langle x, a^* \cdot y\rangle$$

for all $x, y \in X$ and $a \in A$.

Sometimes a C^*-correspondence X over a C^*-algebra A is called a **Hilbert bimodule** over A, or a **correspondence over** A. It is possible to have different C^*-algebras acting on the left and right of the module, and the standard notation

in the literature for this is ${}_BX_A$. However, we will only consider the case where a single C^*-algebra acts on the left or the right of a module X, and so, refer to it as a C^*-correspondence over A, and simply denote the pair by X_A.

Example 2.1.18 ([18, Example 1.2])**.** Let E be a directed graph, and consider the right Hilbert $C_0(E^0)$-module $X(E)$ described in Example 2.1.10. For $x \in C_c(E^1)$ and $a \in C_0(E^0)$, we define a function $a \cdot x : E^1 \to \mathbb{C}$ by $a \cdot x(e) = a(r(e))x(e)$. We claim this function satisfies $\|a \cdot x\|_{X(E)} \leq \|a\|_\infty \|x\|_{X(E)}$. We have

$$\begin{aligned}
\|a \cdot x\|_{X(E)} &= \|\langle a \cdot x, a \cdot x\rangle\|_\infty \\
&= \sup_{v \in E^0} |\langle a \cdot x, a \cdot x\rangle(v)| \\
&= \sup_{v \in E^0} \Big| \sum_{s(e)=v} \overline{a(r(e))x(e)} a(r(e))x(e) \Big| \\
&\leq \|a\|_\infty \sup_{v \in E^0} \Big| \sum_{s(e)=v} \overline{x(e)} x(e) \Big| \\
&= \|a\|_\infty \|x\|_{X(E)}
\end{aligned}$$

Hence, there is a unique function $\phi(a) : x \mapsto a \cdot x$ on $X(E)$, such that for $e \in E^1$, $a \cdot x(e) = a(r(e))x(e)$. Since the algebraic operations are defined pointwise, for $a, b \in C_0(E^0)$, $\alpha \in \mathbb{C}$ we have $\phi(\alpha a + b) = \alpha\phi(a) + \phi(b)$ and $\phi(ab) = \phi(a)\phi(b)$. We have

$$\langle a \cdot x, y\rangle(v) = \sum_{s(e)=v} \overline{a(r(e)x(e)} y(e) = \sum_{s(e)=v} \overline{x(e)}\ \overline{a(r(e))} y(e) = \langle x, a^* \cdot y\rangle(v).$$

Hence $\phi(a)$ is adjointable with $\phi(a^*) = \phi(a)^*$. We call $X(E)$ the graph correspondence of E.

Graph correspondences are useful, as they can be used to construct several C^*-algebras that we will study over the course of this document. We first consider the Toeplitz algebra. We will then develop the necessary theory to discuss Cuntz-Pimsner algebras, which will be of more use in the context of topological graphs, and will feature more in later chapters.

Definition 2.1.19. *Let X be a C^*-correspondence over A. A **Toeplitz representation** of X in a C^*-algebra B is a pair (ψ, π) such that $\psi : X \to B$ is linear and $\pi : A \to B$ is a homomorphism of C^*-algebras satisfying*

1. $\psi(a \cdot x) = \pi(a)\psi(x)$
2. $\pi(\langle x, y\rangle) = \psi(x)^*\psi(y)$
3. $\psi(x \cdot a) = \psi(x)\pi(a)$

for all $x, y \in X$ and $a \in A$.

Condition (2) implies that ψ is bounded. In fact, given a C^*-correspondence over A and a Toeplitz representation (ψ, π) of X in B, we have

$$\|\psi(x)\|^2 = \|\psi(x)^*\psi(x)\| = \|\pi(\langle x, x\rangle)\| \leq \|\langle x, x\rangle\| = \|x\|^2 \tag{2.6}$$

for $x \in X$. If π is injective, this forces equality throughout (2.6), and so ψ is isometic, and hence injective.

Theorem 2.1.20 ([46, Theorem 3.4][18, Proposition 1.3])**.** *Let X be a C^* - correspondence over A. Then there exists a Toeplitz representation (ι_X, ι_A) into a C^*-algebra $\mathcal{T}(X)$ which is universal in the following sense: for any other Toeplitz representation $(\psi, \pi) : X_A \to B$, there exists a homomorphism $(\psi \times \pi) : \mathcal{T}(X) \to B$ such that $(\psi \times \pi) \circ \iota_X = \psi$ and $(\psi \times \pi) \circ \iota_A = \pi$.*

We illustrate the importance of Theorem 2.1.20 by outlining a few key examples.

Example 2.1.21. Let E be a directed graph as in Example 2.1.10, and let $X(E)$ be as defined in Example 2.1.18. Consider the Hilbert space $\mathcal{H} = \ell^2(E^*)$, where E^* is the set of all paths of finite length in E (for those unfamiliar with this terminology, see Definition 2.2.8 in Section 2.2). So the basis elements of $\mathcal{H}$ are

δ_μ, for some $\mu \in E^*$. Since E^0, E^1 are discrete, $C_0(E^0)$ and $X(E)$ are spanned by point mass functions. For $v \in E^0, e \in E^1$ and $\mu \in E^*$, let

$$S_e(\delta_\mu) := \begin{cases} \delta_{e\mu} & \text{if } s(e) = r(\mu) \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad P_v(\delta_\mu) := \begin{cases} \delta_\mu & \text{if } r(\mu) = v \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{P_v \mid v \in E^0\}$ is a set of mutually orthogonal projections. There exist functions $\psi : X(E) \to \mathcal{B}(\ell^2(E^*))$ and $\pi : C_0(E^0) \to \mathcal{B}(\ell^2(E^*))$ such that $\psi(\delta_e) = S_e$ and $\pi(\delta_v) = P_v$.

Routine calculations show that (ψ, π) is a Toeplitz representation in $\mathcal{B}(\ell^2(E^*))$. Condition (2) of Definition 2.1.19 implies that $S_e^* S_e = P_{s(e)}$, which implies that $\{S_e \mid e \in E^1\}$ is a collection of partial isometries. These conditions are reminiscient of the Cuntz-Krieger relations for directed graphs, except that

$$\Big(P_v - \sum_{r(e)=v} S_e S_e^*\Big)\delta_v = \delta_v$$

for $v \in E^0$, so $P_v > \sum_{r(e)=v} S_e S_e^*$. This means the Toeplitz algebra $\mathcal{T}(X(E))$ associated to the graph correspondence $X(E)$ is larger than the Cuntz-Krieger algebra $C^*(E)$, in the sense that $C^*(E)$ is the quotient by the non-zero ideal generated by the projections $\big(P_v - \sum_{r(e)=v} S_e S_e^*\big)$ for v such that $0 < |r^{-1}(v)| < \infty$. In the remainder of the section, we will discuss Katsura's construction of a C^*-algebra $\mathcal{O}_X$ from a C^*-correspondence X, such that when X is the C^*-correspondence of Example 2.1.10 associated to a directed graph E, the C^*-algebra $\mathcal{O}_{X(E)}$ is the Cuntz-Krieger algebra of E, $C^*(E)$.

Example 2.1.22 ([36, Chapter 4],[26, Section 4])**.** Let X, Y be C^*-correspondences over A, and denote the algebraic tensor product by $X \odot Y$. Then for $a \in A$, the formula

$$(x, y) \mapsto (x, y \cdot a)$$

defines a bilinear map on $X \times Y$, and hence induces a linear map on $X \odot Y$, and so a right action of A on $X \odot Y$. That is to say, $X \odot Y$ is a right A-module.

Fix $x_2 \in X$, $y_2 \in Y$. Then the map

$$(x_1, y_1) \mapsto \langle y_2, \langle x_2, x_1\rangle \cdot y_1\rangle \tag{2.7}$$

defines a bilinear map on $X \times Y$ (because $\langle \cdot, \cdot\rangle : X \times X \to A$ and $\langle \cdot, \cdot\rangle : Y \times Y \to A$ are linear in the second variable), and so induces a linear map on $X \odot Y$. Similarly,

$$(x_1, y_1) \mapsto \langle y_1, \langle x_1, x_2\rangle \cdot y_2\rangle^* \tag{2.8}$$

also defines a bilinear map, and so induces a linear map on $X \odot Y$. Combining (2.7) and (2.8) we obtain a sesquilinear form on $X \odot Y$,

$$(x_1 \otimes y_1, x_2 \otimes y_2) \mapsto \langle x_1 \otimes y_1, x_2 \otimes y_2\rangle := \langle y_1, \langle x_1, x_2\rangle \cdot y_2\rangle.$$

Now, for $a \in A$, $x_1, x_2 \in X$ and $y_1, y_2 \in Y$,

$$\begin{aligned}
\langle x_1 \otimes y_1, (x_2 \otimes y_2) \cdot a\rangle &= \langle x_1 \otimes y_1, x_2 \otimes (y_2 \cdot a)\rangle \\
&= \langle y_1, \langle x_1, x_2\rangle \cdot y_2 \cdot a\rangle \\
&= \langle y_1, \langle x_1, x_2\rangle \cdot y_2\rangle a \\
&= \langle x_1 \otimes y_1, x_2 \otimes y_2\rangle a
\end{aligned}$$

and

$$\begin{aligned}
\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle^* &= \langle y_1, \langle x_1, x_2\rangle \cdot y_2\rangle^* \\
&= \langle \langle x_1, x_2\rangle \cdot y_2, y_1\rangle \\
&= \langle y_2, \langle x_2, x_1\rangle \cdot y_1\rangle \\
&= \langle x_2 \otimes y_2, x_1 \otimes y_1\rangle,
\end{aligned}$$

so $\langle \cdot, \cdot\rangle : X \odot Y \times X \odot Y \to A$ satisfies (3) and (4) of Definition 2.1.1. However, we have

$$\langle (x \cdot a) \otimes y - x \otimes (a \cdot y), (x \cdot a) \otimes y - x \otimes (a \cdot y)\rangle$$

$$
\begin{aligned}
&= \langle (x\cdot a)\otimes y, (x\cdot a)\otimes y\rangle - \langle (x\cdot a)\otimes y, x\otimes (a\cdot y)\rangle \\
&\qquad - \langle x\otimes (a\cdot y), (x\cdot a)\otimes y\rangle + \langle x\otimes (a\cdot y), x\otimes (a\cdot y)\rangle \\
&= \langle \langle x\cdot a, x\cdot a\rangle \cdot y, y\rangle - \langle \langle x, x\cdot a\rangle \cdot y, a\cdot y\rangle \\
&\qquad - \langle \langle x\cdot a, x\rangle \cdot (a\cdot y), y\rangle + \langle \langle x, x\rangle \cdot (a\cdot y), (a\cdot y)\rangle \\
&= \langle a^*\langle x, x\rangle a\cdot y, y\rangle - \langle a^*\langle x, x\rangle a\cdot y, y\rangle \\
&\qquad - \langle a^*\langle x, x\rangle a\cdot y, y\rangle + \langle a^*\langle x, x\rangle a\cdot y, y\rangle \\
&= 0,
\end{aligned}
$$

so $\langle \cdot, \cdot \rangle$ has nontrivial kernel, and so is not an A-valued inner-product. Let

$$N := \operatorname{span}\{x\otimes y \in X\odot Y \mid \langle x\otimes y, w\otimes z\rangle = 0 \text{ for all } w\otimes z\in X\odot Y\},$$

then $\langle \cdot, \cdot \rangle$ descends to an A-valued inner-product on $(X\odot Y)/N$. We denote the completion of $(X\odot Y)/N$ in the norm coming from the A-valued inner-product by $X\otimes Y$, and call it the **balanced tensor product of X and Y**. In addition, the left action of A on X determines a left action of A on $X\odot Y$ in the same way as discussed above. Further, for $x_1, x_2 \in X$ and $y_1, y_2 \in Y$ we have

$$
\begin{aligned}
\langle a\cdot (x_1\otimes y_1), x_2\otimes y_2\rangle &= \langle (a\cdot x_1)\otimes y_1, x_2\otimes y_2\rangle \\
&= \langle y_1, \langle a\cdot x_1, x_2\rangle \cdot y_2\rangle \\
&= \langle y_1, \langle x_1, a^*\cdot x_2\rangle \cdot y_2\rangle \\
&= \langle x_1\otimes y_1, (a^*\cdot x_2)\otimes y_2\rangle \\
&= \langle x_1\otimes y_1, a^*\cdot (x_2\otimes y_2)\rangle .
\end{aligned}
$$

So the left action of A on $X\otimes Y$ is adjointable. Hence, $X\otimes Y$ is a C^*-correspondence over A.

Let $X^{\otimes 1} = X$, and for $n \geq 2$, we recursively define a C^*-correspondence by $X^{\otimes n} := X\otimes X^{\otimes (n-1)}$. Define $X^{\otimes 0} = A$ as a C^*-correspondence over A, with left and right actions given by multiplication and $\langle a, b\rangle = a^*b$.

We then define the **Fock space of** X to be the C^*-correspondence

$$\mathcal{F}(X) := \bigoplus_{n=0}^{\infty} X^{\otimes n}.$$

That is,

$$\mathcal{F}(X) = \Big\{ (x_n)_{n=0}^{\infty} \mid x_i \in X^{\otimes i}, \sum_{n=0}^{\infty} \langle x_n, x_n \rangle \text{ converges in } A \Big\}.$$

We claim that $\mathcal{F}(X)$ is a right-Hilbert module over A, with inner-product and right action such that

$$\langle \eta, \nu \rangle = \sum_{n=0}^{\infty} \langle \eta_n, \nu_n \rangle \text{ and } \eta \cdot a = (\eta_n \cdot a)_{n=0}^{\infty},$$

for $\eta = (\eta_n)_{n=0}^{\infty}, \nu = (\nu_n)_{n=0}^{\infty} \in \mathcal{F}(X)$. To do this, we follow the program outlined in [52].

The first thing we should do is check the formula makes sense. By definition, for $\eta, \nu \in \mathcal{F}(X)$, the sums $\sum_{i=0}^{\infty} \langle \eta_i, \eta_i \rangle, \sum_{i=0}^{\infty} \langle \nu_i, \nu_i \rangle$ are Cauchy. So for $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that for $m, n > N$, $\| \sum_{i=0}^{\infty} \langle \eta_i, \eta_i \rangle \|, \| \sum_{i=0}^{\infty} \langle \nu_i, \nu_i \rangle \| < \epsilon$. Suppose without loss of generality that $m < n$. Then,

$$\begin{aligned} \Big\| \sum_{i=0}^{n} \langle \eta_i, \nu_i \rangle - \sum_{i=0}^{m} \langle \eta_i, \nu_i \rangle \Big\|^2 &= \Big\| \sum_{i=n+1}^{m} \langle \eta_i, \nu_i \rangle \Big\|^2 \\ &\leq \Big\| \sum_{i=n+1}^{m} \langle \eta_i, \eta_i \rangle \Big\| \Big\| \sum_{i=n+1}^{m} \langle \nu_i, \nu_i \rangle \Big\| \quad \text{by Proposition 2.1.2} \\ &< \epsilon^2. \end{aligned}$$

Hence $\langle \eta, \nu \rangle = \sum_{i=0}^{\infty} \langle \eta_i, \nu_i \rangle \in A$. Also, since

$$\sum_{i=0}^{\infty} \langle \eta_i \cdot a, \eta_i \cdot a \rangle = \sum_{i=0}^{\infty} a^* \langle \eta_i, \eta_i \rangle a = a^* \Big(\sum_{i=0}^{\infty} \langle \eta_i, \eta_i \rangle \Big) a \in A,$$

we have $\eta \cdot a = (\eta_i \cdot a)_{i=0}^{\infty} \in \mathcal{F}(X)$. All that remains is to check (1), (2), (3) and (4) of Definition 2.1.1. Firstly,

$$\langle \eta, \alpha\nu + \xi \rangle = \sum_{n=0}^{\infty} \langle \eta_n^*, \alpha\nu_n + \xi_n \rangle = \sum_{n=0}^{\infty} \alpha\langle \eta_n, \nu_n \rangle + \langle \eta_n, \xi_n \rangle = \alpha\langle \eta, \nu \rangle + \langle \nu, \xi \rangle,$$

so $\langle \cdot, \cdot \rangle$ is linear in the second variable. Further,

$$\langle \eta, \eta \rangle = \sum_{n=0}^{\infty} \langle \eta_n, \eta_n \rangle = 0$$

if and only if $\eta_n = 0$ for all $n \in \mathbb{N}$, so (2) holds. We also have

$$\langle \eta, \nu \rangle^* = \sum_{n=0}^{\infty} \langle \eta_n, \nu_n \rangle^* = \sum_{n=0}^{\infty} \langle \nu_n, \eta_n \rangle = \langle \nu, \eta \rangle$$

and

$$\langle \eta, \nu \cdot a \rangle = \sum_{n=0}^{\infty} \langle \eta_n, (\nu_n \cdot a) \rangle = \sum_{n=0}^{\infty} \langle \eta_n, \nu_n \rangle a = \langle \eta, \nu \rangle a,$$

so (3) and (4) also hold.

We show that $\mathcal{F}(X)$ is complete with respect to the norm coming from the inner-product. Fix a Cauchy sequence $\{x^m\}_{m=0}^{\infty} = \{(x_k^m)_{k=0}^{\infty}\}_{m=0}^{\infty} \in \mathcal{F}(X)$. Since the map $q_k : (x_n)_{n=0}^{\infty} \to x_k$ is norm decreasing for each $k \in \mathbb{N}$, the sequence $(x_k^m)_{m=0}^{\infty}$ converges in $X^{\otimes k}$. Let $x_k = \lim_{m \to \infty} x_k^m$, and $x = (x_k)_{k=0}^{\infty}$. We claim that $x \in \mathcal{F}(X)$. Fix $\epsilon > 0$, and choose $m \in \mathbb{N}$ such that for $l \geq m$, $\|x^m - x^l\| < \epsilon$. For each $N \in \mathbb{N}$,

$$\begin{aligned}
\Big\| \sum_{n=0}^{N} \langle x_n - x_n^m, x_n - x_n^m \rangle \Big\| &= \lim_{l \to \infty} \Big\| \sum_{n=0}^{N} \langle x_n^l - x_n^m, x_n^l - x_n^m \rangle \Big\| \\
&\leq \lim_{l \to \infty} \Big\| \sum_{n=0}^{\infty} \langle x_n^l - x_n^m, x_n^l - x_n^m \rangle \Big\| \\
&= \lim_{l \to \infty} \|x_n^l - x_n^m\|^2 \\
&< \epsilon^2.
\end{aligned}$$

Since our choice of $N \in \mathbb{N}$ was arbitrary, the sequence $\left(\sum_{n=0}^{N}\langle x_n - x_n^m\rangle\right)_{N\in\mathbb{N}}$ is a bounded nondecreasing sequence of positive elements of A, and hence convergent in A ([52, Page 237]). Hence $x \in \mathcal{F}(X)$ and $x^m \to x$. So, $\mathcal{F}(X)$ is a Hilbert module. Define $\phi : A \to \mathcal{F}(X)$ by

$$a \cdot \xi = (a \cdot \xi_n)_{n=0}^{\infty}$$

for $a \in A$ and $\xi \in \mathcal{F}(X)$. Then for all $a \in A$, $\phi(a) \in \mathcal{L}(\mathcal{F}(X))$ since for $\xi, \eta \in \mathcal{F}(X)$,

$$\langle \phi(a)\xi, \eta\rangle = \sum_{n=0}^{\infty}\langle a \cdot \xi_n, \eta_n\rangle = \sum_{n=0}^{\infty}\langle \xi_n, a^* \cdot \eta_n\rangle = \langle \xi, \phi(a^*)\eta\rangle;$$

this also shows that $\phi(a)^* = \phi(a^*)$. It is routine to check that $\phi : A \to \mathcal{F}(X)$ is a homomorphism. Again, we denote $\phi(a)\xi = a \cdot \xi$ for $\xi \in \mathcal{F}(X)$.

For $x \in X$ and $\xi \in \mathcal{F}(X)$, let $\psi(x)\xi$ be

$$\eta_i = \begin{cases} 0 & \text{if } i = 0 \\ x \cdot \xi_0 & \text{if } i = 1 \\ x \otimes \xi_{i-1} & \text{if } i \geq 2. \end{cases}$$

Then $\psi(x)\xi \in (\mathcal{F}(X))$, because

$$\begin{aligned} \|\psi(x)\xi\|^2 &= \left\|\langle x \cdot \xi_0, x \cdot \xi_0\rangle + \sum_{n=1}^{\infty}\langle x \otimes \xi_n, x \otimes \xi_n\rangle\right\| \\ &= \left\|\langle x \cdot \xi_0, x \cdot \xi_0\rangle + \sum_{n=1}^{\infty}\langle \xi_n, \langle x, x\rangle \cdot \xi_n\rangle\right\| \\ &\leq \|x \cdot \xi_0\|^2 + \|x\|^2\left\|\sum_{n=1}^{\infty}\langle \xi_n, \xi_n\rangle\right\| \\ &\leq \|x \cdot \xi_0\|^2 + \|x\|^2\|\xi\|^2, \end{aligned}$$

which is finite. This also shows that $\psi(x) : \xi \mapsto \psi(x)\xi \in \mathcal{B}(\mathcal{F}(X))$. Further, $\psi(x)$ is adjointable, with adjoint satisfying

$$\psi(x)^*\xi = \Big((\langle x, \xi_{n,1}\rangle) \cdot \xi_{n,2} \otimes \cdots \otimes \xi_{n,n}\Big)_{n=1}^{\infty}$$

when $\xi_i = \xi_{i,1} \otimes \cdots \otimes \xi_{i,i}$. Let $a, b \in A$ and $x, y \in X$. Let $\xi \in \mathcal{F}(X)$, and $\pi = \phi$. Then for $i \in \mathbb{N}$ we have

$$
\begin{aligned}
(\psi(a \cdot x \cdot b)\xi)_i &= \begin{cases} 0 & \text{if } i = 0 \\ (a \cdot x \cdot b) \cdot \xi_0 & \text{if } i = 1 \\ (a \cdot x \cdot b) \otimes \xi_{i-1} & \text{if } i \geq 2 \end{cases} \\
&= \begin{cases} 0 & \text{if } i = 0 \\ (a \cdot x \cdot b) \cdot \xi_0 & \text{if } i = 1 \\ (a \cdot x) \otimes b \cdot \xi_{i-1} & \text{if } i \geq 2 \end{cases} \\
&= \pi(a)\psi(x)\pi(b)\xi,
\end{aligned}
$$

so $\pi(a)\psi(x)\pi(b) = \psi(a \cdot x \cdot b)$. Hence (ψ, π) satisfies (1) and (3) of Definition 2.1.19. Let $\nu = \psi(x)\xi$, and $\eta = \psi(y)^*\nu$. Then $\eta_i = \langle y, x\rangle \cdot \xi_i$ for all $i \in \mathbb{N}$. Hence $\psi(y)^*\psi(x) = \pi(\langle y, x\rangle)$, which is (2) of Defintion 2.1.19. So (ψ, π) is a Toeplitz representation of X on $\mathcal{B}(\mathcal{F}(X))$. Since π is injective, it is injective on $(\psi(\mathcal{F}(X)))^\perp$, so [18, Theorem 2.1] states that $\psi \times \pi$ is injective.

For $n \in \mathbb{N}$, there exists a linear map $\psi^{\otimes n} : X^{\otimes n} \to \mathcal{L}(\mathcal{F}(X))$ such that

$$
\psi^{\otimes n}(x_1 \otimes \cdots \otimes x_n) = \psi(x_1) \cdots \psi(x_n),
$$

with the convention that $\psi^{\otimes 0} = \pi$. For elementary tensors $x \in X^{\otimes m}$, $y \in X^{\otimes n}$, $z \in X^{\otimes k}$ and $w \in X^{\otimes l}$, we have

$$
\begin{aligned}
&\psi^{\otimes m}(x)\psi^{\otimes n}(y)^*\psi^{\otimes k}(z)\psi^{\otimes l}(w)^* \\
&= \begin{cases} \psi^{\otimes m}(x)\pi(\langle y, z_1 \otimes \cdots \otimes z_n\rangle)\psi^{\otimes k-n}(z_{n+1} \otimes \cdots \otimes z_k)\psi^{\otimes l}(w)^* & \text{if } n \leq k \\ \psi^{\otimes m}(x)\psi^{\otimes n-k}(y_{k+1} \otimes \cdots \otimes y_n)^*\pi(\langle y_1 \otimes \cdots \otimes y_k, z\rangle)\psi^{\otimes l}(w)^* & \text{if } k \leq n \end{cases} \\
&= \begin{cases} \psi^{\otimes m+k-n}(x \cdot \langle y, z_1 \otimes \cdots \otimes z_n\rangle \otimes z_{n+1} \otimes \cdots \otimes z_k)\psi^{\otimes l}(w)^* & \text{if } n \leq k \\ \psi^{\otimes m}(x)\psi^{\otimes n-k+l}(w \cdot \langle z, y_1 \otimes \cdots y_k\rangle \otimes y_{k+1} \otimes \cdots \otimes y_n)^* & \text{if } k \leq n. \end{cases}
\end{aligned}
$$

Hence

$$
\mathcal{T}(X) \cong \overline{\operatorname{span}}\{\psi^{\otimes m}(x)\psi^{\otimes n}(y)^* : m, n \geq 0, x \in X^{\otimes m}, y \in X^{\otimes n}\}.
$$

Theorem 2.1.23. ([46, Corollary 3.7], [18, Proposition 1.6], [47, Proposition 8.11]) *Let X be a C^*-correspondence over A, and suppose that (ψ, π) is a Toeplitz representation in a C^*-algebra B. Then there exists a faithful homomorphism $(\psi, \pi)^{(1)} : \mathcal{K}(X) \to B$, such that*

$$(\psi, \pi)^{(1)}(\Theta_{x,y}) = \psi(x)\psi(y)^*,$$

for $x, y \in X$. If $\phi : B \to C$ is a homomorphism of C^*-algebras, then $\phi \circ (\psi, \pi)^{(1)} = (\phi \circ \psi, \phi \circ \pi)^{(1)}$.

Given a C^*-algebra A, and an ideal I of A, we denote

$$I^{\perp} := \{a \in A : ab = 0 \text{ for all } b \in I\}.$$

Observe that $I^{\perp}$ is itself an ideal of A.

Definition 2.1.24. *Let X be a C^*-correspondence over A, with left action denoted by ϕ, and let (ψ, π) be a Toeplitz representation of X_A onto B. We say that (ψ, π) is* ***covariant*** *if*

$$(\psi, \pi)^{(1)}(\phi(a)) = \pi(a)$$

for all $a \in \phi^{-1}(\mathcal{K}(X)) \cap \ker(\phi)^{\perp}$.

Given a C^*-correspondence X over A, we refer to the ideal of $\mathcal{T}(X)$ generated by $\left\{(\iota_X, \iota_A)^{(1)}(\phi(a)) - \iota_A(a) : a \in \phi^{-1}(\mathcal{K}(X)) \cap \ker(\phi)^{\perp}\right\}$ as the **Katsura ideal**, and denote it by J_X.

Definition 2.1.25 ([46, Theorem 3.13])**.** *Let X be a C^*-correspondence over a C^*-algebra A. The* ***Cuntz-Pimsner algebra*** *$\mathcal{O}_X$ is given by*

$$\mathcal{O}_X := \mathcal{T}(X)/J_X.$$

A consequence of the universality of $\mathcal{T}_X$ is that $\mathcal{O}_X$ is the universal C^*-algebra generated by covariant representations of X.

The following example illustrates why we are preoccupied with Cuntz-Pimsner algebras, which will become more apparent next chapter.

Example 2.1.26 ([47, Example 8.13], [18, Proposition 4.4])**.** Let E be a directed graph. Let (ψ, π) be a Toeplitz representation as in Example 2.1.21, with the additional hypothesis that (ψ, π) is covariant. For $v \in E^0$, we have

$$\phi(\delta_v) = \delta_v \circ r = \sum_{r(f)=v} \Theta_{\delta_f, \delta_f}, \tag{2.9}$$

and so $\phi(\delta_v) \in \mathcal{K}(X(E)) = \overline{\operatorname{span}}\{\Theta_{\delta_e,\delta_f} : e, f \in E^1\}$ if $|r^{-1}(v)| < \infty$. Suppose that v receives infinitely many edges. Since $X(E) = \overline{\operatorname{span}}\{\delta_e : e \in E^1\}$, for any $K \in \mathcal{K}(X(E))$, there exists a finite $F \subset E^1$ and scalars $c_{e,f} \in \mathbb{C}$ such that

$$\Big\| K - \sum_{e,f \in F} c_{e,f} \theta_{\delta_e, \delta_f} \Big\| < \frac{1}{2}.$$

Since $r^{-1}(v)$ is infinite, there exists $g \in E^1 \setminus F$ such that $r(g) = v$. Then

$$\begin{aligned}
\Big\| \phi(\delta_v) - \sum_{e,f\in F} c_{e,f}\theta_{\delta_e,\delta_f} \Big\| &= \sup_{\|x\|=1} \Big\| \Big(\phi(\delta_v) - \sum_{e,f\in F} c_{e,f}\theta_{\delta_e,\delta_f} \Big)(x) \Big\| \\
&\geq \Big\| \phi(\delta_v)(\delta_g) - \sum_{e,f\in F} c_{e,f}\theta_{\delta_e,\delta_f}(\delta_g) \Big\| \\
&= \Big\| \delta_g - \sum_{e,f\in F} c_{e,f}\delta_e \cdot \langle \delta_f, \delta_g \rangle \Big\| \\
&= \|\delta_g\| \\
&= 1,
\end{aligned}$$

so $\phi(\delta_v) \notin \mathcal{K}(X(E))$. Hence $\delta_v \in \phi^{-1}(\mathcal{K}(X(E)))$ if and only if $|r^{-1}(v)| < \infty$.

We have

$$\ker(\phi) = \{a \in C_0(E^0) : a \circ r \equiv 0\} = \{a \in C_0(E^0) : a(v) = 0 \text{ for } v \in r(E^1)\},$$

so it follows that

$$\ker(\phi)^{\perp} = \{a \in C_0(E^0) : a(v) = 0 \text{ for all } v \in E^0 \setminus r(E^1)\}. \tag{2.10}$$

Combining (2.9) and (2.10), we see that $\phi^{-1}(\mathcal{K}(X(E)) \cap \ker(\phi)^{\perp}$ is the ideal generated by $\{\delta_v : 0 < |r^{-1}(v)| < \infty\}$. Hence, for $v \in E^0$ such that $0 < |r^{-1}(v)| < \infty$, we have

$$\begin{aligned} P_v &= \pi(\delta_v) \\ &= (\psi, \pi)^{(1)}(\phi(\delta_v)) \\ &= (\psi, \pi)^{(1)}\Big(\sum_{r(e)=v} \Theta_{\delta_e, \delta_e}\Big) \\ &= \sum_{r(e)=v} \psi(\delta_e)\psi(\delta_e)^* \\ &= \sum_{r(e)=v} S_e S_e^*. \end{aligned}$$

So $\{S, P\}$ is a Cuntz-Krieger E-family ([47, Chapter 1]). Hence $C^*(E)$ is generated by a covariant Toeplitz-representation of $X(E)$. By the universal properties of $C^*(E)$ and $\mathcal{O}_{X(E)}$, we conclude that $C^*(E) = \mathcal{O}_{X(E)}$.

2.2 Topological Graphs

We use the definition of topological graphs introduced by Katsura in [25]. We start by defining some essential parts of topological graphs analogous to parts of directed graphs. Our aim is to build on the theory established in [25, 27, 28, 29].

Definition 2.2.1 ([25, Definition 2.1])**.** *A **topological graph** E is a quadruple $E = (E^0, E^1, r, s)$, where E^0, E^1 are locally compact Hausdorff spaces, and $r : E^1 \to E^0$ is continuous, and $s : E^1 \to E^0$ is a local homeomorphism.*

Topological graphs are in some respects similar to a directed graphs. We think of E^0 as the set of verticies and E^1 as the set of edges of E, and the maps r and s giving the directions of the edges.

Example 2.2.2 ([25, Chapter 2])**.** Let E^0, E^1 be countable sets equipped with the discrete topology, and take $r, s : E^1 \to E^0$. Then $E = (E^0, E^1, r, s)$ is a directed graph, and also a topological graph.

Example 2.2.3. Let $r : \mathbb{T} \to \mathbb{T}$ be the function $r : z \mapsto e^{2i\pi\theta} z$ for some $\theta \in [0, 1)$. Then the quadruple $E_\theta = (\mathbb{T}, \mathbb{T}, r, \mathrm{id}_{\mathbb{T}})$ is a topological graph.

Example 2.2.4. The requirement that s is a local-homeomorphism can be quite deceptive. The quadruple $E = (E^0 = [-1, 1], E^1 = [0, 1], r : z \mapsto z, s : z \mapsto z)$ where E^0, E^1 have the induced topology from $\mathbb{R}$, is not a topological graph. The map $s : z \mapsto z$ is not a local homeomorphism, since $s([0, 1]) = [0, 1] \subseteq E^0$ is not open, so s is not an open map and therefore not a local homeomorphism.

Definition 2.2.5 ([39])**.** *Let $E = (E^0, E^1, r, s)$ be a topological graph. We call an open set $U \subseteq E^1$ an **s-section** if $s|_U$ is a homeomorphism.*

Definition 2.2.6. *A topological graph E is said to be **compact** if E^0, E^1 are compact.*

Observe that if E is a compact topological graph, then r, s are proper maps.

We use the following terminology and notation reminiscent of directed graphs.

Notation 2.2.7 ([25, Definition 2.6])**.** *Let E be a topological graph. Then we define*

$$E^0_{sce} := E^0 \setminus \overline{r(E^1)}.$$

*We refer to elements of E^0_{sce} as **sources**. The collection of **essentially finite receivers** is denoted by*

$$E^0_{fin} := \left\{ v \in E^0 \mid \textit{There exists a compact neighbourhood } V \textit{ of } v \textit{ such that } r^{-1}(V) \textit{ is compact} \right\}.$$

Finally, the ***regular verticies*** *are elements of the set*

$$E^0_{rg} := E^0_{\mathit{fin}} \setminus \overline{E^0_{sce}}.$$

For a topological graph E, each of the sets E^0_{sce}, E^0_{fin} and E^0_{rg} are open. The claim that E^0_{sce} is open is self-evident. For the other two, it suffices to show that E^0_{fin} is open. Fix $v \in E^0_{fin}$. Let U be a precompact neighbourhood of v such that $r^{-1}(\overline{U})$ is compact. For each $w \in U$, $w \in \overline{U}$, and so $U \subseteq E^0_{fin}$. Hence E^0_{fin} is a neighbourhood of each of its points, and is open.

Definition 2.2.8 ([25, Section 2], [47, Chapter 9])**.** *Let E be a topological graph. A* ***path*** *is a sequence of edges $\mu = \mu_1\mu_2\cdots\mu_n$ such that $s(\mu_i) = r(\mu_{i+1})$ for $i = 1, 2, \cdots, n-1$.*

For a path $\mu = \mu_1\mu_2\cdots\mu_n$ in a topological graph, we denote the length by $|\mu| = n$. We use the convention that for all $v \in E^0$, $|v| = 0$. The collection of paths of length n is denoted by E^n, and $E^* := \bigcup_{n\in\mathbb{N}} E^n$. Given $\mu = \mu_1\mu_2\cdots\mu_{|\mu|} \in E^*$, we may define a range and source of μ by $r(\mu) = r(\mu_1)$ and $s(\mu) = s(\mu_{|\mu|})$.

Quite shamelessly, we steal the following notation from the literature of k-graphs ([43]). For $v \in E^0$, we denote the set of paths of length n, with source v by $E^n v$. The collection of paths of length n with range v by vE^n. Again, we use E^*v and vE^* for paths of arbitrary length. We also combine the two conventions to obtain $wE^n v := \{\mu \in wE^n \cap E^n v\}$.

Proposition 2.2.9 ([47, Chapter 8])**.** *Let E be a topological graph. Then E^n is a closed subset of the product space $\Pi_{i=1}^n E^1$.*

Proof. Fix $\mu = \mu_1\mu_2\cdots\mu_n \in \Pi_{i=1}^n E^1 \setminus E^n$. Our aim is to find an open neighbourhood of μ whose intersection with E^n is empty. Since $\mu \notin E^n$, there exists $i < n$ such that $r(\mu_{i+1}) \neq s(\mu_i)$.

Choose disjoint open sets A, B such that $r(\mu_{i+1}) \in A$ and $s(\mu_i) \in B$. Let U be an s-section containing μ_i and $s(U) \subseteq B$. Then

$$U \cap s^{-1}(A) = \emptyset.$$

Since $r(\mu_{i+1}) \neq s(\mu_i)$, we have

$$\mu_{i+1} \in r^{-1}(r(\mu_{i+1})) \text{ but } \mu_{i+1} \notin r^{-1}(s(\mu_i)).$$

Since E^1 is locally compact and Hausdorff (hence regular), there exist open disjoint $C, D \subseteq E^1$ such that $\mu_{i+1} \in C$ and $r^{-1}(s(\mu_i)) \subseteq D$. Then

$$\mu \in Z(U, i) \cap Z(C, i+1),$$

where $Z(U, i) := \{(\mu_j)_{j=1}^{\infty} \in \Pi_{i=1}^{\infty} E^1 : \mu_i \in U\}$. However,

$$\big(Z(U, i) \cap Z(C, i+1)\big) \cap E^n = \emptyset,$$

as we aimed to show. □

As one might expect, we aim to assign C^*-algebras to topological graphs, and we do this via Katsura's construction of Cuntz-Pimsner algebras. In order to do this we need to assign a Hilbert module to each topological graph.

Proposition 2.2.10 ([25, Lemma 1.5, Lemma 1.7, Proposition 1.10 & Remark 1.18])**.** *Let E be a topological graph. Then for all $v \in E^0$, $e \in E^1$, $x, y \in C_c(E^1)$ and $a \in C_0(E^0)$*

$$(x \cdot a)(e) = x(e)a(s(e)) \text{ and } \langle x, y\rangle(v) = \sum_{e \in E^1 v} \overline{x(e)}y(e)$$

define a $C_0(E^0)$-valued inner-product over $C_c(E^1)$. The completion of $C_c(E^1)$, which we will denote $X(E)$ is a Hilbert-module, and is equal to

$$\{x \in C(E^1) \mid \langle x, x\rangle \in C_0(E^0)\}. \tag{2.11}$$

Furthermore, there exists a homomorphism $\phi : C_0(E^0) \to \mathcal{L}(X(E))$ such that for $x \in X(E), a \in C_0(E^0)$ and $e \in E^1$,

$$\phi(a)(x)(e) = a \circ r(e)x(e)$$

turning $X(E)$ into a C^*-correspondence over $C_0(E^0)$.

Proof. First, we show that the sum in the definition of $\langle x, y\rangle(v)$ is finite. Fix $x, y \in C_c(E^1)$. Since $\mathrm{supp}(x)$ is compact, we can cover it with finitely many s-sections U_i. Then for any $v \in s(\mathrm{supp}(x))$

$$\langle x, y\rangle = \sum_{i:v\in s(U_i)} \overline{(x \circ (s|_{U_i}))}(y \circ (s|_{U_i}))$$

is a finite sum of continuous functions, so $\langle x, y\rangle$ is continuous at v. If $v \notin s(\mathrm{supp}(x))$, then there exists a neighbourhood U of v such that $U \cap s(\mathrm{supp}(x)) = \emptyset$, and so $\langle x, y\rangle|_U \equiv 0$, so $\langle x, y\rangle$ is continuous at v. Now, if $v \in \mathrm{supp}(\langle x, y\rangle)$, then

$$0 \neq \langle x, y\rangle(v) = \sum_{e\in E^1 v} \overline{x(e)}y(e),$$

so $s^{-1}(v) \cap \mathrm{supp}(x) \cap \mathrm{supp}(y) \neq \emptyset$. Hence, $\mathrm{supp}(\langle x, y\rangle) \subseteq s(\mathrm{supp}(x) \cap \mathrm{supp}(y))$, and so is compact. Hence $\langle x, y\rangle \in C_c(E^0)$.

We now check that $\langle \cdot, \cdot\rangle$ satisfies (1), (2), (3) and (4) of 2.1.1. Fix $x, y \in C_c(E^1), a \in C_0(E^0)$ and $v \in E^0$. Then

$$\langle x, x\rangle(v) = \sum_{e\in E^1 v} \overline{x(e)}x(e) = \sum_{e\in E^1 v} |x(e)|^2 \geq 0,$$

and clearly $\langle x, x\rangle = 0$ only if $x \equiv 0$, so (2) is satisfied.

For (3), we have

$$(\langle x, y\rangle(v))^* = \sum_{e\in E^1 v} (\overline{x(e)}y(e))^* = \sum_{e\in E^1 v} x(e)\overline{y(e)} = \langle y, x\rangle(v).$$

We also have

$$\langle x, y \cdot a\rangle(v) = \sum_{e\in E^1 v} \overline{x(e)}y \cdot a(e)$$

$$= \sum_{e \in E^1 v} \overline{x(e)} y(e) a(s(e))$$
$$= \sum_{e \in E^1 v} (\overline{x(e)} y(e)) a(v)$$
$$= (\langle x, y \rangle a)(v),$$

so (4) holds. So $\langle \cdot, \cdot \rangle$ is a $C_0(E^0)$-valued inner-product.

Fix $x, y \in C(E^1)$ satisfying Equation (2.11). Then, by Proposition 2.1.2, we have

$$\|\langle x, y \rangle^* \langle x, y \rangle\| \leq \|\langle x, x \rangle\| \|\langle y, y \rangle\| < \infty,$$

so $\|x\|, \|y\| < \infty$.

Fix $f \in C_0(\overline{s(E^1)})^+$. We aim to find a sequence of tuples $(x_n, y_n) \in C_c(E^1)^+ \times C_c(E^1)^+$ such that $\|f - \langle x_n, y_n \rangle\| < \frac{1}{n}$ for all $n \in \mathbb{N}$. Since $\overline{C_c(\overline{s(E^1)})} = C_0(\overline{s(E^1)})$, there exists a sequence $g_n \in C_c(E^0)^+$ such that for all $n \in \mathbb{N}$, $\|f - g_n\| < \frac{1}{n}$. Since for each $n \in \mathbb{N}$, $s^{-1}(\mathrm{supp}(g_n))$ is compact, there exist finitely many s-sections covering $s^{-1}(\mathrm{supp}(g_n))$, which we will denote $\{U_{n,i}\}_i$. Let $\xi_{n,i}$ be a partition of unity subordinate to $\{U_{n,i}\}$. For $e \in E^1$, define $x_{n,i}(e) := \sqrt{g_n(s(e))\xi_i(e)}$. Then for $e \in E^1$,

$$\langle x_{n,i}, x_{n,i} \rangle(v) = \sum_{e \in E^1 v} \overline{x_{n,i}(e)} x_{n,i}(e) = \sum_{e \in E^1 v} g_n(v) \xi_{n,i}(e).$$

Hence,

$$\left\| f - \sum_i \langle x_{n,i}, x_{n,i} \rangle \right\| < \frac{1}{n}$$

for each $n \in \mathbb{N}$. Since the inner-product $\langle \cdot, \cdot \rangle$ is sesquilinear the result extends to $C_0(\overline{s(E^1)})$. Hence

$$X(E) = \left\{ x \in C(E^1) : \langle x, x \rangle \in C_0(E^0) \right\}.$$

Given $a \in C_0(E^0)$, the map $a \mapsto a \circ r$ is a homomorphism since it takes values in $C_0(E^1)$. For $a \in C_0(E^0), x \in X(E)$ and $e \in E^1$, define $(\phi(a)x)(e) = a \circ r(e) x(e)$.

For $x, y \in X(E)$, $a \in C_0(E^0)$ and $v \in E^0$,

$$\langle \phi(a)x, y\rangle(v) = \sum_{e \in E^1 v} \overline{a \circ r(e) x(e)} y(e) = \sum_{e \in E^1 v} \overline{x(e)} (\overline{a \circ r(e)} y(e)) = \langle x, \phi(a^*)y\rangle(v),$$

so $\phi(a)$ is adjointable with adjoint $\phi(a^*)$. So $X(E)$ equipped with the homomorphism ϕ is a C^*-correspondence over $C_0(E^0)$. □

Example 2.2.11. We note that the Hilbert-module norm on $X(E)$ is not the same as the supremum norm on functions on E^1. Consider the topological graph, $E = (\mathbb{T}, \mathbb{T}, id_{\mathbb{T}}, s)$, where $s(z) = z^2$ for all $z \in \mathbb{T}$. Consider the function $\mathbb{1} \in C(\mathbb{T})$ such that $\mathbb{1} : z \mapsto 1$. We have

$$\|\mathbb{1}\|^2_{X(E)} = \|\langle \mathbb{1}, \mathbb{1}\rangle\|_{C_0(E^0)} = \sup_{v \in \mathbb{T}} |\langle \mathbb{1}, \mathbb{1}\rangle(v)| = 2,$$

but $\|\mathbb{1}\|^2_\infty = 1$.

Notation 2.2.12. *Let E be a topological graph. Reminiscient of the notation used in the directed graph case, we use $C^*(E)$ to denote the universal Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$. Using the convention introduced by Katsura in [25], we use $\mathcal{T}(E)$ for the C^*-algebra $\mathcal{T}(X(E))$.*

We seek a characterisation of the Katsura ideal.

Theorem 2.2.13 ([25, Proposition 1.24], [47, Proposition 9.2])**.** *Let E be a topological graph, and let $f \in C_0(E^0)$ be such that* $\operatorname{supp}(f) \subseteq E^0_{fin}$. *Then the left action of f on $X(E)$ is by compact operators. Moreover, there exist $\nu_i, \xi_i \in C_c(E^1)$ such that for $x \in X(E)$,*

$$f \cdot x = \sum_{i=1}^{n} \Theta_{\nu_i, \xi_i}(x).$$

Further, $f \cdot x \equiv 0$ for all $x \in X(E)$ if $\operatorname{supp}(f) \subseteq E^0_{sce}$.

Proof. Fix a compact $K \subseteq E^0_{\mathrm{fin}}$. Then there exist finitely many $\{v_i\}_{i=1}^n \in E^0_{\mathrm{fin}}$ with precompact neighbourhoods V_i such that $v_i \in V_i$, $r^{-1}(V_i)$ is compact for all

i, and $K \subseteq \bigcup_{i=1}^n V_i$. Then $\bigcup_{i=1}^n r^{-1}(V_i)$ is compact, and $r^{-1}(K) \subseteq \bigcup_{i=1}^n r^{-1}(V_i)$, so must be compact. Hence, for $\operatorname{supp}(f \circ r)$ is compact. For $e \in \operatorname{supp}(f \circ r)$, let U_e be an precompact s-section containing e, in the sense that $\overline{U_e}$ is a compact set. Observe that $\bigcup_{e \in \operatorname{supp}(f \circ r)} U_e$ is an open cover for $\operatorname{supp}(f \circ r)$, so we may choose finitely many e_i such that $U_i := U_{e_i}$ cover $\operatorname{supp}(f \circ r)$. Choose a partition of unity $\{g_i\}_{i=1}^n$ such that $\operatorname{supp}(g_i) \subseteq U_i$, and for $e \in \operatorname{supp}(f \circ r)$ take

$$\nu_i(e) = f \circ r(e)\sqrt{g_i}(e) \text{ and } \xi_i(e) = \sqrt{g_i}(e).$$

For all $e \in E^1$, we have

$$f \circ r(e) = f \circ r(e) \sum_{i=1}^n g_i(e) = \sum_{i=1}^n f \circ r(e)\sqrt{g_i}(e)\overline{\sqrt{g_i}}(e) = \sum_{i=1}^n \nu_i(e)\overline{\xi_i(e)}.$$

Since $\operatorname{supp}(g_i) \subseteq U_i$ and s is injective on U_i, we have $\nu_i(e)\xi_i(f) = 0$ when $e \neq f$ and $s(e) = s(f)$. Fix $x \in C_c(E^1)$ and $e \in E^1$. Observe that $\nu_i, \xi_i \in C_c(E^1)$ for all i. Then

$$\begin{aligned}
\sum_{i=1}^n \Theta_{\nu_i,\xi_i}(x)(e) &= \sum_{i=1}^n (\nu_i\langle \xi_i, x\rangle)(e) \\
&= \sum_{i=1}^n \nu_i(e)\langle \xi_i, x\rangle(e) \\
&= \sum_{i=1}^n \nu_i(e)\Bigg(\sum_{f \in E^1(s(e))} \overline{\xi_i(f)}x(f)\Bigg) \\
&= \sum_{i=1}^n \nu_i(e)\overline{\xi_i(e)}x(e) \\
&= f \circ r(e)x(e) \\
&= (f \cdot x)(e).
\end{aligned}$$

This extends by continuity of the action from $C_c(E^1)$ to $X(E)$.

Now choose $f \in C_0(E^0)$, such that $f \cdot x \equiv 0$ for all $x \in X(E)$. Then we have $f \cdot x(e) = f \circ r(e)x(e) = 0$ which implies $f \circ r(e) = 0$ for all $e \in E^1$. Hence $\operatorname{supp}(f) \subseteq E^0 \setminus \overline{r(E^1)} = E^0_{\text{sce}}$. $\square$

Corollary 2.2.14. *Let E be a topological graph, and $\phi : C_0(E^0) \to \mathcal{L}(X(E))$ be the left action of $C_0(E^0)$ on $X(E)$. Then $J_{X(E)}$ is the ideal generated by*

$$\{(\iota_{X(E)}, \iota_{C_0(E^0)})^{(1)}(\phi(f)) - \iota_{C_0(E^0)}(f) \in C_0(E^0) \textit{ such that } \operatorname{supp}(f) \subseteq E^0_{rg}\}.$$

Proof. This follows from the fact that $E^0_{\mathrm{rg}} = E^0_{\mathrm{fin}} \setminus E^0_{\mathrm{sce}}$. □

The following is a variation of the guage invariant uniqueness theorem of an Huef and Raeburn. It is an extremely useful theorem for determining the universal C^*-algebras of graphs.

Theorem 2.2.15 ([25, Theorem 4.5])**.** *Let E be a topological graph, and let (ψ, π) be a covariant representation of $X(E)$ in a C^*-algebra A, such that π is injective. Then $\psi \times \pi : C^*(E) \to A$ is injective if and only if there exists a strongly continuous action $\beta : \mathbb{T} \to \operatorname{Aut}(A)$ such that $\beta_z(\psi^{\otimes n}(x)) = z^n \psi^{\otimes n}(x)$ for $x \in X(E)^{\otimes n}$.*

In addition to a variant of the guage-invariance uniqueness theorem, we seek a Cuntz-Krieger uniqueness theorem.

Definition 2.2.16 ([25, Definition 5.3])**.** *Let E be a topological graph. A path $\mu \in E^* \setminus E^0$ is said to be a **cycle** if $r(\mu) = s(\mu)$, and $\mu_i \neq \mu_j$ for distinct $i, j \leq |\mu|$.*

*A cycle μ is said the have an **entrance** if there exists $e \in E^1$ such that $e \neq \mu_i$ for $i \leq |\mu|$, and $r(e) = r(\mu_i)$ for some $i \leq |\mu|$.*

Definition 2.2.17 ([25, Definition 5.4])**.** *A topological graph is said to be **topologically free** if the set*

$$\{v \in E^0 \mid \textit{every cycle through } v \textit{ has an entrance.}\}$$

is dense in E^0.

The following is a variant of the Cuntz-Krieger uniqueness theorem for directed graphs.

Theorem 2.2.18 ([25, Theorem 5.12]). *Let E be a topologically free topological graph. Suppose that (ψ, π) is a Cuntz-Pimsner convariant representation of the graph correspondence $X(E)$ in a C^*-algebra A. If π is injective, then the homomorphism $\psi \times \pi : C^*(E) \to A$ is injective.*

Theorem 2.2.19 ([27, Theorem 7.1]). *Let E be a topological graph. Then $C^*(E)$ is unital if and only if E^0 is compact.*

We use the following alternate construction of C^*-algebras of topological graphs, established in [39]. As we will see, these conditions, though numerous, are actually quite easy to check in practise.

Definition 2.2.20 ([39, Definition 2.10]). *Let E be a topological graph, and let (ψ, π) be a Toeplitz representation of $X(E)$ in a C^*-algebra B. We say that (ψ, π) is **covariant representation of** E if there exists a collection of non-negative functions $\mathcal{G} \subseteq C_0(E^0_{rg})$ that generate $C_0(E^0_{rg})$, such that for each $f \in \mathcal{G}$, there exists a finite cover $\mathcal{N}_f$ of $r^{-1}(\mathrm{supp}(f))$, by s-sections, and a collection of non-negative continuous functions $\{g^f_U : U \in \mathcal{N}_f\}$ such that*

1. $\mathrm{supp}(g^f_U)^\circ \subseteq U \cap r^{-1}(\mathrm{supp}(f))$
2. $\sum_{U \in \mathcal{N}_f} \left(g^f_U\right)^2 = f \circ r$
3. $\pi(f) = \sum_{U \in \mathcal{N}_f} \psi(g^f_U)\psi(g^f_U)^*$.

Not surprisingly, there exists a universal C^*-algebra associated to these representations, and they are exactly what you would expect.

Theorem 2.2.21 ([39, Theorem 2.12]). *Let E be a topological graph. There exists a covariant representation $(j_{X(E)}, j_{C_0(E^0)})$ of E into a C^*-algebra A that is universal in the sense that if there exists another covariant representation (ψ, π) of E into a C^*-algebra B, there exists a homomorphism $\phi : A \to B$, such that $\phi \circ j_{(X(E))} = \psi$ and $\phi \circ \jmath_{C_0(E^0)} = \pi$. Moreover, the C^*-algebra A is isomorphic to $C^*(E)$.*

We begin with a few examples that demonstrate how nice this definition actually is, despite the heavy looking assumptions.

Example 2.2.22. We will show that in the instance E is a directed graph, Definition 2.2.20 are equivalent the Cuntz-Krieger relations. As we saw earlier, $\psi : \delta_e \mapsto S_e$ and $\pi : \delta_v \mapsto P_v$ is a Toeplitz representation, where S_e and P_v are partial isometry associated to and edge e, and the projection associated to a vertex v respectively. Since (ψ, π) is a Toeplitz representation

$$S_e^* S_e = \psi(\delta_e)^* \psi(\delta_e) = \pi(\langle \delta_e, \delta_e \rangle) = \pi(\delta_{s(e)}) = P_{s(e)}.$$

We have $\mathcal{G} = \overline{\text{span}} \left\{ \delta_v | v \in E^0_{rg} \right\}$. Fix $v \in E^0$. Then for δ_v, $\mathcal{N}_{\delta_v} = \{\{e\} : r(e) = v\}$, which is a collection of s-sections, and $\{\delta_e : \{e\} \in \mathcal{N}_{\delta_v}\}$ is a collection of continuous functions that satisfy (1) of Definition 2.2.20. Also, $\delta_v \circ r = \sum_{\{e\} \in \mathcal{N}_{\delta_v}} \delta_e$, so (2) is satisfied. Finally, if (ψ, π) is covariant, we have

$$P_v = \pi(\delta_v) = \sum_{\{e\} \in \mathcal{N}_{\delta_v}} \psi(\sqrt{\delta_e}) \psi(\sqrt{\delta_e})^* = \sum_{e \in vE^1} S_e S_e^*,$$

which coincides with the Cuntz-Krieger relations.

Example 2.2.23 ([25, Example 2])**.** Let $E = (X, X, r, \mathrm{id}_X)$, where X is a locally compact space, and $r : X \to X$ is a homeomorphism. Suppose that $(\iota_{C_0(X)}, \iota_{\mathbb{Z}})$ is the covariant representation of $(C_0(X), \mathbb{Z}, \alpha)$, where α is the action $\alpha(f) = f \circ r^{-1}$. For $f \in C_0(X)$, let $\psi(f) = \iota_{\mathbb{Z}}(1) \iota_{C_0(X)}(f)$ and $\pi(f) = \iota_{C_0(X)}(f)$. Then, for $x, y \in X(E)$, $f \in C_0(X)$ we have

$$\psi(x)\pi(f) = \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(x)\iota_{C_0(X)}(f) = \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(xf) = \psi(x \cdot f).$$

Moreover,

$$\begin{aligned} \psi(x)^* \psi(y) &= (\iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(x))^* \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(y) \\ &= \iota_{C_0(X)}(\overline{x})\iota_{C_0(X)}(y) \end{aligned}$$

$$
\begin{aligned}
&= \iota_{C_0(X)}(\overline{x}y) \\
&= \pi(\langle x, y\rangle),
\end{aligned}
$$

and finally

$$
\begin{aligned}
\psi(f \cdot x) &= \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(f \cdot x) \\
&= \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(f \circ r)\iota_{C_0(X)}(x) \\
&= \iota_{\mathbb{Z}}(1)\iota_{\mathbb{Z}}(-1)\iota_{C_0(X)}(f)\iota_{\mathbb{Z}}(-1)^*\iota_{C_0(X)}(x) \\
&= \iota_{C_0(X)}(f)\iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(x) \\
&= \pi(f)\psi(x).
\end{aligned}
$$

So (ψ, π) is a Toeplitz representation of $X(E)$.

Since r is a homeomorphism, $E^0_{\mathrm{rg}} = X$. Fix a non-negative $f \in C_0(X)$. Let $g = f \circ r$. Then $\mathrm{supp}(g) \subseteq r^{-1}(\mathrm{supp}(f))$, so (1) and (2) of Definition 2.2.20 are satisfied. Now we have

$$
\begin{aligned}
\psi(\sqrt{g})\psi(\sqrt{g})^* &= \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(\sqrt{g})\iota_{C_0(X)}(\sqrt{g})^*\iota_{\mathbb{Z}}(1)^* \\
&= \iota_{\mathbb{Z}}(1)\iota_{C_0(X)}(g)\iota_{\mathbb{Z}}(1)^* \\
&= \iota_{C_0(X)}(g \circ r^{-1}) \\
&= \iota_{C_0(X)}(f \circ r \circ r^{-1}) \\
&= \iota_{C_0(X)}(f).
\end{aligned}
$$

Hence (ψ, π) is a covariant representation of $X(E)$. By universality of both $C^*(E)$ and $C_0(X) \rtimes_\alpha \mathbb{Z}$, we have

$$
C^*(E) = C_0(X) \rtimes_\alpha \mathbb{Z}.
$$

Example 2.2.24. Following on from Example 2.2.23, for E_θ as described in Example 2.2.3, we have $C^*(E_\theta) \cong C(\mathbb{T}) \rtimes_{r_\theta} \mathbb{Z}$.

2.3 An introduction to KMS-states

KMS-states are studied in the context of mathematical physics because they provide a generalisation for the Gibbs states in the infinite dimensional context. This means KMS states give a good description of equilibrium phenomena. However, we are interested in them from a purely C^*-algebraic perspective. We can think of these KMS states as being traces twisted by a strongly continuous action of the reals.

In this chapter we recall the definition of KMS states, and discuss the elements of the theory that we will require to understand the KMS states of noncommutative solenoids. We recall the basics for the uninitiated, and then examine some examples of historical signifiance in the context of operator theory/symbolic dynamics. These examples serve as a heuristic for the following chapter.

Given a C^*-algebra A, we use $\mathcal{S}(A)$ to denote the **state space** of A.

Definition 2.3.1. *A* ***dynamics*** α ***over*** *A is a pair (A, α), where A is a C^*-algebra, and $\alpha : \mathbb{R} \to \mathrm{Aut}(A)$ is a strongly continuous group homomorphism.*

Definition 2.3.2. *Given a dynamics α over A, we say that $a \in A$ is* α***-analytic*** *if the map $z \mapsto \alpha_z(a)$ extends to an analytic function from $\mathbb{C}$ to A.*

If the dynamics is clear from the context, then given an α-analytic element a, it is common to drop the α and refer to such an element as **analytic**.

Lemma 2.3.3 ([5, Proposition 2.5.22])**.** *Let α be a dynamics over A. The set of analytic elements of A are dense in A.*

Definition 2.3.4. *Let α be a dynamics over A, and fix $\beta \in \mathbb{R}$. If $\phi \in \mathcal{S}(A)$ is such that for all analytic $a, b \in A$,*

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) \tag{2.12}$$

then ϕ is said to satisfy the ***KMS condition at inverse temperature*** *β. For a given $\beta \in \mathbb{R}$ and (A, α), we denote*

$$\mathcal{S}_{\alpha,\beta}(A) := \{\phi \in \mathcal{S}(A) : \phi \textit{ satisfies } (2.12)\}.$$

If the value of β is unknown or arbitrary, then we refer to ϕ satisfying (2.12) as a **KMS state**. If the value of β is fixed, we write either $\phi \in \mathcal{S}_{\alpha,\beta}(A)$ or ϕ is a **KMS$_\beta$-state**. In the instance $\beta = 0$, we take $\mathcal{S}_{\alpha,0}(A)$ to be the set of α-invariant traces on A.

Luckily, we don't have to check condition (2.12) on all analytic elements of a C^*-algebra A — it suffices to check (2.12) for all a, b in some collection of analytic elements that span a dense subset of A. Before proving this however, we require some analytic properties, most importantly α-invariance, of KMS-states.

Lemma 2.3.5 ([6, Proposition 5.3.3])**.** *Let α be a dynamics over A, and suppose that there exists $\beta \in \mathbb{R} \setminus \{0\}$ such that $\phi \in S_{\alpha,\beta}(A)$. Then for all analytic $a \in A$,*

$$\phi(a) = \phi(\alpha_t(a)),$$

and the map $z \mapsto \alpha_z(a)$ is bounded in the strip $\{z \in \mathbb{C} | Im(z) \in [0, \beta]\}$.

Further,

$$\phi(ab) = \phi(\alpha_{\frac{-i\beta}{2}}(b)\alpha_{\frac{i\beta}{2}}(a))$$

for all analytic $a, b \in A$.

Proof. Fix an analytic $a \in A$, and $\phi \in \mathcal{S}_{\alpha,\beta}(A)$. Let $F : \mathbb{C} \to \mathbb{C}$ be the function given by $F(z) = \phi(\alpha_z(a))$. Since a is analytic, F is holomorphic. Further, if we denote the approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$, we have

$$F(z) = \phi(\alpha_z(a)) = \lim_{\lambda \in \Lambda} \phi(\alpha_z(a)e_\lambda) = \lim_{\lambda \in \Lambda} \phi(e_\lambda \alpha_{z+i\beta}(a)) = \phi(\alpha_{z+i\beta}(a)) = F(z + i\beta),$$

so F is periodic with period $i\beta$. Moreover, for $z \in \mathbb{C}$,

$$|F(z)| = |\phi(\alpha_z(a))| = |\phi(\alpha_{i\mathrm{Im}(z)} \circ \alpha_{\mathrm{Re}(z)}(a))| = |\phi(\alpha_{i\mathrm{Im}(z)}(a))|,$$

so $F(z)$ depends only on $\mathrm{Im}(z)$. Since F is continuous, F is bounded on the strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \in [0, \beta]\}$, and hence bounded on $\mathbb{C}$. Loiville's Theorem ([50], Theorem 10.23) then implies F is a constant function.

Since $\phi \in \mathcal{S}_{\alpha,\beta}(A)$, for analytic $a, b \in A$, ϕ satisfies $\phi(ab) = \phi(b\alpha_{i\beta}(a))$. Since ϕ is α-invariant, we then have

$$\phi(ab) = \phi(b\alpha_{i\beta}(a)) = \phi(\alpha_{\frac{-i\beta}{2}}(b\alpha_{i\beta}(a))) = \phi(\alpha_{\frac{i\beta}{2}}(b)\alpha_{\frac{i\beta}{2}}(a)),$$

giving us the second part of the lemma. □

Lemma 2.3.6. *Let α be a dynamics over A, and fix $\beta \in \mathbb{R}$. Suppose that $\mathcal{F} \subseteq A$ is a collection of analytic elements such that $\overline{span}(\mathcal{F}) = A$. If $\phi \in \mathcal{S}(A)$ is such that* (2.12) *holds for all $a, b \in \mathcal{F}$, then $\phi \in \mathcal{S}_{\alpha,\beta}(A)$.*

Proof. Fix analytic $a, b \in A$, and let $\phi \in \mathcal{S}(A)$ be such that ϕ satisfies (2.12) for all $x, y \in \mathcal{F}$. For $n \in \mathbb{N}$, let $a_n = \sum_{k=1}^{M_n} c_k x_k$ and $b_n = \sum_{l=1}^{N_n} d_l y_l$, for $c_k, d_l \in \mathbb{C}$, and $x_k, y_l \in \mathcal{F}$, such that $\|a - a_n\| < \frac{1}{n}$ and $\|b - b_n\| < \frac{1}{n}$. Then

$$|\phi(ab) - \phi(a_n b_n)| \leq |\phi(ab) - \phi(ab_n)| + |\phi(ab_n) - \phi(a_n b_n)| \leq \frac{\|a\| + \|b_n\|}{n}. \quad (2.13)$$

Now, consider the map $F_n : \mathbb{R} \to A$ where $F_n(t) = \alpha_t(a) - \alpha_t(a_n)$. Note that since a, a_n are analytic in A, then F_n has an analytic extension to $\mathbb{C}$ such that $F_n(z) = \alpha_z(a) - \alpha_z(a_n)$. Since $\alpha_t(a) - \alpha_t(a_n) = \alpha_t(a - a_n)$ for all $t \in \mathbb{R}$, and analytic continuations are unique, then $F_n(z) = \alpha_z(a - a_n)$; so, $\alpha_z(a) - \alpha_z(a_n) = \alpha_z(a - a_n)$ for all $z \in \mathbb{C}$. Further, since α is bounded on the strip $\{z \in \mathbb{C} : \mathrm{Im}(z) \in [0, \beta]\}$ by Lemma 2.3.5, we may assume that $\|\alpha_{i\beta}\| \leq M$ for some $M \in \mathbb{R}$. Hence

$$\begin{aligned}
|\phi(b_n\alpha_{i\beta}(a_n)) - \phi(b\alpha_{i\beta}(a))| &\leq |\phi(b_n\alpha_{i\beta}(a_n)) - \phi(b_n\alpha_{i\beta}(a))| \\
&\quad + |\phi(b_n\alpha_{i\beta}(a)) - \phi(b\alpha_{i\beta}(a))| \\
&\leq \|b_n\|\|\alpha_{i\beta}(a_n) - \alpha_{i\beta}(a)\| + \|\alpha_{i\beta}(a)\|\|b_n - b\| \\
&\leq \frac{1}{n}\Big(\|b_n\|M + \|\alpha_{i\beta}(a)\|\Big).
\end{aligned}$$

Combining this with (2.13), we obtain

$$\begin{aligned}|\phi(ab) - \phi(b\alpha_{i\beta}(a))| &\leq |\phi(ab) - \phi(a_n b_n)| + |\phi(b_n \alpha_{i\beta}(a_n))\phi(b\alpha_{i\beta}(a))| \\ &\leq \frac{1}{n}\Big(\|a\| + \|b_n\| + M\|b_n\| + \|\alpha_{i\beta}(a)\|\Big),\end{aligned}$$

which tends to 0 as $n \to \infty$. Hence $\phi(ab) = \phi(b\alpha_{i\beta}(a))$ for all analytic $a, b \in A$, so $\phi \in \mathcal{S}_{\alpha,\beta}(A)$. □

Proposition 2.3.7 ([6, Proposition 5.3.23])**.** *Let α be a dynamics over a unital C^*-algebra A. Let $(\beta_\lambda)_{\lambda\in\Lambda} \in \mathbb{R}$ be a net converging to $\beta \in \mathbb{R}$, and $\phi_\lambda \in \mathcal{S}_{\alpha,\beta_\lambda}(A)$ be such that $\phi_\lambda \to \phi$. Then $\phi \in \mathcal{S}_{\alpha,\beta}(A)$.*

Proof. Since $\mathcal{S}(A)$ is closed, ϕ is a state, so it suffices to show that ϕ satisfies (2.12). For analytic $a \in A$, the map $z \mapsto \alpha_z(a)$ is analytic and therefore continuous. Hence $\alpha_{i\beta_\lambda}(a) \to \alpha_{i\beta}(a)$. Then, for analytic $a, b \in A$,

$$\phi(ab) = \lim_{\lambda\in\Lambda} \phi_\lambda(ab) = \lim_{\lambda\in\Lambda} \phi_\lambda(b\alpha_{i\beta_\lambda}(a)) = \phi(b\alpha_{i\beta}(a)),$$

so $\phi \in \mathcal{S}_{\alpha,\beta}(A)$. □

Theorem 2.3.8 ([6, Proposition 5.3.30])**.** *Let α be a dynamics over A, and fix $\beta \in \mathbb{R}$. Then $\mathcal{S}_{\alpha,\beta}(A)$ is a choquet simplex.*

Given that $\mathcal{S}_{\alpha,\beta}(A)$ is a choquet simplex, its set of extreme points $\partial\mathcal{S}_{\alpha,\beta}(A)$ has the property that for any $\phi \in \mathcal{S}_{\alpha,\beta}(A)$, there exists a unique $m \in \mathcal{M}^1(\partial\mathcal{S}_{\alpha,\beta}(A))$ such that

$$\phi(a) = \int_{\partial\mathcal{S}_{\alpha,\beta}(A)} \omega(a)\,\mathrm{d}m(\omega)$$

for all $a \in A$. We occasionally write $\phi = \int_{\partial\mathcal{S}_{\alpha,\beta}(A)} \omega\,\mathrm{d}m$ when this happens.

It should be noted that KMS-states need not exist for all values of β.

Example 2.3.9 ([15, Proposition 2.2], [42])**.** Fix $n \in \mathbb{N}$. Let $X = \{1, 2, \cdots, n\}$, and let X^* denote the set of words of arbitrary but finite length over the alphabet

X. Consider the Hilbert space $\ell^2(X^*)$, and for $j \leq n$ and $\mu \in X^*$, write $S_j\delta_\mu = \delta_{j\mu}$, where $j\mu$ is the concatination of j, and the word $\mu \in X^*$. Given this convention, if we have $\mu = \mu_1\mu_2\cdots\mu_{|\mu|} \in X^*$, we define $S_\mu := S_{\mu_1}S_{\mu_2}\cdots S_{\mu_{|\mu|}}$. For $\mu \in X^*$ and $k, l \leq n$ we have

$$S_k^*S_l\delta_\mu = \delta_{k,l}\delta_\mu.$$

Similar calculations reveal that for $\mu, \nu \in X^*$,

$$S_\mu^*S_\nu = \begin{cases} S_{\nu'} & \text{if } \nu = \mu\nu', \\ S_{\mu'}^* & \text{if } \mu = \nu\mu', \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

Let $\mathcal{TO}_n$ be the C^*-algebra generated by $\{S_j : j \leq n\}$. Calculations similar to (2.14) show that $\mathcal{TO}_n = \overline{\operatorname{span}}\{S_\mu S_\nu^* : \mu, \nu \in X^*, \mu_{|\mu|} = \nu_{|\nu|}\}$. The C^*-algebra $\mathcal{TO}_n$ carries a natural action of $\mathbb{T}$, such that for $z \in \mathbb{T}$,

$$\gamma_z(S_\mu S_\nu^*) = z^{|\mu|-|\nu|}S_\mu S_\nu^*$$

for $\mu, \nu \in X^*$. We lift this to $\mathbb{R}$ in search of KMS states. For $t \in \mathbb{R}$, let

$$\alpha_t(S_\mu S_\nu^*) = \gamma_{e^{it}}\big(S_\mu S_\nu^*\big) = e^{it(|\mu|-|\nu|)}S_\mu S_\nu^*.$$

Suppose that there exists $\beta \in \mathbb{R}$ such that there exists $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n)$. Then for $\mu, \nu \in X^*$,

$$\phi(S_\mu S_\nu^*) = \phi(S_\nu^*\alpha_{i\beta}(S_\mu)) = e^{-\beta|\mu|}\phi(S_\nu^*S_\mu) = \delta_{\mu,\nu}e^{-\beta|\mu|}. \tag{2.15}$$

So, we aim to find a state ϕ satisfying (2.15) for some value of β. Consider the sum $\sum_{\mu\in X^*} e^{-\beta|\mu|}$, and suppose it converges. We have

$$|X^{k+1}| = |\{xi : x \in X^k, i \in X\}| = n|X^k| = n^{k+1},$$

so $\sum_{\mu\in X^*} e^{-\beta|\mu|} = \sum_{k=0}^\infty ne^{-\beta k} = \frac{1}{1-e^{-\beta}}$ only if $\beta > \ln n$. Fix $\beta > \ln n$. Then for $a \in \mathcal{TO}_n$,

$$\phi_\beta(a) = \sum_{\mu\in X^*} e^{-\beta|\mu|}(a\delta_\mu|\delta_\mu).$$

So ϕ_β satisfies (2.15). Further, $\phi_\beta \in \mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n)$ is unique, since, if there exists another $\psi_\beta \in \mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n)$ then, for $\mu, \nu \in X^*$, by (2.15) we have

$$\phi_\beta(S_\mu S_\nu^*) = \delta_{\mu,\nu} e^{-\beta|\mu|} = \psi_\beta(S_\mu S_\nu^*),$$

which forces ϕ_β and ψ_β to agree on a dense subspace of $\mathcal{TO}_n$. Now, if we choose a sequence of numbers $\beta_k \to \ln n$ such that $\beta_k > \ln n$ for all k, then there exists a convergent subsequence β_{k_l} such that $\phi_{\beta_{k_l}}$ converges to some state ϕ as $l \to \infty$. Then

$$\phi(1 - \sum_{j=1}^{n} S_j S_j^*) = \lim_{l\to\infty} \phi_{\beta_{k_l}}(1 - \sum_{j=1}^{n} S_j S_j^*) = \lim_{l\to\infty} 1 - ne^{-\beta_{k_l}} = 1 - ne^{-\ln n} = 0.$$

Hence $\phi \in \mathcal{S}_{\alpha,\ln n}(\mathcal{TO}_n)$. Suppose there exists $\beta < \ln n$ such that there exists $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n)$. We have, by (2.15),

$$\phi(1 - \sum_{j=1}^{n} S_j S_j^*) = 1 - ne^{-\beta} < 0,$$

which contracts the assertion that $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n) \subset \mathcal{S}(\mathcal{TO}_n)$, and is hence positive. So, for $\beta < \ln n$, we have $\mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n) = \emptyset$.

For the following example, we require a lemma.

Lemma 2.3.10 ([1, Lemma 6.2])**.** *Let α be a dynamics over A, and let J be an ideal in A generated by a set of positive elements P that are fixed by α. Denote the quotient map from A to A/J by q. Suppose that there is a family of analytic elements $\mathcal{F}$ such that $\overline{span}\{\mathcal{F}\} = A$, and for each analytic element in $a \in \mathcal{F}$, there is a scalar valued analytic function f_a satisfying $\alpha_z(a) = f_a(z)a$. If $\phi \in \mathcal{S}_{\alpha,\beta}(A)$, and $\phi(p) = 0$ for all $p \in P$, then ϕ factors through to some $\tilde{\phi} \in \mathcal{S}_{\alpha,\beta}(A/J)$.*

Example 2.3.11. Let $X = \{1, 2, \cdots, n\}$. We denote by X^* the words of arbitrary finite length, as in Example 2.3.9, the set of infinite words $X^\infty = \{\mu_1\mu_2 \cdots : \mu_i \in X\}$, and consider the Hilbert space $\ell^2(X^\infty)$. For $\mu \in X^*, \nu \in X^\infty$ let T_μ be the

isometry such that $T_\mu \delta_\nu = \delta_{\mu\nu}$ where $\mu\nu \in X^\infty$ is the concatination of words μ and ν. Then for $\mu, \nu \in X^*$, $T_\mu^* T_\nu$ collapses in a calculation similar to (2.14). Let $\mathcal{O}_n = \overline{\text{span}}\{T_\mu T_\nu^* : \mu, \nu \in X^*\}$. Similarly to $\mathcal{TO}_n$, $\mathcal{O}_n$ carries a natural action γ of $\mathbb{T}$, such that for $z \in \mathbb{T}$,

$$\gamma_z(S_\mu S_\nu^*) = z^{|\mu|-|\nu|} S_\mu S_\nu^*.$$

We lift γ to $\mathbb{R}$ by defining $\alpha_t = \gamma_{e^{it}}$. Since $\mathcal{O}_n$ is isomorphic to the quotient of $\mathcal{TO}_n$ by the ideal generated by the projection $(1 - \sum_{j=1}^n S_j S_j^*)$ ([8, Proposition 3.1]), to understand the KMS states arising from the action α, we aim to apply Lemma 2.3.10. We have

$$\alpha_t(1 - \sum_{j=1}^n T_j T_j^*) = 1 - e^{it(|j|-|j|)} \sum_{j=1}^n T_j T_j^* = 1 - \sum_{j=1}^n T_j T_j^*,$$

so $(1 - \sum_{j=1}^n T_j T_j^*)$ is fixed by α, and for $\mu, \nu \in X^*$, the function $f_{T_\mu T_\nu^*}(z) = e^{iz(|\mu|-|\nu|)}$ is analytic and is such that $f_{T_\mu T_\nu^*}(z) T_\mu T_\nu^* = \alpha_z(T_\mu T_\nu^*)$. Hence, it suffices to find some $\beta \in \mathbb{R}$ and $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{TO}_n)$ such that $\phi(1 - \sum_{j=1}^n T_j T_j^*) = 0$. From Example 2.3.9, we know that the simplex $\mathcal{S}_{\alpha,\ln n}(\mathcal{TO}_n)$ consists of a single state, which we shall denote by ϕ, and is such that $\phi(1 - \sum_{j=1}^n T_j T_j^*) = 0$. So $\phi \in \mathcal{S}_{\alpha,\ln n}(\mathcal{TO}_n)$ factors through to $\mathcal{S}_{\alpha,\ln n}(\mathcal{O}_n)$ by Lemma 2.3.10. For $\beta \neq \ln n$, (2.12) implies that

$$\phi(1 - \sum_{j=1}^n T_j T_j^*) = 1 - \sum_{j=1}^n \phi(T_j T_j^*) = 1 - \sum_{j=1}^n e^{-\beta} \phi(T_j^* T_j) = 1 - ne^{-\beta} \neq 0,$$

so $\mathcal{S}_{\alpha,\beta}(\mathcal{O}_n) = \emptyset$ for $\beta \neq \ln n$.

We finish with this classical result due to Enomoto, Fujii and Watatani.

Example 2.3.12 ([13]). Let $A \in M_n(\{0,1\})$, and let $X = \{1, 2, \cdots, n\}$. Let $\mathcal{O}_A$ be the universal C^*-algebra generated by partial isometries $\{s_j : j \leq n\}$ subject to

1. $1 = \sum_{j=1}^n s_j s_j^*$,

2. $s_k^* s_k = \sum_{j=1}^n A(k,j) s_j s_j^*$.

The universal property of $\mathcal{O}_A$ means that there is an action $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{O}_A)$ such that $\alpha_t(s_j) = e^{it} s_j$ for all $j \leq n$. We then have $s_\mu = s_{\mu_1} s_{\mu_2} \cdots s_{\mu_{|\mu|}}$ for $\mu \in X^*$. It is straightforward to show that elements of the form $s_\mu s_\nu^*$ are analytic.

Now, $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{O}_A)$ if and only if there exists a non-negative vector v such that $Av = e^\beta v$. If A is irreducible and not a permutation, then this happens only when $\beta = \ln \rho(A)$, where $\rho(A)$ is the spectral radius of A. Hence, there is a unique KMS-state on $\mathcal{O}_A$.

In the examples presented, the KMS states are supported on a commutative sub-C^*-algebra. This is a recurring theme we will see throughout the next chapter, and it will feature heavily in our description of the sets of KMS states we will encounter. We will also see that Toeplitz algebras give a much richer theory of KMS states than their Cuntz-Pimsner counterparts. This is due to the quotient imposing strict conditions on the KMS states.

Chapter 3

Maps Between Topological Graphs and Homomorphisms Between Their C^*-Algebras

3.1 Maps between Topological Graphs

The purpose of this chapter is to develop some theory analagous to projective limits of topological spaces, and extend it to the theory of C^*-algebras of topological graphs. Obviously, there is a little more machinery we have to be careful of here, since our aim is to preserve both the graph structure and C^*-correspondence structure. Morphsims of topological graphs were described in [27] and [12]. The treatment of this chapter is based on and generalises the covering maps introduced in [12]. It was unknown to the author at the time that a more general definition and theory are developed in [27]; we will show that in some situations, these two approaches coincide.

Definition 3.1.1. *Let $E = (E^0, E^1, r_E, s_E)$ and $F = (F^0, F^1, r_F, s_F)$ be topological graphs. An s-**injective graph morphism** between topological graphs is a pair*

$p = (p^0, p^1) : E \to F$ such that

$$p^0 : E^0 \to F^0 \text{ and } p^1 : E^1 \to F^1$$

are proper surjective local homeomorphisms such that for $v \in E^0$ and $e \in E^1$ satisfy

1. *$p^0 \circ r_E(e) = r_F \circ p^1(e)$ and $p^0 \circ s_E(e) = s_F \circ p^1(e)$;*

2. *p^1 is **locally injective**, in the sense that*

$$p^1 : s_E^{-1}(v) \to s_F^{-1}(p^0(v))$$

   is injective.

*If p^0 and p^1 are homeomorphisms, we say that p is a **graph isomorphism**, and the topological graphs E and F are isomorphic.*

A pair of maps (not necessarily proper local homeomorphisms) that only satisfies (1) is said to be a **graph morphism**.

Given an s-injective graph morphism $p : E \to F$, we define p^0-sections and p^1-sections in a similar way to s-sections; for $i \in \{0, 1\}$, a **p^i-section** is an open set U such that $p^i|_U$ is a homeomorphism.

Our hypothesis that the maps p^0 and p^1 are proper is motivated by the C^*-algebras of topological graphs, and will be discussed in Section 3.2. The assumption that these maps are surjective is not strictly necessary, but it simplifies the situation and cover the examples that we will study later in the thesis.

Katsura, in [27] introduced a similar notion for topological graphs. His approach seems somewhat different from ours, but we will show the equivalence of these two definitions (given that we are working with surjective local homeomorphsims). For the following definition, given a graph E, we will use the notation

$\tilde{E}$ to denote the graph $\tilde{E} = (\tilde{E}_0, \tilde{E}^1, \tilde{r}, \tilde{s})$, where $\tilde{E}^i$ is the one-point compactification of E^i, and $\tilde{r}$ is the function such that $\tilde{r}|_{E^1} = r$, and $\tilde{r}(\infty) = \infty$, and $\tilde{s}$ is defined similarly.

Definition 3.1.2 ([27, Definition 2.1])**.** *Let E, F be topological graphs. Then a **factor map** $m = (m^0, m^1)$ from E to F is a pair of continuous maps $m^i : \tilde{E}_i \to \tilde{F}_i$ such that $m^i(\infty) = \infty$, and satisfies*

1. *$m^0 \circ r_E(e) = r_F \circ m^1(e)$ and $m^0 \circ s_E(e) = s_F \circ m^1(e)$ for $e \in \tilde{E}^1$;*
2. *if $v \in E^0$ and $f \in F^1$ satisfy $m^0(v) = s_F(f)$, then there exists a unique $e \in E^1$ such that $s_E(e) = v$ and $m^1(e) = f$.*

We refer to (2) of Definiton 3.1.2 as **path lifing**.

Lemma 3.1.3. *Let E, F be topological graphs. Let $p = (p^0, p^1)$ be an s-injective graph morphism from E to F. Then p^0, p^1 have extensions $\tilde{p}^0, \tilde{p}^1$ where $\tilde{p^i}(\infty) = \infty$ for $i \in \{0, 1\}$, such that $\tilde{p} = (\tilde{p}^0, \tilde{p}^1)$ is a factor map.*

Moreover, if $m = (m^0, m^1)$ is a factor map from E to F such that m^0, m^1 are surjective local homeomorphisms, then $q = (q^0 = m^0|_{E^0}, q^1 = m^1|_{E^1})$ is an s-injective graph morphism.

Proof. For $i = 0, 1$, $\tilde{p}^i$ is a local homeomorphism, and hence continuous. Choose $v \in E^0$ and $f \in F^1$ such that $p^0(v) = s_F(f)$. Since is surjective and locally injective, there exists a unique $e \in (p^1)^{-1}(f)$ such that $s_E(e) = v$.

For the second part of the lemma, fix $i \in \{0, 1\}$. Fix a compact set $K \subseteq F^i$. Then K is closed, and $(q^i)^{-1}(K) = (m^i)^{-1}(K) \subseteq \tilde{E}^i$ is closed, and hence a compact set, so q^i is proper. We show local injectivity by a contrapositive argument. Suppose q^1 is such that q does not satisfy (2). That is, there exists some $v \in E^0$ such that there are distinct $e, f \in s_E^{-1}(v)$ such that $q^1(e) = q^1(f)$.

Now, $q^0(v) = s_F \circ q^1(e)$ and $s_E(e) = v$. Also $q^0(v) = s_F \circ q^1(f)$ and $s_E(f) = v$. Hence, q does not have the path lifting property. $\square$

An s-injective graph morphism between topological graphs $E = (E^0, E^1, r_E, s_E)$ and $F = (F^0, F^1, r_F, s_F)$ gives the following commuting diagram.

$$\begin{tikzcd}
E^1 \arrow[d, "r_E"'] & F^1 \arrow[l, "p^1"'] \arrow[d, "r_F"] \\
E^0 & F^0 \arrow[l, "p^0"'] \\
E^1 \arrow[u, "s_E"] & F^1 \arrow[l, "p^1"] \arrow[u, "s_F"']
\end{tikzcd}$$

Example 3.1.4. Using the notation introduced in Example 2.2.3, consider the topological graphs E_θ and $E_{\frac{\theta}{n}}$, for some $\theta \in [0, 1)$. Let $p^0 = p^1 : z \to z^n$. Then $p = (p^0, p^1) : E_{\frac{\theta}{n}} \to E_\theta$ is an s-injective graph morphism.

Proposition 3.1.5 ([27, Lemma 2.7])**.** *Let $p : E \to F$ be an s-injective graph morphism. Then $F^0_{sce} \subseteq p^0(E^0_{sce})$, $p^0(E^0_{fin}) \subseteq F^0_{fin}$ and $F^0_{rg} \subseteq p^0(E^0_{rg})$.*

Proof. It suffices to show $F^0_{\mathrm{sce}} \subseteq p^0(E^0_{\mathrm{sce}})$ and $p^0(E^0_{\mathrm{fin}}) \subseteq F^0_{\mathrm{fin}}$. Let $v \in \overline{r_E(E^1)}$. Then $p^0(v) \in \overline{r_F(F^1)}$. Hence $p^0(v) \in F^0_{\mathrm{sce}}$ implies $v \in E^0_{\mathrm{sce}}$. We show $F^0_{\mathrm{sce}} \subseteq p^0(E^0_{\mathrm{sce}})$ by a contrapositive argument. Fix $v \in \overline{r_E(E^1)}$. Then

$$p^0(v) \in \overline{p^0(r_E(E^1))} = \overline{r_F(p^1(E^1))} = \overline{r_F(F^1)}.$$

Hence, if $v \notin p^0(E^0_{\mathrm{sce}})$ then $p^0(v) \notin F^0_{\mathrm{sce}}$.

Let $v \in E^0_{\mathrm{rg}}$. Then there exists a compact neighbourhood V of v, such that $r_E^{-1}(V)$ is compact. Then $p^0(V)$ is a compact neighbourhood of $p^0(v)$. Since $r_F^{-1}(p^0(V)) = p^1(r_E^{-1}(V))$ which is compact, $p^0(v) \in F^0_{\mathrm{rg}}$. $\square$

Proposition 3.1.6 ([27, Proposition 2.4])**.** *Let E, F and G be topological graphs. Let $q : E \to F$ and $p : F \to G$ be s-injective graph morphisms. Then $p \circ q := (p^0 \circ q^0, p^1 \circ q^1) : E \to G$ is an s-injective graph morphism.*

Proof. First, we show that $p^0 \circ q^0$ is a local homeomorphism. Fix $v \in E^0$, and choose an open neighbourhood W of v such that $p^0|_W$ is a homeomorphism. Choose an open neighbourhood V of $q^0(v)$, such that $q^0|_V$ is a homeomorphism. Then $W \cap (q^0)^{-1}(V)$ is an open neightbourhood of v, and $p^0 \circ q^0|_{W\cap(q^0)^{-1}(V)}$ is a homeomorphism. Further, $p^0 \circ q^0$ is proper, since both p^0 and q^0 are proper. A similar argument shows that $p^1 \circ q^1$ is a proper local homeomorphism.

Fix $e \in E^1$. Then

$$p^0 \circ q^0(r_E(e)) = p^0(r_F(q^1(e))) = r_G(p^1 \circ q^1(e)).$$

Similarly, $p^0 \circ q^0(s_E(e)) = s_G(p^1 \circ q^1(e))$.

Finally, for all $v \in E^0$, the map

$$p^1 \circ q^1 : s_E^{-1}(v) \to s_G^{-1}(p^0 \circ q^0(v))$$

is injective, since it is the composition of injective maps. □

Since s-injective graph morphisms behave well with respect to composition, we can form projective sequences of these morphisms. We will investigate the implications of this throughout the chapter.

Definition 3.1.7 ([27, Definition 4.1])**.** *A* ***projective system of topological graphs*** *is a pair of sequences $(E_n, p_n)_{n=1}^{\infty}$ such that each $p_n : E_{n+1} \to E_n$ is an s-injective graph morphism.*

A projective sequence of Topological graphs can be visualised as a commuting diagram.

$$
\begin{tikzcd}
E_0^1 \arrow[d, "r_0"'] & E_1^1 \arrow[l, "p_0^1"'] \arrow[d, "r_1"] & \cdots \arrow[l] & E_n^1 \arrow[l] \arrow[d, "r_n"'] & E_{n+1}^1 \arrow[l, "p_n^1"'] \arrow[d, "r_{n+1}"] & \cdots \arrow[l] \\
E_0^0 & E_1^0 \arrow[l, "p_0^0"'] & \cdots \arrow[l] & E_n^0 \arrow[l] & E_{n+1}^0 \arrow[l, "p_n^0"'] & \cdots \arrow[l] \\
E_0^1 \arrow[u, "s_0"] & E_1^1 \arrow[l, "p_0^1"] \arrow[u, "s_1"'] & \cdots \arrow[l] & E_n^1 \arrow[l] \arrow[u, "s_n"] & E_{n+1}^1 \arrow[l, "p_n^1"] \arrow[u, "s_{n+1}"'] & \cdots \arrow[l]
\end{tikzcd}
$$

Figure 3.1: A Projective Sequence of Topological Graphs

Example 3.1.8. Let $X = \{0, 1\}$, for $n \in \mathbb{N}$, $X^n = \Pi_{i=0}^n X$, and $X^* = \bigcup_{n \in \mathbb{N}} X^n$. Let $\omega : X^* \to X^*$ be the map such that

$$\omega(\alpha_0\alpha_1 \cdots \alpha_n) = \begin{cases} 1\alpha_1 \cdots \alpha_n & \text{if } \alpha_0 = 0 \\ 0\omega(\alpha_1 \cdots \alpha_n) & \text{if } \alpha_0 = 1. \end{cases}$$

For each $n \in \mathbb{N}$, let $E_n = (X^n, X^n, \omega, \mathrm{id})$. Then E_n is a directed graph when X^n is equipped with the discrete topology. For example, when $n = 2$, we obtain 3.2.

$$
\begin{tikzcd}
00 \arrow[r, bend left] & 10 \arrow[d, bend left] \\
11 \arrow[u, bend left] & 01 \arrow[l, bend left]
\end{tikzcd}
$$

Figure 3.2: The Graph E_2

We let $p_n^0 : X^{n+1} \to X^n$ be the map such that $p_n : \alpha_0 \cdots \alpha_n\alpha_{n+1} \mapsto \alpha_0 \cdots \alpha_n$. Then for $\alpha_0 \cdots \alpha_n\alpha_{n+1} \in X^{n+1}$ we have

$$\omega \circ p_n(\alpha_0 \cdots \alpha_n\alpha_{n+1}) = \omega(\alpha_0 \cdots \alpha_n) = p_n \circ \omega(\alpha_0 \cdots \alpha_n\alpha_{n+1}),$$

and

$$\mathrm{id} \circ p_n(\alpha_0 \cdots \alpha_n \alpha_{n+1}) = p_n(\alpha_0 \cdots \alpha_n \alpha_{n+1}) = \mathrm{id} \circ p_n(\alpha_0 \cdots \alpha_n \alpha_{n+1}).$$

Since each E_n^j is finite and equipped with the discrete topology, each p_n is a proper, local homeomorphism, and hence continuous. Further, each p_n is surjective, and has the path lifting property. Hence $p_n = (p_n, p_n)$ is an s-injective graph morphism, and so $(E_n, p_n)_{n=1}^{\infty}$ is a projective sequence of topological graphs.

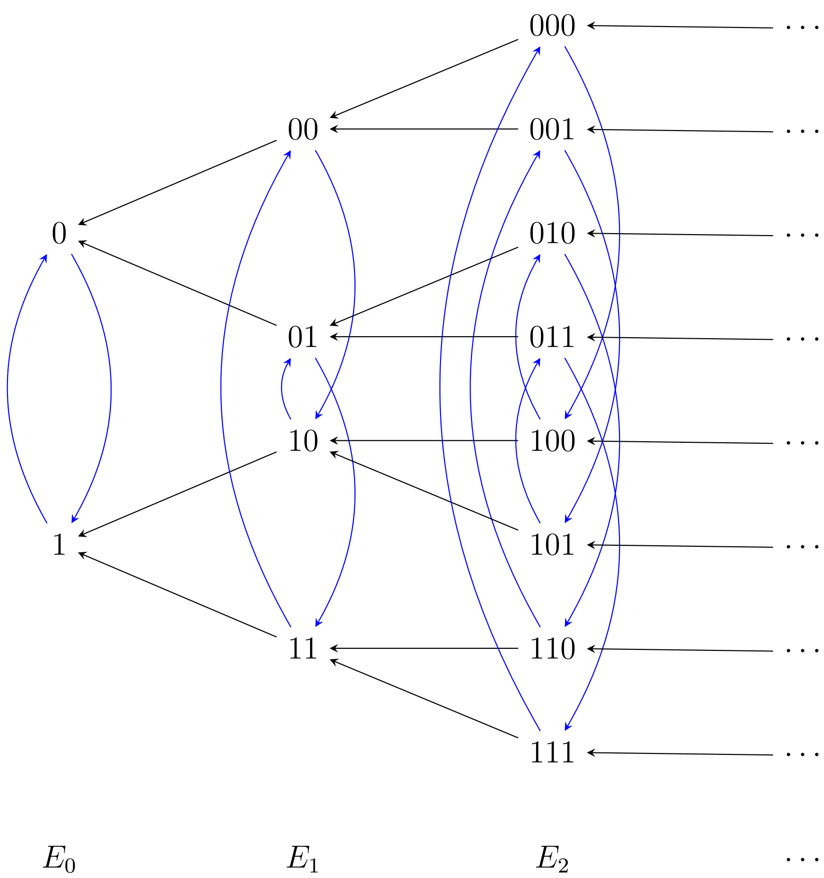


Figure 3.3: An Example of a Projective Sequence of Topological Graphs

In Figure 3.3, the edges in each E_i graph are denoted by a blue edge, and the maps between verticies by black edges. The verticies are mapped in an obvious

way: the source of a black arrow is sent to the range of the arrow. Under these graph morphisms, an edge in E_{i+1} is sent to the edge in E_i whose source is the image of the source of the original vertex. The sequence continues on in this fashion.

Example 3.1.9. Fix $n \in \mathbb{N}$ and $\theta \in [0,1) \setminus \mathbb{Q}$. Let $\left(\theta_i\right)_{i=0}^{\infty}$ be such that $\theta_0 = \theta$ and inductively define $n\theta_{j+1} = \theta_j + k_j$ for some $k_j \in \{0, \cdots n-1\}$. Let $p_j : E_{\theta_j} \to E_{\theta_{j+1}}$ be the pair of maps (p_j^0, p_j^1) such that $p_j^i(z) = z^n$. Observe that p_j^i is proper and surjective for $i = 0, 1$. Then $(E_j, p_j)_{j \in \mathbb{N}}$ is a projective sequence of topological graphs. A slightly modified version will be discussed in great detail in Section 4.1.

As one might expect, the idea of a projective system of topological graphs is accompanied by a limiting structure, which mirrors the projective limits of topological spaces.

Definition 3.1.10. *Let (E_n, p_n) be a projective system of topological graphs. A **projective limit** is a pair (E, P_i) such that E is a topological graph and $(P_i)_{i=1}^{\infty}$ is a sequence of s-injective graph morphisms such that $P_i = p_i \circ P_{i+1}$ for all $i \in \mathbb{N}$.*

Another way of expressing this relationship is the following commuting diagram.

$$\begin{tikzcd}
E_0 & \cdots \arrow[l] & E_j \arrow[l] & E_{j+1} \arrow[l, "p_n"] & \cdots \arrow[l] & E \arrow[lllll, bend right, "P_0"'] \arrow[lll, bend right, "P_j"'] \arrow[ll, bend right, "P_{j+1}"']
\end{tikzcd}$$

Theorem 3.1.11 ([27, Propositions 4.3 & 4.5])**.** *Let $(E_i, p_i)_{i=0}^{\infty}$ be a sequence of topological graphs. Suppose that E_i^j is second countable for each $i \in \mathbb{N}$ and $j \in$*

$\{0,1\}$, *then*

$$E_\infty := (\varprojlim(E_i^0, p_i^0), \varprojlim(E_i^1, p_i^1), (r_i)_{i=0}^\infty, (s_i)_{i=0}^\infty)$$

is a topological graph. If $(E_i, p_i)_{i=0}^\infty$ is a sequence of compact topological graphs, then E_∞ is a compact topological graph. If each r_i is proper and surjective, then $(r_i)_{i\in\mathbb{N}}$ is proper and surjective.

Remark 3.1.12. *For the sake of simplicity, we use the notation $E_\infty^0 = \varprojlim(E_i^0, p_i^0)$ and $E_\infty^1 = \varprojlim(E_i^1, p_i^1)$. We also use the notation r_∞ for $(r_i)_{i=1}^\infty$ and similarly $s_\infty = (s_i)_{i=1}^\infty$.*

Proof of Theorem 3.1.11. By [44, Corollary 3.3], the product space $\Pi_{n=1}^\infty E_\infty^0$ is locally compact, since each E_n^0 is a locally compact, Hausdorff, second countable topological space and the product is countable. Since E_∞^0 is a closed subset of $\Pi_{n=1}^\infty E_\infty^0$, then E_∞^0 is also locally compact. Similarly, E_∞^1 is locally compact. If each E_n is compact, then E_∞ is compact by Tychonoff's Theorem.

Now we show that r_∞ is continuous. Since the topology on E_∞^0 is the relative topology inherited from $\Pi_{n=1}^\infty E_\infty^0$, the cylinder sets form a base for the topology. Fix $n \in \mathbb{N}$, and let $U \subseteq E_n^0$ be open. Then

$$r_\infty^{-1}(Z(U,n) \cap E_\infty^0) = Z(r_n^{-1}(U), n) \cap E_\infty^1,$$

which is open, so r_∞ is continuous.

Fix $e = (e_n)_{n=1}^\infty \in E_\infty^1$. We aim to prove that s_∞ is a local homeomorphism, so we must find a neighbourhood W of e such that $s_\infty|_W$ is a homeomorphism. Fix an open neighbourhood U of e_1 such that $s_1|_U$ is a homeomorphism. Let $f = (f_n)_{n=1}^\infty, g = (g_n)_{n=1}^\infty \in Z(U,1) \cap E_\infty^1$ be such that $s_\infty(f) = s_\infty(g)$. Then $f_1 = g_1$, since $f_1, g_1 \in U$. Now, we have $s_2(f_2) = s_2(g_2)$ and $p_1^1(f_2) = p_1^1(g_2)$. Hence, $f_2 = g_2$ since $p_1^1|_{s_2^{-1}(s_2(f_2))}$ is injective. Proceeding by induction we find $f_i = g_i$ for all $i \in \mathbb{N}$. Hence, $s_\infty|_{Z(U,1)}$ is injective. It then follows that $s_\infty(Z(U,1)) =$

$Z(s_1(U), 1)$. Similarly, for an s_i-section $V \subseteq E_i^0$, $s_\infty(Z(U,i)) = Z(s_i(U_i, i))$, which is open. Since $\left\{Z(U,i) \mid U \subseteq E_i^0 \text{ is an } s\text{-section}\right\}$ is a subbase for the topology on E_∞^0, the map s_∞ is open. Since s_∞ is open and locally injective, it is a local homeomorphism.

For the remainder of the proof, we will assume that each r_i is proper and surjective. Fix $v = (v_i)_{i=1}^\infty \in E_\infty^0$. Since r_1 is surjective, we can choose $e_1 \in E_1^1$ such that $r_1(e_1) = v_1$. Since p_1^1 is surjective, there exists $e_2 \in E_2^1$ such that $r_2(e_2) = v_2$ and $p_1^1(e_2) = e_1$. Proceeding in this fashion, we let for each $i > 1$, e_i be an element of $((p_{i-1}^1)^{-1}(e_{i-1})) \cap r_{i-1}(v_i)$. Then $e = (e_i)_{i=1}^\infty$ belongs to E_∞^1 such that $r_\infty(e) = v$, and so r_∞ is surjective.

We show that r_∞ is proper. Fix a compact set $V \subseteq E_\infty^0$. Since the projection maps $\pi_n : (v_i)_{i=1}^\infty \mapsto v_n$ are continuous for all n, the $\pi_n(V)$ are compact. Since r_n is proper for all n, the $r_n^{-1}(\pi_n(V))$ are also compact, and so, by Tychonoff's Theorem, $\Pi_{n=1}^\infty r_n^{-1}(\pi_n(V))$ is compact. Our aim is to show that $r_\infty^{-1}(V) \subset \Pi_{n=1}^\infty r_n^{-1}(\pi_n(V))$ is closed. To do this, we show that for each $e \in \Pi_{n=1}^\infty r_n^{-1}(\pi_n(V)) \setminus r_\infty^{-1}(V)$, there exists an open neighbourhood W of e such that $W \cap r_\infty^{-1}(V) = \emptyset$. Fix $e \in \Pi_{n=1}^\infty r_n^{-1}(\pi_n(V)) \setminus r_\infty^{-1}(V)$. Then there exists $i \in \mathbb{N}$ such that $p_i^1(e_{i+1}) \neq e_i$. Since p_i^1 is proper, $(p_i^1)^{-1}(e_i)$ is compact, and so there are open disjoint $U, V \subseteq E_\infty^1$ such that $(p_i^1)^{-1}(e_i) \subseteq U$ and $e_{i+1} \in V$. Now, choose an open $B \subseteq E_i^1$ such that $p_i^1(e_{i+1}) \notin B$ and $e_i \in B$. Then $e \in Z(U, i+1) \cap Z(B, i)$ but

$$(Z(U, i+1) \cap Z(B, i)) \cap r_\infty^{-1}(V) = \emptyset,$$

as required. □

We refer to E_∞ as the **projective limit**. The following theorem justifies why we use a definite article to refer to it.

Theorem 3.1.13. *Let $(E_n, p_n)_{n=1}^\infty$ be a projective system of topological graphs, with projective limit E_∞ as in Theorem 3.1.11. Suppose that $Z = (Z^0, Z^1, r_Z, s_Z)$*

is a topological graph and for each $n \in \mathbb{N}$, $\psi_n : Z \to E_n$ is an s-injective graph morphism such that $\psi_{n+1} = p_n \circ \psi_n$. Then there exists a graph morphism Ψ consisting of local homeomorphisms (not necessarily s-injective) $\Psi : Z \to E_\infty$. If Z is compact, then Ψ is an s-injective graph morphism.

Proof. Observe that for $i \in \{0,1\}$ (Z^i, ψ_n^i) is a projective limit for the projective system of topological spaces (E_n^i, p_n^i). Hence, by Theorem A.1.3, there exists a pair of continuous maps $\Psi^j : Z^j \to E_\infty^j$ for $j \in \{0,1\}$. Fix $e \in Z^1$. Then,

$$r_\infty \circ \Psi^1(e) = (r_i \circ \psi_i(e))_{i=1}^\infty = (\psi_i^0 \circ r_Z(e))_{i=1}^\infty = \Psi^0 \circ r_Z(e)$$

and

$$s_\infty \circ \Psi^1(e) = (s_i \circ \psi_i(e))_{i=1}^\infty = (\psi_i^0 \circ s_Z(e))_{i=1}^\infty = \Psi^0 \circ s_Z(e).$$

Hence $\Psi = (\Psi^0, \Psi^1)$ is a graph morphism. Further, for $i \in \{0,1\}$ and $z \in Z^i$, we have for each $n \in \mathbb{N}$,

$$P_n^i \circ \Psi(z) = P_n^i\big((\psi_j^i(z))_{j=0}^\infty\big) = \psi_n^i(z).$$

So $P_n \circ \Psi = \psi_n$ for all $n \in \mathbb{N}$.

Fix $n \in \mathbb{N}$, $z \in Z^0$ and an open $U \subseteq Z^0$ such that $\psi_n^0|_U$ is a homeomorphism. Fix an open neighbourhood V of $\Psi^0(z)$ such that $\pi_n^0|_V$ is a homeomorphism. Then $U \cap (\Psi^0)^{-1}(V)$ is an open nesighbourhood of z, and $\psi_n^0|_{U\cap(\Psi^0)^{-1}(V)} = (\pi_n^0 \circ \Psi^0)_{U\cap(\Psi^0)^{-1}(V)}$ is a homeomorphism. Hence $\Psi_{U\cap(\Psi^0)^{-1}(V)}$ is a homeomorphism, and so, Ψ^0 is a local homeomorphism. A similar argument shows that Ψ^1 is a local homeomorphism too.

For the final part of the proof, we show that Ψ is locally injective in the sense of Definition 3.1.1. For $v \in Z^0$, and $f \in Z^1 v$, we have $s_\infty \circ \Psi^1(f) = (s_i \circ \psi_i^1(f))_{i=0}^\infty = (\psi_i^0 \circ s_Z(f))_{i=0}^\infty = \Psi^0(v)$. Hence $\Psi^1(Z^1 v) \subseteq E_\infty^1 \Psi^0(v)$. For $f, g \in Z^1(v)$ such that $\Psi(f) = \Psi(g)$, we have

$$(\psi_i^1(f))_{i=0}^\infty = (\psi_i^1(g))_{i=0}^\infty.$$

In particular, $\psi_1^1(f) = \psi_1^1(g)$. Since $\psi_1^1|_{s_Z^{-1}(v)}$ is injective, this forces $g = f$. $\square$

Example 3.1.14. Consider the projective sequence described in Example 3.1.8. Then $E_\infty^0 = X^\infty = E_\infty^1$, where

$$X^\infty := \Big\{ (\alpha_i)_{i=0}^\infty \mid \alpha_i \in \{0,1\} \Big\},$$

equipped with the product topology. The set X^∞ is homeomorphic to the Cantor middle thirds set [49, Page 81]. The range map r_∞ satisfies

$$r_\infty(\alpha_0\alpha_1\alpha_2\cdots) = \begin{cases} 1\alpha_1\alpha_2\cdots & \text{if } \alpha_0 = 0 \\ 0r_\infty(\alpha_1\alpha_2\cdots) & \text{if } \alpha_0 = 1. \end{cases}$$

That is, r_∞ is the odometer action. The source map is $s_\infty = \mathrm{id}_{X^\infty}$, since $s_n = \mathrm{id}_{X^n}$. So

$$E_\infty = (X^\infty, X^\infty, r_\infty, \mathrm{id}_{X^\infty})$$

is the binary adding machine.

This example makes it clear as to why it is necessary to consider topological graphs to deal with projective sequences of graphs: we started out with a sequence (E_n, p_n) of discrete graphs, objects whose vertex and edge sets were finite, and contained very little topological information. We finished with a graph E_∞ that is not a directed graph, and whose edge and vertex sets contain a rich topological structure.

Example 3.1.15. Consider the projective sequence of Example 3.1.8. Then (Z, ψ) is a projective limit for $(E_i, p_i)_{i=0}^\infty$, where

$$Z = (\mathbb{N}, \mathbb{N}, \rho : n \mapsto n+1, \mathrm{id}_{\mathbb{N}})$$

and $\psi_j = (\psi_j^0, \psi_j^1) : Z \to E_j$ such that $\psi_j^0 = \psi_j^1$ and

$$\psi_j(n) = \alpha_0\alpha_1\cdots\alpha_j,$$

where $\alpha_0\alpha_1\cdots\alpha_j$ is the unique element of X^* such that $n \mod 2^j = \sum_{i=0}^{j-1}\alpha_i \cdot 2^i$. The map $\Psi^0 : Z^0 \to E^0_\infty$ is not surjective. To see this, fix $n \in \mathbb{N}$. Choose $j \in \mathbb{N}$ such that $2^j > n$. Then $\Psi^0_j(n) = \alpha_0 \cdot \alpha_j$. Then for $k > j$, we have $\Psi^0_j(n) = \alpha_0 \cdots \alpha_j 00 \cdots 0$. Hence, $(\alpha_i)_{i=0}^\infty$ such that $\alpha_i = 1$ for all $i \in \mathbb{N}$ is not in the image of Ψ^0. Hence Ψ^0 is not surjective, and so Ψ is not an s-injective graph morphism.

3.2 C^*-algebras associated to projective sequences of Topological graphs

Our aim for this section is to examine the relationship between the C*-algebras of topological graphs, and the projective limit construction discussed in the previous section.

Definition 3.2.1. *Let X and Y be C^*-correspondences over A and B respectively. A* ***correspondence map*** *from $X \to Y$ is a pair (ρ, κ) such that $\rho : X \to Y$ is a linear map and $\kappa : A \to B$ is a homomorphism satisfying*

1. $\kappa(a) \cdot \rho(x) = \rho(a \cdot x)$
2. $\rho(x) \cdot \kappa(a) = \rho(x \cdot a)$
3. $\langle \rho(x), \rho(y)\rangle_B = \kappa(\langle x, y\rangle_A)$

for all $a \in A$ and $x, y \in X$.

Example 3.2.2. Let E, F be topological graphs, and let $p : F \to E$ be a s-injective graph morphism. Then p induces a pair of maps $p^* = ((p^0)^*, (p^1)^*)$ such that

1. $(p^0)^*(a)(v) = a \circ p^0(v)$ for all $a \in C_0(E^0), v \in F^0$,

2. $(p^1)^*(x)(f) = x \circ p^1(f)$ for all $x \in C_c(E^1), f \in F^1$.

Since addition and multiplication of functions are defined pointwise, it is simple to check that $(p^0)^*$ is a homomorphism, and $(p^1)^*$ is linear. Further, since p^0, p^1 are proper, then $(p^0)^*(f) = f \circ p^0 \in C_0(F^0)$ for all $f \in C_0(E^0)$ and $(p^1)^*(x) = x \circ p^1 \in C_c(F^1)$ for all $x \in C_c(E^1)$ (the need for this assumption will be made explicit in Example 3.2.3). Since both p^0 and p^1 are surjective, it follows that $((p^0)^*, (p^1)^*)$ is a pair of injective homomorphisms. In particular, $(p^0)^* : C_0(E^0) \to C_0(F^0)$ is isometric. Now we check (1), (2) and (3) of definition 3.2.1. Fix $a \in C_0(E^0), x, y \in C_c(E^1)$ and $v \in F^0, f \in F^1$. Then

$$\begin{aligned}
(p^1)^*(a \cdot x)(f) &= (a \cdot x)(p^1(f)) \\
&= a(r_E(p^1(f)))x(p^1(f)) \\
&= a(p^0(r_F(f)))x(p^1(f)) \\
&= (p^0)^*(a)(r_F(f))(p^1)^*(x)(f) \\
&= ((p^0)^*(a) \cdot (p^1)^*(a))(f).
\end{aligned}$$

A similar calculation shows $(p^1)^*(x \cdot a)(f) = ((p^1)^*(x) \cdot (p^0)^*(a))(f)$, so we have (1) and (2). For (3),

$$\begin{aligned}
\langle (p^1)^*(x), (p^1)^*(y) \rangle_{X(F)}(v) &= \sum_{f \in F^1 v} \overline{(p^1)^*(x)(f)}(p^1)^*(y)(f) \\
&= \sum_{s_E(p^1(f)) = p^0(v)} \overline{x(p^1(f))}y(p^1(f)) \\
&= \sum_{e \in E^1(p^0(v))} \overline{x(e)}y(e) \\
&= \langle x, y \rangle_{C_0(E^0)}(p^0(v)) \\
&= (p^0)^*(\langle x, y \rangle_{C_0(E^0)})(v).
\end{aligned}$$

For $x \in C_c(F^1)$, we have

$$\|(p^1)^*(x)\|^2 = \|\langle (p^1)^*(x), (p^1)^*(x) \rangle\| = \|\langle x, x \rangle \circ p^0\| = \|\langle x, x \rangle\| = \|x^2\|,$$

so $(p^1)^*$ is isometric and hence bounded, and therefore extends to $X(E)$.

As discussed earlier, the assumption that p is proper is necessary. This is to ensure we get a map between the assigned C^*-correspondences and coefficient algebras, rather than preserving the structure of the graph. The following example illustrates this.

Example 3.2.3. Let $E = (\mathbb{T}, \mathbb{T}, \mathrm{id}_{\mathbb{T}}, \mathrm{id}_{\mathbb{T}}), F = (\mathbb{R}, \mathbb{R}, \mathrm{id}_{\mathbb{R}}, \mathrm{id}_{\mathbb{R}})$, and let $p : F \to E$ be the pair $p^0 = p^1 : x \mapsto e^{ix}$. Observe that the p^0, p^1 are not proper maps. Let $\mathbb{1}_{\mathbb{T}} : \mathbb{T} \to \mathbb{C}$ be the constant function such that $\mathbb{1}_{\mathbb{T}}(z) = 1$, and similarly for $\mathbb{1}_{\mathbb{R}}$. Note that $\mathbb{1}_{\mathbb{T}} \in C(\mathbb{T}), X(F)$. Then for $i \in \{0, 1\}$

$$(p^i)^*(\mathbb{1}_{\mathbb{T}}) = \mathbb{1}_{\mathbb{T}} \circ p^i = \mathbb{1}_{\mathbb{R}},$$

which is not an element of either $C_0(E^0)$ or $X(E)$.

We aim to show that p^* gives an injective homomorphism between the C^*-algebras of topological graphs. First, we must show there exists maps between Toeplitz algebras.

Proposition 3.2.4. *Let X and Y be C^*-correspondences over A and B respectively, with a correspondence map $(\rho, \kappa) : X \to Y$. Let (ψ_Y, π_B) be a Toeplitz representation of Y in a C^*-algebra D. Then $(\psi_Y \circ \rho, \pi_B \circ \kappa)$ is a Toeplitz representation of X in D. In particular, $(\iota_Y \circ \rho, \iota_B \circ \kappa)$ is a Toeplitz representation of X in $\mathcal{T}(Y)$.*

Proof. Fix $x, y \in X$ and $a \in A$. Then

$$\begin{aligned}
\psi_Y \circ \rho(x \cdot a) &= \psi_Y(\rho(x \cdot a)) \\
&= \psi_Y(\rho(x)\kappa(a)) \\
&= (\psi_Y(\rho(x)))(\pi_B(\kappa(a)))
\end{aligned}$$

$$= (\psi_Y \circ \rho(x))(\pi_B \circ \kappa(a)),$$

and a similar argument shows that $\psi_Y \circ \rho(a \cdot x) = \pi_B \circ \kappa(a)\psi_Y \circ \rho(x)$. We also have

$$\pi_B \circ \kappa(\langle x, y\rangle_A) = \pi_B(\langle \rho(x), \rho(y)\rangle_B) = \psi_Y \circ \rho(x)^* \psi_Y \circ \rho(y).$$

Hence $(\psi_Y \circ \rho, \pi_B \circ \kappa)$ is a Toeplitz representation of X in D. If $\psi_Y = \iota_Y$ and $\pi_B = \iota_B$, we obtain the Toeplitz representation $(\psi_Y \circ \rho, \pi_B \circ \kappa)$ of X in $\mathcal{T}(Y)$. $\square$

Proposition 3.2.5. *Let X, Y be C^*-correspondences over A, B respectively, and let $(\rho, \kappa) : X \to Y$ be a bimodule map. Then there exists a homomorphism $(\iota_Y \circ \rho) \times (\iota_B \circ \kappa) : \mathcal{T}(X) \to \mathcal{T}(Y)$. If κ is injective, $(\iota_Y \circ \rho) \times (\iota_B \circ \kappa)$ is injective.*

Proof. By proposition 3.2.4, $(\iota_Y \circ \rho, \iota_B \circ \kappa)$ is a Toeplitz representation of X in $\mathcal{T}(Y)$. The universal property of $\mathcal{T}(X)$ implies there exists a homomorphism $(\iota_Y \circ \rho) \times (\iota_B \circ \kappa) : \mathcal{T}(X) \to \mathcal{T}(Y)$. If κ is injective, then $\iota_B \circ \kappa$ is injective, it follows that $(\iota_Y \circ \rho) \times (\iota_B \circ \kappa)$ is injective by [18, Theorem 2.1]. $\square$

If (ψ_X, π_A) is a Toeplitz representation of X in a C^*-algebra C, and (ψ_Y, π_B) is a Toeplitz representation in a C^*-algebra D, we obtain Figure 3.4, by Proposition 3.2.4 and Proposition 3.2.5.

$$
\begin{tikzcd}
\mathcal{T}(X) \arrow[rr, "(\iota_Y \circ \rho) \times (\iota_B \circ \kappa)"] \arrow[dd, bend right=60, "\psi_X \times \pi_A"'] & & \mathcal{T}(Y) \arrow[dd, bend left=60, "\psi_Y \times \pi_B"] \\
X_A \arrow[u, "(\iota_X, \iota_A)"'] \arrow[rr, "(\rho, \kappa)"] \arrow[d, "(\psi_X, \pi_A)"'] & & Y_B \arrow[u, "(\iota_Y, \iota_B)"] \arrow[d, "(\psi_Y, \pi_B)"] \\
C & & D
\end{tikzcd}
$$

Figure 3.4: Propositions 3.2.4 and 3.2.5

Corollary 3.2.6 ([27, Proposition 2.3]). *Let E, F be topological graphs such that $p : E \to F$ is an s-injective graph morphism. Then p induces an injective homomorphism $p^* : \mathcal{T}(F) \to \mathcal{T}(E)$, such that for $x \in X(F)$, $a \in C_0(F^0)$,*

$$p^*(\iota_{X(F)})(x) = \iota_{X(E)}(x \circ p^1) \text{ and } p^*(\iota_{C_0(F^0)})(a) = \iota_{C_0(E^0)}(a \circ p^0).$$

Proof. Apply Proposition 3.2.5 to Example 3.2.2. $\square$

Corollary 3.2.7 ([27, Proposition 2.4]). *Let E, F, G be topological graphs, and let $p : E \to F$, $m : F \to G$ be s-injective graph morphisms. Then*

$$(m \circ p)^* = p^* \circ m^* : \mathcal{T}(G) \to \mathcal{T}(E),$$

such that for $x \in X(G)$ and $a \in C_0(G^0)$

$$p^* \circ m^*(\iota_{X(G)})(x) = \iota_{X(E)}(x \circ m^1 \circ p^1) \text{ and } p^*(\iota_{C_0(G^0)})(a) = \iota_{C_0(E^0)}(a \circ m^0 \circ p^0).$$

Proof. Applying Corollary 3.2.6 to Proposition 3.1.6 gives the homomorphism $(m \circ p)^* : \mathcal{T}(G) \to \mathcal{T}(E)$. Then, for $x \in X(G)$ and $a \in C_0(G^0)$ we have

$$(m \circ p)^*(\iota_{X(G)}(x)) = \iota_{X(E)}(x \circ m^1 \circ p^1)$$

$$
\begin{aligned}
&= p^*(\iota_{X(F)}(x \circ m^1)) \\
&= p^* \circ m^*(\iota_{X(G)}(x)),
\end{aligned}
$$

and

$$
\begin{aligned}
(m \circ p)^*(\iota_{C_0(G^0)}(a)) &= \iota_{C_0(E^0)}(a \circ m^0 \circ p^0) \\
&= p^*(\iota_{C_0(F^0)}(a \circ m^0)) \\
&= p^* \circ m^*(\iota_{C_0(G^0)}(a)).
\end{aligned}
$$

Since $(m \circ p)^* = p^* \circ m^*$ on the sets $\iota_{X(G)}(X(G))$ and $\iota_{C_0(G^0)}(C_0(G^0))$ which generate $\mathcal{T}(G)$, the result follows. □

Corollary 3.2.8. *Let E, F be topological graphs, and $p : E \to F$ a graph isomorphism. Then $p^* : \mathcal{T}(F) \to \mathcal{T}(E)$ as defined in Corollary 3.2.6 is an isomorphism of C^*-algebras.*

Proof. Since $p = (p^0, p^1)$ is a pair of homeomorphisms, $h := \big((p^0)^{-1}, (p^1)^{-1}\big) : F \to E$ is a graph isomorphism. By Corollary 3.2.6, p, h induce injective homomorphisms $p^* : \mathcal{T}(F) \to \mathcal{T}(E)$ and $h^* : \mathcal{T}(E) \to \mathcal{T}(F)$. Then for $x \in X(E)$ and $a \in C_0(E^0)$, we have

$$
p^* \circ h^*(\iota_{X(E)})(x) = \iota_{X(E)}(x \circ (p^1)^{-1} \circ p^1) = \iota_{X(E)}(x),
$$

and

$$
p^* \circ h^*(\iota_{C_0(E^0)})(a) = \iota_{C_0(E^0)}(a \circ (p^1)^{-1} \circ p^1) = \iota_{C_0(E^0)}(a).
$$

So $p^* \circ h^* = \mathrm{id}_{\mathcal{T}(E)}$. Similar calulations show that $h^* \circ p^* = \mathrm{id}_{\mathcal{T}(F)}$. □

Proposition 3.2.9. *Let E be a topological graph, and fix $f \in C_c(E^1)$. Then there exist finitely many s-sections K_i covering* $\mathrm{supp}(f)$ *and functions $g_i \in C_0(K_i)$ such that $\sum_i g_i = f$. Moreover, on each K_i, we have*

$$
\| \cdot \|_{X(E)} = \| \cdot \|_\infty .
$$

Proof. For $x \in \operatorname{supp}(f)$, let U_x be an neighbourhood of x that is an s-section, and let V_x be a precompact neighbourhood of x. Then let

$$K_x := U_x \cap V_x.$$

Then K_x is a precompact s-section. Since $\operatorname{supp}(f)$ is compact, there exist a finite set I such that for $i \in I$, there exists $x_i \in \operatorname{supp}(f)$ so that K_{x_i} cover $\operatorname{supp}(f)$. Denote $K_i := K_{x_i}$. Let $\{\xi_i\}$ be a partition of unity such that ξ_i is subordinate to $\overline{K_i}$. Then define

$$g_i(e) := f(e)\xi_i(e)$$

for $e \in E^1$. Then $\sum_{i \in I} g_i = f$. Finally, for $i \in I$,

$$\begin{aligned}
\|g_i\|^2_{X(E)} &= \|\langle g_i, g_i \rangle\|_\infty \\
&= \sup_{v \in E^0} \left|\langle g_i, g_i \rangle(v)\right| \\
&= \sup_{v \in E^0} \left| \sum_{e \in E^1 v} \left|g_i(e)\right|^2 \right| \\
&= \|g_i\|^2_\infty
\end{aligned}$$

as claimed. □

Proposition 3.2.10. *Let $(E_i, p_i)_{i=0}^\infty$ be a projective sequence of topological graphs. For $\epsilon > 0$ and $f \in C_c(E^1_\infty)$, there exists an $n \in \mathbb{N}$ and $g \in C_c(E^1_n)$ such that*

$$\left\|f - \left(P^1_n\right)^*(g)\right\|_\infty < \epsilon.$$

Moreover, $\left\|f - \left(P^0_n\right)^*(g)\right\|_{X(E_\infty)} < \epsilon$.

Proof. Fix $\epsilon > 0$. Cover $\operatorname{supp}(f)$ with finitely many precompact s-sections $\{U_i\}_{i \in I}$ for some finite set I. Let $\{\xi_i\}$ be a partition of unity subordinate to $\overline{U_i}$, and for $e \in E^1_\infty$, define

$$g_i(e) := f(e)\xi_i(e).$$

Then $\sum_{i\in I} g_i = f$.

Since $\operatorname{supp}(g_i) \subseteq \overline{U_i}$, which is compact, we many cover it with finitely many $\{Z(V_k, m_k)\}$, such that $V_k \subseteq E^1_{m_k}$ is open. The Stone-Weierstrass Theorem implies that

$$\left\{ \bigcup_{j=0}^{\infty} \left(P^0_j\right)^*(h) \mid h \in C(E^1_n) \text{ and } \operatorname{supp}\left(\left(P^0_n\right)^*(h)\right) \subseteq \overline{U_i} \right\}$$

is dense in $C_0(U_i)$, for each $i \in I$. Hence, we can find $h_i \in C_c(E^1_{m_i})$, such that $\left\|g_i - \left(P^0_{m_i}\right)^*(h_i)\right\| < \frac{\epsilon}{|I|}$.

Let $M = \max\{m_i \,: i \in I\}$. Then

$$\begin{aligned}
\left\|f - \sum_{i\in I} \left(P^0_{m_i}\right)^*(h_i)\right\| &= \left\|\sum_{i\in I} g_i - \left(P^0_M\right)^* \circ \left(p^0_{M-1}\right)^* \circ \cdots \left(p^0_{m_i}\right)^*(h_i)\right\| \\
&\le \sum_{i\in I} \left\|g_i - \left(P^0_M\right)^* \circ \left(p^0_{M-1}\right)^* \circ \cdots \left(p^0_{m_i}\right)^*(h_i)\right\| \\
&< \sum_{i\in I} \frac{\epsilon}{|I|} \\
&= \epsilon.
\end{aligned}$$

Moreover, since $s_\infty|_{\operatorname{supp}(g_i)}$ for each $i \in I$, we have

$$\left\|f - \sum_{i\in I} \left(P^0_{m_i}\right)^*(h_i)\right\| < \epsilon$$

by Proposition 3.2.9. $\square$

Theorem 3.2.11. *Let $(E_i, p_i)_{i=0}^{\infty}$ be a projective sequence of topological graphs. Then*

$$X(E_\infty) = \overline{\bigcup_{i=1}^{\infty} \left(P^0_n\right)^*(X(E_n))}.$$

Proof. Fix $f \in X(E_\infty)$ and $\epsilon > 0$. Then choose $g \in C_c(E^1_\infty)$ such that $\|f - g\| < \frac{\epsilon}{2}$. By Proposition 3.2.9, we can find $\{g_i\}_{i\in I} \in C_c(E^1_\infty)$, where I is a finite set, such that $s_\infty|_{\operatorname{supp}(g_i)}$ is a homeomorphism and $g = \sum_{i\in I} g_i$. By Proposition 3.2.10, for each g_i, there exists $n_i \in \mathbb{N}$, $h_i \in C_c(E^1_{n_i})$ such that $\left\|g_i - \left(P^0_{n_i}\right)^*(h_i)\right\| < \frac{\epsilon}{2|I|}$.

Then

$$\begin{aligned}
\Big\| f - \sum_{i\in I} \big(P^0_{m_i}\big)^*(h_i) \Big\| &\le \|f-g\| + \Big\| g - \sum_{i\in I} \big(P^0_{m_i}\big)^*(h_i) \Big\| \\
&< \frac{\epsilon}{2} + \sum_{i\in I} \Big\| g_i - \big(P^0_{m_i}\big)^*(h_i) \Big\| \\
&< \frac{\epsilon}{2} + \sum_{i\in I} \frac{\epsilon}{2|I|} \\
&= \epsilon,
\end{aligned}$$

which proves the claim. □

We now prove that, given a projective sequence of topological graphs, we obtain a direct sequence of the Toeplitz algebras of topological graphs, and this satisfies a relationship with $\mathcal{T}(E_\infty)$ that is analagous to Theorem A.1.9 for commutative C^*-algebras. We will then prove that this isomorphism factors through to the direct sequence of Cuntz-Pimsner algebras, and we obtain a similar result for $C^*(E_\infty)$ (Theorem 3.2.19).

Theorem 3.2.12 ([27, Proposition 4.6])**.** *Let $(E_i, p_i)_{i=0}^\infty$ be a projective sequence of topological graphs. Then Figure 3.5 commutes, and the homomorphism P_∞ induced by the universal property of $\varinjlim(\mathcal{T}(E_j), p_j^*)$ is an isomorphsim of $\varinjlim(\mathcal{T}(E_j), p_j^*)$ onto $\mathcal{T}(E_\infty)$.*

$$
\begin{tikzcd}
\mathcal{T}(E_0) \arrow[r] \arrow[rrrrr, bend left, "\Phi_0"] \arrow[drrrrr, bend right, "P_0^*"'] & \cdots \arrow[r] & \mathcal{T}(E_j) \arrow[r, "p_j^*"] \arrow[rrr, bend left, "\Phi_j"] \arrow[drrr, bend right, "P_j^*"'] & \mathcal{T}(E_{j+1}) \arrow[r] \arrow[rr, bend left, "\Phi_{j+1}"] \arrow[drr, bend right, "P_{j+1}^*"'] & \cdots & \varinjlim \big(\mathcal{T}(E_j), p_j^*\big) \arrow[d, "P_\infty"] \\
& & & & & \mathcal{T}(E_\infty)
\end{tikzcd}
$$

Figure 3.5: Direct Limit of Toeplitz Algebras of Topological Graphs

Proof. By Theorem 3.1.11, (E_∞, P_i) is a projective limit for $(E_i, p_i)_{i=0}^{\infty}$, and so induces a sequence of injective homomorphisms $P_i^* : \mathcal{T}(E_i) \to \mathcal{T}(E_\infty)$ such that $P_j^* = P_{j+1}^* \circ p_j^*$ for all $j \in \mathbb{N}$. Hence, there exists a homomorphism $P_\infty : \varinjlim \big(\mathcal{T}(E_i), p_i^*\big) \to \mathcal{T}(E_\infty)$, such that $P_j^* = P_\infty \circ \Phi_j$, and so Figure 3.5 commutes. We claim P_∞ is a bijection.

By [18, Theorem 1.3], $\mathcal{T}(E_\infty)$ is generated as a C^*-algebra by $\iota_{X(E_\infty)}(X(E_\infty))$ and $\iota_{C_0(E_\infty^0)}(C_0(E_\infty^0))$. Fix $x \in X(E_\infty)$ and $\epsilon > 0$. By Theorem 3.2.11, there exists $m \in \mathbb{N}$ and $y \in X(E_m)$ such that $\left\|x - \big(P_m^0\big)^*(y)\right\|_{X(E_\infty)} < \frac{\epsilon}{2}$. Then there exists $f \in C_c(E_m^1)$ such that $\left\|y - f\right\|_{X(E_m)} < \frac{\epsilon}{2}$. Then

$$
\begin{aligned}
&\left\|\iota_{X(E_\infty)}(x) - P_\infty \circ \Phi_m(f)\right\| \\
&= \left\|\iota_{X(E_\infty)}(x) - P_m^*(f)\right\| \\
&\leq \left\|\iota_{X(E_\infty)}(x) - \iota_{X(E_\infty)}((P_m^0)^*(y))\right\| + \left\|P_m^*(\iota_{X(E_m)}(y)) - P_m^*(\iota_{X(E_m)}(f))\right\| \\
&< \epsilon.
\end{aligned}
$$

So $\iota_{X(E_\infty)}(X(E_\infty))$ is in the range of P_∞. A similar argument shows that the range of P_∞ contains $\iota_{C_0(E^0_\infty)}(C_0(E^0_\infty))$, so P_∞ is surjective. $\square$

We now shift our attention to C^*-algebras of topological graphs. In order to prove that we obtain a map between the Cuntz-Pimsner algebras, we need to obtain a map from the ideal J_X to J_Y. To do this, we show that a bimodule map induces a homomorphism between the compact operators. This requires a few technical lemmas first. We use the notation $M_n(\mathbb{C})$ to denote the set of $n \times n$ matricies with complex entries.

This lemma was asserted in the proofs of [22, Proposition 1.18] and [21, Lemma 2.1], and has been included for the sake of completeness.

Lemma 3.2.13 ([21, 22])**.** *Let A be a C^*-algebra, and X be a right-Hilbert A-module. Then $X \otimes \mathbb{C}^n$ is a right-Hilbert $A \otimes M_n(\mathbb{C})$-module and*

$$\mathcal{L}(X) \otimes \mathbb{C} \cong \mathcal{L}(X \otimes \mathbb{C}^n),$$

via the isomorphism

$$a \mapsto a \otimes I$$

for $a \in \mathcal{L}(X)$.

Proof. The fact that $X \otimes \mathbb{C}^n$ is a right-Hilbert $A \otimes M_n(\mathbb{C})$-module follows from the fact that X is a right A-module and $\mathbb{C}^n$ is a right $M_n(\mathbb{C})$-module.

Fix $T \in \mathcal{L}(X \otimes \mathbb{C}^n)$. We aim to show that T can be represented as a matrix. Since

$$X \otimes \mathbb{C}^n = \overline{\operatorname{span}}\big\{x \otimes e_i : x \in X, i \leq n \text{ and } e_i \text{ is a standard basis vector for } \mathbb{C}^n\big\},$$

it suffices to show that this is true for $x \otimes e_i$, where $x \in X, e_i \in \mathbb{C}^n$ are fixed. Now,

$$T(x \otimes e_i) = T \begin{pmatrix} 0_1 \\ 0_2 \\ \vdots \\ x_i \\ \vdots \\ 0_n \end{pmatrix} = \begin{pmatrix} (Tx)_1 \\ (Tx)_2 \\ \vdots \\ (Tx)_i \\ \vdots \\ (Tx)_n \end{pmatrix} = \sum_{i=1}^{n} \left(Tx\right)_i \otimes e_i,$$

where the subscript on the matrix entries denotes the position (we will use this convention for the remainder of the proof), and $(Tx)_i \in X$. Since the formula $x \mapsto (Tx)_i \otimes e_i$ is linear, it follows that each of the $(Tx)_i$ is linear.

Now suppose that T has a matrix representation, say

$$A_T = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & \ddots & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix}.$$

Then

$$A_T(x \otimes e_i) = \begin{pmatrix} a_{1,i}x \\ a_{2,i}x \\ \vdots \\ a_{n,i}x \end{pmatrix}.$$

So, by comparing operators, we obtain $(Tx)_1 = a_{1,i}x$ for all $x \in X$, $(Tx)_2 = a_{2,i}x$ for all $x \in X$, and so on. We let $a_{j,i} \in \mathcal{L}(X) \otimes \mathbb{C}$ be such that $a_{j,i}x := (Tx)_j$ for all $x \in X$.

Since T is adjointable, T^* has a matrix representation A_{T^*}, say

$$A_{T^*} = \begin{pmatrix} b_{1,1} & \cdots & b_{1,n} \\ \vdots & \ddots & \vdots \\ b_{n,1} & \cdots & b_{n,n} \end{pmatrix},$$

where $b_{k,m} \in \mathcal{L}(X) \otimes \mathbb{C}$ for all $k, m \leq n$. Fix $y \in X$, and $j \leq n$. Then

$$A_{T^*}(y \otimes e_j) = \begin{pmatrix} b_{1,j}y \\ b_{2,j}y \\ \vdots \\ b_{n,j}y \end{pmatrix}.$$

Since $T \in \mathcal{L}(X \otimes \mathbb{C}^n)$, we have

$$\langle T(x \otimes e_i), y \otimes e_j \rangle_{A \otimes M_n(\mathbb{C})} = \langle x \otimes e_i, T^*(y \otimes e_j) \rangle_{A \otimes M_n(\mathbb{C})}.$$

Hence

$$\langle A_T(x \otimes e_i), y \otimes e_j \rangle_{A \otimes M_n(\mathbb{C})} = \langle x \otimes e_i, A_{T^*}(y \otimes e_j) \rangle_{A \otimes M_n(\mathbb{C})}.$$

So, our aim is to perform these calculations and compare entries in the resulting matricies.

We have

$$\begin{aligned} \left\langle A_T(x \otimes e_i), y \otimes e_j \right\rangle_{A \otimes M_n(\mathbb{C})} &= \left(\left\langle (A_T(x \otimes e_i))_k, (y \otimes e_j)_l \right\rangle_A \right)_{k,l=1}^n \\ &= \begin{pmatrix} 0 & \cdots & \langle a_{1,i}x, y \rangle_A & \cdots & 0 \\ 0 & \cdots & \langle a_{2,i}x, y \rangle_A & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ 0 & \cdots & \langle a_{n,i}x, y \rangle_A & \cdots & 0 \end{pmatrix} \\ &= C, \end{aligned}$$

where the $\langle a_{k,i}x, y \rangle_A$ occur in the jth column for $k \leq n$. Now,

$$\begin{aligned} \left\langle x \otimes e_i, A_{T^*}(y \otimes e_j) \right\rangle_{A \otimes M_n(\mathbb{C})} &= \left(\left\langle (x \otimes e_i)_k, (A_{T^*}(y \otimes e_j))_l \right\rangle_A \right)_{k,l=1}^n \\ &= \begin{pmatrix} 0 & \cdots & & 0 \\ \vdots & \ddots & & \vdots \\ \langle x, b_{1,j}y \rangle_A & \langle x, b_{2,j}y \rangle_A & \cdots & \langle x, b_{n,j}y \rangle_A \\ \vdots & \cdots & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} \end{aligned}$$

$$= D,$$

where the $\langle x, b_{k,j}y\rangle_A$ occur in the ith row for $k \leq n$. By comparing C and D we find

$$\langle a_{k,i}x, y\rangle_A = \langle x, b_{k,j}y\rangle_A = 0$$

for $k \neq i$ and $m \neq j$. Since the only entry C and D both have non-zero entries is the i, j-th coordinate, we find

$$\langle a_{i,i}x, y\rangle_A = \langle x, b_{j,j}y\rangle_A$$

for all $i, j \leq n$. Hence, $(a_{i,i})^* = b_{j,j}$ for all $i, j \leq n$. This forces $a_{i,i} = a_{k,k}$ for all $i, k \leq n$, so

$$A_T = \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a \end{pmatrix}$$

for some $a \in \mathcal{L}(X) \otimes \mathbb{C}$ with adjoint

$$A_{T^*} = A_T^* = \begin{pmatrix} a^* & 0 & \cdots & 0 \\ 0 & a^* & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a^* \end{pmatrix}.$$

Therefore, $\mathcal{L}(X) \otimes \mathbb{C}$ is isomorphic to $\mathcal{L}(X \otimes \mathbb{C}^n)$ via the map

$$a \mapsto \begin{pmatrix} a & 0 & \cdots & 0 \\ 0 & a & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & \cdots & a \end{pmatrix} = a \otimes I$$

for $a \in \mathcal{L}(X) \otimes \mathbb{C}$. □

A proof was first presented in [21]. Details that were missing from the original proof have been added for clarity.

Lemma 3.2.14 ([21, Lemma 2.1]). *Let A be a C^*-algebra, and X a right Hilbert A-module. Then, for $x_1, x_2, \cdots x_n, y_1, y_2, \cdots, y_n \in X$ we have*

$$\Big\| \sum_{i=1}^{n} \theta_{x_i,y_i} \Big\| = \Big\| \big((\langle x_i, x_j\rangle_A)_{i,j=1}^n\big)^{\frac{1}{2}} \big((\langle y_i, y_j\rangle_A)_{i,j=1}^n\big)^{\frac{1}{2}}\big) \Big\|$$

where the norm on the right is the C^*-norm on $M_n(A)$.

Proof. For $x, y \in X$ we have

$$\begin{aligned}
\|\theta_{x,y}\|^2 &= \|\theta_{x,y}^* \theta_{x,y}\| \\
&= \|\theta_{y,x}\theta_{x,y}\| \\
&= \|\theta_{y\cdot\langle x,x\rangle_A, y}\| \\
&= \|\theta_{y\cdot(\langle x,x\rangle_A)^{\frac{1}{2}}, y\cdot(\langle x,x\rangle_A)^{\frac{1}{2}}}\| \text{ by the Continuous Functional Calculus} \\
&= \|\langle y \cdot (\langle x, x\rangle_A)^{\frac{1}{2}}, y \cdot (\langle x, x\rangle_A)^{\frac{1}{2}}\rangle_A\| \\
&= \|\big((\langle x, x\rangle_A)^{\frac{1}{2}}\big)^* \langle y, y\rangle_A (\langle x, x\rangle_A)^{\frac{1}{2}}\| \\
&= \|(\langle x, x\rangle_A)^{\frac{1}{2}} \langle y, y\rangle_A (\langle x, x\rangle_A)^{\frac{1}{2}}\| \\
&= \|\big((\langle y, y\rangle_A \langle x, x\rangle_A)^{\frac{1}{2}}\big)^* (\langle y, y\rangle_A \langle x, x\rangle_A)^{\frac{1}{2}}\| \\
&= \|\langle y, y\rangle_A \langle x, x\rangle_A\|^2
\end{aligned}$$

Now consider $Y = X \oplus X \oplus \cdots \oplus X = X \otimes \mathbb{C}^n$ as a right $A \otimes M_n(\mathbb{C})$ module. For $x = (x_1, \cdots, x_n), y = (y_1, \cdots, y_n) \in Y$, we put

$$\langle x, y\rangle_{A\otimes M_n(\mathbb{C})} = (\langle x_i, y_j\rangle_A)_{i,j=1}^n.$$

Let $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_{A\otimes M_n(\mathbb{C})}$ for all $x, y, z \in Y$. Then

$$\|\Theta_{x,y}\| = \Big\| \Big(\sum_{i=1}^{n} \theta_{x_i,y_i}\Big) \otimes I \Big\|,$$

by Lemma 3.2.13. Therefore, we have

$$\Big\| \sum_{i=1}^{n} \theta_{x_i,y_i} \Big\| = \|\Theta_{x,y}\|$$

$$
\begin{aligned}
&= \left\| \langle x, x\rangle_{A\otimes M_n(\mathbb{C})} \langle y, y\rangle_{A\otimes M_n(\mathbb{C})} \right\| \\
&= \left\| \big((\langle x_i, x_j\rangle_A)_{i,j=1}^n\big)^{\frac{1}{2}} \big((\langle y_i, y_j\rangle_A)_{i,j=1}^n\big)^{\frac{1}{2}} \right\|,
\end{aligned}
$$

as required. □

The following lemma follows from a [46, Lemma 3.2] and [21, Lemma 2.2]. The full statement of the result, however, appears in [24].

Lemma 3.2.15 ([24, Proposition 2.1])**.** *Suppose (ρ, κ) is a bimodule map from X_A to Y_B. Then there is a homomorphism $(\rho, \kappa)^{(1)} : \mathcal{K}(X) \to \mathcal{K}(Y)$ satisfying*

$$(\rho, \kappa)^{(1)}(\theta_{x,y}) = \theta_{\rho(x),\rho(y)}$$

for all $x, y \in X$.

Sometimes this homomorphism is denoted by either $\kappa^{(1)}$ or $\rho^{(1)}$, but the author feels this is inappropriate, as a bimodule map is a pair of maps, so the homomorphism is induced from a pair, and as such, should reflect that.

Proof of Lemma 3.2.15. First, we show

$$\Big\| \sum_{i=1}^{n} \theta_{\rho(x_i),\rho(y_i)} \Big\| \le \Big\| \sum_{n=1}^{n} \theta_{x_i,y_i} \Big\|$$

By Lemma 3.2.13 we have

$$
\begin{aligned}
&\Big\| \sum_{n=1}^{n} \theta_{\rho(x),\rho(y)} \Big\| \\
&= \Big\| \big(\langle \rho(x_i), \rho(x_j)\rangle\big)_{i,j=1}^n \big(\langle \rho(y_i), \rho(y_j)\rangle\big)_{i,j=1}^n \Big\| \text{ by Lemma 3.2.13} \\
&= \Big\| \big(\kappa(\langle (x_i), (x_j)\rangle)_{i,j=1}^n\big)^{\frac{1}{2}} \big(\kappa(\langle y_i, y_j\rangle)\big)_{i,j=1}^n\big)^{\frac{1}{2}} \Big\| \\
&= \Big\| \kappa\big((\langle x_i, x_j\rangle)_{i,j=1}^n\big)^{\frac{1}{2}} \kappa\big((\langle y_i, y_j\rangle)\big)_{i,j=1}^n\big)^{\frac{1}{2}} \Big\| \text{ by Continuous Functional Calculus} \\
&= \Big\| \kappa\Big(\big(\langle x_i, x_j\rangle_{i,j=1}^n\big)^{\frac{1}{2}} \big(\langle y_i, y_j\rangle_{i,j=1}^n\big)^{\frac{1}{2}} \Big) \Big\| \\
&\le \Big\| \big(\langle x_i, x_j\rangle\big)_{i,j=1}^n \big(\langle y_i, y_j\rangle\big)_{i,j=1}^n \Big\|
\end{aligned}
$$

$$= \Big\| \sum_{n=1}^{n} \theta_{x,y} \Big\| \text{ by Lemma 3.2.13.}$$

Therefore, $\big\| \sum_{i=1}^{n} \theta_{\rho(x_i),\rho(y_i)} \big\| \leq \big\| \sum_{n=1}^{n} \theta_{x_i,y_i} \big\|$ as we aimed to show. We claim this implies there is a unique homomorphism such that

$$(\rho,\kappa)^{(1)}(\theta_{x,y}) = \theta_{\rho(x),\rho(y)}$$

First, we check the formula is well defined. Suppose that

$$\sum_{i=1}^{n} \theta_{x_i,y_i} = \sum_{j=1}^{m} \theta_{w_j,z_j}.$$

Our aim is to show

$$\sum_{i=1}^{n} \theta_{\rho(x_i),\rho(y_i)} = \sum_{j=1}^{m} \theta_{\rho(w_j),\rho(z_j)}.$$

Let $a_i = x_i$ for $i \leq n$ and $a_i = -w_{i-n}$ for $n+1 \leq i \leq n+m$. Similarly define $b_i = y_i$ for $i \leq n$ and $b_i = -z_{i-n}$ for $n+1 \leq i \leq n+m$. Then

$$0 = \sum_{i=1}^{n} \theta_{x_i,y_i} - \sum_{j=1}^{m} \theta_{w_j,z_j} = \sum_{i=1}^{n+m} \theta_{a_i,b_i}$$

Hence

$$\Big\| \sum_{i=1}^{n} \theta_{\rho(x_i),\rho(y_i)} - \sum_{j=1}^{m} \theta_{\rho(w_j),\rho(z_j)} \Big\| = \Big\| \sum_{i=1}^{n+m} \theta_{\rho(a_i),\rho(b_i)} \Big\| \leq \Big\| \sum_{i=1}^{n+m} \theta_{a_i,b_i} \Big\| = 0.$$

So, $\theta_{x,y} \mapsto \theta_{\rho(x),\rho(y)}$ is well-defined. Since we have shown this map is bounded, it extends to a continuous map, which we check is a homomorphism.

Fix $\alpha \in \mathbb{C}$, $S, T \in \mathcal{K}(X)$, and choose sequences S_n, T_n converging to S and T respectively, such that $S_n, T_n \in \operatorname{span}\{\theta_{x,y} : x, y \in X\}$ for all n. Then

$$\begin{aligned}
(\rho,\kappa)^{(1)}(\alpha S + T) &= \lim_{n\to\infty} (\rho,\kappa)^{(1)}(\alpha S_n + T_n) \\
&= \lim_{n\to\infty} \alpha(\rho,\kappa)^{(1)}(S_n) + (\rho,\kappa)^{(1)}(T_n) \\
&= \alpha(\rho,\kappa)^{(1)}(S) + (\rho,\kappa)^{(1)}(T).
\end{aligned}$$

Now, fix $x, y, z, w \in X$. Then

$$\begin{aligned}
(\rho,\kappa)^{(1)}(\theta_{x,y}\theta_{w,z}) &= (\rho,\kappa)^{(1)}(\theta_{\theta_{x,y}(w),z}) \\
&= \theta_{\theta_{\rho(x),\rho(y)}(\rho(w)),\rho(z)} \\
&= \theta_{\rho(x),\rho(y)}\theta_{\rho(w),\rho(z)} \\
&= (\rho,\kappa)^{(1)}(\theta_{x,y})(\rho,\kappa)^{(1)}(\theta_{w,z}).
\end{aligned}$$

Also,

$$\begin{aligned}
(\rho,\kappa)^{(1)}(\theta_{x,y}^*) &= (\rho,\kappa)^{(1)}(\theta_{y,x}) \\
&= \theta_{\rho(y),\rho(x)} \\
&= \theta_{\rho(x),\rho(y)}^* \\
&= (\rho,\kappa)^{(1)}(\theta_{x,y})^*,
\end{aligned}$$

completing the proof. □

Lemma 3.2.16 ([27, Proposition 2.9]). *Let $p : E \to F$ be an s-injective graph morphism. Then p induces an injective homomorphism of C^*-algebras*

$$\tilde{p}^* : C^*(F) \to C^*(E),$$

such that for $x \in X(F)$, $a \in C_0(F^0)$,

$$\tilde{p}^* \circ j_{X(F)}(x) = j_{X(E)}(x \circ p^1) \text{ and } \tilde{p}^* \circ j_{C_0(F^0)}(a) = j_{C_0(E^0)}(a \circ p^0).$$

Moreover, Figure 3.6 commutes, where q_E and q_F are the quotient maps of their respective algebras.

$$\begin{tikzcd}
\mathcal{T}(F) \arrow[r, "p^*"] \arrow[d, "q_F"'] & \mathcal{T}(E) \arrow[d, "q_E"] \\
C^*(F) \arrow[r, "\tilde{p}^*"'] & C^*(E)
\end{tikzcd}$$

Figure 3.6: Lemma 3.2.16

Proof. We begin by showing that $\big(j_{X(F)} \circ \big(p^1\big)^*, j_{C_0(F^0)} \circ \big(p^0\big)^*\big)$ is a covariant representation of $X(F)$ in $C^*(E)$. By Proposition 3.2.4, $\big(j_{X(F)} \circ \big(p^1\big)^*, j_{C_0(F^0)} \circ \big(p^0\big)^*\big)$ is a Toeplitz representation. Fix $a \in C_0(F^0_{\mathrm{rg}})$. Then

$$\begin{aligned}
j_{C_0(F^0)} \circ \big(p^0\big)^*(a) &= j_{C_0(F^0)}(a \circ p^0) \\
&= (j_{X(E)}, j_{C_0(F^0)})^{(1)}(a \circ p^0 \circ r_E) \\
&= (j_{X(E)}, j_{C_0(F^0)})^{(1)}(a \circ r_F \circ p^1) \\
&= p^*(j_{X(E)}, j_{C_0(F^0)})^{(1)}(a \circ r_F) \text{ by Lemma 3.2.15} \\
&= (p^* \circ j_{X(E)}, p^* \circ j_{C_0(E^0)})^{(1)}(a \circ r_F).
\end{aligned}$$

So $\big(j_{X(F)} \circ \big(p^1\big)^*, j_{C_0(F^0)} \circ \big(p^0\big)^*\big)$ is a covariant Toeplitz representations. Moreover, since $\big(p^0\big)^*$ is injective, so $j_{C_0(F^0)} \circ \big(p^0\big)^*$ is injective. Hence, by Theorem 2.2.15, $\tilde{p}^*$ is injective. For $x \in X(F)$ and $a \in C_0(E^0)$, we have

$$\tilde{p}^* \circ q_F\big(\iota_{X(F)}(x)\big) = j_{X(F)}(x \circ p^1) = q_E \circ p^*\big(\iota_{X(F)}(x)\big),$$

and

$$\tilde{p}^* \circ q_F\big(\iota_{C_0(E^0)}(a)\big) = j_{X(F)}(a \circ p^0) = q_E \circ p^*\big(\iota_{C_0(E^0)}(a)\big),$$

so Figure 3.6 commutes. $\square$

Corollary 3.2.17 ([27, Proposition 2.10])**.** *Let E, F, G be topological graphs, and let $p : E \to F$ and $m : F \to G$ be s-injective graph morphisms. Then*

$$\big((m \circ p)^*\big)^{\sim} = \tilde{p}^* \circ \tilde{m}^* : C^*(G) \to C^*(E).$$

Proof. Applying Lemma 3.2.16 to Proposition 3.1.6, we obtain the homomorphism $\big((m \circ p)^*\big)^{\sim} : C^*(G) \to C^*(E)$. For $x \in X(G)$ and $a \in C_0(G^0)$, we have

$$\begin{aligned}\big((m \circ p)^*\big)^{\sim}(j_{X(G)}(x)) &= j_{X(E)}(x \circ m^1 \circ p^1)\\ &= \tilde{p}^*(j_{X(F)}(x \circ m^1))\\ &= \tilde{p}^* \circ \tilde{m}^*(j_{X(G)}(x)),\end{aligned}$$

and

$$\begin{aligned}\big((m \circ p)^*\big)^{\sim}(j_{X(G)}(x)) &= j_{X(E)}(x \circ m^1 \circ p^1)\\ &= \tilde{p}^*(j_{X(F)}(x \circ m^0))\\ &= \tilde{p}^* \circ \tilde{m}^*(j_{C_0(G^0)}(a)).\end{aligned}$$

Since $\big((m \circ p)^*\big)^{\sim} = \tilde{p}^* \circ \tilde{m}^*$ on the sets $j_{X(G)}(X(G))$ and $j_{C_0(G^0)}(C_0(G^0))$ which generate $C^*(G)$, the result follows. $\square$

Corollary 3.2.18. *Let E and F be topological graphs, such that $p : E \to F$ is a graph isomorphism. Then $\tilde{p}^* : C^*(F) \to C^*(E)$ is an isomorphism.*

Theorem 3.2.19 ([27, Proposition 4.13])**.** *Let $(E_i, p_i)_{i=0}^{\infty}$ be a projective sequence of topological graphs. Then the homomorphism P_∞ of Theorem 3.2.12 descends to an isomorphism of $\varinjlim \big(C^*(E_j), \tilde{p}_j^*\big)$ onto $C^*(E_\infty)$.*

Proof. For $j \in \mathbb{N} \cup \{\infty\}$, we denote the quotient map by $q_j : \mathcal{T}(E_j) \to C^*(E_j)$. By Corollary 3.2.17, the projective sequence (E_i, p_i) the commuting diagram Figure 3.7.

$$
\begin{tikzcd}
\mathcal{T}(E_0) \arrow[r] \arrow[d, "q_0"'] \arrow[rrrrr, bend left=40, "P_0^*"] & \cdots \arrow[r] & \mathcal{T}(E_j) \arrow[r, "p_j^*"'] \arrow[d, "q_j"'] \arrow[rrr, bend left=40, "P_j^*"] & \mathcal{T}(E_{j+1}) \arrow[r] \arrow[d, "q_{j+1}"] \arrow[rr, bend left=30, "P_{j+1}^*"] & \cdots & \mathcal{T}(E_\infty) \arrow[d, "q_\infty"] \\
C^*(E_0) \arrow[r] \arrow[rrrrr, bend right=40, "\tilde{P}_0^*"'] & \cdots \arrow[r] & C^*(E_j) \arrow[r, "\tilde{p}_j^*"] \arrow[rrr, bend right=40, "\tilde{P}_j^*"'] & C^*(E_{j+1}) \arrow[r] \arrow[rr, bend right=30, "\tilde{P}_{j+1}^*"'] & \cdots & C^*(E_\infty)
\end{tikzcd}
$$

Figure 3.7: Direct sequences induced by $(E_i, p_i)_{i=0}^\infty$

We denote by $\Phi_j : \mathcal{T}(E_j) \to \varinjlim \big(\mathcal{T}(E_i), p_i^*\big)$ the homomorphisms such that $\varinjlim \big(\mathcal{T}(E_i), p_i^*\big) = \overline{\bigcup_{i=0}^\infty \Phi_i(\mathcal{T}(E_i))}$. Similarly, $\Psi_j : C^*(E_j) \to \varinjlim \big(C^*(E_i), \tilde{p}_i^*\big)$ are the homomorphisms such that $\varinjlim \big(C^*(E_i), \tilde{p}_i^*\big) = \overline{\bigcup_{i=0}^\infty \Psi_i(C^*(E_i))}$. Since Figure 3.7 commutes, the universal property of $\varinjlim \big(C^*(E_j), \tilde{p}_j^*\big)$ implies there exists a homomorphism $\tilde{P}_\infty : \varinjlim \big(C^*(E_i), \tilde{p}_i^*\big) \to C^*(E_\infty)$ such that $\tilde{P}_j^* = \tilde{P}_\infty \circ \Psi_j$ for each $j \in \mathbb{N}$. Since each $\tilde{P}_j^*$ is injective, $\tilde{P}_\infty$ is injective. For each $j \in \mathbb{N}$, we have $\Psi_j \circ q_j = \big(\Psi_{j+1} \circ q_{j+1}\big) \circ p_j^*$, and so there exists $q : \varinjlim \big(\mathcal{T}(E_j), p_j^*\big) \to \varinjlim \big(C^*(E_j), \tilde{p}_j^*\big)$ such that $q \circ \Phi_j = \Psi_j \circ q_j$ for all $j \in \mathbb{N}$. This yields the commuting diagram of Figure 3.8.

$$\begin{tikzcd}
\varinjlim \left(\mathcal{T}(E_j), p_j^*\right) \arrow[r, "P_\infty"] \arrow[d, "q"'] & \mathcal{T}(E_\infty) \arrow[d, "q_\infty"] \\
\varinjlim \left(C^*(E_j), \tilde{p}_j^*\right) \arrow[r, "\tilde{P}_\infty"'] & C^*(E_\infty)
\end{tikzcd}$$

Figure 3.8: The Relationship Between the Direct Limits

Since q_∞ is surjective, it follows that $\tilde{P}_\infty$ is surjective, and therefore an isomorphism of C^*-algebras. $\square$

Example 3.2.20. Consider the projective sequence of directed graphs $(E_n, p_n)_{n=1}^\infty$ of Example 3.1.8, and the projective limit E_∞ of Example 3.1.14. Then for each $n \in \mathbb{N}$, $p_n : C^*(E_n) \to C^*(E_{n+1})$ is such that for $x \in X^n$, $p_n(S_x) = S_{x0} + S_{x1}$ and $p_n^*(P_x) = P_{x0} + P_{x1}$. By Theorem 3.1.11, $E_\infty = (X^\infty, X^\infty, r_\infty, \mathrm{id})$, so Example 2.2.23 then implies that $C^*(E_\infty) = C(X^\infty) \rtimes_{r_\infty^{-1}} \mathbb{Z}$. The homomorphisms $P_n^* : C^*(E_n) \to C^*(E_\infty)$ are such that $P_n^*(S_x) = \iota_\mathbb{Z}(1)\iota_{C(X^\infty)}(\chi(Z(x, n)))$, where $(\iota_{C(X^\infty)}, \iota_\mathbb{Z})$ is the universal covariant representation of $(C(X^\infty), \mathbb{Z}, r_\infty)$.

We present an alternative description of $C^*(E_\infty)$. We make use of the result that $C^*(E_n) = M_{2^n}(C(\mathbb{T})) = M_{2^n}(\mathbb{C}) \otimes C(\mathbb{T})$ ([47, Example 2.14]). We denote by $\Theta_{i,j}$ the element of $M_n(\mathbb{C})$ such that $(\Theta_{i,j})_{k,l=0}^{n-1} = \delta_{i,k}\delta_{j,l}$. Let $\mathbb{1}, \iota \in C(\mathbb{T})$ be the functions such that $\mathbb{1} : z \mapsto 1$ and $\iota : z \mapsto z$. For $n \in \mathbb{N}$, $M_{2^n}(C(\mathbb{T}))$ is generated by $\Theta_{i,j} \otimes \mathbb{1}$ for $i, j \in \{0, 1, \cdots, 2^{n-1}\}$, and $I \otimes \iota$, since $C(\mathbb{T})$ is generated by ι. Then $p_n^* : M_{2^n}(C(\mathbb{T})) \to M_{2^{n+1}}(C(\mathbb{T}))$ is such that

$$p_n^*(\Theta_{i,j} \otimes \mathbb{1}) = \Theta_{i,j} \otimes \mathbb{1} + \Theta_{i+2^n, j+2^n} \otimes \mathbb{1}$$

for $i, j \in \{0, 1, \cdots, 2^n - 1\}$, and

$$p_n^*(I \otimes \iota) = \sum_{i=0}^{2^n} (\Theta_{i+2^n, i} \otimes \mathbb{1} + \Theta_{i, i+2^n} \otimes \iota),$$

which are the homomorphsims of [10, Chapter V.3]. Hence, $C^*(E_\infty) = \mathcal{B}(\{2^n\})$, the Bunce-Deddens algebra of type 2^n.

The consequence of this is $\mathcal{B}(\{2^n\}) = C(X^\infty) \rtimes_{r_\infty^{-1}} \mathbb{Z}$, which is precisely what [10, Theorem VIII.4.1] states.

Chapter 4

Construction of Noncommutative Solenoids via Topological Graph C^*-algebras

4.1 Construction of Noncommutative Solenoids via Topological graph C^*-algebras

For $\theta \in [0, 1)$, we write $\mathcal{A}_\theta$ for the universal C^*-algebra generated by unitaries U_θ and V_θ subject to

$$U_\theta V_\theta = e^{2i\pi\theta} V_\theta U_\theta. \tag{4.1}$$

We call $\mathcal{A}_\theta$ the **rotation algebra of** θ. If $\theta \notin \mathbb{Q}$, we call $\mathcal{A}_\theta$ an **irrational rotation algebra**. For further reading, see [10, Chapter VI]. Using the topological graph

$$E_\theta := (\mathbb{T}, \mathbb{T}, r_\theta : z \mapsto e^{2i\pi\theta} z, \mathrm{id}_\mathbb{T} : z \mapsto z),$$

from Example 2.2.3, by [10, Example VIII.1.1] and Example 2.2.24 we have $C^*(E_\theta) = \mathcal{A}_\theta$. Where it is unambiguous, we write U for U_θ, and V for V_θ.

We dedicate this chapter to the construction of noncommutative solenoids [38, Definition 3.1]. The way we shall do this is via the direct limit of rotation algebras, as shown in [38, Theorem 3.7]. Given the results of Chapter 3, most notably Theorems 3.1.11 and 3.2.19, and the fact that there exist topological graphs E such that $C^*(E) = \mathcal{A}_\theta$, it seems possible that we could find a sequence of topological graphs and s-injective graph morphisms (E_n, p_n) such that the C^*-algebra $C^*(E_\infty)$ is a noncommutative solenoid. Given that we have description of noncommutative solenoids as a direct limit of topological graph algebras (Definition 4.1.1), and a theory of direct limits of topological graph C^*-algebras, it hints at a method to this construction.

The following definition comes from [38]. The original definition presented was different, but was shown to agree with the following construction.

Definition 4.1.1 ([38, Theorem 3.7])**.** *Fix $\theta_0 \in [0, 1)$, and let N be an integer such that $N \geq 2$. Let $(k_n)_{n=0}^\infty$ be such that $k_n \in \{0, 1, \cdots, N^2 - 1\}$ for each $n \in \mathbb{N}$, and let $\theta = (\theta_n)_{n=0}^\infty$ be such that $N^2\theta_{n+1} = \theta_n + k_n$. Let $\psi_n : \mathcal{A}_{\theta_n} \to \mathcal{A}_{\theta_{n+1}}$ be the homomorphism such that*

$$\psi_n : \begin{cases} U_{\theta_n} & \mapsto \quad U_{\theta_{n+1}}^N \\ V_{\theta_n} & \mapsto \quad V_{\theta_{n+1}}^N. \end{cases}$$

We denote $\mathcal{A}_\theta^{\mathscr{S}} = \varinjlim(\mathcal{A}_{\theta_n}, \psi_n)$.

The C^*-algebra $\mathcal{A}_\theta^{\mathscr{S}}$ is called the ***noncommutative solenoid of*** *θ.*

We then have the following commuting diagram.

$$\begin{tikzcd}
\mathcal{A}_{\theta_0} \arrow[r] \arrow[rrrrr, bend left, "\psi_{0,\infty}"] & \cdots \arrow[r] & \mathcal{A}_{\theta_k} \arrow[r, "\psi_k"'] \arrow[rrr, bend left, "\psi_{k,\infty}"] & \mathcal{A}_{\theta_{k+1}} \arrow[r] \arrow[rr, bend left, "\psi_{k+1,\infty}"] & \cdots & \mathcal{A}_\theta^{\mathscr{S}}
\end{tikzcd}$$

We begin by examining what possible graph homomorphisms we can find.

Theorem 4.1.2. *Fix $\theta \in [0,1)$ and let $\sigma \in [0,1)$ be such that $p : E_\theta \to E_\sigma$ is an s-injective graph morphism. Then $p^0 = p^1 : e^{2i\pi t} \mapsto e^{2i\pi f(t)}$ for some continuous surjective $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$, subject to $f(t+\theta) = f(t) + \sigma$.*

Moreover, if $\theta \in [0,1) \setminus \mathbb{Q}$, then $f(t) = \alpha t + b$ for some $b \in [0,1)$ such that $\alpha = \frac{\sigma - b}{\theta} \in \mathbb{Z} \setminus \{0\}$.

Proof. Suppose that $p^0 : e^{2i\pi t} \mapsto e^{2i\pi f(t)}$ and $p^1 : e^{2i\pi t} \mapsto e^{2i\pi g(t)}$ for some continuous surjective $f, g : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$. We then have

$$e^{2i\pi f(t)} = p^0 \circ \mathrm{id}_{\mathbb{T}}(e^{2i\pi t}) = \mathrm{id}_{\mathbb{T}} \circ p^1(e^{2i\pi t}) = e^{2i\pi g(t)}.$$

Hence, $f \equiv g$. We also have

$$e^{2i\pi f(t+\theta)} = p^0 \circ r_\theta(e^{2i\pi t}) = r_\sigma \circ p^1(e^{2i\pi t}) = e^{2i\pi (f(t)+\sigma)},$$

giving us $f(t+\theta) = f(t) + \sigma$. Since for all $z \in \mathbb{T}$, $\mathrm{id}^{-1}(z) = \{z\}$, and $\mathrm{id}^{-1}(p^0(z)) = \{p^0(z)\}$, so $p^1 : \mathrm{id}^{-1}(z) \to \mathrm{id}^{-1}(p^0(z))$ must be injective. Hence (p^0, p^1) is s-injective.

For the remainder of the proof we require $\theta \in [0,1)\mathbb{Q}$ (we make use of the fact that $\{n\theta : n \in \mathbb{N}\}$ is dense in $\mathbb{T}$). For $n \in \mathbb{Z}, f(n\theta) = f(0) + n\sigma$. Hence, for $n, m \in \mathbb{Z}$

$$f(n\theta) + f(m\theta) = n\sigma + m\sigma + 2f(0) = (n+m)\sigma + 2f(0) = f((m+n)\theta) + f(0).$$

So, for $x, y \in [0,1)$ with $x + y < 1$, fix sequences n_i, m_i of natural numbers such that $n_i\theta \pmod 1 \to x$ and $m_i\theta \pmod 1 \to y$ as $i \to \infty$. Continuity of f then implies

$$f(x+y) = \lim_{i\to\infty} f((n_i + m_i)\theta) = \lim_{i\to\infty} f(n_i\theta) + f(m_i\theta) - f(0) = f(x) + f(y) - f(0).$$

So if $f' : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the function such that $f'(x) = f(x) - f(0)$ then f' satisfies

$$f'(x+y) = f(x+y) - f(0) = f(x) + f(y) - 2f(0) = f'(x) + f'(y).$$

Hence f' is affine with $f'(0) = 0$, so $f(t) = \alpha t + b$ for some $\alpha, b \in \mathbb{R}$. Continuing from our calculation previously, we find that

$$f(t+\theta) = f(t) + \sigma \iff \alpha(t+\theta) + b = \alpha t + \sigma \iff \alpha\theta = \sigma - b.$$

The continuity of f forces $f(x_n) \to f(0)$ as $x_n \to 1$, so $\alpha \in \mathbb{Z} \setminus \{0\}$ since p^0, p^1 are surjective. Hence $\frac{\sigma - b}{\theta} \in \mathbb{Z} \setminus \{0\}$. $\square$

We seek a pair of elements $a, b \in C^*(E_\theta)$ that generate $\mathcal{A}_\theta$ as a C^*-algebra.

Proposition 4.1.3. *Fix $\theta \in [0,1)$, and let $(j_{X(E_\theta)}, j_{C(\mathbb{T})})$ be the universal covariant Toeplitz representation of $X(E_\theta)$ into $C^*(E_\theta) = \mathcal{A}_\theta$. Let $\mathbb{1}, \iota \in C(\mathbb{T})$ be such that $\mathbb{1} : z \mapsto 1$ and $\iota : z \mapsto z$ for all $z \in \mathbb{T}$. Then $(j_{X(E_\theta)}(\mathbb{1}), j_{C(\mathbb{T})}(\iota))$ is a pair of unitaries satisfying* (4.1)*, and $C^*(j_{X(E_\theta)}(\mathbb{1}), j_{C(\mathbb{T})}(\iota)) = \mathcal{A}_\theta$.*

Proof. Since $j_{C(\mathbb{T})}$ is a homomorphism, we have

$$j_{C(\mathbb{T})}(\iota) j_{C(\mathbb{T})}(\iota)^* = j_{C(\mathbb{T})}(\iota\bar{\iota}) = j_{C(\mathbb{T})}(\mathbb{1}) = 1 = j_{C(\mathbb{T})}(\bar{\iota}\iota) = j_{C(\mathbb{T})}(\iota)^* j_{C(\mathbb{T})}(\iota),$$

so $j_{C(\mathbb{T})}(\iota)$ is unitary. Also,

$$j_{X(E_\theta)}(\mathbb{1})^* j_{X(E_\theta)}(\mathbb{1}) = j_{C(\mathbb{T})}(\langle \mathbb{1}, \mathbb{1} \rangle) = j_{C(\mathbb{T})}(\mathbb{1}) = 1.$$

Since $(j_{X(E_\theta)}, j_{C(\mathbb{T})})$ is covariant, we have

$$j_{X(E_\theta)}(\mathbb{1}) j_{X(E_\theta)}(\mathbb{1})^* = (j_{X(E_\theta)}, j_{C(\mathbb{T})})^{(1)}(\Theta_{\mathbb{1},\mathbb{1}}) = j_{C(\mathbb{T})}(\phi(\mathbb{1})) = 1$$

hence $j_{X(E_\theta)}(\mathbb{1})$ is also unitary. Finally,

$$j_{C(\mathbb{T})}(\iota) j_{X(E_\theta)}(\mathbb{1}) = j_{X(E_\theta)}(\iota \cdot \mathbb{1}) = j_{X(E_\theta)}(e^{2i\pi\theta} \mathbb{1} \cdot \iota) = e^{2i\pi\theta} j_{X(E_\theta)}(\mathbb{1}) j_{C(\mathbb{T})}(\iota),$$

which is (4.1), as claimed. Since ι generates the C^*-algebra $C(\mathbb{T})$, $j_{C(\mathbb{T})}$ is generated by $j_{C(\mathbb{T})}(\iota)$. Then, for any $f \in C(\mathbb{T})$, $j_{X(E_\theta)}(\mathbb{1})j_{C(\mathbb{T})}(f) = j_{X(E_\theta)}(\mathbb{1} \cdot f) = j_{X(E_\theta)}(f)$. Hence, the pair $(j_{X(E_\theta)}(\mathbb{1}), j_{C(\mathbb{T})}(\iota))$ generate the images of $j_{X(E_\theta)}$ and $j_{C(\mathbb{T})}$, and therefore $C^*(E_\theta)$. Hence $C^*(j_{X(E_\theta)}(\mathbb{1}), j_{C(\mathbb{T})}(\iota)) = \mathcal{A}_\theta$. □

Proposition 4.1.3 shows that one of the two generators comes from the Hilbert module, the other from the coefficient algebra. This is the source of the issues we will encounter.

Proposition 4.1.4. *Fix $\theta \in [0,1)$, and let $\sigma = n\theta$ for some $n \in \mathbb{N}$. Let $p : E_\theta \to E_\sigma$ be an s-injective graph morphism. Then*

$$p^*(j_{X(E_\theta)}(\mathbb{1})) = j_{X(E_\sigma)}(\mathbb{1}).$$

Morover, if $p^0 : z \mapsto z^n$, then

$$p^*(j_{C(\mathbb{T})}(\iota)) = j_{C(\mathbb{T})}(\iota)^n.$$

Proof. Firstly, we have

$$p^*(j_{X(E_\theta)}(\mathbb{1})) = j_{X(E_\sigma)}(\mathbb{1} \circ p^1) = j_{X(E_\sigma)}(\mathbb{1}). \tag{4.2}$$

Now, if $p^0(z) = z^n$ for all $z \in \mathbb{T}$, then $\iota \circ p^0(z) = \iota(z^n) = z^n$. So

$$p^*(j_{C(\mathbb{T})}(\iota)) = j_{C(\mathbb{T})}(\iota \circ p^0) = j_{C(\mathbb{T})}(\iota)^n,$$

as claimed. □

If we consider the case where $p^0 = p^1 : z \mapsto z^n$ and denote by $U_\theta := j_{X(E_\theta)}(\mathbb{1})$ and $V_\theta := j_{C(\mathbb{T})}(\iota)$, we get

$$p^* : \begin{cases} U_\theta & \mapsto & U_\sigma \\ V_\theta & \mapsto & V_\sigma^n, \end{cases}$$

and it becomes apparent that the linking maps in Definition 4.1.1 do not correspond to any s-injective graph morphism $p : E_\theta \to E_\sigma$. In particular, Equation (4.2) shows that for any s-injective graph morphism $p : E_\theta \to E_\sigma$, the induced homomorphism $p^* : C^*(E_\sigma) \to C^*(E_\theta)$ carries $j_{X(E_\theta)}(\mathbb{1})$ to $j_{X(E_\sigma)}(\mathbb{1})$. That

is, the homomorphism $p^*(U_\theta) = U_\sigma$. We require a more subtle approach. Since one of the generating unitaries is associated to the edge set, we require some method of manipulating it so that we can obtain a homomorphism ϕ such that $\phi(j_{X(E_\theta)}(\mathbb{1})) = j_{X(E_\sigma)}(\mathbb{1})^n$.

Notation 4.1.5. *Fix $N \in \mathbb{N}$, and a topological graph E. We denote the N**-th higher power graph** by*

$$E^{(N)} := \left(E^0, E^N, r^N, s^N\right),$$

where the range and source maps are given by $r^N : e_1 e_2 \cdots e_N \mapsto r(e_1)$ and $s^N : e_1 e_2 \cdots e_N \mapsto s(e_N)$ respectively.

Lemma 4.1.6. *Fix $\theta \in [0,1)$, $N \in \mathbb{N}$ and let $k \in \{0, 1 \cdots, N-1\}$. Then $h = (h^0, h^1) : E^{(N)}_{\frac{\theta+k}{N}} \to E_\theta$ where*

$$h^0 : z \mapsto z \text{ and } h^1 : z_1 z_2 \cdots z_N \mapsto z_N,$$

is a graph isomorpism.

Proof. That h^0, h^1 are homeomorphisms because $r_\theta, \mathrm{id}_{\mathbb{T}}$ are homeomorphisms. Fix $z_1 z_2 \cdots z_N \in E^N$. Straightforward calculations yield

$$h^0 \circ r^N_{\frac{\theta+k}{N}}(z_1 z_2 \cdots z_N) = r_\theta \circ h^1(z_1 z_2 \cdots z_N),$$

and

$$h^0 \circ s^N_{\frac{\theta+k}{N}}(z_1 z_2 \cdots z_N) = \mathrm{id}_{\mathbb{T}} \circ h^1(z_1 z_2 \cdots z_N).$$

The s-injectivity follows from the fact that $s^N_{\frac{\theta+k}{N}}$ is homeomorphism. $\square$

Lemma 4.1.7. *Let E be a topological graph, and let E^* denote its path space. There exists a linear map $\lambda^1 : X(E) \to \mathcal{B}(\ell^2(E^*))$ and a homomorphism $\lambda^0 : C_0(E^0) \to \mathcal{B}(\ell^2(E^*))$ such that for all $x \in X(E)$, $f \in C_0(E^0)$ and $\mu \in E^*$,*

$$\lambda^1(x)\delta_\mu = \sum_{e \in E^1 r(\mu)} x(e)\delta_{e\mu} \text{ and } \lambda^0(f)\delta_\mu = f(r(\mu))\delta_\mu.$$

Then the pair (λ^1, λ^0) is a Toeplitz representation of $X(E)$ onto $\mathcal{B}(\ell^2(E^*))$, and the homomorphism $\lambda^1 \times \lambda^0 : \mathcal{T}(E) \to \mathcal{B}(\ell^2(E^*))$ is faithful.

Proof. We first check that λ^0 is well-defined. Fix $f \in C_0(E^0)$ and a finite set I. For each $i \in I$, let $\alpha_i \in \mathbb{C}$ and $\mu_i \in E^*$. Then

$$\Big\|\lambda^0(f)\Big(\sum_i \alpha_i\delta_{\mu_i}\Big)\Big\| = \Big\|\sum_i \alpha_i f(r(\mu_i))\delta_{\mu_i}\Big\| \leq \sum_i |\alpha_i|\|f\|,$$

so λ^0 is norm decreasing on finite sums and hence well-defined. For $f, g \in C_0(E^0), \mu \in E^*$ and $\alpha \in \mathbb{C}$, we have

$$\lambda^0(\alpha f+g)\delta_\mu = (\alpha f+g)(r(\mu))\delta_\mu = (\alpha(f(r(\mu)))+g(r(\mu)))\delta_\mu = (\alpha\lambda^0(f)+\lambda^0(g))\delta_\mu.$$

Moreover,

$$\lambda^0(fg)\delta_\mu = (fg)(r(\mu))\delta_\mu = \big(f(r(\mu))\big)\big(g(r(\mu))\big)\delta_\mu = \lambda^0(f)\lambda^0(g)\delta_\mu.$$

Hence λ^0 is a homomorphism.

Fix $x \in X(E)$, and for a finite set I, and for each $i \in I$, let $\alpha_i \in \mathbb{C}$ and $\mu_i \in E^*$. Then

$$\begin{aligned}
\Big\|\lambda^1(x)\Big(\sum_{i\in I}\alpha_i\delta_{\mu_i}\Big)\Big\|^2 &= \Big\|\sum_{i,j\in I}\Big(\alpha_i\lambda^1(x)\delta_{\mu_i}\Big|\alpha_j\lambda^1(x)\delta_{\mu_j}\Big)\Big\| \\
&= \Big\|\sum_{i,j\in I}\alpha_i\alpha_j\Big(\sum_{e\in E^1r(\mu_i)} x(e)\delta_{e\mu_i}\Big|\sum_{f\in E^1r(\mu_j)} x(f)\delta_{f\mu_j}\Big)\Big\| \\
&= \Big\|\sum_{i,j\in I}\sum_{e\in E^1r(\mu_i)}\sum_{f\in E^1r(\mu_j)}\alpha_i\alpha_j x(e)\overline{x(f)}(\delta_{e\mu_i}|\delta_{f\mu_j})\Big\| \\
&= \Big\|\sum_{i\in I}|\alpha_i|^2\sum_{e\in E^1r(\mu_i)}|x(e)|^2\Big\| \\
&= \Big\|\sum_{i\in I}|\alpha_i|^2\langle x, x\rangle(r(\mu_i))\Big\| \\
&\leq \sum_{i\in I}|\alpha_i|^2\|x\|^2,
\end{aligned}$$

Hence $x \mapsto \lambda^1(x)$ is norm decreasing and well-defined. Further, for finitely many $x_i \in X(E)$ and $\mu \in E^*$,

$$\begin{aligned}
\Big\|\lambda^1\big(\sum_i x_i\big)\delta_\mu\Big\|^2 &= \sum_{i,j}\sum_{e,f\in E^1 r(\mu)} \overline{x_i(e)}x_j(f)(\delta_{e\mu}|\delta_{f\mu}) \\
&= \sum_{i,j}\sum_{e\in E^1 r(\mu)} \overline{x_i(e)}x_j(e) \\
&= \sum_{i,j}\langle x_i, x_j\rangle(r(\mu)) \\
&\le \sum_{i,j}\|\langle x_i, x_j\rangle\| \\
&\le \sum_{i,j}\|x_i\|\|x_j\|,
\end{aligned}$$

hence $x \mapsto \lambda^1(x)$ is well-defined. Further, for $x, y \in X(E)$, $\mu \in E^*$ and $\alpha \in \mathbb{C}$,

$$\begin{aligned}
\lambda^1(\alpha x + y)\delta_\mu &= \sum_{e\in E^* r(\mu)} (\alpha x + y)(e)\delta_{e\mu} \\
&= \alpha \sum_{e\in E^* r(\mu)} x(e)\delta_{e\mu} + \sum_{e\in E^* r(\mu)} y(e)\delta_{e\mu} \\
&= (\alpha\lambda^1(x) + \lambda^1(y))\delta_\mu,
\end{aligned}$$

so λ^1 is linear and hence $\lambda^1(x) \in \mathcal{B}(\ell^2(E^*))$ for $x \in X(E)$, with adjoint

$$\lambda^1(x)^*\delta_\mu = \begin{cases} \overline{x(e)}\delta_{\mu'} & \text{if } |\mu| \ge 1 \text{ and } \mu = e\mu' \\ 0 & \text{if } |\mu| = 0. \end{cases}$$

Finally, we check that (λ^1, λ^0) is a Toeplitz representation. Fix $x, y \in X(E)$, $f \in C_0(E^0)$ and $\mu \in E^*$. Then

$$\begin{aligned}
\lambda^1(x)\lambda^0(f)\delta_\mu &= \lambda^1(x)(f(r(\mu))\delta_\mu) \\
&= \sum_{e\in E^1 r(\mu)} x(e)(f(r(\mu))\delta_{e\mu}) \\
&= \sum_{e\in E^1 r(\mu)} x(e)(f(s(e)))\delta_{e\mu})
\end{aligned}$$

$$= \sum_{e \in E^1 r(\mu)} x \cdot f(e)\delta_{e\mu}$$
$$= \lambda^1(x \cdot f)\delta_\mu.$$

Also,

$$\begin{aligned}
\lambda^1(x)^*\lambda^1(y)\delta_\mu &= \sum_{e \in E^1 r(\mu)} \lambda^1(x)^* y(e)\delta_{e\mu} \\
&= \sum_{e \in E^1 r(\mu)} \overline{x(e)} y(e)\delta_\mu \\
&= \langle x, y\rangle(r(\mu))\delta_\mu \\
&= \lambda^0(\langle x, y\rangle)\delta_\mu.
\end{aligned}$$

For the final part of the calculation, we have

$$\begin{aligned}
\lambda^0(f)\lambda^1(x)\delta_\mu &= \sum_{e \in E^1 r(\mu)} \lambda^0(f)x(e)\delta_{e\mu} \\
&= \sum_{e \in E^1 r(\mu)} f(r(e\mu))x(e)\delta_{e\mu} \\
&= \sum_{e \in E^1 r(\mu)} f(r(e))x(e)\delta_{e\mu} \\
&= \sum_{e \in E^1 r(\mu)} (f \cdot x)(e)\delta_{e\mu} \\
&= \lambda^1(f \cdot x)\delta_\mu.
\end{aligned}$$

Hence (λ^1, λ^0) is a Toeplitz representation.

For the final part of the proof, we claim that the homomorphism $\lambda^1 \times \lambda^0$ is injective. By [18, Proposition 1.6], it suffices to show that $\lambda^0|_{\ell^2(E^*)}$ is faithful. Suppose that $f, g \in C_0(E^0)$ are such that $\lambda^0(f) = \lambda(g)$. Then, for all $v \in E^0$, we have

$$f(v)\delta_v = \lambda^0(f)\delta_v = \lambda^0(g)\delta_v = g(v)\delta_v.$$

Hence $f = g$. □

Corollary 4.1.8. *Fix $N \in \mathbb{N}$, and let E be a topological graph, and $E^{(N)}$ be its N-th higher power graph. Let $(\iota_{X(E)}, \iota_{C_0(E^0)})$ be the universal Toeplitz representation of $X(E)$ into $\mathcal{T}(E)$, and similiarly let $(\iota_{X(E^{(N)})}, \iota^N_{C_0(E^0)})$ be the universal Toeplitz representation of $X(E^{(N)})$ into $\mathcal{T}(E^{(N)})$. Then there exists an injective homomorphism*

$$\phi : \mathcal{T}(E^{(N)}) \to \mathcal{T}(E)$$

such that for $x \in X(E^{(N)})$ and $a \in C_0(E^0)$,

$$\phi \circ \iota_{X(E^{(N)})}(x) = \iota^{\otimes N}_{X(E)}(x) \text{ and } \phi \circ \iota^N_{C_0(E^0)}(a) = \iota_{C_0(E^0)}(a).$$

Proof. Given that $X(E)^{\otimes N} = X(E^{\otimes N})$ by [47, Proposition 9.7], and $X(E^{\otimes N}) = X(E^{(N)})$, we have $X(E^{(N)}) = X(E)^{\otimes N}$.

In a similar method as in Lemma 4.1.7, we can define a Toeplitz representation (ρ^1, ρ^0) of $X(E^{(N)})$ into $\mathcal{B}(\ell^2((E^{(N)})^*))$, where

$$\rho^1(f)\delta_\mu = \sum_{e_1\cdots e_N \in E^N r(\mu)} f(e_1 \cdots e_N)\delta_{e_1\cdots e_N\mu} \text{ and } \rho^0(g)\delta_\mu = g(r(\mu))\delta_\mu$$

for all $f \in C_c(E^{(N)})$ and $g \in C_0(E^0)$. Then the map $\rho^1 \times \rho^0 : \mathcal{T}_{E^{(N)}}$ is injective.

The pair $(\iota^{\otimes N}_{X(E)}, \iota_{C_0(E^0)})$ is a Toeplitz representation of $X(E^{(N)})$ into $\mathcal{T}(E)$ (the properties for this follow from the fact that $(\iota_{X(E)}, \iota_{C_0(E^0)})$ is a Toeplitz representation). We claim that $\phi := \iota^{\otimes N}_{X(E)} \times \iota_{C_0(E^0)} : \mathcal{T}(E^{(N)}) \to \mathcal{T}(E)$ is the desired homomorphism. To show injectivity, we show $(\lambda^1 \times \lambda^0) \circ (\iota^{\otimes N}_{X(E)} \times \iota_{C_0(E^0)}) = (\rho^1 \times \rho^0)$.

Fix $x = x_1 \otimes \cdots \otimes x_N \in X(E^{(N)})$ and $\mu \in E^*$. Then

$$\begin{aligned}
(\lambda^1 \times \lambda^0) \circ (\iota^{\otimes N}_{X(E)} \times \iota^N_{C_0(E^0)})(\iota^{\otimes N}_{X(E)}(x))\delta_\mu &= (\lambda^1 \times \lambda^0)(\iota_{X(E)}(x_1) \cdots \iota_{X(E)}(x_N))\delta_\mu \\
&= \lambda^1(x_1) \cdots \lambda^1(x_N)\delta_\mu \\
&= \sum_{e_1\cdots e_N \in E^N r(\mu)} x(e_1 \cdots e_N)\delta_{e_1\cdots e_N\mu} \\
&= \rho^1(x)\delta_\mu
\end{aligned}$$

$$= ((\rho^1 \times \rho^0) \circ \iota_{X(E)}^{\otimes N}(x))\delta_\mu$$

Moreover, for $a \in C_0(E^0)$ we have

$$\begin{aligned}(\lambda^1 \times \lambda^0) \circ (\iota_{X(E)}^{\otimes N} \times \iota_{C_0(E^0)}^N)(\iota_{C_0(E^0)}^N(a))\delta_\mu &= \lambda^0(a)\delta_\mu \\ &= \rho^0(a)\delta_\mu \\ &= (\rho^1 \times \rho^0) \circ \iota_{C_0(E^0)}^N(a)\delta_\mu.\end{aligned}$$

Hence, the subspace $\ell^2((E^{(N)})^*)$ of $\ell^2(E^*)$ is invariant under the map $(\lambda^1 \times \lambda^0) \circ (\iota_{X(E^{(N)})} \times \iota_{C_0(E^0)}^N) = \rho^1 \times \rho^0$. Hence ϕ is injective. $\square$

Corollary 4.1.9. *Let E be a topological graph such that r is a homeomorphism, and fix $N \in \mathbb{N}$. Let $\phi : \mathcal{T}(E^{(N)}) \to \mathcal{T}(E)$ be the injective homomorphism described in Corollary 4.1.8. Then ϕ descends to an injective homomorphism $\tilde{\phi} : C^*(E^{(N)}) \to C^*(E)$.*

Proof. We show that the image of the Katsura ideal $J_{X(E^{(N)})}$ under ϕ is contained within $J_{X(E)}$. By Proposition 2.2.13, $\iota_{C_0(E^0)}^N(a) \in J_{E^{(N)}}$ if and only if $\mathrm{supp}(a) \subseteq E^0_{\mathrm{rg}}$ and $a \not\equiv 0$. Since r is a homeomorphism, we have $E^0_{\mathrm{rg}} = (E^{(N)})^0_{\mathrm{rg}}$. Hence $\iota_{C_0(E^0)}(a) \in J_{X(E)}$ if and only if $\iota_{C_0(E^0)}^N(a) \in J_{E^{(N)}}$. $\square$

Proposition 4.1.10. *Fix $\theta \in [0, 1)$, $N \in \mathbb{N}$ such that $N \geq 2$, and $k_1, k_2 \in \{0, 1, \cdots, N-1\}$. Let $p : E_{\frac{\theta+k_1}{N}} \to E_\theta$ be the s-injective graph morphism where $p^0 = p^1 : z \mapsto z^N$. Let h be the graph isomorphism described in Lemma 4.1.6, and $\tilde{\phi}$ as described in Corollary 4.1.9.*

Then $\psi = \tilde{\phi} \circ h^* \circ p^* : C^*(E_\theta) \to C^*\left(E_{\frac{\theta+Nk_1+k_2}{N^2}}\right)$ satisfies

$$\psi : \begin{cases} U_\theta & \mapsto & U^N_{\frac{\theta+Nk_1+k_2}{N^2}} \\ V_\theta & \mapsto & V^N_{\frac{\theta+Nk_1+k_2}{N^2}} \end{cases},$$

where U_θ, V_θ generate $C^*(E_\theta)$, and similarly for $U_{\frac{\theta+Nk_1+k_2}{N^2}}, V_{\frac{\theta+Nk_1+k_2}{N^2}}$ and $C^*\left(E_{\frac{\theta+Nk_1+k_2}{N^2}}\right)$.

Proof. Let $(j_{X(E_\theta)}, j_{C(\mathbb{T})})$ be the covariant representation generating $C^*(E_\theta)$. By Proposition 4.1.3, it suffices check the images of $j_{X(E_\theta)}(\mathbb{1})$ and $j_{C(\mathbb{T})}(\iota)$ under ψ. We have

$$\begin{aligned}
\psi \circ j_{X(E_\theta)}(\mathbb{1}) &= \tilde{\phi} \circ h^* \circ p^* \circ j_{X(E_\theta)}(\mathbb{1}) \\
&= \tilde{\phi} \circ h^* \circ j_{X\left(E_{\frac{\theta+k_1}{N}}\right)}(\mathbb{1} \circ p^1) \\
&= \tilde{\phi} \circ h^* \circ j_{X\left(E_{\frac{\theta+k_1}{N}}\right)}(\mathbb{1}) \\
&= \tilde{\phi} \circ \left(j_{X\left(E_{\frac{\theta+k_1}{N}}\right)}(\mathbb{1}) \otimes \cdots \otimes j_{X\left(E_{\frac{\theta+k_1}{N}}\right)}(\mathbb{1}) \right) \\
&= \left(j_{X\left(E_{\frac{\theta+Nk_1+k_2}{N^2}}\right)}(\mathbb{1}) \right)^N,
\end{aligned}$$

and

$$\begin{aligned}
\psi \circ j_{C(\mathbb{T})}(\iota) &= \tilde{\phi} \circ h^* \circ p^* \circ j_{C(\mathbb{T})}(\iota) \\
&= \tilde{\phi} \circ h^* \circ j_{C(\mathbb{T})}(\iota \circ p^0) \\
&= \tilde{\phi} \circ h^* \circ j_{C(\mathbb{T})}(\iota^N) \\
&= \tilde{\phi} \circ \left(j_{C(\mathbb{T})}(\iota^N) \right) \\
&= \left(j_{C(\mathbb{T})}(\iota) \right)^N,
\end{aligned}$$

which is the required homomorphism. □

Proposition 4.1.10 can be expressed as the following commuting diagram.

$$
\begin{tikzcd}
C^*(E_\theta) \arrow[r, "\psi"] \arrow[ddr, "p^*"'] & C^*\Big(E_{\frac{\theta+k_1+k_2}{N^2}}\Big) \\
& C^*\Big(E^{(N)}_{\frac{\theta+k_1+k_2}{N^2}}\Big) \arrow[u, "\tilde{\phi}"'] \\
& C^*\Big(E_{\frac{\theta+k_1}{N}}\Big) \arrow[u, "h^*"']
\end{tikzcd}
$$

We make use of this in the following theorem.

Theorem 4.1.11. *Fix an integer N such that $N \geq 2$, and $\theta_0 \in [0,1)$. Let $\theta = (\theta_n)_{n=0}^{\infty}$, where $N\theta_{n+1} = \theta_n + k_n$ for some $k_n \in \{0, 1, \cdots, N-1\}$ for $n \in \mathbb{N}$. Let $\widetilde{\psi_n} : C^*(E_{\theta_{2n}}) \to C^*(E_{\theta_{2(n+1)}})$ be the homomorphism ψ in Proposition 4.1.10. Then $\varinjlim(C^*(E_{\theta_{2n}}), \widetilde{\psi_n}) = \mathcal{A}_\theta^{\mathscr{S}}$.*

Proof. Repeated applications of Proposition 4.1.10 gives us the sequence as described in Definiton 4.1.1. It follows that $\varinjlim(C^*(E_{\theta_{2n}}), \widetilde{\psi_n}) = \mathcal{A}_\theta^{\mathscr{S}}$. $\square$

$$
\begin{tikzcd}
C^*(E_{\theta_0}) \arrow[r] \arrow[rrrrr, bend left, "\widetilde{\psi_{0,\infty}}"] & \cdots \arrow[r] & C^*\big(E_{\theta_{2n}}\big) \arrow[r, "\widetilde{\psi_n}"'] \arrow[rrr, bend left, "\widetilde{\psi_{n,\infty}}"] \arrow[ddr, "p_n^*"'] & C^*\big(E_{\theta_{2(n+1)}}\big) \arrow[r] \arrow[rr, bend left, "\widetilde{\psi_{n+1,\infty}}"] & \cdots & \mathcal{A}_\theta^{\mathscr{S}} \\
& & & C^*\big(E^{(N)}_{\theta_{2(n+1)}}\big) \arrow[u, "\tilde{\phi}_n"'] & & \\
& & & C^*\big(E_{\theta_{2n+1}}\big) \arrow[u, "h_n^*"'] & &
\end{tikzcd}
$$

Since the maps used respect the Toeplitz algebras involved, we get the following that we will make use of in Chapter 6. The reader will notice that we do change convention slightly between the Toeplitz noncommutative solenoid and the noncommutative solenoid. This is not done with malice, but rather to make later chapters more legible (see Chapter 6).

Definition 4.1.12. *Fix an integer N such that $N \geq 2$ and $\theta_0 \in [0,1)$. Let $k_n \in \{0, 1, \cdots, N^2-1\}$ for $n \in \mathbb{N}$, and $\theta = (\theta_n)_{n=0}^{\infty}$ where $N^2\theta_{n+1} = \theta_n + k_n$. Let $p_n = (p_n^0, p_n^1) : E_{\theta_{n+1}}^{(N)} \to E_{\theta_n}$ be the s-injective graph morphism such that*

$$p_n^0 : z \mapsto z^N \text{ and } p^1 : z_1 z_2 \cdots z_N \mapsto z_N^N,$$

and let ϕ_n be the homomorphism of Corollary 4.1.8. Let $\psi_n = \phi_n \circ p_n^*$. We denote by $\mathcal{T}_\theta^{\mathscr{S}} := \varinjlim(\mathcal{T}(E_{\theta_n}), \psi_n)$, called the ***Toeplitz noncommutative solenoid****.*

This will be of particular importance in Chapter 6, in the context of KMS-states.

$$\begin{tikzcd}
\mathcal{T}(E_{\theta_0}) \arrow[r] \arrow[rrrrr, bend left, "\psi_{0,\infty}"] & \cdots \arrow[r] & \mathcal{T}(E_{\theta_n}) \arrow[r, "\psi_n"'] \arrow[dr, "p_n^*"'] \arrow[rrr, bend left, "\psi_{n,\infty}"] & \mathcal{T}\big(E_{\theta_{n+1}}\big) \arrow[r] \arrow[rr, bend left, "\psi_{n+1,\infty}"] & \cdots & \mathcal{T}_\theta^{\mathscr{S}} \\
 & & & \mathcal{T}\big(E^{(N)}_{\theta_{n+1}}\big) \arrow[u, "\phi_n"'] & &
\end{tikzcd}$$

Remark 4.1.13. *The maps $\psi_j = \phi_j \circ p_j^*$ of Definition 4.1.12 are injective, since ϕ_j is injective by Corollary 4.1.8, and p_j^* is injective by Corollary 3.2.6.*

Remark 4.1.14. *Let $\iota_{C_0(E^0_{\theta_j})}(f) \in J_{X(E_{\theta_j})}$. Then*

$$\widetilde{\psi_j} \circ q_j(\iota_{C_0(E^0_{\theta_j})}(f)) = \widetilde{\psi_j}(j_{C_0(E^0_{\theta_j})}(f))$$

$$
\begin{aligned}
&= j_{C_0(E^0_{\theta_j})}(f \circ p) \\
&= q_{j+1}(\iota_{C_0(E^0_{\theta_j})}(f \circ p)) \\
&= q_{j+1} \circ \psi_j(\iota_{C_0(E^0_{\theta_j})}(f)).
\end{aligned}
$$

Combining this with Proposition 4.1.10 gives the following commuting diagram.

$$
\begin{tikzcd}
J_{X(E_{\theta_j})} \arrow[r, "\psi_j"] \arrow[d, "\iota"'] & J_{X(E_{\theta_{j+1}})} \arrow[d, "\iota"] \\
\mathcal{T}(E_{\theta_j}) \arrow[r, "\psi_j"] \arrow[d, "q_j"'] & \mathcal{T}(E_{\theta_{j+1}}) \arrow[d, "q_{j+1}"] \\
C^*(E_{\theta_j}) \arrow[r, "\widetilde{\psi_j}"'] & C^*(E_{\theta_{j+1}})
\end{tikzcd}
$$

Hence $\mathcal{A}_\theta^{\mathscr{S}}$ is a proper quotient of $\mathcal{T}_\theta^{\mathscr{S}}$.

Chapter 5

The KMS-States of Compact Topological Graph C^*-Algebras

5.1 The KMS states of a Topological Graph algebra

In this section we investigate the KMS states of topological graph C^*-algebras, and the inverse temperatures β for which they exist. We are motivated by applications to KMS states of noncommutative solenoids in the next chapter. It is for this reason we will occasionally forsake generality for practical and concrete examples.

To begin, we need an appropriate dynamics over either $C^*(E)$ and $\mathcal{T}(E)$. The obvious candidate is the gauge action, since $(\mathcal{T}(E), \mathbb{T}, \gamma)$ is a C^*-dynamical system, in the sense of [54]. However, this is not a dynamics over $\mathbb{R}$, so we require a slight modification.

Notation 5.1.1. *Let E be a topological graph, and let $\gamma : \mathbb{T} \to \mathrm{Aut}(\mathcal{T}(E))$ be the gauge action described in 2.2.15. Then define $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}(E))$ by $\alpha_t = \gamma_{e^{it}}$ for $t \in \mathbb{R}$. It should be clear that $\alpha_t \circ \alpha_s = \alpha_{t+s}$ from the definition. Further,*

given that $t \mapsto e^{it}$ is continuous, then α is continuous, as it is the composition of continuous functions.

We then obtain a dynamics α on $\mathcal{T}(E)$. Similarly, we obtain a dynamics also denoted by α on $C^*(E)$.

To get some KMS states from these systems though, we need a set of elements of $\mathcal{T}(E)$ that span a dense subspace of $\mathcal{T}(E)$ and that are α-analytic. Given Example 2.1.22, we have an obvious place to start looking.

Proposition 5.1.2. *Let E be a topological graph. Let $\alpha : \mathbb{R} \to \mathrm{Aut}(\mathcal{T}(E))$ such that for $t \in \mathbb{R}$, $\alpha_t = \gamma_{e^{it}}$. Then the set*

$$\big\{\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^* \mid k, l \geq 0 \text{ and } x \in X(E)^{\otimes k}, y \in X(E)^{\otimes l}\big\}$$

consists of elements that are α-analytic.

Proof. Since $\alpha_t(\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*) = e^{it(m-n)}\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*$, and $z \mapsto e^{iz}$ is analytic, the map $z \mapsto e^{iz(m-n)}\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*$ is a $\mathcal{T}(E)$ valued analytic function. □

We now turn our hand to an algebraic characterisation of the KMS condition for topological graph algebras. We will use this throughout the chapter.

This theorem also appears in a more general form in [1, Proposition 3.1]. It was proved independently by both parties.

Theorem 5.1.3. *Let E be a topological graph, $\phi \in \mathcal{S}(\mathcal{T}(E))$, and fix $\beta > 0$. Then $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ if and only if*

$$\phi\big(\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*\big) = \delta_{m,n}e^{-\beta m}\phi\big(\iota_{C_0(E^0)}(\langle y, x\rangle)\big) \tag{5.1}$$

for $x \in X(E)^{\otimes m}, y \in X(E)^{\otimes n}$ such that $m, n \geq 0$.

Proof. First, suppose that $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$. Then ϕ is invariant under α by Lemma 2.3.5. So, for $x \in X(E)^{\otimes m}, y \in X(E)^{\otimes n}$,

$$\begin{aligned}\phi(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*) &= \int_{\mathbb{T}} \phi(\alpha_z(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*))\,\mathrm{d}z \\ &= \int_{\mathbb{T}} z^{m-n}\phi(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*)\,\mathrm{d}z \\ &= \delta_{m,n}\phi(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*).\end{aligned}$$

Combining this with the KMS condition (2.12) implies that

$$\begin{aligned}&\phi\big(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)\big) \\ &\qquad = \delta_{m,n}e^{-\beta m}\phi\big(\iota^{\otimes n}_{X(E)}(y)^*\iota^{\otimes m}_{X(E)}(x)\big) = \delta_{m,n}e^{-\beta m}\phi\big(\iota_{C_0(E^0)}(\langle y, x\rangle)\big)\end{aligned}$$

as claimed.

Now suppose that ϕ satisfies (5.1). We must show that ϕ satisfies

$$\begin{aligned}&\phi\big(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*\iota^{\otimes j}_{X(E)}(a)\iota^{\otimes k}_{X(E)}(b)^*\big) \\ &\qquad = e^{-\beta(m-n)}\phi\big(\iota^{\otimes j}_{X(E)}(a)\iota^{\otimes k}_{X(E)}(b)^*\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*\big)\end{aligned} \tag{5.2}$$

for $x \in X(E)^{\otimes m}, y \in X(E)^{\otimes n}, a \in X(E)^{\otimes j}$ and $b \in X(E)^{\otimes k}$. By Example 2.1.22, we have

$$\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*\iota^{\otimes j}_{X(E)}(a)\iota^{\otimes k}_{X(E)}(b)^* = \iota^{\otimes p}_{X(E)}(w)\iota^{\otimes q}_{X(E)}(z)^*$$

where $p - q = m - n + j - k$. So, both sides of (5.2) are 0 unless $m - n = k - j$, so we will assume this for the remainder of the proof.

Since $\phi \in \mathcal{S}(\mathcal{T}(E))$, we have $\phi(c) = \overline{\phi(c^*)}$ for all $c \in \mathcal{T}(E)$. We see that (5.2) is equivalent to

$$\begin{aligned}&\phi\big(\iota^{\otimes k}_{X(E)}(b)\iota^{\otimes j}_{X(E)}(a)^*\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*\big) \\ &\qquad = e^{-\beta(k-j)}\phi\big(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*\iota^{\otimes k}_{X(E)}(b)\iota^{\otimes j}_{X(E)}(a)^*\big).\end{aligned}$$

The case where $k \leq m$ (which implies $j \leq n$) is then equivalent to the case where $m \leq k$ (which implies $n \leq j$). It therefore suffices to consider the case where $m \leq k$, and hence $n \leq j$.

Since each $X(E)^{\otimes l} = \overline{\operatorname{span}}\{x_1 \otimes \cdots \otimes x_l : x_i \in X(E)\}$, and $n-j, k-m \geq 0$, we have $a = a_1 \otimes a_2 \in X(E)^{\otimes n} \otimes X(E)^{\otimes n-j}$, and $b = b_1 \otimes b_2 \in X(E)^{\otimes m} \otimes X(E)^{\otimes k-m}$. Then

$$\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*\iota_{X(E)}^{\otimes j}(a)\iota_{X(E)}^{\otimes k}(b)^* = \iota_{X(E)}^{\otimes m+j-n}(x \otimes \langle y, a_1\rangle \cdot a_2)\iota_{X(E)}^{\otimes k}(b)^* \tag{5.3}$$

Similarly,

$$\iota_{X(E)}^{\otimes j}(a)\iota_{X(E)}^{\otimes k}(b)^*\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^* = \iota_{X(E)}^{\otimes j}(a)\iota_{X(E)}^{\otimes n-m+k}(y \cdot \langle x, b_1\rangle \otimes b_2)^* \tag{5.4}$$

In the following calculations we will make use of the identity

$$\langle w_1 \otimes w_2, z_1 \otimes z_2\rangle = \langle w_2, \langle w_1, z_1\rangle \cdot z_2\rangle \tag{5.5}$$

for $w_1, z_1 \in X(E)^{\otimes s}$, $w_2, z_2 \in X(E)^{\otimes t}$, and $s, t \geq 0$. We have

$$\begin{aligned}
&\phi\big(\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*\iota_{X(E)}^{\otimes j}(a)\iota_{X(E)}^{\otimes k}(b)^*\big)\\
&\qquad = \phi\big(\iota_{X(E)}^{\otimes m+j-n}(x \otimes \langle y, a_1\rangle \cdot a_2)\iota_{X(E)}^{\otimes k}(b)^*\big) \text{ by (5.3)}\\
&\qquad = e^{-\beta k}\phi\big(\iota_{C_0(E^0)}(\langle b, x \otimes \langle y, a_1\rangle \cdot a_2\rangle)\big)\\
&\qquad = e^{-\beta k}\phi\big(\iota_{C_0(E^0)}(\langle b_1 \otimes b_2, x \otimes \langle y, a_1\rangle \cdot a_2\rangle)\big)\\
&\qquad = e^{-\beta k}\phi\big(\iota_{C_0(E^0)}(\langle b_2, \langle b_1, x\rangle \cdot \big(\langle y, a_1\rangle \cdot a_2\rangle\big)\big)\big) \text{ by (5.5)}\\
&\qquad = e^{-\beta k}\phi\big(\iota_{C_0(E^0)}(\langle\langle x, b_1\rangle \cdot b_2, \langle y, a_1\rangle \cdot a_2\rangle)\big)\\
&\qquad = e^{-\beta k}\phi\big(\iota_{C_0(E^0)}(\langle y \otimes \langle x, b_1\rangle \cdot b_2, a_1 \otimes a_2\rangle)\big) \text{ by (5.5)}\\
&\qquad = e^{-\beta m}\phi\big(\iota_{C_0(E^0)}(\langle y \otimes \langle x, b_1\rangle \cdot b_2, a\rangle)\big)\\
&\qquad = e^{-\beta(m-n)}\phi\big(\iota_{X(E)}^{\otimes j}(a)\iota_{X(E)}^{\otimes n+k-m}(y \otimes \langle x, b_1\rangle \cdot b_2)\big)\\
&\qquad = e^{-\beta(m-n)}\phi\big(\iota_{X(E)}^{\otimes j}(a)\iota_{X(E)}^{\otimes k}(b)^*\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*\big) \text{ by (5.4)},
\end{aligned}$$

which is (5.2). This completes the proof. $\square$

Calculating these KMS states is actually quite difficult in general, so we will restrict ourselves to a nice, well behaved class of examples — the set of compact topological graphs (in the sense of Definition 2.2.6) — with the intention of better understanding the noncommutative solenoid.

Proposition 5.1.4. *Let E be a compact topological graph. For $f \in C(E^0)$, there exists a function $\varsigma(f) : E^0 \to \mathbb{C}$ such that*

$$\varsigma(f)(v) := \sum_{e \in E^1 v} f \circ r(e). \tag{5.6}$$

For all $f \in C(E^0)$, $\varsigma(f) \in C(E^0)$.

Moreover, if $f \in C(E^0)^+$, there exist a finite set I, s-sections $U_i, i \in I$ and functions $g_i^f \in C_0(U_i) \subseteq C(E^1)$ such that

$$\sum_{i \in I} \langle g_i^f, g_i^f \rangle = \varsigma(f). \tag{5.7}$$

For $f \in C(E^0)^+ \setminus \{0\}$, we have

$$\iota_{C_0(E^0)}(f) > \sum_{i \in I} \iota_{X(E)}(g_i^f) \iota_{X(E)}(g_i^f)^*. \tag{5.8}$$

Proof. Since s is a local homeomorphism and E^1 is compact, each $E^1 v$ is finite, and so (5.7) defines a function $\varsigma(f) : E^0 \to \mathbb{C}$. Since the inner-product in $X(E)$ takes values in $C(E^0)$, Equation (5.8) will guarantee that $\varsigma(f) \in C(E^0)$ whenever f is positive, and it will follow that $\varsigma(f) \in C(E^0)$ for all $f \in C(E^1)$, since $C(E^0)$ is spanned by positive elements, and the formula $\varsigma(f)$ is linear in f.

Fix $f \in C(E^0)^+$. Since E is compact, r is proper, so $r^{-1}(\mathrm{supp}(f))$ is compact. Choose a finite set I such that for $i \in I$ there exists an open s-section U_i such that $r^{-1}(\mathrm{supp}(f)) \subseteq \bigcup_{i \in I} U_i$. Let $\{\xi_i\}_{i \in I}$ be a partition of unity such that ξ_i is based in U_i. Then, for $e \in E^1$, define

$$g_i^f(e) := \sqrt{f \circ r(e) \xi_i(e)}.$$

For $v \in E^0$,

$$\sum_{i \in I} \langle g_i^f, g_i^f \rangle(v) = \sum_{i \in I} \sum_{e \in E^1 v} f \circ r(e) \xi_i(e) = \sum_{e \in E^1} f \circ r(e) = \varsigma(f)(v).$$

Hence (5.7) holds, and so $\varsigma(f) \in C(E^0)$ for all $f \in C(E^0)$.

To show (5.8), it suffices to find a Toeplitz representation (ψ, π) of $X(E)$ such that for all $f \in C(E^0)^+$,

$$\pi(f) - \sum_{i \in I} \psi(g_i^f)\psi(g_i^f)^* > 0.$$

Let (ψ, π) be the Toeplitz representation of Lemma 4.1.7. Let $f \in C(E^0)^+$. Then, for $\mu \in E^*$ such that $\mu = e\mu'$ for some $e \in E^1$, $\mu' \in E^*$,

$$\begin{aligned}
\Big(\pi(f) - \sum_{i \in I} \psi(g_i^f)\psi(g_i^f)^*\Big)\delta_\mu &= f(r(\mu))\delta_\mu - \sum_{i \in I} \overline{g_i^f(e)}\psi(g_i^f)\delta_{\mu'} \\
&= f(r(\mu))\delta_\mu - \sum_{i \in I} \sum_{e' \in E^1 r(\mu')} \overline{g_i^f(e)} g_i^f(e')\psi(g_i^f)\delta_{e'\mu'} \\
&= f(r(\mu))\delta_\mu - \sum_{i \in I} \sum_{e' \in E^1 r(\mu')} |g_i^f(e)|^2 \delta_{e\mu'} \\
&= 0,
\end{aligned}$$

but for $v \in E^0$,

$$\Big(\pi(f) - \sum_{i \in I} \psi(g_i^f)\psi(g_i^f)^*\Big)\delta_v = f(r(\mu))\delta_v - 0 = f(r(\mu))\delta_v.$$

So, for $h \in \ell^2(E^*)$, we have

$$\Big(\big(\pi(f) - \sum_{i \in I} \psi\big(g_i^f\big)\psi\big(g_i^f\big)^*\big)h \mid h\Big) = \sum_{v \in E^0} f(v)|h_v|^2,$$

giving $\pi(f) \geq \sum_{i \in I} \psi\big(g_i^f\big)\psi\big(g_i^f\big)^*$, with equality only if $f \equiv 0$. $\square$

Denote the set of regular unsigned Borel measures of E^0 by $\mathcal{M}(E)$. We use the notation

$$\mathcal{M}^1(X) = \{m \in \mathcal{M}(X) : m(X) = 1\}.$$

Let $A_E : \mathcal{M}(E) \to \mathcal{M}(E)$ be the measure transformation defined by

$$A_E(m)(U) = \int_{E^0} \varsigma(\chi_U)\, \mathrm{d}m$$

for all Borel sets $U \subseteq E^0$.

We recall a theorem that we will make use of throughout the remaining chapters.

Theorem 5.1.5 ([50, Theorem 6.19])**.** *Let X be a locally compact Hausdorff space. Then for any $\phi \in \mathcal{S}(C_0(E^0))$, there exists a unique measure $m^\phi \in \mathcal{M}^1(X)$ such that*

$$\phi(f) = \int_X f \,\mathrm{d}m^\phi$$

for all $f \in C_0(X)$.

Given Theorem 5.1.5, if E is a compact topological graph, and $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$, then $\phi \circ \iota_{C(E^0)} \in \mathcal{S}(C(E^0))$, and so there exists $m^\phi \in \mathcal{M}^1(E^0)$ such that

$$\phi \circ \iota_{C(E^0)}(f) = \int_{E^0} f \,\mathrm{d}m^\phi \tag{5.9}$$

for all $f \in C(E^0)$.

Theorem 5.1.6. *Let E be a compact topological graph, and $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ for some $\beta \in \mathbb{R}$. Then the measure m^ϕ associated to ϕ as in* (5.9) *satisfies*

$$\int_{E^0} f \,\mathrm{d}(A_E m^\phi) \leq e^\beta \int_{E^0} f \,\mathrm{d}m^\phi \tag{5.10}$$

for all $f \in C(E^0)^+$.

Proof. Fix $f \in C(E^0)^+$. By Theorem 5.1.5, we have

$$\begin{aligned}
\int_{E^0} f \,\mathrm{d}m^\phi &= \phi(\iota_{C(E^0)}(f)) \\
&\geq \phi\Big(\sum_{i \in I} \iota_{X(E)}(g_i^f)\iota_{X(E)}(g_i^f)^*\Big) \text{ by (5.8),} \\
&= \sum_{i \in I} e^{-\beta}\phi\Big(\iota_{C(E^0)}(\langle g_i^f, g_i^f\rangle)\Big) \\
&= e^{-\beta}\phi(\varsigma(f)) \\
&= e^{-\beta}\int_{E^0} \varsigma(f)\,\mathrm{d}m^\phi
\end{aligned}$$

$$= e^{-\beta} \int_{E^0} f \, \mathrm{d}(A_E m^\phi)$$

as claimed. $\square$

We call condition (5.10) **subinvariance**. It is a recurring theme in the theory of KMS states and the Toeplitz C^*-algebras of dynamical systems, (see [34, Theorem 2.1]). When we take a quotient to obtain a Cuntz-Pimsner algebra, we obtain something much more restrictive (similarly, see [34, Theorem 2.5]). We call this condition **invariance** (5.11).

Theorem 5.1.7. *Let E be a compact topological graph. A state $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ factors through to $\mathcal{S}_{\alpha,\beta}(C^*(E))$ if and only if*

$$\int_{E^0} f \, \mathrm{d}(A_E m^\phi) = e^\beta \int_{E^0} f \, \mathrm{d}m^\phi \tag{5.11}$$

for all $f \in C(E^0_{rg})^+$.

Proof. We use the notation implemented in the proof of Proposition 5.1.4. For any $f \in C(E^0_{\mathrm{rg}})^+$, each g_i^f is supported on an s-section, and for $e \in E^0$,

$$\sum_{i\in I} \big(g_i^f\big)^2(e) = \sum_{i\in I} \Big(\sqrt{f \circ r(e)\xi_i(e)}\Big)^2 = f \circ r(e).$$

Hence, by Theorem 2.2.21, a state $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ factors through $C^*(E)$ if and only if

$$\phi\Big(\iota_{C(E^0)}(f) - \sum_{i\in I} \iota_{X(E)}(g_i^f)\iota_{X(E)}(g_i^f)^*\Big) = 0 \tag{5.12}$$

for all $f \in C(E^0_{\mathrm{rg}})^+$. We show that for all $f \in C(E^0_{\mathrm{rg}})^+$, (5.12) holds if and only if $(A_E m^\phi)(U) = e^\beta m^\phi(U)$ for all Borel $U \subseteq E^0_{\mathrm{rg}}$. Fix $f \in C(E^0_{\mathrm{rg}})^+$, and $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$. Then

$$\begin{aligned}\phi\Big(&\iota_{C(E^0)}(f) - \sum_{i\in I} \iota_{X(E)}(g_i^f)\iota_{X(E)}(g_i^f)^*\Big)\\ &= \phi(\iota_{C(E^0)}(f)) - \sum_{i\in I} \phi\big(\iota_{X(E)}(g_i^f)\iota_{X(E)}(g_i^f)^*\big)\end{aligned}$$

$$
\begin{aligned}
&= \phi(\iota_{C(E^0)}(f)) - e^{-\beta} \sum_{i\in I} \phi\big(\iota_{C(E^0)}(\langle g_i^f, g_i^f\rangle)\big) \\
&= \phi(\iota_{C(E^0)}(f)) - e^{-\beta} \phi\big(\iota_{C(E^0)}(\varsigma(f))\big) \\
&= \int_{E^0} f \,\mathrm{d}m^\phi - e^{-\beta} \int_{E^0} \varsigma(f)\,\mathrm{d}m^\phi \\
&= \int_{E^0} f \,\mathrm{d}m^\phi - e^{-\beta} \int_{E^0} f \,\mathrm{d}(A_E m^\phi).
\end{aligned}
$$

Hence $\phi\Big(\iota_{C(E^0)}(f) - \sum_{i\in I} \iota_{X(E)}(g_i^f)\iota_{X(E)}(g_i^f)^*\Big) = 0$ for all $f \in C(E^0_{\mathrm{rg}})$ if and only if $e^\beta m^\phi(U) = A_E m^\phi(U)$ for all Borel $U \subseteq E^0_{\mathrm{rg}}$. $\square$

Remark 5.1.8. *We will be interested in computing the operator $(1 - e^{-\beta} A_E)^{-1}$ for suitable values of β, so we need to calculate the spectral radius of A_E, which we will denote $\rho(A_E)$. This will tell us for which values of β the set $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ is nonempty. Since $\rho(A_E) = \lim_{n\to\infty} \|A_E^n\|^{\frac{1}{n}}$, we start by calculating $\|A_E\|$. We have*

$$
\begin{aligned}
\|A_E\| &= \sup_{\|m\|=1} \|A_E m\| \\
&= \sup_{m\in\mathcal{M}^1(E^0)} \Big| \int_{E^0} \varsigma(\chi_{E^0})\,\mathrm{d}m \Big| \\
&= \sup_{m\in\mathcal{M}^1(E^0)} \Big| \int_{E^0} \sum_{e\in E^1 v} \chi_{E^0}(r(e))\,\mathrm{d}m(v) \Big| \\
&= \sup_{m\in\mathcal{M}^1(E^0)} \Big| \int_{E^0} \big|E^1 v\big| \,\mathrm{d}m(v) \Big|.
\end{aligned}
$$

Since E^0 is compact, there exists $v_0 \in E^0$ such that $|s^{-1}(v_0)| \geq |s^{-1}(v)|$ for all $v \in E^0$. Observe that for all $m \in \mathcal{M}^1(E^0)$,

$$
\Big| \int_{E^0} \big|E^1 v\big| \,\mathrm{d}m(v) \Big| \leq \Big| \int_{E^0} \big|E^1 v_0\big| \,\mathrm{d}m \Big| \leq \big|E^1 v_0\big| = \Big| \int_{E^0} \big|E^1 v\big| \,\mathrm{d}\delta_{v_0}(v) \Big|.
$$

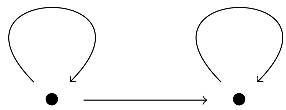
Since $\delta_{v_0} \in \mathcal{M}^1(E^0)$, we have

$$
\|A_E\| = \max_{v\in E^0} \big|E^1 v\big| = \max_{v\in E^0} |s^{-1}(v)|.
$$

Hence each $\|A_E^n\| = \max_{v\in E^0} |s^{-n}(v)|$, and so

$$
\rho(A_E) = \lim_{n\to\infty} \max_{v\in E^0} |s^{-n}(v)|^{\frac{1}{n}}.
$$

Example 5.1.9. Consider the following graph.



Calculating the spectral radius from the vertex matrix gives us $\rho(A_E) = 1$. The formula given in Remark 5.1.8 gives us

$$\rho(A_E) = \lim_{n\to\infty} (n+1)^{\frac{1}{n}} = 1,$$

as expected.

Theorem 5.1.10. *Let E be a compact topological graph, and fix $\beta > \ln(\rho(A_E))$. For each $\epsilon \in \mathcal{M}(E^0)$ satisfying*

$$\int_{E^0} \sum_{\mu\in E^*v} e^{-\beta|\mu|} \, \mathrm{d}\epsilon(v) = 1, \tag{5.13}$$

there exists $\phi_\epsilon \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ such that for $x \in X(E)^{\otimes k}$, $y \in X(E)^{\otimes l}$

$$\phi_\epsilon(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^*) = \delta_{k,l} e^{-\beta k} \int_{E^0} \sum_{\mu\in E^*v} e^{-\beta|\mu|} \langle y, x\rangle(r(\mu)) \, \mathrm{d}\epsilon(v). \tag{5.14}$$

Proof. Again, we let (ψ, π) be the Toeplitz representation of $X(E)$ onto $\mathcal{B}(\ell^2(E^*))$ used in Lemma 4.1.7. Let $\epsilon \in \mathcal{M}(E^0)$ satisfy (5.14). For each $x, y \in X(E)^{\otimes k}$, the map $v \mapsto \sum_{\mu\in E^*v} e^{-\beta|\mu|}\langle y, x\rangle(r(\mu))$ is continuous and bounded by the convergent sum $\sum_{\mu\in E^*v} e^{-\beta|\mu|}\|\langle y, x\rangle\|$, and hence integrable. For $a \in \operatorname{span}\{\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^*\}$ the map

$$v \mapsto \sum_{\mu\in E^*v} e^{-\beta|\mu|}\big((\psi\times\pi)(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes k}(y)^*)\delta_\mu|\delta_\mu\big) \tag{5.15}$$

is then continuous, and therefore integrable. The map (5.15) is also bounded. When $k \neq l$, we have

$$\big((\psi\times\pi)(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes k}(y)^*)\delta_\mu|\delta_\mu\big) = 0.$$

In the case where $k = l$,

$$\begin{aligned}
\Big| \int_{E^0} &\sum_{\mu \in E^* v} e^{-\beta|\mu|} \big((\psi \times \pi)(\iota^{\otimes k}_{X(E)}(x) \iota^{\otimes k}_{X(E)}(y)^*) \delta_\mu | \delta_\mu \big) \, \mathrm{d}\epsilon(v) \Big| \\
&\leq \int_{E^0} \Big| \sum_{\mu \in E^* v} e^{-\beta|\mu|} \big((\psi \times \pi)(\iota^{\otimes k}_{X(E)}(x) \iota^{\otimes k}_{X(E)}(y)^*) \delta_\mu | \delta_\mu \big) \Big| \, \mathrm{d}\epsilon(v) \\
&\leq \int_{E^0} \sum_{\mu \in E^* v} \|\langle y, x \rangle\| \mathrm{d}\epsilon(v) \\
&= \|\langle y, x \rangle\|.
\end{aligned}$$

For $a \in \mathcal{T}(E)$, $\varphi(a)$ is the pointwise limit of integrable functions, and therefore integrable, and $|\varphi(a)| \leq \|a\|$.

All that remains is to show that φ satisfies Equation (5.1). First,

$$\varphi(\iota_{C(E^0)} \langle y, x \rangle) = \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \langle y, x \rangle (r(\mu)) \, \mathrm{d}\epsilon(v).$$

Now,

$$\begin{aligned}
\varphi(\iota^{\otimes k}_{X(E)}(x) \iota^{\otimes k}_{x(E)}(y)^*) &= \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \left(\psi^{\otimes m}(x) \psi^{\otimes m}(y)^* \delta_\mu | \delta_\mu \right) \, \mathrm{d}\epsilon(v) \\
&= \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \left(\psi^{\otimes k}(y)^* \delta_\mu | \psi^{\otimes k}(x)^* \delta_\mu \right) \, \mathrm{d}\epsilon(v) \\
&= \int_{E^0} \sum_{\substack{\mu \in E^* v \\ \mu' \in E^k r(\mu)}} e^{-\beta|\mu' \mu|} \overline{y(\mu')} x(\mu') \, \mathrm{d}\epsilon(v) \\
&= e^{-\beta k} \int_{E^0} \sum_{\substack{\mu \in E^* v \\ \mu' \in E^k r(\mu)}} e^{-\beta|\mu|} \overline{y(\mu')} x(\mu') \, \mathrm{d}\epsilon(v) \\
&= e^{-\beta k} \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \langle y, x \rangle (r(\mu)) \, \mathrm{d}\epsilon(v) \\
&= e^{-\beta k} \varphi(\iota_{C(E^0)}(\langle y, x \rangle)),
\end{aligned}$$

completing the proof. $\square$

Given a compact topological graph, for $v \in E^0$, we use the notation

$$\phi_v := \phi_{(\sum_{\mu \in E^* v} e^{-\beta|\mu|})^{-1}\delta_v} \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E)). \tag{5.16}$$

That is, ϕ_v is the unique KMS$_\beta$ state arising from the Dirac measure based at v, and satisfying the hypothesis of Theorem 5.1.10.

Theorem 5.1.11. *Let E be a compact topological graph, and fix $\beta > \ln(\rho(A_E))$. Then the map $\Omega \mapsto \varphi_\Omega$ where*

$$\varphi_\Omega(a) = \int_{E^0} \phi_v(a) \,\mathrm{d}\Omega(v)$$

for all $a \in \mathcal{T}(E)$ is an affine isomorphism of $\mathcal{M}^1(E^0)$ onto $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$.

Proof. Fix $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$, and denote by m^ϕ the probability measure on E^0 associated to ϕ, so that

$$\phi(\iota_{C(E^0)}(f)) = \int_{E^0} f \,\mathrm{d}m^\phi$$

for all $f \in C(E^0)$. Denote $\epsilon := (1 - e^{-\beta} A_E)m^\phi$, and let $U \subseteq E^0$ be a Borel set. Then

$$(1 - e^{-\beta} A_E)^{-1}\epsilon(U) = (1 - e^{-\beta} A_E)^{-1}(1 - e^{-\beta} A_E)m^\phi(U) = m^\phi(U).$$

It follows that for $x \in X(E)^{\otimes m}, y \in X(E)^{\otimes n}$,

$$\begin{aligned}
\phi_\epsilon(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*) &= \delta_{m,n} e^{-\beta m} \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \langle y, x\rangle(r(\mu)) \,\mathrm{d}\epsilon(v) \\
&= \delta_{m,n} e^{-\beta m} \int_{E^0} \langle y, x\rangle \,\mathrm{d}m^\phi \\
&= \delta_{m,n} e^{-\beta m} \phi(\iota_{C(E^0)}(\langle y, x\rangle)) \\
&= \phi(\iota^{\otimes m}_{X(E)}(x)\iota^{\otimes n}_{X(E)}(y)^*) \text{ by Equation (5.1).}
\end{aligned}$$

Since ϕ and ϕ_ϵ agree on a dense subset of $\mathcal{T}(E)$, they coincide by continuity.

Equation (5.14) implies that $\epsilon \mapsto \phi_\epsilon$ is injective, and continuous with respect to the weak* topology. Observe that $\{\epsilon \in \mathcal{M}(E^0) \mid \epsilon \text{ satisfies (5.14)}\}$ is a closed

subset of a compact set, and therefore compact. Since the topology on the state space is Hausdorff, the map $\epsilon \mapsto \phi_\epsilon$ is a homeomorphism.

It now suffices to show that the map $\Omega \mapsto \varphi_\Omega \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ is surjective, since it is clearly continuous and injective. Fix $\epsilon \in \mathcal{M}(E^0)$ satisfying Equation (5.13), and denote the KMS$_\beta$ state associated to ϵ be ϕ_ϵ. Define $\Omega \in \mathcal{M}^1(E^0)$ by

$$\Omega(U) = \int_U \sum_{\mu \in E^* v} e^{-\beta|\mu|} \, \mathrm{d}\epsilon(v).$$

Then for $x \in X(E)^{\otimes m}, y \in X(E)^{\otimes n}$ we have

$$\begin{aligned}
\varphi_\Omega(\iota_{X(E)}^{\otimes m}(x)\iota_{x(E)}^{\otimes n}(y)^*) &= \int_{E^0} \delta_{m,n} e^{-\beta m} \phi_v(\iota_{C(E^0)}(\langle y, x\rangle)) \, \mathrm{d}\Omega(v) \\
&= \delta_{m,n} e^{-\beta m} \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \langle y, x\rangle(r(\mu)) \Big(\sum_{\nu \in E^* v} e^{-\beta|\nu|} \Big)^{-1} \mathrm{d}\Omega(v) \\
&= \delta_{m,n} e^{-\beta m} \int_{E^0} \sum_{\mu \in E^* v} e^{-\beta|\mu|} \langle y, x\rangle(r(\mu)) \, \mathrm{d}\epsilon(v) \\
&= \phi_\epsilon(\iota_{X(E)}^{\otimes m}(x)\iota_{X(E)}^{\otimes n}(y)^*),
\end{aligned}$$

so $\Omega \mapsto \varphi_\Omega$ is surjective onto $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$, and hence a homeomorphism. □

Proposition 5.1.12. *For a compact topological graph E, $\mathcal{S}_{\alpha,\ln(\rho(A_E))}(\mathcal{T}(E)) \neq \emptyset$. For each $\phi \in \mathcal{S}_{\alpha,\ln(\rho(A_E))}(\mathcal{T}(E))$, the measure m^ϕ associated to ϕ satisfies*

$$\int_{E^0} f \, \mathrm{d}(A_E m^\phi) \leq \rho(A_E) \int_{E^0} f \, \mathrm{d}m^\phi$$

for all $f \in C(E^0)^+$.

Proof. Fix a sequence $\{\beta_n\}$ converging to $\ln(\rho(A_E))$ from above. Theorem 5.1.10 implies that there exists $\phi_n \in \mathcal{S}_{\alpha,\beta_n}(\mathcal{T}(E))$ for each $n \in \mathbb{N}$. Since $\mathcal{S}(\mathcal{T}(E))$ is compact, there exists a subsequence converging to a state, say $\phi_{n_j} \to \phi$ as $j \to \infty$. For $x \in X(E)^{\otimes k}, y \in X(E)^{\otimes l}$, we have

$$\phi(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^*) = \lim_{j\to\infty} \phi_{n_j}(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^*)$$

$$= \lim_{j\to\infty} \delta_{k,l} e^{-\beta_{n_j} k} \phi_{n_j}(\iota_{C(E^0)}(\langle y, x\rangle))$$
$$= \delta_{k,l}\rho(A_E)^{-k}\phi(\iota_{C(E^0)}(\langle y, x\rangle))$$

implying $\phi \in \mathcal{S}_{\alpha,\ln(\rho(A_E))}(\mathcal{T}(E))$ by Theorem 5.1.3. The final assertion follows from Theorem 5.1.6. $\square$

Theorem 5.1.13. *Let E be the topological graph $E = (X, X, h, id)$, where X is a compact Hausdorff, and $h : X \to X$ is a homeomorphism. Then $\mathcal{S}_{\alpha,0}(\mathcal{T}(E)) \neq \emptyset$, and for $\phi \in \mathcal{S}_{\alpha,0}(\mathcal{T}(E))$, the measure associated to ϕ is invariant for h.*

Proof. We have $\rho(A_E) = 1$ by definition. So Propostition 5.1.12 shows that $\mathcal{S}_{\alpha,0}(\mathcal{T}(E)) \neq \emptyset$. Fix $\phi \in \mathcal{S}_{\alpha,0}(\mathcal{T}(E))$, and let $m^\phi \in \mathcal{M}^1(E^0)$ be the associated probability measure. Then, Proposition 5.1.12 shows that for Borel $U \subseteq E^0$,

$$\int_X \varsigma(\chi_U)\,\mathrm{d}m \leq e^{\ln 1}\int_X \chi_U\,\mathrm{d}m = m(U),$$

and hence

$$m(U) \geq \int_X \varsigma(\chi_U)\mathrm{d}m = m(s(r^{-1}(U))) = m(h^{-1}(U)).$$

Let $\{U_i\}_{i\in\mathbb{N}}$ be a covering of X by disjoint Borel sets. Then

$$\begin{aligned} 1 = m(E^0) &= m\Big(\bigcup_{i\in\mathbb{N}} U_i\Big) \\ &= \sum_{i\in\mathbb{N}} m(U_i) \\ &\geq \sum_{i\in\mathbb{N}} m(h^{-1}(U_i)) \\ &= m(h^{-1}(E^0)) = 1, \end{aligned}$$

which forces equality throughout the calculation. Further, since $m(h^{-1}(U_i)) \leq m(U_i)$ for each i, this forces equality for each i. That is, m is invariant with respect to h. $\square$

Corollary 5.1.14. *Fix $\theta \in [0,1) \setminus \mathbb{Q}$. Then $\mathcal{S}_{\alpha,0}(\mathcal{T}(E_\theta))$ consists of a single element ϕ which factors through to the unique element of $\mathcal{S}_{\alpha,0}(C^*(E_\theta))$.*

Proof. By Theorem 5.1.13, there exists $\phi \in \mathcal{S}_{\alpha,0}(\mathcal{T}(E_\theta))$, and associated to ϕ is a measure m^ϕ that is invariant with respect to the range map r_θ. Since there exists a unique rotationally invariant measure on $\mathbb{T}$, the Haar measure, there exists a unique $\phi \in \mathcal{S}_{\alpha,0}(\mathcal{T}(E_\theta))$. Observe that the Haar measure satisfies (5.11), and therefore factors through to $\mathcal{S}_{\alpha,0}(C^*(E_\theta))$. □

Proposition 5.1.15. *Fix $\theta \in [0,1)$, and let $\beta < 0$. Then $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_\theta)) = \emptyset$.*

Proof. We prove the contrapositive statement. Suppose that $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_\theta)) \neq \emptyset$, and fix $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_\theta))$. Since $1 \in C(E^0)^+$, Equation (5.10) gives

$$1 = \phi(\iota_{C(E^0)}(1 \circ r_\theta)) = \phi(\iota_{C(E^0)}(\varsigma(1))) \leq e^\beta \phi(\iota_{C(E^0)}(1)) = e^\beta.$$

So $\beta \geq 0$. □

5.2 Constructions and Examples

We begin by testing our formula derived in Theorem 5.1.11 against results present in the literature.

Example 5.2.1. Let E be a finite directed graph. Let A denote the vertex matrix, that is, $A(v,w) = |vE^1w|$ for $v,w \in E^0$, and $A^n(v,w) = |vE^nw|$. Fix $\mu,\nu \in E^*$, $\beta > \ln(\rho(A))$, and let $s_\mu = \iota_{X(E)}^{\otimes|\mu|}(\delta_\mu)$. Further, let $\epsilon \in \mathcal{M}(E^0)$ be such that $m^\epsilon := (1 - e^{-\beta}A_E)^{-1}\epsilon \in \mathcal{M}^1(E^0)$. For $v \in E^0$, let $\epsilon_v = \epsilon(\{v\})$ and $m_v^\epsilon := m^\epsilon(\{v\})$.

Theorem 5.1.10 gives a state $\phi_\epsilon \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ satisfying

$$\begin{aligned}\phi_\epsilon(s_\mu s_\nu^*) &= e^{-\beta|\mu|} \int_{E^0} \sum_{\lambda \in E^* v} e^{-\beta|\lambda|} \langle \delta_\nu, \delta_\mu \rangle (r(\lambda)) \, \mathrm{d}\epsilon(v) \\ &= \delta_{\mu,\nu} e^{-\beta|\mu|} \int_{E^0} \sum_{\lambda \in E^* v} e^{-\beta|\lambda|} \delta_{s(\mu)}(r(\lambda)) \, \mathrm{d}\epsilon(v)\end{aligned}$$

$$
\begin{aligned}
&= \delta_{\mu,\nu} e^{-\beta|\mu|} \sum_{v \in E^0} \sum_{\lambda \in E^* v} e^{-\beta|\lambda|} \delta_{s(\mu)}(r(\lambda)) \epsilon(\{v\}) \\
&= \delta_{\mu,\nu} e^{-\beta|\mu|} \sum_{v \in E^0} \sum_{n=0}^{\infty} e^{-\beta n} |s(\mu) E^n v| \epsilon_v \\
&= \delta_{\mu,\nu} e^{-\beta|\mu|} \sum_{v \in E^0} \sum_{n=0}^{\infty} e^{-\beta n} A^n(s(\mu), v) \epsilon_v \\
&= \delta_{\mu,\nu} e^{-\beta|\mu|} \Big(\sum_{n=0}^{\infty} e^{-\beta n} A^n \epsilon \Big)_{s(\mu)} \\
&= \delta_{\mu,\nu} e^{-\beta|\mu|} ((1 - e^{-\beta} A)^{-1} \epsilon)_{s(\mu)} \\
&= \delta_{\mu,\nu} e^{-\beta|\mu|} m^{\epsilon}_{s(\mu)},
\end{aligned}
$$

which is precisely the formula derived in [2, Theorem 3.1(b)]

Example 5.2.2. Let X be a compact topological space, and $h : X \to X$ a surjective local homeomorphism. Consider the topological graph $E = (X, X, \mathrm{id}, h)$. Then

$$
\begin{aligned}
\ln(\rho(A_E)) &= \ln \lim_{n \to \infty} \max_{v \in X} |s^{-n}(v)|^{\frac{1}{n}} \\
&= \lim_{n \to \infty} n^{-1} \ln \max_{v \in X} |s^{-n}(v)| \\
&= \limsup_{n \to \infty} n^{-1} \ln \max_{v \in X} |s^{-n}(v)|,
\end{aligned}
$$

which is precisely the number β_c of [1, Theorem 4.2]. Now fix $\beta > \beta_c$, and choose $\epsilon \in \mathcal{M}(X)$ such that $\lambda := (1 - e^{-\beta} A_E)^{-1} \epsilon \in \mathcal{M}^1(X)$. For $x \in X(E)^{\otimes k}$, $y \in X(E)^{\otimes l}$, we have

$$
\begin{aligned}
\phi_\epsilon(\iota^{\otimes k}_{X(E)}(x) \iota^{\otimes l}_{X(E)}(y)^*) &= \delta_{k,l} e^{-\beta k} \int_X \sum_{\mu \in E^* v} e^{-\beta|\mu|} \langle y, x \rangle (r(\mu)) \, \mathrm{d}\epsilon(v) \\
&= \delta_{k,l} e^{-\beta k} \int_X \langle y, x \rangle \, \mathrm{d}\Big(\sum_{n=0}^{\infty} e^{-\beta n} A_E^n \epsilon \Big) \\
&= \delta_{k,l} e^{-\beta k} \int_X \langle y, x \rangle \, \mathrm{d}((1 - e^{-\beta} A_E)^{-1} \epsilon)
\end{aligned}
$$

$$= \delta_{k,l} e^{-\beta k} \int_X \langle y, x \rangle \,\mathrm{d}\lambda,$$

which is the formula (5.1) of [1, Theorem 5.1].

The following example will provide us with a template to follow for Chapter 6. Given the topological graph E_∞ associated to a projective sequence of compact topological graphs (E_n, p_n), our aim is to examine $\partial\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_\infty))$, from the perspective of the underlying projective sequence.

Example 5.2.3. Let (E_n, p_n) be a projective sequence of compact topological graphs, with projective limit E_∞. Let $P_n : E_\infty \to E_n$ the s-injective graph morphisms described in Theorem 3.1.11. Fix $\beta > \ln(\rho(A_{E_0}))$. For $n \in \mathbb{N} \cup \{\infty\}$ and $v \in E_n^0$, we write $\phi_v^n \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_n))$ for the state corresponding to $\epsilon := (\sum_{\mu \in E_n^* v} e^{-\beta|\mu|})^{-1}\delta_v$, in Theorem 5.1.10. Fix $v \in E_\infty^0$, and $n \in \mathbb{N}$. For $x \in X(E_n)^{\otimes k}$ and $y \in X(E_n)^{\otimes l}$,

$$\begin{aligned}
&\phi_v^\infty \circ P_n^*(\iota_{X(E_n)}^{\otimes k}(x)\iota_{X(E_n)}^{\otimes l}(y)^*) \\
&\qquad = \delta_{k,l} e^{-\beta k} \phi_v^\infty(\iota_{C(E_\infty^0)}(\langle y, x\rangle \circ P_n^0)) \\
&\qquad = \delta_{k,l} e^{-\beta k} \int_{E_\infty^0} \sum_{\mu \in E_\infty^* w} e^{-\beta|\mu|} \langle y, x\rangle \circ P_n^0(r_\infty(\mu)) \,\mathrm{d}\Big(\sum_{\nu \in E_\infty^* v} e^{-\beta|\nu|}\Big)^{-1} \delta_v \\
&\qquad = \delta_{k,l} e^{-\beta k} \int_{E_\infty^0} \sum_{\mu \in E_\infty^* w} e^{-\beta|\mu|} \langle y, x\rangle \circ r_n(P_n^1(\mu)) \,\mathrm{d}\Big(\sum_{\nu \in E_\infty^* v} e^{-\beta|\nu|}\Big)^{-1} \delta_v \\
&\qquad = \delta_{k,l} e^{-\beta k} \Big(\sum_{\nu \in E_\infty^* v} e^{-\beta|\nu|}\Big)^{-1} \sum_{\mu \in E_\infty^* v} e^{-\beta|\mu|} \langle y, x\rangle \circ r_n(P_n^1(\mu)) \\
&\qquad = \delta_{k,l} e^{-\beta k} \Big(\sum_{\nu \in E_n^*(P_n^0(v))} e^{-\beta|\nu|}\Big)^{-1} \sum_{\mu \in E_n^*(P_n^0(v))} e^{-\beta|\mu|} \langle y, x\rangle (r_n(\mu)) \\
&\qquad = \delta_{k,l} e^{-\beta k} \phi_{P_n^0(v)}^n(\iota_{C(E_\infty^0)}(\langle y, x\rangle)) \\
&\qquad = \phi_{P_n^0(v)}^n(\iota_{X(E_n)}^{\otimes k}(x)\iota_{X(E_n)}^{\otimes l}(y)^*).
\end{aligned}$$

Further, for $v \in E_{n+1}^0$ and x, y as above we have

$$\phi_v^{n+1} \circ p_n^*(\iota_{X(E_n)}^{\otimes k}(x)\iota_{X(E_n)}^{\otimes l}(y)^*) = \phi_{p_n^0(v)}^n(\iota_{X(E_n)}^{\otimes k}(x)\iota_{X(E_n)}^{\otimes l}(y)^*).$$

So, the projective limit structure is such that the extreme points of the simplex $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_n))$ are sent to the extreme points of $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_{n+1}))$ and $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_\infty))$.

$$\begin{tikzcd}
\mathcal{T}(E_n) \arrow[rr, "p_n^*"] \arrow[dr, "\phi^n_{P_n^0(v)}"] \arrow[ddr, "P_n^*"'] & & \mathcal{T}(E_{n+1}) \arrow[dl, "\phi^{n+1}_{P_{n+1}^0(v)}"'] \arrow[ddl, "P_{n+1}^*"] \\
& \mathbb{C} & \\
& \mathcal{T}(E_\infty) \arrow[u, "\phi_v^\infty"'] &
\end{tikzcd}$$

Chapter 6

The KMS-States of Noncommutative Solenoids

6.1 The KMS states of Noncommutative Solenoids

We now return to the example of the noncommutative solenoid $\mathcal{A}_\theta^{\mathscr{S}}$ and its Toeplitz extension $\mathcal{T}_\theta^{\mathscr{S}}$, first presented in Chapter 4. As we saw in Chapter 5.1, the Toeplitz algebra of a topological graph has a much richer supply of KMS states than it's Cuntz-Pimsner counterpart. This would seem to indicate that $\mathcal{T}_\theta^{\mathscr{S}}$ has a much risher supply of KMS states than $\mathcal{A}_\theta^{\mathscr{S}}$. In Theorem 6.1.9 we show that under a positivity assumption, the boundary of the simplex of KMS states is homeomorphic to a solenoid. First we need some machinery, given that we have not described $\mathcal{T}_\theta^{\mathscr{S}}$ as a topological graph C^*-algebra. We will attempt to use the program outlined in Example 5.2.3, however, quickly run into some issues.

Firstly, to save the reader flipping back, we recall the notation used in Chapter 4. We will use it thoughout the remainder of the section.

Notation 6.1.1. *Fix an integer $n \geq 2$, and $\theta_0 \in [0,1)$. Fix a sequence of integers $k_j \in \{0, 1, \cdots, n^2 - 1\}$, and inductively define $\theta_{j+1} \in \mathbb{R}$ by $n^2\theta_{j+1} = \theta_j + k_j$.*

Let $\theta = (\theta_j)_{j=0}^{\infty}$. We let $p_j^0 : E^0_{\theta_{j+1}} \to E^0_{\theta_j}$ and $p_j^1 : E^n_{\theta_{j+1}} \to E^1_{\theta_{j+1}}$ be such that $p^0(z) = z^n$ and $p^1(z_1 \cdots z_n) = z_n$.

We denote by $\mathcal{T}_\theta^{\mathscr{S}}$ the direct limit $\varinjlim(\mathcal{T}(E_{\theta_j}), \psi_j)$, where $\psi_j : \mathcal{T}(E_{\theta_j}) \to \mathcal{T}(E_{\theta_{j+1}})$ is the injective homomorphism from Definition 4.1.12, such that for $a \in C(\mathbb{T})$ and $x \in X(E_{\theta_j})$,

$$\psi_j : \begin{cases} \iota_{X(E_{\theta_j})}(x) & \mapsto & \iota_{X(E_{\theta_{j+1}})}(x \circ p_j^1) \\ \iota_{C(\mathbb{T})}(a) & \mapsto & \iota_{C(\mathbb{T})}(a \circ p_j^0), \end{cases}$$

as in Chapter 4.

To study KMS states, we need an action α of $\mathbb{R}$ on $\mathcal{T}_\theta^{\mathscr{S}}$. A natural approach would be to try $\alpha_t \circ \psi_{j,\infty} = \psi_{j,\infty} \circ \gamma_{j,e^{it}}$, where γ_j is the guage action on $\mathcal{T}(E_{\theta_j})$, but this fails since $\gamma_{j+1,z} \circ \psi_j \neq \psi_j \circ \gamma_{j,z}$. However, we can modify this idea to constuct a "gauge like" action on $\mathcal{T}_\theta^{\mathscr{S}}$, in the sense that it resticts to a rescaled gauge action on each of the approximating subalgebras.

Proposition 6.1.2. *There exists a dynamics α_∞ on $\mathcal{T}_\theta^{\mathscr{S}}$ such that*

$$\alpha_{\infty,t}(\psi_{j,\infty}(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^*) = e^{\frac{it}{n^j}}(\psi_{j,\infty}(\iota_{X(E)}^{\otimes k}(x)\iota_{X(E)}^{\otimes l}(y)^*)$$

for all $x \in X(E_{\theta_j})^{\otimes k}$, $y \in X(E_{\theta_j})^{\otimes l}$.

Proof. We construct α_∞ from gauge acions. For $t \in \mathbb{R}$, define $\alpha_{j,t} \in \operatorname{Aut}(\mathcal{T}(E_{\theta_j}))$ by

$$\alpha_{t,j}\Big(\iota_{X(E_{\theta_j})}^{\otimes k}(x)\iota_{X(E_{\theta_j})}^{\otimes l}(y)^*\Big) = \gamma_{e^{\frac{it}{n^j}}}\Big(\iota_{X(E_{\theta_j})}^{\otimes k}(x)\iota_{X(E_{\theta_j})}^{\otimes l}(y)^*\Big).$$

We claim that the diagram

$$\begin{tikzcd}
\mathcal{T}(E_{\theta_1}) \arrow[r] \arrow[d, "\alpha_{1,t}"'] & \cdots \arrow[r] & \mathcal{T}(E_{\theta_j}) \arrow[r, "\psi_j"] \arrow[d, "\alpha_{j,t}"'] & \mathcal{T}(E_{\theta_{j+1}}) \arrow[r] \arrow[d, "\alpha_{j+1,t}"'] & \cdots \\
\mathcal{T}(E_{\theta_1}) \arrow[r] & \cdots \arrow[r] & \mathcal{T}(E_{\theta_j}) \arrow[r, "\psi_j"'] & \mathcal{T}(E_{\theta_{j+1}}) \arrow[r] & \cdots
\end{tikzcd}$$

commutes.

To see this, fix $j \in \mathbb{N}$. For $x \in X(E_{\theta_j})^{\otimes k}$, $y \in X(E_{\theta_j})^{\otimes l}$,

$$\begin{aligned}
\psi_j \circ \alpha_{j,t}(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*) &= \psi_j(\gamma_{e^{\frac{it}{n^j}}}\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*) \\
&= \psi_j(e^{\frac{it(k-l)}{n^j}}\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*) \\
&= e^{\frac{it(k-l)}{n^j}}\iota^{\otimes kn}_{X(E_{\theta_{j+1}})}(x \circ p^1_j)\iota^{\otimes ln}_{X(E_{\theta_{j+1}})}(y \circ p^1_j)^* \\
&= e^{\frac{it(kn-ln)}{n^{j+1}}}\iota^{\otimes kn}_{X(E_{\theta_{j+1}})}(x \circ p^1_j)\iota^{\otimes ln}_{X(E_{\theta_{j+1}})}(y \circ p^1_j)^* \\
&= \alpha_{j+1,t}\big(\iota^{\otimes kn}_{X(E_{\theta_{j+1}})}(x \circ p^1_j)\iota^{\otimes ln}_{X(E_{\theta_{j+1}})}(y \circ p^1_j)^*\big) \\
&= \alpha_{j+1,t} \circ \psi_j(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*).
\end{aligned}$$

By the universal property of $\mathcal{T}^{\mathscr{S}}_\theta$, we obtain a homomorphism $\alpha_{\infty,t} : \mathcal{T}^{\mathscr{S}}_\theta \to \mathcal{T}^{\mathscr{S}}_\theta$.

For $s, t \in \mathbb{R}$,

$$\begin{aligned}
\alpha_{\infty,t} \circ \alpha_{\infty,s}\Big(\psi_{j,\infty}\big(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*\big)\Big) &= e^{\frac{is(k-l)}{n^j}}\psi_{j,\infty}(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*) \\
&= e^{\frac{i(t+s)(k-l)}{n^j}}\psi_{j,\infty}(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*) \\
&= \alpha_{\infty,t+s}\Big(\psi_{j,\infty}\big(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*\big)\Big).
\end{aligned}$$

So the map $t \mapsto \alpha_{\infty,t}$ is a group homomorphism. Moreover, for all $t \in \mathbb{R}$, $\alpha_{\infty,t} \circ \alpha_{\infty,-t} = \mathrm{id}_{\mathcal{T}^{\mathscr{S}}_\theta} = \alpha_{\infty,-t} \circ \alpha_{\infty,t}$. Hence $\alpha_{\infty,t}$ is an isomorphism of $\mathcal{T}^{\mathscr{S}}_\theta$ onto itself, and so $\alpha_{\infty,t} \in \mathrm{Aut}(\mathcal{T}^{\mathscr{S}}_\theta)$.

All that remains is to show $t \mapsto \alpha_{\infty,t}$ is strongly continuous. Fix $\epsilon > 0$, $t \in \mathbb{R}$ and $a \in \mathcal{T}^{\mathscr{S}}_\theta$. There exists a finite sum $a_0 = \sum_{h=1}^N c_h \psi_{j,\infty}\Big(\iota^{\otimes k_h}_{X(E_{\theta_j})}(x_h)\iota^{\otimes l_h}_{X(E_{\theta_j})}(y_h)^*\Big)$ with $x_h \in X(E_{\theta_j})^{\otimes k_h}$, $y_h \in X(E_{\theta_j})^{\otimes l_h}$ and $c_h \in \mathbb{C}$ satisfying $\|a - a_0\| < \frac{\epsilon}{3}$. Choose $s \in \mathbb{R}$ such that for $h \leq N$, $\left|e^{\frac{it(k_h-l_h)}{n^j}} - e^{\frac{is(k_h-l_h)}{n^j}}\right| < \frac{\epsilon}{3\|a_0\|}$. Then

$$\begin{aligned}
&\|\alpha_{\infty,t}(a) - \alpha_{\infty,s}(a)\| \\
&\qquad \leq \|\alpha_{\infty,t}(a) - \alpha_{\infty,t}(a_0)\| + \|\alpha_{\infty,t}(a_0) - \alpha_{\infty,s}(a_0)\| + \|\alpha_{\infty,s}(a_0) - \alpha_{\infty,s}(a)\| \\
&\qquad < \frac{2\epsilon}{3} + \Big\|\sum_{h=1}^N \Big(\Big(e^{\frac{it(k_h-l_h)}{n^j}} - e^{\frac{is(k_h-l_h)}{n^j}}\Big)c_h\psi_{j,\infty}\Big(\iota^{\otimes k_h}_{X(E_{\theta_j})}(x_h)\iota^{\otimes l_h}_{X(E_{\theta_j})}(y_h)^*\Big)\Big)\Big\|
\end{aligned}$$

$$
\begin{aligned}
&< \frac{2\epsilon}{3} + \frac{\epsilon\|a_0\|}{3\|a_0\|} \\
&= \epsilon.
\end{aligned}
$$

Hence α_∞ is a dynamics over $\mathcal{T}_\theta^{\mathscr{S}}$. $\square$

Proposition 6.1.2 can be expressed in the following commuting diagram.

$$
\begin{tikzcd}
\mathcal{T}(E_{\theta_0}) \arrow[r] \arrow[d, "\alpha_{0,t}"'] \arrow[rrrrr, bend left=30, "\psi_{0,\infty}"] & \cdots \arrow[r] & \mathcal{T}(E_{\theta_j}) \arrow[r, "\psi_j"'] \arrow[d, "\alpha_{j,t}"'] \arrow[rrr, bend left=25, "\psi_{j,\infty}"] & \mathcal{T}(E_{\theta_{j+1}}) \arrow[r] \arrow[d, "\alpha_{j+1,t}"'] \arrow[rr, bend left=20, "\psi_{j+1,\infty}"] & \cdots & \mathcal{T}_\theta^{\mathscr{S}} \arrow[d, "\alpha_{\infty,t}"] \\
\mathcal{T}(E_{\theta_0}) \arrow[r] \arrow[rrrrr, bend right=30, "\psi_{0,\infty}"'] & \cdots \arrow[r] & \mathcal{T}(E_{\theta_j}) \arrow[r, "\psi_j"] \arrow[rrr, bend right=25, "\psi_{j,\infty}"'] & \mathcal{T}(E_{\theta_{j+1}}) \arrow[r] \arrow[rr, bend right=20, "\psi_{j+1,\infty}"'] & \cdots & \mathcal{T}_\theta^{\mathscr{S}}
\end{tikzcd}
$$

Figure 6.1: The dynamics α_∞ over $\mathcal{T}_\theta^{\mathscr{S}}$.

By construction we have

$$
\alpha_{\infty,t}\big(\psi_{j,\infty}(\iota_{X(E_{\theta_j})}^{\otimes k}(x)\iota_{X(E_{\theta_j})}^{\otimes l}(y)^*)\big) = e^{\frac{it(k-l)}{n^j}}\psi_{j,\infty}\big(\iota_{X(E_{\theta_j})}^{\otimes k}(x)\iota_{X(E_{\theta_j})}^{\otimes l}(y)^*\big).
$$

Each α_j is periodic with period $2n^j\pi$, since it is the lift of an action of $\mathbb{T}$. However, the action α_∞ is not periodic, and thus not a lift of $\mathbb{T}$. It is a genuine $\mathbb{R}$-action.

Proposition 6.1.3. *The set*

$$
\Big\{\psi_{j,\infty}(\iota_{X(E_{\theta_j})}^{\otimes k}(x)\iota_{X(E_{\theta_j})}^{\otimes k}(y)^*) \;\Big|\; k,l \geq 0 \text{ and } x \in X(E_{\theta_j})^{\otimes k}, y \in X(E_{\theta_j})^{\otimes l}\Big\}
$$

consists of α_∞-analytic elements.

Proof. By construction, for any $t \in \mathbb{R}$, $j, k, l \in \mathbb{N}$, $x \in X(E_{\theta_j})^{\otimes k}$ and $y \in X(E_{\theta_j})^{\otimes l}$, we have

$$\alpha_{t,\infty} \circ \psi_{j,\infty}\big(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*\big) = \psi_{j,\infty} \circ \gamma_{e^{\frac{it}{n^j}}}\big(\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*\big).$$

Since elements of the form $\iota^{\otimes k}_{X(E_{\theta_j})}(x)\iota^{\otimes l}_{X(E_{\theta_j})}(y)^*$ are α_j-analytic, they must also be α_∞-analytic by Lemma 2.3.3. $\square$

Proposition 6.1.4. *Fix $\beta > 0$, and for each $j \in \mathbb{N}$, let $\beta_j = \frac{\beta}{n^j}$. Suppose that $(\phi_j)_{j=0}^\infty$ is a sequence such that $\phi_j \in \mathcal{S}_{\alpha_j,\beta_j}(\mathcal{T}(E_{\theta_j}))$ for each $j \in \mathbb{N}$, and $\phi_j = \phi_{j+1} \circ \psi_j$ for all j. Then there exists $\phi_\infty \in \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}^{\mathscr{S}}_\theta)$, such that $\phi_\infty \circ \psi_{j,\infty} = \phi_j$ for all j.*

Proof. Let $k, l \in \mathbb{N}$, and without loss of generality, assume that $k \geq l$. For $a \in \mathcal{T}(E_{\theta_k})$ and $b \in \mathcal{T}(E_{\theta_l})$ such that $\psi_{k,\infty}(a) = \psi_{l,\infty}(b)$, we have

$$\phi_k(a) - \phi_l(b) = \phi_k(a) - \phi_k \circ \psi_{k-1} \cdots \circ \psi_l(b) = \phi_k\big(a - \psi_{k-1} \cdots \circ \psi_l(b)\big). \quad (6.1)$$

Since ψ_j is injective (Remark 4.1.13) for all j, $\psi_{j,\infty}$ is injective for all j. We have

$$\|a - \psi_{k-1} \cdots \circ \psi_l(b)\| = |\phi_k\big(a - \psi_{k-1} \cdots \circ \psi_l(b)\big)| = |\phi_\infty(\psi_{k,\infty}(a) - \psi_{l,\infty}(b))| = 0,$$

and so (6.1) gives $\phi_k(a) = \phi_l(b)$. Hence there exists a well-defined map $\phi_\infty : \bigcup_{j \in \mathbb{N}} \psi_{j,\infty}(\mathcal{T}(E_{\theta_j})) \to \mathbb{C}$ such that $\phi_\infty \circ \psi_{j,\infty} = \phi_j$. Observe that

$$|\phi_\infty(\psi_{j,\infty}(a))| = |\phi_j(a)| \leq \|a\|,$$

as ϕ_j is a state, so ϕ_∞ is bounded, and hence extends to $\mathcal{T}^{\mathscr{S}}_\theta$. We now check that (2.12) is satisfied. It suffices to check this on analytic $a, b \in \mathcal{T}(E_{\theta_j})$, since if $a \in \mathcal{T}(E_{\theta_k})$ and $b \in \mathcal{T}(E_{\theta_j})$ for $k \leq j$, we have $\psi_{j-1} \circ \cdots \circ \psi_k(a) \in \mathcal{T}(E_{\theta_j})$. Fix analytic $a, b \in \mathcal{T}(E_{\theta_j})$. We have

$$\phi_\infty(\psi_{j,\infty}(a)\psi_{j,\infty}(b)) = \phi_\infty(\psi_{j,\infty}(ab))$$

$$
\begin{aligned}
&= \phi_j(ab) \\
&= \phi_j(b\alpha_{i\beta,j}(a)) \\
&= \phi_\infty(\psi_{j,\infty}(b)\psi_{j,\infty} \circ \alpha_{j,i\beta_j}(a)) \\
&= \phi_\infty(\psi_{j,\infty}(b)\alpha_{\infty,i\beta} \circ \psi_{j,\infty}(a)) \text{ by Figure 6.1,}
\end{aligned}
$$

which is precisely the KMS condition. □

Proposition 6.1.5. *Let α be a dynamics over A, and take $\phi \in \mathcal{S}_{\alpha,\beta}(A)$. Let B be a α-invariant C^*-subalgebra of A. Then $\phi|_B \in \mathcal{S}_{\alpha,\beta}(B)$.*

Proof. Fix analytic $c, d \in B$. Then c, d are analytic in A, and

$$\phi|_B(cd) = \phi(cd) = \phi(d\alpha_{i\beta}(c)) = \phi|_B(d\alpha_{i\beta}(c))$$

which is precisely the KMS condition. □

For each $j \in \mathbb{N}$, the subalgebra $\psi_{j,\infty}(\mathcal{T}(E_{\theta_j}))$ is an α_∞-invariant subalgebra of $\mathcal{T}_\theta^{\mathscr{S}}$. So, for $\phi_\infty \in \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$, $\phi_\infty \circ \psi_{k,\infty} \in \mathcal{S}_{\alpha,\beta_j}(\mathcal{T}(E_{\theta_k}))$ where $\beta_j = \frac{\beta}{n^j}$, since $\mathcal{T}(E_{\theta_j}) \cong \psi_{j,\infty}(\mathcal{T}(E_{\theta_j}))$. From Section 5.1, we already understand $\mathcal{S}_{\alpha,\beta}(\mathcal{T}(E_{\theta_k}))$, and we will use this to investigate $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$, making use of Propositions 6.1.4 and 6.1.5. The obvious place is to start with finding appropriate values of β.

Corollary 6.1.6. *For $\beta < 0$, $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}}) = \emptyset$.*

Proof. Fix $\beta \in \mathbb{R}$, and suppose that $\phi_\infty \in \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$. For $j \in \mathbb{N}$, let $\beta_j = \frac{\beta}{n^j}$. Then $\phi_\infty \circ \psi_{j,\infty} \in \mathcal{S}_{\alpha,\beta_j}(\mathcal{T}(E_{\theta_j}))$. By Proposition 5.1.15, $\beta_j \geq 0$, which implies $\beta = n^j\beta_j \geq 0$. □

Given Propositions 6.1.4 and 6.1.5, we can think of each KMS$_\beta$ state on $\mathcal{T}_\theta^{\mathscr{S}}$ as a sequence of KMS$_{\beta_j}$ states. We use the following lemma to define such a sequence of KMS states that we will make use of in Theorem 6.1.9.

Lemma 6.1.7. *Let $(X, \mathcal{M})$ be a compact Borel measurable space. Let $h : X \to X$ be a surjective local homeomorphism. Let $\{I_j\}_{j=1}^N$ be a disjoint family of Borel sets*

such that $h : I_j \to X$ is a bijection onto X for each j. If m, n are measures on X such that $m = n(h^{-1}(\cdot))$, and $n(U) = n(U \cap I_j)$ for some j and all Borel U, then $m \circ h = n$.

Proof. Fix $j \le N$, and a Borel $U \subseteq I_j$. Then

$$m \circ h(U) = n(h^{-1}(h(U))) = n(h^{-1}(h(U)) \cap I_j) = n(U),$$

as claimed. $\square$

Given $\beta > 0$, each $\phi_\infty \in \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$ restricts to a sequence $\phi_j := \phi_\infty \circ \psi_{j,\infty} \in \mathcal{S}_{\alpha,\beta_j}(\mathcal{T}(E_{\theta_j}))$ such that $\phi_{j+1} \circ \psi_j = \phi_\infty \circ \psi_{j+1,\infty} \circ \psi_j = \phi_\infty \circ \psi_{j,\infty} = \phi_j$. In light of Theorem 5.1.3, this sequence corresponds to a sequence of probability measures $\{m_j\}_{j=0}^\infty$ on $\mathbb{T}$ satisfying $m_j = m_{j+1}(p^{-1}(\cdot))$. By Lemma 6.1.7, we can characterise m_{j+1} based on m_j.

Using the characterisation of the KMS states of $\mathcal{T}(E_{\theta_j})$ in terms of an $\epsilon_j \in \mathcal{M}(E^0_{\theta_j})$ as in Theorem 5.1.11, we obtain

$$(1 - e^{-\beta_j} A_{E_{\theta_j}})^{-1}\epsilon_j = (1 - e^{-\beta_{j+1}} A_{E_{\theta_{j+1}}})^{-1}\epsilon_{j+1}(p^{-1}(\cdot)).$$

Hence,

$$\begin{aligned}\epsilon_{j+1}(p^{-1}(\cdot)) &= (1 - e^{-\beta_{j+1}} A_{E_{\theta_{j+1}}})(1 - e^{-\beta_j} A_{E_{\theta_j}})^{-1}\epsilon_j \\ &= \sum_{k=0}^{\infty} \Big(e^{-\beta_j k}\epsilon_j(r_j^{-k}) + e^{-(\beta_j k + \beta_{j+1})}\epsilon_j(r_j^{-k} \circ r_{j+1}^{-1}) \Big).\end{aligned}$$

Since the $A_{E_{\theta_j}}$ pairwise commute, we also have

$$\begin{aligned}\epsilon_j &= (1 - e^{-\beta_{j+1}} A_{E_{\theta_{j+1}}})^{-1}(1 - e^{-\beta_j} A_{E_{\theta_j}})\epsilon_{j+1}(p^{-1}(\cdot)) \\ &= (1 - e^{-\beta_{j+1}} A_{E_{\theta_{j+1}}})^{-1}(1 - e^{-\beta_{j+1}^n} A^n_{E_{\theta_{j+1}}})\Big(\sum_{k=0}^{n^2-1} e^{-\beta_{j+1} k} A^k_{\theta_{j+1}}\Big)\epsilon_{j+1}(p^{-1}(\cdot)) \\ &= \sum_{k=0}^{n-1} e^{-\beta_{j+1} k} A^k_{\theta_{j+1}}\epsilon_{j+1}(p^{-1}(\cdot))\end{aligned}$$

$$= \sum_{k=0}^{n-1} e^{-\beta_{j+1}k} \epsilon_{j+1}(p^{-1}(\cdot)) \circ r_{j+1}^{-k}. \tag{6.2}$$

For $j \in \mathbb{N}$, we define

$$I_j := \Big\{ z \in \mathbb{T} : \operatorname{Arg}(z) \in \Big[0, \frac{2\pi}{n^j}\Big) \Big\}. \tag{6.3}$$

We also use the notation

$$\varprojlim(\mathbb{T}, p) := \big\{ (z_j)_{j\in\mathbb{N}} \mid z_j \in \mathbb{T} \text{ and } z_j = p(z_{j+1}) \text{ for all } j \in \mathbb{N} \big\}.$$

This, of course, is the topological solenoid.

We will prove in Theorem 6.1.9 that this space is homeomorphic to the boundary of $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$, assuming a positivity condition.

Notation 6.1.8. *Fix $z = (z_k)_{k=0}^\infty \in \varprojlim(\mathbb{T}, p)$. Let $\{\epsilon_{z,k}\}_{k=0}^\infty$ be the sequence of measures such that*

$$\epsilon_{z,0} := (1 - e^{-\beta_0})\delta_{z_0},$$

where δ_{z_0} is the Dirac measure based at z_0; and for $k \geq 1$, $\epsilon_{z,k}$ is the measure on $\mathbb{T}$ such that

$$\epsilon_{z,k}(U) := \sum_{j=0}^{\infty} \Big(e^{-\beta_0} \epsilon_{z,0} \circ r_0^{-j}(U \cap z_k \cdot I_k) - e^{-\beta_k + \beta_0 j} \epsilon_{z,0} \circ r_k^{-1} \circ r_0^{-j}(U \cap z_k \cdot I_k) \Big)$$

for all measurable $U \subseteq \mathbb{T}$.

We denote by $\phi_{z,\infty}$ the element of $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$ such that $\phi_{z,\infty} \circ \psi_{k,\infty} = \phi_{\epsilon_{z,k}}$, for $\phi_{\epsilon_{z,k}} \in \mathcal{S}_{\alpha,\beta_k}(\mathcal{T}(E_{\theta_k}))$.

We will assume that for $z \in \varprojlim(\mathbb{T}, p)$, the sequence of measures $(\epsilon_{z,k})_{k=0}^\infty$ is a sequence of positive measures. We will call this the ***positivity assumption****.*

We will proceed with the analysis as if the sequences of measures in question were positive, and clearly state where this assumption is being used. By construction of Notation 6.1.8, for $z \in \varprojlim(\mathbb{T}, p)$ and $j \in \mathbb{N}$, we have

$$\phi_{z,\infty} \circ \psi_{j,\infty} = \phi_{\epsilon_{z,j}} = \phi_{\epsilon_{z,j+1}} \circ \psi_j.$$

So, $\left(\phi_{\epsilon_{z,j}}\right)_{j=0}^{\infty}$ satisfy the hypothesis of Proposition 6.1.4.

We are now ready to state the main theorem of this thesis. In an attempt to make the statement self-contained, so we will explicitly recall all the standing notation we have been using, and state all assumptions.

Theorem 6.1.9. *Fix $n \in \mathbb{N}$ such that $n \geq 2$, and let $p : \mathbb{T} \to \mathbb{T}$ be the map $p(z) = z^n$. Fix a sequence $(k_j)_{j=0}^{\infty}$ such that $k_j \in \{0, 1, \cdots, n^2 - 1\}$. Fix $\theta_0 \in [0, 1)$ and inductively define $n^2\theta_{j+1} = \theta_j + k_j$ for all $j \in \mathbb{N}$. Let $\psi_j : \mathcal{T}(E_{\theta_j}) \to \mathcal{T}(E_{\theta_{j+1}})$ be as in Notation 6.1.1, and let $\mathcal{T}_\theta^{\mathscr{S}}$ denote the direct limit $\varinjlim(\mathcal{T}(E_{\theta_j}), \psi_j)$. Let α_∞ be the dynamics on $\mathcal{T}_\theta^{\mathscr{S}}$ described in Proposition 6.1.2. Assume that the positivity assumption of Notation 6.1.8 holds.*

Fix $\beta > 0$. There exists a continuous injection $\Gamma : \varprojlim(\mathbb{T}, p) \to \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$ such that

$$\Gamma(z) = \phi_{z,\infty},$$

where $\phi_{z,\infty}$ is the KMS$_\beta$ state introduced in Notation 6.1.8 under the positivity assumption. Moreover, Γ is a homeomorphism of the solenoid onto the boundary $\partial\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$.

Before we prove this theorem, we require some technical estimates. It should be noted that the measure m in the following lemma may not be rotationally invariant.

Lemma 6.1.10. *Let $f \in C(\mathbb{T})^+$. Let m be a finite measure on $\mathbb{T}$, and for $\omega \in \mathbb{T}$, let R_ω be the map $z \mapsto \omega z$. Suppose that there exists $C \in \mathbb{R}$ such that $\left|\int_{\mathbb{T}} f \circ R_\omega \,\mathrm{d}m\right| < C$ for all $\omega \in \mathbb{T}$. Then*

$$\int_{\mathbb{T}} f \,\mathrm{d}((1 - e^{-\beta_0} A_{E_{\theta_0}})^{-1}(1 - e^{-\beta_j} A_{E_{\theta_j}})m) \leq \frac{1 + e^{-\beta_j}}{1 - e^{-\beta_0}} C$$

Proof. Using the series expansion of $(1 - e^{-\beta} A_E)^{-1}$, we have

$$(1 - e^{-\beta_j} A_{E_{\theta_j}})(1 - e^{-\beta_0} A_{E_{\theta_0}})^{-1} m = \sum_{k=0}^{\infty} e^{-\beta_0 k} A_{E_{\theta_0}}^k m - e^{-(\beta_j + \beta_0 k)} A_{E_{\theta_0}}^k A_{E_{\theta_j}} m.$$

We also have

$$\int_{\mathbb{T}} f \,\mathrm{d}A^k_{E_{\theta_l}} m = \int_{\mathbb{T}} f \circ r_l^{-k} \,\mathrm{d}m.$$

Since $r_l^{-k} = R_{e^{\frac{2i\pi\theta_k}{n^l}}}$, our hypothesis implies that for $l \in \mathbb{N}$, we have $\left|\int_{\mathbb{T}} f \,\mathrm{d}A_{E_{\theta_l}} m\right| < C$. Similarly, we have $\left|\int_{\mathbb{T}} f \,\mathrm{d}A_{E_{\theta_0}} A_{E_{\theta_l}} m\right| < C$. Therefore,

$$\begin{aligned}
&\left|\int_{\mathbb{T}} f \,\mathrm{d}((1 - e^{-\beta_j} A_{E_{\theta_j}})(1 - e^{-\beta_0} A_{E_{\theta_0}})^{-1} m)\right| \\
&\quad \le \sum_{k=0}^{\infty} \left(e^{-\beta_0 k} \left|\int_{\mathbb{T}} f \,\mathrm{d}(A^k_{E_{\theta_0}} m)\right| + e^{-(\beta_j + \beta_0 k)} \left|\int_{\mathbb{T}} f \,\mathrm{d}(A^K_{E_{\theta_0}} A_{E_{\theta_j}} m)\right| \right) \\
&\quad < \sum_{k=0}^{\infty} e^{-\beta_0 k} C + e^{-(\beta_j + \beta_0 k)} C \\
&\quad = \frac{1 + e^{-\beta_j}}{1 - e^{-\beta_0}} C,
\end{aligned}$$

as we aimed to show. □

We require another approximation (see Lemma 6.1.12), which also requires some preliminary technical results.

Proposition 6.1.11. *Let X be a compact Hausdorff space, and let m be a finite Borel measure on X. Then there exists a sequence $\{m_n\}_{n=1}^{\infty}$ of finite linear combinations of Dirac measures such that $m_n \to m$, in the sense that for all $f \in C(X)$*

$$\int_X f \,\mathrm{d}m_n \to \int_X f \,\mathrm{d}m.$$

Proof. If $m(X) = 0$, the statement is trivial, so we will suppose otherwise. Fix a sequence $f_i \in C(X)$, such that $\overline{\{f_i : i \in \mathbb{N}\}} = C(X)$. For each $x \in X$, let $B(f(x), \frac{1}{n \cdot m(X)})$ denote the open ball centred around $f(x)$ of radius $\frac{1}{n \cdot m(X)}$, and denote $U_{x,n} := \bigcap_{i \le n} f_i^{-1}(B(f(x); \frac{1}{n \cdot m(X)}))$. Since X is compact, we can choose finitely many x_j such that $X = \bigcup_{j=1}^{J(n)} U_{x_j,n}$. Let $V_{x_j,n} = U_{x_j,n} \setminus \bigcup_{k=1}^{j-1} U_{x_k,n}$. By construction, the $V_{x_j,n}$ are disjoint Borel sets and cover X. Let $m_n := \sum_{j=1}^{J(n)} m(V_{x_j,n}) \delta_{x_j}$. Then

$$m_n(X) = \sum_{j=1}^{J(n)} m(V_{x_j,n}) = m\Big(\bigcup_{j=1}^{J(n)} V_{x_j,n}\Big) = m(X).$$

We have for each $n \in \mathbb{N}$, $i \leq n$ and j,

$$\Big|\int_{V_{x_j,n}} f_i \,\mathrm{d}m_n - \int_{V_{x_j,n}} f_i \,\mathrm{d}m\Big| \leq \frac{2}{m(X)n} m(V_{x_j,n}),$$

so

$$\Big|\int_X f_i \,\mathrm{d}m_n - \int_X f_i \,\mathrm{d}m\Big| \leq \sum_{j=1}^{J(n)} \frac{2}{m(X)n} m(V_{x_j,n}) = \frac{2}{n}.$$

Fix $\epsilon > 0$, and let $g \in C(X)$. Then there exists some $i \in \mathbb{N}$ such that $\|g - f_i\|_\infty \leq \frac{\epsilon}{3m(X)}$. Choose $n \in \mathbb{N}$ such that $n > \max\{\frac{3}{2\epsilon}, i\}$. Then

$$\begin{aligned}\Big|\int_X g \,\mathrm{d}m - \int_X g \,\mathrm{d}m_n\Big| &\leq \Big|\int_X g \,\mathrm{d}m - \int_X f_i \,\mathrm{d}m\Big| + \Big|\int_X f_i \,\mathrm{d}m - \int_X f_i \,\mathrm{d}m_n\Big| \\ &\quad + \Big|\int_X f_i \,\mathrm{d}m_n - \int_X g \,\mathrm{d}m_n\Big| \\ &< \frac{\epsilon}{3} + \frac{2}{n} + \frac{\epsilon}{3} \\ &< \epsilon.\end{aligned}$$

Hence $m_n \to m$ as claimed. $\square$

Lemma 6.1.12. *Let E be a topological graph, fix $\beta > \ln(\rho(A_E))$, $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$, $a_1, \cdots a_j \in \mathcal{T}(E)$, and $C > 0$. For each $v \in E^0$, let ϕ_v be the state of Equation (5.16). Then there exists a finite set $V \subseteq E^0$ and scalars $\{c_v : v \in E^0\}$ satisfying $\sum_{v \in V} c_v = 1$ such that*

$$\Big|\phi(a_i) - \sum_{v \in V} c_v \phi_v(a_i)\Big| < C \tag{6.4}$$

for $i \leq j$.

Proof. Fix $C > 0$. For each $i \leq j$, there exists a finite collection of elements $x_{l,i} \in X(E)^{\otimes m_{l,i}}, y_{l,i} \in X(E)^{\otimes n_{l,i}}$ such that $F_i := \sum_{l=1}^{L_i} \iota_{X(E)}^{\otimes m_{l,i}}(x_{l,i}) \iota_{X(E)}^{\otimes n_{l,i}}(y_{l,i})^*$ satisfies $\|a_i - F_i\| < \frac{C}{3}$. Let $J_i := \{l : m_{l,i} = n_{l,i}\}$. If $J_i = \emptyset$, then $\phi(F_i) = 0$ for all $\phi \in \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$ by Theorem 5.1.3. Suppose that $J_i \neq \emptyset$.

By Theorem 5.1.11, there exists a unique Borel measure ϵ on E^0 such that $\phi = \phi_\epsilon$. The operator $(1-e^{-\beta}A_E)^{-1}$ has finite norm since it is invertable. By Proposition 6.1.11 there exists a finite set $V \subseteq E^0$ and a set of scalars $\{c_v : v \in V \text{ and } c_v \in [0,1]\}$ with $\sum_{v\in V} c_v = 1$ such that

$$\Big|\int_{E^0} \langle y_{l,i}, x_{l,i}\rangle \,\mathrm{d}\epsilon - \sum_{v\in V}\int_{E^0} \langle y_{l,i}, x_{l,i}\rangle \,\mathrm{d}\Big(c_v\Big(\sum_{\mu\in E^*v} e^{-\beta|\mu|}\Big)^{-1}\delta_v\Big)\Big| < \frac{C}{3\|(1-A_E)^{-1}\||J_i|}$$

for each $i \leq j$ and $l \in J_i$.

Now, we have

$$\begin{aligned}
&\Big|\phi(a_i) - \sum_{v\in V} c_v\phi_v(a_i)\Big| \\
&\quad \leq \Big|\phi(a_i)-\phi(F_i)\Big| + \Big|\phi(F_i) - \sum_{v\in V} c_v\phi_v(F_i)\Big| + \Big|\sum_{v\in V} c_v\phi_v(F_i) - \sum_{v\in V} c_v\phi_v(a_i)\Big| \\
&\quad < 2\,\|a_i - F_i\| + \Big|\phi(F_i) - \sum_{v\in V} c_v\phi_v(F_i)\Big| \qquad (6.5)\\
&\quad < \frac{2C}{3} + \sum_{l\in J_i} e^{-\beta m_{l,i}}\Big|\phi(\iota_{C(E^0)}(\langle y_{l,i}, x_{l,i}\rangle)) - \sum_{v\in V} c_v\phi_v(\iota_{C(E^0)}(\langle y_{l,i}, x_{l,i}\rangle))\Big| \\
&\quad < \frac{2C}{3} + \sum_{l\in J_i} e^{-\beta m_{l,i}}\frac{C}{3|J_i|} \\
&\quad < C,
\end{aligned}$$

as claimed. If $J_i = \emptyset$, we have $\big|\phi(a_i) - \sum_{v\in V} c_v\phi_v(a_i)\big| < 2\|a_i - F_i\| < \frac{2C}{3}$ from Equation (6.5). $\square$

The following lemma is required to prove surjectivity of Γ onto $\partial\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$, given the positivity assumption of Notation 6.1.8. The implication here is the boundary of a simplex is not unique — which should raise some eyebrows.

Lemma 6.1.13. *Assume positivity assumption of Notation 6.1.8 holds. Take $\omega \in \mathbb{T} = E^0_{\theta_k}$, $k \in \mathbb{N}$ and fix $\beta > 0$. Let $\phi_\omega \in \mathcal{S}_{\alpha,\beta_k}(\mathcal{T}(E_{\theta_k}))$ be as in Equation* (5.16)*. There exists $J \in \mathbb{N}$, such that for each $j \leq J$, there exists $z_j \in \varprojlim(\mathbb{T}, p)$ and*

$\{c_j\}_{j=1}^J \in [0,1]$ *satisfying* $\sum_{j=1}^J c_j = 1$ *such that*

$$\sum_{j=1}^{J} c_j \Gamma(z_j) \circ \psi_{k,\infty}(a) = \phi_\omega(a)$$

for all $a \in \mathcal{T}(E_{\theta_k})$.

Proof. For $w \in \varprojlim(\mathbb{T}, p)$, $\Gamma(w) \circ \psi_{j,\infty} \in \mathcal{S}_{\alpha,\beta_j}(\mathcal{T}(E_{\theta_j}))$ for each $j \in \mathbb{N}$. Let $m_j \in \mathcal{M}^1(\mathbb{T})$ be such that for all $f \in C(\mathbb{T})^+$, $\Gamma(w) \circ \psi_{j,\infty}(\iota_{C(\mathbb{T})}(f)) = \int_{\mathbb{T}} f \, \mathrm{d}m_j$. Let $\epsilon_j = (1 - e^{-\beta_j} A_{E_{\theta_j}})m_j$.

From Equation (6.2), we find

$$\epsilon_0 = \sum_{j=0}^{n^k-1} e^{-\beta_k j} A^j_{E_{\theta_k}} \epsilon_k(p^{-k}(\cdot)).$$

Now, if $\epsilon_k = (1 - e^{-\beta_k})\delta_z$ for some $z \in \mathbb{T}$, then

$$\epsilon_0 = (1 - e^{-\beta_k}) \sum_{j=0}^{n^k-1} e^{-\beta_k j} \delta_z(p^{-k}(\cdot)) \circ r_k^{-j}.$$

Choose $\omega \in p^{-k}(\{z\})$. Then $\delta_\omega = \delta_z(p^{-k})$. Hence,

$$\begin{aligned}
\epsilon_0 &= \sum_{j=0}^{n^k-1} (e^{-\beta_k j} - e^{-\beta_k (j+1)}) \delta_\omega \circ r_k^{-j} \\
&= \sum_{j=0}^{n^k-1} (e^{-\beta_k j} - e^{-\beta_k (j+1)}) \delta_{r_k^j(\omega)} \\
&= \sum_{j=0}^{n^k-1} \frac{(e^{-\beta_k j} - e^{-\beta_k (j+1)})}{1 - e^{-\beta_0}} (1 - e^{-\beta_0}) \delta_{r_k^j(\omega)}.
\end{aligned}$$

Choose $z_j = (z_{j,l})_{l=0}^\infty \in \varprojlim(\mathbb{T}, p)$ such that $z_{j,0} = r_k^j(\omega)$ and $z \in z_{j,k} \cdot I_k$. Then

$$\sum_{j=0}^{n^k-1} \frac{e^{-\beta_k j} - e^{-\beta_k j+1}}{1 - e^{-\beta_0}} \Gamma(z_j) \circ \psi_{k,\infty} = \phi_{(1-e^{-\beta_k})\delta_z} = \phi_z \in \mathcal{S}_{\alpha,\beta_k}(\mathcal{T}(E_{\theta_k})),$$

which is what we wanted. ∎

Proof of Theorem 6.1.9. Suppose that $z = (z_j)_{j=0}^\infty, w = (w_j)_{j=0}^\infty \in \varprojlim(\mathbb{T}, p)$ are distinct. Then there exists $k \in \mathbb{N}$ such that $z_k \neq w_k$. Hence $\epsilon_{z,k} \neq \epsilon_{w,k}$ since the set I_k of Notation 6.1.8 satisfies $z_k \cdot I_k \neq w_k \cdot I_k$. Therefore $\phi_{z,\infty} \circ \psi_{k,\infty} \neq \phi_{w,\infty} \circ \psi_{k,\infty}$ by Theorem 5.1.11. So Γ is injective.

To show that Γ is continuous, fix $\delta > 0$, $z = (z_j)_{j=0}^\infty \in \varprojlim(\mathbb{T}, p)$ and $a \in \mathcal{T}_\theta^{\mathscr{S}}$. Let $z_k = (z_{k,j})_{j=0}^\infty \to z$ as $k \to \infty$. Choose finitly many $x_l \in X(E_{\theta_K})^{\otimes h_l}$ and $y_l \in X(E_{\theta_K})^{\otimes i_l}$ such that $F := \sum_l \psi_{K,\infty}\big(\iota_{X(E_{\theta_K})}^{\otimes h_l}(x_l)\iota_{X(E_{\theta_K})}^{\otimes i_l}(y_l)^*\big)$ satisfies $\|a - F\| < \frac{\delta}{3}$.

Let $J = \{l : h_l = i_l\}$, and for each $l \in J$ let $\tilde{F}_l := \psi_{K,\infty}(\iota_{C(E_{\theta_K}^0)}(\langle y_l, x_l\rangle))$. Linearity and the triangle inequality imply that

$$|\phi_{z_k,\infty}(F) - \phi_{z,\infty}(F)| \leq \sum_{l\in J} e^{-\beta_K h_l} \left|\phi_{z_k,\infty}(\tilde{F}_l) - \phi_{z,\infty}(\tilde{F}_l)\right|.$$

We assume that $J \neq \emptyset$. The case where $J = \emptyset$ is a special case to be discussed later in the proof. Since $\mathbb{T}$ is compact, $\langle y_l, x_l\rangle$ is a uniformly continuous function for each $l \in J$. Then, for each $l \in J$, there exists $\kappa_l > 0$ such that for any $z, w \in \mathbb{T}$ satisfying $|z - w| < \kappa_l$ we have

$$|\langle y_l, x_l\rangle(z) - \langle y_l, x_l\rangle(w)| < \frac{(1 - e^{-\beta_0})^2\delta}{6(1 + e^{-\beta_K})|J|}.$$

Let $\kappa = \min\{\kappa_l : l \in J\}$, and for $\omega \in \mathbb{T}$, let $R_\omega : \mathbb{T} \to \mathbb{T}$ be the rotation map such that $R_\omega(z) = \omega z$. Since $z_k \to z$ pointwise, there exists $L \in \mathbb{N}$ such that for $k > L$, $|z_0 - z_{0,k}| < \kappa$, and hence $|R_\omega(z_0) - R_\omega(z_{0,k})| < \kappa$ for all $\omega \in \mathbb{T}$. So, for $k > L$,

$$\left|\int_{\mathbb{T}} \langle y_l, x_l\rangle \circ R_\omega \,\mathrm{d}(\epsilon_{z_k,0} - \epsilon_{z,0})\right| < \frac{(1 - e^{-\beta_0})^2\delta}{6(1 + e^{-\beta_K})|J|} \tag{6.6}$$

for all $\omega \in \mathbb{T}$. The measure ϵ_K satisfies

$$\epsilon_{z,K}(p^{-K}) = \sum_{j=0}^\infty \left(e^{-\beta_0 j}\epsilon_{z,0} \circ r_0^{-j} + e^{-(\beta_K + \beta_0 j)}\epsilon_{z,0} \circ r_0^{-j} \circ r_K^{-1}\right)$$

from Equation (6.2). Hence for each Borel $U \subseteq \mathbb{T}$,

$$\epsilon_{z,K}(U) = \sum_{j=0}^\infty \epsilon_{z,0}(r_0^{-j})(p^K(U \cap z_K \cdot I_K))$$

$$
\begin{aligned}
&\qquad + e^{-(\beta_K+\beta_0 j)}\epsilon_{z,0}(r_0^{-j}(r_K^{-1}))(p^K(U\cap z_K\cdot I_K))\\
&= (1-e^{-\beta_0})\Big(\sum_{j=0}^{\infty}\delta_{r_0^j(z_0)}(p^K(U\cap z_K\cdot I_K))\\
&\qquad + e^{-(\beta_K+\beta_0 j)}\delta_{r_0^j(r_K(z_0))}(p^K(U\cap z_K\cdot I_K))\Big).
\end{aligned}
$$

For $f\in C(\mathbb{T})$, we then have

$$
\begin{aligned}
\int_{\mathbb{T}} f\,\mathrm{d}\epsilon_{z,K} &= \int_{z_K\cdot I_K} f\circ p^K\,\mathrm{d}\left((1-e^{-\beta_0})\sum_{j=0}^{\infty}\left(e^{-\beta_0 j}\delta_{r_0^j(z_0)} + e^{-(\beta_K+\beta_0 j)}\delta_{r_K(r_0^j(z_0))}\right)\right)\\
&= \int_{\mathbb{T}} f\,\mathrm{d}\left((1-e^{-\beta_0})\sum_{j=0}^{\infty}\left(e^{-\beta_0 j}\delta_{r_0^j(z_0)} + e^{-(\beta_K+\beta_0 j)}\delta_{r_K(r_0^j(z_0))}\right)\right)\\
&= \sum_{j=0}^{\infty}\left(e^{-\beta_0 j}\int_{\mathbb{T}} f\circ R_{e^{2i\pi\theta}}\,\mathrm{d}\epsilon_{z,0} + e^{-(\beta_K+\beta_0 j)}\int_{\mathbb{T}} f\circ R_{e^{2i\pi(\theta j+\theta_K)}}\,\mathrm{d}\epsilon_{z,0}\right).
\end{aligned}
$$

Hence, for $l\in J$,

$$
\begin{aligned}
&\left|\phi_{z,\infty}(\tilde{F}_l) - \phi_{(z_k,\infty}(\tilde{F}_l)\right|\\
&\qquad = \left|\int_{\mathbb{T}}\langle y_l, x_l\rangle\,\mathrm{d}(\epsilon_{z,K}-\epsilon_{z_k,K})\right|\\
&\qquad \le \sum_{h=0}^{\infty}\Big(e^{-\beta_0 h}\left|\int_{\mathbb{T}}\langle y_l,x_l\rangle\circ R_{e^{2i\pi\theta h}}\,\mathrm{d}(1-e^{-\beta_K}A_{E_{\theta_K}})(\epsilon_{z,0}-\epsilon_{z_k,0})\right|\\
&\qquad\qquad + e^{-(\beta_0 h+\beta_K)}\left|\int_{\mathbb{T}}\langle y_l,x_l\rangle\circ R_{e^{2i\pi(\theta h+\frac{\theta}{n^{2K}})}}\,\mathrm{d}(1-e^{-\beta_K}A_{E_{\theta_K}})(\epsilon_{z,0}-\epsilon_{z_k,0})\right|\Big).
\end{aligned}
$$

We know from Equation (6.6) that $\left|\int_{\mathbb{T}}\langle y_l,x_l\rangle\,\mathrm{d}(\epsilon_{z,0}-\epsilon_{z_k,0})\right| < \frac{(1-e^{-\beta_0})^2\delta}{6(1+e^{-\beta_K})|J|}$, so applying Lemma 6.1.10 with $C=\frac{(1-e^{-\beta_0})^2\delta}{6(1+e^{-\beta_K})|J|}$, we obtain

$$
\begin{aligned}
&\left|\int_{\mathbb{T}}\langle y_l,x_l\rangle\,\mathrm{d}((1-e^{-\beta_0}A_{E_{\theta_0}})(1-A_{E_{\theta_K}})^{-1}(\epsilon_{z,0}-\epsilon_{z_k,0}))\right|\\
&\qquad\qquad < \sum_{h=0}^{\infty} e^{-\beta_0 h}C + e^{-(\beta_0 h+\beta_K)}C\\
&\qquad\qquad < \frac{2C}{1-e^{-\beta_0}}.
\end{aligned}
$$

Hence

$$\left|\phi_{z,\infty}(\tilde{F}_l) - \phi_{z_k,\infty}(\tilde{F}_l)\right| < \frac{2}{1-e^{-\beta_0}} \cdot \frac{1+e^{-\beta_K}}{1-e^{-\beta_0}} \cdot \frac{(1-e^{-\beta_0})^2\delta}{6(1+e^{-\beta_K})|J|}$$
$$= \frac{\delta}{3|J|},$$

by Lemma 6.1.10. Then

$$\begin{aligned}
&\left|\phi_{z,\infty}(a) - \phi_{z_k,\infty}(a)\right| \\
&\quad \leq |\phi_{z,\infty}(a) - \phi_{z,\infty}(F)| + |\phi_{z,\infty}(F) - \phi_{z_k,\infty}(F)| + |\phi_{z_k,\infty}(F) - \phi_{z_k,\infty}(a)| \\
&\quad \leq 2\,\|a - F\| + \sum_{l\in J} e^{-\beta_K h_l} \left|\phi_{z_k,\infty}(\tilde{F}_l) - \phi_{z,\infty}(\tilde{F}_l)\right| \\
&\quad < \frac{2\delta}{3} + \sum_{l\in J} \frac{e^{-\beta_K h_l}\delta}{3|J|} \\
&\quad < \delta.
\end{aligned}$$

In the case where $J = \emptyset$, we have $|\phi_{z,\infty}(F) - \phi_{z_k,\infty}(F)| = 0$, so the above calculation yields $|\phi_{z,\infty}(a) - \phi_{z_k,\infty}(a)| < \frac{2\delta}{3} < \delta$, so the map Γ is continuous. Since Γ is injective and $\varprojlim(\mathbb{T}, p)$ is compact, Γ is a homeomorphism onto its range.

Next we show for any $\phi_\infty \in \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$, there exists a sequence $\{\phi_j\}_{j\in\mathbb{N}} \in \mathrm{conv}(\Gamma(\varprojlim(\mathbb{T}, p)))$ such that $\phi_j \to \phi_\infty$ pointwise.

Fix $\phi_\infty \in \mathcal{S}_{\alpha_\infty,\beta}$. Since each $\mathcal{T}(E_{\theta_j})$ is separable, there exists a sequence $(a_{l,j})_{l=0}^\infty \in \mathcal{T}(E_{\theta_j})$ such that $\overline{\{a_{l,j} : l \in \mathbb{N}\}} = \mathcal{T}(E_{\theta_j})$. Given that we have a countable collection of dense sequences in $\bigcup_{j=0}^\infty \psi_{j,\infty}(\mathcal{T}(E_{\theta_j}))$, a standard diagonal argument then shows that there exists a sequence $(a_j)_{j=0}^\infty \in \bigcup_{k\in\mathbb{N}} \psi_{k,\infty}(\mathcal{T}(E))$ such that $\overline{\{a_j : j \in \mathbb{N}\}} = \mathcal{T}_\theta^{\mathscr{S}}$. For each $j \in \mathbb{N}$, there exists N_j such that $a_1, \cdots, a_j \in \psi_{N_j,\infty}(\mathcal{T}(E_{\theta_{N_j}}))$. Applying Lemma 6.1.12 to the preimages of $a_1, \cdots, a_j \in \mathcal{T}(E_{\theta_{N_j}})$ and Lemma 6.1.13 we see that there exist $z_{i,j} \in \varprojlim(\mathbb{T}, p)$ and $c_{i,j} \in [0, 1]$ with $\sum_i c_{i,j} = 1$ such that $\phi_j := \sum_i c_{i,j}\Gamma(z_{i,j})$ satisfies

$$|\phi_\infty(a_l) - \phi_j(a_l)| < \frac{1}{j}$$

for $l \leq j$.

Fix $a \in \mathcal{T}_\theta^{\mathscr{S}}$ and $\delta > 0$. Since $\{a_j : j \in \mathbb{N}\}$ is dense, there exists $J \in \mathbb{N}$ such that $\|a - a_J\| < \frac{\delta}{3}$. Now, for $j \geq \max\{J, \frac{3}{\delta}\}$,

$$\begin{aligned}|\phi_\infty(a) - \phi_j(a)| &\leq |\phi_\infty(a) - \phi_\infty(a_J)| + |\phi_\infty(a_J) - \phi_j(a_J)| + |\phi_j(a_J) - \phi_j(a)| \\ &\leq 2\|a - a_J\| + |\phi_\infty(a_J) - \phi_j(a_J)| \\ &< \frac{2\delta}{3} + \frac{1}{j} \\ &< \delta.\end{aligned}$$

Hence, $\phi_j \to \phi_\infty$ pointwise.

We have just shown $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}}) = \overline{\operatorname{conv}}(\Gamma(\varprojlim(\mathbb{T}, p)))$. By the Krien-Milman Theorem ([51, Theorem 3.23]), $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}}) = \overline{\operatorname{conv}}(\partial\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}}))$, so it suffices to show that for each $z \in \varprojlim(\mathbb{T}, p)$, the image $\Gamma(z)$ is an extreme point.

Fix $z \in \varprojlim(\mathbb{T}, p)$, and suppose that for all $a \in \mathcal{T}_\theta^{\mathscr{S}}$, we have $\Gamma(z)(a) = \int_{\varprojlim(\mathbb{T},p)} \Gamma(\omega)(a)\, \mathrm{d}\Omega(\omega)$ for some $\Omega \in \mathcal{M}^1(\varprojlim(\mathbb{T}, p))$. We then have

$$0 = \int_{\varprojlim(\mathbb{T},p)} (\Gamma(\omega) - \Gamma(z))(a)\ \mathrm{d}\Omega(\omega).$$

for all $a \in \mathcal{T}_\theta^{\mathscr{S}}$. In particular,

$$\begin{aligned}0 &= \int_{\varprojlim(\mathbb{T},p)} (\Gamma(\omega) - \Gamma(z))(\psi_{j,\infty}(\iota_{X(E_{\theta_j})}^{\otimes k}(x)\iota_{X(E_{\theta_j})}^{\otimes l}(y)^*))\, \mathrm{d}\Omega(\omega) \\ &= \int_{\varprojlim(\mathbb{T},p)} (\Gamma(\omega) - \Gamma(z))(\delta_{k,l} e^{\frac{-\beta l}{n^j}} \psi_{j,\infty}(\iota_{C(\mathbb{T})}(\langle y, x\rangle))\, \mathrm{d}\Omega(\omega)\end{aligned}$$

for all $j \in \mathbb{N}$, $x \in X(E_{\theta_j})^{\otimes k}$, $y \in X(E_{\theta_j})^{\otimes l}$. In particular, this happens for all $f \in C(\mathbb{T})^+$. Hence $\Omega = \delta_z$. Hence $\Gamma(z) \in \partial\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$, and so $\Gamma(\varprojlim(\mathbb{T}, p)) = \partial\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$. ☐

Corollary 6.1.14. *Resume the notation of Theorem 6.1.9, including the positivity assumption of Notation 6.1.8.*

1. *For $\beta > 0$, Γ induces an affine isomorphism $\Gamma_* : \mathcal{M}^1(\varprojlim(\mathbb{T}, p)) \to \mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$ such that*
$$\Gamma_*(\Omega)(a) = \int_{\varprojlim(\mathbb{T},p)} \phi_{z,\infty}(a)\,\mathrm{d}\Omega(z).$$
for $\Omega \in \mathcal{M}^1(\varprojlim(\mathbb{T}, p))$ and $a \in \mathcal{T}_\theta^{\mathscr{S}}$.

2. *If $\theta_0 \notin \mathbb{Q}$, then $\mathcal{S}_{\alpha_\infty,0}(\mathcal{T}_\theta^{\mathscr{S}})$ consists of a single trace.*

3. *For $\beta < 0$, $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}}) = \emptyset$.*

4. *For $\beta \neq 0$, $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{A}_\theta^{\mathscr{S}}) = \emptyset$. If $\theta_0 \in [0, 1) \setminus \mathbb{Q}$, then there exists a unique $\phi \in \mathcal{S}_{\alpha_\infty,0}(\mathcal{T}_\theta^{\mathscr{S}})$ that factors through to $\mathcal{A}_\theta^{\mathscr{S}}$.*

Proof. We begin by showing (1). Fix $\Omega \in \mathcal{M}^1(\varprojlim(\mathbb{T}, p))$. Then for analytic $a, b \in \mathcal{T}_\theta^{\mathscr{S}}$,

$$\begin{aligned}\Gamma_*(\Omega)(ab) &= \int_{\varprojlim(\mathbb{T},p)} \phi_{z,\infty}(ab)\,\mathrm{d}\Omega(z)\\ &= \int_{\varprojlim(\mathbb{T},p)} \phi_{z,\infty}(b\alpha_{i\beta,\infty}(a))\,\mathrm{d}\Omega(z)\\ &= \Gamma_*(\Omega)(b\alpha_{i\beta,\infty}(a)),\end{aligned}$$

which is the KMS$_\beta$-condition of (2.12). Since integrals are linear, Γ_* is affine.

By Proposition 2.3.7, $\mathcal{S}_{\alpha_\infty,0}(\mathcal{T}_\theta^{\mathscr{S}}) \neq \emptyset$. When θ_0 is irrational, for each $j \in \mathbb{N}$, $\theta_j \notin \mathbb{Q}$, and so $\mathcal{S}_{\alpha,0}(\mathcal{T}(E_{\theta_j}))$ contains a unique trace by Corollary 5.1.14. Hence $\mathcal{S}_{\alpha_\infty,0}(\mathcal{T}_\theta^{\mathscr{S}})$ consists of a unique trace ϕ, and so (2) holds. Moreover, for each $j \in \mathbb{N}$, $\phi_j \in \mathcal{S}_{\alpha,0}(\mathcal{T}(E_{\theta_j}))$ factors through to $\mathcal{S}_{\alpha,0}(\mathcal{A}_{\theta_j})$, so ϕ factors through to $\mathcal{S}_{\alpha_\infty,0}(\mathcal{A}_\theta^{\mathscr{S}})$, which is (4).

Finally, since $\mathcal{S}_{\alpha,\beta}(C^*(E_{\theta_j})) = \emptyset$ for $\beta \neq 0$, $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{A}_\theta^{\mathscr{S}}) = \emptyset$ for $\beta \neq 0$ by Proposition 6.1.5, as claimed in (3). ☐

Observation 6.1.15. *It has come to the author's attention that the sequences of measures in Notation 6.1.8 are not positive. To see this, fix $z = (z_j)_{j=0}^\infty \in \varprojlim(\mathbb{T}, p)$,*

and fix $\theta_0 \in [0,1)\backslash\mathbb{Q}$*. Inductively define* $\theta = (\theta_j)_{j=0}^{\infty}$ *as described in Notation 6.1.1. Then*

$$\begin{aligned}\epsilon_{z,1}(\{e^{2i\pi\theta_1}z_0\}) &= \sum_{j=0}^{\infty}\Big(e^{-\beta_0}\epsilon_{z,0}\circ r_0^{-j}(\{e^{2i\pi\theta_1}z_0\}\cap z_k\cdot I_k) \\ &\qquad\qquad - e^{-\beta_k+\beta_0 j}\epsilon_{z,0}\circ r_k^{-1}\circ r_0^{-j}(\{e^{2i\pi\theta_1}z_0\}\cap z_k\cdot I_k)\Big) \\ &= 0 - e^{-\beta_1} \\ &= -e^{\beta_1}.\end{aligned}$$

Hence, the measure $\epsilon_{z,1}$ *is not positive. So, the sequences of measures in Notation 6.1.8 are not sequences of positive measures. Example 6.1.16 will demonstrate why this is such an important property.*

Example 6.1.16. Let E be the following directed graph.

$$\begin{tikzcd} v \arrow[r, bend left, "f"] & w \arrow[l, bend left, "e"] \end{tikzcd}$$

Fix $\beta > \ln(\rho(A_E)) = 0$. Consider the measure ϵ on E^0 such that $\epsilon(\{w\}) = 1$ and $\epsilon(\{v\}) = e^{-\beta}$. Then

$$(1 - e^{-\beta}A_E)^{-1}\epsilon = \begin{pmatrix} 1 & -e^{-\beta} \\ -e^{-\beta} & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ -e^{-\beta}\end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

which is a probability measure on E^0. Hence, for $\mu, \nu \in E^*$,

$$\phi_\epsilon(S_\mu S_\nu^*) := \delta_{\mu,\nu}e^{-\beta|\mu|}\Big((1 - e^{-\beta}A_E)^{-1}\epsilon\Big)_{s(\mu)}$$

is a linear functional on $\mathcal{T}(E)$ such that $\phi_\epsilon(1) = 1$. However, if ϕ_ϵ was a KMS state, then the resulting state would satisfy the subinvariance condition of Theorem 5.1.6. However,

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \not\leq e^{\beta}\begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where the inequality denotes the operation applied pointwise, and so are not subinvariant. Hence $\phi_\epsilon \notin \mathcal{S}_{\alpha,\beta}(\mathcal{T}(E))$.

Obervation 6.1.15 seems to indicate that if $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$ is nonempty, then we require that the KMS_β state restricts to a sequence of rotationally invariant measures. Of course, there is only one invariant measure on $\mathbb{T}$ — the Haar measure. This leads us to the following conjecture.

Conjecture 6.1.17. *Resume the notation of Theorem 6.1.9, except for the positivity assumption of Notation 6.1.8. For $\beta > 0$, $\mathcal{S}_{\alpha_\infty,\beta}(\mathcal{T}_\theta^{\mathscr{S}})$ consists of a unique state ϕ, such that for $j \in \mathbb{N}$, $x \in X(E_{\theta_j})^{\otimes k_1}$, $y \in X(E_{\theta_j})^{\otimes k_2}$,*

$$\phi \circ \psi_j\left(\iota_{X(E_{\theta_j})}^{\otimes k_1}(x)\iota_{X(E_{\theta_j})}^{\otimes k_2}(y)^*\right) = \delta_{k_1,k_2} \int_{\mathbb{T}} \sum_{l=0}^{\infty} e^{2i\pi\theta_k l}\langle y, x\rangle(e^{2i\pi\theta_j l}v)\,\mathrm{d}(1-e^{-\beta_j})m(v),$$

where m is the Haar measure.

It appears that the analysis required to prove this result is vastly different from the course undertaken, and will be a project for future inverstigation.

Appendix A

Projective Limits and Direct Limits

A.1 Projective limits and Direct Limits

In this section we provide a brief overview of projective sequences of locally compact Hausdorff spaces, as well as direct limits of commutative C^*-algebras. We include the necessary notation to follow Chapter 3. For a more detailed approach, to projective limits, see [20, 53], and for more on direct limits of C^*-algebras, see [41, 48]

Definition A.1.1 ([53, Definition 29.9]). *A* ***projective sequence*** *is a pair of sequences $(X_i, \psi_i)_{i=0}^{\infty}$, such that for each $i \in \mathbb{N}$, X_i is a locally compact Hausdorff space, and $\psi_i : X_{i+1} \to X_i$ is continuous.*

Definition A.1.2. *Let $(X_i, \psi_i)_{i=0}^{\infty}$ be a projective sequence. A* ***projective limit*** *is a pair (X_∞, Ψ), where X_∞ is a nonempty topological space and $\Psi = (\Psi_i)_{i=0}^{\infty}$ is a sequence of continuous maps $\Psi_i : X_\infty \to X_i$ such that $\Psi_i = \psi_i \circ \Psi_{i+1}$.*

Theorem A.1.3 ([53, 29 C], [20, §24 Proposition 14])**.** *Let $(X_i, \psi_i)_{i=0}^{\infty}$ be a projective sequence. Then define*

$$\varprojlim(X_i, \psi_i) := \Big\{(x_i)_{i=0}^{\infty} \mid x_i \in X_i \text{ such that } x_i = \psi_i(x_{i+1})\Big\},$$

and for each $i \in \mathbb{N}$, $\Psi_i : \varprojlim(X_i, \psi_i) \to X_i$ by $\Psi_i\big((x_j)_{j=0}^{\infty}\big) = x_i$. Then the pair $(\varprojlim(X_i, \psi_i), \Psi)$ is a projective limit for $(X_i, \psi_i)_{i=0}^{\infty}$. Sets of the form

$$Z(U, n) := \big\{(x_i)_{i=0}^{\infty} \mid U \subseteq X_n \text{ is open and } x_n \in U\big\}$$

form a base for the topology on $\varprojlim(X_i, \psi_i)$. The pair $\big(\varprojlim(X_i, \psi_i), \Psi\big)$ is universal in the sense that if (X_∞, Φ) is another projective limit for $(X_i, \psi_i)_{i=0}^{\infty}$, then there exists a continuous map $\pi : X_\infty \to \varprojlim(X_i, \psi_i)$, such that Firgure A.1 commutes.

If ϕ is a sequence of continuous surjections, and X_∞ is compact, π is a continuous surjection.

$$\begin{tikzcd}
X_0 & \cdots \arrow[l] & X_j \arrow[l] & X_{j+1} \arrow[l, "\psi_n"'] & \cdots \arrow[l] & \varprojlim(X_i, \phi_i) \arrow[lllll, bend right, "\Psi_0"'] \arrow[lll, bend right, "\Psi_j"'] \arrow[ll, bend right, "\Psi_{j+1}"'] \\
 & & & & & X_\infty \arrow[u, "\pi"'] \arrow[ulllll, bend left, "\Phi_0"] \arrow[ulll, bend left, "\Phi_j"] \arrow[ull, "\Phi_{j+1}"]
\end{tikzcd}$$

Figure A.1: Theorem A.1.3

We call $\big(\varprojlim(X_i, \psi_i), \Psi\big)$ **the projective limit of** (X_i, ψ_i), because of the universal property.

Proposition A.1.4 ([20, §24 Proposition 13]). *Let $(X_i, \psi_i)_{i=0}^{\infty}$ be a projective sequence of Hausdorff spaces. The space $\varprojlim(X_i, \psi_i)$ is closed in the induced topology inherited from $\Pi_{i=0}^{\infty} X_i$.*

Theorem A.1.5 ([53, Theorem 29.13]). *Let $(X_i, \psi_i)_{i=0}^{\infty}$ be a projective sequence such that for each $i \in \mathbb{N}$, X_i is compact and ψ_i is surjective. Then $\varprojlim(X_i, \psi_i)$ is compact. Suppose that (X_∞, ϕ) is another projective limit of $(X_i, \psi_i)_{i=0}^{\infty}$, in which each $\phi_i : X_\infty \to X_i$ is surjective. If X_∞ is compact, then $\pi : X_\infty \to \varprojlim(X_i, \psi_i)$ is a surjection.*

Definition A.1.6 ([41, Chapter 6]). *A **directed sequence** $\big(A_i, \psi_i\big)_{n=0}^{\infty}$ of C^*-algebras consists of a sequence of C^*-algebras $\big(A_i\big)_{i=0}^{\infty}$ and a sequence of homomorphisms $\psi_i : A_i \to A_{i+1}$.*

*A **direct limit** is a pair (A_∞, Ψ) consisting of a C^*-algebra A_∞ and a sequence of homomorphisms $(\Psi_j)_{j=0}^{\infty}$ such that $\Psi_j = \Psi_{j+1} \circ \psi_j$ for all $j \in \mathbb{N}$.*

Theorem A.1.7 ([41, Theorem 6.1.2]). *Let $(A_i, \psi_i)_{i=0}^{\infty}$ be a direct sequence of C^*-algebras. Then there exists a direct limit $\big(\varinjlim(A_i, \psi_i), \Psi\big)$ such that $\varinjlim(A_i, \psi_i) = \overline{\bigcup_{i=0}^{\infty} \Psi_i(A_i)}$, that is universal in the sense that given another direct limit (A_∞, Φ), then there exists a homomorphism $\Phi_\infty : \varinjlim(A_i, \psi_i) \to A_\infty$ such that $\Phi_j = \Phi_\infty \circ \Psi_j$ for all $j \in \mathbb{N}$.*

Proposition A.1.8 ([20, §66 Theorem 8]). *Let X, Y be locally compact spaces, and $f : X \to Y$ be continuous. Then f induces a homomorphism $f^* : C_0(Y) \to C_0(X)$. Moreover, if $\psi : C_0(Y) \to C_0(X)$ is a homomorphism, then ψ induces a continuous map $\psi^* : X \to Y$.*

If X, Y, Z are locally compact spaces and $f : X \to Y$ and $g : Y \to Z$ are continuous, then $(g \circ f)^* = f^* \circ g^* : C_0(Z) \to C_0(X)$. Similarly, if $\phi : C_0(Z) \to C_0(Y)$ and $\psi : C_0(Y) \to C_0(X)$, then $(\psi \circ \phi)^* = \phi^* \circ \psi^* : X \to Z$.

Theorem A.1.9. *Let $(X_i, \psi_i)_{i=0}^{\infty}$ be a projective sequence of locally compact spaces. Then $(C_0(X_i), \psi_i^*)_{i=0}^{\infty}$ is a direct sequence of commutative C^*-algebras, and*

$$C_0(\varprojlim(X_i, \psi)) = \varinjlim(C_0(X_i), \psi_i^*).$$

Proof. By Proposition A.1.8, $(C_0(X_i), \psi_i^*)_{i=0}^{\infty}$ is a direct sequence of commutative C^*-algebras, and has direct limit $\big(\varinjlim(C_0(X_i), \psi_i^*), \Phi\big)$. Hence $\varinjlim(C_0(X_i), \psi_i^*)$ is a commutative C^*-algebra, and therefore isomorphic to $C_0(\Omega)$, for some locally compact Ω, by the Gelfand Naimark Theorem [41, Theorem 2.1.10]. By Proposition A.1.8, each Φ_i induces a continuous map $\Phi_i^* : \Omega \to X_i$ such that $\Phi_i^* = \psi_i \circ \Phi_{i+1}^*$ for all $i \in \mathbb{N}$. Then (Ω, Φ^*) is a projective limit for $(X_i, \psi_i)_{i=0}^{\infty}$. Hence, by Theorem A.1.5, there exists a continuous map $\pi : \Omega \to \varprojlim(X_i, \psi_i)$, such that $\Phi_i^* = \Psi_i \circ \pi$. So π induces a homomorphism of C^*-algebras, $\pi^* : C_0(\varprojlim(X_i, \psi_i)) \to C_0(\Omega)$, such that $\pi^* \circ \Psi_j^* = \Phi_j$. We have

$$\Phi_j = \pi^* \circ \Psi_j^* = \pi^* \circ \Psi_\infty \circ \Phi_j,$$

for all $j \in \mathbb{N}$. Since $\bigcup_{i=0}^{\infty} \Phi_i(C_0(X_i))$ is dense in $\varinjlim(C_0(X_i), \psi_i^*)$, π^* is an isomorphism. That is to say, Figure A.2 commutes. $\square$

$$
\begin{tikzcd}
C_0(X_0) \arrow[r] \arrow[rrrrr, bend left=40, "\Phi_0"] \arrow[rrrrrd, bend right=30, "\Psi_0^*"'] & \cdots \arrow[r] & C_0(X_j) \arrow[r, "\psi_j^*"'] \arrow[rrr, bend left=30, "\Phi_j"] \arrow[rrrd, bend right=25, "\Psi_j^*"'] & C_0(X_{j+1}) \arrow[r] \arrow[rr, bend left=20, "\Phi_{j+1}"] \arrow[rrd, bend right=15, "\Psi_{j+1}^*"'] & \cdots & \varinjlim\left(C_0(X_i), \psi_i^*\right) \arrow[d, bend left, "\Psi_\infty"] \\
 & & & & & C_0\big(\varprojlim(X_i, \psi_i)\big) \arrow[u, bend left, "\pi^*"]
\end{tikzcd}
$$

Figure A.2: Theorem A.1.9

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