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The noncommutative Gohberg-Krein theorem

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THE NONCOMMUTATIVE GOHBERG-KREIN THEOREM

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CERTIFICATION

I, Andreas Andersson, declare that this thesis, submitted in fulfilment of the requirements for the award of Doctor of Philosophy, in the School of Mathematics and Applied Statistics, Faculty of Engineering and Information Sciences, University of Wollongong, is wholly my own work unless otherwise referenced or acknowledged. The document has not been submitted for qualifications at any other academic institution.

Andreas Andersson
1 Oct 2015

Dedicated to

those with a strong interest in mathematics

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ABSTRACT

We prove a noncommutative higher-dimensional generalization of the classical Gohberg-Krein theorem. The latter says that the index of a Toeplitz operator acting on Hardy space is equal to minus the winding number of its symbol (a function on the circle). In the process we construct an explicit realization of the KK Thom class in terms of a Dirac operator.

KEYWORDS: K-theory, KK-theory, spectral triples, Thom isomorphism, spectral flow, Fredholm modules, crossed products.

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Chapter 1

Introduction

1.1 Overview

The classical Gohberg-Krein theorem was the first result showing that an analytically defined "index" could be identified with a "topological index" and calculated via a local formula. Generalizations of this theorem, in particular the Atiyah-Singer index theorem, have played an important role in many areas of mathematics. In this thesis we consider a noncommutative generalization which captures the fact that the original results involve an action of the real line $\mathbb{R}$ on a C^*-algebra. In the process we construct explicit (unbounded and bounded) representatives of the so-called Thom classes in Kasparov KK-theory. The construction also provides examples for spectral flow and index pairings in the more recent setting of nonunital and semifinite spectral triples. This thesis is based on [3].

1.2 The classical Gohberg-Krein theorem

Any square-integrable function f on the unit circle $\mathbb{S}^1 \subset \mathbb{C}$ has a Fourier-series expansion

$$f(z) = \sum_{k=-\infty}^{\infty} f_k z^k, \tag{1.1}$$

and the **Hardy space** $H^2(\mathbb{S}^1)$ of the circle is the subspace of $L^2(\mathbb{S}^1)$ spanned by the functions (1.1) with $f_k = 0$ for $k < 0$. The **Hardy projection** P, i.e. the projection onto the Hardy subspace of $L^2(\mathbb{S}^1)$, coincides with the projection onto the nonnegative spectrum

of the unbounded selfadjoint operator D on $L^2(\mathbb{S}^1)$ defined by

$$De_k := k\,e_k, \qquad \forall k \in \mathbb{Z}$$

in the orthonormal basis $e_k(z) := z^k$ for $L^2(\mathbb{S}^1)$. Note that, writing $z = e^{i\theta}$, we have

$$D = \frac{1}{i}\frac{d}{d\theta}.$$

Any function $f \in C(\mathbb{S}^1)$ acts on $L^2(\mathbb{S}^1)$ as a multiplication operator on $L^2(\mathbb{S}^1)$, which we denote by $\pi(f)$,

$$(\pi(f)\psi)(z) := f(z)\psi(z), \qquad \forall \psi \in L^2(\mathbb{S}^1).$$

For later comparison, we shall view the C^*-algebra $C(\mathbb{S}^1)$ as the unitization of $A := C_0(\mathbb{R})$. So we write $A^\sim := C(\mathbb{S}^1)$.

Definition 1.2.1. A **Toeplitz operator** is an operator on $H^2(\mathbb{S}^1)$ of the form $T_a := P\pi(a)P$ for some $a \in A^\sim$. In particular, the generator z of $A^\sim$ defines the **unilateral shift**

$$S := P\pi(z)P,$$

and the C^*-algebra $\mathcal{T} = C^*(S)$ generated by S (equivalently, by all Toeplitz operators) is called the **Toeplitz algebra**.

The important property of the present setup is that

($\star$) the commutator $[P, \pi(a)] := P\pi(a) - \pi(a)P$ is compact for all $a \in A^\sim$,

as can be seen by looking at the monomials $a = z^m$ and recalling the Stone-Weierstrass theorem. Property ($\star$) makes the triple $(\pi, L^2(\mathbb{S}^1), P)$ a so-called odd Fredholm module. In general, an odd Fredholm module over an algebra B is a triple $(\pi, \mathcal{H}, P)$ where $\pi : B \to \mathcal{B}(\mathcal{H})$ is a representation and $P \in \mathcal{B}(\mathcal{H})$ is a projection such that $\star$ holds (with B instead of $A^\sim$).

For any infinite-dimensional separable Hilbert space $\mathcal{H}$, we denote by $\mathcal{K} = \mathcal{K}(\mathcal{H})$ the ideal of compact operators on $\mathcal{H}$. The significance of ($\star$) is that it gives

$$(P\pi(a)P)(P\pi(b)P) + \mathcal{K} = P\pi(a)\pi(b)P + \mathcal{K}, \qquad \forall a, b \in A^\sim,$$

so $P\pi(a)P$ is invertible modulo $\mathcal{K}$ iff a is invertible. Thus we see the relevance of the following notion.

Definition 1.2.2. A bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is **Fredholm** if T is invertible modulo $\mathcal{K}$. If T is Fredholm, the **Fredholm index** $\mathrm{Index}_{\mathrm{Tr}}(T)$ of T is defined to be the dimension of the kernel of T minus the dimension of the cokernel of T,

$$\mathrm{Index}_{\mathrm{Tr}}(T) = \dim \mathrm{Ker}(T) - \dim \mathrm{Ker}(T^*) \in \mathbb{Z}.$$

The subscript Tr here refers to the operator trace $\mathrm{Tr} : \mathcal{B}(\mathcal{H})_+ \to [0, \infty)$, which is a well-defined functional on the set $\mathcal{B}(\mathcal{H})_+$ of positive elements $\mathcal{B}(\mathcal{H})$. The dependence of the index on Tr comes from the fact that if $P \in \mathcal{B}(\mathcal{H})$ is a projection onto a subspace $\mathcal{H}_0 \subset \mathcal{H}$ then $\dim \mathcal{H}_0 = \mathrm{Tr}(P)$. Therefore, if $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$ denote also the *projections* onto the spaces $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$ respectively, we have

$$\mathrm{Index}_{\mathrm{Tr}}(T) = \mathrm{Tr}(\mathrm{Ker}(T)) - \mathrm{Tr}(\mathrm{Ker}(T^*)).$$

The fact that a Fredholm operator has finite-dimensional kernel and cokernel is called **Atkinson's theorem**. A Fredholm module $(\pi, \mathcal{H}, P)$ over B will thus assign an integer $\mathrm{Index}_{\mathrm{Tr}}(P\pi(u)P) \in \mathbb{Z}$ to each invertible $u \in B$. One may ask how this integer depends on u, i.e. we consider the task of determining when two invertible elements u and v in B give rise to the same integer. The answer is that it is precisely when $[u] = [v]$ in odd K-theory $K_1(B)$ that we get

$$\mathrm{Index}_{\mathrm{Tr}}(P\pi(u)P) = \mathrm{Index}_{\mathrm{Tr}}(P\pi(v)P),$$

and this is the motivation for the definition of K_1 (to be recalled in §2.2).

We would also like to know what Fredholm modules $(\pi', \mathcal{H}', P')$ are such that the integer $\mathrm{Index}_{\mathrm{Tr}}(P'\pi(u)P')$ is equal to $\mathrm{Index}_{\mathrm{Tr}}(P\pi(u)P)$ for all invertible $u \in B$. The odd K-homology group $K^1(B)$ is defined as the set of equivalence classes of Fredholm modules $(\pi, \mathcal{H}, P)$, the equivalence relations being chosen such that the Fredholm index of $P\pi(u)P$ for invertible u is independent of the representative of the class in $K^1(B)$ (see §2.3).

Example 1.2.3. The unilateral shift $S : H^2(\mathbb{S}^1) \to H^2(\mathbb{S}^1)$, defined in the canonical basis

by $Se_k := e_{k+1}$, has Fredholm index

$$\mathrm{Index}_{\mathrm{Tr}}(S) = -1.$$

In particular, S is not of the form normal plus compact (since a normal operator has zero index and the index is stable under compact perturbations). Yet S is **essentially normal** in the sense that the failure of normality

$$[S^*, S] = \mathbf{1} - SS^* = |e_0\rangle\langle e_0|$$

is compact. Using

$$S^m[S^*, S]S^{*n} = S^m|e_0\rangle\langle e_0|S^{*n} = |e_m\rangle\langle e_n|,$$

it follows that every compact operator on Hardy space belongs to $\mathcal{T}$. In fact, we have the short exact sequence (the **Toeplitz extension**) [38, 39]

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C(\mathbb{S}^1) \longrightarrow 0. \tag{1.2}$$

Halmos proposed the idea of classifying which essentially normal operators T are of the form normal plus compact [57]. That motivated Brown-Douglas-Fillmore [24] to look at extensions

$$0 \longrightarrow \mathcal{K} \longrightarrow C^*(T, \mathbf{1}, \mathcal{K}) \longrightarrow C(X) \longrightarrow 0 \tag{1.3}$$

for the compact Hausdorff space X which is the essential spectrum of T, generalizing the Toeplitz extension (1.2). The solution to Halmos' problem is obtained in the form of the extension group $\mathrm{Ext}(X)^{-1}$, the property of being normal plus compact being related to the possibility of splitting the exact sequence (1.3). The group $\mathrm{Ext}(X)^{-1}$ can be shown to be isomorphic to the K-homology group $K_1(X) = K^1(C(X))$ of X. This useful extension picture of K-homology was rapidly shown to live on in Kasparov's bivariant K-theory [66], and plays an important role also in the generalization of the Gohberg-Krein theorem that is the topic of this thesis.

As a result of Example 1.2.3 and the fact that $S = T_z = P\pi(z)P$ and z generate $\mathcal{T}$ and $A^\sim = C(\mathbb{S}^1)$ respectively, we have the following.

Theorem 1.2.4. *The Toeplitz operator T_a is Fredholm iff its "symbol" $a \in A^\sim$ is invertible*

and, in that case, the Fredholm index of T_a is equal to minus the winding number of a,

$$\mathrm{Index}_{\mathrm{Tr}}(T_a) = -\mathrm{wind}(a).$$

For invertible u in the smooth subalgebra $\mathcal{A}^{\sim} := C^{\infty}(\mathbb{S}^1)$ we have the "local" formula

$$\mathrm{Index}_{\mathrm{Tr}}(T_u) = -\frac{1}{2\pi i}\int u^{-1}\, du.$$

A fundamental property of the Fredholm index $\mathrm{Index}_{\mathrm{Tr}}(T_u)$ is that it is stable under compact perturbations and homotopies. Therefore, we can regard the Gohberg-Krein result as a K-theoretical pairing

$$K_1(A) \times K^1(A) \to \mathbb{Z}, \qquad ([u],[P]) \to \mathrm{Index}_{\mathrm{Tr}}(P\pi(u)P)$$

between the K-theory class of u and the K-homology class $[P] = [\pi, L^2(\mathbb{S}^1), P]$. Such pairings can be generalized further using Kasparov KK-theory; the above example is a special case of the Kasparov product,

$$\otimes_A : KK^1(\mathbb{C}, A) \otimes KK^1(A, \mathbb{C}) \to KK^0(\mathbb{C}, \mathbb{C}) = \mathbb{Z}.$$

There are isomorphisms $KK^0(\mathbb{C},\mathbb{C}) \cong K_0(\mathbb{C}) \cong \mathbb{Z}$. The operator trace Tr induces an isomorphism of $K_0(\mathbb{C})$ with $\mathbb{Z}$ which assigns an integer (namely the Fredholm index of T_u) to the difference between the K-classes of $\mathrm{Ker}(P\pi(u)P)$ and $\mathrm{Ker}(P\pi(u^{-1})P)$.

Gohberg and Krein proved Theorem 1.2.4 in the framework of integral equations of Wiener-Hopf type [51]. Sometimes the terms "Toeplitz operator" and "Wiener-Hopf operator" are used interchangeably, but in the latter case one usually has in mind an operator T_f on $L^2(a,b)$ of the form

$$(T_f\psi)(t) := \int_a^b f(t-s)\psi(s)\, ds, \qquad \forall \psi \in L^2(a,b),$$

for some $a < b \in \mathbb{R}$. One has $T_f = P\pi(f)P$ where $\pi(f)$ is the operator of convolution with f on $L^2(\mathbb{R})$ and P is the projection which multiplies with the characteristic function of the interval (a,b). Now for $a = 0$ and $b = \infty$, so that P projects onto the positive half-axis $\mathbb{R}_+$, the operator T_f is unitarily equivalent to the operator T_f on $H^2(\mathbb{S}^1)$ which we considered

before. These are the operators considered in [51, Section 10].

Before formulating Theorem 1.2.4 more as it appears in [51], we recall some facts about compactification and unitization which facilitate comparison with Theorem 1.2.4. If X is a locally compact Hausdorff space, the minimal unitization $A^{\sim} = A \times \mathbb{C}$ of the C^*-algebra $A := C_0(X)$ can be identified with the C^*-algebra of continuous functions on the one-point compactification $X \cup \{\infty\}$ of X. An element $f + \lambda \mathbf{1} \in A^{\sim}$, with $f \in C_0(X)$ and $\lambda \in \mathbb{C}$, is regarded as a function on $X \cup \{\infty\}$ by setting $(f + \lambda \mathbf{1})(x) := f(x) + \lambda$ with the convention that $f(\infty) := 0$.

Now for $X = \mathbb{R}$ we can make the identification $\mathbb{R} \cup \{\infty\} = \mathbb{S}^1$. One may have that in mind when comparing Theorem 1.2.4 with the following result, which we refer to as the **Gohberg-Krein theorem**.

Theorem 1.2.5 (Gohberg-Krein [51, §10])**.** *Let $A := C_0(\mathbb{R})$ and let $f + \lambda \mathbf{1} \in A^{\sim}$ be invertible. Then the Fredholm index of the Wiener-Hopf operator T_f equals minus the winding number of f about $0 \in \mathbb{R}$:*

$$\mathrm{Index}_{\mathrm{Tr}}(T_f) = -\frac{1}{2\pi i} \int_{\mathbb{R}} f^{-1}(t) \frac{df(t)}{dt} dt.$$

The Gohberg-Krein theorem can be reformulated as a Toeplitz extension very similar to (1.2), as shown by Phillips-Raeburn [98, Section 4(a)]. The same approach is then taken in the noncommutative generalization, so it will be useful to begin by recalling it here. We note that the C^*-algebra $\mathcal{K}$ of compact operators (on any countably infinite-dimensional Hilbert space) is isomorphic to the crossed product

$$B := C_0(\mathbb{R}) \rtimes_\alpha \mathbb{R} \cong \mathcal{K}$$

of $A := C_0(\mathbb{R})$ by the action α of $\mathbb{R}$ by translations. Set $\mathfrak{H} := L^2(\mathbb{R})$. There is a representation of A on $L^2(\mathbb{R}, \mathfrak{H})$ given by

$$(\pi_\alpha(a)\psi)(t,s) := a(s-t)\psi(t), \qquad \forall\, \psi \in L^2(\mathbb{R}, \mathfrak{H}),\ t, s \in \mathbb{R},$$

which can be written as

$$(\pi_\alpha(a)\psi)(t) = \alpha_{-t}(a)\psi(t), \qquad \forall\, \psi \in L^2(\mathbb{R}, \mathfrak{H}),\ t \in \mathbb{R}, \tag{1.4}$$

and the crossed product B acts via the integrated representation

$$\hat{\pi}_\alpha(f) := \int_{\mathbb{R}} \pi_\alpha(f(t)) e^{2\pi i t D}\, dt, \tag{1.5}$$

where $D = -\sqrt{-1}d/dt \otimes \mathbf{1}$ is the generator of the unitary group on $L^2(\mathbb{R}) \otimes \mathfrak{H}$ implementing the translation action α.

The operator trace is a semifinite trace $\hat{\tau} = \mathrm{Tr}$ on the von Neumann algebra $\mathcal{N} := B'' = \mathcal{B}(L^2(\mathbb{R}))$, and it is in fact the **dual** trace of the Lebesgue integral τ on A, in the sense that

$$\hat{\tau}(\hat{\pi}(f^*)\hat{\pi}(g)) = \tau(\langle f|g\rangle_A), \qquad \forall\, f, g \in L^1(\mathbb{R}, A) \cap L^2(\mathbb{R}, \mathfrak{H}),$$

where $\langle f|g\rangle_A := \int_{\mathbb{R}} f(t)^* g(t)\, dt \in A$. Note that $\hat{\tau}$ restricts to $\tau \circ \pi_\alpha^{-1}$ on $\pi_\alpha(A)$. Note also that the Fredholm index of a Fredholm operator T on $L^2(\mathbb{R}, \mathfrak{H})$ can be written as

$$\mathrm{Index}_{\hat{\tau}}(T) := \hat{\tau}\big(\mathrm{Ker}(T)\big) - \hat{\tau}\big(\mathrm{Ker}(T^*)\big), \tag{1.6}$$

where $\mathrm{Ker}(T)$ and $\mathrm{Ker}(T^*)$ denote the kernel projections.

The Gohberg-Krein theorem is then the statement that for u in the unitization $A^\sim$ of A, we have

$$\mathrm{Index}_{\hat{\tau}}(P\pi_\alpha(u)P) = -\frac{1}{2\pi i}\tau(\delta(u)u^{-1}), \tag{1.7}$$

where $\delta := \partial/\partial t$ is the generator of the action α. One may also show that there is an extension

$$0 \longrightarrow A \rtimes_\alpha \mathbb{R} \longrightarrow \mathcal{T} \longrightarrow A \longrightarrow 0, \tag{1.8}$$

where $\mathcal{T}$ is the C^*-algebra generated by $B = A \rtimes_\alpha \mathbb{R} \cong \mathcal{K}$ and the Toeplitz operators T_a for all $a \in A^\sim$. Indeed, the extension (1.8) is isomorphic to Coburn's Toeplitz extension (1.2).

Note however that in this formulation, the trace τ is not finite on all of the C^*-algebra $A = C_0(\mathbb{R})$, complicating the task of generalizing the result to noncommutative algebras.

One more thing we can learn from this simplest example is that the index can be calculated as a residue:

$$\mathrm{Index}_{\hat{\tau}}(P\pi_\alpha(u)P) = -\operatorname*{Res}_{s=1} \hat{\tau}(\pi_\alpha(u^{-1})[D, \pi_\alpha(u)](\mathbf{1} + D^2)^{-s/2}). \tag{1.9}$$

To see this, look first at the function $u(z) = z^\kappa$ for some $\kappa \in \mathbb{N}$. For this choice of u, the index of the Toeplitz operator $P\pi_\alpha(u)P$ is equal to $-\kappa$ (minus the winding number of u), by the Gohberg-Krein result. Moreover, $[D, \pi_\alpha(u)] = \kappa\, \pi_\alpha(u)$, so

$$\mathrm{Tr}\left(\pi_\alpha(u^{-1})[D, \pi_\alpha(u)](\mathbf{1} + D^2)^{-s/2}\right) = \kappa\, \mathrm{Tr}((\mathbf{1} + D^2)^{-s/2}) = \kappa \sum_{j\in\mathbb{Z}} \frac{1}{(1 + j^2)^{s/2}}$$

which is finite for all $s > 1$. The function $s \to \sum_{j\in\mathbb{Z}}(1 + j^2)^{-s/2}$ differs from the Riemann zeta function $\zeta(s)$ by an entire function. It is well known that $\mathrm{Res}_{s=1}\, \zeta(s) = 1$. Thus formula (1.9) holds when u is a monomial. The general result follows since $C(\mathbb{S}^1)$ is generated by the function $z \to z$.

Moreover, using some facts about the dual trace $\mathrm{Tr} = \hat{\tau}$ the winding number formula in Theorem 1.2.5 can be derived directly from (1.9). The residue formula (1.9) will have an analogue in higher dimensions and for not necessarily commutative algebras A, and from it one derives a "local" formula which generalizes that for the winding number.

1.3 Noncommutative Toeplitz extension

The Gohberg-Krein theorem dates to 1957 [51] and is the first example of "topological index equals analytic index" theorem. In the subsequent years, generalizations in several directions were obtained, all of which were finally superseded by the Atiyah-Singer index theorem. In another direction, Matthias Lesch sought for a noncommutative analogue of the Gohberg-Krein theorem, i.e. a similar "local" index formula for Toeplitz operators with symbols in a not necessarily commutative algebra [79]. He considered a concrete unital C^*-algebra $A \subset \mathcal{B}(\mathfrak{H})$ with an action α of $\mathbb{R}$ by $*$-automorphisms. The algebra A embeds into the multiplier algebra of the crossed product $B := A \rtimes_\alpha \mathbb{R}$ by defining $\pi_\alpha(a)$ exactly as in (1.4) and representing B on $L^2(\mathbb{R}, \mathfrak{H})$ via the integrated representation (1.5). He assumes the existence of an α-invariant finite trace τ on A and considers its dual trace $\hat{\tau}$ characterized by (1.7). To obtain a numerical index in general, when the weak closure $\mathcal{N} := B''$ of the crossed product is not necessarily equal to all of $\mathcal{B}(L^2(\mathbb{R}, \mathfrak{H}))$, we have to replace the usual Fredholm index with the **Breuer-Fredholm index** with respect to $\hat{\tau}$, defined exactly as in (1.6). Again one has Atkinson's theorem and stability with respect to the ideal of so-called $\hat{\tau}$-compact operators in $\mathcal{N}$.

Theorem 1.3.1 ([79]). *Suppose that there is an α-invariant finite trace $\tau : A \to \mathbb{C}$ and let $\hat{\tau}$ be the dual trace extending τ to the weak closure of the crossed product. Then for a unitary $u \in \mathcal{A}$, the $\hat{\tau}$-index of $T_u \in \mathcal{T}$ is given by formula* (1.7)*, where δ is the generator of α.*

Moreover, Phillips-Raeburn showed how to extend this to the case when A is nonunital and τ is merely densely defined and lower semicontinuous [98].

So at present we have the Gohberg-Krein theorem for higher dimensions in the commutative case (this is the Boutet de Monvel theorem [18] which we have not discussed here), and for noncommutative algebras in the 1-dimensional case. We still need $\mathbb{R}^n$-actions on noncommutative algebras.

1.4 Statement of the result

The Lesch-Phillips-Raeburn formula has been put into the context of spectral triples [27]. Having such a close relation to KK-theory and the general local index formula is important for the higher-dimensional generalization discussed next.

1.4.1 Toeplitz algebra from $\mathbb{R}^n$-actions

Let A be a separable C^*-algebra and let α be a strongly continuous action of $\mathbb{R}^n$ on A. Briefly, we say that $(A, \mathbb{R}^n, \alpha)$ is a C^*-**dynamical system**. Set $B := A \rtimes_\alpha \mathbb{R}^n$. It turns out that the K-theoretical constructions that we want require the crossed product $B = A \rtimes_\alpha \mathbb{R}^n$ to be represented not on $L^2(\mathbb{R}^n, \mathfrak{H})$ but rather on an amplification thereof. We consider the Hilbert space

$$\mathcal{H} := \mathbb{C}^N \otimes L^2(\mathbb{R}^n, \mathfrak{H}),$$

where $\mathbb{C}^N$ carries an irreducible representation of the n-dimensional complex Clifford algebra $\mathbb{C}_n$. Explicitly,

$$N := \begin{cases} 2^{n/2} & \text{if } n \text{ is even,} \\ 2^{(n-1)/2} & \text{if } n \text{ is odd.} \end{cases}$$

The n-dimensional complex Clifford algebra $\mathbb{C}_n$ can then be identified with

$$\mathbb{C}_n \cong \begin{cases} \mathrm{M}_N(\mathbb{C}) & \text{if } n \text{ is even,} \\ \mathrm{M}_N(\mathbb{C}) \oplus \mathrm{M}_N(\mathbb{C}) & \text{if } n \text{ is odd.} \end{cases}$$

The irreducible representation of $\mathbb{C}_n$ for even n is on $\mathbb{C}^N$. For odd n there are two irreducible representations, given by sending the first respectively the second $\mathrm{M}_N(\mathbb{C})$-summand in $\mathbb{C}_n$ to the fundamental representation of $\mathrm{M}_N(\mathbb{C})$ on $\mathbb{C}^N$.

The representation of A is the diagonal one,

$$\pi_\alpha(a) := 1_N \otimes \pi_\alpha(a),$$

where 1_N is the identity matrix of size $N \times N$. The selfadjoint generators $D_1, \dots, D_n$ of the unitary group implementing α on $L^2(\mathbb{R}^n, \mathfrak{H})$ can be used to define the Dirac operator (the tensor product implicit)

$$\not{D} := \sum_{k=1}^{n} \gamma^k D_k, \tag{1.10}$$

where $\gamma^1, \dots, \gamma^n$ are Hermitian $N \times N$ matrices representing the generators of $\mathbb{C}_n$ on $\mathbb{C}^N$, satisfying therefore the Clifford relations $\gamma^j \gamma^k + \gamma^k \gamma^j = 2\delta^{jk}$.

Since $\not{D}$ is not invertible, yet one more "doubling-up" trick is necessary. Consider $\boldsymbol{\mathcal{H}} := \mathcal{H} \otimes \mathbb{C}^2$ and the operator

$$\boldsymbol{\not{D}} := \begin{pmatrix} \not{D} & 0 \\ 0 & -\not{D} \end{pmatrix} + m \begin{pmatrix} 0 & \mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}$$

for some arbitrary $m > 0$. We let $\boldsymbol{\not{P}}$ denote the spectral projection of $\boldsymbol{\not{D}}$ corresponding to the interval $[0, +\infty)$. We represent the algebra $A^\sim$ on $\boldsymbol{\mathcal{H}}$ by setting

$$\boldsymbol{\pi}_\alpha(a + \lambda \mathbf{1}) := \begin{pmatrix} \pi_\alpha(a) + \lambda \mathbf{1} & 0 \\ 0 & \lambda \mathbf{1} \end{pmatrix}$$

for $a \in A$ and $\lambda \in \mathbb{C}$. We write $\boldsymbol{B} := B \otimes \mathbb{C}^2$. The **Toeplitz algebra** is the C^*-subalgebra $\boldsymbol{\mathcal{T}}$ of $\mathcal{B}(\boldsymbol{\mathcal{H}})$ generated by $\mathrm{M}_N(\boldsymbol{B})$ together with elements of the form

$$T_a := \boldsymbol{\not{P}} \boldsymbol{\pi}_\alpha(a) \boldsymbol{\not{P}}$$

for $a \in A^{\sim}$. Our first result is the following Toeplitz extension.

Proposition 1.4.1. *There is a semisplit short exact sequence*

$$0 \longrightarrow \mathrm{M}_N(\boldsymbol{B}) \longrightarrow \boldsymbol{\mathcal{T}} \longrightarrow A \longrightarrow 0.$$

The operator $T_a \in \boldsymbol{\mathcal{T}}$ is Fredholm (as an operator on $\boldsymbol{\not{P}\mathcal{H}}$) relative to $\mathrm{M}_N(\boldsymbol{B})$ iff a is invertible in $A^{\sim}$.

As we shall discuss in §2.5.2, the triple $(\boldsymbol{\pi}_\alpha, \mathbf{M}_N(B), \boldsymbol{\not{P}})$ carries the same K-theoretical information as the triple $(\pi_\alpha, \mathrm{M}_N(B), \not{P})$, where $\not{F} := \not{D}(1+\not{D}^2)^{-1/2}$.

1.4.2 Thom classes

An odd Kasparov module from A to B is a generalization of an odd Fredholm module, with B replacing the complex numbers. Thus it involves a representation $\pi : A \to \mathcal{M}(B \otimes \mathcal{K})$ of A in the multiplier algebra of the stabilization of B, and a projection $P \in \mathcal{M}(B \otimes \mathcal{K})$ such that $[P, \pi(A)] \subset B \otimes \mathcal{K}$. It turns out that this is exactly what we have here. The triple $(\pi_\alpha, \mathcal{M}(\mathrm{M}_N(B)), P)$ defines a class $\mathbf{t}_\alpha$ in the Kasparov group $KK^1(A, B)$, which is defined as the set of equivalence classes of such Kasparov modules, for equivalence relations perfectly analogous to the case $B = \mathbb{C}$. Also here we have the extension picture $KK^1(A, B) \cong \mathrm{Ext}(A, B)^{-1}$, the group of classes of extensions of A by B, so the fact that we have a KK-class is expected from the existence of the Toeplitz extension. We refer to

$$\mathbf{t}_\alpha := [\boldsymbol{\pi}_\alpha, \mathrm{M}_N(\boldsymbol{B}), \boldsymbol{\not{P}}]$$

as the **Thom element** of $(A, \mathbb{R}^n, \alpha)$.

Remark 1.4.2. Consider the explicit expressions for the Dirac operator in low dimensions $n = 1, 2, 3$ given by

$$\not{D} = -D_1 \qquad \text{if } n = 1,$$

$$\not{D} = \begin{pmatrix} 0 & iD_1 + D_2 \\ -iD_1 + D_2 & 0 \end{pmatrix} \qquad \text{if } n = 2,$$

$$\not{D} = \begin{pmatrix} D_3 & iD_1 + D_2 \\ -iD_1 + D_2 & -D_3 \end{pmatrix} \qquad \text{if } n = 3,$$

where for both $n = 2$ and $n = 3$, the γ^k's in (3.15) are the Pauli matrices. There is one important difference between the Dirac operator for $n = 2$ compared to that for $n = 1$ and $n = 3$. Namely, the even-dimensional Dirac operator is what physicists call a "supercharge" [113, §5]: for even n we can always find a grading operator Γ on $\mathcal{H}$ such that $\Gamma\pi_\alpha(a) = \pi_\alpha(a)\Gamma$ for all $a \in A$ and $\Gamma\not{D} = -\not{D}\Gamma$. In the example $n = 2$ we can take Γ to be the third Pauli matrix $\gamma^3 = \operatorname{diag}(1, -1)$ (the diagonal matrix with eigenvalues 1 and -1). In general we can take $\Gamma = (-i)^{n/2}\gamma^1 \cdots \gamma^n$. We write

$$\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$$

for even n, with $\mathcal{H}_\pm$ the ±1-eigenspace of the grading operator Γ.

In this introduction we shall mostly discuss the results for odd n. The even case is similar but involves the index of $\boldsymbol{\pi}_\alpha(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_\alpha(e)$ for projections e over A, where $\not{\boldsymbol{R}}_+$ is the restriction of $\not{\boldsymbol{R}} := \not{\boldsymbol{D}}|\not{\boldsymbol{D}}|^{-1}$ to $\mathcal{H}_+$.

The $K_0(B)$**-valued index** $\operatorname{Index}(T_u)$ of for an invertible u in $A^\sim$ is defined as the difference

$$\operatorname{Index}(T_u) := [\operatorname{Ker}(T_u)] - [\operatorname{Ker}(T_u^*)]$$

between the K-theory classes of the kernel and cokernel projections of T. The fact that $[\operatorname{Ker}(T_u)]-[\operatorname{Ker}(T_u^*)]$ is a well-defined element of $K_0(B)$ follows from the long exact sequence in K-theory obtained from the Toeplitz extension given in Proposition 3.4.3 (a generalized Atkinson result).

It is also possible to view $\operatorname{Index}(T_u)$ as the index of a Fredholm operator on the right Hilbert $\boldsymbol{B}$-module $X = \boldsymbol{B}_B$ which is just $\boldsymbol{B}$ itself with the $\boldsymbol{B}$-valued inner product $\langle b_1|b_2\rangle_B := b_1^* b_2$ and the right action given by multiplication. Let $\boldsymbol{\pi}_B(u)$ be the action of u by left multiplication by $\boldsymbol{\pi}_\alpha(u)$ on this module. One shows that $\operatorname{Index}(T_u)$ is invertible modulo compact operators on X.

The data $(\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}})$ thus provides a homomorphism from $K_1(A)$ to $K_0(B)$, sending the class $[u]$ of a unitary over A to the $K_0(B)$-valued Fredholm index of T_u. Had X been a Hilbert space, i.e. a Hilbert module over the algebra $\mathbb{C}$, then T_u would have been an ordinary Fredholm operator. In that case, $(\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}})$ defines a class in the odd K-homology $K^1(A)$. The case of general Hilbert modules is taken care of by the bivariant K-functor $KK^1(A, B)$ which is part of Kasparov KK-theory.

Replacing A by $\mathrm{M}_r(A)$ for any $r \in \mathbb{N}$ one obtains a similar Toeplitz extension with the same formula for $u \in \mathrm{M}_r(\mathcal{A}^{\sim})$, tensoring τ with the standard trace on $\mathrm{M}_r(\mathbb{C})$. To simplify notation, for $x \in \mathrm{M}_r(A) = \mathrm{M}_r(\mathbb{C}) \otimes A$, we write

$$\boldsymbol{\pi}_B(x) := (\mathrm{id} \otimes \boldsymbol{\pi})(x), \qquad \not{\boldsymbol{R}}\boldsymbol{\pi}_B := (1_r \otimes \not{\boldsymbol{R}})\boldsymbol{\pi}_B(x)$$

as operators on $\mathbb{C}^r \otimes \boldsymbol{B} = \boldsymbol{B}^{\oplus r}$. We are going to prove the following.

Theorem 1.4.3. *Suppose that n is odd. The data $(\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}})$ is an odd Kasparov A-B-module, so it represents a class $\mathbf{t}_\alpha = [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}]$ in $KK^1(A, B)$. The map*

$$\partial : K_1(A) \to K_0(B), \qquad \partial([u]) := [u] \otimes_A \mathbf{t}_\alpha,$$

which takes $[u] \in K_1(A) = K^1(\mathbb{C}, A)$ to the Kasparov product with $\mathbf{t}_\alpha$, implements the Thom isomorphism. One has explicitly that

$$[u] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}] = \mathrm{Index}(\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}})$$

is the $K_0(B)$-valued index of the Toeplitz operator $T_u = \not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}}$.

1.4.3 The local index formula

Kasparov KK-theory can be made even more powerful by looking at "unbounded representatives" of KK-classes. The unbounded manipulations are usually carried out using a triple ("spectral triple") $(\mathcal{C}, \mathcal{H}, \not{D})$ consisting of a dense $*$-subalgebra $\mathcal{C}$ of A, a Hilbert space $\mathcal{H}$ and a selfadjoint operator $\not{D}$ on $\mathcal{H}$ satisfying certain conditions which were motivated from examples in classical geometry where $\not{D}$ is a Dirac-type operator on a smooth manifold. It is nontrivial but possible to prove that these conditions allow the construction of a Kasparov A-B-module $(\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}})$ for a suitable C^*-algebra B. Here one uses a double-up construction, just as in our example coming from $\mathbb{R}^n$-actions, to obtain an invertible operator $\not{\boldsymbol{D}}$, and $\not{\boldsymbol{R}} := \not{\boldsymbol{D}}|\not{\boldsymbol{D}}|^{-1}$ is the "phase" of $\not{\boldsymbol{D}}$. The algebra B is generated by commutators of the form $[\boldsymbol{\pi}_\alpha(a), \not{\boldsymbol{R}}]$, the continuous functions of $\not{\boldsymbol{D}}$ and some more elements. In our example, the algebra B does not exactly coincide with the crossed product but plays the same role, whence the notation. The construction of B shows that every spectral triple defines a KK-class. Moreover, every KK-class arises this way.

The interest in having an unbounded representative $(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$ for an element of $KK^\bullet(A, B)$ is that the index pairing may be computed explicitly, under some extra hypotheses on the spectral triple. Suppose that $\mathcal{C}$ has the property that every class in $K_\bullet(A)$ can be represented by a matrix over $\mathcal{C}^\sim$. Then it is enough to deduce an explicit formula for the index of T_u when u is a unitary over $\mathcal{C}^\sim$ (usually it is impossible to achieve better than that). To do so it is useful to systematically find the "smoothness" and "summability" properties on $(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$, and importantly the interplay between smoothness and integration theory, required for an explicit index formula. This leads to a rich theory of noncommutative integration theory and pseudodifferential calculus. The state of the art here is provided by [28, 29] where the most general "'local" (explicitly computable) index formula for spectral triples was deduced. In particular it allows the smoothness and summability conditions to be "relative to" a von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ and a choice of semifinite trace τ on $\mathcal{N}$. We will provide the background leading to these results in this thesis.

Going back to our example of the index theory connected to a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$, the main obstacle in applying the general local index formula is to show that there is a dense $*$-subalgebra $\mathcal{C}$ of A fulfilling the desired hypotheses. We shall have the following result.

Theorem 1.4.4. *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system. Then there is a dense $*$-subalgebra $\mathcal{C}$ of A such that $(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$ is a smoothly summable $(\mathcal{N}, \hat{\tau})$-semifinite spectral triple over A and the inclusion $\mathcal{C} \hookrightarrow A$ induces an isomorphism on K-theory.*

From this result, general facts about spectral triples (and properties of the dual trace) allow us to deduce the correct local formula for the index without much trouble.

(i) The $*$-algebra $\mathcal{C}$ is a complete locally convex algebra and the K-theoretical "Chern character" is a map $\mathrm{Ch} : K_\bullet(A) \to HP_\bullet(\mathcal{C})$ from K-theory to the continuous periodic cyclic homology of $\mathcal{C}$.

(ii) The finitely summable spectral triple $(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$ comes with a Chern character $\mathrm{Ch}(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$, which represents a class in $HP^\bullet(\mathcal{C})$ (continuous periodic cyclic cohomology).

(iii) (a) If n is even and $e \in \mathrm{M}_\infty(\mathcal{C}^\sim)$ is a projection, evaluating $\mathrm{Ch}(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$ on $\mathrm{Ch}(e)$ gives the index of the Fredholm operator $\boldsymbol{\pi}_B(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(e)$.

(b) If n is odd and $u \in \mathrm{U}_\infty(\mathcal{C}^\sim)$ is a unitary, evaluating $\mathrm{Ch}(\mathcal{C}, \mathcal{H}, \not{\mathcal{D}})$ on $\mathrm{Ch}(u)$ gives the index of the Fredholm operator $\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}}$.

(iv) The above pairing is independent of the cohomology class of the Chern character $\mathrm{Ch}(\mathcal{C}, \mathcal{H}, \not{D})$. Therefore, as long as we stay within the same cohomology class, we can use a more computable cocycle to calculate the same index.

The last point requires some explanation. If n is even, let $\Gamma = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$ be the grading operator on $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$. If n is odd, set $\Gamma := \sqrt{2i}\mathbf{1}$. The Chern character $\mathrm{Ch}(\mathcal{C}, \mathcal{H}, \not{D})$ is an $(n+1)$-multilinear functional on $\mathcal{C}$ given by

$$\mathrm{Ch}(\mathcal{C}, \mathcal{H}, \not{D})(a_0, \ldots, a_n) := \frac{\Gamma(n/2+1)}{n!} \hat{\tau}(\Gamma \not{\boldsymbol{R}}[\not{\boldsymbol{R}}, \pi(a_0)][\not{\boldsymbol{R}}, \boldsymbol{\pi}(a_1)] \cdots [\not{\boldsymbol{R}}, \boldsymbol{\pi}(a_n)])$$

for all $a_0, \ldots, a_n \in \mathcal{C}$, where $\Gamma(s)$ is the gamma function. Unfortunately, the Chern character is hard to compute with; in the commutative case the operator $\not{\boldsymbol{R}}$ is typically a singular-integral operator. We want to replace commutators with $\not{\boldsymbol{R}}$ by commutators with $\not{\boldsymbol{D}}$. Using the path

$$[0,1] \ni t \to \not{\boldsymbol{D}}_t := \not{\boldsymbol{D}}|\not{\boldsymbol{D}}|^{-t},$$

which starts at $\not{\boldsymbol{D}}_0 = \not{\boldsymbol{D}}$ and ends at $\not{\boldsymbol{D}}_1 = \not{\boldsymbol{R}}$, one can after much work find a useful cocycle in the same class as the Chern character. This is the **residue cocycle**

$$\phi_n(a_0, a_1, \ldots a_n) := \frac{\sqrt{\pi}}{n!} \operatorname*{Res}_{s=n} \hat{\tau}(\Gamma \boldsymbol{\pi}_\alpha(a_0)[\not{\boldsymbol{D}}, \boldsymbol{\pi}_\alpha(a_1)] \cdots [\not{\boldsymbol{D}}, \boldsymbol{\pi}_\alpha(a_n)](1 + \not{\boldsymbol{D}}^2)^{-n/2-s}).$$

The residue cocycle exists for spectral triples with "isolated spectral dimension". It turns out that the triple $(\mathcal{C}, \mathcal{H}, \not{D})$ associated with a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$ has isolated spectral dimension. However, to conclude that we need a second way of calculating the index pairing. A solution to this problem is provided by the "resolvent cocycle" Φ, which is a finite sequence $(\Phi_m)_{m=\bullet}^n$ of cocycles with values in the space of meromorphic functions on $\mathbb{C}$. However, under the sole assumption that the spectral triple is smoothly summable, $\Phi = \Phi(r)$ (viewed as a meromorphic function of $r \in \mathbb{C}$) has a simple pole at $r = (1-n)/2$ (where n is the spectral dimension) and

$$\operatorname*{Res}_{r=(1-n)/2} \Phi(r)(\mathrm{Ch}(u)) = -\sqrt{2\pi i}\ \mathrm{Index}_{\hat{\tau}}(\not{\boldsymbol{P}} \boldsymbol{\pi}_B(u) \not{\boldsymbol{P}}), \qquad \forall\, [u] \in K_1(A)$$

for n odd, and similarly the residue gives the index of $\boldsymbol{\pi}_B(e) \not{\boldsymbol{R}}_+ \boldsymbol{\pi}_B(e)$ in the even case by pairing with $\mathrm{Ch}(e)$.

Now some basic facts about the dual trace $\hat{\tau}$ can be used to produce an index formula which involves the original trace τ on A. To present the result we introduce a shorthand notation. For $a, b \in \mathcal{A}$ and $m = 1, \ldots, n$, we write

$$(a\delta(b))^m := \sum_{\varepsilon} (-1)^{\varepsilon} \prod_{k=1}^{m} a\delta_{\varepsilon(k)}(b),$$

where the sum is over all permutations ε of $\{1, \ldots, n\}$ and $(-1)^{\varepsilon}$ is the sign of ε. Here and sometimes later on, we shall simplify the formulas by assuming that u belongs to $\mathcal{A}^{\sim}$ and not some matrix algebra over $\mathcal{A}$. One should have in mind however that most interesting K-theory classes have representatives only in matrix algebra over $\mathcal{A}$.

Theorem 1.4.5. *Let n be odd. For each unitary u in $\mathcal{C}^{\sim}$,*

$$\mathrm{Index}_{\hat{\tau}}(T_u) = -\frac{2^{(n-1)/2}(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^n n!} \tau\big((u^*\delta(u))^n\big).$$

Let n be even. For each projection e in $\mathcal{C}$,

$$\mathrm{Index}_{\hat{\tau}}(\boldsymbol{\pi}_B(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(e)) = \frac{(-1)^{n/2}}{(n/2)!}\frac{2^n}{(2\pi i)^n}\tau\big((e\delta(e)\delta(e))^{n/2}\big).$$

In the next chapter we try to put together all the background needed to present the main result of [28]. This index theory has many aspects, as it uses both K- and KK-theory, semifinite Fredholm theory, the most general framework for spectral triples, cyclic cohomology and requires various extra tools for coping with the nonunital setting. Having all this material put together in one place, we can spend the third chapter on index pairings for $\mathbb{R}^n$-actions and show how they fit into the general index theory. In the final chapter we come to the original motivation for this work, namely to obtain index pairings for "Rieffel deformations". These deformations provide examples of C^*-algebras whose structure is intimately related to $\mathbb{R}^n$-actions.

Chapter 2

Index pairings

2.1 Notation

- $\mathbb{N} := \{1, 2, 3, \dots\}$ and $\mathbb{N}_0 := \{0, 1, 2, \dots\}$.
- $\mathcal{A}_+ :=$ positive cone in a $*$-algebra.
- $\mathcal{B}(\mathcal{H}) :=$ algebra of bounded operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$.
- $\mathcal{K} := \mathcal{K}(\mathcal{H}) :=$ algebra of compact operators on $\mathcal{H}$.
- $\mathbf{1} :=$ identity operator on a Hilbert space or unit in an algebra.
- $\mathcal{A}^{\sim} := \mathcal{A} \times \mathbb{C} :=$ minimal unitization of an algebra $\mathcal{A}$.
- $\mathcal{M}(\mathcal{A}) :=$ multiplier algebra of a nondegenerate algebra $\mathcal{A}$.
- $\mathcal{Q}(\mathcal{A}) := \mathcal{M}(\mathcal{A})/\mathcal{A}$.
- $A'' := (A')' :=$ double commutant of a C^*-algebra A.
- $X_B :=$ right Hilbert module over a C^*-algebra B.
- $\mathcal{L}_B(X) :=$ algebra of adjointable operators on a Hilbert module $X = X_B$.
- $\mathcal{K}_B(X) :=$ algebra of compact adjointable operators on a Hilbert module $X = X_B$.
- $S^n A := C_0(\mathbb{R}^n) \otimes A =:$ suspension of A.

- $\mathrm{M}_r(\mathbb{C})$:= algebra of $r \times r$ matrices.
- $\mathrm{M}_r(A) := \mathrm{M}_r(\mathbb{C}) \otimes A$ = algebra of $r \times r$ matrices with entries in an algebra A.
- $\mathrm{diag}(a, b)$:= diagonal matrix of size $(m + l) \times (m + l)$ with two blocks $a \in \mathrm{M}_m(A)$ and $b \in \mathrm{M}_l(A)$.
- $C(X, V)$:= continuous functions on a space X with values in a vector space A.

All algebras are over the field $\mathbb{C}$ of complex numbers in this thesis. By "nonunital" we mean "not necessarily unital" and by "noncommutative" we mean "not necessarily commutative".

2.2 K-theory

The facts recalled in this section with be essential for all parts of the sections and often used implicitly. Our main sources are [93] and [14].

2.2.1 Basic definitions

Definition 2.2.1 ([93, Def. 2.1.6])**.** Let A be a C^*-algebra. The **minimal unitization** of A is the algebra $A^\sim$ whose elements are of the form $(a, \lambda) \in A \times \mathbb{C}$ with multiplication

$$(a, \lambda)(b, \mu) := (ab + \lambda b + \mu a, \lambda\mu), \qquad \forall a, b \in A,\ \lambda, \mu \in \mathbb{C}.$$

We identify A with its image in $A^\sim$ under the isometric embedding $a \to (a, 0)$. We use the notations $a + \lambda$ and (a, λ) interchangeably and write $\mathbf{1} := (0, 1)$ (which is the identity element in $A^\sim$). The projection of $A^\sim$ onto $\mathbb{C} = \{(0, \lambda) \in A^\sim |\ \lambda \in \mathbb{C}\}$ is denoted by $\varepsilon : A^\sim \to \mathbb{C}$.

One has $A \cong A^\sim/\mathbb{C}$ (as vector spaces). If A already has a unit then $A^\sim \cong A \oplus \mathbb{C}$ is just a direct sum of C^*-algebras. See [93, Prop. 2.1.7].

Notation 2.2.2. Let A be a C^*-algebra. We let $\mathrm{M}_n(A)$ denote the C^*-algebra of $n \times n$ matrices with entries in A. The inductive limit $\mathrm{M}_\infty(A) = \bigcup_n \mathrm{M}_n(A)$ is defined by inserting $a \in \mathrm{M}_n(A)$ as the upper left corner of the matrix $\left(\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}\right)$ in $\mathrm{M}_{n+1}(A)$.

We shall write $1_r := \mathrm{diag}(\mathbf{1}, \dots, \mathbf{1}, 0, 0, \dots)$ for the diagonal matrix in $\mathrm{M}_\infty(A^\sim)$ whose first r entries equal the identity $\mathbf{1} \in A^\sim$ while all other entries are zero.

A projection in $\mathrm{M}_n(A)$ is referred to as a projection **over** A.

Definition 2.2.3. Let a, b be elements of $\mathrm{M}_r(A)$. We say that a and b are **stably unitarily equivalent**, written $a \sim_u b$, if there is an invertible $u \in \mathrm{GL}_{r+1}(A^\sim)$ such that

$$b = u^{-1}au.$$

It is a well-known fact that if $a \sim_u b$ then $b = u^{-1}au$ for some *unitary* u [93, Lemma 5.2.4].

Definition 2.2.4. If p is a projection in $\mathrm{M}_\infty(A^\sim)$, let $[p]$ denote its $\sim_u$-equivalence class. The set of such classes $[p]$ is an Abelian semigroup $V(A^\sim)$ under addition. The K_0-group of A is defined to be

$$K_0(A) := \mathrm{Ker}(K_{00}(A^\sim) \to \mathbb{Z}),$$

where $K_{00}(A^\sim)$ is the Grothendieck group of $V(A^\sim)$.

Remark 2.2.5. (i) A basic property of K-theory is that $K_0(A \oplus B) \cong K_0(A) \oplus K_0(B)$ for any two C^*-algebras A and B. If A is unital then $A^\sim \cong A \oplus \mathbb{C}$, so $K_0(A) = K_{00}(A)$ is the Grothendieck group of $V(A)$ [93, Prop. 6.2.2]. In general one has $K_0(A^\sim) \cong K_0(A) \oplus \mathbb{Z}$.

(ii) The equivalence relation $\sim_u$ can be replaced by stable homotopy equivalence, yielding the same equivalence classes of projections [14, Prop. 4.4.1]. This is true only because we allow for projection *over* $A^\sim$ and not only in $A^\sim$.

(iii) Every element of $K_0(A)$ can be written as a differences $[e] - [f]$ of equivalence classes of projections $e, f \in \mathrm{M}_\infty(A^\sim)$ with $e - f \in \mathrm{M}_\infty(A)$ [93, Prop. 6.2.7.1].

(iv) Every element of $K_0(A)$ can also be written as $[e] - [1_r]$ for some $r \in \mathbb{N}$, where e is a projection over $A^\sim$ (of matrix size $\geq r$) such that $e - 1_r \in \mathrm{M}_\infty(A)$ [93, Prop. 6.2.7.2]. This is the **standard picture** of $K_0(A)$ [14, Def. 5.5.1].

Elements of $K_0(A)$ are thus formal differences $[e] - [f]$ of classes of projections $e, f \in \mathrm{M}_\infty(A^\sim)$.

For reasons that will become clear in Section 2.2.5, $K_0(A)$ is often called the **even** K-theory of A. To introduce the "odd" K-theory, define the inductive limit $\mathrm{GL}_\infty(A^\sim) := \bigcup_r \mathrm{GL}_r(A^\sim)$ by mapping $v \in \mathrm{GL}_r(A^\sim)$ to $\left(\begin{smallmatrix} v & 0 \\ 0 & \mathbf{1} \end{smallmatrix}\right)$ in $\mathrm{GL}_{r+1}(A^\sim)$.

Definition 2.2.6. The **odd K-theory** of a C^*-algebra A is defined by

$$K_1(A) := \frac{\mathrm{GL}_\infty(A^\sim)}{\mathrm{GL}_\infty(A^\sim)_0},$$

where $\mathrm{GL}_\infty(A^\sim)_0$ is the connected component of the identity in $\mathrm{GL}_\infty(A^\sim)$.

In other words, $K_1(A)$ can be described as the set of connected components in $\mathrm{GL}_\infty(A^\sim)$, or as the set of homotopy equivalence classes of elements in $\mathrm{GL}_\infty(A^\sim)$.

Both $K_0(A)$ and $K_1(A)$ are Abelian groups. Any homomorphism $\phi : A \to B$ of C^*-algebras induces homomorphisms

$$\phi_* : K_0(A) \to K_0(B), \qquad \phi_* : K_1(A) \to K_1(B),$$

defined by applying ϕ entrywise to representative matrices. Obviously, the identity map induces the identity, and $(\phi \circ \psi)_* = \phi_* \circ \psi_*$, so both K_0 and K_1 are covariant functors from the category of C^*-algebras to the category of Abelian groups.

Remark 2.2.7 (K-theory of spaces)**.** The K-theory of C^*-algebras defined here generalizes topological K-theory of spaces: if X is a locally compact Hausdorff space then

$$K^0(X) \cong K_0(C_0(X)), \tag{2.1}$$

where the left-hand side is the topological K-theory of X and $C_0(X)$ is the C^*-algebra of all continuous functions on X vanishing at infinity. Recall that if X is compact then $K^0(X)$ is a group of equivalence classes of "virtual" vector bundles over X [4]. The isomorphism (2.1) is then a direct consequence of the Serre-Swan theorem [49, Cor. 3.21]. For locally compact X, one defines the "K-theory with compact support"

$$K^0(X) := \mathrm{Ker}(K^0(X \cup \{\infty\}) \to \mathbb{Z}),$$

where $\mathbb{Z} \cong K^0(\mathbb{C})$ is the subgroup coming from the added point $\{\infty\}$ in the one-point compactification of X. The isomorphism (2.1) now follows from Remark 2.2.5 and its counterpart for spaces.

The group structure on $K^0(X)$ comes from the direct-sum operation on vector bundles. Furthermore, the tensor-product operation on vector bundles makes $K^0(X)$ a commutative

ring. There is no ring structure on $K_0(A)$ in general.

Remark 2.2.8 (Topological versus algebraic)**.** There is also an *algebraic* K-theory defined for any C^*-algebra A [106], since A is in particular a ring. In fact, the algebraic and topological K_0-groups of A coincide.

Going over to K_1, algebraic and topological K-theory differ. The reason why topological K-theory is usually preferred is that it is homotopy invariant: if two homomorphisms $\varphi, \phi : A \to B$ are homotopic then the induced maps $\varphi_*, \phi_* : K_\bullet(A) \to K_\bullet(B)$ are equal. The properties of being homotopy invariant, half-exact, and stable implies that "Bott periodicity" holds in topological K-theory (see §2.2.5 below), and the long exact sequence (Theorem 2.2.12 below) reduces to a 6-term cyclic sequence. It should be noted however that algebraic $K_1(A)$, defined as the largest Abelianization of $\mathrm{GL}_\infty^\sim(A)$, is a finer invariant than topological $K_1(A)$. If A is a von Neumann algebra then A has so many unitaries that $K_1(A) = 0$, whereas there are some von Neumann algebras with nonzero algebraic K_1 [81].

Remark 2.2.9 (K-theory of non-C^*-algebras)**.** Topological K-theory can be defined not only for C^*-algebras but more generally for "local Banach algebras" $\mathcal{C}$ [14]. We will indeed consider local Banach algebras later on when we discuss spectral triples. As for $K_0(\mathcal{C})$ one obtains the same thing as when considering $\mathcal{C}$ as a ring (i.e. algebraic K_0), and we can take that as the definition. The local Banach algebras $\mathcal{C}$ that we need will always be dense $*$-subalgebras of a given C^*-algebra A, and they will satisfy $K_0(\mathcal{C}) \cong K_0(A)$ (see §2.6). Moreover, we will have $K_1(A) \cong K_1(\mathcal{C}) := \mathrm{GL}_\infty(\mathcal{C}^\sim)/\,\mathrm{GL}_\infty(\mathcal{C}^\sim)_0$. Therefore, we will only need to consider K-theory of C^*-algebras.

2.2.2 Normalization

Here we want to mention that in the odd K-theory of C^*-algebras, one has the luxury to work with unitaries instead of invertibles, and that each K_1-class has a representative which is normalized in a convenient way.

We write, as in [93, Def. 4.1.1],

$$\mathrm{GL}_r^\sim(A) := \{a \in \mathrm{GL}_r(\mathbb{C}) \otimes A^\sim \,|\, (\mathrm{id} \otimes \varepsilon)(a) = 1_r\},$$

where $1_r \in \mathrm{M}_r(\mathbb{C})$ is the unit matrix, $\varepsilon : A^\sim \to \mathbb{C}$ is the augmentation and $\mathrm{id} : \mathrm{GL}_r(\mathbb{C}) \to \mathrm{GL}_r(\mathbb{C})$ is the identity map. The inductive system which defines $\mathrm{GL}_\infty(A^\sim)$ maps the sub-

group $\mathrm{GL}_r^{\sim}(A) \subset \mathrm{GL}_r(A^{\sim})$ into $\mathrm{GL}_r^{\sim}(A)$, so we may define the inductive limit $\mathrm{GL}_\infty^{\sim}(A) := \bigcup_r \mathrm{GL}_r^{\sim}(A)$.

Let $\mathrm{U}_r(\mathbb{C})$ be the unitary group of $n \times n$ matrices, and write $\mathrm{U}_r(A^{\sim}) = \mathrm{U}_r(\mathbb{C}) \otimes A^{\sim}$ for the corresponding group of unitary matrices over $A^{\sim}$. We also have the "normalized" subgroup

$$\mathrm{U}_r^{\sim}(A) := \{a \in \mathrm{U}_r(\mathbb{C}) \otimes A^{\sim} | \, (\mathrm{id} \otimes \varepsilon)(a) = 1_r\}.$$

Again, the same inductive system $v \to \left(\begin{smallmatrix} v & 0 \\ 0 & 1 \end{smallmatrix}\right)$ can be used to define inductive limits $\mathrm{U}_\infty(A^{\sim})$ and $\mathrm{U}_\infty^{\sim}(A)$.

Proposition 2.2.10 ([93, Prop. 4.2.6]). *If G is a topological group, let G_0 denote the connected component of the identity element. For any C^*-algebra A and each $r \in \mathbb{N} \cup \{\infty\}$, there are group isomorphisms*

$$\frac{\mathrm{GL}_r(A^{\sim})}{\mathrm{GL}_r(A^{\sim})_0} \cong \frac{\mathrm{GL}_r^{\sim}(A)}{\mathrm{GL}_r^{\sim}(A)_0} \cong \frac{\mathrm{U}_r(A^{\sim})}{\mathrm{U}_r(A^{\sim})_0} \cong \frac{\mathrm{U}_r^{\sim}(A)}{\mathrm{U}_r^{\sim}(A)_0}.$$

In particular, for $r = \infty$ this gives four ways of describing $K_1(A)$.

The idea here is that polar decomposition makes $\mathrm{U}_\infty(A^{\sim})$ a deformation retract of $\mathrm{GL}_\infty(A^{\sim})$.

As a consequence of Proposition 2.2.10, if A is already unital then

$$K_1(A) \cong \frac{\mathrm{GL}_r(A)}{\mathrm{GL}_r(A)_0} \cong \frac{\mathrm{U}_r(A)}{\mathrm{U}_r(A)_0}.$$

Indeed, in that case $A^{\sim} \cong A \oplus \mathbb{C}$, and matrices over $A^{\sim}$ are of the form $a \oplus 1_r$ for some $a \in \mathrm{M}_r(A)$ and some $r \in \mathbb{N}_0$. Such a matrix is invertible iff a is invertible.

2.2.3 Stabilization

Instead of using matrix algebras over the C^*-algebra A, one can use the **stabilization** of A, i.e. the C^*-algebraic tensor product $A \otimes \mathcal{K}$ of A with the algebra $\mathcal{K} = \mathcal{K}(\mathcal{H})$ of compact operators on an infinite-dimensional separable Hilbert space $\mathcal{H}$. Indeed, let $(e_m)_{m\in\mathbb{N}}$ be an orthonormal basis for $\mathcal{H}$, and let P_m be the projection onto the subspace spanned by $e_1, \ldots, e_m$. Then $P_m \mathcal{B}(\mathcal{H}) P_m \cong \mathrm{M}_m(\mathbb{C})$ is a finite-dimensional subalgebra of $\mathcal{K}$. The sequence $(P_m)_{m\in\mathbb{N}_0}$ is an approximate identity for $\mathcal{K}$, implying that the union $\mathrm{M}_\infty(\mathbb{C}) =$

$\bigcup_m \mathrm{M}_m(\mathbb{C})$ is dense in $\mathcal{K}$. In other words, $\mathcal{K}$ is the norm completion of $\mathrm{M}_\infty(\mathbb{C})$ [93, Prop. 1.10.2.2].

Since $\mathrm{M}_m(A) = \mathrm{M}_m(\mathbb{C}) \otimes A$, one immediately obtains that $A \otimes \mathcal{K}$ is the norm completion of $\mathrm{M}_\infty(A)$ for each C^*-algebra A. Now, evidently

$$\mathrm{M}_m(A \otimes \mathcal{K}) \cong A \otimes \mathcal{K}$$

for all m, so it is not hard to obtain the following picture of K-theory.

Lemma 2.2.11. *For any C^*-algebra A, the group $K_0(A)$ is consists of unitary (or homotopy) equivalence classes of projections in $A \otimes \mathcal{K}$. The group $K_1(A)$ consists of equivalence classes of invertibles in $A \otimes \mathcal{K}$ [93, Cor. 7.1.10]:*

$$K_1(A) = \frac{\mathrm{GL}_1^\sim(A \otimes \mathcal{K})}{\mathrm{GL}_1^\sim(A \otimes \mathcal{K})_0}.$$

Later on we will often make the identification $A \otimes \mathcal{K} = \mathcal{K}_A(\ell^2(\mathbb{N}; A))$ of the stabilization with the C^*-algebra of compact operators on the "standard Hilbert A-module" $\ell^2(\mathbb{N}; A)$ introduced in particular Example 2.4.10.

Any minimal projection e in $\mathcal{K}$ defines an inclusion $A \hookrightarrow A \otimes \mathcal{K}$ of A into its stabilization by sending $a \in A$ to $a \otimes e$. It follows from Lemma 2.2.11 that the K-functors are **stable**, i.e. the inclusion $A \hookrightarrow A \otimes \mathcal{K}$ induces an isomorphism $K_\bullet(A) \cong K_\bullet(A \otimes \mathcal{K})$ for all C^*-algebras A.

2.2.4 Abstract index

Recall that for $n \in \mathbb{N}$, we write 1_n for the identity matrix of size $n \times n$ embedded into $\mathrm{M}_\infty(A^\sim)$ by adding zeros. Thus, when we write $a1_n$ for $a \in \mathrm{M}_m(A^\sim)$, we mean multiplication in some $\mathrm{M}_r(A^\sim)$ for large enough $r \geq \max\{m, n\}$.

Theorem 2.2.12 (Long exact sequence in K-theory)**.** *Let J be a closed two-sided ideal in a C^*-algebra A. Then from the exact sequence $0 \longrightarrow J \xrightarrow{i} A \xrightarrow{q} A/J \longrightarrow 0$ of C^*-algebras we have a long exact sequence*

$$K_1(J) \xrightarrow{i_*} K_1(A) \xrightarrow{q_*} K_1(A/J) \xrightarrow{\delta} K_0(J) \xrightarrow{i_*} K_0(A) \xrightarrow{q_*} K_0(A/J)$$

*in K-theory, with the **abstract index map** $\delta : K_1(A/J) \to K_0(J)$ defined by*

$$\delta([u]) := [w^{-1}1_n w] - [1_n], \qquad [u] \in K_1(A/J), \tag{2.2}$$

where $w \in \mathrm{GL}^{\sim}_{\infty}(A)$ is any invertible lift of $\mathrm{diag}(u, u^{-1})$. Moreover, δ is unique up to a sign.

Definition 2.2.13. Let $0 \longrightarrow J \longrightarrow A \xrightarrow{q} A/J \longrightarrow 0$ be a short exact sequence of C^*-algebras. An element $a \in A$ is **Fredholm** relative to J if the image of a in the quotient A/J is invertible. If $a \in A$ is Fredholm relative to J, the **abstract index** of a is defined as

$$\mathrm{ind}(a) := \delta([q(a)]),$$

where $\delta : K_1(A/J) \to K_0(J)$ is the connecting homomorphism (2.2).

Part of Theorem 2.2.12 states that $\delta \circ q_* = 0$, so that if u is already invertible in A then $\mathrm{ind}(a) = 0$. The index is designed to detect only the invertibles in A/J which are not coming from invertibles in A.

For the usual Fredholm index $\mathrm{Ind}_{\mathrm{Tr}}(T) = \mathrm{Tr}(\mathrm{Ker}(T^*T)) - \mathrm{Tr}(\mathrm{Ker}(TT^*))$ of a Fredholm operator T, it is obvious that the index is zero if T is normal, i.e. if $TT^* = T^*T$. One would therefore guess that $\mathrm{ind}(a) = 0$ when $a \in A$ is normal, but that is in fact false [86].

Proposition 2.2.14 (Additivity of index)**.** *Let $a, b \in A$ be invertible modulo J. Then*

$$\mathrm{ind}(ab) = \mathrm{ind}(a) + \mathrm{ind}(b).$$

Proof. We have $[q(ab)] = [q(a)][q(b)]$. Thus the result follows from the fact that δ is a homomorphism. □

2.2.5 Bott periodicity

The long exact sequence in K-theory (Theorem 2.2.12) does not rely on homotopy invariance of the $K_\bullet$-functors. An analogous result holds in algebraic K-theory. The most important consequence of homotopy invariance is that

$$K_\bullet(C[0,1) \otimes A) = 0, \qquad \forall \bullet = 0, 1, \tag{2.3}$$

as follows directly from the contractibility of the **cone** $CA := C[0,1)\otimes A$. To see why (2.3) is important, consider the short exact sequence

$$0 \longrightarrow SA \longrightarrow CA \longrightarrow A \longrightarrow 0,$$

where $SA := C_0(\mathbb{R}) \otimes A \cong C(0,1) \otimes A$ is the **suspension** of A. It follows from this short exact sequence (and Theorem 2.2.12) that the K_1-group of a C^*-algebra A is isomorphic to K_0 of its suspension,

$$K_0(SA) \cong K_1(A).$$

It is harder to show, but still true due to homotopy invariance, that $K_1(SA) \cong K_0(A)$. These two isomorphisms together give **Bott periodicity**,

$$K_{\bullet+2}(A) \cong K_{\bullet}(A), \tag{2.4}$$

where we define $K_m(A) := K_0(S^m A)$, with $S^m A := C_0(\mathbb{R}^m) \otimes A$ and $S^0 A := A$. The isomorphisms (2.4) are unique up to sign, as a consequence of the absence of natural transformations of the K-functors.

If we identify the m-fold suspension SA with the space of A-valued loops based at $0 \in A$, and more generally

$$S^m A \cong \{f \in C(\mathbb{S}^{m-1}) |\ f(1) = 0\},$$

the K-groups can be identified with the homotopy groups of $\mathrm{GL}_\infty(A^\sim)$:

$$K_m(A) = \pi_m(\mathrm{GL}_\infty(A^\sim)), \qquad \forall m \in \mathbb{N}_0.$$

The periodicity (2.4) then takes the form $\pi_{\bullet+2}(\mathrm{GL}_\infty(A^\sim)) \cong \pi_\bullet(\mathrm{GL}_\infty(A^\sim))$, which is how it was originally formulated by Bott for $A = \mathbb{C}$.

2.2.6 Traces on K-theory

From the definition of K-theory in terms of (stably) unitarily equivalent elements in matrix algebras over the C^*-algebra A, a necessary requirement for a functional $\phi : A \to \mathbb{C}$ to

induce a functional $\phi_* : K_0(A) \to \mathbb{C}$ is that

$$\phi(uau^{-1}) = \phi(a), \qquad \forall u \in \mathrm{GL}_1(A^\sim),$$

i.e. ϕ has to be a trace. Being a trace is also sufficient, for if ϕ is a trace then the componentwise application of ϕ also respects unitary equivalence of matrices over A. So a trace induces a map on K_0, and via Bott periodicity also a map on K_1. With a trace τ at hand, evaluating τ on an abstract index $\mathrm{ind}(a)$ gives a numerical quantity $\tau(\mathrm{ind}(a)) \in \mathbb{R}$. We will usually need to consider functionals τ which are allowed to take the value $+\infty$ on some elements (see Section 2.3.2).

Definition 2.2.15. Let $J \subset A$ be an ideal and let τ be a trace on J which is finite on all projections in J. If $a \in A$ is invertible modulo J then the τ**-index** of a is the real value defined by

$$\mathrm{ind}_\tau(a) := \tau_*(\mathrm{ind}(a)),$$

where the abstract index $\mathrm{ind}(a)$ is as in Definition 2.2.13.

Corollary 2.2.16. *Let $a, b \in A$ be Fredholm relative to J. Then*

$$\mathrm{ind}_\tau(ab) = \mathrm{ind}_\tau(a) + \mathrm{ind}_\tau(b).$$

Proof. We apply τ_* to the K_0-class $\mathrm{ind}(ab)$ and use Theorem 2.2.14. $\square$

Now look again at the long exact sequence given in Theorem 2.2.12. In the special case that a unitary $u \in \mathrm{GL}_n^+(A/J)$ lifts to a partial isometry $v \in \mathrm{M}_n(A)$ (as in the case $A = \mathcal{B}(\mathcal{H}), J = \mathcal{K}(\mathcal{H})$), the index map gives the difference $[1 - v^*v] - [1 - vv^*]$ between the classes of the kernel and cokernel projections of v, i.e. the Fredholm index of v relative to J. This is seen by noting that $\mathrm{diag}(u, u^{-1})$ lifts to

$$w = \begin{pmatrix} v & 1 - vv^* \\ 1 - v^*v & v^* \end{pmatrix},$$

(which is invertible since v is a partial isometry) so that

$$[w1_nw^{-1}] - [1_n] = \left[\begin{pmatrix} v & 1_n - vv^* \\ 1_n - v^*v & v^* \end{pmatrix}\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} v^* & v^*v - 1_n \\ vv^* - 1_n & v \end{pmatrix}\right] - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right]$$
$$= \left[\begin{pmatrix} vv^* & 0 \\ 0 & 1_n - v^*v \end{pmatrix}\right] - \left[\begin{pmatrix} 1_n & 0 \\ 0 & 0 \end{pmatrix}\right] = [1 - v^*v] - [1 - vv^*],$$

i.e.

$$\delta([u]) = [1 - v^*v] - [1 - vv^*]. \tag{2.5}$$

If there is a trace τ on J then applying the induced homomorphism $\tau_* : K_0(J) \to \mathbb{R}$ one obtains the numerical index

$$\mathrm{ind}_\tau(v) = \tau(1 - v^*v) - \tau(1 - vv^*).$$

The prototype example is given by

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H}) \longrightarrow 0$$

where $\mathcal{Q}(\mathcal{H}) = \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin algebra. In the polar decomposition $T = V|T|$ of a Fredholm operator T on $\mathcal{H}$, the positive part $|T|$ does not contribute to the index. Applying the operator trace $\tau = \mathrm{Tr}$ to $\mathbf{1} - V^*V - (\mathbf{1} - VV^*)$ gives the Tr-Fredholm index, which in this case is an integer and coincides precisely with the Fredholm index of T in the usual sense.

2.3 K-homology

Recall that an operator T on a Hilbert space $\mathcal{H}$ is **Fredholm** if the kernel and range projections $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$ have finite operator trace. This implies that the range of T is closed, which is usually included in the definition; see [49, §4.1].

To obtain an index pairing between K-theory and some kind of group coming from classes of "Fredholm modules", to be defined shortly, we need the following. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space and write it as a direct sum $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ of two infinite-dimensional separable Hilbert spaces $\mathcal{H}_\pm$.

Let A be a C^*-algebra with a representation $\pi : A \to \mathcal{B}(\mathcal{H})$ on $\mathcal{H}$ by **even** operators, i.e. for all $a \in A$, the operator $\pi(a)$ takes the form

$$\pi(a) = \begin{pmatrix} \pi_+(a) & 0 \\ 0 & \pi_-(a) \end{pmatrix} \in \mathcal{B}(\mathcal{H}_+ \oplus \mathcal{H}_-)$$

for some representations $\pi_\pm : A \to \mathcal{B}(\mathcal{H}_\pm)$. Let $F = \left(\begin{smallmatrix} 0 & F_- \\ F_+ & 0 \end{smallmatrix}\right)$ be an operator on $\mathcal{H}_+ \oplus \mathcal{H}_-$ such that $U := F_+ : \mathcal{H}_+ \to \mathcal{H}_-$ is a unitary operator. Supposed that

$$[\pi(a), F] \in \mathcal{K}(\mathcal{H}), \qquad \forall a \in A. \tag{2.6}$$

Let $\tau :=$ Tr denote the operator trace on $\mathcal{K}$ and let ind_τ denote the associated numerical index (recall Definition 2.2.15). We argue that condition (2.6) will allow the data $(\pi, \mathcal{H}, F)$ to "pair" with $K_0(A)$. Indeed (omitting π from the notation), if $[a, U] \in \mathcal{K}$ for all $a \in A$ then if e is a projection (*in* and not over A for simplicity) and $f = u^*eu$ for some unitary $u \in A$, we get that

$$\begin{aligned} fUf &= u^*euUu^*eu = u^*euu^*Ueu + \text{compacts} \\ &= u^*eUeu + \text{compacts}\,. \end{aligned} \tag{2.7}$$

So $\pi(fUf) \sim_u \pi(eUe)$ in $\mathcal{Q}(\mathcal{H})$ via the unitary $\pi(u)$, and the same is true on $e\mathcal{H}$. Under the assumption (2.6) we have

$$(fUf)(fU^*f) + \mathcal{K} = e + \mathcal{K} = (fU^*f)(eUe) + \mathcal{K},$$

which says that fUf is a Fredholm operator on $e\mathcal{H}$. Then from Theorem 2.2.12 we get that

$$\begin{aligned} \mathrm{ind}_\tau(fUf|_{e\mathcal{H}}) &= \mathrm{ind}_\tau(eUe|_{e\mathcal{H}}) \\ &= \tau(e - eU^*eUe) - \tau(e - eUeU^*e), \end{aligned}$$

so the index of the operator eUe depends only on the K-theory class of e. With these observations we are ready to define K-homology.

2.3.1 Fredholm modules

Definition 2.3.1 ([62, Def. 8.1.1])**.** A **Fredholm module** over a C^*-algebra A is a triple $(\pi, \mathcal{H}, F)$ where $\pi : A \to \mathcal{B}(\mathcal{H})$ is a $*$-representation of A on an infinite-dimensional Hilbert space $\mathcal{H}$ and F is a selfadjoint operator on $\mathcal{H}$ such that

$$[F, \pi(A)] \subset \mathcal{K}, \qquad \pi(A)(F^2 - \mathbf{1}) \subset \mathcal{K}.$$

The Fredholm module is said to be **even** if there is a decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with respect to which F is odd and $\pi(A)$ is even. Otherwise $(\pi, \mathcal{H}, F)$ is said to be **odd**.

Two such Fredholm modules $(\pi_1, \mathcal{H}_1, F_1)$ and $(\pi_2, \mathcal{H}_2, F_2)$ are said to be **unitarily equivalent** if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$ such that $F_1 = U^* F_2 U$ and $\pi_1(a) = U^* \pi_2(a) U$ for all $a \in A$.

We say that $(\pi, \mathcal{H}, F')$ is a **compact perturbation** of $(\pi, \mathcal{H}, F)$ if

$$\pi(a)(F - F') \in \mathcal{K}, \qquad \forall a \in A.$$

Definition 2.3.2 ([62, §8.2])**.** Let A be a C^*-algebra. Define two Fredholm modules over A to be **equivalent** if the first is unitarily equivalent to a compact perturbations of the other, as specified in Definition 2.3.1. By the **even K-homology** of A, denoted by $K^0(A)$, we mean the Grothendieck group of the semigroup of equivalence classes of Fredholm modules over A. The group operation is the one coming from the direct sum $\oplus$.

Remark 2.3.3. The Grothendieck construction can be replaced by taking the addition modulo degenerate modules, i.e. those for which $F^* = F$, $F^2 = \mathbf{1}$ and $[\pi(A), F] = \{0\}$. Indeed, if $(\pi, \mathcal{H}, F)$ is a degenerate Fredholm module over A then the infinite direct sum $(\pi^{(\infty)}, \mathcal{H}^{(\infty)}, F^{(\infty)})$ of $(\pi, \mathcal{H}, F)$ with itself is also a Fredholm module over A (this is not true for any Fredholm module). Now $(\pi^{(\infty)}, \mathcal{H}^{(\infty)}, F^{(\infty)}) \oplus (\pi, \mathcal{H}, F)$ is unitarily equivalent to $(\pi, \mathcal{H}, F)$, so the class of $(\pi, \mathcal{H}, F)$ in $K^0(A)$ must be zero (every group has cancellation) [62, Prop. 8.2.8].

Remark 2.3.4. The Fredholm module is said to be **normalized** if $F^2 = \mathbf{1}$. Sometimes Fredholm modules with $F^2 \neq \mathbf{1}$ are called "pre-Fredholm modules" [28]. We prefer to just say "Fredholm module" and add the term "normalized" in case $F^2 = \mathbf{1}$ because the same terminology is often used for Kasparov modules (which will be defined in §2.4.4).

Remark 2.3.5 ([43, Lemma I.A.2.1])**.** Let $(\pi, \mathcal{H}, F)$ be a Fredholm module over A.

(i) Suppose that $F^2 - \mathbf{1}$ is compact. Extend π to the unitization $A^\sim = A \times \mathbb{C}$ by setting $\pi(a+\lambda\mathbf{1}) := \pi(a)+\lambda\mathbf{1}$ for all $a \in A$ and $\lambda \in \mathbb{C}$. Then $(\pi, \mathcal{H}, F)$ is a Fredholm module over $A^\sim$.

(ii) Let the elements of the algebra $\mathrm{M}_r(\mathbb{C}) \otimes A$ act in $\mathbb{C}^r \otimes \mathcal{H}$ via the representation by $\mathrm{id} \otimes \pi$. Then $(\mathrm{id} \otimes \pi, \mathbb{C}^r \otimes \mathcal{H}, \mathbf{1} \otimes F)$ is a Fredholm module over $\mathrm{M}_r(A)$.

With the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ associated to an even Fredholm module $(\pi, \mathcal{H}, F)$, we write

$$F_+ : \mathcal{H}_+ \to \mathcal{H}_-$$

for the restriction of $F = \left(\begin{smallmatrix} 0 & F_- \\ F_+ & 0 \end{smallmatrix}\right)$ to $\mathcal{H}_+$.

Proposition 2.3.6. *Let* $\tau = \mathrm{Tr}$ *denote the operator trace on* $\mathcal{B}(\mathcal{H})$*. If* $(\pi, \mathcal{H}, F')$ *is a compact perturbation of* $(\pi, \mathcal{H}, F)$ *then*

$$\mathrm{ind}_\tau(\pi(e)F'_+\pi(e)) = \mathrm{ind}_\tau(\pi(e)F_+\pi(e))$$

for all projections $e \in \mathrm{M}_\infty(A^\sim)$.

Proof. We have $\pi(e)F'\pi(e) + \mathcal{K} = \pi(e)F\pi(e) + \mathcal{K}$. To obtain the τ-index we simply have to, by definition, lift this coset from $\mathcal{Q}(\mathcal{H})$ to $\mathcal{B}(\mathcal{H})$. ☐

This means that a compact perturbation of $(\pi, \mathcal{H}, F)$ induces the same pairing with $K_0(A)$ as $(\pi, \mathcal{H}, F)$ itself. It is also clear that if $(\pi_1, \mathcal{H}_1, F_1)$ and $(\pi_2, \mathcal{H}_2, F_2)$ are unitarily equivalent then $\mathrm{ind}_\tau(\pi_1(e)F\pi_1(e)) = \mathrm{ind}_\tau(\pi_2(e)F\pi_2(e))$ (cf. the discussion in the beginning of this section).

We thus have a well-defined pairing of K-theory and K-homology

$$K_0(A) \times K^0(A) \to \mathbb{R}, \qquad ([\pi, \mathcal{H}, F], [e]) \to \mathrm{ind}_\tau(\pi(e)F_+\pi(e)).$$

Remark 2.3.7. The long exact sequence in K-theory (Theorem 2.2.12) that we use for the pairing between K_0 and K^0 comes from the sequence

$$0 \longrightarrow \mathcal{K}(\mathcal{H}) \longrightarrow \mathcal{B}(\mathcal{H}) \longrightarrow \mathcal{Q}(\mathcal{H}) \longrightarrow 0.$$

Operators of the form $\pi(e)F_+\pi(e)$ define elements of $K_1(\mathcal{Q})$, for $e \in A$, and map via the connecting homomorphism to their index in $K_0(\mathcal{K})$.

2.3.1.1 Odd Pairing

Definition 2.3.8. The equivalence classes of odd Fredholm modules for unitary equivalence and compact perturbations give rise, via the Grothendieck construction, to the Abelian group $K^1(A)$, the **odd K-homology** of A. The group operation is induced from taking direct sums.

If $(\pi, \mathcal{H}, F)$ is an odd Fredholm module over A then we can form the operator $P := (F+\mathbf{1})/2$ which is a projection modulo compacts. In fact, P will usually be the operator of more direct interest, so we sometimes write $(\pi, \mathcal{H}, P)$ instead of $(\pi, \mathcal{H}, F)$ for odd Fredholm modules. Since compact perturbations do not affect the class in $K^1(A)$, each class can be represented by a Fredholm module $(\pi, \mathcal{H}, P)$ where P is a true projection. However, other properties of the representative may be impossible to obtain if we insist that P is a projection. We will therefore need to discuss the full generality where $P - P^2$ and $P - P^*$ can be nonzero elements of $\mathcal{K}$. This is the reason why somewhat awkward expressions such as $P\pi(u)P - (\mathbf{1} - P)$ will appear in the following.

So in the odd case we pair a class $[u] \in K_1(A)$ with an odd Fredholm module $(\pi, \mathcal{H}, P)$ over A by sending these classes to the τ-index of the operator $P\pi(u)P$. Again we need the connecting homomorphism $\delta : K_1(\mathcal{Q}) \to K_0(\mathcal{K})$. We obtain a pairing

$$K_1(A) \times K^1(A) \to \mathbb{R}, \qquad ([u], [\pi, \mathcal{H}, P]) \to \mathrm{ind}_\tau(P\pi(u)P - \mathbf{1} + P)$$

where again $\tau = \mathrm{Tr}$ is the operator trace.

If $(\pi_1, \mathcal{H}_1, P_1)$ is unitarily equivalent to $(\pi_2, \mathcal{H}_2, P_2)$ then $\mathrm{ind}_\tau(P_1\pi_1(u)P_1) = \mathrm{ind}_\tau(P_2\pi_2(u)P_2)$, which is seen just as for projections e, f over A in the even case. For compact perturbations we can also do as in the even case.

It remains to show that $\mathrm{ind}_\tau(P\pi(u)P - \mathbf{1} + P)$ is independent of the representative of the class of u. If $v = x^{-1}ux$ then, since we have $[P, \pi(a)] \in \mathcal{K}$ for all $a \in A$,

$$P\pi(v)P + \mathcal{K} = P\pi(x^{-1}ux)P + \mathcal{K} = \pi(x)^{-1}P\pi(u)P\pi(x) + \mathcal{K}.$$

This means that $P\pi(v)P + \mathcal{K}$ lifts from $\mathcal{Q}(\mathcal{H})$ to $\pi(x)^{-1}P\pi(u)P\pi(x) \in \mathcal{B}(\mathcal{H})$. The kernel

and cokernel projections of $\pi(x)^{-1}P\pi(u)P\pi(x)$ and $P\pi(u)P$ differ only by conjugation by $\pi(x)$, so when we apply τ to $\pi(x)^{-1}P\pi(u)P\pi(x)$ we get the same index, as required.

Remark 2.3.9. We also mention that Higson has developed a "dual-algebra" approach to K-homology [59], [62, Section 8.4], where $K^0(A)$ is defined as the K_1-group of a certain C^*-algebra associated with A, and vice versa for K^1, K_0. We can anticipate this result from the way a normalized even Fredholm module involves a unitary $U = F_+ \in \mathcal{B}(\mathcal{H})$ (as would be used to define a K_1-class) while an odd Fredholm module involves a projection $P = 2F + \mathbf{1} \in \mathcal{B}(\mathcal{H})$ (as in K_0).

2.3.2 Weights on operator algebras

We summarize some facts about weights on C^*-algebras from [73], [112, §7.4], [109]. Let A be a C^*-algebra and let A_+ be its positive cone. We write $\mathbb{R}_+ := [0, \infty)$.

Definition 2.3.10. A function $\varphi : A_+ \to [0, +\infty]$ on A is called a **weight** if $\varphi(x + y) = \varphi(x) + \varphi(y)$ and $\varphi(\lambda x) = \lambda\varphi(x)$ for all $x, y \in A_+$ and $\lambda \in \mathbb{R}_+$.

Definition 2.3.11. The **half-domain** of a weight φ is the vector space

$$\mathrm{Dom}^{1/2}(\varphi) := \{a \in A |\ \varphi(a^*a) < +\infty\},$$

while **positive domain** of φ is

$$\mathrm{Dom}(\varphi)_+ := \{a \in A_+ |\ \varphi(a) < +\infty\}.$$

The **domain** of φ is then defined as the linear span of the positive domain:

$$\mathrm{Dom}(\varphi) := \mathrm{span}_{\mathbb{C}} \mathrm{Dom}(\varphi)_+ = \mathrm{span}\{a^*b \in A |\ a, b \in \mathrm{Dom}^{1/2}(\varphi)\}.$$

Every weight φ has an extension to a continuous linear functional $\varphi : \mathrm{Dom}(\varphi) \to \mathbb{C}$.

The inequality $(ax)^*ax \leq \|a\|^2 x^*x$ shows that $A\,\mathrm{Dom}^{1/2}(\varphi) \subset \mathrm{Dom}^{1/2}(\varphi)$. Moreover, if $a \leq b$ in A_+ then the property $\varphi(a) \leq \varphi(b) = \varphi(a) + \varphi(b - a)$ for a weight shows that sums of elements in $\mathrm{Dom}^{1/2}(\varphi)$ belong to $\mathrm{Dom}^{1/2}(\varphi)$. Hence $\mathrm{Dom}^{1/2}(\varphi)$ is a left ideal in A. This left ideal is not closed unless φ belongs to A^* (the continuous dual space). Importantly, if

φ is a trace then $\mathrm{Dom}^{1/2}(\varphi)$ is a two-sided ideal, and this is what makes Fredholm theory possible.

Definition 2.3.12. A weight φ on A is **densely defined** if $\mathrm{Dom}(\varphi)_+$ is dense in A_+, and **faithful** if the kernel $N_\varphi := \{x \in A_+ | \varphi(x) = 0\}$ is zero. A weight φ is **norm semifinite** if $\mathrm{Dom}^{1/2}(\varphi)$ is dense in A. It is called **lower semicontinuous** if the set $\{a \in A_+ | \varphi(a) \leq \lambda\}$ is norm closed for all $\lambda \in \mathbb{R}_+$.

For a lower semicontinuous weight φ on A we have [73]

$$\varphi(a) = \sup\{\omega(a) |\ \omega \in A_+^*,\ \omega \leq \varphi\}.$$

2.3.2.1 GNS representation of a C^*-algebraic weight

The GNS construction for a weight $\varphi : A_+ \to [0, +\infty]$ is defined similarly to the case of a bounded functional. The main difference being that an approximate identity in A does not provide a cyclic vector unless $\mathbf{1}$ is in the half-domain of the weight φ (which is the case iff φ is bounded). We write the GNS representation as a triple $(\mathfrak{H}_\varphi, \pi_\varphi, \Lambda_\varphi)$. Its construction is carried out by letting $\Lambda_\varphi(a)$ denote the image of $a \in \mathrm{Dom}^{1/2}(\varphi)$ in the quotient $\mathrm{Dom}^{1/2}(\varphi)/N_\varphi$ and then defining $\mathfrak{H}_\varphi$ to be the Hilbert space completion of $\mathrm{Dom}^{1/2}(\varphi)/N_\varphi$ in the inner product

$$\langle \Lambda_\varphi(a) | \Lambda_\varphi(b) \rangle := \varphi(a^* b), \qquad \forall a, b \in \mathrm{Dom}^{1/2}(\varphi).$$

The representation $\pi_\varphi : A \to \mathcal{B}(\mathfrak{H}_\varphi)$ is given by $\pi_\varphi \Lambda_\varphi(b) := \Lambda_\varphi(ab)$ for all $a \in A$ and $b \in \mathrm{Dom}^{1/2}(\varphi)$. For each weight φ, the GNS construction is unique up to unitary equivalence.

If φ is lower semicontinuous then π_φ is nondegenerate. If φ is faithful then so is π_φ, but the latter can be faithful even if φ is not [15, II.6.7.8].

For a bounded weight φ, i.e. for $\varphi \in A^*$, the map Λ_φ is bounded on all of A, extends to a map on $\pi_\varphi(A)$, and there is a vector $\Omega \in \mathfrak{H}_\varphi$ such that $\Lambda_\varphi(a) = a\Omega$ for all $a \in \pi_\varphi(A)$. Namely Ω is the limit of $\Lambda_\varphi(e_\lambda)$ for some approximate identity $(e_\lambda)_\lambda$ in A.

Any densely defined lower semicontinuous weight φ on a separable C^*-algebra is of the form

$$\varphi = \sum_{k \in \mathbb{N}} \varphi_k$$

for some sequence $(\varphi_k)_{k\in\mathbb{N}}$ of positive functionals $\varphi_k : A \to \mathbb{C}$ [82, Lemma C.1]. A consequence of this fact is that the GNS space $\mathfrak{H}_\varphi$ of φ embeds as a subspace of the direct sum $\mathfrak{H}_\varphi = \bigoplus_k \mathfrak{H}_{\varphi_k}$ of GNS spaces of bounded functionals [82, Thm. C.2]. In particular, $\mathfrak{H}_\varphi$ is separable.

The properties of the GNS map $\Lambda_\varphi : \mathrm{Dom}^{1/2}(\varphi) \to \mathfrak{H}_\varphi$ characterize the weight φ. Namely, let A be a C^*-algebra and let $\pi : A \to \mathcal{B}(\mathfrak{H})$ be a representation of A on a separable Hilbert space $\mathfrak{H}$. Let $\Lambda : \mathrm{Dom}(\Lambda) \to \mathfrak{H}$ be a closed (for the strong operator topology on $\pi(A)$ and norm on $\mathfrak{H}$) map with dense domain $\mathrm{Dom}(\Lambda) \subset \pi(A)$ such that

$$\Lambda(ab) = a\Lambda(b), \qquad \forall a \in \pi(A),\ b \in \mathrm{Dom}(\Lambda).$$

Then there are vectors $\Omega_k \in \mathfrak{H}$ such that $\pi(A)\Omega_k$ is orthogonal to $\pi(A)\Omega_l$ for all $k \neq l$ and

$$\Lambda(a) = \sum_{k\in\mathbb{N}} a\Omega_k, \qquad \forall a \in \mathrm{Dom}(\Lambda).$$

The map Λ extends to a map $\Lambda'' : \mathrm{Dom}(\Lambda'')\pi(A)'' \to \mathfrak{H}$ by the same formula, and if we set

$$\varphi(a^*a) := \begin{cases} \langle \Lambda''(a) | \Lambda''(a) \rangle, & \text{if } a \in \mathrm{Dom}(\Lambda'') \\ +\infty, & \text{otherwise} \end{cases} \tag{2.8}$$

then φ is a **normal** (i.e. ultraweakly continuous) semifinite weight on $\pi(A)''$ which restricts to a densely defined lower semicontinuous weight on $\pi(A)$ [82, Thm. C.3].

On the von Neumann-algebraic level, this leads us into the theory of Hilbert algebras. It is well known that any faithful normal semifinite weight φ on a standardly represented von Neumann algebra $\mathcal{N} \subset \mathcal{B}(\mathfrak{H})$ is given by (2.8) for some densely defined map Λ'' of $\mathcal{N}$ into $\mathfrak{H}$.

Thus, a densely defined lower semicontinuous weight $\varphi : A_+ \to [0, +\infty]$ has an extension to a normal semifinite weight on the von Neumann algebra $\pi_\varphi(A)''$. In fact, one sees that φ extends to the universal enveloping von Neumann algebra A^{**}, as follows [73, Section 2]. Every $\omega \in A^*$ has a continuous extension ω^{**} to A^{**} by definition, and then one defines the extension φ^{**} of φ by

$$\varphi^{**}(a) := \sup\{\omega^{**}(a) |\ \omega \in A^*_+,\ \omega \leq \varphi\}, \qquad \forall a \in A^{**},$$

which is unique, and we get that φ^{**} is normal and semifinite. The GNS representation of φ^{**} can be realized in $\mathfrak{H}_\varphi$ as well.

2.3.3 Semifinite Fredholm theory

The algebra $\mathcal{B}(\mathcal{H})$ of all bounded operators on a Hilbert space is a von Neumann algebra, and it equals the multiplier algebra $\mathcal{M}(\mathcal{K})$ of its norm-closed two-sided ideal $\mathcal{K} = \mathcal{K}(\mathcal{H})$ of compact operators. A possible definition of a **Fredholm operator** is as an element of $\mathcal{B}(\mathcal{H})$ which is invertible modulo $\mathcal{K}$. One may therefore say that the usual Fredholm property is "relative to the ideal $\mathcal{K}$". It is sometimes useful to discuss more general von Neumann algebras $\mathcal{N} \subset \mathcal{B}(\mathcal{H})$ which possess a norm-closed two-sided ideal $\mathcal{K}_\mathcal{N} \subset \mathcal{N}$. The existence of such an ideal is granted if $\mathcal{N}$ possesses a faithful densely defined ultraweakly continuous functional τ satisfying $\tau(ST) = \tau(TS)$ for all S, T in its domain. The pair $(\mathcal{N}, \tau)$ of a von Neumann algebra possessing such a trace (briefly, a **semifinite** trace) generalizes the pair $(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$, where Tr is the operator trace. In fact, the whole Fredholm theory can be transferred very satisfactorily to this more general setting, with a τ-dependent Fredholm index which is stable under homotopies and "compact" perturbations. However, the index is not $\mathbb{Z}$-valued in general, but $\mathbb{R}$-valued.

In this more general framework, we speak of "semifinite" Fredholm theory, to stress that we are not specializing to the case where $\mathcal{N} = \mathcal{B}(\mathcal{H})$ and $\mathrm{Tr} = \tau$.

2.3.3.1 Fredholm operators in semifinite von Neumann algebras

Definition 2.3.13. A von Neumann algebra $\mathcal{N}$ is **semifinite** if it possesses a faithful normal semifinite trace $\tau : \mathcal{N}_+ \to [0, +\infty]$.

Definition 2.3.14. Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra equipped with a fixed semifinite trace τ. An element $T \in \mathcal{N}$ is τ-**Fredholm** if

$$\tau(\mathrm{Ker}(T)) < +\infty,$$

and if there exists a τ-finite projection E_T for which

$$\mathrm{Ran}(\mathbf{1} - E_T) \subseteq \mathrm{Ran}(T). \tag{2.9}$$

Remark 2.3.15. If $\mathcal{N}$ is a factor, there is only one choice of trace τ, and τ-finiteness is the same thing as finiteness in the sense of Murray and von Neumann. In this case the notion of τ-Freholmness coincides with the Fredholmness defined by Breuer [22], so one may refer to τ-Fredholm elements in a factor as **Breuer-Fredholm** operators.

Remark 2.3.16. It is clear that (2.9) is stronger than the requirement that $\mathrm{Coker}(T) = \mathrm{Ker}(T^*)$ is τ-finite. So if $T \in \mathcal{N}$ is τ-Fredholm then [7, Lemma 1.5.3]

$$\tau(\mathrm{Ker}(T^*)) < +\infty.$$

Moreover, in the situation that τ-finiteness of a projection implies finite-dimensionality of its range, condition (2.9) implies that T has closed range [22, Section 3].

Definition 2.3.17. An operator $K \in \mathcal{N}$ is **τ-compact** if there is a sequence $(K_j)_{j\in\mathbb{N}}$ of elements in $\mathcal{N}$ with $\tau(\mathrm{Ran}(K_j)) < +\infty$ for all $j \in \mathbb{N}$ and $\lim_j \|K_j - K\| = 0$.

It is clear from the definition that the set of all τ-compact operators, denoted by $\mathcal{K}(\mathcal{N}, \tau)$ or $\mathcal{K}_{\mathcal{N}}$, is a norm-closed subspace of $\mathcal{N}$, i.e. an operator space in $\mathcal{N}$. Moreover, $\mathcal{K}(\mathcal{N}, \tau)$ contains the identity only if $\mathcal{N}$ is a finite von Neumann algebra.

Theorem 2.3.18 (Generalized Atikinson theorem [35, Lemma 3.15], [7, Section 1.5.3])**.** *The operator space $\mathcal{K}(\mathcal{N}, \tau)$ is a two-sided ideal in $\mathcal{N}$. An element $T \in \mathcal{N}$ is τ-Fredholm if and only if it is invertible modulo $\mathcal{K}(\mathcal{N}, \tau)$.*

Lemma 2.3.19 ([64, Lemma 6.4])**.** *The trace τ induces a homomorphism $\tau_* : K_0(\mathcal{K}_{\mathcal{N}}) \to \mathbb{R}$, explicitly given by*

$$\tau_*([e + \lambda\mathbf{1}] - [f + \mu\mathbf{1}]) = (\mathrm{Tr}\otimes\tau)(e) - (\mathrm{Tr}\otimes\tau)(f)$$

for all τ-finite projections e, f and all $[\lambda] = [\mu] \in K_0(\mathbb{C})$. Here $\mathrm{Tr}\otimes\tau$ is the trace on $\mathcal{K} \otimes \mathcal{K}_{\mathcal{N}}$.

Proof. For a projection $e \in \mathcal{K} \otimes \mathcal{N}$ we have $(\mathrm{Tr}\otimes\tau)(e) < +\infty$ iff e is $\mathrm{Tr}\otimes\tau$-compact [11, Lemma 3]. Thus $K_0(\mathcal{K}_{\mathcal{N}}) = K_0(\mathcal{F}_{\mathcal{N}})$, where $\mathcal{F}_{\mathcal{N}}$ is the $*$-ideal of τ-finite elements in $\mathcal{N}$. The tracial property of τ ensures that τ_* is well-defined (recall that the equivalence relation in K_0 is unitary equivalence). □

2.3.3.2 Semifinite Fredholm modules

The following is a generalization of Definition 2.3.1. Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra.

Definition 2.3.20. A $(\mathcal{N}, \tau)$**-semifinite Fredholm module** over a C^*-algebra A is a triple $(\pi, \mathcal{H}, F)$ where $\pi : A \to \mathcal{N}$ is a $*$-representation of A in $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$ and F is a selfadjoint operator on $\mathcal{H}$ such that $[F, \pi(A)] \subset \mathcal{K}(\mathcal{N}, \tau)$ and $\pi(A)(F^2 - \mathbf{1}) \subset \mathcal{K}(\mathcal{N}, \tau)$. The Fredholm module is said to be **even** if there is a decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$ with respect to which F is odd and $\pi(A)$ is even. Otherwise $(\pi, \mathcal{H}, F)$ is said to be **odd**.

Unitary equivalence and compact perturbation are defined in the same way as in Definition 2.3.1 but with $\mathcal{K}$ replaced by $\mathcal{K}(\mathcal{N}, \tau)$ for the latter. The Grothendieck group $K^1(A; \mathcal{N})$ of the semigroup of equivalence classes for these relations is called the **semifinite K-homology** of A relative to $\mathcal{N}$.

The construction of the pairings in the $\mathcal{B}(\mathcal{H})$-case now carry over directly to

$$K_0(A) \times K^0(A; \mathcal{N}) \to \mathbb{R}, \qquad ([\pi, \mathcal{H}, F], [e]) \to \mathrm{ind}_\tau(\pi(e) F_+ \pi(e)),$$

$$K_1(A) \times K^1(A; \mathcal{N}) \to \mathbb{R}, \qquad ([u], [\pi, \mathcal{H}, P]) \to \mathrm{ind}_\tau(P \pi(u) P).$$

A pairing between certain "semifinite K-theory" and semifinite K-homology groups was considered in [97].

2.4 Kasparov KK-theory

2.4.1 Hilbert modules

Definition 2.4.1. Let B be a C^*-algebra. A (right) **pre-Hilbert B-module** $(X, \langle\cdot|\cdot\rangle_B)$ is a right B-module X together with a map ("rigging")

$$\langle\cdot|\cdot\rangle_B : X \times X \to B,$$

linear in the second argument and conjugate-linear in the first, such that for all $x, y \in X$ and $b \in B$,

$$\langle x|y\rangle_B^* = \langle y|x\rangle_B, \qquad \langle x|yb\rangle_B = \langle x|y\rangle_B b,$$

If in addition X is complete in the topology of the norm $\|x\| := \|\langle x|x\rangle_B\|_B^{1/2}$ then $(X, \langle\cdot|\cdot\rangle_B)$ is a (right) **Hilbert B-module**.

Example 2.4.2. Throughout this thesis, we will denote by $X = B_B$ (or just B) the right Hilbert B-module which is equal to B as a set, with right B-action given by multiplication in B and with the inner product

$$\langle x|y\rangle_B := x^*y, \qquad \forall x, y \in B_B.$$

For any two Hilbert B-modules $(X, \langle\cdot|\cdot\rangle_{X,B})$ and $(Y, \langle\cdot|\cdot\rangle_{Y,B})$, a B-linear mapping $T : X \to Y$ is called **adjointable** if there exists an B-linear operator $T^* : Y \to X$ such that

$$\langle y|Tx\rangle_{Y,B} = \langle T^*y|x\rangle_{X,B}, \qquad \forall x \in X,\ y \in Y.$$

Boundedness and B-linearity of T do *not* ensure the existence of T^* in general [93, Section 15.2], as one might have thought after working with the special case $B = \mathbb{C}$. The space of adjointable maps $T : X \to Y$ is denoted by $\mathcal{L}_B(X, Y)$ and we write $\mathcal{L}_B(X) := \mathcal{L}_B(X, X)$.

Definition 2.4.3 ([67, Def. 5])**.** A Hilbert B-module X is

(i) **countably generated** if there is a sequence $(x_m)_{m\in\mathbb{N}}$ of elements in X such that finite sums of the form $\sum_j x_j b_j$ with $b_j \in B$ are dense in X.

(ii) **finitely generated** if there is an $r \in \mathbb{N}$ and an r-tuple of elements $x_1, \ldots, x_r$ in X such that the sums $\sum_j x_j b_j$ with $b_j \in B$ are dense in X.

Remark 2.4.4. It is immediate from the definitions that if B is a unital C^*-algebra then $K_0(B)$ is the Grothendieck group of unitary equivalence classes of finitely generated Hilbert B-modules [62, Prop. A.4.6]. For nonunital B we obtain $K_0(B)$ as in the kernel of the map $K_0(B^\sim) \to K_0(\mathbb{C})$ as before.

Definition 2.4.5 ([49, Def. 3.4], [46, Def. 2.2, Prop. 2.3])**.** A Hilbert module X over a C^*-algebra B is

(i) of B-**finite rank** if $X \cong eB^r$ for some $r \in \mathbb{N}$ and some idempotent $e \in \mathrm{M}_r(B)$,

(ii) **finite projective** if X is isomorphic to a complemented subspace of B^r for some $r \in \mathbb{N}$.

If B is unital then the notions of "B-finite rank" and "finite projective" coincide [49, Prop. 3.9]. In any case, a B-finite rank Hilbert module X is a finite projective Hilbert $B^\sim$-module.

Definition 2.4.6 ([67, Def. 5])**.** Let A and B be C^*-algebras. If X is a Hilbert A-module, Y is a Hilbert B-module and $\rho : A \to \mathcal{L}_B(Y)$ is a $*$-homomorphism, endow the algebraic tensor product $X \odot Y$ with the inner product specified by

$$\langle x \otimes y | x' \otimes y' \rangle_B^\rho := \langle y | \rho(\langle x | x' \rangle_{X,A}) y' \rangle_{Y,B}.$$

The **balanced tensor product** of X and Y under ρ is the Hilbert B-module $X \otimes_\rho Y$ obtained from $(X \odot Y, \langle \cdot | \cdot \rangle_B^\rho)$ by dividing out vectors of length zero and completing. If $A = B$ and $\rho = \mathrm{id} : A \to A$ is the identity then we write $X \otimes_A Y := X \otimes_{\mathrm{id}} Y$.

Example 2.4.7 ([14, Example 13.5.2])**.** Consider a $*$-homomorphism $\rho : A \to B$ of C^*-algebras. Then the balanced tensor product $A \otimes_\rho B$ makes sense, and is isomorphic to the closed right ideal in B generated by $\rho(A)$. So if $\rho(\mathbf{1}) = \mathbf{1}$ then $A \otimes_\rho B \cong B$, and this is true more generally if ρ is "essential" in the sense that the ideal $\rho(A)$ generates B.

2.4.2 Multiplier algebras

Definition 2.4.8. Let B be a C^*-algebra. The **multiplier algebra** of B is the unital C^*-algebra

$$\mathcal{M}(B) := \mathcal{L}_B(B)$$

of adjointable operators on the standard Hilbert B-module B.

Evidently, $\mathcal{M}(B) = B$ iff B is unital. The multiplier algebra $\mathcal{M}(B)$ is also characterized as [93, Section 2.2]

$$\mathcal{M}(B) = \{T \in \mathcal{B}(\mathfrak{H}) |\ TB \subseteq B \supseteq TB\}$$

in any faithful representation $B \subset \mathcal{B}(\mathfrak{H})$ where B acts nondegenerately (i.e. $B\mathfrak{H}$ is dense in $\mathfrak{H}$). From this definition we see that B is an ideal in $\mathcal{M}(B)$ and that $TB \neq \{0\}$ for all nonzero $T \in \mathcal{M}(B)$, i.e. the ideal B is **essential**. In fact, $\mathcal{M}(B)$ can be equivalently defined as the largest C^*-algebra containing A as an essential ideal.

The multiplier algebra $\mathcal{M}(B)$ is closed in the **strict topology**, determined by semi-

norms

$$|T|_{a,b} := \|Ta\| + \|Tb\|, \qquad \forall T \in \mathcal{M}(B)$$

(parameterized by $a, b \in B$), which is weaker than the norm topology (unless A is unital). If B is nonunital then $\mathcal{M}(B)$ is not separable. Nevertheless, $\mathcal{M}(B)$ is not closed in the weak or strong operator topologies, so $\mathcal{M}(B) \subset B''$.

Example 2.4.9. For a commutative C^*-algebra $B = C_0(X)$, the multiplier algebra is $\mathcal{M}(C_0(X)) = C_b(X) = C(\beta X)$, the C^*-algebra of all bounded continuous functions on the underlying space X, identifiable with the continuous functions on the Stone-Čech compactification βX of X.

In K-theory we often need the multiplier algebra $\mathcal{M}(B \otimes \mathcal{K})$ of the stabilization $B \otimes \mathcal{K}$ of a C^*-algebra B. Therefore, multiplier algebras are important also for unital C^*-algebras, since $\mathcal{M}(B \otimes \mathcal{K}) \neq B \otimes \mathcal{K}$ even if B is unital. The inclusion

$$\mathcal{M}(B^{\sim} \otimes \mathcal{K}) \subseteq \mathcal{M}(B \otimes \mathcal{K})$$

is proper precisely when B has no unit.

Example 2.4.10. Let B be a C^*-algebra and let $\ell^2(\mathbb{N}; B)$ be the Hilbert B-module of all sequences $b_\bullet = (b_m)_{m\in\mathbb{N}}$ satisfying $\sum_m \|b_m\|^2 < +\infty$. If B has a countable approximate identity then $\ell^2(\mathbb{N}; B)$ has a countable basis. The compact operators on $\ell^2(\mathbb{N}; B)$ identify with the stabilization $B \otimes \mathcal{K}$ of B, and the algebra $\mathcal{L}_B(\ell^2(\mathbb{N}; B))$ of all adjointable operators identify with the **stable multiplier algebra** $\mathcal{M}(B \otimes \mathcal{K})$ of B.

The following result is known as **Kasparov's stabilization theorem**.

Theorem 2.4.11 ([67, Thm. 2])**.** *Every countably generated Hilbert B-module X is isomorphic to a direct summand of $\ell^2(\mathbb{N}; B)$.*

The following is perhaps the most suggestive result for regarding K-theory as an abstract generalization of Fredholm theory. Write $\mathcal{Q}(B \otimes \mathcal{K}) := \mathcal{M}(B \otimes \mathcal{K})/(B \otimes \mathcal{K})$.

Lemma 2.4.12 ([14, Prop. 12.2.1], [93, Cor. 10.3])**.** *For any C^*-algebra B,*

$$K_0(B) \cong K_1(\mathcal{Q}(B \otimes \mathcal{K})), \qquad K_1(B) \cong K_0(\mathcal{Q}(B \otimes \mathcal{K})).$$

Thus, a class in $K_0(B)$ can be represented by a multiplier of $B \otimes \mathcal{K}$ which is invertible modulo $B \otimes \mathcal{K}$. A class in $K_1(B)$ can be represented by a multiplier of $B \otimes \mathcal{K}$ which is a projection modulo $B \otimes \mathcal{K}$.

Finally we shall recall that a general lifting result for ideals in C^*-algebras [14, Prop. 3.4.6] implies that every homotopy in $\mathcal{Q}(B \otimes \mathcal{K})$ can be lifted to a homotopy in $\mathcal{M}(B \otimes \mathcal{K})$.

Lemma 2.4.13. *Suppose that $q(T)$ and $q(S)$ are homotopic elements of $\mathcal{Q}(B \otimes \mathcal{K})$. Then T and S are homotopic in $\mathcal{M}(B \otimes \mathcal{K})$.*

2.4.3 Fredholm operators on Hilbert modules

There is a Fredholm theory for operators on Hilbert modules which generalizes Fredholm theory for operators on Hilbert spaces. It involves a K-theory-valued index which we will recall in this section. We will also discuss how it relates to the abstract index coming from the long exact sequence in K-theory.

We use [93, Section 17], [49, Section 4.3], [46] (the latter two covering nonunital algebras) and [85] (from where most of the results originate).

Definition 2.4.14 ([85, Section 1.1])**.** Let B be a C^*-algebra and let X and Y be two Hilbert B-modules. An adjointable operator $T : X \to Y$ is **Fredholm** if there exists an adjointable operator $S : Y \to X$ (a **parametrix** for T) for which

$$\mathbf{1} - ST \in \mathcal{K}_B(X), \qquad \mathbf{1} - TS \in \mathcal{K}_B(Y).$$

The motivation for Definition 2.4.14 is of course that if $B = \mathbb{C}$ then it reduces to the standard notion of Fredholm operators on Hilbert spaces.

Let now $T \in \mathcal{L}_B(X, Y)$ be a Fredholm operator. The most pleasing definition of an "index" of T would be as the element $[\mathrm{Ker}(T)] - [\mathrm{Ker}(T^*)]$ in $K_0(B)$. Unfortunately, neither $\mathrm{Ker}(T)$ nor $\mathrm{Ker}(T^*)$ is guaranteed to have finite rank.

Moreover, also in contrast to the case $B = \mathbb{C}$, a Fredholm operator on a Hilbert module over a general C^*-algebra need not have closed range. The following is the desired generalization of the property of having closed range. It gives an algebraic characterization of the closed-range property when $B = \mathbb{C}$.

Definition 2.4.15 ([46, Def. 3.2])**.** Let B be a C^*-algebra and let X and Y be two Hilbert B-modules. An operator $T \in \mathcal{L}_B(X, Y)$ is **regular** if there exists an $S \in \mathcal{L}_B(Y, X)$ (a

pseudo-inverse of T) such that

$$TST = T, \qquad STS = S.$$

Any pseudo-inverse S of T allows us to express the projections $\mathrm{Ker}(T)$ and $\mathrm{Ran}(T)$ as

$$\mathrm{Ker}(T) = \mathbf{1} - ST, \qquad \mathrm{Ker}(T^*) = \mathbf{1} - TS. \tag{2.10}$$

Lemma 2.4.16. *Let B be any C^*-algebra and let X be a Hilbert B-module. Then a projection $e \in \mathcal{L}_B(X)$ has finite rank if and only if e is compact.*

Proof. See [46, Prop. 2.3] and the paragraph after [49, Corollary 3.10] for the nonunital case. □

Proposition 2.4.17 ([46, Prop. 3.3])**.** *If $T \in \mathcal{L}_B(X, Y)$ is both regular and Fredholm then* $\mathrm{Ker}(T)$ *and* $\mathrm{Ker}(T^*)$ *are projections of finite rank.*

Proof. Let S_1 be a parametrix for T and let S_2 be a pseudo-inverse of T. Then $(\mathbf{1}-S_1T)(\mathbf{1}-S_2T) = \mathbf{1} - S_2T$. Since $\mathbf{1} - S_1T$ is compact, and since $\mathcal{K}_B(X)$ is an ideal in $\mathcal{L}_B(X)$, we see that $(\mathbf{1} - S_1T)(\mathbf{1} - S_2T) = \mathbf{1} - S_2T$ is compact as well. So $\mathbf{1} - S_2T$ is a compact projection on X, hence a projection with finite rank by Lemma 2.4.16. The argument for $\mathbf{1} - TS_2$ is identical (in particular, every pseudo-inverse of T is a parametrix for T). □

Corollary 2.4.18. *Let B be a C^*-algebra. Then $T \in \mathcal{L}_B(X, Y)$ is Fredholm if and only if there exists a parametrix $S \in \mathcal{L}_B(Y, X)$ such that $\mathbf{1} - ST$ and $\mathbf{1} - TS$ are not only compact, but in fact finite-rank operators.*

Proof. Apply [49, Lemma. 4.4] to $A = \mathcal{L}_B(X, Y)$ and $J = \mathcal{K}_B(X, Y)$. □

Note that if $T \in \mathcal{L}_B(X, Y)$ is Fredholm, then T is also Fredholm as an element of $\mathcal{L}_{B^\sim}(X, Y)$, i.e. as an operator on Hilbert modules over the unitization $B^\sim = B \times \mathbb{C}$.

Lemma 2.4.19 ([49, Prop. 4.10], [46, Lemma 3.8])**.** *Let B be a C^*-algebra and let X and Y be two Hilbert B-modules. Every Fredholm operator $T \in \mathcal{L}_B(X, Y)$ has a* ***regular amplification****, i.e. there is an $r \in \mathbb{N}$ and a Fredholm operator $\tilde{T} \in \mathcal{L}_{B^\sim}(X^r, Y^r)$ which restricts to $T \in \mathcal{L}_{B^\sim}(X, Y)$ and is regular.*

Proof. By Corollary 2.4.18, we can find a parametrix $S \in \mathcal{L}_B(Y, X)$ such that $\mathbf{1} - ST$ and $\mathbf{1} - TS$ are finite rank operators. So there are $x_1, \ldots, x_r \in X$ and $y_1, \ldots, y_r \in Y$ such that

$$(\mathbf{1} - ST)x' = \sum_{j=1}^{r} y_j \langle x_j | x' \rangle, \qquad \forall x' \in X.$$

Consider the operators

$$\Omega_x : A^r \to X, \qquad \omega_x(a_1, \ldots, a_r) := \sum_{j=1}^{r} x_j a_j,$$

$$\Omega_y : A^r \to Y, \qquad \omega_y(a_1, \ldots, a_r) := \sum_{j=1}^{r} y_j a_j.$$

Define $\tilde{T}$ as the operator from $X^r = X \oplus (B^\sim)^r$ to $Y^r = X \oplus (B^\sim)^r$ given by

$$\tilde{T} := \begin{pmatrix} T & 0 \\ \Omega_x^* & 0 \end{pmatrix}$$

Now $\tilde{T}$ is a regular operator, because $\tilde{S} := \left(\begin{smallmatrix} S & \Omega_y \\ 0 & 0 \end{smallmatrix}\right)$ is an explicit pseudo-inverse for $\tilde{T}$. Moreover, we have

$$\mathbf{1} - \tilde{T}\tilde{S} = \begin{pmatrix} \mathbf{1} - TS & -T\Omega_y \\ -\Omega_x^* S & \mathbf{1} - \Omega_x^* \Omega_y \end{pmatrix}, \qquad \mathbf{1} - \tilde{S}\tilde{T} = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{1} \end{pmatrix}.$$

Since $B^\sim$ is unital, the Hilbert $B^\sim$-module $(B^\sim)^r$ is finite projective, with $\mathbf{1} \in \mathcal{L}_{B^\sim}((B^\sim)^r) \cong \mathrm{M}_r(B^\sim)$ given by the finite-rank operator $\mathbf{1} = \mathrm{diag}(\mathbf{1}, \ldots, \mathbf{1})$. So $\mathbf{1} - \tilde{T}\tilde{S}$ and $\mathbf{1} - \tilde{S}\tilde{T}$ are finite-rank operators. We conclude that $\tilde{T}$ is both regular and Fredholm. $\square$

Lemma 2.4.20 ([46, Prop. 3.9])**.** *Let $\tilde{T} : X \to Y$ be a regular Fredholm operator between Hilbert $B^\sim$-modules. Then the difference $[\mathrm{Ker}(\tilde{T})] - [\mathrm{Ker}(\tilde{T}^*)]$, a priori an element of $K_0(B^\sim)$, belongs to $K_0(B)$.*

Definition 2.4.21 ([49, Def. 4.4])**.** Let B be a C^*-algebra and let X and Y be two Hilbert B-modules. The **Mingo index** of a Fredholm operator $T \in \mathcal{L}_B(X, Y)$ is defined as

$$\mathrm{Index}(T) := [\mathrm{Ker}(\tilde{T})] - [\mathrm{Ker}(\tilde{T}^*)] \in K_0(B)$$

where $\tilde{T}$ is some regular amplification of T (as in Lemma 2.4.20).

That Index(T) does not depend on the choice of regular amplification is a consequence of the fact (which we will not go into detail here) that

$$[\mathrm{Ker}(\tilde{T}) + K] - [\mathrm{Ker}(\tilde{T}^*) + K] = [\mathrm{Ker}(\tilde{T})] - [\mathrm{Ker}(\tilde{T}^*)]$$

for all regular Fredholm operators $\tilde{T}$ and all compact adjointable operators K [49, Prop. 4.9].

All Hilbert modules that we are going to discuss in this work will be countably generated. By Kasparov stabilization (Theorem 2.4.11), we may regard a Hilbert B-module X as a direct summand of the standard Hilbert B-module $\ell^2(\mathbb{N}, B)$. Having this in mind we shall now discuss Fredholm operators on the standard Hilbert B-module $\ell^2(\mathbb{N}, B)$.

Recall that $\mathcal{L}_B(\ell^2(\mathbb{N}, B)) \cong \mathcal{M}(B \otimes \mathcal{K})$ and $\mathcal{K}_B(\ell^2(\mathbb{N}, B)) \cong B \otimes \mathcal{K}$. Write

$$\mathcal{F}_A \subset \mathcal{M}(B \otimes \mathcal{K})$$

for the set of Fredholm operators on $\ell^2(\mathbb{N}, B)$. It is shown in [85] that the index map

$$\mathcal{F}_A \to K_0(B), \qquad T \to \mathrm{Index}(T) \tag{2.11}$$

descends to an isomorphism

$$\pi_0(\mathcal{F}_A) \cong K_0(B),$$

where $\pi_0(\mathcal{F}) = \mathcal{F}_A/(\mathcal{F}_A)_0$ is the group of path-connected components in $\mathcal{F}_A$. Therefore, an element of $K_0(B)$ can be viewed as an index of a Fredholm operator on $\ell^2(\mathbb{N}, B)$.

Recall that we denote by $\mathcal{Q}(B \otimes \mathcal{K})$ the Calkin algebra $\mathcal{M}(B \otimes \mathcal{K})/B \otimes \mathcal{K}$. Let $q : \mathcal{M}(B \otimes \mathcal{K}) \to \mathcal{Q}(B \otimes \mathcal{K})$ be the Calkin projection. Then, by Definition 2.4.14, the set of Fredholm operators on $\ell^2(\mathbb{N}; B)$ is equal to

$$\mathcal{F}_A = \{T \in \mathcal{M}(B \otimes \mathcal{K}) |\ q(T) \text{ is invertible}\}.$$

So $K_0(B) \cong \pi_0(\mathcal{F}_A)$ is the group of connected components of the group of invertible ele-

ments in $\mathcal{Q}(B \otimes \mathcal{K})$. In view of Lemma 2.2.11, we conclude that

$$K_0(B) \cong K_1(\mathcal{Q}(B \otimes \mathcal{K})),$$

as we recorded already in Lemma 2.4.12. It is thus the K_0-valued index map 2.11 which underlies this isomorphism.

Lemma 2.4.12 can be proven without referring to Fredholm operators. Mingo showed that the unitary group of the stabilized multiplier algebra $\mathcal{M}(B \otimes \mathcal{K})$ is contractible [85] when B has a countable approximate identity consisting of projections. This implies that $K_1(\mathcal{M}(B \otimes \mathcal{K})) = 0$. However, it is true for any C^*-algebra B that $K_1(\mathcal{M}(B \otimes \mathcal{K})) = 0$, as shown in [93, Thm. 10.2] (and this sharpening still does not rely on Hilbert modules). Therefore, for any C^*-algebra B, the short exact sequence

$$0 \longrightarrow B \otimes \mathcal{K} \longrightarrow \mathcal{M}(B \otimes \mathcal{K}) \longrightarrow \mathcal{Q}(B \otimes \mathcal{K}) \longrightarrow 0 \tag{2.12}$$

gives a long exact sequence in K-theory such that the abstract index map $\delta : K_1(\mathcal{Q}(B \otimes \mathcal{K})) \to K_0(B \otimes \mathcal{K})$ is an isomorphism.

If $T \in \mathcal{M}(B \otimes \mathcal{K})$ is invertible modulo $B \otimes \mathcal{K}$ (i.e. if T is Fredholm) then we can choose a pseudo-inverse $S \in \mathcal{M}(B \otimes \mathcal{K})$ of T. Defining (cf. [49, Prop. 4.8])

$$V := \begin{pmatrix} T & \mathbf{1} - TS \\ \mathbf{1} - ST & S \end{pmatrix},$$

it follows from the defining formula (2.2) of the abstract index map $\delta : K_1(\mathcal{Q}(B \otimes \mathcal{K})) \to K_0(B \otimes \mathcal{K})$ that

$$\delta([q(T)]) = [V 1_1 V^*] - [1_1],$$

where $1_1 = \operatorname{diag}(1, 0) \in \mathrm{M}_2(B \otimes \mathcal{K})$ is the identity of $B \otimes \mathcal{K}$ embedded into a larger matrix

algebra. Explicitly, if T is also assumed to be regular then

$$\begin{aligned}\delta([q(T)]) &= \left[\begin{pmatrix} T & \mathbf{1}-TS \\ \mathbf{1}-ST & S \end{pmatrix}\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} S & \mathbf{1}-ST \\ \mathbf{1}-TS & T \end{pmatrix}\right] - \left[\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix}\right] \\ &= \left[\begin{pmatrix} TS-\mathbf{1} & 0 \\ 0 & \mathbf{1}-ST \end{pmatrix}\right] \\ &= [\mathrm{Ker}(T)] - [\mathrm{Ker}(T^*)],\end{aligned}$$

where the last equality comes from formula (2.10). In other words, the connecting homomorphism δ maps the Calkin element $q(T) \in \mathrm{GL}_1(\mathcal{Q}(B \otimes \mathcal{K}))$ to the Mingo index of T:

$$\delta([q(T)]) = \mathrm{Index}(T) \in K_0(B).$$

We summarize the discussion in the following lemma.

Lemma 2.4.22. *Let B be a C^*-algebra and let X be a countable Hilbert B-module. View X as embedded into the standard Hilbert B-module $\ell^2(\mathbb{N}; B)$. If T is a regular Fredholm operator, write*

(i) $\mathrm{Index}(T)$ *for the Mingo index of T, and*

(ii) $\mathrm{ind}(T)$ *for the abstract index of T, coming from the short exact sequence* (2.12).

Then we have equality

$$\mathrm{Index}(T) = \mathrm{ind}(T)$$

in $K_0(B)$.

On the other hand, let

$$0 \longrightarrow B \longrightarrow D \longrightarrow A \longrightarrow 0 \tag{2.13}$$

be a short exact sequence of C^*-algebras. Then we have the abstract index map $\delta : K_1(A) \to K_0(B)$. Consider the right Hilbert B-module B_B (recall Example 2.4.2) and let $\pi_B : D \to \mathcal{L}_B(B_B)$ be the action by multiplication from the right. Since B is an ideal in D, we have $D \subset \mathcal{M}(B)$ and hence $D \subset \mathcal{L}_B(B_B)$. That an element $T \in D$ is Fredholm as an element of $\mathcal{L}_B(B_B)$ means precisely that T is invertible modulo B, i.e. that the image

of T in $D/B \cong A$ is invertible. Under the assumption that B_B is a countably generated B-module (which happens iff $\mathcal{K}_B(B) = B$ has a countable approximate identity), we have an embedding $B_B \subset \ell^2(\mathbb{N}; B)$ and we can carry out the same calculation as above to show that $\mathrm{Index}(T) = \delta([q(T)])$ whenever T is regular. Indeed, the quotient map $q : D \to D/B$ is just the restriction of the Calkin projection $q : \mathcal{M}(B \otimes \mathcal{K}) \to \mathcal{M}(B \otimes \mathcal{K})/B$.

Thus, for a short exact sequence (2.13) such that B_B is a countably generated module, an abstract index $\mathrm{ind}(T)$ always has an interpretation as the Mingo index of a Fredholm operator on a Hilbert module. Conversely, the Mingo index of a Fredholm operator is just an instance of the abstract index of a C^*-algebraic element. From now on, we will therefore only speak of the "abstract" (or "$K_0(B)$-valued) index".

2.4.4 The KK-groups

In this section, A is a separable C^*-algebra and B is a C^*-algebra with a countable approximate identity (briefly, B is σ-**unital**). The following definition generalizes the notion of Fredholm module (Definition 2.3.1).

Definition 2.4.23. A **Kasparov** A-B-module is a triple (π, X_B, F) of a representation

$$\pi : A \to \mathcal{L}_B(X) \subseteq \mathcal{M}(B \otimes \mathcal{K})$$

of A as adjointable operators on a B-submodule $X = X_B$ of $\ell^2(\mathbb{N}; B)$ (cf. Example 2.4.10) and an $F \in \mathcal{L}_B(X)$ such that

$$\pi(a)(F^2 - \mathbf{1}), \qquad \pi(a)(F - F^*), \qquad [F, \pi(a)]$$

belong to $B \otimes \mathcal{K}$, for each $a \in A$.

The notions of "even" and "odd" Kasparov modules are as in Definition 2.3.1. Then the even and odd **Kasparov** KK**-groups** $KK^0(A, B)$ and $KK^1(A, B)$ are defined in the same way as $K^0(A)$ and $K^1(A)$ respectively.

Remark 2.4.24 ("The operator K-functor" [66]). For each fixed separable C^*-algebra A, the assignment $B \to KK^0(A, B)$ gives rise to a covariant functor with values in the category of Abelian groups. Indeed, if $\rho : B \to C$ is a $*$-homomorphism, we can define the

group morphism

$$\rho_* = KK^0(A,\rho) : KK^0(A,B) \to KK^0(A,C), \qquad \rho_*[\pi, X_B, F] := [\pi, X_B \otimes_\rho C, F \otimes \mathbf{1}],$$

where $X_B \otimes_\rho C$ is the balanced tensor product (Definition 2.4.6). The functoriality of $KK^0(A,\cdot)$ is clear from the definitions. Moreover, the functor $KK^0(A,\cdot)$ is **homotopy invariant** in the sense that if $\rho : B \to C$ and $\sigma : B \to C$ are homotopic homomorphisms then $\rho_* = \sigma_*$ [14, Prop. 17.9.1].

Similarly, for each fixed σ-unital C^*-algebra B, the map $A \to KK^0(A,B)$ defines a contravariant functor on separable C^*-algebras. This functor takes a $*$-homomorphism $\rho : C \to A$ to the group morphism

$$\rho^* = KK^0(\rho, B) : KK^0(A,B) \to KK^0(C,B), \qquad \rho^*[\pi, X_B, F] := [\pi \circ \rho, X_B, F].$$

Thus KK^0 is a bivariant functor from C^*-algebras to Abelian groups, homotopy invariant in both variables. Similar remarks apply to KK^1.

Remark 2.4.25 (The Kasparov product)**.** A $*$-homomorphism $\rho : C \to A$ defines an even Kasparov C-A module $(A \oplus A, \rho, F)$ with the operator $F := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Denote the corresponding KK class by $[\rho] := [A \oplus A, \rho, F] \in KK^0(C,A)$. Then for each C^*-algebra B and each class $[\pi, X_B, F] \in KK^0(A,B)$, the pullback discussed in Remark 2.4.24, viz.

$$[\rho] \otimes_A [\pi, X_B, F] := \rho^*[\pi, X_B, F] \in KK^0(C,B),$$

is a special instance of the “Kasparov product” [14, Section 18]

$$\otimes_A : KK^0(C,A) \otimes KK^0(A,B) \to KK^0(C,B).$$

Similarly, if $\rho : B \to C$ is a $*$-homomorphism then the the map $\rho_* : KK^0(A,B) \to KK^0(C,A)$ is given by Kasparov product with the class $[\rho] \in KK^0(B,C)$, but now from the right:

$$\rho_*[\pi, X_B, F] = [\pi, X_B, F] \otimes_B [\rho].$$

We will only need the Kasparov product in this simplest form. The Kasparov product between two general Kasparov modules (neither coming from a $*$-homomorphism) is ex-

tremely hard to compute.

When we take $A = \mathbb{C}$ and $B = \mathbb{C}$ we obtain functors $KK^\bullet(\mathbb{C}, \cdot)$ and $KK^\bullet(\cdot, \mathbb{C})$ which are familiar from previous sections.

Proposition 2.4.26 ([14, Prop. 17.5]). *Kasparov's KK-theory generalizes both K-theory and K-homology:*

$$KK^0(\mathbb{C}, B) \cong K_0(B), \qquad KK^1(\mathbb{C}, B) \cong K_1(B),$$

$$KK^0(A, \mathbb{C}) = K^0(A), \qquad KK^1(A, \mathbb{C}) = K^1(A).$$

Proof. The equality with K-homology is by definition. We prove the isomorphism with K-theory. It will be useful to view the K-groups as consisting of equivalence classes of elements $B \otimes \mathcal{K}$ instead of matrices over B (Lemma 2.2.11).

Let (π, X_B, F) be a Kasparov $\mathbb{C}$-B-module. Since $\pi : \mathbb{C} \to \mathcal{L}_B(X)$ is required to be a homomorphism, $\pi(1)$ is a projection. Since $\mathbb{C}$ is generated by 1, we may without loss of generality replace X by $\pi(1)X$. Then $\pi : \mathbb{C} \to \mathcal{L}_B(X)$ is a unital $\mathbb{C}$-linear map and there is only one such map, namely $\pi(\lambda) := \lambda\mathbf{1}$. So we can discard π from the data. A Kasparov $\mathbb{C}$-B-module is therefore specified by an element $F \in \mathcal{M}(B \otimes \mathcal{K})$ whose image $q(F)$ in $\mathcal{Q}(B \otimes \mathcal{K})$ is selfadjoint and unitary.

For an even module X, we have a decomposition $X = X_+ \oplus X_-$ under which the operator F takes the form

$$F = \begin{pmatrix} 0 & F_+ \\ F_- & 0 \end{pmatrix},$$

with $q(F_+)q(F_-) = \mathbf{1} = q(F_-)q(F_+)$. Thus, by Lemma 2.4.13 an element of $KK^0(\mathbb{C}, B)$ is represented by a homotopy equivalence class of unitaries in $\mathcal{Q}(B \otimes \mathcal{K})$. In other words,

$$KK^0(\mathbb{C}, B) \cong K_1(\mathcal{Q}(B \otimes \mathcal{K})$$

and we are done in the even case by Lemma 2.4.12.

Anyway, we would like to give an alternative proof of the isomorphism $KK^0(\mathbb{C}, B) \cong K_0(B)$ which does not use Lemma 2.4.13. A class in $K_0(B)$ is represented by two projections

e, f in $\mathrm{M}_r(B^\sim)$ with $e - f \in \mathrm{M}_r(B)$. We define

$$X = X_+ \oplus X_- := B^{\oplus r} \oplus B^{\oplus r},$$

and the representation $\pi : \mathbb{C} \to \mathcal{L}_B(X)$ takes $1 \in \mathbb{C}$ to $e \oplus f$. Finally, letting $F := \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ we get that (π, X_B, F) defines a Kasparov $\mathbb{C}$-B-module, representing an element of $KK^0(\mathbb{C}, B)$. Due to the fact that the equivalence relations in KK involves $B \otimes \mathcal{K}$, they are easily seen to coincide with those in K_0.

Next we prove the isomorphism $KK^1(\mathbb{C}, B) \cong K_1(B)$. If $F \in \mathcal{M}(B \otimes \mathcal{K})$ defines an odd Kasparov $\mathbb{C}$-B-module then $P := (\mathbf{1} + q(F))/2$ is a projection in $\mathcal{Q}(B \otimes \mathcal{K})$. Since the equivalence relations defining of KK is that of homotopy in $\mathcal{M}(B \otimes \mathcal{K})$, Lemma 2.4.13 shows that the group $KK^1(\mathbb{C}, B)$ consists of homotopy equivalence classes of projections in $\mathcal{Q}(B \otimes \mathcal{K})$. So

$$KK^1(\mathbb{C}, B) \cong K_0(\mathcal{Q}(B \otimes \mathcal{K})),$$

and the latter group is isomorphic to $K_1(B)$ (Lemma 2.4.12). □

Remark 2.4.27 (Formal Bott periodicity)**.** The odd KK-group can also be obtained as

$$KK^1(A, B) \cong KK^0(A \otimes \mathbb{C}_1, B) \cong KK^0(A, B \otimes \mathbb{C}_1), \tag{2.14}$$

where $\mathbb{C}_1 \cong \mathbb{C} \oplus \mathbb{C}$ is the Clifford algebra over $\mathbb{C}$. In fact, (2.14) was Kasparov's original definition of KK^1. Similarly one defines

$$KK^p(A, B) = KK^0(A \otimes \mathbb{C}_p, B) = KK^0(A, B \otimes \mathbb{C}_p), \qquad p \in \mathbb{N}_0,$$

with $KK^0 := KK$. Recall that $\mathbb{C}_{p+2} \cong \mathbb{C}_p$ for all $p \in \mathbb{N}$. The **formal Bott periodicity** (2.14) can be used to prove Bott periodicity for the KK-bifunctor [14, §19].

Remark 2.4.28 (Grading [66])**.** Gradings of C^*-algebras and Hilbert modules [14, §14] play an important role in KK theory, but here we work with ungraded C^*-algebras and mostly we need only to distinguish an even Kasparov module from an odd (i.e. not even) Kasparov module. If B is a C^*-algebra acting on a $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, with grading operator $\Gamma = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$ then B carries the **standard even grading** $B = B_+ \oplus B_-$

where

$$B_{\pm} := \{b \in B |\ \Gamma b = \pm b\Gamma\}.$$

Suppose further that $(\pi, \mathcal{H}, F)$ is an even Fredholm module over a C^*-algebra A, and that both $\pi(A)$ and F) are multipliers of B. Then the even grading on B gives the Kasparov A-B-module B_B an even grading, a fact that we will often need.

If B is ungraded, the algebra $\mathbb{C}_1 \otimes B$ (graded tensor product) has an **odd grading**, i.e. there is a decomposition $B \oplus B = (B \oplus B)_+ \oplus (B \oplus B)_-$ where

$$(B \oplus B)_+ := \{(b, b) \in B \oplus B\}, \qquad (B \oplus B)_- := \{(b, -b) \in B \oplus B\}.$$

This grading on $\mathbb{C}_1 \otimes B$ is not even, because $(B \oplus B)_{\pm}$ are not the $\pm$-eigenspaces for the adjoint action of a selfadjoint unitary $\Gamma \in \mathcal{M}(B)$. It is convenient to just talk about gradings in the even case, and regard tensoring an even algebra by $\mathbb{C}_1$ (or $\mathbb{C}_p$ for odd p) as spoiling the (even) grading.

2.4.5 Extension picture of KK

The classical Toeplitz extension is an example of the kind of extensions studied by Brown-Douglas-Fillmore,

$$0 \longrightarrow \mathcal{K} \longrightarrow D \longrightarrow C(X) \longrightarrow 0,$$

namely extensions of commutative C^*-algebras by the C^*-algebra $\mathcal{K}$ of compact operators. If we want a Toeplitz extension with noncommutative algebra A of symbols, replacing $C(X)$, we need more general extensions

$$0 \longrightarrow B \longrightarrow D \longrightarrow A \longrightarrow 0 \tag{2.15}$$

(in our terminology, an **extension** of a C^*-algebra A by a C^*-algebra B is a short-exact sequence of the form (2.15)). In order to classify such extensions we shall need the **corona** (or **outer multiplier algebra**) of B, which is the quotient (already introduced in Lemma 2.4.12)

$$\mathcal{Q}(B) := \mathcal{M}(B)/B.$$

Given a homomorphism $\gamma : A \to \mathcal{Q}(B)$ we obtain an extension of A by B as the pullback

$$D_\gamma := \mathcal{M}(B) \oplus_\gamma A := \{(T, a) \in \mathcal{M}(B) \oplus A |\ q(T) = \gamma(a)\}, \tag{2.16}$$

where $q : \mathcal{M}(B) \to \mathcal{Q}(B)$ is the quotient map. Elements of the form $(b, 0)$ with $b \in B$ are included in D_γ, and $(b, 0)(T, a) = (bT, 0)$ is again of this form for any $T \in \mathcal{M}(B)$ and $a \in A$. So B is an ideal in D_γ and if we define a map from D_γ to A by sending (T, a) to a then we have our extension.

Lemma 2.4.29 ([94, Prop. 2.2.14])**.** *Let B be a norm-closed two-sided ideal in a C^*-algebra D. Then there exists a unique $*$-homomorphism $\sigma : D \to \mathcal{M}(B)$ restricting to the identity on $B \subset D$. The kernel of σ is the annihilator of B,*

$$\mathrm{Ker}(\sigma) = \{T \in D |\ Tb = 0 \text{ for all } b \in B\},$$

so σ is injective iff B is essential in D.

Given an extension (2.15) we thus obtain a homomorphism $\sigma : D \to \mathcal{M}(B)$. Composing σ with the quotient map $q : \mathcal{M}(B) \to \mathcal{Q}(B)$ we obtain a homomorphism $\gamma : A \to \mathcal{Q}(B)$ (using that $D/B \cong A$) such that $D \cong D_\gamma$.

Definition 2.4.30. The map $\gamma : A \to \mathcal{Q}(B)$ such that $D = D_\gamma$ is the **Busby invariant** of the extension (2.15).

The most obvious way of putting an equivalence relation on extensions (2.15) is to say that they are equivalent if they are isomorphic, i.e. there is an isomorphism $D_1 \to D_2$ giving a commutative diagram connecting $B \to D_1 \to A$ with $B \to D_2 \to A$. The set of isomorphism classes of extensions is therefore equal to $\mathrm{Hom}(A, \mathcal{Q}(B))$, by identifying an isomorphism class of extensions with its Busby invariant [26, Thm. 4.3]. However, we need a weaker notion of equivalence for the set of equivalence classes to form a nice structure. In other words, we are putting an equivalence relation on $\mathrm{Hom}(A, \mathcal{Q}(B))$.

Definition 2.4.31. Two extensions $\gamma_1 : A \to \mathcal{Q}(B)$ and $\gamma_2 : A \to \mathcal{Q}(B)$ are strongly unitarily equivalent if there exists a unitary $U \in \mathcal{Q}(B)$ such that

$$\gamma_2(a) = U\gamma_1(a)U^{-1}, \qquad \forall a \in A.$$

We shall consider extensions of the form

$$0 \longrightarrow B \otimes \mathcal{K} \longrightarrow D \longrightarrow A \longrightarrow 0. \tag{2.17}$$

In fact, since $\mathcal{Q}(B) \subseteq \mathcal{Q}(B \otimes \mathcal{K})$, any extension $\gamma : A \to \mathcal{Q}(B)$ defines an extension $\gamma : A \to \mathcal{Q}(B) \subset \mathcal{Q}(B \otimes \mathcal{K})$ [66, Remark 7.2]. The reason for looking at maps into the corona of $B \otimes \mathcal{K}$ instead of B is similar to why we look at projections *over* A when defining $K_0(A)$. In particular, if $\gamma_1 : A \to \mathcal{Q}(B \otimes \mathcal{K})$ and $\gamma_2 : A \to \mathcal{Q}(B \otimes \mathcal{K})$ are two extensions then

$$\gamma_1 \oplus \gamma_2 : A \to \mathcal{Q}(B \otimes \mathcal{K}) \oplus \mathcal{Q}(B \otimes \mathcal{K}) \subset \mathrm{M}_2(\mathcal{Q}(B \otimes \mathcal{K})) \cong \mathcal{Q}(B \otimes \mathcal{K})$$

is determined up to strong unitary equivalence.

Definition 2.4.32. Let $\mathrm{Ext}(A, B)$ denote the quotient of the set of all equivalence classes $[\gamma]$ of extensions under strong unitary equivalence by the set of "trivial" extensions (see below). Equip $\mathrm{Ext}(A, B)$ with the structure of an Abelian semigroup induced from the direct-sum operation $\oplus$.

Remark 2.4.33 (Zero element)**.** The zero element in $\mathrm{Ext}(A, B)$ is represented by the **trivial** extensions, i.e. those $\gamma : A \to \mathcal{Q}(B \otimes \mathcal{K})$ which lift to homomorphisms $\tilde{\gamma} : A \to \mathcal{M}(B \otimes \mathcal{K})$. If γ is trivial then the C^*-algebra D_γ in (2.16) takes the form

$$D_\gamma = \{(T, a) \in \mathcal{M}(B \otimes \mathcal{K}) \oplus A |\ T - \tilde{\gamma}(a) \in B \otimes \mathcal{K}\},$$

and the corresponding extension of A by B is **split**, i.e. there is a homomorphic right inverse $\varsigma : A \to D_\gamma$ for the surjection $\pi : D_\gamma \to A$, meaning that $\pi \circ \varsigma = \mathrm{id}$. Indeed, we can take $\varsigma(a) := (\tilde{\gamma}(a), a)$. Note that if (2.15) is a split extension then $D \cong A \oplus B$. The map $\sigma : D \to \mathcal{M}(B)$ guaranteed by Lemma 2.4.29 can be composed with the splitting ς to produce a lifting $\sigma \circ \varsigma : A \to \mathcal{M}(B)$ of the Busby invariant of any split extension. Hence, split extensions are exactly the trivial ones [26, Prop. 5.3], and the zero element in $\mathrm{Ext}(A, B)$ is represented by all elements of $\mathrm{Hom}(B, \mathcal{M}(A))$ (and there are no other representatives).

There are some extensions γ which possesses an "inverse" in the sense that there is an extension γ^{-1} such that $[\gamma]+[\gamma^{-1}] = 0$ in $\mathrm{Ext}(A, B)$. The subset $\mathrm{Ext}(A, B)^{-1}$ of (equivalence classes of) invertible extensions is an Abelian group. If A is nuclear then every extension

is invertible, $\mathrm{Ext}(A,B)^{-1} = \mathrm{Ext}(A,B)$ [14, Thm. 15.8.3].

The invertible extensions allow an alternative, more concrete description. For, suppose that $\gamma \oplus \gamma^{-1}$ represents $0 \in \mathrm{Ext}(A,B)$. This says that $\gamma \oplus \gamma^{-1} : A \to \mathrm{M}_2(\mathcal{Q}(B \otimes \mathcal{K}))$ lifts to a $*$-homomorphism

$$\pi = \begin{pmatrix} \pi_{11} & \pi_{12} \\ \pi_{21} & \pi_{22} \end{pmatrix} : A \to \mathrm{M}_2(\mathcal{M}(B \otimes \mathcal{K})),$$

satisfying $q \circ \pi_{1,1} = \gamma$ and $q \circ \pi_{2,2} = \gamma^{-1}$. We may view π as a map into $\mathcal{M}(B \otimes \mathcal{K})$. Since $\pi_{1,1} : A \to \mathcal{M}(B \otimes \mathcal{K})$ is a compression of π, there is then a projection P in $\mathcal{M}(B \otimes \mathcal{K})$ such that

$$\gamma(a) = \pi(P\pi(a)P), \tag{2.18}$$

and the fact that γ is a homomorphism is equivalent to

$$[\pi(a), P] \subset B \otimes \mathcal{K}, \qquad \forall a \in A. \tag{2.19}$$

We summarize these observations.

Lemma 2.4.34. *The class of an invertible extension $\gamma : A \to \mathcal{Q}(B \otimes \mathcal{K})$ can be represented by a pair (π, P) where $\pi : A \to \mathcal{M}(B \otimes \mathcal{K})$ is a homomorphism and P is a projection in $\mathcal{M}(B \otimes \mathcal{K})$ for which* (2.18) *and* (2.19) *hold. If B has a countable approximate identity, then*

$$\mathrm{Ext}(A,B)^{-1} \cong KK^1(A,B).$$

Proof. Let (π, X_B, F) be a Kasparov A-B module. We may regard the Hilbert module X_B as being embedded into $\ell^2(\mathbb{N}; B)$. Then $P := 2F - \mathbf{1} \in \mathcal{M}(B \otimes \mathcal{K})$ is a projection modulo $B \otimes \mathcal{K}$, and without changing the KK^1-class of (π, X_B, F) we may assume that P is a true projection. We can then define an extension $\gamma : A \to \mathcal{Q}(B \otimes \mathcal{K})$ by (2.18). This extension is invertible: $\pi_{2,2}(a) := (\mathbf{1} - P)\psi(a)(\mathbf{1} - P)$ defines a compression of π and applying q we obtain an extension γ^{-1} such that $\gamma \oplus \gamma^{-1}$ is trivial.

It remains to show that the association (2.18) respects equivalence relations. That would require a more detailed discussion about the equivalence relations in KK^1. See [66, Lemma 6.2] for the proof. □

Remark 2.4.35. The extension (2.17) is **semi-split** if the surjection $D \to A$ admits a

completely positive norm-decreasing splitting. Now by the description of extensions given in Lemma 2.4.34, the extension is semi-split if and only if the extension is invertible. Indeed, in that case the Busby invariant $\gamma : A \to \mathcal{Q}(B \otimes \mathcal{K})$ has a completely positive contractive lift

$$\pi_{1,1} : A \to D \subset \mathcal{M}(B \otimes \mathcal{K}), \qquad \pi_{1,1}(a) = P\pi(a)P$$

as in (2.18) above.

2.5 Spectral triples

Kasparov modules give rise to groups $KK^0(A, B)$ whose purpose is to give an abstract and generalized index pairing. This is in particular so for (even) Fredholm modules, whose classes form the K-homology group $K^0(A)$ which pairs with $K_0(A)$. An index pairing with K-theory can also been achieved from the data of a "spectral triple". A spectral triple can be regarded as an "unbounded representative" of a $K^0(A)$-class, because every (even) spectral triple defines an element of $K^0(A)$. One reason for the significance of spectral triples is that they are easier to compute with than a Fredholm module. In recent years, also KK-theory has been studied using unbounded representatives, in order to find ways of computing Kasparov products (see e.g. [21]).

We are going to need a notion of spectral triples which is slightly more general than the original one, and this fact is stressed by adding the term "semifinite". So we shall begin by defining these "semifinite spectral triples".

2.5.1 Semifinite spectral triples

Definition 2.5.1 ([28, Def. 2.1])**.** Let A be a C^*-algebra. A $(\mathcal{N}, \tau)$**-semifinite spectral triple** over A is a triple $(\mathcal{A}, \mathcal{H}, \not{D})$ of a dense $*$-subalgebra $\mathcal{A} \subset A$, a representation $\pi : A \to \mathcal{N} \subset \mathcal{B}(\mathcal{H})$ and a densely defined selfadjoint operator $\not{D}$ affiliated with $\mathcal{N}$ such that for all $a \in \mathcal{A}$,

(i) the operator $\pi(a)$ preserves the domain of $\not{D}$ (implying that the commutator $[\not{D}, \pi(a)]$ is densely defined), $[\not{D}, \pi(a)]$ extends to a bounded operator on $\mathcal{H}$, and

(ii) $\pi(a)(\mathbf{1} + \not{D}^2)^{-1/2}$ belongs to $\mathcal{K}(\mathcal{N}, \tau)$.

If $\mathcal{H}$ has a decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with corresponding grading operator $\Gamma = \operatorname{diag}(\mathbf{1}, -\mathbf{1})$ which commutes with $\pi(\mathcal{A})$ and anticommutes with $\not{D}$ then the spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ is **even**; otherwise **odd**.

Notation 2.5.2. To adhere with standard convention we have referred to the data of a spectral triple as a triple $(\mathcal{A}, \mathcal{H}, \not{D})$. However, we also wrote Kasparov and Fredholm modules as triples (π, X_B, F). To be consistent we should write $(\mathcal{A}, \pi, \mathcal{H}, \not{D})$ because the representation π is still part of the data and the dense $*$-subalgebra $\mathcal{A}$ is one extra input. Anyway, when we write $(\mathcal{A}, \mathcal{H}, \not{D})$, the symbol π will *always* be used for the representation of A on $\mathcal{H}$.

Notation 2.5.3. We will often discuss properties of spectral triples which depend on the parity (even and odd) but work for both, provided that one associates the correct KK-group (KK^0 and KK^1 respectively). In such a situation we write $KK^\bullet(\cdot,\cdot)$ with the interpretation that $\bullet \in \{0, 1\} = \{\text{even}, \text{odd}\}$ and that the relevant spectral triple has parity $\bullet$. Similarly for $K^\bullet(\cdot)$ and $K_\bullet(\cdot)$.

Remark 2.5.4 ([28, Remark 2.2])**.** Condition (ii) is equivalent to

$$\pi(a)(i + \not{D})^{-1} \in \mathcal{K}(\mathcal{N}, \tau), \qquad \forall a \in \mathcal{A}, \tag{2.20}$$

and, moreover, density of $\mathcal{A}$ in A implies that (2.20) holds for all $a \in A$.

When we need no explicit reference to the pair $(\mathcal{N}, \tau)$, we shall simply refer to a $(\mathcal{N}, \tau)$-semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ as a "spectral triple".

It is well known that (even) $(\mathcal{B}(\mathcal{H}), \mathrm{Tr})$-semifinite spectral triples over A represent elements of the K-homology group $K^0(A) = KK^0(A, \mathbb{C})$ [14, Section 17.11]. For general $(\mathcal{N}, \tau)$-semifinite spectral triples, the compactness conditions are not with respect to $\mathcal{K}(\mathcal{H})$ but with respect to $\mathcal{K}(\mathcal{N}, \tau)$, and so these spectral triples do not represent elements of $KK^0(A, \mathbb{C})$. However, KK-theory is sufficiently general to be useful also in this setting. Namely, we shall see that one can find a C^*-algebra B replacing $\mathbb{C}$ such that a semifinite spectral triple over A defines an element of $KK^0(A, B)$, although it is at first sight far from obvious how to do that.

Notation 2.5.5 ([28, Def. 2.5])**.** Let $(\mathcal{A}, \mathcal{H}, D)$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over A. For $\varepsilon > 0$ we set

$$\not{F}_\varepsilon := \not{D}(\varepsilon \mathbf{1} + \not{D}^2)^{-1/2},$$

and define $\mathcal{B} \subset \mathcal{K}(\mathcal{N}, \tau)$ to be the $*$-algebra generated by the operators

$$\not{F}_\varepsilon[\not{F}_\varepsilon, \pi(a)], \qquad [\not{F}_\varepsilon, \pi(a)], \qquad \not{F}_\varepsilon \pi(b)[\not{F}_\varepsilon, \pi(a)], \qquad \pi(a) f(\not{D})$$

for all $a, b \in \mathcal{A}$ and all $f \in C_0(\mathbb{R})$. Let B be the norm closure of $\mathcal{B}$.

Theorem 2.5.6 ([64, Thm. 5.3])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a separable C^*-algebra A. Then $\mathcal{B}$ is separable, contained in $\mathcal{K}(\mathcal{N}, \tau)$ and defines a Kasparov A-B-module $(\pi_B, B_B, \not{F}_\varepsilon)$, where $\pi_B : A \to \mathcal{M}(B)$ is the action of A by left multiplication with elements of $\pi(A)$. If $(\mathcal{A}, \mathcal{H}, \not{D})$ is even then we have $[\pi_B, B_B, \not{F}_\varepsilon] \in KK^0(A, B)$, while if $(\mathcal{A}, \mathcal{H}, \not{D})$ is odd we have $[\pi_B, B_B, \not{F}_\varepsilon] \in KK^1(A, B)$. The class $[\pi_B, B_B, \not{F}_\varepsilon]$ is independent of $\varepsilon > 0$.*

Here we use Remark 2.4.28 to obtain the correct parity (even or odd) for the KK-class $[\pi_B, B_B, \not{F}_\varepsilon]$.

For $\varepsilon = 1$, the operator $\not{F} := \not{F}_1$ is known as the **bounded transform** of $\not{D}$.

Lemma 2.5.7. *Suppose that $\not{D}$ is invertible. Then the class $[\pi_B, B_B, \not{F}] \in KK^\bullet(A, B)$ is also represented by $(\pi_B, B_B, \not{R})$, where $\not{R} := \not{D}|\not{D}|^{-1}$ is the **phase** of $\not{D}$. This representative is normalized, i.e. $\not{R}^2 = \mathbf{1}$.*

Proof. The continuous path $[0, 1] \ni \varepsilon \to F_\varepsilon$ in $\mathcal{L}_B(B)$ defined by

$$F_\varepsilon := \not{D}(\varepsilon + \not{D}^2)^{-1/2}$$

gives an operator homotopy between $(\pi_B, B_B, \not{R})$ and $(\pi_B, B_B, \not{F})$. $\square$

From the description of the isomorphism $KK^1(A, B) \cong \mathrm{Ext}(A, B)^{-1}$ given in the proof of Lemma 2.4.34, we have the following.

Corollary 2.5.8. *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be an odd spectral triple over a separable C^*-algebra A. Suppose that the operator $\not{D}$ is invertible, so that $\not{F} := \not{D}(1 + \not{D}^2)^{-1/2}$ satisfies $\not{F}^2 = \mathbf{1}$ and*

$$\not{P} := \frac{1}{2}(\mathbf{1} + \not{F})$$

is a projection. Then the triple $(\mathcal{A}, \mathcal{H}, \not{D})$ determines an extension

$$0 \longrightarrow B \longrightarrow \mathcal{T} \longrightarrow A \longrightarrow 0$$

where B is as in Notation 2.5.5 and $\mathcal{T}$ is the pullback

$$\mathcal{T} := \{(T, a) \in \mathcal{M}(B) \oplus A |\ q(T) = \gamma(a)\},$$

with the Busby invariant $\gamma : A \to \mathcal{Q}(B)$ given by

$$\gamma(a) := q(\not{P}\pi_B(a)\not{P}), \qquad \forall a \in A.$$

Definition 2.5.9 ([28, Def. 2.12])**.** Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a separable C^*-algebra A. If $(\mathcal{A}, \mathcal{H}, \not{D})$ is even, we denote by

$$([e] - [1_r]) \otimes_A [\pi_B, B_B, \not{F}] \in KK^0(\mathbb{C}, B)$$

the Kasparov product between a K_0-class $[e]-[1_r] \in K_0(A) = KK^0(\mathbb{C}, A)$ and $[\pi_B, B_B, \not{F}] \in KK^0(A, B)$. We then define the **index pairing** between $(\mathcal{A}, \mathcal{H}, \not{D})$ and $[e]$ to be the real number

$$\langle [e] - [1_r], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle := \tau_*([e] - [1_r] \otimes_A [\pi_B, B_B, \not{F}])$$

obtained by applying the homomorphism $\tau_* : K_0(B) \to \mathbb{R}$ induced by the trace $\tau : \mathcal{N}_+ \to [0, +\infty]$ (see Lemma 2.3.19) to the Kasparov product. Similarly, if $(\mathcal{A}, \mathcal{H}, \not{D})$ is odd then $\langle [u], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle$ denotes the index pairing between a K_1-class $[u] \in K^1(\mathbb{C}, A)$ and $[\pi_B, B_B, \not{F}] \in KK^1(A, B)$,

$$\langle [u], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle := \tau_*([u] \otimes_A [\pi_B, B_B, \not{F}]).$$

The name "index pairing" will be justified in §2.5.3.

2.5.2 Doubling up for invertiblity

We now recall from [28, Def. 2.9] how to construct, by doubling-up to 2×2 matrices, a representative $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{\not{D}})$ of the KK-class coming from a spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ over a C^*-algebra A for which the operator $\boldsymbol{\not{D}}$ is invertible. The doubled-up Hilbert space is

$\boldsymbol{\mathcal{H}} := \mathcal{H} \otimes \mathbb{C}^2$ and the new operator is taken to be

$$\boldsymbol{D\!\!\!/} := \begin{pmatrix} D\!\!\!/ & m\mathbf{1} \\ m\mathbf{1} & -D\!\!\!/ \end{pmatrix}, \tag{2.21}$$

for some $m > 0$, while the representation of an element $a \in A^{\sim}$ on $\boldsymbol{\mathcal{H}}$ is

$$\boldsymbol{\pi}(a) := \begin{pmatrix} \pi(a) & 0 \\ 0 & \pi(\varepsilon(a)) \end{pmatrix}, \tag{2.22}$$

where $\varepsilon : A^{\sim} \to \mathbb{C}$ is the projection $\varepsilon(x + \lambda\mathbf{1}) := \lambda$.

We have $\boldsymbol{D\!\!\!/}^2 \geq m^2\mathbf{1}$, so $\boldsymbol{D\!\!\!/}$ is invertible. If $(\mathcal{A}, \mathcal{H}, D\!\!\!/)$ is even, so that there is a grading $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with respect to which $D\!\!\!/$ is odd and $\pi(\mathcal{A})$ is even, we can define

$$\boldsymbol{\mathcal{H}}_+ := \mathcal{H}_+ \oplus \mathcal{H}_-, \qquad \boldsymbol{\mathcal{H}}_- := \mathcal{H}_- \oplus \mathcal{H}_+.$$

If $\Gamma = \operatorname{diag}(\mathbf{1}, -\mathbf{1})$ is the grading operator on $\mathcal{H}$, the grading operator on $\boldsymbol{\mathcal{H}}$ is then $\boldsymbol{\Gamma} = \operatorname{diag}(\Gamma, -\Gamma)$. Then the doubled triple $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{D\!\!\!/})$ is again even.

The Kasparov module defined from $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{D\!\!\!/})$ (cf. Theorem 2.5.6) represents an element of $KK^0(A, B)$. It is given by $(\boldsymbol{\pi}_B, \boldsymbol{B}_B, \boldsymbol{F\!\!\!/})$, where $\boldsymbol{F\!\!\!/} := \boldsymbol{D\!\!\!/}(1 + \boldsymbol{D\!\!\!/}^2)^{-1/2}$ and $\boldsymbol{B} := B \otimes \mathbb{C}^2$. By Lemma 2.5.7, we can replace $\boldsymbol{F\!\!\!/}$ by $\boldsymbol{R\!\!\!/} := \boldsymbol{D\!\!\!/}|\boldsymbol{D\!\!\!/}|^{-1}$ without changing the KK-class.

Lemma 2.5.10 ([28, Lemma 2.10])**.** *The spectral triples $(\mathcal{A}, \mathcal{H}, D\!\!\!/)$ and $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{D\!\!\!/})$ define the same class in $KK^0(A, C)$. A bounded representative of this class is provided by $(\boldsymbol{\pi}_B, \boldsymbol{B}_B, \boldsymbol{R\!\!\!/})$ where $\boldsymbol{R\!\!\!/} = \boldsymbol{D\!\!\!/}|\boldsymbol{D\!\!\!/}|^{-1}$, $\boldsymbol{B} := B \otimes \mathbb{C}^2$ and $\boldsymbol{\pi}_B(a)$ is the action of left multiplication by $\boldsymbol{\pi}(a)$ for all $a \in A$. This representative is normalized, i.e. $\boldsymbol{R\!\!\!/}^2 = \mathbf{1}$.*

We thus have an equality (in the even case)

$$\langle [e] - [1_r], [\mathcal{A}, \mathcal{H}, D\!\!\!/] \rangle = \langle [e] - [1_r], [\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{D\!\!\!/}] \rangle$$

for all $[e] - [1_r] \in K_0(A)$, i.e.

$$\tau_*([e] - [1_r] \otimes_A [\pi_B, B_B, F\!\!\!/]) = \tau_*([e] - [1_r] \otimes_A [\pi_B, \boldsymbol{B}_B, \boldsymbol{R\!\!\!/}])$$

(and similarly in the odd case). Developing $\langle [e] - [1_r], [\mathcal{A}, \mathcal{H}, D\!\!\!/] \rangle$ into a more concrete

expression can only be done using the triple $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{\not{D}})$, unless $\not{D}$ is already invertible. We will see below that $\langle [e] - [1_r], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle$ has many different interpretations.

Remark 2.5.11. One may replace the scalar operator $m\mathbf{1}$ in the double-up construction (2.21) by a more general operator on $\mathcal{H}$ satisfying some compatibility relations with $\not{D}$, such that $\boldsymbol{\not{D}}$ becomes invertible. However, due to [60, Lemma 6.3], little is gained in this more general setting.

The double-up construction also allows us to generalize Corollary 2.5.8 to arbitrary spectral triples of odd parity. Write $\boldsymbol{\not{P}} := (\mathbf{1} + \boldsymbol{\not{R}})/2$ for the projection onto the nonnegative part of the spectrum of the selfadjoint operator $\boldsymbol{\not{D}}$.

Corollary 2.5.12. *Any odd spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ over a separable C^*-algebra A determines an extension*

$$0 \longrightarrow \boldsymbol{B} \longrightarrow \boldsymbol{\mathcal{T}} \longrightarrow A \longrightarrow 0$$

where $\boldsymbol{B} := B \otimes \mathbb{C}^2$ (with B is as in Notation 2.5.5). The C^*-algebra $\boldsymbol{\mathcal{T}}$ is the pullback

$$\boldsymbol{\mathcal{T}} := \{(T, a) \in \mathcal{M}(\boldsymbol{B}) \oplus A |\ q(T) = \boldsymbol{\gamma}(a)\},$$

where the Busby invariant $\boldsymbol{\gamma} : A \to \mathcal{Q}(\boldsymbol{B})$ is given by

$$\boldsymbol{\gamma}(a) := q(\boldsymbol{\not{P}} \boldsymbol{\pi}_B(a) \boldsymbol{\not{P}}), \qquad \forall a \in A.$$

2.5.3 The pairing as an abstract Fredholm index

Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a separable C^*-algebra A, of parity $\bullet \in \{0, 1\} := \{\text{even}, \text{odd}\}$. We shall show that the Kasparov product $[x] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \boldsymbol{\not{R}}]$ with K-theory classes $[x] \in K_\bullet(A)$ is equal to the Mingo index of a Fredholm operator on the Hilbert B-module $\boldsymbol{B}_B$.

Notation 2.5.13. In the following statements we want to allow for matrices over A. In order to make the formulas readable, we shall make the following convention. For $x \in \mathrm{M}_r(A) = \mathrm{M}_r(\mathbb{C}) \otimes A$, we write

$$\boldsymbol{\pi}_B(x) := (\mathrm{id} \otimes \boldsymbol{\pi}_B)(x), \qquad \boldsymbol{\not{R}} \boldsymbol{\pi}_B(x) := (1_r \otimes \boldsymbol{\not{R}}) \boldsymbol{\pi}_B(x)$$

as operators on $\mathbb{C}^r \otimes \boldsymbol{B} = \boldsymbol{B}^{\oplus r}$. There should be no confusion since without this convention, the expression $\boldsymbol{\pi}_B(x)$ etc. does not make sense unless $r = 1$.

Theorem 2.5.14. *Suppose that $(\mathcal{A}, \mathcal{H}, \not{D})$ is odd. Let $u \in \mathrm{U}_r(A^\sim)$ be a unitary over $A^\sim$ and denote by $[u] \in K_1(A)$ the homotopy class of u. Then we have the equality*

$$[u] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}] = \mathrm{Index}(\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}})$$

in $K_0(B)$.

Proof. For ease of notation, assume $r = 1$.

To a unitary $u \in A^\sim$ there corresponds a homomorphism $\rho_u : C_0(\mathbb{R})^\sim \to \mathcal{M}(A)$ which takes $z - \mathbf{1}$ to $u - \mathbf{1}$ under the identification of $K_1(A)$ with $KK^0(C_0(\mathbb{R}), A)$. The Kasparov product of the class of ρ_u with the element $[\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}]$ is given by

$$[\rho_u] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}] = [\boldsymbol{\pi}_B \circ \rho_u, \boldsymbol{B}_B, \not{\boldsymbol{R}}],$$

which is an element of $KK^1(C_0(\mathbb{R}), B)$. A homomorphism such as $\boldsymbol{\pi}_B \circ \rho_u : C_0(\mathbb{R})^\sim \to \mathcal{M}(B \otimes \mathcal{K})$ defines a unitary operator $U = \boldsymbol{\pi}_B \circ \rho_u(z)$ in $\mathcal{M}(B \otimes \mathcal{K})$ (which in the present case is just $\boldsymbol{\pi}_B(u)$), and conversely a unitary in $\mathcal{M}(B \otimes \mathcal{K})$ determines a homomorphism from $C_0(\mathbb{R})^\sim$ to $\mathcal{M}(B \otimes \mathcal{K})$. Homotopy equivalence of homomorphisms from $C_0(\mathbb{R})^\sim$ to $\mathcal{M}(B \otimes \mathcal{K})$ translates into homotopy equivalence of the corresponding unitaries in $\mathcal{M}(B \otimes \mathcal{K})$. So the class $[\boldsymbol{\pi}_B \circ \rho_u, \boldsymbol{B}_B, \not{\boldsymbol{R}}]$ is represented by a unitary $U = \boldsymbol{\pi}_B(u)$ in $\mathcal{M}(B \otimes \mathcal{K})$ which commutes with $\not{\boldsymbol{R}}$ modulo $B \otimes \mathcal{K}$.

Therefore, if we set $\not{\boldsymbol{P}} := (\mathbf{1} + \not{\boldsymbol{R}})/2$ then the operator $\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}}$ is a Fredholm operator, i.e. it is invertible modulo $B \otimes \mathcal{K}$. Using Lemma 2.4.22, our class can be identified with the class $[q(\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}})]$ in $K_1(\mathcal{Q}(B \otimes \mathcal{K}))$, where $q : \mathcal{M}(B \otimes \mathcal{K}) \to \mathcal{Q}(B \otimes \mathcal{K})$ is the quotient map.

What we have done so far is just to trace the fate of the representative $(\boldsymbol{\pi}_B \circ \rho_u, \boldsymbol{B}_B, \not{\boldsymbol{R}})$ under the isomorphism of $KK^1(C_0(\mathbb{R}), B)$ with $K_1(\mathcal{Q}(B \otimes \mathcal{K}))$ [14, Prop. 17.5.7].

As we know from §2.4.3, equivalence classes of Fredholm operators on $\ell^2(\mathbb{N}; B)$ correspond to elements in $K_0(B)$ via the index map. The image of $[\rho_u] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}]$ under this map is the $K_0(B)$-valued index of the Fredholm operator $\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}}$:

$$\delta([q(\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}})]) = [\mathrm{Ker}(\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}})] - [\mathrm{Ker}(\not{\boldsymbol{P}}\boldsymbol{\pi}_B(u^*)\not{\boldsymbol{P}})].$$

□

As mentioned in §2.5.2, if the spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ is even then so is the doubled triple $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{\not{D}})$. The phase $\boldsymbol{\not{R}} = \boldsymbol{\not{D}}|\boldsymbol{\not{D}}|^{-1}$ decomposes in $\boldsymbol{\mathcal{H}} = \boldsymbol{\mathcal{H}}_+ \oplus \boldsymbol{\mathcal{H}}_-$ as

$$\boldsymbol{\not{R}} = \begin{pmatrix} 0 & \boldsymbol{\not{R}}_- \\ \boldsymbol{\not{R}}_+ & 0 \end{pmatrix}$$

with $\boldsymbol{\not{R}}_- = (\boldsymbol{\not{R}}_+)^*$. Under the decomposition $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, the algebra B splits as $B = B_+ \oplus B_-$, and this induces an even grading $\boldsymbol{B} = \boldsymbol{B}_+ \oplus \boldsymbol{B}_-$ of the Hilbert B-module $\boldsymbol{B}$. Here $\boldsymbol{B}_+$ is the part of $\boldsymbol{B}$ commuting with the grading operator $\Gamma = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$ and $\boldsymbol{B}_-$ is the part anti-commuting with Γ (cf. Remark 2.4.28).

Theorem 2.5.15. *Suppose that $(\mathcal{A}, \mathcal{H}, \not{D})$ is even. Let $e, f \in \mathrm{M}_\infty(A^\sim)$ be projections over $A^\sim$ with $[e] - [f] \in K_0(A)$. Then we have the equality*

$$([e] - [f]) \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \boldsymbol{\not{R}}] = \mathrm{Index}(\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_+\boldsymbol{\pi}_B(e)) - \mathrm{Index}(\boldsymbol{\pi}_B(f)\boldsymbol{\not{R}}_+\boldsymbol{\pi}_B(f))$$

in $K_0(B)$, where $\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_+\boldsymbol{\pi}_B(e)$ is viewed as an operator from $\boldsymbol{\pi}_B(e)\boldsymbol{B}_+^{\oplus r}$ to $\boldsymbol{\pi}_B(e)\boldsymbol{B}_-^{\oplus r}$.

Proof. Again we consider matrices of size $r = 1$ for simplicity. So let $e, f \in A^\sim$ be projections with $e - f \in A$.

Under the isomorphism $K_0(A) \cong KK^0(\mathbb{C}, A)$ described in the proof of Proposition 2.4.26, the class $[e] - [f]$ corresponds to a homomorphism ρ from $\mathbb{C}$ to $\mathcal{M}(A \oplus A)$ sending $1 \in \mathbb{C}$ to $e \oplus f$. The Kasparov product with the KK-class of the spectral triple is then the element

$$[\rho] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \boldsymbol{\not{R}}] = [\boldsymbol{\pi}_B \circ \rho, \boldsymbol{B}_B, \boldsymbol{\not{R}}]$$

in $KK^0(\mathbb{C}, B)$. The map $\boldsymbol{\pi}_B \circ \rho$ sends $1 \in \mathbb{C}$ to the operator $\boldsymbol{\pi}_B \circ \rho(1) = \boldsymbol{\pi}_B(e) - \boldsymbol{\pi}_B(f)$. From

$$(\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_+\boldsymbol{\pi}_B(e))(\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_-\boldsymbol{\pi}_B(e)) = \boldsymbol{\pi}_B(e) \mod B \otimes \mathcal{K},$$

$$(\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_-\boldsymbol{\pi}_B(e))(\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_+\boldsymbol{\pi}_B(e)) = \boldsymbol{\pi}_B(e) \mod B \otimes \mathcal{K},$$

we see that $\boldsymbol{\pi}_B(e)\boldsymbol{\not{R}}_+\boldsymbol{\pi}_B(e)$ is Fredholm as an operator from the module $\boldsymbol{\pi}_B(e)\boldsymbol{B}_+$ to the module $\boldsymbol{\pi}_B(e)\boldsymbol{B}_-$.

We identify $q(\boldsymbol{\pi}_B(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(e))$ with a unitary in $\mathcal{Q}(B\otimes\mathcal{K})$ and similarly with f replacing e. The difference $q(\boldsymbol{\pi}_B(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(e)) - q(\boldsymbol{\pi}_B(f)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(f))$ represents the class $[\rho]\otimes_A[\boldsymbol{\pi}_B,\boldsymbol{B}_B,\not{\boldsymbol{R}}]$ under the identification of $KK^0(\mathbb{C},B)$ with $K_1(\mathcal{Q}(B\otimes\mathcal{K}))$.

The group isomorphism $\delta : K_1(\mathcal{Q}(B\otimes\mathcal{K}))\to K_0(B)$ just sends the homotopy class $[q(\boldsymbol{\pi}_B(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(e)) - q(\boldsymbol{\pi}_B(f)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(f))]$ to the $K_0(B)$-valued index of the Fredholm operator $\boldsymbol{\pi}_B(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(e) - \boldsymbol{\pi}_B(f)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_B(f)$, as asserted. □

Remark 2.5.16 (The obstruction to using $\not{F}$)**.** In general we cannot use $\not{F} := \not{D}(1+\not{D}^2)^{-1/2}$ instead of $\not{\boldsymbol{R}}$ in the above pairings. The problem is that $\pi_B(e)\not{F}_+\pi_B(e)$ need not be Fredholm when A is nonunital and $\not{F}^2\neq\mathbf{1}$. We recall the details about this fact from [28, §2.3].

Generally, let A and B be C^*-algebras and let (π_B, X_B, F) be an even Kasparov A-B module. Let $e\in A^\sim$ be a projection. We would like $\pi_B(e)F_+\pi_B(e)$ to be a Fredholm operator from $\pi_B(e)X_+$ to $\pi_B(e)X_-$. So we try to show that $\pi_B(e)F_+\pi_B(e)$ is invertible modulo $\mathcal{K}_B(X)$. We have

$$(\pi_B(e)F_-\pi_B(e))(\pi_B(e)F_+\pi_B(e)) = \pi_B(e)F_-[\pi_B(e),F_+]\pi_B(e)+\pi_B(e)(F_-F_+-\mathbf{1})\pi_B(e)+\pi_B(e).$$

The term $\pi_B(e)F_-[\pi_B(e),F_+]\pi_B(e)$ is compact. Indeed $[\pi_B(a),F_+]$ was required to be compact for $a\in A$ by definition of Kasparov module, and elements of the form $\lambda\mathbf{1}\in A^\sim$ have trivial commutators. The problematic term is $\pi_B(e)(F_-F_+-\mathbf{1})\pi_B(e)$, which is guaranteed to be compact only for $e\in A$.

This is where the condition $F^2=\mathbf{1}$ becomes important. If $F^2=\mathbf{1}$ then $F_-F_+-\mathbf{1}=0$ and so $\pi_B(e)F_+\pi_B(e)$ is Fredholm.

2.5.4 The pairing as a semifinite Fredholm index

Let $(\mathcal{A},\mathcal{H},\not{D})$ be an odd $(\mathcal{N},\tau)$-semifinite spectral triple over a C^*-algebra A. We saw in Theorem 2.5.14 that for any $[u]\in K_1(A)$, the element $T_u := \not{\boldsymbol{P}}\boldsymbol{\pi}_B(u)\not{\boldsymbol{P}}\in\mathcal{M}(B\otimes\mathcal{K})$ is a Fredholm operator on the Hilbert B-module $\ell^2(\mathbb{N};B)$. The projections $\mathrm{Ker}(T_u)$ and $\mathrm{Ker}(T_u^*)$ are of finite rank and can be regarded as elements of $B\otimes\mathcal{K}$. Now the trace τ induces a homomorphism $\tau_* : K_0(B)\to\mathbb{R}$. Therefore, the (Hilbert-module) Fredholm operator T_u is $(\tau\otimes\mathrm{Tr})$-Fredholm in the "semifinite" sense of §2.3.3, where Tr is operator trace on $\mathcal{K}$.

Similarly, if $(\mathcal{A}, \mathcal{H}, \not{D})$ is even then elements of $K_0(A)$ produce $(\tau \otimes \mathrm{Tr})$-Fredholm operators in the semifinite sense.

Recall that the index pairing $\langle [x], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle$ is simply the real number obtained by applying τ_* to the Kasparov product $[x] \otimes_A [\boldsymbol{\pi}_B, \boldsymbol{B}_B, \not{\boldsymbol{R}}]$.

Corollary 2.5.17. *Suppose that $(\mathcal{A}, \mathcal{H}, \not{D})$ is odd. Let $u \in \mathrm{U}_\infty(\mathcal{A}^\sim)$ be a unitary over $\mathcal{A}^\sim$ and denote by $[u] \in K_1(A)$ the homotopy class of u. Then we have the equality*

$$\langle [u], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle = \tau_*(\mathrm{Index}(\not{\boldsymbol{P}} \boldsymbol{\pi}_B(u) \not{\boldsymbol{P}}))$$

in $\tau_*(K_0(B)) \subseteq \mathbb{R}$.

Suppose that $(\mathcal{A}, \mathcal{H}, \not{D})$ is even. Let $e, f \in \mathrm{M}_\infty(\mathcal{A}^\sim)$ be projections over $\mathcal{A}^\sim$ with $[e] - [f] \in K_0(A)$. Then we have the equality

$$\langle [e] - [f], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle = \tau_*(\mathrm{Index}(\boldsymbol{\pi}_B(e) \not{\boldsymbol{R}}_+ \boldsymbol{\pi}_B(e)) - \mathrm{Index}(\boldsymbol{\pi}_B(f) \not{\boldsymbol{R}}_+ \boldsymbol{\pi}_B(f)))$$

in $\tau_*(K_0(B)) \subseteq \mathbb{R}$.

One may phrase the above in terms of semifinite Fredholm modules. In fact, to obtain a Fredholm module one need not pass to the doubled triple.

Lemma 2.5.18 ([28, Lemma 2.8])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a C^*-algebra A and, as before, write $\pi : A \to \mathcal{B}(\mathcal{H})$ for the representation of A as operators on the Hilbert space $\mathcal{H}$. Then $(\pi, \mathcal{H}, \not{F})$ is a $(\mathcal{N}, \tau)$-semifinite Fredholm module over A, with the same parity (odd or even) as $(\mathcal{A}, \mathcal{H}, \not{D})$.*

What Corollary (2.5.17) says is thus that $\langle \cdot, [\mathcal{A}, \mathcal{H}, \not{D}] \rangle$ is exactly the index pairing of the K-homology class of the Fredholm module $(\pi, \mathcal{H}, \not{F})$ with $K_\bullet(A)$, described in §2.3.3.2.

Later on however, when we construct its Chern character in cyclic cohomology, we will anyway need a *normalized* Fredholm module from the spectral triple, so we are forced to use the doubled-up triple $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \not{\boldsymbol{D}})$ in general.

2.5.5 From the double back to the original triple

Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a separable C^*-algebra A. The associated KK-class

$$[\pi_B, B_B, \not{F}] \in KK^\bullet(A, B)$$

was obtained using be the bounded transform $\slashed{F} := \slashed{D}(\mathbf{1}+\slashed{D}^2)^{-1/2}$ of $\slashed{D}$. Since $\slashed{F}^2 \neq \mathbf{1}$, we were forced to use the double $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \boldsymbol{\slashed{D}})$ for constructions which require the normalization property $\slashed{F}^2 = \mathbf{1}$. We now want to find another normalized representative of the class $[\pi_B, B_B, \slashed{F}]$ without having to replace B_B by the doubled module.

Proposition 2.5.19 ([28, Prop. 2.25])**.** *Denote by $\slashed{P}$ the spectral projection of $\slashed{D}$ corresponding to the nonnegative part of the spectrum. Then $(\pi_B, B_B, 2\slashed{P} - \mathbf{1})$ is a normalized Kasparov A-B-module and*

$$[\pi_B, B_B, 2\slashed{P} - \mathbf{1}] = [\pi_B, B_B, \slashed{F}] \in KK^\bullet(A, B).$$

This result has interesting consequences if the spectral triple $(\mathcal{A}, \mathcal{H}, \slashed{D})$ is odd. Indeed, since $\slashed{P}$ is a true projection, we have (cf. Theorem 2.5.14)

$$\operatorname{Index}(\boldsymbol{\slashed{P}}\boldsymbol{\pi}(u))\boldsymbol{\slashed{P}}) = \operatorname{Index}(\slashed{P}\pi(u)\slashed{P}),$$

so that the index pairing can be defined using $\slashed{P}$ instead of $\boldsymbol{\slashed{P}} = 2\boldsymbol{\slashed{R}} - \mathbf{1}$.

2.5.6 Spectral flow

It turns out that the odd index pairing, which produces a real number $\langle [u], [\mathcal{A}, \mathcal{H}, \slashed{D}] \rangle$ from a K_1-class $[u]$ and a spectral triple $(\mathcal{A}, \mathcal{H}, \slashed{D})$, has an intuitive interpretation. It is a non-obvious observation that $\langle [u], [\mathcal{A}, \mathcal{H}, \slashed{D}] \rangle$ coincides with the "spectral flow" of any path of selfadjoint operators starting at $\slashed{D}$ and ending at $u^*\slashed{D}u$. Importantly, this observation leads to some interesting integral formulas for $\langle [u], [\mathcal{A}, \mathcal{H}, \slashed{D}] \rangle$.

Suppose that F_0 and F_1 are two bounded selfadjoint operators on a Hilbert space $\mathcal{H}$ with trivial kernels. Let $\mathcal{B}^1_{\mathbb{R}}(\mathcal{H})$ denote set of bounded selfadjoint operators T on $\mathcal{H}$ with $\dim \operatorname{Ker} T = 1$. Atiyah-Patodi-Singer defined the **spectral flow** of the straight-line path $F_t := (1-t)F_0 + tF_1$ as the number of intersections of the path $F_\bullet$ with $\mathcal{B}^1_{\mathbb{R}}(\mathcal{H})$. It is this kind of quantity that will turn up in the context of spectral triples.

Notation 2.5.20. For two projections P and Q on a Hilbert space $\mathcal{H}$, we denote by $P \cap Q$ the projection onto $P\mathcal{H} \cap Q\mathcal{H}$.

Definition 2.5.21 ([96, Cor. 3.7])**.** Let $(\mathcal{N}, \tau)$ be a semifinite von Neumann algebra and let $[0,1] \ni t \to F_t$ be a norm-continuous function with values in the space of selfadjoint τ-

Fredholm elements in $\mathcal{N} \subseteq \mathcal{B}(\mathcal{H})$. Denote by P_t the spectral projection of F_t corresponding to the interval $[0, \infty)$. Fix an arbitrary partition $0 = t_0 < t_1 < \cdots < t_d = 1$ of the interval $[0, 1]$ such that for all $j = 1, \ldots, d$

$$\|q(P_t) - q(P_s)\| < 1/2, \qquad \forall s, t \in [t_{j-1}, t_j],$$

where $q : \mathcal{N} \to \mathcal{N}/\mathcal{K}_{\mathcal{N}}$ is the Calkin map. For brevity, write $P_j := P_{t_j}$. The **numerical spectral flow** of the path $F_\bullet$ is the real number

$$\mathrm{Sf}(F_\bullet) := \tau\big(\operatorname{Ker}(P_d \cdots P_0) \cap P_0\big) - \tau\big(\operatorname{Ker}(P_d \cdots P_0) \cap P_d\big).$$

The reason why the number $\mathrm{Sf}(F_\bullet)$ in Definition 2.5.21 can give some information about the change in the spectrum under the path $t \to F_t$ is that

(i) $\mathrm{Sf}(F_\bullet)$ is independent of the partition $\{t_j\}_j$ [96, Lemma 1.3], and

(ii) $\mathrm{Sf}(F_\bullet)$ depends only on the homotopy class of $F_\bullet$ [96, Prop. 2.5].

The numerical spectral flow discussed above has an abstract counterpart.

Definition 2.5.22 ([64, Cor. 3.7])**.** Let $[0, 1] \ni t \to F_t$ be a norm-continuous function with values in the space of selfadjoint τ-Fredholm elements in $\mathcal{N}$. Let P_t denote the spectral projection of F_t corresponding to the interval $[0, \infty)$. Fix an arbitrary partition $0 = t_0 < t_1 < \cdots < t_d = 1$ of the interval $[0, 1]$ such that for all $j = 1, \ldots, d$

$$\|q(P_t) - q(P_s)\| < 1/2, \qquad \forall s, t \in [t_{j-1}, t_j],$$

where $q : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the Calkin map. For brevity, write $P_j := P_{t_j}$. The **abstract spectral flow** of the path $F_\bullet$ is the class

$$\mathrm{sf}(F_\bullet) := [\operatorname{Ker}(P_d \cdots P_0) \cap P_0] - [\operatorname{Ker}(P_d \cdots P_0) \cap P_d]$$

in $K_0(\mathcal{K}_{\mathcal{N}})$.

We are interested in the spectral flow of the type

$$[0, 1] \ni t \to \not{D} + t\,\pi(u)^*[\not{D}, \pi(u)] \tag{2.23}$$

for unitaries $u \in \mathcal{A}^{\sim}$, where $(\mathcal{A}, \mathcal{H}, \not{D})$ is a $(\mathcal{N}, \tau)$-semifinite spectral triple over some C^*-algebra A. The **spectral flow** of such a path $\not{D}_\bullet$ is defined to be the spectral flow of the path $\not{F}_\bullet$, where $\not{F}_t := \not{D}_t(\mathbf{1} + \not{D}_t^2)^{-1/2}$ is the bounded transform of $\not{D}_t$. Note that the conditions on a spectral triple are precisely such that we can take the partition $0 = t_0 < t_1 = 1$, i.e. the spectral flow will depend only on the end-points. Namely, on the level of bounded transforms, every element of the path $\not{F}_\bullet$ is a compact perturbation of the initial point $\not{F}_0$,

$$\not{F}_s = (1-s)\not{F}_0 + s\pi(u)^*\not{F}_0\pi(u) = \not{F}_0 + s\,\pi(u)^*[\not{F}_0, \pi(u)],$$

so for $\not{P}_s = (\mathbf{1} + \not{F}_s)/2$ we have

$$[\pi(u), \not{P}_0] \in \mathcal{K}(\mathcal{N}, \tau) \implies q(\not{P}_s) = q(\not{P}_0) \in \mathcal{K}(\mathcal{N}, \tau) \text{ for all } s \in [0,1].$$

Thus, the abstract spectral flow of the path (2.23) is given by the class

$$\mathrm{sf}(\not{D}_\bullet) := \mathrm{Sf}(\not{F}_\bullet) = [(\mathbf{1} - P_1) \cap \not{P}_0] - [(\mathbf{1} - \not{P}_0) \cap \not{P}_1]$$

in $K_0(B)$, with B the C^*-algebra defined in Notation 2.5.5. Now $\not{P} := \not{P}_0$ is the spectral projection of $\not{D}$ corresponding to $[0, \infty)$, while $\not{P}_1 = \pi_B(u)^*\not{P}\pi_B(u)$. So the abstract spectral flow

$$\mathrm{sf}(\not{D}_\bullet) = [\mathrm{Ker}(\not{P}_0\pi_B(u)\not{P}_0)] - [\mathrm{Ker}(\not{P}_0\pi_B(u^*)\not{P}_0)] = \mathrm{Index}(\not{P}\pi_B(u)\not{P}) \tag{2.24}$$

equals the $K_0(B)$-valued index of the Fredholm operator $\not{P}\pi_B(u)\not{P}$ (here we assumed A to be separable).

Theorem 2.5.23 ([64, Thm. 6.9])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a unital $(\mathcal{N}, \tau)$-semifinite spectral triple over a C^*-algebra A. For a unitary $u \in \mathcal{A}$, the specral flow of the path* (2.23) *is given by the τ-Fredholm index*

$$\mathrm{Sf}_\tau(\not{D}_\bullet) = \mathrm{Ind}_\tau(\not{P}u\not{P})$$

of the operator $\not{P}u\not{P}$ on $\not{P}\mathcal{H}$, where $\not{P}$ is the spectral projection of $\not{D}$ corresponding to $[0, \infty)$.

Proof. Just apply τ_* to the formula (2.24). $\square$

In [6], the numerical spectral flow between $\not{D}$ and $\not{D} + \pi(u)^*\not{D}\pi(u)$ is interpreted as the "charge" created by u, since in their setting $\not{P}$ is the Fermi projection of a condensed-matter system.

We now have to discuss the nonunital setting. As always, we simplify the formulas by assuming that u belongs to $\mathcal{A}^\sim$ and not some matrix algebra over $\mathcal{A}$. We just have to remember that the interesting K-theory classes usually have representatives only in matrix algebra over $\mathcal{A}$.

Theorem 2.5.24 ([27, Cor. 4.4])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a C^*-algebra A. For a unitary $u \in \mathcal{A}^\sim$ such that*

$$[\not{D}, \pi(u)](\mathbf{1} + \not{D}^2)^{-1/2} \in \mathcal{K}_{\mathcal{N}},$$

the specral flow of the path

$$\boldsymbol{\not{D}}_t := \boldsymbol{\not{D}} + t\boldsymbol{\pi}(u)^*[\boldsymbol{\not{D}}, \boldsymbol{\pi}(u)],$$

where $(\boldsymbol{\pi}(\mathcal{A}), \boldsymbol{\mathcal{H}}, \boldsymbol{\not{D}})$ is the double of $(\mathcal{A}, \mathcal{H}, \not{D})$ as in Section 2.5.2, is given by the τ-Fredholm index

$$\mathrm{Sf}_\tau(\boldsymbol{\not{D}}_\bullet) = \mathrm{Ind}_\tau(\boldsymbol{\not{P}}\boldsymbol{\pi}(u)\boldsymbol{\not{P}}) \tag{2.25}$$

of the operator $\boldsymbol{\not{P}}\boldsymbol{\pi}(u)\not{P}$, where $\boldsymbol{\not{P}}$ is the spectral projection of $\boldsymbol{\not{D}}$ corresponding to $[0, \infty)$.

In our application (§3) of Theorem 2.5.24 we shall have $[\not{D}, \pi(a)] \in \pi(\mathcal{A}^\sim)$ for all $a \in \mathcal{A}^\sim$ and so formula (2.25) holds for all unitaries over $\mathcal{A}^\sim$.

2.6 Smoothness and summability

2.6.1 Summability of spectral triples

Notation 2.6.1. For $p \in [1, \infty)$ we let $\mathcal{L}^p(\mathcal{N}, \tau)$ be the two-sided ideal in $\mathcal{N}$ of all elements T with

$$\tau(|T|^p) < \infty.$$

Remark 2.6.2. The ideal $\mathcal{L}^p(\mathcal{N}, \tau)$ should not be confused with the noncommuative L^p-space, usually denoted by $L^p(\mathcal{N}, \tau)$, which consists of all τ-measurable operators on $\mathcal{H}$

affiliated with $\mathcal{N}$. In particular, $L^p(\mathcal{N},\tau)$ contains unbounded operators unless $\mathcal{N} = \mathcal{B}(\mathcal{H})$. In general we have $\mathcal{L}^p(\mathcal{N},\tau) = L^p(\mathcal{N},\tau) \cap \mathcal{N}$.

Definition 2.6.3 ([28, Def. 2.15])**.** A $(\mathcal{N},\tau)$-semifinite spectral triple $(\mathcal{A},\mathcal{H},\not{D})$ is **finitely summable** if there is a $p > 0$ such that

$$a(1+\not{D}^2)^{-1/2} \in \mathcal{L}^p(\mathcal{N},\tau), \qquad \forall a \in \mathcal{A}. \tag{2.26}$$

In that case, the **spectral dimension** of $(\mathcal{A},\mathcal{H},\not{D})$ is the number

$$n := \inf\{p > 0 |\ \tau(|a|(1+\not{D}^2)^{-p/2}) \text{ for all } a \in \mathcal{A}\},$$

where $|a| := (a^*a)^{1/2} \in \mathcal{N}$.

Notation 2.6.4. For $T \in \mathcal{N}$ and $k \in \mathbb{N}_0$, we denote by $T^{(k)}$ the kth iterated commutator of T with $\not{D}^2$. In other words, $T^{(0)} := T$ and $T^{(k)} := [\not{D}^2, T^{(k-1)}]$ for $k \in \mathbb{N}$.

Definition 2.6.5 ([28, Def. 3.1])**.** The triple $(\mathcal{A},\mathcal{H},\not{D})$ is said to have **isolated spectral dimension** if, for all $T \in \mathcal{N}$ of the form

$$T = a_0[\not{D}, a_1^{(k_1)}]\cdots[\not{D}, a_r^{(k_r)}], \qquad \text{with } a_0,\ldots,a_r \in \mathcal{A},$$

the zeta function

$$\zeta_T(z) := \mathrm{Tr}(T(\mathbf{1}+\not{D}^2)^{-z})$$

has an analytic continuation to a deleted neighborhood of $z = 0$.

2.6.2 The notion of being "smoothly summable"

To a pair $(\not{D},\tau)$ of a faithful normal semifinite trace τ on a von Neumann algebra $\mathcal{N}$ and a selfadjoint operator $\not{D}$ affiliated to $\mathcal{N}$, it is possible to associate a noncommutative integration theory (this is one of the main achievements in [28, 29]). We shall briefly recall how this works. From now on, such a pair $(\not{D},\tau)$ has been fixed.

We begin by considering the one-parameter family $(\varphi_s)_{s>0}$ of faithful normal semifinite weights on $\mathcal{N}$ defined by

$$\varphi_s(T) := \tau\big((1+\not{D}^2)^{-s/4}T(1+\not{D}^2)^{-s/4}\big), \qquad \forall T \in \mathcal{N}_+.$$

Recall Definition 2.3.11 of the half-domain $\mathrm{Dom}^{1/2}(\varphi)$ of a weight φ.

Definition 2.6.6 ([28, §1.1]). Let $p \geq 1$ be a real number. The algebra of $(\not{D}, \tau, p)$-**square-integrable** elements in $\mathcal{N}$ is the one defined by

$$\mathcal{B}_2(\not{D}, \tau, p) := \bigcap_{s>p} \mathrm{Dom}^{1/2}(\varphi_s) \cap \mathrm{Dom}^{1/2}(\varphi_s)^*.$$

It is discussed in [28, §1.1] how to give $\mathcal{B}_2(\not{D}, \tau, p)$ the structure of a Fréchet algebra. Since this is an essential part of what follows, we briefly summarize the result. The topology on $\mathcal{B}_2(\not{D}, \tau, p)$ is provided by a sequence $(\mathcal{Q}_m)_{m\in\mathbb{N}}$ of norms defined by

$$\mathcal{Q}_m(T) := \sqrt{\|T\|^2 + \varphi_{p+1/m}(|T|^2) + \varphi_{p+1/m}(|T^*|^2)} \tag{2.27}$$

for $T \in \mathcal{B}_2(\not{D}, \tau, p)$. As desired (see Definition 2.6.17), this topology is stronger than the norm topology. Moreover, $\mathcal{B}_2(\not{D}, \tau, p)$ is complete and metrizable.

Proposition 2.6.7 ([28, Prop. 1.6]). *For each $p \geq 1$, the space $\mathcal{B}_2(\not{D}, \tau, p)$ is a Fréchet $*$-algebra.*

Furthermore, the multiplication in $\mathcal{B}_2(\not{D}, \tau, p)$ behaves well with respect to the norms in the sense that [28, Prop. 1.6]

$$\mathcal{Q}_m(TS) \leq \mathcal{Q}(T)\mathcal{Q}(S), \qquad \forall\, S, T \in \mathcal{B}_2(\not{D}, \tau, p).$$

By definition, this says that the locally convex algebra $\mathcal{B}_2(\not{D}, \tau, p)$ is an "m-algebra".

Let $\mathcal{A}$ and $\mathcal{B}$ be locally convex algebras, with topologies specified by seminorms $(\mathcal{Q}_\alpha)_\alpha$ and $(Q_\beta)_\beta$ respectively. Then their **projective tensor product** $\mathcal{A} \otimes \mathcal{B}$ is the completion of the algebraic tensor product $\mathcal{A} \odot \mathcal{B}$ in the topology provided by the seminorms

$$(\mathcal{Q}_\alpha \otimes \mathcal{Q}_\beta)(c) := \inf \Big\{ \sum_{j=1}^{r} \mathcal{Q}_\alpha(a)\mathcal{Q}_\beta(b) \Big\}, \qquad \forall c \in \mathcal{A} \odot \mathcal{B}, \tag{2.28}$$

where the infimum runs over all decompositions $c = \sum_{j=1}^{r} a_j \otimes b_j$.

We always write $\otimes$ for the projective tensor product. We endow $\mathcal{B}_2(\not{D}, \tau, p) \otimes \mathcal{B}_2(\not{D}, \tau, p)$ with the $\otimes$-topology (see [28, §1.2] for details).

Definition 2.6.8 ([28, §1.2]). The algebra of $(\not{D}, \tau, p)$-**integrable** elements in $\mathcal{N}$ is the

subalgebra

$$\mathcal{B}_1(\not{D},\tau,p) \subset \mathcal{B}_2(\not{D},\tau,p)$$

defined as the closure of the image of $\mathcal{B}_2(\not{D},\tau,p) \otimes \mathcal{B}_2(\not{D},\tau,p)$ under the multiplication map.

Definition 2.6.9 ([28, §1.3]). For $k \geq 1$, the algebra of k **times differentiable** elements in $\mathcal{B}_1(\not{D},\tau,p)$ is defined by

$$\mathcal{B}_1^k(D,\tau,p) := \{T \in \mathcal{B}_1(\not{D},\tau,p) |\ \not{\delta}(T),\dots,\not{\delta}^k(T) \in \mathcal{B}_1(\not{D},\tau,p)\},$$

where $\not{\delta}(T) := [|\not{D}|,T]$. We also write $\mathcal{B}_1^0(\not{D},\tau,p) := \mathcal{B}_1(\not{D},\tau,p)$.

Definition 2.6.10. A $(\mathcal{N},\tau)$-semifinite spectral triple $(\mathcal{A},\mathcal{H},\not{D})$ is **smoothly summable** if there is a $p \geq 1$ such that $\pi(a)$ and $[\not{D},\pi(a)]$ belong to $\mathcal{B}^k(\not{D},\tau,p)$ for all $k \geq 0$ and all $a \in \mathcal{A}$. That is, if we have

$$\pi(\mathcal{A}), [\not{D},\pi(\mathcal{A})] \subset \mathcal{B}^\infty(\not{D},\tau,p) := \bigcap_{k \in \mathbb{N}_0} \mathcal{B}^k(\not{D},\tau,p).$$

We will also need an alternative description of $\mathcal{B}_1(\not{D},\tau,p)$. For that we consider the linear span $\mathcal{B}_2(\not{D},\tau,p)^2$ of products of two elements in $\mathcal{B}_2(\not{D},\tau,p)$. Introduce a family $(\mathcal{P}_{m,l})_{m,l\in\mathbb{N}}$ of norms on $\mathcal{B}_2(\not{D},\tau,p)^2$ by

$$\mathcal{P}_{m,l}(c) := \inf\Big\{\sum_{j=1}^r \mathcal{Q}_m(a)\mathcal{Q}_l(b)\Big\}, \qquad \forall c \in \mathcal{B}_2(\not{D},\tau,p)^2, \tag{2.29}$$

where the norms $\mathcal{Q}_m$ are as in (2.27) and the infimum is taken with respect to all decompositions $c = \sum_{j=1}^r a_j b_j$. Comparing with (2.28), the following should be expected.

Proposition 2.6.11 ([28, Thm. 1.10]). *The Fréchet space $\mathcal{B}_1(\not{D},\tau,p)$ coincides with the completion of $\mathcal{B}_2(\not{D},\tau,p)^2$ in the locally convex topology provided by the norms $(\mathcal{P}_{m,l})_{m,l\in\mathbb{N}}$.*

Moreover, $\mathcal{B}_1(\not{D},\tau,p)$ is a $*$-subalgebra of $\mathcal{N}$ [28, Cor. 1.12].

Proposition 2.6.12 ([28, Prop. 2.17]). *Let $(\mathcal{A},\mathcal{H},\not{D})$ be a smoothly summable $(\mathcal{N},\tau)$-semifinite spectral triple over a separable C^*-algebra A, of spectral dimension n. Then*

$$\mathcal{A} \subset \mathcal{B}_1(\not{D},\tau,n)$$

as a $*$-subalgebra.

2.6.3 Pseudodifferential calculus

Let $D\!\!\!/$ be a selfajoint operator on a Hilbert space $\mathcal{H}$. The vector spaces

$$\mathcal{H}_s := \operatorname{Dom}(|D\!\!\!/|^s) = \operatorname{Dom}((\mathbf{1} + D\!\!\!/)^{-s/2}), \qquad s \in \mathbb{R}_+$$

are Hilbert spaces (already complete) for the norm

$$\|\xi\|_s := \sqrt{\|\xi\|^2 + \|(\mathbf{1} + D\!\!\!/)^{-s/2}\xi\|^2},$$

and called the **Sobolev spaces** associated with $D\!\!\!/$ (thinking of $D\!\!\!/$ as a differential operator). Since $D\!\!\!/$ is selfadjoint, the intersection $\mathcal{H}_\infty := \bigcap_s \mathcal{H}_s$ is dense in $\mathcal{H}$. For $r \in \mathbb{R}$, the operator $(\mathbf{1} + D\!\!\!/)^{r/2}$ is continuous as a mapping from $(\mathcal{H}_\infty, \|\cdot\|_{s+r})$ to $(\mathcal{H}_\infty, \|\cdot\|_s)$ for all s (with $s + r \geq 0$). The same is true for $|D\!\!\!/|^r$ for $r \in \mathbb{R}_+$. Observe that $\mathcal{H}_0 = \mathcal{H}$.

Definition 2.6.13. A (not necessarily bounded) operator T on $\mathcal{H}$ has **analytic order** $\leq r \in \mathbb{R}$ if T is continuous as a mapping from $\mathcal{H}_{s+r}$ to $\mathcal{H}_s$ for all s. The linear space of operators with analytic order $\leq r$ is denoted by $\operatorname{op}^r(D\!\!\!/)$.

The analytic order is not unrelated to the "smoothness" of the operator. If T belongs to the smooth domain of the derivation $\delta\!\!\!/(T) := [(\mathbf{1} + D\!\!\!/^2)^{1/2}, T]$ then T has analytic order 0 [34, Cor. 6.6]. The space of **$D\!\!\!/$-pseudodifferential operators** of order $\leq r$ is usually defined as

$$\operatorname{OP}^r(D\!\!\!/) := (\mathbf{1} + D\!\!\!/^2)^{r/2} \bigcap_{k\in\mathbb{N}_0} \operatorname{Dom}(\delta\!\!\!/^k) \subset \operatorname{op}^r(D\!\!\!/)$$

(the domains of the derivations $\delta\!\!\!/$ and $[|D\!\!\!/|, \cdot]$ coincide [34, §2]). The derivation $\delta\!\!\!/$ maps $\operatorname{OP}^r(D\!\!\!/)$ into itself (preserving the order $\leq r$).

When $D\!\!\!/$ satisfies some summability condition, such as $(\mathbf{1}+D\!\!\!/)^{-n/2} \in \mathcal{L}^1(\mathcal{H})$ for some n, then an operator of the form $(\mathbf{1}+D\!\!\!/)^{-r/2}T$ belongs to $\mathcal{L}^{n/r}(\mathcal{H})$. Such situations arise in geometric examples of differential operators on manifolds. Some examples require more general summability conditions on $D\!\!\!/$, such as those which give rise to semifinite spectral triples. To accommodate for this greater generality, it is desirable to make the pseudodifferential calculus depend on a semifinite trace.

Suppose therefore that $\not{D}$ is affiliated to a von Neumann algebra $\mathcal{N}$ equipped with a normal faithful semifinite trace τ. Fix $p \geq 0$.

Definition 2.6.14 ([28, Def. 1.23])**.** For $s \in \mathbb{R}$, the set of (tame) $(\not{D}, \tau, p)$**-pseudodifferential operators** of **order** s is defined as

$$\mathrm{OP}_0^s(\not{D}, \tau, p) := (1 + \not{D}^2)^{s/2} \mathcal{B}_1^\infty(\not{D}, \tau, p).$$

Lemma 2.6.15 ([28, Cor. 1.30])**.** *For $s > p$ we have*

$$\mathrm{OP}_0^{-s}(\not{D}, \tau, p) \subset \mathcal{L}^1(\mathcal{N}, \tau).$$

One application of the integration-based pseudodifferential calculus is to the structure of the algebra $\mathcal{B}_1^\infty(\not{D}, \tau, p)$ which appears in the condition of smooth summability.

Proposition 2.6.16. *Consider the unbounded operators L and R on $\mathcal{N}$ given by*

$$L(T) := (1 + \not{D}^2)^{-1/2}[\not{D}^2, T], \qquad R(T) := [\not{D}^2, T](1 + \not{D}^2)^{-1/2}$$

for $T \in \mathcal{N}$.

(i) The smooth domain of δ coincides with $\bigcap_{j,l \in \mathbb{N}} \mathrm{Dom}(L^j \circ R^l)$ [32, Prop. 6.5]. Moreover, since $R(T)^* = -L(T^*)$, we have $\bigcap_{j,l \in \mathbb{N}} \mathrm{Dom}(L^j \circ R^l) = \bigcap_{j \in \mathbb{N}} \mathrm{Dom}(L^j)$.

(ii) The algebra $\mathcal{B}_1^\infty(\not{D}, \tau, p)$ can be described [28, Lemma 1.29] as

$$\mathcal{B}_1^\infty(\not{D}, \tau, p) = \{T \in \mathcal{B}_1(\not{D}, \tau, p) |\ L^k(T) \in \mathcal{B}_1(\not{D}, \tau, p) \text{ for all } k \in \mathbb{N}\}.$$

(iii) $\mathcal{B}_1^\infty(\not{D}, \tau, p)$ is a two-sided ideal in $\bigcap_k \mathrm{Dom}(\delta^k)$ [28, Lemma 1.31].

2.6.4 Local subalgebras

We now discuss the possibility of representing each K-theory class $[x] \in K_\bullet(A)$ by some matrix $x \in \mathrm{M}_\infty(\mathcal{C}^\sim)$ over a dense $*$-subalgebra $\mathcal{C}$ of the C^*-algebra A. The motivation for this is that such representatives might be much "smoother" or more "integrable" than a general matrix over A.

If A is a Banach algebra, then any $a \in A$ defines a "holomorphic functional calculus", as follows. Write

$$\mathrm{Sp}_A(a) := \{\lambda \in \mathbb{C} |\ a - \lambda \mathbf{1} \text{ has no inverse in } A^{\sim}\}$$

for the **spectrum** of a. Let f be a function which is holomorphic on an open neighborhood Ω of $\mathrm{Sp}_A(a)$ with smooth positively oriented boundary $\partial\Omega$, and suppose that $f(0) = 0$. The **holomorphic functional calculus** takes f to the element $f(a) \in A^{\sim}$ defined via the integral

$$f(a) := \frac{1}{2\pi i} \int_{\partial\Omega} f(z)(z\mathbf{1} - a)^{-1}\, dz. \tag{2.30}$$

If f has a power series expansion $f(z) = \sum_j c_j z^j$ then one simply has $f(a) = \sum_j c_j a^j$. If A is moreover a C^*-algebra then the above map $f \to f(a)$ is subsumed by the continuous functional calculus, i.e. it works when f is merely a continuous function on the spectrum of a.

Definition 2.6.17 ([49, Def. 3.25])**.** Let A be a Banach algebra. A subalgebra $\mathcal{C} \subset A$ is **stable under the holomorphic functional calculus** if

(i) $\mathcal{C}$ is complete in some locally convex topology finer than the norm topology of A, and

(ii) $f(c) \in \mathcal{C}$ for all $c \in \mathcal{C}$ and all functions f which are holomorphic in a neighborhood of $\mathrm{Sp}_A(c)$ and satisfy $f(0) = 0$.

Let $\mathcal{C}$ be a subalgebra of a Banach algebra A. We write $\mathrm{Sp}_{\mathcal{C}}(c)$ for the spectrum of an element $c \in \mathcal{C}$, i.e.

$$\mathrm{Sp}_{\mathcal{C}}(c) := \{\lambda \in \mathbb{C} |\ c - \lambda \mathbf{1} \text{ has no inverse in } \mathcal{C}^{\sim}\}.$$

If $c \in \mathcal{C}$ and $\lambda \in \mathbb{C}$ is such that $c - \lambda\mathbf{1}$ has an inverse in $\mathcal{C}$, then certainly it also has an inverse in $A \supset \mathcal{C}$. Thus $\mathrm{Sp}_A(c) \subseteq \mathrm{Sp}_{\mathcal{C}}(c)$. We say that $\mathcal{C}$ is a **spectrally invariant** subalgebra of A if

$$\mathrm{Sp}_A(c) = \mathrm{Sp}_{\mathcal{C}}(c), \qquad \forall c \in \mathcal{C}.$$

In some standard references such as [42, Section III.C], [71, Section 4.3], condition (i) of Definition 2.6.17 is not included in the definition of "stability under the holomorphic functional calculus". The motivation for condition (i) is that it ensures the folllowing.

Lemma 2.6.18 ([49, §3.8]]). *A subalgebra $\mathcal{C}$ of a Banach algebra A is stable under the holomorphic functional calculus if and only if $\mathcal{C}$ is spectrally invariant.*

Sketch of proof. If $\mathcal{C}$ satisfies condition (i) in Definition 2.6.17 then $\mathrm{Sp}_A(c) = \mathrm{Sp}_{\mathcal{C}}(c)$, so the "only if" statement is clear.

We have to show that conditions (i) and (ii) in Definition 2.6.17 together imply spectral invariance. The tricky part is to show that completeness condition (ii) ensures that $f(a)$, defined as in (2.30), belongs to $\mathcal{C}$ whenever $f(z)(z\mathbf{1} - a)^{-1}$ is in $\mathcal{C}$ for all z. Given this fact, we obtain

$$\mathrm{GL}_1(\mathcal{C}^\sim) = \mathrm{GL}_1(A^\sim) \cap \mathcal{C}^\sim.$$

Thus, $\mathcal{C}$ is stable under the holomorphic functional calculus (if and) only if invertibility in A implies invertibility in $\mathcal{C}$ for elements in $\mathcal{C}$. Thus, only if $\mathrm{Sp}_A(c) = \mathrm{Sp}_{\mathcal{C}}(c)$.

The following is the reason why stability under the holomorphic functional calculus is so important.

Theorem 2.6.19 ([49, Thm. 3.44]). *Let A be a C^*-algebra and let $\mathcal{C}$ be a subalgebra of A. If $\mathcal{C}$ is stable under the holomorphic functional calculus (i.e. if $\mathcal{C}$ is a **pre-C^*-algebra**) and moreover Fréchet, then the inclusion map $\iota : \mathcal{C} \to A$ induces an isomorphism*

$$\iota_* : K_0(\mathcal{C}) \to K_0(A)$$

on K-theory.

We now look at the possibility of representing elements of $K_1(A)$ by unitaries over dense subalgebras.

Lemma 2.6.20 ([4, Lemma A.9]). *If X and Y are compact spaces, let $[X, Y]$ denote the set of homotopy classes of continuous maps from X to Y.*

Let $\iota : \mathcal{C} \to A$ be a continuous linear map of Banach spaces such that $\iota(\mathcal{C})$ is dense in A. Then for any compact space X and any open subset $U \subset A$, the induced map

$$\iota_* : [X, \iota^{-1}(U)] \to [X, U]$$

is a bijection.

Corollary 2.6.21 ([41, §VI.3]). *Let $\mathcal{C}$ be a dense subalgebra of a C^*-algebra A such that* $\mathrm{GL}_\infty(\mathcal{C}^\sim) = \mathrm{GL}_\infty(A^\sim) \cap \mathrm{M}_\infty(\mathcal{C}^\sim)$. *Then the inclusion $\mathcal{C} \hookrightarrow A$ induces an isomorphism*

$$K_1(A) \cong \frac{\mathrm{GL}_\infty(\mathcal{C}^\sim)}{\mathrm{GL}_\infty(\mathcal{C}^\sim)_0}.$$

Proof. For $X = \mathbb{S}^n$ (the n-sphere) we have $[\mathbb{S}^n, U] = \pi_n(U)$, the nth homotopy group of a space U. We apply Lemma 2.6.20 to the one-point space $X = \{*\}$ and the open subset $U = \mathrm{GL}_\infty(A^\sim)$ of the Banach space $\mathrm{M}_\infty(A^\sim)$. The conclusion is that the set

$$K_1(A) = \frac{\mathrm{GL}_\infty(A^\sim)}{\mathrm{GL}_\infty(A^\sim)_0} = \pi_0(\mathrm{GL}_\infty(A^\sim)) = [\{*\}, \mathrm{GL}_\infty(A^\sim)]$$

is bijective to

$$[\{*\}, \iota^{-1}(\mathrm{GL}_\infty(A^\sim))] = [\{*\}, \mathrm{GL}_\infty(\mathcal{C}^\sim)] = \frac{\mathrm{GL}_\infty(\mathcal{C}^\sim)}{\mathrm{GL}_\infty(\mathcal{C}^\sim)_0}.$$

Moreover, it is clear that ι_* is a homomorphism. □

The spectrally invariant subalgebras $\mathcal{C} \subset A$ that we will encounter will in fact be of the following kind.

Definition 2.6.22 ([14, Def. 3.1.1]). A **local C^*-algebra** is a $*$-algebra $\mathcal{C}$ equipped with a C^*-norm such that $\mathrm{M}_r(\mathcal{C})$ is stable under the holomorphic functional calculus in $\mathrm{M}_r(A)$ for each $r \in \mathbb{N}$, where A is the norm completion of $\mathcal{C}$.

Combining Corollary 2.6.21 and Theorem 2.6.19 we can find a sufficient condition on a subalgebra to yield representatives of every K-class of A.

Corollary 2.6.23. *Let A be a C^*-algebra and let $\mathcal{C} \subset A$ be a dense subalgebra which is a local C^*-algebra in the norm of A. Then the inclusion $\mathcal{C} \hookrightarrow A$ induces isomorphisms $K_\bullet(A) \cong K_\bullet(\mathcal{C})$ for all $\bullet \in \{0, 1\}$, where $K_1(\mathcal{C}) := \mathrm{GL}_\infty(\mathcal{C}^\sim)/\mathrm{GL}_\infty(\mathcal{C}^\sim)_0$.*

In the situation of Corollary 2.6.23, we say that $\mathcal{C}$ is a **local subalgebra** of A.

2.6.5 Spectral invariance related to spectral triples

Definition 2.6.24 ([28, §3.5]). Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a C^*-algebra A such that

$$\not{\delta}^k(T) \in \mathcal{N}, \qquad \forall T \in \pi(\mathcal{A}) \cup [\not{D}, \pi(\mathcal{A})],\ k \in \mathbb{N}_0, \tag{2.31}$$

where $\delta(T) := [|\not{D}|, T]$. The δ-**topology** on $\mathcal{A}$ is the topology determined by the family $(\|\cdot\|_k)_{k\in\mathbb{N}_0}$ of norms

$$\|a\|_k := \|\delta^k(\pi(a))\| + \|\delta^k([\not{D}, \pi(a)])\|, \qquad \forall a \in \mathcal{A}.$$

If $(\mathcal{A}, \mathcal{H}, \not{D})$ satisfies (2.31) then the completion $\mathcal{A}_\delta$ of $\mathcal{A}$ in the δ-topology satisfies (2.31) as well. The upshot is that $\mathcal{A}_\delta$ is a Fréchet algebra which is spectrally invariant in A. In particular, the inclusion $\iota_* : K_0(\mathcal{A}) \to K_0(A)$ is an isomorphism by Theorem 2.6.19.

If $(\mathcal{A}, \mathcal{H}, \not{D})$ is "unital" in the sense that $\not{D}$ has compact resolvent, the δ-topology is completely satisfactory. In general however, $\mathcal{A}_\delta$ does not necessarily have the same summability properties as $\mathcal{A}$. We will therefore need the following generalization of the δ-topology.

Definition 2.6.25 ([28, Def. 2.19]). Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a smoothly summable $(\mathcal{N}, \tau)$-semifinite spectral triple over a separable C^*-algebra A. The δ-φ-**topology** on $\mathcal{A}$ is the topology determined by the family $(\|\cdot\|_{m,k})_{m\in\mathbb{N},\ k\in\mathbb{N}_0}$ of norms

$$\|a\|_{m,k} := \mathcal{P}^{m,k}(\pi(a)) + \mathcal{P}^{m,k}([\not{D}, \pi(a)]), \qquad \forall a \in \mathcal{A},$$

where, for all $T \in \mathcal{N}$,

$$\mathcal{P}^{m,k}(T) := \sum_{j=0}^{k} \mathcal{P}_m(\delta^j(T)).$$

Here we use Proposition 2.6.12 and $\mathcal{P}_m := \mathcal{P}_{m,m}$ is the norm on $\mathcal{B}_2(\not{D}, \tau, p)$ defined in (2.29).

We write $\mathcal{A}_{\delta,\varphi}$ for the completion of $\mathcal{A}$ in the δ-φ-topology.

Proposition 2.6.26 ([28, Prop. 2.20]). *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a smoothly summable $(\mathcal{N}, \tau)$-semifinite spectral triple over a separable C^*-algebra A, of spectral dimension n. Then $(\mathcal{A}_{\delta,\varphi}, \mathcal{H}, \not{D})$ is again a smoothly summable semifinite spectral triple with spectral dimension n. In addition, $\mathcal{A}_{\delta,\varphi}$ is a Fréchet algebra which is stable under the holomorphic functional calculus in A.*

2.6.6 Summability of Fredholm modules

If $(\pi, \mathcal{H}, F)$ is a $(\mathcal{N}, \tau)$-semifinite Fredholm module over a C^*-algebra A then, by definition, we have the condition

$$[F, \pi(a)] \in \mathcal{K}(\mathcal{N}, \tau), \qquad \forall a \in A$$

involving the norm-closed ideal $\mathcal{K}(\mathcal{N}, \tau)$ of τ-compact elements in $\mathcal{N}$. In order to associate a "Chern character" to a Fredholm module, and to obtain local formulas for the index pairing $K_0(A) \times K^0(A; \mathcal{N}) \to \mathbb{R}$ described in §2.3.3.2, one needs that the commutators $[F, \pi(a)]$ are not only compact, but belong to one of the (non-closed) ideals $\mathcal{L}^n(\mathcal{N}, \tau) \subset \mathcal{K}(\mathcal{N}, \tau)$, at least for all a in some local subalgebra of A.

Definition 2.6.27 ([43, Def. I.1], [28, Def. 2.7])**.** A $(\mathcal{N}, \tau)$-semifinite Fredholm module $(\pi, \mathcal{H}, F)$ over a C^*-algebra A is p-**summable** if there exists a local subalgebra $\mathcal{C} \subset A$ such that

$$[F, \pi(c)] \in \mathcal{L}^p(\mathcal{N}, \tau), \qquad \pi(c)(\mathbf{1} - F^2) \in \mathcal{L}^{p/2}(\mathcal{N}, \tau)$$

for all $c \in \mathcal{C}$.

Remark 2.6.28. Let $(\pi, \mathcal{H}, F)$ be any Fredholm module over A. Then, for any $p \in \mathbb{N}$, the subalgebra

$$\mathcal{A}_p := \{a \in A | \ [\pi(a), F] \in \mathcal{L}^p(\mathcal{N}, \tau)\}$$

is spectrally invariant in A, and each matrix algebra $\mathrm{M}_r(\mathcal{A}_p)$ is spectrally invariant in $\mathrm{M}_r(A)$ [43, Prop. I.A.3.3]. However, $\mathcal{A}_p$ need not be dense in A (so, not a local subalgebra).

We can now relate the summability of a spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ to the summability of the Fredholm module $(\pi, \mathcal{H}, \not{F})$ associated with the triple in Lemma 2.5.18.

Proposition 2.6.29 ([28, Prop. 2.14])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a $(\mathcal{N}, \tau)$-semifinite spectral triple over a C^*-algebra A and let $(\pi, \mathcal{H}, \not{F})$ be the associated Fredholm module over A (see Lemma 2.5.18). Suppose that $(\mathcal{A}, \mathcal{H}, \not{D})$ has spectral dimension n and that for all $s > n$ we have*

$$[D, \pi(a)](\mathbf{1} + \not{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \tau), \qquad \forall a \mathcal{A}.$$

Then $(\pi, \mathcal{H}, \not{F})$ is $(n + 1)$-summable.

The assumptions in Proposition 2.6.29 will be satisfied if $(\mathcal{A}, \mathcal{H}, \not{D})$ is smoothly summable with spectral dimension n.

2.7 Cyclic cohomology

The classical Chern character is a homomorphism

$$\mathrm{Ch} : K^0(M) \to H^{\mathrm{even}}_{\mathrm{dR}}(M, \mathbb{R})$$

from the K-theory of a manifold M to the even de Rham cohomology of M. A result of Connes says that $H^{\mathrm{even}}_{\mathrm{dR}}(M, \mathbb{R})$ is isomorphic to the continuous even periodic cyclic cohomology of the algebra $\mathcal{A} := C^\infty(M)$. A noncommutative version of de Rham cohomology would then be the cyclic cohomology of a local C^*-algebra $\mathcal{A}$. Dually, the role of de Rham currents is played by *cyclic* cycles $c \in \mathcal{A}^{\otimes(m+1)}$ in noncommutative geometry.

Each cyclic m-cocycle $\psi : \mathcal{A}^{m+1} \to \mathbb{C}$ induces a homomorphism $\psi_* = \langle \cdot, [\psi] \rangle : K_\bullet(A) \to \mathbb{C}$ on the K-theory of the C^*-completion of A, where $\bullet$ is the parity (even or odd) of the integer m. The map ψ_* depends only on the class of ψ in cyclic cohomology. We will use these homomorphisms to obtain new formulas for the index pairings associated with a spectral triple.

2.7.1 Important facts about cyclic cohomology and homology

2.7.1.1 Cohomology

This section gives the background on cyclic cohomology needed for the local index formula. Good references for this material are [80], [42, §III], [71, §3], [49, §10.1].

Let $\mathcal{A}$ be a unital algebra over $\mathbb{C}$ and let $\mathcal{M}$ be an $\mathcal{A}$-bimodule. We write the left and right $\mathcal{A}$-actions on $\mathcal{M}$ simply by juxtaposition, i.e. as axb for $a, b \in \mathcal{A}$ and $x \in \mathcal{M}$. For $m \geq 1$, the space of **Hochschild m-cochains** of $\mathcal{A}$ with coefficients in $\mathcal{M}$ is defined by

$$C^m(\mathcal{A}, \mathcal{M}) := \mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{\otimes m}, \mathcal{M}),$$

while for $m = 0$ we set $C^0(\mathcal{A}, \mathcal{M}) := \mathcal{M}$. The **Hochschild coboundary operator** b maps $C^m(\mathcal{A}, \mathcal{M})$ into $C^{m+1}(\mathcal{A}, \mathcal{M})$ by means of the formula

$$\begin{aligned}(b\psi)(a_1, \dots, a_m) := a_1\psi(a_2, \dots, a_{m+1}) + \sum_{j=1}^{m} (-1)^{j+1} \psi(a_1, \dots, a_j a_{j+1}, \dots, a_{m+1}) \\ + (-1)^{m+1} \psi(a_1, \dots, a_m) a_{m+1}.\end{aligned}$$

One has $b^2 = 0$, so the sequence $(C^m(\mathcal{A}, \mathcal{M}), b)_{m\geq 0}$ is a complex. The **Hochschild cohomology** of $\mathcal{A}$ with coefficients in $\mathcal{M}$ is the cohomology $H^\bullet(\mathcal{A}, \mathcal{M})$ of this complex.

An interesting choice of bimodule $\mathcal{M}$ is the $\mathbb{C}$-linear dual $\mathcal{A}^* := \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}, \mathbb{C})$ of $\mathcal{A}$, with the usual action of $a, b \in \mathcal{A}$ on $\psi \in \mathcal{A}^*$ given by

$$(a\psi b)(c) := \psi(bca), \qquad \forall c \in \mathcal{A}.$$

We can identify $\mathrm{Hom}_{\mathcal{A}}(\mathcal{A}^{\otimes m}, \mathcal{A}^*)$ with $\mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\otimes(m+1)}, \mathbb{C})$. Indeed, for an m-cochain $\psi : \mathcal{A}^{\otimes m} \to \mathcal{A}^*$ and any elements $a_1, \ldots, a_m \in \mathcal{A}$, we can evaluate $\psi(a_1, \ldots, a_m) \in \mathcal{A}^*$ on elements of $\mathcal{A}$. We write

$$C^m := \mathrm{Hom}_{\mathbb{C}}(\mathcal{A}^{\otimes(m+1)}, \mathbb{C})$$

for the space of m-cochains with coefficients in $\mathcal{A}^*$ when we make this identification. The formula for the Hochschild coboundary operator $b : C^m \to C^{m+1}$ becomes

$$\begin{aligned}(b\psi)(a_0, a_1, \ldots, a_m) = &\sum_{j=0}^{m} (-1)^j \psi(a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_{m+1}) \\ &+ (-1)^{m+1} \psi(a_{m+1}, a_0, a_1, \ldots, a_m).\end{aligned}$$

We write $HH^\bullet(\mathcal{A}) := H^\bullet(\mathcal{A}, \mathcal{A}^*)$ for the corresponding cohomology.

Example 2.7.1. (i) A Hochschild 0-cocycle, i.e. a $\psi \in C^0$ with $b\psi = 0$, is precisely a trace on $\mathcal{A}$.

(ii) If τ is a trace on $\mathcal{A}$ and δ is a derivation on $\mathcal{A}$ such that $\tau \circ \delta = 0$ then

$$\psi(a_0, a_1) := \tau(a_0 \delta(a_1)), \qquad \forall a_0, a_1 \in \mathcal{A}$$

defines a Hochschild 1-cocycle on $\mathcal{A}$.

(iii) As a generalization of the last example, let $\delta_1, \ldots, \delta_n$ be mutually commuting derivations on $\mathcal{A}$ such that $\tau \circ \delta_j = 0$ for all $j = 1, \ldots, n$. Define the n-cochain

$$\psi(a_0, a_1, \ldots, a_n) := \sum_{\varepsilon} (-1)^{\varepsilon} \tau(a_0 \delta_{\varepsilon(1)}(a_1) \cdots \delta_{\varepsilon(n)}(a_n)),$$

where the sum runs over all permutations ε on $\{1, \ldots, n\}$. Then ψ is a Hochschild

n-cocycle on $\mathcal{A}$.

The important feature of the choice $\mathcal{M} = \mathcal{A}^*$ is that some elements ψ of C^m are **cyclic** in the sense that

$$\psi(a_m, a_0, \ldots, a_{m-1}) = (-1)^m \psi(a_0, a_1, \ldots, a_m)$$

for all choices of elements $a_0, \ldots, a_m \in \mathcal{A}$. The subspace of cyclic elements in C^m is usually denoted by C^m_λ. In fact, the coboundary operator b maps C^m_λ into C^{m+1}_λ, so $(C^m_\lambda, b)_{m\geq 0}$ is a cochain complex [71, Lemma 3.6.1].

Proposition 2.7.2 ([71, Lemma 3.6.1])**.** *The coboundary operator b maps C^m_λ into C^{m+1}_λ.*

Proof. Define the operator $\lambda : C^\bullet \to C^\bullet$ by

$$(\lambda\psi)(a_0, \ldots, a_m) := (-1)^m \psi(a_m, a_0, \ldots, a_{m-1}),$$

so that $C^\bullet_\lambda = \mathrm{Ker}(1-\lambda)$. Introduce a new coboundary operator $b' : C^\bullet \to C^{\bullet+1}$ by

$$(b'\psi)(a_0, a_1, \ldots, a_m) = \sum_{j=0}^{m} (-1)^j \psi(a_0, a_1, \ldots, a_j a_{j+1}, \ldots, a_{m+1}). \tag{2.32}$$

Then $(1-\lambda)b = b'(1-\lambda)$. So for $\psi \in C^m_\lambda$ we have

$$(1-\lambda)(b\psi) = b'(1-\lambda)\psi = 0,$$

i.e. $b\psi \in \mathrm{Ker}(1-\lambda)$. $\square$

Since b preserves each C^m_λ, the data $(C^m_\lambda, b)_{m\geq 0}$ is a cochain complex. The cohomology of the complex $(C^m_\lambda, b)_{m\geq 0}$ is denoted by $HC^\bullet(\mathcal{A})$ and referred to as the **cyclic cohomology** of the algebra $\mathcal{A}$. From the proof of Propostion 2.7.2, a cyclic m-cocycle over $\mathcal{A}$ is given by a linear functional $\psi : \mathcal{A}^{\otimes(m+1)} \to \mathbb{C}$ such that

$$(1-\lambda)\psi = 0 = b\psi.$$

It is possible to give an alternative description of the cyclic cohomology $HC^\bullet(\mathcal{A})$ in terms of *non*-cyclic cochains on $\mathcal{A}$. For that we need, in addition to the "small b operator" $b : C^\bullet \to C^{\bullet+1}$, to introduce the "big B operator" $B : C^\bullet \to C^{\bullet-1}$. This is the operator

defined by

$$B := NB_0$$

where $B_0 : C^\bullet \to C^{\bullet-1}$ is the operator

$$(B_0\psi)(a_0, \ldots, a_{m-1}) := \psi(\mathbf{1}, a_0, \ldots, a_{m-1}) - (-1)^m \psi(a_0, \ldots, a_{m-1}, \mathbf{1}),$$

and $N : C^\bullet \to C^\bullet$ is the **norm operator** given by $N := \mathbf{1} + \lambda + \lambda^2 + \lambda^3 + \cdots$. The data $(C^\bullet, b, B)$ is a **bicomplex** in the sense that

$$b^2 = 0 = B^2, \qquad bB = -Bb.$$

If ψ is any cochain then the presence of the operator N in B ensures that $B\psi$ is cyclic. In particular, if ψ is already cyclic then $\psi = B\psi'$ for some cochain ψ', and hence $B\psi = 0$. Thus, cyclic cocycles are those cochains ψ satisfying $B\psi = 0 = b\psi$.

To avoid confusion, one sometimes writes $H^\bullet_\lambda(\mathcal{A})$ for the cyclic cohomology when regarded as the cohomology of $C^\bullet_\lambda$, reserving the notation $HC^\bullet(\mathcal{A})$ for the cohomology of the (b, B)-bicomplex.

Note that, while there is an inclusion $C^m_\lambda(\mathcal{A}) \hookrightarrow C^m(\mathcal{A})$, the induced map $I : HC^m(\mathcal{A}) \to HH^m(\mathcal{A})$ is not injective (in general). The failure of injectivity (and surjectivity) of I is expressed in "Connes's short exact sequence" [42, §III.1.γ], [71, §3.7]

$$\cdots \longrightarrow HC^m(\mathcal{A}) \xrightarrow{I} HH^m(\mathcal{A}) \xrightarrow{B} HC^{m-1}(\mathcal{A}) \xrightarrow{S} HC^{m+1}(\mathcal{A}) \longrightarrow \cdots,$$

where B is the coboundary operator and S is the "periodicity operator" discussed in §2.7.1.3 below. So a cyclic m-cocycle can be nontrivial as a cyclic cocycle but trivial as a Hochschild cocycle, and vice versa.

2.7.1.2 Homology

There is a homology theory dual to $HC^\bullet(\mathcal{A})$ [80, Section 2.1]. The underlying chain complex $(C^\lambda_\bullet, b^T)$ is the cyclic part of the Hochschild *chain* complex $(C_\bullet, b^T)$ of $\mathcal{A}$ with values in the bimodule $\mathcal{M} = \mathcal{A}$. In more detail, put

$$C_m := \mathcal{A}^{\otimes(m+1)}.$$

The "duality" of C_m with the cochain complex C^m which underlies $HH^\bullet(\mathcal{A})$ refers to the fact that

$$C^m = \mathrm{Hom}_{\mathbb{C}}(C_m, \mathbb{C}).$$

The boundary operator $b^T : C_m \to C_{m-1}$ satisfies

$$\psi(b^T c) = (b\psi)(c), \qquad \forall \psi \in C^m,\ c \in C_{m+1}.$$

A chain $c \in C_m$ is **cyclic** if it is invariant under the $\mathbb{Z}_{m+1}$-action given on simple tensors by

$$\lambda_m(a_0 \otimes \otimes a_1 \otimes \cdots \otimes a_m) := (-1)^m a_m \otimes a_0 \otimes \cdots \otimes a_{m-1}.$$

The boundary operator b^T maps the image of $1 - \lambda$ into itself: there is an operator $(b')^T$ such that

$$b^T(1-\lambda) = (1-\lambda)(b')^T.$$

The **cyclic Hochschild complex** is the quotient

$$C_\bullet^\lambda := C_\bullet/(1-\lambda).$$

In other words, C_m^λ is the space of "coinvariants" in C_m under the $\mathbb{Z}_{m+1}$-action (compare the fact that the cyclic cochains are obtained as *invariants* under a $\mathbb{Z}_{m+1}$-action, which form a subspace and not a quotient). The boundary map on the complex $C_\bullet^\lambda$ is induced from b^T and denoted by the same symbol. The homology $HC_\bullet(\mathcal{A})$ of the complex $C_\bullet^\lambda$ is the **cyclic homology** of $\mathcal{A}$.

Since the Hochschild chain and cochain complex are related as $C^m = \mathrm{Hom}_{\mathbb{C}}(C_m, \mathbb{C})$, we have a $\mathbb{C}$-valued pairing betwen cocycles $\psi = (\psi_m)_{m\geq 0}$ and cochains $c = (c_m)_{m\geq 0}$, given by

$$\langle c, \psi\rangle := \sum_{m\geq 0} \phi_m(c_m). \tag{2.33}$$

One has $\langle b^T c, \psi\rangle = \langle c, b\psi\rangle$. Consequently, if $\psi - \phi$ is a coboundary and c is a cycle then $\langle c, \psi\rangle = \langle c, \phi\rangle$.

Definition 2.7.3. The **pairing** between $HH^\bullet(\mathcal{A})$ and $HH_\bullet(\mathcal{A})$ is the map

$$\langle\cdot,\cdot\rangle : HH_\bullet(\mathcal{A}) \times HH^\bullet(\mathcal{A}) \to \mathbb{C}, \qquad \langle [c], [\psi]\rangle := \langle c, \psi\rangle,$$

where $\langle c, \psi \rangle$ is the duality pairing (2.33).

2.7.1.3 Periodic cyclic cohomology

The presence of a pairing between cyclic cohomology and K-theory suggests that there should be an analogue of Bott periodicity in cyclic cohomology. That is, the spectrum should degenerate into a theory where there are (at most) two non-isomorphic cohomology group, "even" and "odd". As with Bott periodicity, such a property can be deduced by looking at the trivial algebra $\mathcal{A} = \mathbb{C}$.

By linearity any cochain $\psi \in C^m$ over the algebra $\mathbb{C}$ is determined by its value $\psi(1, \ldots, 1)$, where $1 \in \mathbb{C}$ is the identity. If ψ is cyclic and m is odd then only $\psi(1, \ldots, 1) = 0$ is possible, so $\psi = 0$. If m is even then there is (up to a scalar multiple) a unique nontrivial cyclic m-cocycle. On the level of cohomology, the result is [71, Example 3.6.1]

$$HC^{2k}(\mathbb{C}) \cong \mathbb{C}, \qquad HC^{2k+1}(\mathbb{C}) = 0.$$

Equipped with a certain multiplication (viz. the cup product; see [42, §III.1.α]), $HC(\mathbb{C}) = \bigoplus_m HC^m(\mathbb{C})$ is the polynomial ring $\mathbb{C}[\sigma]$ with one generator σ of degree 2 [43, Cor. II.1.2]. We also write $\sigma : \mathbb{C}^3 \to \mathbb{C}$ for the cyclic 2-cocycle determined by

$$\sigma(1, 1, 1) = 2\pi i,$$

so that σ represents the class $\sigma \in HC^2(\mathbb{C})$. For any complex algebra $\mathcal{A}$, the cup product makes $HC(\mathcal{A})$ into a $HC(\mathbb{C})$-bimodule, with $\sigma \cup \psi = \psi \cup \sigma$ for all $[\psi] \in HC(\mathcal{A})$. We let

$$S : HC^\bullet(\mathcal{A}) \to HC^{\bullet+2}(\mathcal{A}), \qquad S([\psi]) := [\psi \cup \sigma]$$

be the operator which, on the level of cocycles, acts by cup multiplication by σ.

Definition 2.7.4 ([49, Def. 10.5])**.** The **periodic cyclic cohomology** of $\mathcal{A}$ is the inductive limit of the cyclic cohomology under the periodicity map $S : HC^\bullet(\mathcal{A}) \to HC^{\bullet+2}(\mathcal{A})$,

$$HP^0(\mathcal{A}) := \lim_{\rightarrow} HC^{2\bullet}(\mathcal{A}), \qquad HP^1(\mathcal{A}) := \lim_{\rightarrow} HC^{2\bullet+1}(\mathcal{A}).$$

Alternatively, consider the **totalization** of the (b, B)-bicomplex, namely the direct sum

of all even and odd cochains:

$$C^{\text{even}}(\mathcal{A}) := \bigoplus_{k\in\mathbb{N}_0} C^{2k}(\mathcal{A}), \qquad C^{\text{odd}}(\mathcal{A}) := \bigoplus_{k\in\mathbb{N}_0} C^{2k+1}(\mathcal{A}).$$

With the differential $b + B$, we obtain a new complex

$$\cdots \xrightarrow{b+B} C^{\text{odd}}(\mathcal{A}) \xrightarrow{b+B} C^{\text{even}}(\mathcal{A}) \xrightarrow{b+B} C^{\text{odd}}(\mathcal{A}) \xrightarrow{b+B} \cdots .$$

The periodic cyclic cohomology of $\mathcal{A}$ can then be defined as the cohomology of the totalization of this complex:

$$HP^0(\mathcal{A}) = \frac{\operatorname{Ker}(b + B : C^{\text{even}}(\mathcal{A}) \to C^{\text{odd}}(\mathcal{A}))}{\operatorname{Ran}(b + B : C^{\text{odd}}(\mathcal{A}) \to C^{\text{even}}(\mathcal{A}))},$$

$$HP^1(\mathcal{A}) = \frac{\operatorname{Ker}(b + B : C^{\text{odd}}(\mathcal{A}) \to C^{\text{even}}(\mathcal{A}))}{\operatorname{Ran}(b + B : C^{\text{even}}(\mathcal{A}) \to C^{\text{odd}}(\mathcal{A}))}.$$

Definition 2.7.5. An **even** (b, B)-**cocycle** is an cochain $\psi = (\psi_{2k})_{k=0}^r \in C^{\text{even}}(\mathcal{A})$ with $(B + b)\psi = 0$, i.e.

$$b\psi_{2k} + B\psi_{2k+2} = 0.$$

Similarly, an **odd** (b, B)-**cocycle** is an cochain $\psi = (\psi_{2k+1})_{k=0}^r \in C^{\text{odd}}(\mathcal{A})$ with $(B + b)\psi = 0$.

Importantly, the elements of $C^{\text{even}}(\mathcal{A})$ or $C^{\text{odd}}(\mathcal{A})$ have only finitely many nonzero components. The definition of periodic cyclic *homology* is very analogous. Note however that, by duality, the periodic chains can have infinitely many nonzero components.

2.7.1.4 Nonunital algebras

We now extend the definition of Hochschild and cyclic cohomology to a possibly nonunital algebra $\mathcal{A}$. First we need to look at slight modifications of the Hochschild complexes for *unital* algebras.

Let $\mathcal{A}$ be a unital algebra. As far as cohomology is concerned, it is possible to restrict attention to **normalized** cochains, i.e. those $\psi \in C^m(\mathcal{A}, \mathcal{M})$ satisfying $\psi(a_0, a_1, \ldots, a_m) = 0$ whenever $a_j = \mathbf{1}$ for some $j = 1, \ldots, m$ [80, §1.5.7]. Namely, the operator b maps normalized cochains into normalized cochains, and the cohomology of the normalized complex is

again $H^\bullet(\mathcal{A},\mathcal{M})$. In other words, any element of $H^\bullet(\mathcal{A},\mathcal{M})$ has a normalized representative. Similarly, let $\overline{C}_m(\mathcal{A},\mathcal{M})$ be the quotient of $C_m(\mathcal{A},\mathcal{M}) = \mathcal{M}\otimes\mathcal{A}^{\otimes m}$ by the chains $(x, a_1, \ldots, a_m)$ with $a_j = \mathbf{1}$ for at least one j. Then

$$\overline{C}_m(\mathcal{A},\mathcal{M}) \cong \mathcal{M}\otimes\overline{\mathcal{A}}^{\otimes m},$$

where $\overline{\mathcal{A}} := \mathcal{A}/\mathbb{C}\mathbf{1}$ and the homology of the complex $\overline{C}_\bullet(\mathcal{A},\mathcal{M})$ is isomorphic to the homology of $C_\bullet(\mathcal{A},\mathcal{M})$ (i.e. the Hochschild homology of $\mathcal{A}$ with coefficients in $\mathcal{M}$) [80, §1.1.15].

Definition 2.7.6 ([80, §1.4.2])**.** The **reduced Hochschild complex** of $\mathcal{A}$ is defined by $C_m^{\mathrm{red}}(\mathcal{A}) := \overline{C}_m(\mathcal{A})$ for all $m\in\mathbb{N}$ and

$$C_0^{\mathrm{red}}(\mathcal{A}) := \overline{\mathcal{A}} = \mathcal{A}/\mathbb{C}\mathbf{1}$$

(in contrast to $\overline{C}_0 = \mathcal{A}$). The **reduced Hochschild homology** of $\mathcal{A}$ is the homology $\overline{HH}_\bullet(\mathcal{A})$ of the reduced Hochschild complex.

Note that $\overline{HH}_m(\mathcal{A}) = HH_m(\mathcal{A})$ for all $m\geq 2$.

Now let $\mathcal{A}$ be a nonunital (i.e. not necessarily unital) algebra and consider the minimal unitization $\mathcal{A}^\sim = \mathcal{A}\times\mathbb{C}$. We have

$$\overline{HH}_\bullet(\mathcal{A}^\sim) = \operatorname{Coker}\big(HH_\bullet(\mathbb{C}) \to HH_\bullet(\mathcal{A}^\sim)\big),$$

and we take this as the *definition* of the Hochschild homology $HH_\bullet(\mathcal{A})$ of $\mathcal{A}$.

Similarly, for cohomology we consider the Hochschild complex $(C^m(\mathcal{A}^\sim), b)_{m\geq 0}$ of modules over the unitization $\mathcal{A}^\sim$ of $\mathcal{A}$. Let $HH^\bullet(\mathcal{A}^\sim)$ be the cohomology of the latter complex.

Definition 2.7.7. The **Hochschild cohomology** of $\mathcal{A}$ is the kernel $HH^\bullet(\mathcal{A})$ of the map $HH^\bullet(\mathbb{C}) \to HH^\bullet(\mathcal{A}^\sim)$ induced by the inclusion $\mathbb{C}\hookrightarrow\mathcal{A}^\sim$.

2.7.1.5 Continuous version

The cohomology and homology theories that we are going to use will in fact be slightly different from the "algebraic" ones discussed so far.

Definition 2.7.8 ([80, §5.6])**.** Let $\mathcal{A}$ be a locally convex algebra. A multilinear functional

$\psi : \mathcal{A}^{m+1} \to \mathbb{C}$ on $\mathcal{A}$ is **continuous** if

$$|\psi(a_0, \ldots, a_m)| \leq \mathcal{P}(a_0) \cdots \mathcal{P}(a_m), \qquad \forall a_1, \ldots, a_m \in \mathcal{A}$$

for some continuous seminorm $\mathcal{P}$ on $\mathcal{A}$. A continuous functional $\psi : \mathcal{A}^{m+1} \to \mathbb{C}$ is called a **continuous Hochschild** m**-cochain** on $\mathcal{A}$, and (as in the algebraic case) we denote by $C^m(\mathcal{A})$ the space of all such cochains.

Using the same formulas for the differentials b and B as in the algebraic setting, one defines continuous versions of Hochschild cohomology and cyclic cohomology, and we still use the notation $HH^\bullet(\mathcal{A})$ and $HC^\bullet(\mathcal{A})$ for these.

On the homology side, the continuous version is a bit trickier. The reason for this is that the chain complex involves tensor products, and the continuous version should be a completed tensor product.

Definition 2.7.9 ([80, §5.6])**.** Let $\mathcal{A}$ be a locally convex algebra and assume that $\mathcal{A}$ is complete in this topology. The space of **continuous Hochschild** m**-chains** on $\mathcal{A}$ is defined by

$$C_m(\mathcal{A}) := \mathcal{A}^{\otimes(m+1)},$$

where $\otimes$ is the projective tensor product (whose definition was recalled in connection to Equation (2.28)).

Beginning in the next subsection, the algebra $\mathcal{A}$ will be part of the data in a smoothly summable spectral triple. Then $\mathcal{A}$ is locally convex in in the δ-φ-topology (Definition 2.6.27) and can without loss of generality be assumed complete by Proposition 2.6.26. In this setting, all tensor products will be the projective tensor product. That is, we will always use the continuous version of cyclic homology, and also for cohomology.

2.7.2 The Chern character of a spectral triple

Let $(\pi, \mathcal{H}, F)$ be a $(\mathcal{N}, \tau)$-semifinite Fredholm module over a C^*-algebra A such that $F^2 = \mathbf{1}$. We shall denote by τ' the **conditional trace**, which is defined by

$$\tau'(T) := \frac{1}{2}\tau(T + FTF) = \frac{1}{2}\tau\big(F(T + FTF)\big)$$

for all $T \in \mathcal{N}$ with $FT + TF \in \mathcal{L}^1(\mathcal{N}, \tau)$. The property $F^2 = \mathbf{1}$ is needed to ensure that $\tau'(T) = \tau(T)$ whenever T is already in $\mathcal{L}^1(\mathcal{N}, \tau)$. If the Fredholm operator is even, we denote by Γ the grading operator on $\mathcal{H}$. In the even case we write $\Gamma := \sqrt{2i}\mathbf{1}$ so that we can use the same formulas for the even and odd versions.

Suppose further that the Fredholm module is $(n+1)$-summable. Recall that this means there is a local subalgebra $\mathcal{A} \subset A$ and some integer $n \geq 2$ such that

$$[\pi(a), F] \in \mathcal{L}^{n+1}(\mathcal{N}, \tau), \qquad F^2 - \mathbf{1} \in \mathcal{L}^{(n+1)/2}(\mathcal{N}, \tau)$$

for all $a \in \mathcal{A}$. For simplicity we shall assume that n is even if the Fredholm module $(\pi, \mathcal{H}, F)$ is even, and that n is odd if $(\pi, \mathcal{H}, F)$ is odd.

Definition 2.7.10 ([28, Def. 2.22])**.** In the above setting, we define the **Chern character** of $(\pi, \mathcal{H}, F)$ to be the (b, B)-cocycle over $\mathcal{A}$ whose only non-vanishing term is

$$\begin{aligned}\mathrm{Ch}^n_F(a_0, a_1, \ldots, a_n) &:= \frac{\Gamma(n/2+1)}{n!}\tau'(\Gamma\pi(a_0)[F, \pi(a_1)]\cdots[F, \pi(a_n)]) \\ &= \frac{\Gamma(n/2+1)}{n!}\tau(\Gamma F[F, \pi(a_0)][F, \pi(a_1)]\cdots[F, \pi(a_n)])\end{aligned}$$

where $\Gamma \in \mathcal{B}(\mathcal{H})$ is the $\mathbb{Z}_2$-grading which, by convention in this formula, is defined to be $\Gamma := \sqrt{2i}\mathbf{1}$ if n is odd.

By Lemma 2.5.18, any $(\mathcal{N}, \tau)$-semifinite spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$ over A gives rise to a Fredholm module over A, by taking the bounded transform $\not{F} = \not{D}(1 + \not{D}^2)^{1/2}$ of $\not{D}$. If $(\mathcal{A}, \mathcal{H}, \not{D})$ satisfies the assumptions in Proposition 2.6.29 (with spectral dimension n) then $(\pi, \mathcal{H}, \not{F})$ is moreover $(n+1)$-summable. However, we need to pass to the double $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \not{\boldsymbol{D}})$ and the Fredholm module with $\not{\boldsymbol{F}} = \not{\boldsymbol{D}}|\not{\boldsymbol{D}}|^{-1}$ in order to ensure $\not{\boldsymbol{F}}^2 = \mathbf{1}$, which is needed for the definition of the Chern character in cyclic cohomology (or else it gets more complicated, see [52]. Since $(\mathcal{A}, \boldsymbol{\mathcal{H}}, \not{\boldsymbol{D}})$ has the same spectral dimension as the original triple, $(\boldsymbol{\pi}, \boldsymbol{\mathcal{H}}, \not{\boldsymbol{F}})$ is also an $(n+1)$-summable Fredholm module. When $\not{\boldsymbol{F}} = \not{\boldsymbol{D}}|\not{\boldsymbol{D}}|^{-1}$, we will write

$$\mathrm{Ch}(\mathcal{A}, \mathcal{H}, D) := \mathrm{Ch}_{\not{\boldsymbol{F}}}$$

and refer to it as the Chern character of the spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$. We will usually regard $\mathrm{Ch}(\mathcal{A}, \mathcal{H}, \not{D})$ as a representative of a class in $HP^\bullet(\mathcal{A})$.

2.7.3 Chern character on K-theory

Let $e \in \mathrm{M}_r(\mathcal{A})$ be a projection over $\mathcal{A}$ for some $r \in \mathbb{N}$. The property $e^2 = e$ implies that the image of $e^{\otimes(m+1)} \in C_m(\mathrm{M}_r(A))$ under the Hochschild boundary operator b^T is

$$b^T(e^{\otimes(m+1)}) = \begin{cases} e^{\otimes m}, & \text{if } m \text{ is odd} \\ 0, & \text{if } m \text{ is even.} \end{cases}$$

So $e^{\otimes(\bullet+1)}$ is not a Hochschild cycle. However, the situation improves if we regard the chain $e^{\otimes m}$ as a cyclic chain, i.e. as an element of the quotient space $C_m(\mathrm{M}_r(A))/(1-\lambda)$. Since $e^{\otimes m}$ is just the mth tensor power of a single element e, its equivalence class in the quotient $C^\lambda_\bullet(\mathrm{M}_r(A))$ satisfies

$$e^{\otimes m} = (-1)^{m-1} e^{\otimes m},$$

so $e^{\otimes m} = 0$ when m is odd. Thus $b^T(e^{\otimes(m+1)}) = 0$ for all m and $e^{\otimes(\bullet+1)}$ is a cyclic cycle over $\mathrm{M}_r(A)$.

Definition 2.7.11 ([28, Def. 2.23])**.** Let $\mathcal{A}$ be a $*$-algebra. The **Chern characters** of a unitary $u \in \mathrm{GL}^\sim_\infty(\mathcal{A})$ and a projection $e \in \mathrm{M}_\infty(\mathcal{A}^\sim)$ are the cyclic cycles $\mathrm{Ch}(e)$ and $\mathrm{Ch}(u)$ whose components are

$$\mathrm{Ch}_{2k}(e) := (-1)^k \frac{(2k)!}{k!} \operatorname{Tr}\big((e - \mathbf{1}/2) \otimes e \otimes \cdots \otimes e\big) \in (\mathcal{A}^\sim)^{\otimes(2k+1)}, \qquad k \in \mathbb{N}_0,$$

respectively

$$\mathrm{Ch}_{2k+1}(u) := (-1)^k k! \operatorname{Tr}(u^{-1} \otimes u \otimes \cdots \otimes u^{-1} \otimes u) \in (\mathcal{A}^\sim)^{\otimes(2k+2)}, \qquad k \in \mathbb{N}_0.$$

Here Tr is the matrix trace, so if $e_{j,k} \in \mathcal{A}^\sim$ denotes the (j,k) entry of the matrix e, then

$$\mathrm{Ch}_{2k}(e) = (-1)^k \frac{(2k)!}{k!} \sum_{j_1,\dots,j_{2k+1}} (e_{j_1,j_2} - \mathbf{1}\delta_{j_1,j_2}/2) \otimes e_{j_2,j_3} \otimes \cdots \otimes e_{j_{2k},j_{2k+1}}.$$

The Chern character is a (B^T, b^T)-cycle and defines a class in periodic cyclic homology $HP_\bullet(\mathcal{A})$.

Any invertible $v \in \mathrm{GL}_\infty(\mathcal{A}^\sim)$ induces a map v_* on $C_\bullet(\mathcal{A})$, which on the Chern characters

read

$$v_* \operatorname{Ch}_{2k}(e) := \operatorname{Ch}_{2k}(v^{-1}ev).$$

It is a fact that v_* becomes the identity on both Hochschild and cyclic homology [80, §4.1]. Thus, if $[f] = [e]$ in $K_0(\mathcal{A})$ then $[\operatorname{Ch}_{2k}(f)] = [\operatorname{Ch}_{2k}(e)]$ in cyclic homology. One obtains group homomorphisms

$$\operatorname{Ch}_{2k} : K_0(\mathcal{A}) \to HC_{2k}(\mathcal{A}).$$

and

$$\operatorname{Ch}_{2k+1} : K_1(\mathcal{A}) \to HC_{2k+1}(\mathcal{A}), \qquad [u] \to \operatorname{Ch}_{2k+1}(u),$$

which are sometimes also called "Chern characters". Note that for $k = 0$, the Chern character of $e \in \mathrm{M}_\infty(\mathbb{C}) \otimes \mathcal{A}$ is just the $\mathcal{A}$-valued trace

$$\operatorname{Ch}_0(e) = (\mathrm{id} \otimes \operatorname{Tr})(e),$$

and Ch_0 induces a map from $K_0(\mathcal{A})$ to $HC_0(\mathcal{A}) \cong \mathcal{A}/[\mathcal{A}, \mathcal{A}]$. The normalization in Ch_{2k} ensure that the map $\operatorname{Ch} = (\operatorname{Ch}_{2k})_{k \in \mathbb{N}_0}$ descends to a map from $K_0(\mathcal{A})$ to $HP^0(\mathcal{A})$, and similarly in the odd case.

The pairing between homology and cohomology then induces a pairing between K-theory and cyclic cohomology,

$$\langle [e], [\psi] \rangle := \langle [\operatorname{Ch}_{2k}(e)], [\psi] \rangle, \qquad \forall\, [\psi] \in HC^{2k}(\mathcal{A}),\ [e] \in K_0(\mathcal{A}),$$

$$\langle [u], [\psi] \rangle := \langle [\operatorname{Ch}_{2k+1}(u)], [\psi] \rangle, \qquad \forall\, [\psi] \in HC^{2k+1}(\mathcal{A}),\ [u] \in K_1(\mathcal{A}).$$

Explicitly,

$$\langle [e], [\psi] \rangle = \langle [\operatorname{Ch}_{2k}(e)], [\psi] \rangle = (-1)^k \frac{(2k)!}{k!} (\psi \otimes \operatorname{Tr})\big((e - 1/2)(de)^{2k}\big).$$

Proposition 2.7.12 ([42, Prop. III.3.2])**.** *The pairing is invariant under the periodicity operator S in the sense that*

$$\langle [x], [S\psi] \rangle = \langle [x], [\psi] \rangle$$

for all $[x] \in K_\bullet(A)$*. Therefore, we have an induced pairing*

$$K_\bullet(A) \times HP^\bullet(A) \to \mathbb{C}$$

between K*-theory and periodic cyclic cohomology.*

2.7.4 Index pairing as a pairing between cohomology and homology

Theorem 2.7.13 ([43, Thm. I.3.1], [28, Prop. 2.25])**.** *Let* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ *be a* $(\mathcal{N}, \tau)$*-semifinite spectral triple over a* C^**-algebra* A *of spectral dimension* $n \in \mathbb{N}$*. Suppose that* n *has the same parity (even or odd) as* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$*. If* n *is even then for all projections* $e \in \mathrm{M}_\infty(\mathcal{A}^\sim)$*,*

$$\langle \mathrm{Ch}_n(e), \mathrm{Ch}^n(\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = \mathrm{Index}_\tau(\boldsymbol{\pi}(e) \boldsymbol{\not{R}}_+ \boldsymbol{\pi}(e)).$$

If n *is odd then for all unitaries* $u \in \mathrm{U}_\infty(\mathcal{A}^\sim)$*,*

$$\langle \mathrm{Ch}_n(u), \mathrm{Ch}^n(\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle = -\sqrt{2\pi i}\, \mathrm{Index}_\tau(\boldsymbol{\not{P}} \boldsymbol{\pi}(u) \boldsymbol{\not{P}}).$$

Observe that the theorem is purely a statement about the $(n+1)$-summable Fredholm module associated with $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, and could be formulated without explicit reference to the spectral triple.

Thus, the Chern character $\mathrm{Ch}(\mathcal{A}, \mathcal{H}, \mathcal{D})$ can be used to calculate the index pairing. However, even for spectral triples over a commutative C^*-algebra, $\langle \mathrm{Ch}_n(e), \mathrm{Ch}^n(\mathcal{A}, \mathcal{H}, \mathcal{D}) \rangle$ is hard to compute and usually involves singular-integral operators.

One benefit of realizing that the Chern character is a cyclic cocycle is that the pairing depends only on the cohomology class of the cyclic cocycle. This means that we can calculate the pairing from any cyclic cocycle which is homotopic to the Chern character. As we will discuss next, replacing the Chern character with the so-called residue and resolvent cocycles one obtains a "local formula" for the index pairing.

2.8 The local index formula

We have seen how a class in Kasparov's KK-theory allows for an abstract index pairing. The conditions on a representative of a KK-class are arranged precisely to ensure that this is the case. As we saw in the introduction, the numerical index can sometimes be calculated via an explicit formula, namely the integration formula for the winding number. The local index theorem generalizes this feature.

Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a smoothly summable $(\mathcal{N}, \tau)$-spectral triple over a C^*-algebra A with isolated spectral dimension n. We assume that n is odd if $(\mathcal{A}, \mathcal{H}, \not{D})$ is odd and that n is even if $(\mathcal{A}, \mathcal{H}, \not{D})$ is even. Since it is a trivial matter to pass to matrix algebras $\mathrm{M}_n(\mathcal{A})$, we shall write the formulas for elements in $\mathcal{A}$ and not over $\mathcal{A}$. Also the assumption that $(\mathcal{A}, \mathcal{H}, \not{D})$ has the same parity as its dimension is just for convenience.

For $\lambda \in \mathbb{C},\ s \in [0, \infty)$, consider the resolvent $R_s(\lambda) := (\lambda - (\mathbf{1} + s^2 + \not{D}^2))^{-1}$ of the operator $(\mathbf{1} + s^2 + \not{D}^2)$.

Definition 2.8.1 ([28, Def. 3.4])**.** Let $\bullet \in \{0, 1\} = \{\text{even}, \text{odd}\}$ be the parity of $(\mathcal{A}, \mathcal{H}, \not{D})$. The **resolvent cocycle** is the finite sequence $(\Phi_m)_{m=\bullet}^n$ of mappings $r \to \Phi_m^r$ on the upper half plane in $\mathbb{C}$ whose values are functionals $\Phi_m^r : \mathcal{A} \otimes \mathcal{A}^{\otimes m} \to \mathbb{C}$ defined for $a_0, \ldots, a_m \in \mathcal{A}$ by

$$\Phi_m^r(a_0, \ldots, a_m) := (-\sqrt{2i}) 2^{m+1} \frac{\Gamma((m+1)/2)}{\Gamma(m+1)}$$
$$\times \int_0^\infty s^m\, \tau\Big(\frac{1}{2\pi i} \Gamma \int_{\varepsilon + i\mathbb{R}} \lambda^{-n/2-r} \pi(a_0) R_s(\lambda)[\not{D}, \pi(a_1)] R_s(\lambda) \cdots [\not{D}, \pi(a_m)] R_s(\lambda)\, d\lambda\Big)\, ds.$$

Here $\Gamma(z)$ is the gamma function, Γ is the grading operator ($\Gamma := \mathbf{1}$ for $\bullet = 1$) and ε is any number in the range $0 < \varepsilon < 1$.

The resolvent cocycle defines a family, parameterized by r, of elements of the (b, B)-bicomplex. In fact, modulo functions that are holomorphic at $r = (1-n)/2$, the functional $\Phi_m^r : \mathcal{A} \otimes \mathcal{A}^{\otimes m} \to \mathbb{C}$ defines the same class as the function $r \to (r - (1-n)/2)^{-1} \operatorname{Ch}(\mathcal{A}, \mathcal{H}, \not{D})$ [28, Thm. 3.29]. Heuristically, if we expand Φ_m^r as a Laurent series $\sum_k a_k r^k$ around the point $r = (1-n)/2$, where Φ_m^r has a simple pole, then the coefficient a_{-1} of $1/(r - (1-n)/2)$ is precisely the Chern character of the spectral triple $(\mathcal{A}, \mathcal{H}, \not{D})$. Therefore, taking the residue at this point, the resolvent cocycle recovers a cocycle which has values in $\mathbb{C}$ (and not in a space of meromorphic functions).

An important feature of the resolvent cocycle is that the functionals Φ_s^r are well-defined even if $\not{D}$ is not invertible. Nevertheless, most manipulations can be carried out using the doubled operator $\boldsymbol{\not{D}}$, because the resolvent cocycles defined by $\not{D}$ and $\boldsymbol{\not{D}}$ both recover the class of $[\mathrm{Ch}(\mathcal{A}, \mathcal{H}, \not{D})]$ upon taking residues.

Theorem 2.8.2 (Local index formula via resolvent cocycle [28, Thm. 3.33])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a smoothly summable $(\mathcal{N}, \tau)$-semifinite spectral triple with spectral dimension n of the same parity $\bullet \in \{0, 1\}$ as $(\mathcal{A}, \mathcal{H}, \not{D})$. Then for all unitaries $u \in \mathcal{A}^\sim$,*

$$\langle [u], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle = \frac{-1}{\sqrt{2\pi i}} \operatorname{Res}_{r=(1-n)/2} \sum_{m=1,\text{ odd}}^{n} \Phi_m^r(\mathrm{Ch}_m(u)),$$

and for all projections $e \in \mathcal{A}^\sim$,

$$\langle [e], [\mathcal{A}, \mathcal{H}, \not{D}] \rangle = \sum_{m=0,\text{ even}}^{n} \phi_m\big(\mathrm{Ch}_m(e) - \mathrm{Ch}_m(\varepsilon(e))\big),$$

where $\varepsilon(e) \in \mathbb{C}$ is the image of e under the quotient map $\varepsilon : \mathcal{A}^\sim \to \mathbb{C}$.

The function $r \to \sum_{m=\bullet}^{n} \Phi_m^r(\mathrm{Ch}_m(x))$ analytically continues to a deleted neighborhood of the point $r = (1-n)/2$ where it has at worst a simple pole.

Under the extra assumption that $(\mathcal{A}, \mathcal{H}, \not{D})$ has isolated spectral dimension, there is an additional part of the local index formula (which is the one usually thought of as the local index formula).

The **residue cocycle** of $(\mathcal{A}, \mathcal{H}, \not{D})$ [28, §3.1] is a cocycle $\phi = (\phi_m)_{m=0}^n$ whose components $\phi_m : \mathcal{A} \otimes \mathcal{A}^{\otimes m} \to \mathbb{C}$ consists of sums of terms proportional to

$$\operatorname*{Res}_{z=0} z^j \, \tau\big(\boldsymbol{\Gamma\pi}(a_0)[\boldsymbol{\not{D}}, \boldsymbol{\pi}(a_1)]^{(k_1)} \cdots [\boldsymbol{\not{D}}, \boldsymbol{\pi}(a_m)]^{(k_m)} (\mathbf{1} + \boldsymbol{\not{D}}^2)^{-|\mathbf{k}|-m/2-z}\big)$$

for multi-indices $\mathbf{k} = (k_1, \ldots, k_m)$ of length m. Here $|\mathbf{k}| := k_1 + \cdots k_m$ and $T^{(k)}$ is the kth iterated commutator of an operator T with $\not{D}^2$ as in Notation 2.6.4. In our application of the local index formula there will only be one such term contributing to the index, and this term occurs in the component ϕ_n, i.e. $m = n$.

The next part of the local index formula is then obtained by showing that the residue cocycle $(\phi_m)_{m \geq 0}$ defines the same cohomology class as the Chern character $\mathrm{Ch}(\mathcal{A}, \mathcal{H}, \not{D})$.

Theorem 2.8.3 (Simplified local index formula [28, Thm. 3.33])**.** *Let $(\mathcal{A}, \mathcal{H}, \not{D})$ be a*

smoothly summable $(\mathcal{N},\tau)$-semifinite spectral triple with isolated spectral dimension n of the same parity as $(\mathcal{A},\mathcal{H},D)$. Then for all invertible $u\in\mathcal{A}^\sim$,

$$\langle [u],[\mathcal{A},\mathcal{H},\not{D}]\rangle = \frac{-1}{\sqrt{2\pi i}}\sum_{m=1,\text{ odd}}^{n}\phi_m(\mathrm{Ch}_m(u)),$$

and for all projections $e\in\mathcal{A}^\sim$,

$$\langle [e],[\mathcal{A},\mathcal{H},\not{D}]\rangle = \sum_{m=0,\text{ even}}^{n}\phi_m\big(\,\mathrm{Ch}_m(e)-\mathrm{Ch}_m(\varepsilon(e))\big).$$

While it may be hard to prove directly that a given finitely summable spectral triple has isolated spectral dimension, one could use the local index formula with the *resolvent* cocycle to deduce which terms from the *residue* cocycle will contribute.

To show that the Chern character is cohomologous to a cocycle involving $[\boldsymbol{\not{D}},\cdot]$ one uses the path

$$\boldsymbol{\not{D}}_t := \boldsymbol{\not{D}}|\boldsymbol{\not{D}}|^{-t},\qquad t\in[0,1].$$

Indeed, one has $\boldsymbol{\not{D}}_0=\boldsymbol{\not{D}}$ and $\boldsymbol{\not{D}}_1=\boldsymbol{\not{R}}$. It remains then to show that inserting $\boldsymbol{\not{R}}$ instead of $\boldsymbol{\not{D}}$ in the resolvent cocycle gives the Chern character (this is a highly technical task; see [28, §3]). Now $\boldsymbol{\not{R}}$ anti-commutes with all commutators $[\boldsymbol{\not{R}},\boldsymbol{\pi}(a)]$, so $\boldsymbol{\not{R}}$ can be moved to the left. Then the Cauchy integral can be calculated and one obtains the Chern character.

Using an asymptotic expansion for the commutators $[R_s(\lambda),T]$ entering Definition 2.8.1, where $T=[\not{D},\pi(a_j)]$ for $j=1,\dots,m$ by means of pseudodifferential calculus (see [28, Prop. 1.32]), one can move all resolvents appearing in Φ_m^r to the right and perform the Cauchy integral. The result is a sum of zeta functions. If $(\mathcal{A},\mathcal{H},\not{D})$ has isolated spectral dimension, these zeta functions can be analytically continued to a region containing the point $r=(1-n)/2$, where they have simple poles. The whole sum of zeta functions then has a simple pole at $r=(1-n)/2$. Taking the residue at this point one recovers again the pairing $\langle [u],[\mathcal{A},\mathcal{H},\not{D}]\rangle$.

The local index formula is astonishingly general. Here is the simplest possible example, long known (see the introduction to this thesis).

Example 2.8.4 (Gohberg-Krein theorem)**.** Applying the local index formula to the spectral triple $(C^\infty(\mathbb{S}^1),L^2(\mathbb{S}^1),\sqrt{-1}d/dt)$ over the C^*-algebra $C(\mathbb{S}^1)$ one obtains the result, for

unitaries $u \in C^\infty(\mathbb{S}^1)$, that

$$\operatorname*{Res}_{s=0} \operatorname{Tr}\left(u^*[\boldsymbol{D}, u]|\boldsymbol{D}|^{-2s-1}\right) = \frac{1}{2\pi i}\int_{\mathbb{S}^1} u^*(z)\, du(z),$$

where we have used that $\boldsymbol{D} := \sqrt{-1}d/dt + m\mathbf{1}/2$ is invertible for $m \in \mathbb{Z}$ (because the eigenvalues of $D = \sqrt{-1}d/dt$ are integers).

Chapter 3

Index pairings for $\mathbb{R}^n$-actions

3.1 The noncommutative Gohberg-Krein theorem

In this chapter we prove the higher-dimensional noncommutative Gohberg-Krein theorem. We consider an $\mathbb{R}^n$-action α on a separable C^*-algebra A and a lower semicontinuous α-invariant faithful trace τ on A with dense domain $\mathrm{Dom}(\tau) \subset A$. We represent A as a subalgebra of $\mathcal{B}(\mathfrak{H})$ where $\mathfrak{H}$ is a Hilbert space such that α is unitarily implemented. The crossed product $B = A \rtimes_\alpha \mathbb{R}^n$ acts on the Hilbert space $L^2(\mathbb{R}^n, \mathfrak{H})$. For K-theoretical reasons, the Hilbert space we use is $\mathcal{H} = \mathbb{C}^N \otimes L^2(\mathbb{R}^n, \mathfrak{H})$ where $N = 2^{(n-1)/2}$ for odd n and $N = 2^{n/2}$ for even n (recall Section 1.4.1).

Let $D_1, \ldots, D_n$ be generators of the unitary group on $\mathcal{H}$ implementing α in $\mathcal{H}$, i.e.

$$\pi_\alpha(\alpha_t(a)) = e^{2\pi i t\cdot D}\pi_\alpha(a)e^{-2\pi i t\cdot D}, \qquad \forall a \in A,\ t \in \mathbb{R}^n,$$

where $t \cdot D := t_1 D_1 + \cdots + t_n D_n$. We can then form the Dirac operator

$$\not{D} := \sum_{k=1}^{n} \gamma^k \otimes D_k$$

using generators γ^k of the n-dimensional complex Clifford algebra $\mathbb{C}_n$ acting irreducibly on $\mathbb{C}^N$. For even n, the operator $\Gamma := (-i)^{n/2}\gamma^1 \cdots \gamma^n$ gives a $\mathbb{Z}_2$-grading of $\mathcal{H}$, which we write as $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, and $[\Gamma, \pi_\alpha(a)] = 0$ for all $a \in A$ while $\Gamma\not{D} = -\not{D}\Gamma$.

For $k = 1, \ldots, n$, let δ_k denote the infinitesimal generator of α in the kth direction, so

that $\pi_\alpha(\delta_k(a)) = 2\pi i[D_k, \pi_\alpha(a)]$. The smooth domain of δ will be denoted by

$$\mathcal{A} := \bigcap_{k\in\mathbb{N}} \mathrm{Dom}(\delta^k).$$

For $a, b \in \mathcal{A}$ and $m = 1, \ldots, n$ we use the shorthand notation

$$(a\delta(b))^m := \sum_{\varepsilon}(-1)^{\varepsilon}\prod_{k=1}^{m} a\delta_{\varepsilon(k)}(b), \tag{3.1}$$

where the sum is over all permutations ε of $\{1, \ldots, n\}$.

In order to go beyond abstract index theory and be able to talk about real-valued indices, we are going to require that the C^*-algebra A admits a densely defined trace $\tau : A_+ \to [0, +\infty]$. Then there exists a weight $\hat{\tau}$ on $B := A \rtimes_\alpha \mathbb{R}^n$ (the "dual weight") such that

$$\hat{\tau}(\hat{\pi}_\alpha(g)^*\hat{\pi}_\alpha(f)) = \int_{\mathbb{R}^n} \tau\big(f(s)^*g(s)\big)\, ds, \qquad \forall f, g \in C_0(\mathbb{R}^n, \mathrm{Dom}(\tau)) \cap L^2(\mathbb{R}, \mathfrak{H}).$$

We shall need that $\hat{\tau}$ is a trace. This will be the case precisely when τ is invariant under the $\mathbb{R}^n$-action, i.e. $\tau \circ \alpha_t = \tau$ for all $t \in \mathbb{R}^n$ (see Remark 3.2.18 below).

We will find a $*$-subalgebra $\mathcal{C}$ of $\mathcal{A}$ of elements which are both sufficiently "smooth" with respect to $D\!\!\!/$ and "integrable" with respect to $(D\!\!\!/, \hat{\tau})$. It is from elements of this algebra that our K-theoretical quantities can be explicitly calculated.

For notation simplicity we will formulate the result for unitaries u and projections e in $\mathcal{C}^\sim$. It is easily adapted to matrices over $\mathcal{C}^\sim$ as well.

Theorem 3.1.1. *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system, with A separable, and suppose that τ is a faithful densely defined lower semicontiuous α-invariant trace on A. Consider the Hilbert space $\mathcal{H} := \mathbb{C}^N \otimes L^2(\mathbb{R}^n, \mathfrak{H})$ and the Dirac operator $D\!\!\!/ := \sum_k \gamma^k \otimes D_k$. There exists a local subalgebra $\mathcal{C}$ of A such that $(\mathcal{C}, \mathcal{H}, D\!\!\!/)$ is a smoothly summable spectral triple over A, with spectral dimension n.*

Let $B := A \rtimes_\alpha \mathbb{R}^n$ be the crossed product. The Thom class $\mathbf{t}_\alpha \in KK^\bullet(A, B)$ is represented by any of the following Kasparov A-B modules.

(i) $(\pi_B, \mathrm{M}_N(B), F\!\!\!/)$, where $F\!\!\!/ := D\!\!\!/(\mathbf{1} + D\!\!\!/^2)^{-1/2}$ is the bounded transform of $D\!\!\!/$, and the representation $\pi_B : A \to \mathcal{M}(B \otimes \mathcal{K})$ is given by left multiplication via π_α.

(ii) $(\boldsymbol{\pi}_B, \mathrm{M}_N(\boldsymbol{B}), \boldsymbol{R\!\!\!/})$*, which is the Kasparov module obtained by applying the double-up construction to the spectral triple* $(\mathcal{C}, \mathcal{H}, D\!\!\!/)$ *(see §2.5.2).*

(iii) $(\pi_B, \mathrm{M}_N(B), 2P\!\!\!/ - \mathbf{1})$*, where* $P\!\!\!/$ *is the spectral projection of* $D\!\!\!/$ *corresponding to the interval* $\mathbb{R}_+$*.*

In particular, the Thom isomorphism is given by Kasparov product with the class of any of these Kasparov modules.

Suppose that n *is odd. For each unitary* $u \in \mathcal{C}^\sim$*, the* $\hat{\tau}$*-index of the Toeplitz operator* $P\!\!\!/\pi_\alpha(u)P\!\!\!/$ *can be calculated as the spectral flow between* $\boldsymbol{D\!\!\!/}$ *and* $\boldsymbol{\pi}_\alpha(u^*)\boldsymbol{D\!\!\!/}\boldsymbol{\pi}_\alpha(u)$*,*

$$\mathrm{Index}_{\hat{\tau}}(P\!\!\!/\pi_\alpha(u)P\!\!\!/) = \mathrm{Index}_{\hat{\tau}}(\boldsymbol{P\!\!\!/}\boldsymbol{\pi}_\alpha(u)\boldsymbol{P\!\!\!/}) = \mathrm{Sf}(\boldsymbol{D\!\!\!/}, u^*\boldsymbol{D\!\!\!/}u),$$

and as the pairing between the Chern character $\mathrm{Ch}(u) \in HP_1(\mathcal{C})$ *in periodic cyclic homology with the cohomological Chern character* $\mathrm{Ch}(\mathcal{A}, \mathcal{H}, D\!\!\!/) \in HP^1(\mathcal{C})$*,*

$$\begin{aligned}\mathrm{Index}_{\hat{\tau}}(P\!\!\!/\pi_\alpha(u)P\!\!\!/) &= \frac{-1}{\sqrt{2\pi i}}\langle \mathrm{Ch}(u), \mathrm{Ch}(\mathcal{C}, \mathcal{H}, D\!\!\!/)\rangle \qquad (3.2)\\ &= \frac{-1}{2^{2n-1}}\hat{\tau}\big(\boldsymbol{R\!\!\!/}[\boldsymbol{R\!\!\!/}, \boldsymbol{\pi}_\alpha(u^{-1})][\boldsymbol{R\!\!\!/}, \boldsymbol{\pi}_\alpha(u)]\cdots[\boldsymbol{R\!\!\!/}, \boldsymbol{\pi}_\alpha(u^{-1})][\boldsymbol{R\!\!\!/}, \boldsymbol{\pi}_\alpha(u)]\big).\end{aligned}$$

Finally, one has the local formula

$$\mathrm{Index}_{\hat{\tau}}(P\!\!\!/\pi_\alpha(u)P\!\!\!/) = -\frac{2^{(n-1)/2}(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^n n!}\tau\big((u^*\delta(u))^n\big).$$

Suppose that n *is even. Then for each projection* $e \in \mathcal{C}^\sim$*, one has*

$$\begin{aligned}\mathrm{Index}_{\hat{\tau}}(\boldsymbol{\pi}_\alpha(e)\boldsymbol{R\!\!\!/}_+\boldsymbol{\pi}_\alpha(e)) &= \langle \mathrm{Ch}(e), \mathrm{Ch}(\mathcal{C}, \mathcal{H}, D\!\!\!/)\rangle \qquad (3.3)\\ &= \frac{(-1)^{n/2}}{(n/2)!}\frac{2^n}{(2\pi i)^n}\tau\big((e\delta(e)\delta(e))^{n/2}\big),\end{aligned}$$

where $\boldsymbol{R\!\!\!/}_+ : \mathcal{H}_+ \to \mathcal{H}_-$ *is the* $+$*-part of* $\boldsymbol{R\!\!\!/} = \boldsymbol{D\!\!\!/}|\boldsymbol{D\!\!\!/}|^{-1}$ *under the splitting* $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$*.*

In the sense of (3.2) and (3.3) we may say that $(\mathcal{C}, \mathcal{H}, D\!\!\!/)$ is an "unbounded representative" of the Thom class for $(A, \mathbb{R}^n, \alpha)$. Indeed, the left-hand sides of (3.2) and (3.3) are obtained by applying the homomorphism $\hat{\tau}_* : K_0(B) \to \mathbb{R}$ to the Kasparov products $[u] \otimes_A \mathbf{t}_\alpha$ and $[e] \otimes_A \mathbf{t}_\alpha$ (see Corollary 2.5.17).

Theorem 3.1.1 gives a generalization of the $n = 1$ formulae in [27, 79, 98]. If furthermore

$A = C(\mathbb{S}^1)$, $\mathcal{C} = C^\infty(\mathbb{S}^1)$, with $\not{D} = \sqrt{-1}\partial/\partial t$ and τ the Lebesgue integral, we get the classical Gohberg-Krein theorem (see [98, §4(a)]).

3.2 Preliminary facts

3.2.1 Crossed products by $\mathbb{R}^n$

The following material can be found e.g. in [15, §II.10].

Definition 3.2.1. Let A be a C^*-algebra. An **action** of $\mathbb{R}^n$ on A is a homomorphism $\alpha : \mathbb{R}^n \to \mathrm{Aut}(A)$ such that $\mathbb{R}^n \ni t \to \|\alpha_t(a)\|$ is continuous for each $a \in A$ (we say that α is **strongly continuous**). The triple $(A, \mathbb{R}^n, \alpha)$ is a C^*-**dynamical system**.

Let α be an action by $\mathbb{R}^n$ on a C^*-algebra A, and let $\pi : A \to \mathcal{B}(\mathfrak{H})$ be a faithful nondegenerate representation of A. Set $\mathcal{H} := L^2(\mathbb{R}^n) \otimes \mathfrak{H}$. We define representations

$$\pi_\alpha : A \to \mathcal{B}(\mathcal{H}), \qquad \lambda : \mathbb{R}^n \to \mathrm{U}(\mathcal{H})$$

by sending $a \in A$ and $t \in \mathbb{R}^n$ to the operators $\pi_\alpha(a)$ and λ_t acting on $\xi \in \mathcal{H}$ as

$$(\pi_\alpha(a)\xi)(s) := \alpha_{-s}(a)\xi(s), \qquad (\lambda_t\xi)(s) := \xi(s-t) \tag{3.4}$$

for all $s \in \mathbb{R}^n$. Here we identify an element of $\mathcal{H}$ with a (measure class of a) square-integrable function from $\mathbb{R}^n$ to $\mathfrak{H}$. There is a corresponding **integrated representation** $\hat{\pi}_\alpha$ of $L^1(\mathbb{R}^n, A)$ on $\mathcal{H}$, given by

$$\hat{\pi}_\alpha(f) := \int_{\mathbb{R}^n} \pi_\alpha\big(f(t)\big)\lambda_t\, dt, \qquad \forall f \in L^1(\mathbb{R}^n, A). \tag{3.5}$$

We have

$$\hat{\pi}_\alpha(f)\hat{\pi}_\alpha(g) = \hat{\pi}_\alpha(f * g),$$

where $*$ is the α-twisted convolution product

$$(f * g)(t) := \int_{\mathbb{R}^n} f(s)\alpha_s(g(t-s))\, ds,$$

and $\hat{\pi}_\alpha$ respects the involution $f^*(t) := \alpha_t(f(-t))^*$.

Definition 3.2.2. Let α be an action by $\mathbb{R}^n$ on a C^*-algebra A, and let $\pi : A \to \mathcal{B}(\mathfrak{H})$ be a faithful nondegenerate representation of A. The **crossed product** of A by $\mathbb{R}^n$ is the C^*-algebra $A \rtimes_\alpha \mathbb{R}^n$ generated by the operators (3.5).

Since π is faithful, the isomorphism class of the C^*-algebra $A \rtimes_\alpha \mathbb{R}^n$ does not depend on the choice of π.

Remark 3.2.3. A slight modification of Definition 3.2.2 makes sense for general locally compact groups G playing the role of $\mathbb{R}^n$. In that case we should refer to $A \rtimes_\alpha G$ as the *reduced* crossed product, since there is also a "full" crossed product which is isomorphic to $A \rtimes_\alpha G$ iff G is amenable. Since $\mathbb{R}^n$ is amenable, we shall not bother about the term "reduced".

Example 3.2.4. (i) If $A = \mathbb{C}$ then $A \rtimes_\alpha \mathbb{R}^n = C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)$ is the group C^*-algebra of $\mathbb{R}^n$.

(ii) More generally, if α is the trivial action then $A \rtimes_\alpha \mathbb{R}^n = A \otimes C^*(\mathbb{R}^n) = S^n A$ is the n-fold suspension of A.

We denote by $D = (D_1, \ldots, D_n)$ the infinitesimal generators of the group $(\lambda_t)_{t\in\mathbb{R}}$, so that

$$\lambda_t = e^{-2\pi i t \cdot D},$$

where $t \cdot D := t_1 D_1 + \cdots + t_n D_n$. Then the D_k's are "affiliated" with the C^*-algebra $A \rtimes_\alpha \mathbb{R}^n$ in the sense that $f(D)$ is a multiplier of the crossed product for each $f \in C_b(\mathbb{R}^n)$,

$$f(D) \in \mathcal{M}(A \rtimes_\alpha \mathbb{R}^n).$$

Like any unitary representation, λ extends to a representation of the group C^*-algebra $C^*(\mathbb{R}^n) \cong C_0(\mathbb{R}^n)$. We may regard the representations π_α and λ as embeddings

$$\pi_\alpha : A \to \mathcal{M}(A \rtimes_\alpha \mathbb{R}^n), \qquad \lambda : C^*(\mathbb{R}^n) \to \mathcal{M}(A \rtimes_\alpha \mathbb{R}^n)$$

of A and $\mathbb{R}^n$ into the multiplier algebra of the crossed product. In contrast to the von Neumann-algebraic crossed product, which may be regarded as an "extension" of the von Neumann algebra by $C^*(\mathbb{R}^n)$, neither of the algebras A and $C^*(\mathbb{R}^n)$ are included in the crossed product but merely in the multiplier algebra.

Remark 3.2.5 (Weak closure)**.** The weak closure $A'' \subset \mathcal{B}(\mathfrak{H})$ of A can be represented on $L^2(\mathbb{R}^n, \mathfrak{H})$ by the same formula (3.4), and we have

$$\pi_\alpha(A'') = \pi_\alpha(A)''.$$

Remark 3.2.6 (Suspended action)**.** Any action $\alpha : \mathbb{R}^n \to \mathrm{Aut}(A)$ induces an action $S^n\alpha : \mathbb{R}^n \to \mathrm{Aut}(SA)$ on the suspension $S^nA = C_0(\mathbb{R}^n, A)$ by

$$((S^n\alpha)_t)(f)(s) := \alpha_t(f(s)), \qquad \forall\, f \in S^nA,\ t \in \mathbb{R}^n,\ s \in \mathbb{R}.$$

There is an action $\hat{\alpha} : \hat{\mathbb{R}}^n \to \mathrm{Aut}(B)$ of the dual $\hat{\mathbb{R}}^n = \mathbb{R}^n$ of $\mathbb{R}^n$ on the crossed product, called the **dual action** [112, Def. X.2.4], characterized by $(s \in \hat{\mathbb{R}}^n)$

$$\hat{\alpha}_s(\pi_\alpha(a)) := \pi_\alpha(a), \qquad \forall\, a \in A,$$

$$\hat{\alpha}_s(\lambda_t) := e^{-2\pi i s\cdot t}\lambda_t, \qquad \forall\, t \in \mathbb{R}^n.$$

Evidently, the fixed-point subalgebra of B under the action $\hat{\alpha}$ is just $\pi_\alpha(A)$. A fundamental fact is that iterating the crossed-product construction using the dual action gives back A (up to stable isomorphism).

Theorem 3.2.7 (Takesaki-Takai duality [112, Thm. X.2.3], [111])**.** *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system. Then the crossed product of $A \rtimes_\alpha \mathbb{R}^n$ with $\hat{\mathbb{R}}^n$ by the dual action $\hat{\alpha}$ is stably isomorphic to the original algebra A:*

$$(A \rtimes_\alpha \mathbb{R}^n) \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n \cong A \otimes \mathcal{K}.$$

On the level of von Neumann algebras $\mathcal{M} = A''$ and $\mathcal{N} = (A \rtimes_\alpha \mathbb{R}^n)''$, the duality reads

$$\mathcal{N} \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n \cong \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}^n)).$$

The isomorphism can be chosen so that the double dual action $\hat{\hat{\alpha}}$ is intertwined with the action $\alpha \otimes \mathrm{Ad}(\lambda)$ on $A \otimes \mathcal{K}$, where $\mathrm{Ad}(\lambda_t)(T) := \lambda_{-t}T\lambda_t$ for $T \in \mathcal{K}(L^2(\mathbb{R}^n))$.

In view of Theorem 3.2.7, we refer to $(B, \hat{\mathbb{R}}^n, \hat{\alpha})$ as the **dual dynamical system** of $(A, \mathbb{R}^n, \alpha)$.

Lemma 3.2.8 ([115, Cor. 2.48]). *Let $(A, \mathbb{R}^n, \alpha)$ and $(A', \mathbb{R}^n, \alpha')$ be C^*-dynamical systems and let $\rho : A \to A'$ be an* ***equivariant*** *$*$-homomorphism in the sense that $\alpha' \circ \rho = \beta$. We write this as $\rho : (A, \alpha') \to (A, \alpha')$. There exists a $*$-homomorphism*

$$\hat{\rho} : A \rtimes_\alpha \mathbb{R}^n \to A' \rtimes_{\alpha'} \mathbb{R}^n$$

taking the operator $\tilde{\pi}_\alpha(f)$ given by (3.5) to

$$\hat{\rho}(\tilde{\pi}_\alpha(f)) := \int_{\mathbb{R}^n} \pi_\alpha \circ \rho\big(f(t)\big)\lambda_t \, dt. \tag{3.6}$$

In particular,

$$\hat{\rho} \circ \pi_\alpha = \pi_{\alpha'} \circ \rho$$

as maps from A into $\mathcal{M}(A' \rtimes_{\alpha'} \mathbb{R}^n)$.

Since $\hat{\rho}$ leaves λ_t untouched and intertwines π_α with $\pi_{\alpha'}$, we have again an equivariant map

$$\hat{\rho} : (B, \hat{\alpha}) \to (B', \widehat{\alpha'})$$

of the dual dynamical systems. Therefore, we can iterate the process and obtain a map $\hat{\hat{\rho}} : B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n \to B' \rtimes_{\hat{\alpha'}} \hat{\mathbb{R}}^n$ between the iterated crossed products. Under the isomorphism $B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n \cong A \otimes \mathcal{K}$ one checks that $\hat{\hat{\rho}}$ becomes

$$\hat{\hat{\rho}} = \rho \otimes \mathrm{id}\,.$$

3.2.2 Operator-valued weights

Let $\mathcal{M}$ be a Neumann algebra. If $\alpha : \Gamma \to \mathrm{Aut}(\mathcal{M})$ is and action of a *discrete* Abelian group on $\mathcal{M}$ then there exists a normal conditional expectation $E : \mathcal{N} \to \pi_\alpha(\mathcal{M})$ from the crossed product $\mathcal{N} = \mathcal{M} \rtimes_\alpha \Gamma$ to $\pi_\alpha(\mathcal{M})$. Indeed, $\pi_\alpha(\mathcal{M})$ is the fixed-point subalgebra of $\mathcal{N}$ under the dual action $\hat{\alpha}$ of the dual group $G = \hat{\Gamma}$, which is a compact group. The conditional expectation is then obtain by averaging over the group G using the Haar measure dt:

$$E : \mathcal{N} \to \pi_\alpha(\mathcal{M}), \qquad E(T) := \int_G \hat{\alpha}_t(T)\, dt. \tag{3.7}$$

One can use the map E to produce weights on $\mathcal{N}$ from weights on $\mathcal{M}$. Indeed, if φ is a normal faithful semifinite weight on $\mathcal{M}$ then

$$\hat{\varphi} := \varphi \circ \pi_\alpha^{-1} \circ E$$

is a normal faithful semifinite weight on $\mathcal{N}$. In fact, the assignment $\varphi \to \hat{\varphi}$ is a bijection between the set of normal faithful semifinite weights on $\mathcal{M}$ and the set of normal faithful semifinite weights on $\mathcal{N}$ which are invariant under the dual action $\hat{\alpha}$ [55, Cor. 3], [53, Thm. 3.7].

Evidently, the formula (3.7) does not define a conditional expectation on $\mathcal{M} \rtimes_\alpha \Gamma$ when G is noncompact (equivalently, when Γ is not discrete), because then the volume of G is infinite.

Haagerup observed that having an analogue of a conditional expectation at hand also in the case $\Gamma = \mathbb{R}^n = G$ could be used to relate functionals on $\mathcal{M}$ to functionals on $\mathcal{N} = \mathcal{M} \rtimes_\alpha \mathbb{R}^n$ [55]. We have seen that weights $\varphi : \mathcal{M}_+ \to [0, +\infty]$ play an important role in operator theory. These are analogues of continuous functionals $\varphi : \mathcal{M} \to \mathbb{C}$ but the value space $\mathbb{C}$ is replaced by its "extended positive part" $[0, +\infty]$. The "generalized conditional expectation", or "operator-valued weight", from $\mathcal{N}$ to $\mathcal{M}$ will be a conditional expectation where the value space $\mathcal{M}$ is replaced by the "extended positive part" $\hat{\mathcal{M}}_+$ of $\mathcal{M}$.

Definition 3.2.9 ([112, Def. IX.4.4])**.** The **extended positive part** $\hat{\mathcal{M}}_+$ of $\mathcal{M}$ is the set of all maps $\Psi : \mathcal{M}_*^+ \to [0, \infty]$ which are lower semicontinuous in the ultraweak topology and satisfy

$$\Psi(\lambda_1 \varphi_1 + \lambda_2 \varphi_2) = \lambda_1 \Psi(\varphi_1) + \lambda_2 \Psi(\varphi_2)$$

for all $\lambda_i \in \mathbb{R}_+$ and $\varphi_i \in \mathcal{M}_*^+$.

The term "extended" is justified by the fact that $\mathcal{M}_+ \subset \hat{\mathcal{M}}_+$ by letting $x = \int_0^\infty \lambda \, dE^x(\lambda) \in \mathcal{M}_+$ be identified with

$$\Psi_x(\varphi) := \varphi(x) = \int_0^\infty \lambda \, \varphi(dE^x(\lambda)).$$

Then Ψ_x is a "finite" element of $\mathcal{M}_*^+$ in the sense that it never takes the value $+\infty$. More generally, if $\mathcal{M} \subset \mathcal{B}(\mathcal{H})$ then we can regard every positive operator T in $\mathcal{H}$ affiliated with $\mathcal{M}$ as an element m_T of $\hat{\mathcal{M}}_+$ by letting $\Psi_T(\omega_\psi) := +\infty$ on the vector states $\omega_\psi := \langle \psi | \cdot \psi \rangle$

for which $\psi \notin \mathrm{Dom}(T^{1/2})$, while $\Psi_T(\omega_\psi) = \omega_\psi(T)$ as before when ψ is in the domain of $T^{1/2}$. Thus, the map $T \to m_T$ is injective.

In fact, for each $\Psi \in \hat{\mathcal{M}}_+$ one has a spectral decomposition [112, Thm. IX.4.8].

$$\Psi(\varphi) = \infty \cdot \varphi(dE^x(\infty)) + \int_0^\infty \lambda\, \varphi(dE^x(\lambda)), \qquad \forall \varphi \in \mathcal{M}_*^+$$

where $dE^x(\infty) := \mathbf{1} - \lim_{\lambda\to\infty} dE^x(\lambda)$ and $\infty \cdot \varphi(dE^x(\infty))$ means that $\Psi(\varphi) = +\infty$ iff $dE^x(\infty) \neq 0$.

Example 3.2.10 ([55, Example 1.7])**.** Let $\mathcal{N}$ be a commutative von Neumann algebra. Then there is a locally compact space Ω equipped with a Radon measure μ such that $\mathcal{N} \cong L^\infty(\Omega, \mu)$ is the algebra of μ-measure classes of bounded measurable functions on Ω. The extended positive part $\hat{\mathcal{N}}_+$ is then the set of equivalence classes of measurable functions $f : \Omega \to [0, +\infty]$.

Definition 3.2.11 ([55, Def. 2.1])**.** Let $\mathcal{M}$ and $\mathcal{N}$ be von Neumann algebras. An **operator-valued weight** from $\mathcal{M}$ to $\mathcal{N}$ is a map $E : \mathcal{N}_+ \to \hat{\mathcal{M}}$ such that

(i) $E(\lambda T) = \lambda E(T)$ for all $T \in \mathcal{N}_+$, $\lambda \in \mathbb{R}_+$,

(ii) $E(S + T) = E(S) + E(T)$ for all $S, T \in \mathcal{N}_+$, and

(iii) $E(M^*TM) = M^*E(T)M$ for all $T \in \mathcal{N}_+$ and all $M \in \mathcal{M}$.

The notions of "normal", "semifinite" and "faithful" carry over verbatim from the case $\mathcal{M} = \mathbb{C}$.

Example 3.2.12. If an operator-valued weight $E : \mathcal{N}_+ \to \hat{\mathcal{M}}$ satisfies $E(\mathbf{1}) = \mathbf{1}$ then E extends $\mathbb{C}$-linearly to a conditional expectation $E : \mathcal{N} \to \mathcal{M}$.

Operator-valued weights therefore generalize both weights and conditional expectations.

Lemma 3.2.13 ([55, Thm.1.1])**.** *Let $\mathcal{N} = \mathcal{M} \rtimes_\alpha \mathbb{R}^n$ be the crossed product of a von Neumann algebra by an action of $\mathbb{R}^n$, and let $\hat{\alpha} : \mathbb{R}^n \to \mathrm{Aut}(\mathcal{N})$ be the dual action. Then the formula*

$$E(x^*x) := \int_{\mathbb{R}^n} \hat{\alpha}_p(x^*x)\, dp, \qquad \forall x \in \mathcal{N}.$$

defines a normal faithful semifinite operator-valued weight from $\mathcal{N}$ to $\mathcal{M}$.

3.2.3 The dual trace

Again consider a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$ and let $\mathcal{N} := B'' \subseteq \mathcal{B}(L^2(\mathbb{R}^n, \mathfrak{H}))$ be the weak closure of the crossed product $B := A \rtimes_\alpha \mathbb{R}^n$. If $A \subset \mathcal{B}(\mathfrak{H})$ is already concretely represented, and if $\mathcal{M} := A''$, then $\mathcal{N} = \mathcal{M} \rtimes_\alpha \mathbb{R}^n$.

Definition 3.2.14 ([112, Def. X.1.16], [53, 54])**.** Let φ be a faithful semifinite normal weight on $\mathcal{M}$. The semifinite weight on $\mathcal{N}$ **dual** to φ is defined to be

$$\hat{\varphi} := \varphi \circ \pi_\alpha^{-1} \circ E, \tag{3.8}$$

where E is the operator-valued weight from $\mathcal{N}$ to the fixed-point subalgebra $\mathcal{N}^{\hat{\alpha}} = \pi_\alpha(\mathcal{M})$ given by (here $\hat{\alpha}$ is the dual action)

$$E(T^*T) := \int_{\mathbb{R}^n} \hat{\alpha}_p(T^*T)\, dp, \qquad \forall T \in \mathcal{N}.$$

For $f \in L^1(\mathbb{R}^n, \mathrm{Dom}(\varphi)) \cap L^2(\mathbb{R}^n, \mathfrak{H})$, the important formula is [112, Thm. 1.27]

$$\hat{\varphi}(\hat{\pi}_\alpha(f)^* \hat{\pi}_\alpha(f)) = \varphi(\langle f|f\rangle_{\mathcal{M}}), \tag{3.9}$$

where $\langle\cdot|\cdot\rangle_{\mathcal{M}}$ is the $\mathcal{M}$-valued inner product given by

$$\langle f|g\rangle_{\mathcal{M}} := \int_{\mathbb{R}^n} f(s)^* g(s)\, ds, \qquad \forall f, g \in L^2(\mathbb{R}^n, \mathcal{M}). \tag{3.10}$$

Lemma 3.2.15 ([112, Lemma. X.1.18])**.** *The dual weight $\hat{\varphi}$ is uniquely determined by the formula* (3.9)*.*

It follows from the defining formula (3.8) that the dual weight $\hat{\varphi}$ is invariant under the dual action $\hat{\alpha}$. In fact, a faithful weight ψ on $\mathcal{N}$ is the dual of some faithful weight on $\mathcal{M}$ if and only if ψ is invariant under $\hat{\alpha}$ [112, Thm. X.2.3]. We can now show the fruitfulness of applying operator-valued weights to the problem mentioned in the beginning of Section 3.2.2.

Theorem 3.2.16 ([53, Thm. 3.7])**.** *The map $\varphi \to \hat{\varphi}$ is a bijection between the set of normal faithful semifinite weights on $\mathcal{M}$ and the set of normal faithful semifinite weights on $\mathcal{N}$ which are invariant under the dual action $\hat{\alpha}$.*

The name "dual weight" is motivated by the fact that, under the Takesaki-Takai duality isomorphism $\mathcal{N} \rtimes_\alpha \mathbb{R}^n \cong \mathcal{M} \otimes \mathcal{B}(L^2(\mathbb{R}^n))$ (Theorem 3.2.7), the bidual $\hat{\hat{\varphi}}$ of a weight φ on $\mathcal{M}$ is identified with $\varphi \otimes \mathrm{Tr}$, where Tr is the operator trace on $\mathcal{B}(L^2(\mathbb{R}^n))$.

Remark 3.2.17. If $\mathfrak{H} = \mathfrak{H}_\varphi$ is the GNS space of φ then we can also define $\hat{\varphi}$ as the semifinite weight on $\mathcal{N}$ corresponding to the Hilbert algebra $L^1(\mathbb{R}^n, \mathrm{Dom}(\varphi)) \cap L^2(\mathbb{R}^n, \mathfrak{H}_\varphi)$ (embedded in $\mathcal{N}$ via $\hat{\pi}_\alpha$). This is the original approach in [53, Def. 3.1], [112, Def. X.1.6].

Suppose now that the C^*-algebra A has a faithful lower semi-continuous trace τ. A C^*-algebraic version of Definition 3.2.14 has been used for a long time (e.g. [41], [98, Section 2]). It can be defined precisely using the general construction in [**?**, §1], which provides us with a weight $\hat{\tau}$ on $B := A \rtimes_\alpha \mathbb{R}^n$ satisfying

$$\hat{\tau}(\hat{\pi}_\alpha(f)^* \hat{\pi}_\alpha(f)) = \tau(\langle f|f\rangle_A), \qquad \forall f \in C_0(\mathbb{R}^n, \mathrm{Dom}(\tau)) \cap L^2(\mathbb{R}, \mathfrak{H}) \tag{3.11}$$

where $\langle \cdot|\cdot\rangle_A$ is the A-valued inner product given by

$$\langle f|g\rangle_A := \int_{\mathbb{R}^n} f(s)^* g(s)\, ds, \qquad \forall f, g \in L^2(\mathbb{R}^n, A). \tag{3.12}$$

For clarity we denote by $\bar{\tau}$ the normal extension of τ to $\mathcal{M} = A''$. Then we have the dual weight of $\bar{\tau}$ on $\mathcal{N}$, which extends $\hat{\tau} : \mathcal{M}_+ \to [0, \infty]$. We write $\hat{\tau}$ also for this extension.

In view of how strongly our discussion in the last chapter about index theory depended on traces, we would like to know under what circumstances $\hat{\tau}$ is a trace.

Remark 3.2.18. In our generalization of the Gohberg-Krein theorem we shall need to assume τ to be invariant under the $\mathbb{R}^n$-action. To see why, suppose that $\tau(\alpha_t(a)) = \tau(\rho^{it} a)$ for all $a \in A$, $t \in \mathbb{R}$ for some positive invertible operator ρ affiliated to A''. Then the modular automorphism group $\sigma^{\hat{\tau}}$ of $\hat{\tau}$ is nontrivial, namely

$$\sigma_t^{\hat{\tau}}(x) = x, \qquad \sigma_t^{\hat{\tau}}(e^{2\pi i s\cdot D}) = \pi_\alpha(\rho^{it}) e^{2\pi i s\cdot D}, \qquad \forall x \in \mathcal{N},\ s, t \in \mathbb{R}^n.$$

So $\hat{\tau}$ is not a trace in this case. On the other hand, if τ is α-invariant then $\sigma_t^{\hat{\tau}} \equiv \mathrm{id}$, which is equivalent to saying that $\hat{\tau}$ is a trace.

3.2.4 Some more facts about KK

In order to understand the role of the Thom class in KK, we need to recall some more properties of KK-theory.

Definition 3.2.19 ([14, Def. 19.1.1])**.** An element $\mathbf{t} \in KK^0(A, B)$ is a KK**-equivalence** if there exists a two-sided inverse for $\mathbf{t}$, i.e. an element $\mathbf{s} \in KK^0(B, A)$ such that

$$\mathbf{t} \otimes_B \mathbf{s} = 1_A, \qquad \mathbf{s} \otimes_B \mathbf{t} = 1_B.$$

Two C^*-algebras A and B are KK**-equivalent** if a KK-equivalence exists in $KK^0(A, B)$.

In view of the isomorphisms $KK^1(A, B) \cong KK^0(A \otimes \mathbb{C}_1, B) \cong KK^0(A, B \otimes \mathbb{C}_1)$, one may also say that A and B are "KK-equivalent with a degree-1 shift" if there exists a KK-equivalence in $KK^0(A, B \otimes \mathbb{C}_1)$.

Remark 3.2.20. For $A = B$, the Kasparov product gives the group $KK^0(A, A)$ the structure of a ring. The class in $KK^0(A, A)$ defined by $(\mathrm{id}, A_A, 0)$, where $\mathrm{id} : A \to A$ is the identity morphism, is usually denoted by 1_A. It a KK-equivalence and the identity in the ring $KK^0(A, A)$. For a general morphism $\rho : C \to A$ one has

$$[\rho] = \rho^* 1_A$$

where $[\rho]$ is the class in $KK^0(C, A)$ defined in Remark 2.4.25.

It is a basic fact that $KK^0(\mathbb{C}, \mathbb{C}) \cong \mathbb{Z}$ [14, Example 17.3.4] is the cyclic group generated by $[1_\mathbb{C}]$. Moreover, it easy to find representatives of the class $[1_\mathbb{C}]$.

Suppose that V is a "graded Fredholm operator" on an $\mathbb{Z}_2$-graded infinite-dimensional Hilbert space $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$, in the sense that V is odd with respect to the grading,

$$V = \begin{pmatrix} 0 & V_+^* \\ V_+ & 0 \end{pmatrix},$$

and $V_+ : \mathcal{H}_+ \to \mathcal{H}_-$ is Fredholm. If V_+ is a partial isometry then $(\pi, \mathcal{H}, V)$ is a Kasparov $\mathbb{C}$-$\mathbb{C}$-module, where $\pi(\lambda) := \lambda \mathbf{1}$. Suppose moreover that the Fredholm index of V (i.e. the Fredholm index of V_+) is equal to 1. Then $\mathrm{Index}(V^{\oplus k}) = k$ for all $k \in \mathbb{Z}$ and so $(\pi, \mathcal{H}, V)$ represents $[1_\mathbb{C}]$.

We have already mentioned the Kasparov product (also known as the "internal product"). There is another product-type operation on KK-theory, which is more easily computed.

Definition 3.2.21 ([61, Thm. 4.8])**.** Define

$$\boxtimes 1_C : KK^0(A, B) \to KK^0(A \otimes C, B \otimes C), \qquad [\pi, X_B, F] \boxtimes 1_C := [\pi \otimes \mathrm{id}, X_B \otimes C, F \otimes \mathbf{1}],$$

where $\otimes$ is the minimal tensor product. The **external product** of two KK-classes $\mathbf{x} \in KK^0(A', A)$ and $\mathbf{y} \in KK^0(C', C)$ is the class

$$\mathbf{x} \boxtimes \mathbf{y} := (\mathbf{x} \boxtimes 1_{C'}) \otimes_{A \otimes C'} (1_A \boxtimes \mathbf{y})$$

in $KK^0(A' \otimes C', A \otimes C)$.

3.2.5 Thom isomorphisms

For any $n \in \mathbb{N}$, the $2n$-fold suspension of the C^*-algebra $C_0(M)$,

$$S^{2n} C_0(M) = C_0(\mathbb{R}^{2n}) \otimes C_0(M) \cong C_0(\mathbb{R}^{2n} \times M),$$

has the same K-theory as $C_0(M)$, by Bott periodicity. Thus, for the trivial complex vector bundle $M \times \mathbb{C}^n \cong M \times \mathbb{R}^{2n}$, we have

$$K^\bullet(M \times \mathbb{C}^n) \cong K^\bullet(M).$$

One would like to know if there is something preventing such an isomorphism for *nontrivial* complex vector bundles $\Pi : E \to M$. Let E be a complex vector bundle over M. The **Thom isomorphism** [4, §2.7], [65, §IV.1] in K-theory says that

$$K^\bullet(M) \cong K^\bullet(E),$$

which is a generalization of Bott periodicity. Atiyah proved Bott periodicity and the Thom isomorphism using elliptic operators [5], and his approach may be regarded as one of the first uses of KK-theory [107].

Bott periodicity in KK says that $C_0(M)$ and $C_0(\mathbb{R}^{2n} \times M)$ are also KK-equivalent. Moreover, Kasparov proved the Thom isomorphism in KK-theory [66, §5].

One may also ask for a Thom isomorphism for noncommutative C^*-algebras. There is in fact a quite similar phenomenon occuring when taking crossed products. While the analogy with the classical Thom isomorphism is not perfect, they have at least some things in common. Both provide a very important tool for computation and in proving general statements.

3.2.5.1 Connes' analogue of the Thom isomorphism

Connes constructed a natural isomorphism [41]

$$\partial_\bullet : K_\bullet(A) \to K_{\bullet+n}(A \rtimes_\alpha \mathbb{R}^n) \tag{3.13}$$

for any $\mathbb{R}^n$-action α on a C^*-algebra A, which he called the "Thom isomorphism in K-theory of C^*-algebras". The explicit construction was only carried out for $n = 1$ and is quite involved. A concrete realization of ∂_1 is also given in [79], but still only for $n = 1$. An alternative proof of the isomorphism (3.13) is also given in [104], but without any explicit formula for the map.

The idea of Connes' proof was to show that there exist unique K-theory maps $\partial_\bullet$ satisfying a set of axioms.

Definition 3.2.22 ([41, §II]). Let α be an action of $\mathbb{R}$ on a C^*-algebra A. For $\bullet \in \mathbb{Z}_2$, the **Thom map** for (A, α) is the additive map $\partial_\bullet = \partial_\bullet^\alpha$ from $K_\bullet(A)$ to $K_{\bullet+1}(A \rtimes_\alpha \mathbb{R})$ satisfying the following properties.

(i) If $A = \mathbb{C}$, so $\alpha_t = \mathrm{id}$ for all $t \in \mathbb{R}$, then ∂_0 maps the generator of $K_0(\mathbb{C}) \cong \mathbb{Z}$ to the positive generator of $K_1(C^*(\mathbb{R})) \cong K^1(\mathbb{R}) \cong \mathbb{Z}$.

(ii) (Naturality) If B is another C^*-algebra equipped with an $\mathbb{R}$-action β, and if $\phi_* : K_\bullet(A) \to K_\bullet(B)$ is the map induced by a homomorphism $\phi : A \to B$ such that $\phi \circ \alpha = \beta$, then $(\phi \otimes \mathrm{id})_* \circ \partial_\bullet^\alpha = \partial_\bullet^\beta \circ \phi_\bullet$.

(iii) (Suspension) The connecting homomorphisms $K_\bullet(A) \to K_{\bullet+1}(SA)$ intertwine the Thom maps of (A, α) and $(SA, S\alpha)$, where $S\alpha : \mathbb{R} \to \mathrm{Aut}(SA)$ is the suspended action (see Remark 3.2.6).

Suppose that such a map $\partial_\bullet^\alpha$ exists for each dynamical system $(A, \mathbb{R}, \alpha)$. Write $\partial_\bullet := \partial_\bullet^\alpha$ and $\hat{\partial}_\bullet := \partial_\bullet^{\hat{\alpha}}$ where $\hat{\alpha}$ is the dual action. By Takesaki-Takai duality (Theorem 3.2.7), the composition $\hat{\partial}_\bullet \circ \partial_\bullet$ is a map

$$\hat{\partial}_\bullet \circ \partial_\bullet : K_\bullet(A) \to K_{\bullet+2}(A).$$

Since both $\hat{\partial}_\bullet$ and $\partial_\bullet$ are assumed to commute with suspension, so must $\hat{\partial}_\bullet \circ \partial_\bullet$. Together with axioms (i)and (iii), this implies that $\hat{\partial}_\bullet \circ \partial_\bullet$ must coincide with the Bott periodicity isomorphism $K_\bullet(A) \cong K_{\bullet+2}(A)$. In particular, $\partial_\bullet = \partial_\bullet^\alpha$ must be an isomorphism for each α.

So the axioms for the Thom maps forces them to be isomorphisms. The Thom isomorphism theorem therefore amounts to the existence and uniqueness of the Thom map.

Theorem 3.2.23 ([41, Thm. 2])**.** *The Thom map exists and is unique, so we have natural isomorphisms*

$$K_\bullet(A) \cong K_{\bullet+1}(A \rtimes_\alpha \mathbb{R}).$$

The proof in [41] is *not* available for $n \geq 2$. For some interesting remarks about why that is true, see [40, §13].

Connes' Thom isomorphism shows that every action α gives the same K-theory of the crossed product $A \rtimes_\alpha \mathbb{R}^n$. The representatives of the K-classes may be very different however.

Example 3.2.24 ([107, §2.4])**.** Since we are only using the group structure of $\mathbb{R}^{2n}$, we can also regard an $\mathbb{R}^{2n}$-action α as an action by $\mathbb{C}^n$. If α is the trivial action on the C^*-algebra $A = C_0(M)$, where M is a locally compact Hausdorff space, then

$$C_0(M) \rtimes_\alpha \mathbb{C}^n \cong C_0(M) \otimes C_0(\mathbb{C}^n) \cong C_0(M \times \mathbb{C}^n).$$

If we identify $M \times \mathbb{C}^n$ with the trivial rank-n complex vector bundle over M then $K_\bullet(A) \cong K_\bullet(A \rtimes_\alpha \mathbb{C}^n)$ is the classical Thom isomorphism.

Example 3.2.25. Taking α to be the trivial action, the crossed product is equal to $B = C_0(\mathbb{R}) \otimes A = SA$, the suspension of A. Therefore, Connes' Thom isomorphism for the trivial action is exactly Bott periodicity.

3.2.5.2 Thom class in KK for $n = 1$

In order to obtain a concrete realization of Connes' Thom isomorphism for general n, we shall begin with what is known for $n = 1$. It is a result of Fack and Skandalis [47] that, in analogy with the classical Thom isomorphism, $\partial_\bullet$ is given by Kasparov product with a certain "Thom class" $\mathbf{t}_\alpha$ in $KK^1(A, B)$, where $B := A \rtimes_\alpha \mathbb{R}$. The class $\mathbf{t}_\alpha$ is a KK-equivalence with degree shift 1, so the result is stronger than merely isomorphism in K-theory.

Let F be the singular-integral operator on $L^2(\mathbb{R}, \mathfrak{H})$ given by the principal-value

$$F := \frac{1}{i\pi} \,\mathrm{P.V.} \int_{\mathbb{R}} \frac{1}{t} e^{2\pi i t D}\, dt, \tag{3.14}$$

where D is the generator of $\mathbb{R} \ni t \to \lambda_t$. It is possible to show that (π_α, B_B, F) defines an element $\mathbf{t}_\alpha$ in $KK^1(A, B)$ [47, Prop. 1].

The proof in [47] is similar to that of [41]. One defines a "Thom element" as a class in $KK^0(A, B)$ satisfying a set of axioms, and then one shows that these axioms implies that the Thom element is a KK-equivalence. Here one makes use of the Takesaki-Takai duality $B \rtimes_{\hat{\alpha}} \mathbb{R} \cong A \otimes \mathcal{K}$, which ensures that the Thom element $\mathbf{t}_{\hat{\alpha}} \in KK^0(B, B \rtimes_{\hat{\alpha}} \mathbb{R})$ of the dual action can be regarded as an element of $KK^0(B, A)$ and hence potentially provide an inverse for $\mathbf{t}_\alpha$.

In fact, the axioms in [47] are formulated for the morphisms given by Kasparov product with $\mathbf{t}_\alpha$ and $\hat{\mathbf{t}}_\alpha$. The equivalent axioms for the Thom elements themselves were given in [107, §2.4].

Theorem 3.2.26 ([47], [107, §2.4]). *Let $\mathbf{t}_\alpha$ be the class in $KK^1(A, B)$ represented by the odd Kasparov A-B module (π_α, B_B, F). Then the following axioms are satisfied.*

(i) (Normalization) If $A = \mathbb{C}$ (so that α is necessarily trivial) then $\mathbf{t}_\alpha \in KK^1(\mathbb{C}, C_0(\mathbb{R})) \cong \mathbb{Z}$ is the positive generator of this group.

(ii) (Naturality) If $\rho : (A, \alpha) \to (A', \alpha')$ is an equivariant $*$-homomorphism (cf. Lemma 3.2.8) then

$$(\mathrm{id} \rtimes \rho)_*(\mathbf{t}_\alpha) = \rho^*(\mathbf{t}'_\alpha) \in KK^0(A, A' \rtimes_\beta \mathbb{R}^n),$$

(iii) (Compatibility with external products) For all $\mathbf{x} \in KK^0(A', A)$ *and* $\mathbf{y} \in KK^0(C', C)$,

$$\mathbf{y} \boxtimes (\mathbf{x} \otimes_A \mathbf{t}_\alpha) = (\mathbf{y} \boxtimes \mathbf{x}) \otimes_{C \otimes A} \mathbf{t}_{\mathrm{id}_C \otimes \alpha},$$

where $\mathrm{id}_C : \mathbb{R}^n \to \mathrm{Aut}(C)$ *is the trivial action.*

Moreover, these axioms force $\mathbf{t}_\alpha$ *to be a* KK*-equivalence, with inverse* $\hat{\mathbf{t}}_\alpha := \mathbf{t}_{\hat{\alpha}}$*. In particular,* A *and* $A \rtimes_\alpha \mathbb{R}$ *are* KK*-equivalent with a degree-*1 *shift.*

We shall refer to $\hat{\mathbf{t}}_\alpha := \mathbf{t}_{\hat{\alpha}}$ as the **dual Thom element** for $(A, \mathbb{R}, \alpha)$. In this 1-dimensional setting there is no need to distinguish the between $\mathbb{R}$-actions from action by the dual $\hat{\mathbb{R}}$ in constructing the representative (π_α, B_B, F) of $\mathbf{t}_\alpha$ or $\hat{\mathbf{t}}_\alpha$. For $\mathbb{R}^n$-actions with $n \geq 2$ we shall see that it gets more complicated.

Example 3.2.27 ([14, Example 19.3.4(a)]). Let $A = \mathbb{C}$ and let $\alpha = \mathrm{id}$ be the trivial action, so that the crossed product is $B = C_0(\mathbb{R}) \cong C_0(0,1)$. Then the extension associated to the Thom element $\mathbf{t}_\alpha$ is just

$$0 \longrightarrow C_0(0,1) \longrightarrow C[0,1) \longrightarrow \mathbb{C} \longrightarrow 0,$$

and the operator F in (3.14) is the Hilbert transform on $L^2(\mathbb{R})$.

Example 3.2.28 ([14, Example 19.3.4(b)]). Taking $B = C_0(\mathbb{R})$ and the action $\widehat{\mathrm{id}} : \mathbb{R} \to \mathrm{Aut}(B)$ by translations, we have $B \rtimes_{\widehat{\mathrm{id}}} \mathbb{R} \cong \mathcal{K}$. The Thom element is in this case represented by the Toeplitz extension

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \longrightarrow C_0(\mathbb{R}) \longrightarrow 0,$$

where we use the isomorphism $\mathcal{K} \cong C_0(\mathbb{R}) \rtimes_{\widehat{\mathrm{id}}} \mathbb{R}$.

Thom classes in KK for higher n will be discussed in §3.5.

3.3 The Dirac operator

From now on, the C^*-algebra A, the action α and the trace τ are as in the introduction, i.e. α is a strongly continuous automorphic action of $\mathbb{R}^n$ on a separable C^*-algebra A with

smooth subalgebra $\mathcal{A}$, and τ is a faithful norm lower semicontinous α-invariant trace on A with dense domain $\mathrm{Dom}(\tau)$.

Let A be identified with its image $A \subset \mathcal{B}(\mathfrak{H})$ in some a faithful representation such that α is unitarily implemented (for example, this always happens if A'' is in standard form [112, Chapter IX.1]). The Hilbert space $\mathfrak{H}$ determines a representation

$$\pi_\alpha : A \to \mathcal{B}(L^2(\mathbb{R}^n, \mathfrak{H}))$$

where, as in (3.4),

$$(\pi_\alpha(a)\xi)(t) := \alpha_{-t}(a)\xi(t), \qquad \forall a \in A,\ \xi \in L^2(\mathbb{R}^n, \mathfrak{H}),\ t \in \mathbb{R}^n.$$

Again we denote by $\hat{\pi}_\alpha$ the induced representation of $L^1(\mathbb{R}^n, A)$ which defines the crossed product $B := A \rtimes_\alpha \mathbb{R}^n$. The von Neumann algebra $\mathcal{N} := B''$ is independent of the choice of $\mathfrak{H}$, up to isomorphism [112, Thm. X.1.7], and we fix such an $\mathfrak{H}$ and the corresponding π_α.

We need the crossed product $B = A \rtimes_\alpha \mathbb{R}^n$ to be represented not on $L^2(\mathbb{R}^n, \mathfrak{H})$ but rather on an amplification thereof.

We consider the Hilbert space

$$\mathcal{H} := \mathbb{C}^N \otimes L^2(\mathbb{R}^n, \mathfrak{H})$$

where $N = 2^{(n-1)/2}$ for odd n and $N = 2^{n/2}$ for even n, so that in any case we have an irreducible representation of the Clifford algebra $\mathbb{C}_n$ on $\mathcal{H}$ (cf. Section 1.4.1). We obtain a representation of A on $\mathcal{H}$ by sending $a \in A$ to

$$\pi_\alpha(a) := 1_N \otimes \pi_\alpha(a).$$

The selfadjoint generators $D_1, \ldots, D_n$ of the unitary group $\lambda_\bullet$ implementing α on $L^2(\mathbb{R}^n, \mathfrak{H})$ (see (3.4)) can be used to define the Dirac operator (the tensor product implicit)

$$\not{D} := \sum_{k=1}^{n} \gamma^k D_k, \tag{3.15}$$

where $\gamma^1, \ldots, \gamma^k$ are hermitian $N \times N$ matrices representing the generators of $\mathbb{C}_n$ on $\mathbb{C}^N$, satisfying therefore the Clifford relations $\gamma^j\gamma^k + \gamma^k\gamma^j = 2\delta^{jk}$.

We now apply the doubling-up construction of an invertible representative of the operator defining a KK-class coming from a spectral triple (see §2.5.2). Recall that, for the doubled spectral triple, the Hilbert space is $\boldsymbol{\mathcal{H}} := \mathcal{H} \otimes \mathbb{C}^2$ and the operator is

$$\boldsymbol{D\!\!\!/} = \begin{pmatrix} D\!\!\!/ & m \\ m & -D\!\!\!/ \end{pmatrix}, \tag{3.16}$$

while the representation of an element a in the C^*-algebra A on $\boldsymbol{\mathcal{H}}$ is

$$\boldsymbol{\pi}_\alpha(a) := \begin{pmatrix} \pi_\alpha(a) & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.17}$$

We shall use the fact (Lemma 2.5.10) that the index pairings on $\boldsymbol{\mathcal{H}}$ using (3.16) and (3.17) coincide with those on $\mathcal{H}$ using (3.15) and π_α. We will need the doubled triple for the upcoming "Toeplitz extension" but for the local formula for the numerical index we shall use (3.15).

Proposition 3.3.1. *For a in the intersection of the domains* $\mathrm{Dom}(\delta_k)$ *of the generators δ_k of α, we have*

$$[D\!\!\!/, \pi_\alpha(a)] = \frac{1}{2\pi i}\sum_{k=1}^{n} \gamma^k \pi_\alpha(\delta_k(a)).$$

Proof. This is seen as in [27, Prop. 3.3] using

$$[D\!\!\!/, \pi_\alpha(a)] = \sum_{k=1}^{n} \gamma^k [D_k, \pi_\alpha(a)].$$

Namely, if ξ is in the domain of D_k and a is in the domain of δ_k then

$$\begin{aligned}(D_k\pi_\alpha(a)\xi)(t) &= \frac{1}{2\pi i}\frac{\partial}{\partial t_k}(\alpha_{-t}(a)\xi(t)) \\ &= \frac{1}{2\pi i}\alpha_{-t}(\delta_k(a))\xi(t) + \frac{1}{2\pi i}\alpha_{-t}(a)\frac{\partial}{\partial t_k}\xi(t),\end{aligned}$$

so $\pi_\alpha(a)\xi$ is in the domain of D. On the other hand, $(\pi_\alpha(a)D\xi)(t) = (2\pi i)^{-1}\alpha_{-t}(a)\partial\xi(t)/\partial t_k$.

Thus

$$([D_k, \pi_\alpha(a)]\xi)(t) = \frac{1}{2\pi i}\alpha_{-t}(\delta_k(a))\xi(t),$$

and the formula for the commutator with $\not{D} = \sum_k \gamma^k D_k$ follows. $\square$

The following lemma was proven in [30] and is a very important result for the interplay between spectral triples and KK-theory.

Lemma 3.3.2 ([30, Lemma 2.3])**.** *Let $\not{D}$ be an unbounded selfadjoint operator on a Hilbert space $\mathcal{H}$, and let $\mathrm{Dom}(\not{D})$ be the domain of $\not{D}$. Suppose that $T \in \mathcal{B}(\mathcal{H})$ maps $\mathrm{Dom}(\not{D})$ into itself. Then*

$$[T, (\mathbf{1} + \not{D}^2)^{-1}] = \not{D}(\mathbf{1} + \not{D}^2)^{-1}[\not{D}, T](\mathbf{1} + \not{D}^2)^{-1} + (\mathbf{1} + \not{D}^2)^{-1}[\not{D}, T]\not{D}(\mathbf{1} + \not{D}^2)^{-1}$$

is an equality in $\mathcal{B}(\mathcal{H})$.

In the next lemma we write $D := (D_1, \ldots, D_n)$ and $|D| := \sqrt{D_1^2 + \cdots + D_n^2}$.

Lemma 3.3.3. *Let $\not{D}$ be the Dirac operator* (3.15) *associated with the C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$ and define*

$$\not{F} := \not{D}(\mathbf{1} + \not{D}^2)^{-1/2}.$$

Then for each $a \in A$, the commutator $[\not{F}, \pi_\alpha(a)]$ belongs to $\mathrm{M}_N(B)$.

Proof. For $a \in A$, we know e.g. from Proposition 3.3.1 that $[\not{D}, \pi_\alpha(a)]$ is a bounded operator on $\mathcal{H}$, in fact a multiplier of $\mathrm{M}_N(B)$. So by Lemma 3.3.2 we have

$$[\not{F}, \pi_\alpha(a)] = [\not{D}, \pi_\alpha(a)](\mathbf{1} + \not{D}^2)^{-1/2} + \not{D}[(\mathbf{1} + \not{D}^2)^{-1/2}, \pi_\alpha(a)].$$

For every $\varphi \in C_0(\mathbb{R})$, the operator $\varphi(\not{D})$ is in $\mathrm{M}_N(B)$. Therefore, the term $[\not{D}, \pi_\alpha(a)](\mathbf{1} + \not{D}^2)^{-1/2}$ is in $\mathrm{M}_N(B)$. It remains to show that we also have

$$\not{D}[(\mathbf{1} + \not{D}^2)^{-1/2}, \pi_\alpha(a)] \in \mathrm{M}_N(B).$$

For that, we use [30, Remark A.3] to write

$$(\mathbf{1} + \not{D}^2)^{-1/2} = \frac{1}{\pi}\int_0^\infty (\mathbf{1} + \lambda + \not{D}^2)^{-1}\lambda^{-1/2}\, d\lambda,$$

where the right-hand side converges in the norm on $\mathcal{B}(\mathcal{H})$. We then have

$$\begin{aligned}&\not{D}[(\mathbf{1}+\not{D}^2)^{-1/2},\pi_\alpha(a)]\\&=\frac{1}{\pi}\not{D}\int_0^\infty(\mathbf{1}+\lambda+\not{D}^2)^{-1}\big([\not{D},\pi_\alpha(a)]\not{D}+\not{D}[\not{D},\pi_\alpha(a)]\big)(\mathbf{1}+\lambda+\not{D}^2)^{-1}\lambda^{-1/2}\,d\lambda.\end{aligned}$$

Again we have convergence in the operator norm, so we can actually move the prefactor $\not{D}$ under the integral sign to obtain

$$\begin{aligned}\not{D}[(\mathbf{1}+\not{D}^2)^{-1/2},\pi_\alpha(a)]&=\frac{1}{\pi}\int_0^\infty\not{D}(\mathbf{1}+\lambda+\not{D}^2)^{-1}[\not{D},\pi_\alpha(a)]\not{D}(\mathbf{1}+\lambda+\not{D}^2)^{-1}\lambda^{-1/2}\,d\lambda\\&+\frac{1}{\pi}\int_0^\infty\not{D}^2(\mathbf{1}+\lambda+\not{D}^2)^{-1}[\not{D},\pi_\alpha(a)](\mathbf{1}+\lambda+\not{D}^2)^{-1}\lambda^{-1/2}\,d\lambda,\end{aligned}$$

The whole integrand is in $\mathrm{M}_N(B)$ because, for instance, the operator $\not{D}^2(\mathbf{1}+\lambda+\not{D}^2)^{-1}$ is bounded with norm ≤ 1 and a multiplier of $\mathrm{M}_N(B)$. Moreover, the estimates [27, Remark 5]

$$\|(\mathbf{1}+\lambda+\not{D}^2)^{-1}\|\leq\frac{1}{1+\lambda},\qquad\|\not{D}(\mathbf{1}+\lambda+\not{D}^2)^{-1}\|\leq\frac{1}{2\sqrt{1+\lambda}},$$

which follow from functional calculus, show that the integral is norm convergent. That completes the proof. $\square$

Remark 3.3.4. For even n we can always find a grading operator Γ on $\mathcal{H}$ such that $\Gamma\pi_\alpha(a)=\pi_\alpha(a)\Gamma$ for all $a\in A$ and $\Gamma\not{D}=-\not{D}\Gamma$. In the example $n=2$ we can take Γ to be $\mathrm{diag}(1,-1)$. We write

$$\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$$

for even n, with $\mathcal{H}_\pm$ the ±1-eigenspace of the grading operator Γ. Under the decomposition $\mathcal{H}=\mathcal{H}_+\oplus\mathcal{H}_-$, the algebra B splits as $\mathrm{M}_N(B)=\mathrm{M}_N(B)_+\oplus\mathrm{M}_N(B)_-$, and this induces an even grading $\mathrm{M}_N(\boldsymbol{B})=\mathrm{M}_N(\boldsymbol{B})_+\oplus\mathrm{M}_N(\boldsymbol{B})_-$ of the Hilbert B-module $\mathrm{M}_N(\boldsymbol{B})$. Here $\mathrm{M}_N(B)_+$ is the part of $\mathrm{M}_N(B)$ commuting with the grading operator $\Gamma=\mathrm{diag}(\mathbf{1},-\mathbf{1})$ and $\mathrm{M}_N(B)_-$ is the part anti-commuting with Γ (cf. Remark 2.4.28).

We let $\pi_B : A\to\mathrm{M}_N(\mathbb{C})\otimes\mathcal{M}(B)$ be the representation of A which takes $a\in A$ to the operator of left multiplication by the multiplier $1_N\otimes\pi_\alpha(a)$ of $\mathrm{M}_N(\mathbb{C})\otimes B$. The operators $\pi_B(a)$ are even for the grading of the Hilbert B-module $\mathrm{M}_N(B)$, whereas $\gamma^1,\ldots,\gamma^n$ are odd. These observations lead to the following result.

Proposition 3.3.5. *The triple* $(\pi_B, \mathrm{M}_N(B), \not{F})$ *is a Kasparov A-B-module and defines a class*

$$[\pi_B, \mathrm{M}_N(B), \not{F}] \in KK^\bullet(A, B),$$

where $\bullet \in \{0, 1\} = \{\text{even}, \text{odd}\}$ *is the parity of* n.

Let $\boldsymbol{\not{R}} := \boldsymbol{\not{D}}|\boldsymbol{\not{D}}|^{-1}$ *denote the phase of the massive Dirac operator* $\boldsymbol{\not{D}}$. *Then* $(\boldsymbol{\pi}_B, \mathrm{M}_N(\boldsymbol{B}), \boldsymbol{\not{R}})$ *is an even Kasparov A-B-module and defines the same class* $[\pi_B, \mathrm{M}_N(B), \not{F}]$ *in* $KK^\bullet(A, B)$.

Proof. We have seen in Lemma 3.3.3 that $[\not{F}, \pi_\alpha(a)]$ is in $\mathrm{M}_N(B)$ for all $a \in A$. So for the first statement it remains only to show that $\pi_B(A)(\not{F}^2 - \mathbf{1})$ is contained in $\mathrm{M}_N(B)$. For that, let $a \in A$ and write

$$\pi_\alpha(a)(\not{F}^2 - \mathbf{1}) = \pi_\alpha(a)(\mathbf{1} + \not{D}^2)^{-1} = \pi_\alpha(a)\varphi(\not{D})$$

where $\varphi : \mathbb{R} \to \mathbb{C}$ vanishes at infinity. Since $\pi_\alpha(a)$ is a multiplier of B, we have $\pi_B(a)(\not{F}^2 - \mathbf{1}) \in \mathrm{M}_N(B)$.

The same proof as that of Lemma 3.3.3 shows that $[\boldsymbol{\not{R}}, \boldsymbol{\pi}_B(a)]$ belongs to $\mathrm{M}_N(\boldsymbol{B})$ for all $a \in A$. That $[\pi_B, \mathrm{M}_N(B), \not{F}]$ is also represented by $(\boldsymbol{\pi}_B, \mathrm{M}_N(\boldsymbol{B}), \boldsymbol{\not{R}})$ follows from the facts presented in §2.5.2. □

3.4 The Toeplitz algebra

Let $\boldsymbol{\not{P}} := E^{\boldsymbol{\not{D}}}(\mathbb{R}_+)$ denote the projection on $\boldsymbol{\mathcal{H}}$ corresponding to the nonnegative spectrum of the Dirac operator (3.16). Then we have $\boldsymbol{\not{P}} = (\mathbf{1} + \boldsymbol{\not{R}})/2$, where $\boldsymbol{\not{R}} := \boldsymbol{\not{D}}|\boldsymbol{\not{D}}|^{-1}$ is as in Proposition 3.3.5.

Definition 3.4.1. The **Toeplitz algebra** of $(A, \mathbb{R}^n, \alpha)$ is the C^*-subalgebra $\boldsymbol{\mathcal{T}}$ of $\mathcal{B}(\mathcal{H})$ generated by $\mathrm{M}_N(\boldsymbol{B})$ together with elements of the form

$$T_a := \boldsymbol{\not{P}}\boldsymbol{\pi}_\alpha(u)\boldsymbol{\not{P}}$$

for $a \in A^\sim$.

Remark 3.4.2. The kernel of the "symbol map" from $\boldsymbol{\mathcal{T}}$ to A, sending T_a to a, contains the ideal (the "semicommutator ideal") generated by $\{T_aT_b - T_{ab}|\ a, b \in A^\sim\}$. In order to obtain an extension by B instead of this semicommutator ideal, the approach of several

authors [63], [79] has been to replace the projection P onto the nonnegative spectrum of $\not{D}$ by a smoothened version $h(D)$ (for $n = 1$ where $\not{D} = -D := -D_1$), where for some fixed $\varepsilon > 0$, the function $h : \mathbb{R} \to [0, 1]$ is required to be smooth and such that $h(t) = 0$ for $t \leq -\varepsilon$ and $h(t) = 1$ for $t \geq \varepsilon$. One then gets a smooth Toeplitz algebra $\mathcal{T}(\varepsilon)$ depending on ε which is an extension of A by B.

Nevertheless, the result of [98, Lemma 3.2] together with [79, Prop. 3.3, 1] (which we have generalized in Corollary 3.3.5 using the doubling-up construction) shows that, for $n = 1$ (resp. any n), there is no need to use the smoothened version $\mathcal{T}(\varepsilon)$, since $\mathbf{M}_N(B)$ is an ideal (see Prop. 3.4.3 below) in the "true" Toeplitz algebra $\mathcal{T}$ and equal to the semicommutator ideal. The smoothened version has nevertheless been useful; it is instrumental in the proof of the index formula in [79]. The operator $\boldsymbol{\not{D}}$ in (3.16), where $\not{D}$ is defined in (3.15), has a gap $[-m, +m]$ in the spectrum. Hence, using (3.16) the smoothened version $h(\not{D})$ and the true projection $\boldsymbol{\not{P}}$ would actually coincide, provided $\varepsilon < m$.

Proposition 3.4.3. *There is a semisplit short exact sequence*

$$0 \longrightarrow \mathrm{M}_N(\boldsymbol{B}) \longrightarrow \boldsymbol{\mathcal{T}} \longrightarrow A \longrightarrow 0.$$

Proof. Regard $\mathrm{M}_N(\boldsymbol{B})$ as a subalgebra of $B \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on some infinite-dimensional Hilbert space. From Proposition 3.3.5 we know that $\boldsymbol{\not{R}}$ and $\boldsymbol{\not{P}}$ are multipliers of $B \otimes \mathcal{K}$. So we have a projection $\boldsymbol{\not{P}} \in \mathcal{M}(B \otimes \mathcal{K})$ with $[\boldsymbol{\not{P}}, \boldsymbol{\pi}_\alpha(A)] \subset B \otimes \mathcal{K}$. We know that this characterizes an invertible extension. The Busby invariant of this extension (cf. Lemma 2.4.34) is given by $\boldsymbol{\gamma}_\alpha(a) := q(\boldsymbol{\not{P}}\boldsymbol{\pi}_\alpha(a)\boldsymbol{\not{P}})$, where $q : \mathcal{M}(B \otimes \mathcal{K}) \to \mathcal{Q}(B \otimes \mathcal{K})$ is the Calkin map. The proof is complete by noticing that the pullback C^*-algebra associated to $\boldsymbol{\gamma}_\alpha$ as in formula (2.16),

$$\boldsymbol{\mathcal{T}} \cong \{(T, a) \in \mathcal{M}(B \otimes \mathcal{K}) \oplus A |\ q(T) = \boldsymbol{\gamma}_\alpha(a)\},$$

is indeed the Toeplitz C^*-algebra. □

We refer to the exact sequence in Proposition 3.4.3 as the **Toeplitz extension** of $(A, \mathbb{R}^n, \alpha)$. It determines an element of $\mathrm{Ext}(A, B)^{-1}$, the group of semisplit extensions of A by $B \otimes \mathcal{K}$. Recall Lemma 2.4.34, which says that

$$KK^1(A, B) \cong \mathrm{Ext}(A, B)^{-1},$$

where $\mathrm{Ext}(A,B)^{-1}$ is the group if invertible elements in the semigroup $\mathrm{Ext}(A,B)$ of extensions of A by B. As we saw in the proof of Proposition 3.4.3, the Busby invariant of the Toeplitz extension from Proposition 3.4.3 is given by

$$\boldsymbol{\gamma}_\alpha(a) := q(\boldsymbol{P}\!\!\!/\,\boldsymbol{\pi}_\alpha(a)\boldsymbol{P}\!\!\!/\,).$$

For odd n, it follows that the class of the Toeplitz extension identitifes with the element of $KK^1(A,B)$ denoted by $\mathbf{t}_\alpha$ in Proposition 3.3.5. We shall see in the next section that $\mathbf{t}_\alpha$ is in fact the Thom element for $(A,\mathbb{R}^n,\alpha)$, both for even and odd n.

Remark 3.4.4. By Proposition 3.3.5, the ideal of $\boldsymbol{\mathcal{T}}$ generated by the elements

$$\boldsymbol{P}\!\!\!/\,\boldsymbol{\pi}_\alpha(a)\boldsymbol{\pi}_\alpha(b)\boldsymbol{P}\!\!\!/ - \boldsymbol{P}\!\!\!/\,\boldsymbol{\pi}_\alpha(a)\boldsymbol{P}\!\!\!/\,\boldsymbol{\pi}_\alpha(b)\boldsymbol{P}\!\!\!/\,, \qquad a,b \in A^\sim$$

coincides with $\mathrm{M}_N(\boldsymbol{B})$, which is another way of seeing that $\mathrm{M}_N(\boldsymbol{B})$ is an ideal in $\boldsymbol{\mathcal{T}}$.

Lemma 3.4.5. *The operator $T_a \in \boldsymbol{\mathcal{T}}$ is Fredholm as an operator on $\mathrm{M}_N(\boldsymbol{B})$ iff a is invertible in $A^\sim$.*

Proof. From Proposition 3.4.3 it follows that if T_a is invertible modulo $\mathrm{M}_N(\boldsymbol{B})$ then a is invertible. Conversely, if $u \in \mathcal{A}^\sim$ is invertible then, since $[\boldsymbol{P}\!\!\!/\,,\boldsymbol{\pi}_\alpha(u)] \in \mathrm{M}_N(\boldsymbol{B})$ by Lemma 3.3.3, we get

$$(\boldsymbol{P}\!\!\!/\,\boldsymbol{\pi}_\alpha(u)\boldsymbol{P}\!\!\!/\,)(\boldsymbol{P}\!\!\!/\,\boldsymbol{\pi}_\alpha(u^{-1})\boldsymbol{P}\!\!\!/\,) \equiv \boldsymbol{P}\!\!\!/ \mod \mathrm{M}_N(\boldsymbol{B}),$$

and similarly for $u \leftrightarrow u^{-1}$. Now $\boldsymbol{P}\!\!\!/$ is the identity in $\boldsymbol{P}\!\!\!/\,\mathrm{M}_N(\boldsymbol{\mathcal{N}})\boldsymbol{P}\!\!\!/$. $\square$

We can then define a $K_0(B)$-valued index for $\mathrm{M}_N(\boldsymbol{B})$-relative Fredholm operators in $\boldsymbol{\mathcal{T}}$.

3.5 The Thom class

In this section we show that $(\pi_\alpha, \mathrm{M}_N(B), F\!\!\!/\,)$ is a representative of the Thom class for the C^*-dynamical system $(A,\mathbb{R}^n,\alpha)$.

We want the construction of the Kasparov A-B-module $(\pi_\alpha, \mathrm{M}_N(B), F\!\!\!/\,)$ from the data $(A,\mathbb{R}^n,\alpha)$ to be "compatible" with Takesaki-Takai duality. Otherwise the class in $KK^\bullet(B,A)$ associated with the dual dynamical system $(B,\hat{\mathbb{R}}^n,\hat{\alpha})$ will not be an inverse for the class

$[\pi_\alpha, \mathrm{M}_N(B), \not{F}] \in KK^\bullet(A,B)$. To have this compatibility we need to be a little bit more cunning and distinguish between $\mathbb{R}^n$-actions and actions by the dual group $\hat{\mathbb{R}}^n$. Thus, we make the following convention.

Definition 3.5.1. As before, let $\mathbb{C}_n$ be the complex Clifford algebra associated with the vector space $\mathbb{R}^n$ equipped with the standard Euclidean inner product $\langle\cdot|\cdot\rangle$. We let $\mathbb{C}_{-n}$ be complex Clifford algebra associated with $(\mathbb{R}^n, -\langle\cdot|\cdot\rangle)$, i.e. with $\mathbb{R}^n$ equipped with the negative inner product. Let $\hat{\gamma}^1, \dots, \hat{\gamma}^n$ be the skew-Hermitian generators of the irreducible representation of $\mathbb{C}_{-n}$ on $\mathbb{C}^N$.

Let $(B, \hat{\mathbb{R}}^n, \hat{\alpha})$ be a C^*-dynamical system. Let $X_1, \dots, X_n$ be the generators of the unitary group implementing $\hat{\alpha}$ in the representation $\pi_{\hat{\alpha}}$. We define the Kasparov B-$(B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n)$-module

$$(\pi_{\hat{\alpha}}, \mathrm{M}_N(B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n), \not{X}(\mathbf{1} + \not{X}^2)^{-1/2})$$

just as the module $(\pi_\alpha, \mathrm{M}_N(A \rtimes_\alpha \mathbb{R}^n), \not{D}(\mathbf{1} + \not{D}^2)^{-1/2})$ was defined in the case of $\mathbb{R}^n$-actions (§3.3), but now with

$$\not{X} := \sqrt{-1} \sum_{k=1}^{n} \hat{\gamma}^k X_k \tag{3.18}$$

playing the role of $\not{D}$.

The operator $\hat{\gamma}^k$ on $\mathbb{C}^N$ anticommutes with γ^j for each $j, k = 1, \dots, n$. We shall see in the proof of the following why that is important.

Proposition 3.5.2. *The class of the Kasparov A-B module $(\pi_\alpha, \mathrm{M}_N(B), \not{F})$ is the Thom class of the C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$. Namely, the class $[\pi_\alpha, \mathrm{M}_N(B), \not{F}] \in KK^\bullet(A,B)$ is a KK-equivalence between A and B of degree shift n, whose inverse is given by the class $[\pi_{\hat{\alpha}}, \mathrm{M}_N(B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n), \not{X}(\mathbf{1} + \not{X}^2)^{-1/2}] \in KK^\bullet(B, A)$.*

Proof. Let $\mathbf{t}_\alpha := [\pi_\alpha, \mathrm{M}_N(B), \not{F}]$ and $\hat{\mathbf{t}}_\alpha = \mathbf{t}_{\hat{\alpha}} := [\pi_{\hat{\alpha}}, \mathrm{M}_N(B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n), \not{X}(\mathbf{1} + \not{X}^2)^{-1/2}]$. We have to show that $\mathbf{t}_\alpha$ satisfies the axioms of the Thom class similar to those stated in Theorem 3.2.26. Most of the proof is very similar to the case $n = 1$ but worth spelling out in detail.

Normalization. Let $A = \mathbb{C}$, so that α is the trivial $\mathbb{R}^n$-action and $B = C_0(\mathbb{R}^n)$. We need to show that

$$\mathbf{t}_\alpha \otimes_{\mathbb{C}} \hat{\mathbf{t}}_\alpha = 1_{\mathbb{C}}, \qquad \hat{\mathbf{t}}_\alpha \otimes_{\mathbb{C}} \mathbf{t}_\alpha = 1_B. \tag{3.19}$$

But in this case, $\mathbf{t}_\alpha$ and $\hat{\mathbf{t}}_\alpha$ are the "Dirac" and "Dirac-dual" elements for $\mathbb{R}^n$ [68, Def. 4.2] and the equalities (3.19) are equivelent to Bott periodicity in KK [66, Thm. 5.7]. So the result is well known. Let us just sketch the idea, so that we see the motivation for Definition 3.5.1.

Both B and the iterated crossed product $B \rtimes_{\hat{\alpha}} \mathbb{R}^n \cong \mathcal{K}$ act on $L^2(\mathbb{R}^n)$. Let $X_1, \ldots, X_n$ be the generators of the unitary group implementing the dual action $\hat{\alpha}$ in $L^2(\mathbb{R}^n)$, which is the action of $\mathbb{R}^n$ by translations on B.

As defined in the last section, the element $\mathbf{t}_\alpha$ is represented by $(\pi_\alpha, \mathcal{H}, \not{D})$, where $\not{D} = \sum_k \gamma^k D_k$ for unbounded selfadjoint operators $D_1, \ldots, D_n$ on $L^2(\mathbb{R}^n)$ such that $[X_j, D_k] = \sqrt{-1}$. Consider the operator

$$\not{K} := \not{D} + \not{X}.$$

Definition 3.5.1 ensures that $\not{K}^2$ is (minus a bounded normal operator) the n-dimensional harmonic oscillator, which has discrete spectrum. In particular, $(\mathbf{1} + \not{K})^{-1}$ is compact. Thus $(\pi_{\mathbb{C}}, \mathcal{H}, \not{K}(\mathbf{1} + \not{K}^2)^{-1/2})$ is a Kasparov $\mathbb{C}$-$\mathbb{C}$-module, where $\pi_{\mathbb{C}}(\lambda) := \lambda\mathbf{1}$. In fact, $(\pi_{\mathbb{C}}, \mathcal{H}, \not{K}(\mathbf{1}+\not{K}^2)^{-1/2})$ represents the Kasparov product $\mathbf{t}_\alpha \otimes_B \hat{\mathbf{t}}_\alpha$ [66, Thm. 5.7]. Moreover, $\not{K}$ is surjective and its kernel is the 1-dimensional subspace spanned by the vector $\xi_0(t) := e^{-|t|^2}$. So $\not{K}$ is Fredholm, with Fredholm index 1, and it represents the generator $[1_{\mathbb{C}}] \in KK^\bullet(\mathbb{C}, \mathbb{C})$. So $\mathbf{t}_\alpha \otimes_B \hat{\mathbf{t}}_\alpha = 1_{\mathbb{C}}$. The second equality in (3.19) follows from a version of Atiyah's rotation trick [5] or, alternatively, Takesaki-Takai duality (cf. below in the last paragraph in this proof). $\square$

Naturality. Let $\rho : (A, \alpha) \to (A', \alpha')$ be an equivariant homomorphism of C^*-dynamical systems. As in Lemma 3.2.8, define the $*$-homomorphism $\hat{\rho} : B \to B'$ of the crossed products $B := A \rtimes_\alpha \mathbb{R}^n$ and $B' := A' \rtimes_{\alpha'} \mathbb{R}^n$ by formula (3.6). We need to show that

$$\hat{\rho}_*[\pi_\alpha, \mathrm{M}_N(B), \not{F}] := [\pi_\alpha \otimes \mathrm{id}, \mathrm{M}_N(B) \otimes_{\hat{\rho}} \mathrm{M}_N(B'), \not{F} \otimes \mathbf{1}]$$

coincides with

$$\rho^*[\pi_{\alpha'}, \mathrm{M}_N(B'), \not{F}'] := [\pi_{\alpha'} \circ \rho, \mathrm{M}_N(B'), \not{F}'].$$

By definition of the balanced tensor product, $\mathrm{M}_N(B) \otimes_{\hat{\rho}} \mathrm{M}_N(B') = \overline{\hat{\rho}(\mathrm{M}_N(B))}$ is the closed right ideal in $\mathrm{M}_N(B')$ generated by $\hat{\rho}(\mathrm{M}_N(B))$ and $\pi \otimes \mathrm{id}$ becomes the representation $\hat{\rho} \circ \pi_\alpha$. Now recall that $\hat{\rho} \circ \pi_\alpha = \pi_{\alpha'} \circ \rho$. $\square$

Compatibility with external products. We need to show that, for all $\mathbf{x} \in KK^0(A', A)$ and $\mathbf{y} \in KK^0(C', C)$,

$$\mathbf{y} \boxtimes (\mathbf{x} \otimes_A \mathbf{t}_\alpha) = (\mathbf{y} \boxtimes \mathbf{x}) \otimes_{C \otimes A} \mathbf{t}_{\mathrm{id}_C \otimes \alpha}.$$

This property is clearly satisfied by $\mathbf{t}_\alpha = [\pi_\alpha, \mathrm{M}_N(B), \not{F}]$. For instance, take $C = C'$ and $\mathbf{y} = 1_C$. Then $1_C \boxtimes \mathbf{t}_\alpha = \mathbf{t}_{\mathrm{id}_C \otimes \alpha}$. The general case follows by definition of $\boxtimes$. □

Thus, we have shown that $\mathbf{t}_\alpha := [\pi_\alpha, \mathrm{M}_N(B), \not{F}]$ and its dual $\hat{\mathbf{t}}_\alpha := [\pi_{\hat{\alpha}}, \mathrm{M}_N(B \rtimes_{\hat{\alpha}} \hat{\mathbb{R}}^n), \not{X}(\mathbf{1} + \not{X}^2)^{-1/2}]$ satisfy the higher-dimensional analogue of the Fack-Skandalis axioms for the Thom elements. The next task is to show that these axioms implies that $\mathbf{t}_\alpha$ is a KK-equivalence. The proof [14, §19.3], [107, Thm. 2.3], [47] that $\mathbf{t}_\alpha$ is a KK-equivalence carries over completely. For completeness we reproduce the details.

For each $\lambda \in [0, 1]$ we have the rescaled $\mathbb{R}^n$-action

$$\mathbb{R} \ni t \to \alpha_t^\lambda := \alpha_{\lambda t},$$

where $\lambda(t_1, \dots, t_n) := (\lambda t_1, \dots, \lambda t_n)$. We note that for $\lambda = 1$ we have the original action α while α^0 is the trivial action. Consider the C^*-algebra $A' = C([0, 1], A)$ and the $\mathbb{R}^n$-action

$$(\alpha'_t(f))(\lambda) := \alpha_t^\lambda(f(\lambda))$$

on $A' = C([0, 1]) \otimes A$. We use the shorthand notation $B' := A' \rtimes_{\alpha'} \mathbb{R}^n$ and $\mathbf{t}'_\alpha := \mathbf{t}_{\alpha'}$. The evaluation $\rho_\lambda : A' \to A$, given by $\rho_\lambda(f) := f(\lambda)$, is equivariant: $\alpha^\lambda \circ \rho_\lambda = \alpha'$. By naturality of the Thom elements, we therefore have

$$(\rho_\lambda)_*(\hat{\mathbf{t}}'_\alpha) = \hat{\rho}_\lambda^*(\hat{\mathbf{t}}_{\alpha^\lambda}), \qquad (\hat{\rho}_\lambda)_*(\mathbf{t}'_\alpha) = \rho_\lambda^*(\mathbf{t}_{\alpha^\lambda}).$$

Recall from Remark 2.4.25 and Remark 2.4.24 that the maps ρ_λ^* and $(\rho_\lambda)_*$ are given by left and right Kasparov product with a class $[\rho_\lambda] \in KK^0(A', A)$. So by associativity of the

Kasparov product, we have

$$\begin{aligned}(\rho_\lambda)_*(\mathbf{t}'_\alpha \otimes_{B'} \hat{\mathbf{t}}'_\alpha) &= \mathbf{t}'_\alpha \otimes_{B'} (\rho_\lambda)_*(\hat{\mathbf{t}}'_\alpha)\\ &= \mathbf{t}'_\alpha \otimes_{B'} \hat{\rho}^*_\lambda(\hat{\mathbf{t}}_{\alpha^\lambda})\\ &= (\hat{\rho}_\lambda)_*(\mathbf{t}'_\alpha) \otimes_{B'} \hat{\mathbf{t}}_{\alpha^\lambda}\\ &= \rho^*_\lambda(\mathbf{t}_{\alpha^\lambda}) \otimes_{B'} \hat{\mathbf{t}}_{\alpha^\lambda}\\ &= \rho^*_\lambda(\mathbf{t}_{\alpha^\lambda} \otimes_{B'} \hat{\mathbf{t}}_{\alpha^\lambda})\\ &= [\rho_\lambda] \otimes_A (\mathbf{t}_{\alpha^\lambda} \otimes_{B'} \hat{\mathbf{t}}_{\alpha^\lambda}).\end{aligned}$$

The family $(\rho_\lambda)_{\lambda\in[0,1]}$ is a continuous homotopy so each ρ_λ induces the same map $(\rho_\lambda)_*$ on KK. So for any $\lambda \in [0,1]$ we have

$$(\mathbf{t}'_\alpha \otimes_{B'} \hat{\mathbf{t}}'_\alpha) \otimes_{A'} [\rho_0] = [\rho_0] \otimes_A (\mathbf{t}_{\alpha^\lambda} \otimes_{B'} \hat{\mathbf{t}}_{\alpha^\lambda}).$$

Now $\rho_0(f) = f(0)$ is the evaluation at the endpoint, and the map $\iota(a) := a \otimes \mathbf{1}$ is a homotopy inverse to ρ_0. So we have an inverse $\iota^* = [\iota]\otimes_{A'}$ to $\rho_0^* = [\rho_0]\otimes_A$, and we can make the rearrangement

$$[\iota] \otimes_{A'} (\mathbf{t}'_\alpha \otimes_{B'} \hat{\mathbf{t}}'_\alpha) \otimes_{A'} [\rho_0] = \mathbf{t}_{\alpha^\lambda} \otimes_B \hat{\mathbf{t}}_{\alpha^\lambda}.$$

The left-hand side is independent of $\lambda \in [0,1]$, so the right-hand side must be independent of $\lambda \in [0,1]$ as well. But for $\lambda = 0$ we know from the normalization and naturality axioms that $\mathbf{t}_{\alpha^0} \otimes_B \hat{\mathbf{t}}_{\alpha^0} = 1_A$. At $\lambda = 1$ we obtain the desired result $\mathbf{t}_\alpha \otimes_B \hat{\mathbf{t}}_\alpha = 1_A$.

Using Takesaki-Takai duality we have $\hat{\hat{\mathbf{t}}}_\alpha = \mathbf{t}_\alpha \otimes_{\mathbb{C}} 1_{\mathcal{K}}$, so by replacing α with $\hat{\alpha}$ we obtain $\hat{\mathbf{t}}_\alpha \otimes_A \mathbf{t}_\alpha = 1_B$. That finishes the proof. □

3.6 Numerical index

In this section we will prove a formula in the spirit of [27, 79, 98] for the $\hat{\tau}$-index of Toeplitz operators $T_u = P\pi_\alpha(u)P$. We adopt the powerful approach to the case $n = 1$ given in the recent paper [27]. In particular, we will need the local index formula for nonunital semifinite spectral triples described in §2.8.

We want to apply the general version of the local index formula (recall Section 2.8)

to compute the $\hat{\tau}$-index. Thus, first of all we need to find a nonzero $*$-algebra $\mathcal{C} \subset A$ which gives a smoothly summable $(\mathcal{N}, \hat{\tau})$-semifinite spectral triple. Recall that for smooth summability (Definition 2.6.10) we need both a suitable smoothness property of elements a in $\mathcal{C}$ with respect $\not{D}$, as well as a $(\hat{\tau}, \not{D})$-integrability condition on $\pi_\alpha(a)$.

As expected, in our setting the smoothness with respect to $\not{D}$ is tightly related to the smoothness with respect to the $\mathbb{R}^n$-action α. In fact, we get as in [27, Prop. 3.12] that if an element $a \in A$ is smooth for the generator δ of α then $\pi_\alpha(a)$ is smooth for $\not{\partial}$. Since we also need an integrability condition we will not be able to use all of $\mathcal{A}$.

First we shall discuss how integrability properties will be affected by the choice of Hilbert space. Remember that A is acting on a Hilbert space $\mathfrak{H}$ and that

$$\pi_\alpha : A \to L^2(\mathbb{R}^n, \mathfrak{H})$$

is defined in terms of $\mathfrak{H}$. Recall that the dual trace $\hat{\tau}$ on $\mathcal{N}$ is defined in terms of the Hilbert algebra (see Remark 3.2.17)

$$\mathfrak{A}_\tau := L^2(\mathbb{R}^n, \mathfrak{H}_\tau) \cap L^1(\mathbb{R}^n, \mathrm{Dom}(\tau)), \tag{3.20}$$

where $\mathfrak{H}_\tau$ is the GNS space of τ. It is therefore natural to want π_α to be a representation on $L^2(\mathbb{R}^n, \mathfrak{H}_\tau)$, and this was the approach in [27]. The action α is then required to preserve the trace, or else it will not have a unitary implementation. However, the dual trace can also be described (see Definition 3.2.14) as the composition of τ, π_α^{-1} and the operator-valued weight E and as we shall see, we do not need $\mathfrak{H}$ to be $\mathfrak{H}_\tau$. (Again, we *do* assume that α preserves τ in this work but we aim for some flexibility in the choice of $\mathfrak{H}$ that could be useful in the future.) The reason for this is the isomorphism $\mathcal{N} \cong \hat{\pi}_\alpha(\mathfrak{A}_\tau)''$ [112, Lemma X.1.15], where $\mathfrak{A}_\tau$ is the left Hilbert algebra (3.20) which completely defines $\hat{\tau}$.

Again we use $\pi_\alpha(a)$ to denote $1_N \otimes \pi_\alpha(a)$ for $a \in A$ and we write $\hat{\tau}$ for $\mathrm{Tr} \otimes \hat{\tau}$ where Tr is the matrix trace on $\mathrm{M}_N(\mathbb{C})$.

Let A'' be the weak closure of A in its original representation. Then τ extends to a normal trace $\bar{\tau}$ on A'' with the same GNS space as τ. The following lemma is the counterpart of [27, Lemma 3.4].

Lemma 3.6.1. *Let $h \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and let $a \in A''$ be such that a^*a is in* $\mathrm{Dom}(\bar{\tau})$.

If we define

$$x(t) := ah(t),$$

then $\hat{\pi}_\alpha(x) \in \mathcal{N}$ is $\hat{\tau}$-Hilbert-Schmidt and

$$\hat{\tau}(\hat{\pi}_\alpha(x)^*\hat{\pi}_\alpha(x)) = \tau(a^*a)\int_{\mathbb{R}^n} |h(t)|^2\,dt.$$

Proof. We write $\hat{\pi}_\alpha(x) = \int \pi_\alpha(a)h(s)e^{-2\pi i s\cdot D}\,ds$ so that

$$\hat{\alpha}_p(\hat{\pi}_\alpha(x)) = \int_{\mathbb{R}^n} \pi_\alpha(a)h(s)e^{-2\pi i p\cdot s}e^{-2\pi i s\cdot D}\,ds, \qquad \forall p \in \mathbb{R}^n.$$

Since $\hat{\tau} = \bar{\tau}\circ\pi_\alpha^{-1}\circ E$, the assumptions on x give

$$\begin{aligned}
\hat{\tau}(\hat{\pi}_\alpha(x)^*\hat{\pi}_\alpha(x)) &= \bar{\tau}\circ\pi_\alpha^{-1}\Big(\int_{\mathbb{R}^n}\hat{\alpha}_p(\hat{\pi}_\alpha(x^*x))\,dp\Big)\\
&= \bar{\tau}\circ\pi_\alpha^{-1}\Big(\iint_{\mathbb{R}^n\times\mathbb{R}^n}\pi_\alpha(a^*a)|h(s)|^2e^{-2\pi i p\cdot s}e^{-2\pi i s\cdot D}\,ds\,dp\Big)\\
&= \bar{\tau}\circ\pi_\alpha^{-1}\circ\pi_\alpha(a^*a)|h(0)|^2\\
&= \bar{\tau}(a^*a)\int_{\mathbb{R}^n}|\hat{h}(p)|^2\,dp\\
&= \bar{\tau}(a^*a)\int_{\mathbb{R}^n}|h(t)|^2\,dt
\end{aligned}$$

where we used Plancherel's formula in the last line. □

Corollary 3.6.2. *Let $s > n$ and define a weight φ_s on $\mathcal{N}$ by setting*

$$\varphi_s(T) := \hat{\tau}((1+\not{D}^2)^{-s/4}T(1+\not{D}^2)^{-s/4})$$

for all $T \in \mathcal{N}_+$. Then the restriction of φ_s to $\mathcal{M} := \pi_\alpha(A)''$, viewed as a subalgebra of $\mathcal{N}$, is proportional to $\bar{\tau}\circ\pi_\alpha^{-1}$.

Proof. From the Clifford relations we get

$$\not{D}^2 = \sum_{k=1}^n \mathbf{1}\otimes D_k^2,$$

and so if $h_s(t) := (1+|t|^2)^{-s/2}$ then by Lemma 3.6.1 we have for each positive a in the

domain of τ that

$$\begin{aligned}\varphi_s(\pi_\alpha(a)) &= \hat{\tau}((1+\not{D}^2)^{-s/2}\pi_\alpha(a))\\ &= \hat{\tau}(h_s(\not{D})\pi_\alpha(a))\\ &= \mathrm{Tr}(\mathbf{1}_N)\tau(a)\int_{\mathbb{R}^n}|h_s(t)|^2\,dt.\end{aligned}$$

It follows that more generally that $\varphi_s = \|h_s\|_2^2\bar{\tau}$ holds on $\pi_\alpha(\mathrm{Dom}(\bar{\tau})_+) \subset \mathcal{M}_+$. That $\varphi_s(a) = +\infty$ whenever $\bar{\tau}(a) = +\infty$ can be seen as in Corollary 3.5 of [27]. □

Let δ_τ denote the restriction of the generators $\delta = (\delta_1, \ldots, \delta_n)$ to $\mathrm{Dom}(\tau)$ and let $\mathrm{Dom}(\delta_\tau) \subset \mathrm{Dom}(\tau)$ denote the domain of δ_τ.

Lemma 3.6.3. *Let $\mathcal{C}$ be the $*$-subalgebra of $\mathrm{Dom}(\tau)$ generated by the set*

$$\{a = bc \in \mathcal{A}|\ \delta^k(b), \delta^k(c) \in \mathrm{Dom}(\delta_\tau) \text{ for all } k \in \mathbb{N}_0\}. \tag{3.21}$$

Then $(\mathcal{C}, \mathcal{H}, \not{D})$ is a $(\mathcal{N}, \hat{\tau})$-semifinite spectral triple which is smoothly summable. That is, there is a $p \geq 1$ such that $\pi_\alpha(a)$ and $[\not{D}, \pi_\alpha(a)]$ belong to $\mathcal{B}_1^k(\not{D}, \hat{\tau}, p)$ for all $k \geq 0$ and all $a \in \mathcal{C}$.

Proof. We anticipate that $p = n$ will suffice. Consider the unbounded operator L on $\mathcal{N}$ given by

$$L(T) := (1+\not{D}^2)^{-1/2}[\not{D}^2, T]$$

for $T \in \mathcal{N}$. We shall use Proposition 2.6.16(ii) to conclude the result. We begin by showing that $\pi_\alpha(\mathcal{A})$ is contained in the smooth domain of L.

Write $\not{F} := \not{D}(1+\not{D}^2)^{-1/2} = (1+\not{D}^2)^{-1/2}\not{D}$ and $F_k := D_k(1+\not{D}^2)^{-1/2}$ for $k = 1, \ldots, n$.

Using Proposition 3.3.1 and we see that

$$\begin{aligned}
L(\pi_\alpha(a)) &= (1+\not{D}^2)^{-1/2}(\not{D}[\not{D},\pi_\alpha(a)] + [\not{D},\pi_\alpha(a)]\not{D}) \\
&= \frac{1}{2\pi i}\sum_{k=1}^n \not{F}\gamma^k \pi_\alpha(\delta_k(a)) + \frac{1}{2\pi i}(1+\not{D}^2)^{-1/2}\sum_{k=1}^n \gamma^k\pi_\alpha(\delta_k(a))\not{D} \\
&= \frac{1}{2\pi i}\sum_{k=1}^n \not{F}\gamma^k \pi_\alpha(\delta_k(a)) + \frac{1}{2\pi i}(1+\not{D}^2)^{-1/2}\sum_{k=1}^n \big([\gamma^k\pi_\alpha(\delta_k(a)),\not{D}] + \not{D}\gamma^k\pi_\alpha(\delta_k(a))\big) \\
&= \frac{1}{\pi i}\sum_{k=1}^n \not{F}\gamma^k \pi_\alpha(\delta_k(a)) + \frac{1}{4\pi^2}(1+\not{D}^2)^{-1/2}\sum_{l,k=1}^n \big(\gamma^l\gamma^k\pi_\alpha(\delta_l\delta_k(a)) + [\gamma^l,\gamma^k]D_k\pi_\alpha(\delta_k(a))\big) \\
&= \frac{1}{\pi i}\sum_{k=1}^n \big(\not{F}\gamma^k + [\gamma^l,\gamma^k]F_k\big)\pi_\alpha(\delta_k(a)) + \frac{1}{4\pi^2}(1+\not{D}^2)^{-1/2}\sum_{l,k=1}^n \gamma^l\gamma^k\pi_\alpha(\delta_l\delta_k(a))
\end{aligned}$$

whenever a belongs to the domain of δ^2.

This shows that $\mathrm{Dom}(L) \subset \pi_\alpha(\mathrm{Dom}(\delta^2))$. Since L^j is defined using the derivation $[\not{D}^2,\cdot]$, for all $f,g \in L^\infty(\mathbb{R}^n)$ and $T \in \mathrm{Dom}(L^j)$ one has

$$L^j(f(D)Tg(D)) = f(D)L^j(T)g(D).$$

Note that $L(\pi_\alpha(a))$ is of the form $f(D)Tg(D)$ with $f,g \in L^\infty(\mathbb{R}^n)$ and $T \in \mathrm{Dom}(L)$. Thus, the action of L^j for $j \in \mathbb{N}$ can be deduced by just repeating the above calculation with $\delta_k(a)$ and $\delta_l\delta_k(a)$ instead of a, provided that a belongs to the domain of δ^{2j}. Thus, an element a in $\bigcap_{r\in\mathbb{N}}\mathrm{Dom}(\delta^r) = \mathcal{A}$ will belong to $\mathrm{Dom}(L^j)$ for all $j \in \mathbb{N}$.

From Corollary 3.6.2 we have

$$\pi_\alpha(\mathrm{Dom}^{1/2}(\tau)) \subset \mathcal{B}_2(\not{D},\hat{\tau},n).$$

From [28, Prop. 1.19] we know that, since each φ_s is tracial on $\mathcal{M} := \pi_\alpha(A)''$, the space $\mathcal{B}_1(\not{D},\hat{\tau},n)\cap\mathcal{M}$ is equal to the intersection of trace-ideals $\mathcal{L}^1(\mathcal{M},\varphi_s) = \mathrm{Dom}(\varphi_s)$,

$$\mathcal{B}_1(\not{D},\hat{\tau},n)\cap\mathcal{M} = \bigcap_{s>n}\mathcal{L}^1(\mathcal{M},\varphi_s).$$

But φ_s is proportional to $\bar{\tau}\circ\pi_\alpha^{-1}$ on $\mathcal{M}$ for all $s > n$, so we obtain

$$\mathcal{B}_1(\not{D},\hat{\tau},n)\cap\mathcal{M} = \pi_\alpha(\mathrm{Dom}(\bar{\tau})).$$

Finally, on the C^*-level this yields

$$\mathcal{B}_1(\not{D}, \hat{\tau}, n) \cap \pi_\alpha(A) = \pi_\alpha(\mathrm{Dom}(\tau)).$$

In particular, $\pi_\alpha(\mathcal{C}) \subset \mathcal{B}_1(\not{D}, \hat{\tau}, n)$. From the definition of $\mathcal{C}$ and our calculation of $L(\pi_\alpha(\mathcal{C}))$ it follows that $L^k(\pi_\alpha(\mathcal{C})) \subset \mathcal{B}_1(\not{D}, \hat{\tau}, n)$ for all $k \in \mathbb{N}$. Moreover, we have $[\not{D}, \pi_\alpha(a)] \in \mathrm{M}_N(\mathbb{C}) \otimes \pi_\alpha(\mathcal{C})$ for all $a \in \mathcal{C}$ (Proposition 3.3.1). Thus, $L^k([\not{D}, \pi_\alpha(\mathcal{C})]) \subset \mathcal{B}_1(\not{D}, \hat{\tau}, n)$ for all $k \in \mathbb{N}$ as well. That finishes the proof. $\square$

Corollary 3.6.4. *For all $a \in \mathcal{C}$ we have*

$$\pi_\alpha(a)(1 + \not{D}^2)^{-s/2} \in \mathcal{L}^1(\mathcal{N}, \hat{\tau}), \qquad \forall s > n.$$

Proof. As we have seen, for $a \in \mathcal{C}$ we have $\pi_\alpha(a) \in \mathcal{B}_1^\infty(\not{D}, \hat{\tau}, n) = \mathrm{OP}_0^0(\not{D}, \hat{\tau}, n)$. Therefore, $\pi_\alpha(a)(1 + \not{D}^2)^{-s/2}$ belongs to $\mathrm{OP}_0^{-s}(\not{D}, \hat{\tau}, n)$ for all s. Now $\mathrm{OP}_0^{-s}(\not{D}, \hat{\tau}, n) \subset \mathcal{L}^1(\mathcal{N}, \hat{\tau})$ for $s > n$ by Lemma 2.6.15. $\square$

In the following we use the notation (3.1).

Proposition 3.6.5. *For n odd and a unitary $u \in \mathcal{C}^\sim$, we have*

$$\mathrm{Index}_{\hat{\tau}}(T_u) = -\frac{2^{(n-1)/2}(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^n n!} \tau\big((u^*\delta(u))^n\big).$$

Proof. The proof of Lemma 3.6.3 shows that $(\mathcal{C}, \mathcal{H}, \not{D})$ has spectral dimension n.

From the odd part of the local index formula in term of the resolvent cocycle (Theorem 2.8.2) we have

$$\mathrm{Index}_{\hat{\tau}}(T_u) = \frac{-1}{\sqrt{2\pi i}} \operatorname*{Res}_{r=(1-n)/2} \sum_{m=1,\text{ odd}}^{n} \Phi_m^r(\mathrm{Ch}_m(u)), \tag{3.22}$$

where we recall that the Chern character $\mathrm{Ch}_m(u)$ is the cycle

$$\mathrm{Ch}_m(u) := (-1)^{(m-1)/2}((m-1)/2)!\, u^{-1} \otimes u \otimes \cdots \otimes u^{-1} \otimes u \in (\mathcal{C}^\sim)^{\otimes(m+1)}.$$

Now $\Phi_m^r(\mathrm{Ch}_m(u))$ is, up to some constants (see Definition 2.8.1), the integral over $s \in \mathbb{R}_+$

of the function

$$(\mathrm{Tr} \otimes \hat{\tau})\Big(\frac{1}{2\pi i} s^m \Gamma \int_{\varepsilon + i\mathbb{R}} \lambda^{-n/2-r} \pi_\alpha(u^{-1}) R_s(\lambda) [\not{D}, \pi_\alpha(u)] R_s(\lambda) \cdots [\not{D}, \pi_\alpha(u)] R_s(\lambda)\, d\lambda\Big)$$

where $R_s(\lambda) := (\lambda - (\mathbf{1} + s^2 + \not{D}^2))^{-1}$. There is a product of m commutators $[\not{D}, \pi_\alpha(u)] = (2\pi i)^{-1} \sum_k \gamma^k [D_k, \pi_\alpha(u)]$ in the above expression, and hence a factor $\mathrm{Tr}(\Gamma \gamma^{k_1} \cdots \gamma^{k_m})$. Only a product of n Clifford generators γ^k has nonzero graded trace [12, Prop. 3.21] (note that $\Gamma = \mathbf{1}$ here because n is odd, but the mentioned fact is true for even n as well). Therefore, only the nth component in right-hand side of (3.22) survives. Theorem 2.8.2 says that the function $r \to \Phi_n^r(\mathrm{Ch}_n(u))$ can be analytically continued to a deleted neighborhood of $r = (1-n)/2$ where it has at worst a simple pole.

The fact that only one term $\Phi_n^r(\mathrm{Ch}_n(u))$ survives and has a well-defined residue at $r = (1-n)/2$ allows the proof of [28, Prop. 3.20] to be carried out without the hypothesis of isolated spectral dimension. The result is that $\mathrm{Index}_\tau(T_u)$ equals $-(2\pi i)^{-1/2}$ times the value of the residue cocycle

$$\phi_n(a_0, a_1, \ldots a_n) := \frac{\sqrt{2\pi i}}{n!} \operatorname*{Res}_{s=n} \hat{\tau}(\boldsymbol{\pi}_\alpha(a_0)[\boldsymbol{\not{D}}, \boldsymbol{\pi}_\alpha(a_1)] \cdots [\boldsymbol{\not{D}}, \boldsymbol{\pi}_\alpha(a_n)](1 + \boldsymbol{\not{D}}^2)^{-n/2-s})$$

on the cycle $\mathrm{Ch}_n(u)$. That is,

$$\mathrm{Index}_{\hat{\tau}}(T_u) = -\frac{(-1)^{(n-1)/2}((n-1)/2)!}{\sqrt{2\pi i}}\, \phi_n(u^*, u, \ldots, u^*, u),$$

Recall the explicit expression for the commutators $[\not{D}, \pi_\alpha(a)]$ from Proposition 3.3.1. In the notation (3.1), Lemma 3.6.1 shows that

$$\phi_n(u^*, u, \ldots, u^*, u) = \frac{2^{(n-1)/2}\sqrt{2\pi i}}{(2\pi i)^n n!} \tau\Big(u^*(\delta(u)\delta(u^*))^{(n-1)/2}\delta(u)\Big) \operatorname*{Res}_{s=n} \int_{\mathbb{R}^n} (1 + |t|^2)^{-s/2}\, dt$$

(the factors of $1/2\pi i$ come from Proposition 3.3.1 while the factor $2^{(n-1)/2}$ is the trace of the product of all γ matrices). Since δ is a derivation, $uu^{-1} = \mathbf{1}$ gives

$$\delta(u^{-1}) = -u^{-1}\delta(u)u^{-1},$$

and so

$$u^*\big((\delta(u)\delta(u^*)\big)^{(n-1)/2}\delta(u) = u^*\big(\delta(u^*)u^*\delta(u)u^*\big)^{(n-1)/2}\delta(u) = (u^*\delta(u))^n,$$

from which

$$\phi_n(u^*, u, \ldots, u^*, u) = \frac{2^{(n-1)/2}\sqrt{2\pi i}}{(2\pi i)^n n!}\tau\big((u^*\delta(u))^n\big)\operatorname*{Res}_{s=n}\int_{\mathbb{R}^n}(1+|t|^2)^{-s/2}\,dt.$$

□

Proposition 3.6.6. *For n even and a projection $e \in \mathcal{C}^\sim$, we have*

$$\mathrm{Index}_{\hat{\tau}}(\boldsymbol{\pi}_\alpha(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_\alpha(e)) = \frac{(-1)^{n/2}}{(n/2)!}\frac{2^n}{(2\pi i)^n}\tau\big((e\delta(e)\delta(e))^{n/2}\big).$$

Proof. For the same reason as in Proposition 3.6.5, we obtain the relation

$$\mathrm{Index}_{\hat{\tau}}(\boldsymbol{\pi}_\alpha(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_\alpha(e)) = \phi_0(e) + \frac{(-1)^{n/2}}{(n/2)!}\phi_n(e, \ldots, e).$$

The zeroth term

$$\phi_0(e) = \operatorname*{Res}_{z=0}\frac{1}{z}\operatorname{Tr}(\boldsymbol{\Gamma}\boldsymbol{\pi}_\alpha(e)(\mathbf{1}+\not{\boldsymbol{D}}^2)^{-z})$$

is 0 because the grading $\Gamma = \mathrm{diag}(\mathbf{1}, -\mathbf{1})$ gives $\operatorname{Tr}(\boldsymbol{\Gamma}\boldsymbol{\pi}_\alpha(e)(\mathbf{1}+\not{\boldsymbol{D}}^2)^{-z}) = 0$. Now the expression from Lemma 3.6.1,

$$\phi_n(e, \ldots, e) = \frac{2^n}{(2\pi i)^n n!}\tau\big(e\delta(e)\cdots\delta(e)\big)\operatorname*{Res}_{s=n}\int_{\mathbb{R}^n}(1+|t|^2)^{-s/2}\,dt,$$

can rearranged using $e(\delta(e))^{n-1} = (e\delta(e)\delta(e))^{n/2}$, which follows from idempotency of e. □

By Proposition 2.6.26, the completion $\mathcal{C}_{\delta,\varphi}$ of $\mathcal{C}$ in the δ-φ-topology is a dense $*$-subalgebra of A such that the inclusion $\mathcal{C}_{\delta,\varphi} \hookrightarrow A$ induces isomorphisms on both K-groups and $(\mathcal{C}_{\delta,\varphi}, \mathcal{H}, \not{D})$ is again a smoothly summable spectral triple over A. Therefore, given *any class* $[x] \in K_\bullet(A)$ there is a representative $x \in \mathcal{C}_{\delta,\varphi}$ such that a matrix analogue of one of the formulas (depending on the parity $\bullet$ of n) in Proposition 3.6.5 or Proposition 3.6.6 holds.

3.7 Another choice of projection

We now construct a Toeplitz extension without doubling up the Hilbert space. We shall use the same notation throughout, since it will be clear from the context which of the Toeplitz extensions is considered. Recall that $\mathcal{H} := \mathbb{C}^N \otimes L^2(\mathbb{R}^n, \mathfrak{H})$.

Let $\mathcal{T}$ be the C^*-subalgebra of $\mathcal{B}(\mathcal{H})$ generated by $\mathrm{M}_N(B)$ and the Toeplitz operators

$$T_a := \not{P}\pi_\alpha(a)\not{P}, \qquad a \in A^\sim,$$

where $\not{P}$ is the spectral projection onto the nonnegative part of the spectrum of the massless Dirac operator $\not{D}$.

Proposition 3.7.1. *For n odd, there is a semi-split extension*

$$0 \longrightarrow \mathrm{M}_N(B) \longrightarrow \mathcal{T} \longrightarrow A \longrightarrow 0.$$

The triple $(\pi_\alpha, \mathrm{M}_N(B), 2\not{P} - \mathbf{1})$ is a Kasparov A-B module representing the same class as the double $(\boldsymbol{\pi}_\alpha, \mathbf{M}_N(B), \not{\boldsymbol{R}})$. In particular, for all $[u] \in K_1(A)$ we have

$$\mathrm{Index}(\not{P}\pi_\alpha(u)\not{P}) = \mathrm{Index}(\not{\boldsymbol{P}}\boldsymbol{\pi}_\alpha(u)\not{\boldsymbol{P}})$$

and for all $[e] \in K_0(A)$ we have

$$\mathrm{Index}(\pi_\alpha(e)(2\not{P} - \mathbf{1})_+\pi_\alpha(e)) = \mathrm{Index}(\boldsymbol{\pi}_\alpha(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}_\alpha(e)).$$

Proof. By Proposition 2.5.19, $(\pi_\alpha, \mathrm{M}_N(B), 2\not{P} - \mathbf{1})$ is a Kasparov A-B module representing the same KK-class as $(\pi_\alpha, \mathrm{M}_N(B), \not{F})$. The extension associated with $(\pi_\alpha, \mathrm{M}_N(B), 2\not{P} - \mathbf{1})$ under the isomorphism $KK^1(A, B) \cong \mathrm{Ext}(A, B)$ gives the Toeplitz extension in the statement. $\square$

Since the KK-classes of $(\boldsymbol{\pi}_\alpha, \mathrm{M}_N(\boldsymbol{B}), \not{\boldsymbol{R}})$ and $(\pi_\alpha, \mathrm{M}_N(B), \not{F})$ coincide. Therefore, $\not{P}$ and $2\not{F} - \mathbf{1}$ also defined the same element in $KK^\bullet(A, B)$, where $\bullet \in \{0, 1\}$ is equal to 0 if n is even and equal to 1 if n is odd.

Chapter 4

Rieffel deformations

With an action α of $\mathbb{R}^n$ on a C^*-algebra A and a skew-symmetric $n \times n$ matrix Θ one can consider the Rieffel deformation A_Θ of A, which is a C^*-algebra generated by the α-smooth elements of A with a new multiplication $\times_\Theta$ depending on Θ. The purpose of this chapter is to obtain explicit formulas for K-theoretical quantities defined by elements of A_Θ. Our approach relies on the smoothly summable spectral triple $(\mathcal{C}, \mathcal{H}, \not{D})$ associated with the C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$ as in the last chapter.

Rieffel showed that the K-theories of A and A_Θ are isomorphic [105]. However, there is no explicit description for the generators of $K_\bullet(A_\Theta)$ even when the generators of $K_\bullet(A)$ are known. A projection $e \in \mathrm{M}_\infty(A_\Theta)$ (so that $e \times_\Theta e = e$) need not be a projection in $\mathrm{M}_\infty(A)$ (i.e. $e^2 = e$ may not hold) and vice versa. Similar remarks hold for unitaries.

We shall use three different pictures of Rieffel deformation, and all of them will be needed to obtain a full understanding of the relation between the index pairings for A and A_Θ.

4.1 Rieffel deformations in three ways

4.1.1 Noncommutative quantization

Motivated by the mathematical theory of quantization, Rieffel introduced a way of deforming a C^*- or Fréchet algebra by changing the multiplication [101, 103]. He shows that the resulting algebra is a C^*-algebra and that the construction is functorial in a certain sense. His approach is very analytical and technical, based on operator-valued oscillatory

integrals.

Let us recall why Rieffel deformations are natural objects to consider. For a nice phase-space function $f : \mathbb{R}^{2n} \to \mathbb{C}$, its "Weyl quantization" $\mathrm{Op}(f)$ is an operator on $L^2(\mathbb{R}^n)$. For $f, g \in \mathcal{S}(\mathbb{R}^{2n})$ (Schwartz space), the product $\mathrm{Op}(f)\,\mathrm{Op}(g)$ is again a Weyl operator. Therefore, composition of Weyl operators defines implicitly a noncommutative product $\times_\Theta$ on an algebra of functions on $\mathbb{R}^n \times \mathbb{R}^n$:

$$\mathrm{Op}(f)\,\mathrm{Op}(g) = \mathrm{Op}(f \star g). \tag{4.1}$$

This product (called the "Moyal product") has an explicit integral formula

$$(f \star g)(t) = \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} f(t + \Theta z) g(t + s) e^{2\pi i z\cdot s}\, dz\, ds$$

where $\Theta = \left(\begin{smallmatrix} 0 & \mathbf{1} \\ -\mathbf{1} & 0 \end{smallmatrix}\right)$ is the standard symplectic structure on $\mathbb{R}^n$. If one defines an action α of $\mathbb{R}^{2n}$ on $\mathcal{S}(\mathbb{R}^{2n})$ by translations,

$$(\alpha_s(g))(t) := g(t + s),$$

then $f \star g$ can be expressed as

$$f \star g = \iint_{\mathbb{R}^{2n}\times\mathbb{R}^{2n}} \alpha_{\Theta z}(f)\alpha_s(g) e^{2\pi i z\cdot s}\, dz\, ds.$$

Rieffel deformation amounts to defining such a deformed product on a general C^*-algebra A. He shows that it has nice properties and makes sense also if A is not commutative.

Let A be a C^*-algebra and let $C_b(\mathbb{R}^n, A)$ be the C^*-algebra of bounded continuous A-valued functions on $\mathbb{R}^n$ equipped with the supremum norm. There is an action of $\mathbb{R}^n$ on $C_b(\mathbb{R}^n, A)$ by translation. The subalgebra $\mathcal{B}^A(\mathbb{R}^n)$ of smooth elements for this action can be deformed in the following way. Let Θ be a fixed real skew-symmetric $n \times n$ matrix. For $f, g \in \mathcal{B}^A(\mathbb{R}^n)$ we define

$$(f \times_\Theta g)(t) := \iint_{\mathbb{R}^n\times\mathbb{R}^n} f(t + \Theta z) g(t + s) e^{2\pi i z\cdot s}\, dz\, ds,$$

where the integral has to be understood in the sense of [101, Prop. 1.6]. Denote by $\mathcal{B}^A_\Theta(\mathbb{R}^n)$ the algebra $\mathcal{B}^A(\mathbb{R}^n)$ equipped with the new multiplication $\times_\Theta$.

Now let $\mathcal{S}^A(\mathbb{R}^n)$ be the space of A-valued Schwartz functions. We let $f \in \mathcal{B}^A_\Theta(\mathbb{R}^n)$ act on $\mathcal{S}^A(\mathbb{R}^n)$ as

$$\pi^\Theta(f)g := f \times_\Theta g, \qquad \forall g \in \mathcal{S}^A(\mathbb{R}^n).$$

There is an A-valued inner product on $\mathcal{S}^A(\mathbb{R}^n)$, given by

$$\langle f|g\rangle_A := \int_{\mathbb{R}^n} f(s)^* g(s)\, ds, \qquad \forall f, g \in \mathcal{S}^A(\mathbb{R}^n). \tag{4.2}$$

We denote by X the completion of $\mathcal{S}^A(\mathbb{R}^n)$ in the norm $\|f\|_A := \sqrt{\langle f|f\rangle_A}$. Then X is a right Hilbert A-module and $\pi^\Theta(f)$ is an adjointable operator on X, with adjoint $\pi^\Theta(f)^* = \pi^\Theta(f^*)$ [101, Prop. 4.2]. Moreover, $\pi^\Theta(f)$ is a bounded operator [101, Thm. 4.6].

Write $\mathcal{B}^A_\Theta(\mathbb{R}^n)$ for the algebra $\mathcal{B}^A(\mathbb{R}^n)$ equipped with the product $\times_\Theta$ and the pre-C^*-norm

$$\|f\|_\Theta := \|\pi^\Theta(f)\|, \qquad \forall f \in \mathcal{B}^A_\Theta(\mathbb{R}^n)$$

where $\|\cdot\|$ is the operator norm on $\mathcal{L}_A(X)$. The completion of $\mathcal{B}^A_\Theta(\mathbb{R}^n)$ in this norm is a C^*-algebra, which we denote by $B^A_\Theta(\mathbb{R}^n)$. Similarly, let $\mathcal{S}^A_\Theta(\mathbb{R}^n)$ be the algebra $\mathcal{S}^A(\mathbb{R}^n)$ regarded as a subalgebra of $\mathcal{B}^A_\Theta(\mathbb{R}^n)$. Then $\mathcal{S}^A_\Theta(\mathbb{R}^n)$ is a pre-C^*-algebra and in fact [101, Prop. 3.3] a $*$-ideal in $\mathcal{B}^A_\Theta(\mathbb{R}^n)$.

Definition 4.1.1 ([101, Def. 4.9])**.** Suppose that $(A, \mathbb{R}^n, \alpha)$ is a C^*-dynamical system and let $\mathcal{A} \subset A$ be the subalgebra of smooth elements for the action α. For $a \in A$, let $\alpha(a) \in B^A_\Theta(\mathbb{R}^n)$ be the function $\alpha(a)(t) := \alpha_{-t}(a)$. For $a \in \mathcal{A}$ we have $\alpha(a) \in \mathcal{B}^A_\Theta(\mathbb{R}^n)$. Let $\pi^\Theta : \mathcal{A} \to \mathcal{L}_A(X)$ be the map which takes $a \in \mathcal{A}$ to the operator $\pi^\Theta(a)$ given by

$$(\pi^\Theta(a)g)(t) := (\alpha(a) \times_\Theta g)(t) = \iint_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{-t+\Theta z}(a) g(t+s) e^{2\pi i z \cdot s}\, dz\, ds$$

for all $g \in \mathcal{S}^A(\mathbb{R}^n)$. The **Rieffel deformation** of A with respect to (α, Θ) is the C^*-algebra A_Θ obtained by completing $\mathcal{A}$ in the norm

$$\|a\|_\Theta := \|\pi^\Theta(a)\|,$$

where $\|\cdot\|$ is the operator norm on $\mathcal{L}_A(X)$.

Thus, A_Θ is a C^*-algebra with multipliaction given by

$$a \times_\Theta b := \iint_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{\Theta z}(a)\alpha_s(b)e^{2\pi i z\cdot s}\, dz\, ds$$

for a, b in the dense subalgebra $\mathcal{A}_\Theta$ (we use the subscript Θ on $\mathcal{A}$ when equipped with the product $\times_\Theta$).

Remark 4.1.2. (i) The map $A_\Theta \ni a \to \alpha(a) \in B^A_\Theta(\mathbb{R}^n)$ is a faithful $*$-homomorphism and intertwines the action α on A with the translation action on $B^A_\Theta(\mathbb{R}^n)$. It is sometimes useful to regard A_Θ as a subalgebra of $B^A_\Theta(\mathbb{R}^n)$ in this way.

(ii) The action α induces a strongly continuous action on A_Θ [101, Prop. 4.9], which we sometimes denote by α^Θ for clarity.

(iii) The norm completion of $\mathcal{S}^A_\Theta(\mathbb{R}^n)$ is an ideal in the C^*-algebra $B^A_\Theta(\mathbb{R}^n)$ and coincides with the algebra $\mathcal{K}_A(X)$ of compact operators on X.

(iv) If Θ is invertible, the norm completion of $\mathcal{S}^A_\Theta(\mathbb{R}^n)$ is isomorphic to $A \otimes \mathcal{K}$, where $\mathcal{K}$ is the C^*-algebra of compact operators on an infinite-dimensional Hilbert space [101, Lemma 5.1].

4.1.2 Warped convolutions

Suppose that $(A, \mathbb{R}^n, \alpha)$ is a C^*-dynamical system and that $\pi : A \to \mathcal{B}(\mathcal{H})$ is a representation of the C^*-algebra A. We are now interested in the following task: use π to construct an explicit representation of the Rieffel deformation A_Θ on the same Hilbert space $\mathcal{H}$.

Buchholz, Lechner and Summers introduced a way of deforming an operator T on a Hilbert space $\mathfrak{H}$ to what they called a "warped convolution" of the operator [25]. The idea is as follows. For some positive integer n, consider an n-tuple of commuting selfadjoint operators $P = (P_\mu)_\mu = (P_0, P_1, \ldots, P_{n-1})$ in $\mathfrak{H}$. The notation here is taken from the motivating example of the relativistic momentum operator. There is an associated action

$$\alpha_t(T) := e^{it\cdot P}Te^{-it\cdot P} \tag{4.3}$$

of $\mathbb{R}^n$ on $\mathcal{B}(\mathcal{H})$. Fix a real antisymmetric $n \times n$ matrix Θ. For a bounded operator T which is smooth with respect to the action (4.3), the **warped convolution** (or just "warping")

of T with respect to (α, Θ) can be defined as the oscillatory integral

$$T^{\Theta} := \int_{\mathbb{R}^n} \alpha_{\Theta s}(T)\, dE^P(s), \tag{4.4}$$

where $dE^P(s)$ is the joint spectral measure of the P_μ's and Θ is an $n \times n$ skew-symmetric matrix. In fact, (4.4) makes sense also for certain unbounded operators [87, 1, 89], but we shall only need this fact once (in Theorem 4.1.10).

Warped convolution turns out to be related to the deformed products developed by Rieffel. In fact, if $\times_\Theta$ denotes the Rieffel product defined by a unitarily implemented action (4.3) and the same matrix Θ, then for α-smooth operators $S, T \in \mathcal{B}(\mathfrak{H})$ one has [25]

$$S^{\Theta} T^{\Theta} = (S \times_\Theta T)^{\Theta}, \tag{4.5}$$

which is a generalization of (4.1).

Lemma 4.1.3 ([25, Thm. 2.8])**.** *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system and let $\pi : A \to \mathcal{B}(\mathcal{H})$ be a representation in which α is unitarily implemented, i.e. there are selfadjoint operators $D_1, \ldots, D_n$ on $\mathcal{H}$ such that*

$$\pi(\alpha_t(a)) = e^{2\pi i t \cdot D} \pi(a) e^{-2\pi i t \cdot D}, \qquad \forall a \in A,\ t \in \mathbb{R}^n.$$

Fix a real skew-symmetric $n \times n$ matrix Θ and define a homomorphism $\pi^\Theta : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ by

$$\pi^{\Theta}(a) := \pi(a)^{\Theta}, \qquad \forall a \in \mathcal{A},$$

where T^Θ is the warped convolution of an operator $T \in \mathcal{B}(\mathcal{H})$ with respect to (α, Θ). Then π^Θ extends to a representation of the Rieffel deformation A_Θ on $\mathcal{H}$. Moreover, π^Θ is faithful iff π is faithful, and

$$\pi^{\Theta}(\alpha_t(a)) = e^{2\pi i t \cdot D} \pi^{\Theta}(a) e^{-2\pi i t \cdot D}, \qquad \forall a \in A_\Theta,\ t \in \mathbb{R}^n.$$

In particular, if we have a concrete C^*-algebra $A \subset \mathcal{B}(\mathfrak{H})$ equipped with a strongly continuous $\mathbb{R}^n$-action α, the C^*-algebra generated by $\mathcal{A}^\Theta := \{a^\Theta |\ a \in \mathcal{A}\}$ is isomorphic to the Rieffel deformation A_Θ. Here and below, $\mathcal{A} \subset A$ denotes the subalgebra of A which is smooth under the action.

Example 4.1.4. Suppose that the $\mathbb{R}^n$-action is periodic, so it can be regarded as an action of the n-dimensional torus $\mathbb{T}^n \cong \mathbb{R}^n/\mathbb{Z}^n$. Warped convolution have been used quite a lot in this setting (without identifying it with a warped convolution). If α is a unitarily implemented action on $\mathcal{B}(\mathfrak{H})$ then $\mathfrak{H}$ decomposes into spectral subspaces $\mathfrak{H}^{(r)}$ for $r \in \mathbb{Z}^n$. For a $\mathbb{T}^n$-homogeneous operator $T \in \mathcal{B}(\mathfrak{H})$ of degree r, i.e. $\alpha_s(T) = e^{2\pi i s\cdot r}T$, the warped convolution of T with respect to (α, Θ) is the operator T^Θ which acts as

$$T^\Theta \xi = e^{2\pi i r\cdot\Theta s}\xi, \qquad \forall\, \xi \in \mathfrak{H}^{(s)},\ s \in \mathbb{Z}^n;$$

see [75, §2], [116].

So let $A \subset \mathcal{B}(\mathfrak{H})$ be a concrete C^*-algebra such that the action (4.3) is strongly continuous on A. Then A_Θ is generated by the operators

$$a^\Theta = \int_{\mathbb{R}^n} e^{i\Theta s\cdot P} a e^{-i\Theta s\cdot P}\, dE^P(s), \qquad a \in \mathcal{A}. \tag{4.6}$$

Whenever we are discussing Rieffel deformations we have a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$. Recall that the C^*-algebraic crossed product $B := A \rtimes_\alpha \mathbb{R}^n$ acts on the Hilbert space $L^2(\mathbb{R}^n, \mathfrak{H})$ if $A \subset \mathcal{B}(\mathfrak{H})$. Let $t \to \lambda_t = e^{-2\pi i t\cdot D}$ be a unitary implementation of α in $L^2(\mathbb{R}^n, \mathfrak{H})$. Under the embedding $\pi_\alpha : A \to \mathcal{M}(B)$ of A into the multiplier algebra of the crossed product $B := A \rtimes_\alpha \mathbb{R}^n$ we have

$$\pi_\alpha(\alpha_t(a)) = \pi_\alpha(e^{is\cdot P} a e^{-is\cdot P}) = \lambda_t^* \pi_\alpha(a)\lambda_t = e^{2\pi i t\cdot D}\pi_\alpha(a)e^{-2\pi i t\cdot D}.$$

Identifying A_Θ with its concrete image in $\mathcal{B}(\mathfrak{H})$ (the C^*-algebra generated by the warpings a^Θ), the C^*-dynamical system $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$ gives rise to a crossed product $B_\Theta := A_\Theta \rtimes_{\alpha^\Theta} \mathbb{R}^n$ which is represented on the same space $L^2(\mathbb{R}^n, \mathfrak{H})$ by the map $\pi_{\alpha^\Theta} : A_\Theta \to \mathcal{M}(B_\Theta)$. For $T \in A_\Theta$, the operator $\pi_{\alpha^\Theta}(T)$ is given by pointwise multiplication by the operator-valued function $\alpha^\Theta(T)$,

$$\pi_{\alpha^\Theta}(T)\xi = \alpha^\Theta(T)\xi, \qquad \forall\, \xi \in L^2(\mathbb{R}^n, \mathfrak{H}).$$

In particular, for a warping $a^\Theta \in \mathcal{A}_\Theta$ and a nice vector $\xi \in \mathcal{S}^A(\mathbb{R}^n, \mathfrak{H})$ we have

$$\begin{aligned}(\pi_{\alpha^\Theta}(a^\Theta)\xi)(t) &= (\alpha^\Theta(a^\Theta)\xi)(t) \\ &= (\alpha(a) \times_\Theta \xi)(t) \\ &:= \iint_{\mathbb{R}^n \times \mathbb{R}^n} \alpha_{-t+\Theta z}(a)\xi(t+s)e^{2\pi i z\cdot s}\, dz\, ds \\ &=: (\pi^\Theta(a)\xi)(t),\end{aligned}$$

so that $\pi_{\alpha^\Theta}(a^\Theta)$ is the operator of "left Rieffel multiplication" by the function $\alpha(a)$. Recall that π^Θ appeared also in Definition 4.1.1 as a representation of A_Θ on the Hilbert A-module X. If we use the identification of the internal tensor product $X \otimes_A \mathfrak{H}$ with $L^2(\mathbb{R}^n, \mathfrak{H})$ then π_{α^Θ} is the representation on $L^2(\mathbb{R}^n, A)$ induced by $\pi^\Theta : A \to \mathcal{L}_A(X)$. We will sometimes identify π_{α^Θ} and π^Θ in this way.

Remark 4.1.5. Note that for $\Theta = 0$ (the zero matrix) we have (rewriting the Rieffel product slightly using the Fourier transform)

$$\begin{aligned}(\pi^\Theta(a)\xi)(t) &= \int_{\mathbb{R}^n} \alpha_{-t+\Theta s}(a)\hat{\xi}(s)e^{2\pi i s\cdot t}\, ds \\ &= \alpha_{-t}(a) \int_{\mathbb{R}^n} \hat{\xi}(s)e^{2\pi i s\cdot t}\, ds \\ &= \alpha_{-t}(a)\xi(t) \\ &= (\pi_\alpha(a)\xi)(t).\end{aligned}$$

That is,

$$\pi^{\Theta=0} = \pi_\alpha,$$

so that when we discuss π^Θ with general skew-symmetric Θ we automatically include the case π_α.

The operator $\pi^\Theta(a)$ acts by left Rieffel mutliplication with $\alpha(a)$. Since $\pi_\alpha(a)$ is the operator of left multiplication with $\alpha(a)$, this means that

$$\pi^\Theta(a) = \pi_\alpha(a)^\Theta$$

is the warped convolution of $\pi_\alpha(a)$ with respect to (α, Θ), and we have an example of

Lemma 4.1.3 with $\pi = \pi_\alpha$. Since $\pi_\alpha(a)$ acts on $L^2(\mathbb{R}^n, \mathfrak{H})$, the warped convolution (4.6) takes the form

$$\pi_\alpha(a)^\Theta = \int_{\mathbb{R}^n} e^{2\pi i \Theta s \cdot D} \pi_\alpha(a) e^{-2\pi i \Theta s \cdot D} \, dE^D(s). \tag{4.7}$$

We recollect these observations.

Theorem 4.1.6. *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system and let Θ be a real skew-symmetrix $n \times n$ matrix. Then the operator $\pi_\alpha(a^\Theta)$ on $L^2(\mathbb{R}^n, \mathfrak{H})$ is the warped convolution of $\pi_\alpha(a) \in \mathcal{A}$ using generators D and matrix Θ.*

Using $\pi^\Theta(a) = \pi_{\alpha^\Theta}(a^\Theta) = \pi_\alpha(a^\Theta)$, we will be able to obtain a formula for the index of operators of the form $\not{P}\pi^\Theta(u)\not{P}$ in terms of the warpings $u^\Theta \in \mathcal{A}_\Theta \subset \mathcal{B}(\mathfrak{H})$.

Index pairings for Rieffel deformations were the original motivation for considering crossed products. The idea was inspired by the third picture of Rieffel deformations, which we recall next.

4.1.3 Kasprzak deformations

Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system. Denote by $\hat{\alpha} : \mathbb{R}^n \to \mathrm{Aut}(B)$ the dual action on the crossed product $B := A \rtimes_\alpha \mathbb{R}^n$. In Kasprzak's approach to Rieffel deformations, the deforming parameter is (a priori) not a matrix Θ but a continuous 2-cocycle

$$\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{U}(1)$$

on the group $\mathbb{R}^n$ with values in the circle group $\mathrm{U}(1)$. For later comparison we shall label the upcoming deformed objects by Θ and not by Φ. For each $t \in \mathbb{R}^n$ we have the function $\Phi_t(s) := \Phi(t, s)$ on $\mathbb{R}^n$. Then $\lambda(\Phi_t)$ is an element of $\mathcal{M}(B)$, where $\lambda : L^1(\mathbb{R}^n) \to \mathcal{M}(B)$ is the embedding. Kasprzak noticed [69, Thm. 3.1] that

$$\hat{\alpha}^\Theta_s(T) := \lambda(\Phi_s)^* \hat{\alpha}_s(T) \lambda(\Phi_s), \qquad \forall\, T \in B \tag{4.8}$$

defines a strongly continuous action of $\mathbb{R}^n$ on B. Moreover,

$$\hat{\alpha}^\Theta_s(\lambda_t) = e^{-2\pi i s \cdot t} \lambda_t, \qquad \forall\, s, t \in \mathbb{R}^n,$$

just as the original dual action $\hat{\alpha}$. The idea is now to apply Landstad's theory of crossed products [94, §7.8]. Let us recall the minimal facts needed to understand Kasprzak's deformation.

Definition 4.1.7 ([94, Def. 7.8.2])**.** An $\mathbb{R}^n$**-product** is a triple $(B, \lambda, \hat{\alpha})$ where

(i) B is a C^*-algebra,

(ii) $\lambda : \mathbb{R}^n \to \mathcal{M}(B)$ is a homomorphism such that $\mathbb{R} \ni t \to \lambda_t(b)$ is continuous for each $b \in B$,

(iii) $\hat{\alpha} : \hat{\mathbb{R}}^n \to \in \mathrm{Aut}(B)$ is a strongly continuous action such that

$$\hat{\alpha}_s(\lambda_t) = e^{-2\pi i s\cdot t}\lambda_t, \qquad \forall\, s \in \hat{\mathbb{R}}^n,\ t \in \mathbb{R}^n.$$

We extend λ to a representation $\lambda : C^*(\mathbb{R}^n) \to \mathcal{M}(B)$. Given an $\mathbb{R}^n$-product $(B, \lambda, \hat{\alpha})$, an element $T \in \mathcal{M}(B)$ satisfies the **Landstad conditions** if

(i) $\hat{\alpha}_s(T) = T$ for all $s \in \hat{\mathbb{R}}^n$,

(ii) $\lambda(f)T \in B$ and $T\lambda(f) \in B$ for all $f \in L^1(\mathbb{R}^n)$, and

(iii) The map $\mathbb{R}^n \ni t \to \alpha_t(T) := \lambda_t T \lambda_{-t}$ is continuous.

The set of elements in $\mathcal{M}(B)$ satisfying the Landstad conditions form a C^*-algebra (the **Landstad C^*-algebra** of the $\mathbb{R}^n$-product). In fact, we have the following characterization.

Theorem 4.1.8 ([94, Thm. 7.8.8])**.** *A triple $(B, \lambda, \hat{\alpha})$ is an $\mathbb{R}^n$-product if and only if there is a C^*-dynamical system $(A, \mathbb{R}^n, \alpha)$ such that $B = A \rtimes_\alpha \mathbb{R}^n$ and $\hat{\alpha}$ is the dual action. The system $(A, \mathbb{R}^n, \alpha)$ is unique and $\pi_\alpha(A)$ can be defined as the set of elements in $\mathcal{M}(B)$ that satisfy the Landstad conditions.*

Coming back to our cocycle-deformed dual action $\hat{\alpha}^\Theta$, we see that $(B, \lambda, \hat{\alpha}^\Theta)$ is an $\mathbb{R}^n$-product.

Definition 4.1.9. Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system and let Φ be a 2-cocycle on $\mathbb{R}^n$. The **Kasprzak deformation** of A with respect to (α, Φ) is the Landstad C^*-algebra A_Θ of the $\mathbb{R}^n$-product $(A \rtimes_\alpha \mathbb{R}^n, \lambda, \hat{\alpha}^\Theta)$.

Consequently, the Kasprzak deformation A_Θ satisfies

$$A_\Theta \rtimes_{\alpha^\Theta} \mathbb{R}^n \cong A \rtimes_\alpha \mathbb{R}^n \tag{4.9}$$

where α^Θ is the "same" action as α but on a different algebra (namely on A_Θ instead of A). The algebra A_Θ was called the "Rieffel deformation" of A [69, §3].

Kasprzak formulated his deformation for locally compact Abelian groups (not necessarily $\mathbb{R}^n$) [69] and his approach extend to not necessarily Abelian groups [13] and even to locally compact quantum groups [91].

4.1.4 Comparison of deformations

Having described three different ways of deforming a C^*-algebra equipped by an $\mathbb{R}^n$-action we now show that it is possible to pass from one to another.

Since Kasprzak used the term "Rieffel deformation" in his approach, several workers tried to elucidate the relation to Rieffel's deformation by actions of $\mathbb{R}^n$ and the Kasprzak deformation [13, 58, 108], with some success. It was shown in [90] that the deformed algebra A_Θ of Rieffel's satisfies (4.9) and is isomorphic to the Kasprzak deformation of A for a canonical choice of 2-cocycle Φ, whence the notation A_Θ for both Rieffel and Kasprzak deformations.

That is, if $(A, \mathbb{R}^n, \alpha)$ is a C^*-dynamical system and A_Θ is a Rieffel deformation of A for some choice of matrix Θ, the result of [90] is that the crossed products $B := A \rtimes_\alpha \mathbb{R}^n$ and $B_\Theta := A_\Theta \rtimes_\alpha \mathbb{R}^n$ are isomorphic. On the level of smooth crossed products [45], the explicit isomorphism $\mathcal{S}^A(\mathbb{R}^n) \ni f \to f^\Theta \in \mathcal{S}^A_\Theta(\mathbb{R}^n)$ which underlies (4.9) is given by [90]

$$f^\Theta(t) := \int_{\mathbb{R}^n} \alpha_{\Theta s}(\hat{f}(s)) e^{2\pi i t\cdot s}\, ds. \tag{4.10}$$

We denote by $\hat{\pi}_\alpha$ and $\hat{\pi}^\Theta$ the representations of B and B_Θ induced by π_α and π^Θ respectively. The important relation is [90]

$$\hat{\pi}^\Theta(f) = \hat{\pi}_\alpha(f^\Theta), \qquad \forall\, f \in \mathcal{S}^\mathcal{A}(\mathbb{R}^n)$$

where f on the left-hand side is viewed as an element of B_Θ and on the right-hand side as $f \in B$.

The notation f^Θ is used here to stress the similarity with warped convolution. The function f^Θ defined in (4.10) is the Fourier transform of $s \to \alpha_{\Theta s}(\hat{f}(s))$. So in the spectral representation of the D_k's, the operator $\hat{\pi}_\alpha(f^\Theta)$ acts as multiplication by the function $s \to \pi_\alpha\big(\alpha_{\Theta s}(\hat{f}(s))\big)$. Therefore,

$$\begin{aligned}\hat{\pi}_\alpha(f^\Theta) &= \int_{\mathbb{R}^n} \pi_\alpha\big((f^\Theta(t))\big) e^{-2\pi i t\cdot D}\, dt \\ &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} e^{2\pi i \Theta s\cdot D} \pi_\alpha\big(\hat{f}(s)\big) e^{-2\pi i \Theta s\cdot D} e^{2\pi i t\cdot s} e^{-2\pi i t\cdot D}\, ds\, dt,\end{aligned}$$

and if $E^D(s)$ is the spectral measure of D then we can write

$$\begin{aligned}\hat{\pi}_\alpha(f^\Theta) &= \iint_{\mathbb{R}^n\times\mathbb{R}^n} \pi_\alpha\big((f^\Theta(t))\big) e^{-2\pi i t\cdot s}\, dE^D(s)\, dt \\ &= \int_{\mathbb{R}^n} e^{2\pi i \Theta s\cdot D} \pi_\alpha\big(\hat{f}(s)\big) e^{-2\pi i \Theta s\cdot D}\, dE^D(s).\end{aligned}$$

Recall now (4.7), which says that (under the identification $\alpha^\Theta = \alpha$)

$$\pi_\alpha(a^\Theta) = \int_{\mathbb{R}^n} e^{2\pi i \Theta s\cdot D} \pi_\alpha(a) e^{-2\pi i \Theta s\cdot D}\, dE^D(s).$$

We see that $\hat{\pi}_\alpha(f^\Theta)$ is a very close analogue to $\pi_\alpha(a^\Theta)$, so the notation seems appropriate. In this sense, the notion of warped convolution extends to the crossed product by means of the formula (4.10). By considering $\pi_\alpha(A)$ instead of A we can use the isomorphism $B_\Theta \cong B$ etc., and things simplify. The idea is thus to obtained a local formula for Fredholm operators related to the warped convolutions $\pi_\alpha(a^\Theta)$ by viewing the operator $\pi_\alpha(a^\Theta)$ as a multiplier of the crossed product.

The relation $\pi_\alpha(a^\Theta) = \pi^\Theta(a)$ is the multiplier analogue of the relation (4.10). Note that this gives

$$\pi_\alpha(a^\Theta b^\Theta) = \pi_\alpha(a^\Theta)\pi_\alpha(b^\Theta) = \pi^\Theta(a)\pi^\Theta(b) = \pi^\Theta(a \times_\Theta b) = \pi_\alpha((a \times_\Theta b)^\Theta). \tag{4.11}$$

In the following theorem we consider warped convolution with respect to (α, Θ) for unbounded operators T acting on $L^2(\mathbb{R}^n, \mathfrak{H})$ and denote by T^Θ the resulting operator. For the proof, cf. [1, 2, 87].

Theorem 4.1.10. *Let $X_1, \ldots, X_n$ denote the generators of the unitary group implementing*

the dual action $\hat{\alpha}$ on B. Then for all $j \in \{1, \dots, n\}$,

$$X_j^\Theta = X_j + 2\pi \sum_{k=1}^{n} \Theta_{j,k} D_k.$$

We have yet to mention what cocycle should be used in Kasprzak's deformation to obtain the Rieffel deformation A_Θ. If $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{U}(1)$ is a 2-cocycle, let $\Phi_t : \mathbb{R}^n \to \mathrm{U}(1)$ be the function $\Phi_t(s) := \Phi(t, s)$. As before we have the multiplier $\lambda(\Phi_t)$ of the crossed product B.

Proposition 4.1.11 ([90, Thm. 2.3])**.** *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system and let Θ be a real skew-symmetrix $n \times n$ matrix. Let $\Phi : \mathbb{R}^n \times \mathbb{R}^n \to \mathrm{U}(1)$ be the 2-cocycle $\Phi(t, s) := e^{-2\pi i t \cdot \Theta s}$, so that*

$$\lambda(\Phi_t) = e^{-2\pi i t \cdot \Theta D}.$$

Then the Kasprzak deformation of A by (α, Φ) is isomorphic to the Rieffel deformation of A by (α, Θ).

Combining Proposition 4.1.11 with Theorem 4.1.10 we conclude yet another relation between warping and Kasprzak deformation.

Corollary 4.1.12. *Let $X_1, \dots, X_n$ be the generators of the dual action $\hat{\alpha} : \mathbb{R}^n \to \mathrm{Aut}(B)$ and let $X_1^\Theta, \dots, X_n^\Theta$ be their warped convolutions with respect to (α, Θ). Then the Kasprzak deformation A_Θ of A is the Landstad C^*-algebra of $(B, \lambda, \hat{\alpha}^\Theta)$, where $\hat{\alpha}^\Theta$ is the action generated by $X_1^\Theta, \dots, X_n^\Theta$.*

Proof. The Heisenberg commutation relations $[X_j, D_k] = \sqrt{-1}\delta_{jk}$ imply that the cocycle which intertwines the unitary groups generated by X_j and X_j^Θ is simply $\mathbb{R}^n \ni s \to e^{-is \cdot \Theta D}$. But then the transformation $X \to X^\Theta$ is the Kasprzak approach because changing the dual action as in (4.8) corresponds exactly to the addition of the terms $\Theta_{j,k} D_k$. □

4.2 Deformed index pairings

Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system. For any real skew-symmetric $n \times n$ matrix Θ, the automorphisms α_t act as automorphisms also for the new multiplication on $\mathcal{A}_\Theta$ [101, Prop. 2.5]. As mentioned in Remark 4.1.2, $\alpha : \mathbb{R}^n \to \mathrm{Aut}(\mathcal{A}_\Theta)$ extends to a strongly continuous

action α^Θ on A_Θ, so we have a new C^*-dynamical system $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$. Moreover, if τ is an α-invariant faithful trace on A then τ induces an α^Θ-invariant faithful trace on A_Θ [102, Thm. 4.1]. We can therefore apply the results of the last chapter to the deformed system $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$.

On the other hand, we need to choose a Hilbert-space representation of A_Θ in which α^Θ is unitarily implemented. Warped convolutions allow us to use any representation of the undeformed algebra A in which α is unitarily implemented. In that way, we can use the undeformed embedding $\pi_\alpha : A \to \mathcal{M}(B)$ to pass to crossed products. Doing so there might be a chance of obtaining a formula for the index of $\not{P}\pi^\Theta(u)\not{P}$ in terms of the warped convolution u^Θ. By staying in the original representation π_α we could use the relation $\pi^\Theta(a) = \pi_\alpha(a^\Theta)$. This works well, except for the fact that a^Θ is only defined as a multiplier of $\mathcal{S}^A(\mathbb{R}^n)$. We will indicate the required modifications in §4.2.2.

4.2.1 The deformed Thom element

We can use Corollary 4.1.12 to deduce a representative of the Thom element for $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$ in terms of that of $(A, \mathbb{R}^n, \alpha)$.

Corollary 4.2.1. *The Thom element $\hat{\mathbf{t}}_\alpha^\Theta$ of the dynamical system $(B, \hat{\mathbb{R}}^n, \hat{\alpha}^\Theta)$ is represented by the operator*

$$\not{X}^\Theta = \sum_{j,k=1}^{n} \gamma^k (X_k + 2\pi\Theta_{j,k} D_k).$$

Let $\mathbf{t}_\alpha^\Theta$ be the Thom element for $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$. Then

$$\mathbf{t}_\alpha^\Theta \otimes_B \hat{\mathbf{t}}_\alpha^\Theta = 1_{A_\Theta}.$$

Proof. The first statement comes from Corollary 4.1.12. The last statement holds because under the isomorphism $B \cong B_\Theta$ induced by $f \to f^\Theta$, the action $\hat{\alpha}^\Theta$ is intertwined with $\hat{\alpha} = \widehat{\alpha^\Theta}$ [90, Thm. 3.3], and $(B_\Theta, \hat{\mathbb{R}}^n, \widehat{\alpha^\Theta})$ is the ordinary "dual" Thom element of $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$. ∎

4.2.2 Numerical index for $\Theta \neq 0$

We now try to extend the local index formula from the last chapter to Rieffel deformations. Let X be Hilbert A-module obtained by completing $\mathcal{S}^A(\mathbb{R}^n)$ in the inner product (4.2). For

nonzero Θ, the best way of taking the trace of the elements a^Θ seem to be by viewing a^Θ as an adjointable operator on X.

There is a general construction for extending traces to operators on a given Hilbert module (see [74, §1]). In the present case it means that we have to replace $\tau(a^\Theta)$ for $a \in \mathcal{A}_+$ by

$$\tilde{\tau}(a^\Theta) := \sup_{\mathcal{I}} \sum_{\phi \in \mathcal{I}} \tau(\langle \phi | a^\Theta \phi \rangle_A), \tag{4.12}$$

where the supremum is taken over all finite subsets $\mathcal{I}$ of X for which it holds $\sum_{\phi \in \mathcal{I}} \phi\phi^* \leq \mathbf{1}$, where $\phi\phi^*$ is regarded as a compact operator on $\mathcal{S}^A(\mathbb{R}^n)$. We denote by $\tilde{\tau}$ this extension of τ to the C^*-algebra $\mathcal{L}_A(X)$ of adjointable operators on X. Note that $\tilde{\tau}$ also extends the trace $\bar{\tau} : A''_+ \to [0, +\infty]$.

In the following we endow the smooth subalgebra $\mathcal{A}$ with the Fréchet topology given by the seminorms

$$\|a\|_m := \sum_{k_1 + \cdots + k_n \leq m} \frac{1}{k_1! \cdots k_n!} \|\delta_1^{k_1} \circ \cdots \circ \delta_n^{k_n}(a)\|, \qquad m \in \mathbb{N}_0$$

where $\delta_1, \ldots, \delta_n$ are the generators of the action α.

Lemma 4.2.2. *The α-smooth subalgebra $\mathcal{A}$ has an approximate identity $(e_k)_{k\in\mathbb{N}}$ consisting of positive elements of A. Moreover, $(e_k)_{k\in\mathbb{N}}$ is a bounded approximate identity also for the deformed product $\times_\Theta$ on $\mathcal{A}$ for any Θ.*

Proof. This is [101, Props. 2.17, 2.18]. □

Lemma 4.2.3. *Let $\tilde{\tau}$ be the extension of τ to $\mathcal{L}_A(X)$ as above. Then for all $a, b \in \mathcal{A}_+$ we have*

$$\tilde{\tau}(a^\Theta b^\Theta) = \tau(a \times_\Theta b).$$

Proof. Let $(e_k)_{k\in\mathbb{N}}$ be a bounded approximate identity for $\mathcal{A}$ and let $(f_k)_{k\in\mathbb{N}}$ be an approximate identity for the convolution algebra $\mathcal{S}(\mathbb{R}^n)$, where the latter implies that $f_k \geq 0$ and that the Fourier transform of f_k satisfies $\hat{f}_k(0) = 1$ for all k. Then we get an approximate identity $(\phi_k)_{k\in\mathbb{N}}$ for $\mathcal{S}^A(\mathbb{R}^n)$ by setting

$$\phi_k(t) := f_k(t) e_k, \qquad \forall t \in \mathbb{R}^n.$$

For all $a \in \mathcal{A}$,

$$\begin{aligned}\lim_k \langle \phi_k | a^\Theta \phi_k \rangle &= \lim_k \int_{\mathbb{R}^n} |f_k(t)|^2 e_k \, a \times_\Theta e_k \, dt \\ &= \lim_k \widehat{|f_k|^2}(0) e_k (a \times_\Theta e_k) \\ &= a\end{aligned}$$

where we used Lemma 4.2.2 in the last line. Hence we get a back when we apply $\tilde{\tau}$ to a^Θ, even if τ is not invariant under the $\mathbb{R}^n$-action. On the other hand, for $a, b \in \mathcal{A}_+$ we have $a^\Theta b^\Theta = (a \times_\Theta b)^\Theta$. □

From the proof of Lemma 4.2.3 we obtain the following.

Corollary 4.2.4. *The extension $\tilde{\tau}$ of τ is finite on every element in the set*

$$\mathrm{Dom}(\tau)^\Theta := \{a^\Theta | \ a \in \mathrm{Dom}(\tau)\}.$$

Lemma 4.2.5. *Let $x(t) := ah(t)$ with $h \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and α-smooth $a \in A''$ with $a^* a \in \mathrm{Dom}(\bar{\tau})$. Then $\hat{\pi}^\Theta(x)$ is $\hat{\tau}$-Hilbert-Schmidt and*

$$\hat{\tau}(\hat{\pi}^\Theta(x)^* \hat{\pi}^\Theta(x)) = \tau(a^* \times_\Theta a) \int_{\mathbb{R}^n} |h(t)|^2 \, dt.$$

Proof. If h is in $\mathcal{S}(\mathbb{R}^n)$ then x is in $\mathcal{S}^\mathcal{A}(\mathbb{R}^n)$ and we can define x^Θ explicitly. This is again an element of $\mathcal{S}^\mathcal{A}(\mathbb{R}^n)$ and hence $\hat{\pi}^\Theta(x)^* \hat{\pi}^\Theta(x)$ is $\hat{\tau}$-traceable. For general $h \in L^\infty(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ we have $x \in L^2(\mathbb{R}^n, \mathcal{A})$ and we define x^Θ by approximation with Schwartz functions. In this way x^Θ is again in $L^2(\mathbb{R}^n, \mathcal{A})$ and hence $\hat{\pi}_\alpha(x^\Theta) = \hat{\pi}^\Theta(x)$ is Hilbert-Schmidt for $\hat{\tau}$. Proceeding as in Lemma 3.6.1 one obtains the formula. □

Remark 4.2.6. The action α on $\mathcal{S}^\mathcal{A}(\mathbb{R}^n)$ is just the translation action. From the invariance of the Lebesgue integral under translations one obtains [101, Prop. 3.6]

$$\int_{\mathbb{R}^n} (f \times_\Theta g)(t) dt = \int_{\mathbb{R}^n} f(t) g(t) \, dt,$$

holds for all $f, g \in \mathcal{S}^\mathcal{A}(\mathbb{R}^n)$. Rieffel showed furthermore in [102, Thm. 4.1] that any

α-invariant trace τ on A satisfies

$$\tau(a \times_\Theta b) = \tau(ab)$$

for all positive and smooth elements a, b. However, this does *not* imply that $\tau(a \times_\Theta b \times_\Theta c) = \tau(abc)$ and so on, and hence the indices in Theorem 3.1.1 cannot, for $n \geq 2$, be expressed in terms of the undeformed product in general.

Our result from the last chapter (Theorem 3.1.1) can in particular be applied the deformed system $(A_\Theta, \mathbb{R}^n, \alpha^\Theta)$. We want to rewrite the resulting formula by replacing producs $\times_\Theta$ by the ordinary operator multiplication in $A \subset \mathcal{B}(\mathfrak{H})$. Let $a, b, c \in \mathcal{A}$. Using

$$a^\Theta b^\Theta c^\Theta = (a \times_\Theta b)^\Theta c^\Theta = (a \times_\Theta b \times_\Theta c)^\Theta$$

we deduce the relation $\tau(a \times_\Theta b \times_\Theta c) = \tilde{\tau}(a^\Theta b^\Theta c^\Theta)$ as in Lemma 4.2.3, and similarly for products of n elements in $\mathcal{A}$. This gives the following result.

Theorem 4.2.7. *Let $(A, \mathbb{R}^n, \alpha)$ be a C^*-dynamical system as in Theorem 3.1.1 and adapt the notation introduced there.*

If n is odd and $u \in \mathcal{C}^\sim$ is unitary, then

$$\begin{aligned}\mathrm{Index}_{\hat{\tau}}(\not{P}\pi^\Theta(u)\not{P}) &= \frac{-1}{\sqrt{2\pi i}}\hat{\tau}\big(\not{P}[\not{P}, \pi_\alpha(u^{\Theta*})][\not{P}, \pi_\alpha(u^\Theta)] \cdots [\not{P}, \pi_\alpha(u^{\Theta*})][\not{P}, \pi_\alpha(u^\Theta)]\big)\\ &= -\frac{2^{(n-1)/2}(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^n n!}\tilde{\tau}\big((u^{\Theta*}\delta(u^\Theta))^n\big),\end{aligned}$$

where $\tilde{\tau}$ is the extension of τ to $\mathcal{L}_A(X)$ defined by (4.12). *If n is even then for each projection $e \in \mathcal{C}^\sim$ one has*

$$\begin{aligned}\mathrm{Index}_{\hat{\tau}}(\boldsymbol{\pi}^\Theta(e)\not{\boldsymbol{R}}_+\boldsymbol{\pi}^\Theta(e)) &= \frac{1}{2}\hat{\tau}\big(\not{\boldsymbol{R}}\Gamma[\not{\boldsymbol{R}}, \boldsymbol{\pi}_\alpha(e^\Theta)] \cdots [\not{\boldsymbol{R}}, \boldsymbol{\pi}_\alpha(e^\Theta)]\big)\\ &= \frac{(-1)^{n/2}}{(n/2)!}\frac{2^n}{(2\pi i)^n}\big((e^\Theta\delta(e^\Theta)\delta(e^\Theta))^{n/2}\big).\end{aligned}$$

We stress again that these formulas require that we have a representation (namely the warped convolution) of A_Θ on the same Hilbert space $\mathfrak{H}$ as A in the first place. Only then do we have $\pi_{\alpha^\Theta} = \pi_\alpha$ as maps from A_Θ into $\mathcal{B}(L^2(\mathbb{R}^n, \mathfrak{H}))$.

4.3 Some applications

Our original motivation for the present work was to obtain an explicit index pairing for Rieffel deformations. The relevance of such deformations to physics is that they appear when modeling interactions between quantum systems using quantum measurement theory (see [1, 2]).

First we give an (counter)example which illustrates the need of an even more general index theory than the one used in this thesis.

Example 4.3.1 (κ-Minkowski space)**.** The Lebesgue integral τ defines a trace on the Schwarz algebra $\mathcal{S}(\mathbb{R}^2)$. A certain star-product $\star_\kappa$ put on a subalgebra $\mathcal{A}$ of $\mathcal{S}(\mathbb{R}^2)$ leads to the noncommutative space called "κ-Minkowski space". This can be described as a Rieffel deformation of $\mathcal{A}$ [84, 6] using an action which does not leave τ invariant. It is a beautiful fact that τ is not a trace on $\mathcal{A}_\Theta = (\mathcal{A}, \star_\kappa)$ but rather a KMS weight with respect to a group of automorphisms of $\mathcal{A}$ (we recommend [83] for details). The ideas presented in this thesis could be a step towards index pairings for κ-Minkowski.

The next example discusses a very well-established application of index pairings in physics, where Rieffel deformations could provide a new tool.

Example 4.3.2 (Quantum Hall effect)**.** Consider the commutative C^*-algebra $A = C_0(\Omega)$ where (Ω, μ) is a probability measure space. Let $\tau(f) := \int_\Omega f(t)\, d\mu(t)$ be the trace given by integration on (Ω, μ) (note that τ is here finite on all of A). Let α be an action of $\mathbb{R}^n$ on $\mathcal{A} = C_c^\infty(\Omega)$. Suppose $X_1, \ldots, X_n$ are generators of a unitary group implementing α in $L^2(\mathbb{R}^n \times \Omega, \mu)$. On the crossed product $L^\infty(\Omega, \mu) \rtimes_\alpha \mathbb{R}^n$ there is a weight $\hat{\tau}$ dual to τ which is a trace if τ is invariant under α; let us assume that this is the case. We let an element $x \in L^\infty(\Omega, \mu) \rtimes \mathbb{R}^n$ be written formally as

$$x \sim \int_{\mathbb{R}^n} x(p) e^{ip \cdot X}$$

where each $x(p) : \Omega \to \mathbb{C}$ belongs to $L^\infty(\Omega, \mu)$. Then the dual weight is given by (see e.g. [72])

$$\hat{\tau}(x) = \tau(x(0)) = \int_\Omega (x(0))(\omega)\, d\mu(\omega).$$

It is also possible to consider a crossed product $\mathcal{N} := L^\infty(\Omega, \mu) \rtimes_{\alpha, \tilde{B}} \mathbb{R}^n$ twisted by a 2-cocycle $(s, t) \to e^{is \cdot \tilde{B} t}$ on $\mathbb{R}^n$; the dual-trace construction works in this case as well [110].

The C^*-algebra of interest is then a subalgebra $B = C_0(\Omega; \tilde{B}) \subset \mathcal{N}$ with the same product.

Let $\not{X}$ denote the Dirac operator formed as in (3.15) from the generators $X_1, \ldots, X_n$ of the unitary group implementing α on $\mathcal{N}$. Take n odd, for example, and set

$$\lambda_n := -\frac{2^{(n-1)/2}(-1)^{(n-1)/2}((n-1)/2)!}{(2\pi i)^n n!}.$$

Then for a unitary $u \in \mathcal{A}$ we get a formula for the spectral flow $\mathrm{Sf}(\not{X}, u^*\not{X}u)$ from Theorem 3.1.1,

$$\begin{aligned}\mathrm{Sf}(\not{X}, u^*\not{X}u) &= \lambda_n \sum_\epsilon (-1)^\epsilon \tau\Big(\prod_{k=1}^n u^{-1}\sqrt{-1}[X_{\epsilon(k)}, u]\Big) \\ &= \lambda_n \sum_\epsilon (-1)^\epsilon \prod_{k=1}^n \int_\Omega u^{-1}(\omega)\sqrt{-1}[X_{\epsilon(k)}, u](\omega)\, d\mu(\omega),\end{aligned}$$

the sum running over all permutations ϵ of $\{1, \ldots, n\}$ with sign $(-1)^\epsilon$. According to Theorem 4.2.7, deforming with a matrix Θ to incorporate the effect of some external interaction, the spectral flow becomes

$$\mathrm{Sf}^\Theta(\not{X}, u^*\not{X}u) = \lambda_n \sum_\epsilon (-1)^\epsilon \prod_{k=1}^n \int_\Omega u^*(\omega) \times_\Theta \sqrt{-1}[X_{\epsilon(k)}, u](\omega)\, d\mu(\omega).$$

The reader may recognize that what we are discussing here is the setting of the extremely elegant and successful formulation of the integral quantum Hall effect using noncommutative geometry, due to Bellissard et al. [8]. There, the matrix $\tilde{B}$ which defines the 2-cocycle $(s,t) \to e^{is\cdot\tilde{B}t}$ is given by $\tilde{B}t := B \wedge t$, where $B = (B_1, \ldots, B_n)$ is the constant magnetic vector field. It has been realized [70] that the Bellissard approach is related to the more recent magnetic pseudodifferential calculus of [78]. Twisted crossed product are very similar to Rieffel deformation but still different [9]. Using the results of this paper we can reproduce the quantum Hall algebra and the operators whose Fredholm indices give the quantized conductance, modulo the distinction between crossed products and Rieffel deformation.

In [8] the $X_k's$ play the role of position operators, generators of momentum translations, and a spectral triple is defined using $\not{X}$. Most prominently, the index of the bounded transform of $\not{X}$ (compressed with the Fermi projection) has been used to calculate the Hall conductivity when $n = 2$. In [99], this was generalized to the construction of a spectral

triple from $\not{X}$ to any even $n \geq 1$, and for even n only. The index of Toeplitz operators PuP was used in [6] for a mathematical formulation of physical processes, including the integer Hall effect. Theorem 3.1.1 shows that the X_k's appear also here, and even though the approaches [8] and [6] seem very different at first, the distinction mainly comes from “even versus odd”. For even dimensions the odd pairing can be used by considering unitaries in $C_0(\mathbb{R}, A)^\sim$, which is also what is done in [6] for $n = 2$ (see also [50]).

There is also a paper [100] showing the relevance of the Bellissard approach also to odd case. Moreover, results very similar to those of this thesis but applicable to twisted crossed products are discussed in [19]. In the same work [19] appears an extensive up-to-date discussion about the C^*-approach to topological condensed-matter systems such as the quantum Hall effect.

One reason why we are more attracted to the use of Rieffel deformation than twisted crossed products is the direct relation between Rieffel deformation to interactions as they are usually described in quantum physics [1, 2]. If the D_k's are position operators then the dual action $\hat{\alpha} : \mathbb{R}^n \to \mathrm{Aut}(\mathcal{N})$, implemented by a unitary group $e^{iv\cdot P}$, can be interpreted as the group of spacetime translations. Thus $P = (P_1, \ldots, P_n)$ are the energy-momenta. Performing a Rieffel deformation gives that the P_k's are changed by a term coming from the D_k's as we saw in Theorem 4.1.10. So the deformation is like adding an external term to the energy or to the momenta, interpreted suitably as coming from the interaction with another quantum system. Note that, by choice of gauge, a transient external electric field can be incorporated either via a potential energy term added to the Hamiltonian, or via an external vector potential term added to $P_1, \ldots, P_n$ [17]. We know that either of these can be obtained from Rieffel deformation [2].

On the other hand, if we have an action α generated by the momenta $(D_1, \ldots, D_n) = (P_1, \ldots, P_n)$, then $\not{D} = \gamma^k P_k$ is a Dirac operator in the physical sense, and the positive projection P singles out the states of positive energy. The spectral flow between $\not{D}$ and $u^* \not{D} u$ is then like the amount of charge transferred due to the operation u. This could be any real number, although it may be possible to obtain further restrictions on its possible values in specific examples.

Use of such $\not{D}$ is not limited to condensed matter physics. In fact, (Lorentzian) spectral triples have been used to define a CAR algebra when the field operators act as multiplication operators with a Moyal product (i.e. the special kind of Rieffel product when the initial

algebra $\mathcal{A}$ is commutative) [16, 114]. The relevant algebra is thus a Rieffel deformation of a commutative algebra like $\mathcal{S}(\mathbb{R}^n)$ and the present paper strongly suggests that Connes'-type pairings can be used also in this setting.

There is another interesting field of research where both Rieffel deformations and index theory have already been very useful, namely noncommutative gauge theory [20, 76]. The C^*-algebra deformed in this context is the C^*-algebra $A = C(M)$ of continuous functions on a smooth compact manifold M equipped with an action of some torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ (which lifts to an action α of $\mathbb{R}^n$). The vector bundles consider in [20, 76] are $\mathbb{T}^n$-equivariant, and this ensures that the associated projections over A are fixed by the $\mathbb{T}^n$-action and hence defines projections in A_Θ as well. Thus the setting is rather different from that of the present thesis, where the Dirac operator is formed from the generators of the deforming action and we look for new projections in the Rieffel-deformed algebra A_Θ. It would be very interesting to see if something complementary to [20, 76] could be obtained with this alternative Dirac operator. We should mention that the spectral triple of a Dirac operator acting on a principal $\mathbb{T}^n$-bundle can be "factorized" into two parts, where one is the Dirac operator coming from the $\mathbb{T}^n$-action [48].

As a final example of an already known index pairing of a Rieffel-deformed algebra we mention the noncommutative Chern-Simons action for the noncommutative 3-torus $A_\Theta = C(\mathbb{T}^3_\Theta)$ [95]. The algebra $C(\mathbb{T}^3_\Theta)$ is generated by canonical unitaries u_p paramterized by $p \in \mathbb{Z}^n$ and obtained from the commutative C^*-algebra $C(\mathbb{T}^3)$ by Rieffel-deforming with the action α by $\mathbb{R}^3$ defined by

$$\alpha_t(u_p)(s) = u_p(s+t) = e^{2\pi i p\cdot t} u_p(s), \qquad \forall p \in \mathbb{Z}^3,\ t, s \in \mathbb{R}^3,$$

where we denote by u_p also the function on $\mathbb{R}^3$ induced by $u_p \in C(\mathbb{T}^3)$. The Dirac operator of the spectral triple used in [95] is $\not{X} = \sum_{k=1}^3 X_k \otimes \gamma^k$, where X_1, X_2, X_3 are the generators of the action dual to the translation action α. We cannot completely fit this example into the framework of the present thesis because the operators X_1, X_2, X_3 (used for index pairings) do not implement the action α (used for deformation). The strong similarity between the setting in [95] and that discussed here is however promising, and it is possible that a slight reformulation could lead to a unification. In that case it seems likely that more general and systematic results would follow.

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