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UNIFORM REGULARITY IN A WEDGE AND REGULARITY OF TRACES OF CR FUNCTIONS

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Abstract. We discuss in Section 1 the property of regularity at the boundary of separately holomorphic functions along families of discs and apply, in Section 2, to two situations. First, let $\mathcal{W}$ be a wedge of $\mathbb{C}^n$ with C^ω, generic edge $\mathcal{E}$: a holomorphic function f on $\mathcal{W}$ has always a generalized (hyperfunction) boundary value $bv(f)$ on $\mathcal{E}$ and this coincides with the collection of the boundary values along the discs which have C^ω transversal intersection with $\mathcal{E}$. Thus Section 1 can be applied and yields the uniform continuity at $\mathcal{E}$ of f when $bv(f)$ is (separately) continuous. When $\mathcal{W}$ is only smooth an additional property, the temperateness of f at $\mathcal{E}$ characterizes the existence of boundary value $bv(f)$ as a distribution on $\mathcal{E}$. If $bv(f)$ is continuous, this operation is consistent with taking limits along discs (Theorem 2.8). By Section 1, this yields again the uniform continuity at $\mathcal{E}$ of tempered holomorphic functions with continuous bv. This is the theorem by Rosay in [12] in whose original proof the method of "slicing" by discs is not used.

As related literature we mention, among others, Sato-Kashiwara-Kawai [13], Komatsu [11], Hörmander [10], Cordaro-Treves [9], Baouendi-Ebenfelt-Rothschild [3] and Berhanu-Hounie [8].
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1. Uniform regularity of separately holomorphic functions on families of discs

Let $\mathbb{C}$ be the complex plane with coordinate $\tau=\rho+i\sigma$, Δ the open unit disc, Δ^+ the open upper half-disc, I the unit interval in $\mathbb{R}$ and f a function holomorphic in Δ^+. We say that f admits on I a boundary value $bv(f)$ which is a measure if for any $\varphi\in C^0_c(I)$ there exists the limit $\lim\limits_{\sigma\to 0}\int_I f(\rho+i\sigma)\varphi(\rho)d\rho$ and

$$\lim_{\sigma\to 0}\int_I f(\rho+i\sigma)\varphi(\rho)d\rho\le k||\varphi||_{C^0(I)}, \tag{1.1}$$

where k is independent of φ. We denote by $\langle bv(f),\varphi\rangle$ the limit above. By Cauchy formula the limit remains unchanged if we approach I not along the level lines $\sigma=$ const but any sequence of curves C^1-converging to I. Also, it is readily seen from [10] Theorem 3.1.14 that (1.1) implies that f is tempered with growth 1 that is

$$|f|\underset{\sim}{<}\sigma^{-1}. \tag{1.2}$$

And conversely, by [10] Theorem 3.1.11, (1.2) implies the analogous of (1.1) with $||\varphi||_{C^0(I)}$ replaced by $||\varphi||_{C^2(I)}$.

Remark 1.1. There is always a generalized boundary value at I in the sense of hyperfunctions; when this happens to be a measure, then f must satisfy (1.2) and the boundary value coincides with the above limit. In fact, by [10] Theorem 8.4.15 there must be a holomorphic function which satisfies (1.2) and has the same boundary value. By uniqueness this is f itself. Thus the limit (1.1) exists and it is a standard fact that it coincides with the generalized boundary value.

Remark 1.2. Let A^+ be a domain of $\mathbb{C}$ and $\mathcal{I}$ a C^1 piece of its boundary, a graph in ρ, the real part of $z = \rho + i\sigma$, with A^+ being in the upper half side. For a holomorphic function f on A^+ which is bounded in a neighborhood of $\partial\mathcal{I}$, we can define a generalized boundary value by
(1.3)

$$\langle bvf, \varphi\rangle = \lim_{\epsilon\to 0}\int_{\mathcal{I}_\epsilon} f(z+i\epsilon)\varphi(z)dz \quad \text{for any } \varphi \text{ holomorphic in } \overset{\circ}{A}{}^+ \text{ and continuous in } \bar{A}^+,$$

where $\mathcal{I}_\epsilon \subset\subset \mathcal{I}$ so that $\mathcal{I}_\epsilon + i\epsilon \subset\subset A^+$ and $\mathcal{I}_\epsilon \nearrow \mathcal{I}$. In fact, by taking a cut off function $\chi \in C^1_c(\mathcal{I})$ with $0 \leq \chi \leq \eta$ and $\chi \equiv \eta$ in the part of $\mathcal{I}$ where there is no control of boundedness of f, we have

$$\begin{aligned}\int_{\mathcal{I}_\epsilon} f(\rho+i\epsilon)\varphi(z)dz &= \int_{\mathcal{I}_\epsilon} f(z+i\epsilon+i\chi)\varphi(z+i\chi)dz\\ &\to \int_{\mathcal{I}} f(z+i\chi)\varphi(z+i\chi)dz.\end{aligned}$$

The first equality follows from Stokes formula whereas the existence of the second limit is a consequence of the fact that f is bounded in a neighborhood of $\mathcal{I}\setminus \operatorname{supp}\chi$. If this boundary value satisfies (1.1), then we have again (1.2) as we can check by applying (1.1) to φ which is Cauchy's kernel. It follows that the present boundary value is indeed a boundary value in the sense of the measure. In case $\mathcal{I}$ is C^ω, this gives an easy explanation of the conclusion of Remark 1.1. Otherwise, (1.3) defines a very general boundary value. Notice that, though there is a requirement of boundedness of f at $\partial\mathcal{I}$, the resulting boundary value has a well defined action over functions φ which are no more holomorphic on a neighborhood of $\mathcal{I}$ (as was the case of $\varphi \in C^\omega$ and $\mathcal{I} \in C^\omega$) but just on one side $\overset{\circ}{A}{}^+$.

A special interest relies in the case when $bv(f)$ is C^0 and thus it is well a measure and (1.1) holds. In this case we have, at it is well known, coincidence of bv with usual limit.

Proposition 1.3. *Let $f \in hol(\Delta^+)$ satisfy $bv(f) \in C^0(I)$; then f is uniformly continuous on Δ^+ up to I.*

Proof. We take a function χ in $C^\infty_c(I)$ with $\chi \equiv 1$ in $I' \subset\subset I$, and extend from I to $\mathbb{C}$ so that $\bar\partial\chi = O(|\sigma|^k)$. Under this choice we have that $\bar\partial_\tau(\chi f)$ is uniformly bounded. We

write $F(\zeta) := \chi f(\rho + i\zeta\sigma)$ and apply Cauchy formula to the function $\frac{F(\zeta)}{\zeta+1}$ at $\zeta = 1$ for the half-plane $\Pi^+ = \{\mathsf{Re}\,\zeta > 0\}$. We get, after substituting $w = u + iv$ for $i\zeta$,
(1.4)

$$\chi f(\rho + i\sigma) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\chi bv(f)\,(\rho + u\sigma)}{u^2+1} dv - \frac{2}{\pi}\iint_{\Pi^+} -i\bar\partial_\tau(\chi f)(\rho + w\sigma)\sigma\frac{1}{w^2+1} du \wedge dv.$$

Recall that $\frac{1}{\pi}\int_{-\infty}^{+\infty}\frac{1}{u^2+1}du = 1$. Hence (1.4) implies

$$\chi f(\rho + i\sigma) - \chi f(\rho) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{\chi bv(f)\,(\rho + u\sigma) - \chi bv(f)\,(\rho)}{u^2+1} du + O(\sigma), \tag{1.5}$$

where the error $O(\sigma)$ is uniform for ρ ranging on compact subsets of I. The continuity of f on I yields the conclusion. $\square$

Remark 1.4. The proposition remains true for any simply connected domain A^+ at a C^1 piece of its boundary $\mathcal{I}$: if $bv(f) \in C^0$, then f is uniformly continuous at $\mathcal{I}$.

We pass to consider functions which are holomorphic over families of discs $\{A^+_\xi\}_\xi$ where ξ ranges through a set Λ of real parameters. For the purpose of our applications, Λ can be assumed to be a smooth manifold.

Proposition 1.5. *Let $\{\varphi_\xi\}_\xi$ be a family of subharmonic functions on Δ^+, continuous on $\bar\Delta^+$ such that for any $\eta > 0$ and for suitable c_η*

$$\begin{cases} (i)\ \varphi_\xi \le c_\eta \quad \text{for } \sigma \ge \eta \text{ and for } \sigma = 0, \\ (ii)\ \varphi_\xi \underset{\sim}{<} \sigma^{-1+\epsilon} \quad \text{at } +1,\ -1 \text{ uniformly with respect to } \xi, \\ (iii)\ \underset{\xi\to\xi_o}{lim\ sup}\,\varphi_\xi(\rho) = -\infty \quad \text{for } \rho \in I. \end{cases}$$

Then

$$\underset{\xi\to\xi_o}{lim\ sup}\ \sup_{\Delta^+\cup I}\ \varphi_\xi = -\infty.$$

Proof. Select a sequence $\xi_\nu \to \infty$. For $M >> 1$, set

$$E_j = \{\rho \in I :\ \varphi_{\xi_\nu} \ge -M \text{ for some } \nu \ge j\}.$$

For large j_o we have that the Lebesgue measure $\lambda(E_{j_o})$ is arbitrarily small. We also set $E_\eta = \{z \in \partial\Delta^+ :\ 0 < \sigma < \eta\}$ and denote by $P = P_\tau(\zeta)$ the Poisson kernel of Δ^+. We have

$$\begin{aligned}\varphi_{\xi_\nu}(\tau) &\le \int_{\partial\Delta^+} P_\tau(\zeta)\varphi_{\xi_\nu}(\zeta)d\lambda(\zeta)\\ &= \int_{(\partial\Delta^+\setminus I)\setminus E_\eta} * + \int_{E_\eta} * + \int_{I\setminus E_{j_o}} * + \int_{E_{j_o}} *\,,\end{aligned}$$

where the first inequality is a consequence of the subharmonicity of the φ_{ξ_ν}'s. Now, for any small α we have, uniformly with respect to $\nu \geq j_o$

$$\begin{aligned}
\int_{(\partial\Delta^+\setminus I)\setminus E_\eta} * &\leq c_\eta \sigma \quad \text{for } |\tau| \leq 1-\alpha,\\
\int_{E_\eta} * &\lesssim \int_0^\eta \sigma^{-1+\epsilon} d\sigma \lesssim \eta^\epsilon < \alpha,\\
\int_{I\setminus E_{j_o}} * &\leq -M+\alpha,\\
\int_{E_{j_o}} * &\leq c_o \lambda(E_{j_o}) \leq \alpha \quad \text{for large } j_o;
\end{aligned}$$

here the first line follows from (i) (uniform boundedness in $(\partial\Delta^+ \setminus I) \setminus E$), the second from (ii), the third from the definition of E_{j_o} and the fourth again from (i) (uniform boundedness on I). In conclusion, for $\nu \geq j_o$:

$$\varphi_{\xi_\nu} \leq -M + 3\alpha + c_\eta \sigma \quad \text{uniformly on } \Delta^+.$$

□

Proposition 1.6. *Let $\{f_\xi(\tau)\}$, $(\xi,\tau) \in \Lambda \times \Delta^+$, be a family of functions holomorphic in τ which satisfy*

(i) $|f_\xi| \lesssim e^{\frac{1}{\sigma^{1-\epsilon}}}$ *at $\tau = +1, -1$ uniformly in ξ,*

(ii) $|f_\xi| \leq c$ *on compact subsets of $\bar{\Delta}^+ \setminus I$ and on I,*

(iii) $\rho \mapsto bv_I(f_\xi)(\rho)$ *is continuous and uniformly bounded,*
(iv) $\xi \mapsto f_\xi(\rho)$ *is continuous.*

Then

$$\lim_{\xi\to\xi_o} f_\xi = f_{\xi_o} \quad \textit{with uniform convergence on } \Delta^+ \cup I.$$

We will refer to the property contained in (i) as *subexponential growth.*

Proof. First, on account of (iii), we get from Proposition 1.3 that each f_ξ, for fixed ξ, is uniformly continuous in $\bar{\Delta}^+$ (where maybe the radius needs to be shrunck from 1 to $1-\alpha$). The conclusion then follows by applying Proposition 1.5 to the family of subharmonic functions $\varphi_\xi := \log|f_\xi|$.

□

We denote by $f = f_\xi(\tau)$ the collection of the f_ξ's in $\Lambda \times \Delta$; a better way of stating our conclusion is

Theorem 1.7. *Let $\{f_\xi\}_\xi$ be a family of functions on Δ^+ such that*

(i) $|f_\xi| \underset{\sim}{<} e^{\frac{1}{\sigma^{1-\epsilon}}}$ *at* $\tau = +1, -1$ *uniformly in* ξ,

(ii) *each* f_ξ *is holomorphic,*

(iii) *by collecting the separate boundary values* $bv_I(f_\xi)$ *it is well defined a function in* $\Lambda \times I$ *which is bounded and separately continuous in* ξ *and* ρ,

(iv) f *is continuous for* $(\xi, \tau) \in \Lambda \times \Delta^+$.

Then

$$f \text{ is uniformly continuous in } \Lambda \times (\Delta^+ \cup I).$$

Proof. We write

$$\begin{aligned} |f_\xi(\tau) - f_{\xi_o}(\tau_o)| &\leq |f_\xi(\tau) - f_{\xi_o}(\tau)| + |f_{\xi_o}(\tau) - f_{\xi_o}(\tau_o)| \\ &=: (I) + (II), \end{aligned}$$

where the second line serves as a definition of (I) and (II). Now, by applying Proposition 1.5 to $\varphi_\xi := \log |f_\xi|$ we get that (I) converges to 0 uniformly with respect to $\tau \in \Delta^+ \cup I$. For the continuity of f_{ξ_o} on $\Delta^+ \cup I$ for fixed ξ_o, which follows from Proposition 1.3 as already noticed, we have that (II) also converges to 0.

□

Remark 1.8. In most cases the f_ξ's glue up to a holomorphic function in a wedge $\mathcal{W} \subset \mathbb{C}^n$ of dimension $2n$ with generic edge $\mathcal{E}$; in this case f is uniformly bounded on compact subsets of $\overline{\mathcal{W}} \setminus \mathcal{E}$. If, moreover, $f|_{\mathcal{E}}$ is C^0 and f has subexponential growth, then Theorem 1.7 can be applied. In general, if $\mathcal{W}$ is not of full dimension $2n$, the assumption of uniform boundedness on compact subsets of $\overline{\mathcal{W}} \setminus \mathcal{E}$ cannot be dispensed of.

For example, consider in $\mathbb{R} \times \mathbb{C}$ the function

$$f(x_1, z_2) = \begin{cases} x_1 \sin \frac{z_2}{x_1} & \text{if } x_1 \neq 0, \\ 0 & \text{if } x_1 = 0. \end{cases}$$

This is the separate holomorphic extension from $\mathbb{R} \times \mathbb{R}$ to $\mathbb{R} \times \mathbb{C}$ of the continuous function $x_1 \sin \frac{x_2}{x_1}$. However, (ii) of Proposition 1.6 (or (iv) of Theorem 1.7) do not hold. And in fact there is not there defined any uniformly continuous function.

2. Regularity of holomorphic functions at the edge of a wedge

In this section, we consider a smooth manifold $\mathcal{M} \subset \mathbb{C}^n$, a submanifold $\mathcal{E} \subset \mathcal{M}$ and a *wedge* $\mathcal{W}$ of $\mathcal{M}$ with *edge* $\mathcal{E}$. These are obtained as deformations of the linear models

$$\begin{aligned} M &= \mathbb{C}^{n-(l_1+l_2)} \times \mathbb{C}^{l_1} \times \mathbb{R}^{l_2}, \\ E &= \mathbb{C}^{n-(l_1+l_2)} \times \mathbb{R}^{l_1} \times \mathbb{R}^{l_2}, \\ W &= \mathbb{C}^{n-(l_1+l_2)} \times (\mathbb{R}^{l_1} + i\Gamma) \times \mathbb{R}^{l_2}, \end{aligned}$$

where Γ is a cone in the plane $\mathbb{R}^{l_1}$. We denote by $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{W}$ the images of M, E and W under a smooth diffeomorphism of $\mathbb{R}^{2n} = \mathbb{C}^n$ whose differential at 0 is the identity and which is holomorphic with respect to $w \in \mathbb{R}^{l_1} + i\Gamma$. This is the standard definition. We need another presentation of these sets which is consistent with the setting of Section 1. For this, we write

$$\Lambda = \mathbb{C}^{n-(l_1+l_2)} \times (\mathbb{R}^{l_1-1} + i(\Gamma \cap S^{l_1-1})) \times \mathbb{R}^{l_2}.$$

We denote by $\xi = (z, s', v, a)$ the coordinate in Λ; we identify the point $s' \in \mathbb{R}^{l_1-1}$ to a point in the normal plane to v in $\mathbb{R}^{l_1}$, and denote by $A^+_\xi = \Phi(\{\xi\} \times \Delta^+)$ the ξ-slice of $\mathcal{W}$; we also set $I_\xi := \Phi(\{\xi\} \times I)$. By means of the family of Riemann mappings

$$\Phi_\xi : \Delta^+ \overset{\sim}{\to} A^+_\xi,$$

we may set up a smooth diffeomorphism

$$\begin{aligned} \Phi :\ & \Lambda \times \Delta^+ \overset{\sim}{\to} \mathcal{W} \\ & (\xi, \tau) \mapsto \Phi_\xi(\tau), \end{aligned}$$

which is holomorphic in τ. In fact, the Riemann mappings can be obtained as follows. We take a projection $\pi : \mathbb{C}^{l_1} \to \mathbb{C}$ whose fiber has intersection of dimension 1 with $\mathbb{R}^l + i\Gamma$ and set $\pi_\xi := \pi|_{A^+_\xi}$ and $\tilde{A}^+_\xi := \pi(A^+_\xi)$; thus π_ξ is a diffeomorphism between A^+_ξ and its image $\tilde{A}^+_\xi$. We then first solve, for a fixed $a \in \tilde{A}^+_\xi$, the Dirichlet problem

$$\begin{cases} \Delta g(\tau) = 0, & \tau \in \tilde{A}^+_\xi, \\ g(\tau) = \log(|\tau - a|), & \tau \in \partial\tilde{A}^+_\xi, \end{cases} \tag{2.1}$$

and next take the harmonic conjugate to $g(\tau) - \log(|\tau - a|)$. On the other hand, this latter can be found as

$$Tg + \arg(\tau - a),$$

where T is the Hilbert transform. Since $g - \log|\tau - a| \leq 0$, then $\Phi^{-1}_\xi := exp(g - \log|\tau - a| + ih)$ is the desired mapping (cf. [15] p. 323). Note that, since both, the solution of the Dirichlet problem (2.1) and the functional T depend smoothly on the parameters ξ, the same is true of Φ^{-1}_ξ (and there is also smooth dependence on $\tau \in \partial A^+_\xi$).

We call proper subwedge $\mathcal{W}' << \mathcal{W}$ the image $\mathcal{W}' := \Phi(\Lambda' \times \Delta^+)$ where Λ' is obtained from Λ by shrinking Γ to a proper subcone $\Gamma' << \Gamma$. Let f be a function in $\mathcal{W}$ and set $f_\xi := f|_{A^+_\xi}$; applying to the function $f \circ \Phi$ in the product $\Lambda \times \Delta^+$ the conclusions of Theorem 1.7, we get

Theorem 2.1. *Let f be a C^0 function in $\mathcal{W}$, separately holomorphic along each A^+_ξ, whose boundary values at I_ξ along each A^+_ξ glue on to a bounded function in $\mathcal{E}$ separately continuous in ξ and ρ. We also assume $|f| \underset{\sim}{<} e^{\frac{1}{\sigma^{1-\epsilon}}}$ on $\Phi(\Lambda \times (\{+1\} \cup \{-1\}))$ uniformly with respect to ξ. Then $f \in C^0(\mathcal{W}' \cup \mathcal{E})$ for any $\mathcal{W}' << \mathcal{W}$.*

Remark 2.2. By Cauchy formula in polydiscs and Fubini's Theorem, continuity and separate analyticity is equivalent to analyticity in $\mathcal{W}$. We keep the setting of separate analyticity to emphasize that the boundary value is taken disc by disc. In next statements, Theorems 2.4 and 2.9, we will assume directly $f \in \text{hol}(\mathcal{W})$.

Proof. For any fixed ξ, we insert A^+_ξ into a manifold with boundary obtained by selecting in Λ a subset of parameters whose dimension is $\dim(\mathcal{E}) - 1$. For instance, we point our attention to the component v in $\Gamma \cap S^{l_1}$ of ξ and set $\Lambda_v := \mathbb{C}^{n-(l_1+l_2)} \times (\mathbb{R}^{l_1-1} + i\{v\}) \times \mathbb{R}^{l_2}$ and $\mathcal{W}_v := \Phi(\Lambda_v \times \Delta^+)$. We apply Theorem 1.7 to $\mathcal{W}_v$ and conclude that $f \in C^0(\mathcal{W}_v \cup \mathcal{E})$. We repeat this operation for a family of vectors $v_1, ..., v_N$ such that the cone $\mathcal{C}(v_1, ..., v_N)$ spanned by the v_j's is a polyhedral approximation of the directional cone Γ of $\mathcal{W}$: thus f is C^0 in each $\mathcal{W}_{v_j} \cup \mathcal{E}$.

By the edge of the wedge theorem of [1], the function f, continuous CR on each $\mathcal{W}_{v_j}$ extends as a continuous CR function on $\mathcal{W}'$ any proper subwedge with directional cone $\Gamma' << \mathcal{C}(v_1, ..., v_N)$.

$\square$

Remark 2.3. Direct inspection of the proof shows that for uniform continuity in $\mathcal{W}' \cup \mathcal{E}$ with $\Gamma' << \mathcal{C}(v_1, ..., v_N)$, we do not need to assume $f \in \text{hol}(\mathcal{W})$ but just $f \in CR(\mathcal{W}_{v_j}) \cap C^0(\mathcal{W}_{v_j})$ for any j.

Also, separate continuity of $f \circ \Phi$ in the subsets $\Lambda_{v_j} \times I$ suffice. But, if the rank of the v_j's is l_1 we can say more. By a recourrence argument on the planes $L_\nu = \mathbb{C}^{n-(l_1+l_2)} \times (\sum_{j=1}^{\nu} \mathbb{C}v_j) \times \mathbb{R}^{l_2}$ for $\nu = 2, ...$, separate continuity can be reduced from the sets $\Lambda_{v_j} \times I$ to $(\mathbb{C}^{n-(l_1+l_2)} \times \mathbb{R}v_j \times \mathbb{R}^{l_2}) \times I$.

In the proof of Theorem 2.1, the continuity of $bv(f)|_{I_\xi}$ yields uniform continuity in each $A^+_\xi \cup I_\xi$. Next, this continuity is uniformized to continuity in each $\mathcal{W}_{v_j} \cup \mathcal{E}$ by the aid of Theorem 1.7. Last, by the edge of the wedge theorem, the continuity on each $\mathcal{W}_{v_j} \cup \mathcal{E}$ is brought to continuity on $\mathcal{W}' \cup \mathcal{E}$ for any subwedge $\mathcal{W}'$ with directional cone $\Gamma' << \mathcal{C}(v_1, ..., v_N)$.

We change our setting and point our attention to the boundary value of f on the whole $\mathcal{E}$, if it exists, instead of the collection of the boundary values at I_ξ along each disc A^+_ξ. We have to note first that if $\dim(\mathcal{W}) = 2n$ and $\mathcal{E}$ is C^ω, this boundary value $bv_{\mathcal{E}}(f)$ always exists, independently of the behavior of f at $\mathcal{E}$, in the space of CR hyperfunctions. These are the hyperfunctions on $\mathcal{E}$ which solve the tangential $\bar\partial$-system (or equivalently the cohomology of the sheaf of holomorphic function on $\mathbb{C}^n$ with support in $\mathcal{E}$ in degree l_1+l_2). This boundary value accepts restriction $(bv_{\mathcal{E}}(f))|_{I_\xi}$ to each I_ξ. In fact, since each complex plane $\xi =$ const cuts $\mathcal{W}$ along a disc A^+_ξ, one defines this restriction as $bv_{I_\xi}^{A^+_\xi}(f|_{A^+_\xi})$. In other terms we have, merely by definition,

$$(bv_{\mathcal{E}}(f))|_{I_\xi} = bv_{I_\xi}^{A_\xi^+}(f|_{A_\xi^+}). \tag{2.2}$$

In this situation we have a better restatement of Theorem 2.1

Theorem 2.4. *Let $\mathcal{W} \subset \mathbb{C}^n$ be a wedge of dimension $2n$ and C^ω edge $\mathcal{E}$, choose vectors $v_j \in \Gamma \cap S^{l_1-1}$, $j = 1, ..., N$ of rank l_1 and denote by $\mathcal{C}(v_1, ..., v_N)$ the cone spanned by the v_j's. Assume*

(i) *$f \in hol(\mathcal{W})$,*

(ii) *$bv_{\mathcal{E}}(f)$ is bounded and separately continuous in $T^{\mathbb{C}}\mathcal{E}$ and $\mathbb{R}v_j$ for any j.*

Then $f \in C^0(\mathcal{W}' \cup \mathcal{E})$ for any $\mathcal{W}' << \mathcal{W}$ with $\Gamma' << \mathcal{C}(v_1, ..., v_N)$.

Proof. (a) The function f is tempered at $\mathcal{E}$: we owe to Paulo Cordaro the guidelines of the proof. In fact, assume first $\mathcal{E}$ totally real maximal in $\mathbb{C}^n$: it is not restrictive to suppose that $\mathcal{E} = \mathbb{R}^n$. The wave front set $WF(bv(f))$ is the same in the sense of hyperfunctions or distributions. Since it is controlled by the polar cone Γ^*, then $bv(f)$ is in fact the boundary value of a tempered holomorphic function in $W' << W$ (cf. e.g. [10]). By uniqueness this function must be f itself. If $\mathcal{E}$ is no more totally real, we consider the commutative diagram

$$\begin{array}{ccc} \mathcal{E} & \overset{j}{\hookrightarrow} & \mathbb{C}^n \\ \downarrow & \nearrow & \uparrow p \\ & \tilde{j} & \\ \mathcal{E}^{\mathbb{C}} & \underset{j^{\mathbb{C}}}{\hookrightarrow} & \mathbb{C}^n \times \bar{\mathbb{C}}^n, \end{array}$$

where $\mathcal{E}^{\mathbb{C}} \simeq \mathbb{C}^{2n-l_1}$ is the complexification of $\mathcal{E}$, $j^{\mathbb{C}}$ the complexification of the embedding $j : \mathcal{E} \to \mathbb{C}^n$, p the projection $\mathbb{C}^n \times \bar{\mathbb{C}}^n \to \mathbb{C}^n$ and $\tilde{j}$ the composition $\tilde{j} := p \circ j^{\mathbb{C}}$. We set

$$\tilde{\mathcal{W}} := \tilde{j}^{-1}(\mathcal{W}) \quad \text{and} \quad \tilde{f} := f \circ \tilde{j}.$$

Now, $\tilde{\mathcal{W}}$ is a wedge with totally real edge $\mathcal{E}$ in $\mathcal{E}^{\mathbb{C}}$. Hence $\tilde{f}$ is tempered by what we have just seen and thus f itself is tempered. In particular, it satisfies the much weaker condition of Theorem 2.1.
(b) By the commutation formula (2.2) we may apply Theorem 2.1 taking also into account the second half part of Remark 2.3.

□

Corollary 2.5. *Let $\mathcal{W} = \mathbb{R}^n + i\Gamma \subset \mathbb{C}^n$ where Γ is a conic neighborhood of the closed quadrant $\{y : y_j \geq 0$ for any $j\} \setminus \{0\}$. Let $f \in hol(\mathcal{W})$ have boundary value $bv_{\mathbb{R}^n}(f)$ which is bounded and separately continuous in x_j for any j. Then f is uniformly continuous in $\mathcal{W}' \cup \mathbb{R}^n$ for any $\mathcal{W}' << \mathcal{W}$. In particular, f is continuous in $\mathbb{R}^n$.*

Remark 2.6. There are bounded separately continuous functions in $\mathbb{R}^n$ which are not continuous: the function $f(x) = \frac{\underset{j}{\Pi} x_j}{(\sum\limits_j x_j^2)^{\frac{n}{2}}}$ is an easy example. The point is that they do not extend holomorphically to a quadrant (or equivalently their wave front set is not contained in a quadrant since quadrant is stable under "polarization").

Remark 2.7. Let $\mathcal{W}$ be a wedge of dimension $2n$ but not necessarily endowed with a C^ω edge $\mathcal{E}$. We assume that $\mathcal{E}$ is embedded as a compact subset in a manifold $\tilde{\mathcal{E}}$ of the same dimension. We also suppose that $\tilde{\mathcal{E}}$ is a graph, e.g. over (z', x''), choose $|y''| = 1$ and assume that $\mathcal{E}_{\epsilon_\sigma} + i\sigma y'' \subset \mathcal{W}$. If f is holomorphic in $\mathcal{W}$ and bounded in a neighborhood of $\partial\mathcal{E}$, there is well defined an action over $\varphi \in \mathrm{hol}(\mathcal{W}) \cap C^0(\bar{\mathcal{E}})$ by

$$\langle bvf, \varphi\rangle = \lim_\sigma \int_{\mathcal{E}} f(z + i\sigma y'')\varphi(z) dz' \wedge d\bar{z}' \wedge dz''.$$

This is analogous to Remark 1.2. By Stokes formula, the limit exists and is independent of the choice of y''. We conjecture that if the limit is controlled by $||\varphi||_{C^0}$, then f is tempered with growth $|y''|^{-\dim\mathcal{E}}$. Notice that the converse is true: temperateness of index $\dim\mathcal{E}$ implies the control of the limits by $||\varphi||_{C^{\dim\mathcal{E}+1}}$. This can be proved in the same way as in [10] Theorem 3.1.15 by replacing the functions $\Psi = \sum_\alpha \frac{\partial^\alpha\varphi(x) i y^\alpha}{\alpha!}$ by new functions Ψ satisfying $\bar{\partial}_{z_i}\Psi|_{\mathcal{E}} = O^{N+1}$ and $\Psi|_{\mathcal{E}} = \varphi$.

We stay in the general situation in which $\mathcal{E}$ is not necessarily C^ω and also assume that the dimension of $\mathcal{W}$ is not necessarily $2n$. Hyperfunctions do not exist any more. The more general situation to try to start from are ultradistributions $(G^d)'$ dual to Gevrey functions G^d which are defined as soon as $\mathcal{W}$ has class G^d.

We do not know in general the following.

(i) Let σ denote the Euclidean distance to $\mathcal{E}$ (which was coinciding with $\mathsf{Im}\,\tau$ in Section 1). When $|f| \underset{\sim}{<} e^{\sigma^{\frac{1}{1-d}}}$, is there defined $bv_{\mathcal{E}}(f)$ as a ultradistribution in $(G^d)'$? This is true when $\mathcal{E} = \mathbb{R}^n$ and even the converse is true: if the hyperfunction boundary value is in $(G^d)'$, then $|f| \underset{\sim}{<} e^{\sigma^{\frac{1}{1-d}}}$.

(ii) Does this new boundary value satisfy the commutation relation (2.2)?

If we have positive answer to these questions, we can generalize Theorem 2.4 to manifolds of class G^d for $d > 2$. Note in fact that it is this restraint for d which makes the expression $e^{\sigma^{\frac{1}{1-d}}}$ equal to $e^{\frac{1}{\sigma^{1-\epsilon}}}$ for some $\epsilon > 0$.

However, in the "limit" case $d = \infty$, that is, $G^d = C^\infty$, we have positive answer. In this situation, the growth at $\mathcal{E}$ which characterizes the existence of boundary value in the space of distributions is the temperateness described by

$$|f| \underset{\sim}{<} \sigma^{-k} \quad \text{for some } k \tag{2.3}$$

(cf. [10]). Indeed the necessity of (2.3) for existence of distribution boundary value is only true when $\dim \mathcal{W} = 2n$ (but we do not need this for our discussion). So, let $\mathcal{W}$ be a wedge with edge $\mathcal{E}$ contained in a submanifold $\mathcal{M} \subset \mathbb{C}^n$ of codimension l_2; here both $\mathcal{M}$ and $\mathcal{E}$ are assumed to be C^∞. Under (2.3) it is well defined $bv_{\mathcal{E}}(f)$ as the distribution acting on $\varphi \in C^\infty_c(\mathcal{E})$ by

$$\langle bv_{\mathcal{E}}(f), \varphi \rangle = \lim_{\sigma \to 0} \int_{\mathcal{E}_\sigma} f\, \varphi \circ \pi_\sigma dv,$$

where $\mathcal{E}_\sigma$ are manifolds of the same dimension as $\mathcal{E}$ such that there is a submersion $\mathcal{M} \twoheadrightarrow \mathcal{E}$ which induces diffeomorphisms $\mathcal{E}_\sigma \overset{\pi_\sigma}{\tilde{\to}} \mathcal{E}$ with π_σ C^k-converging to te identity. Here k is the constant of (2.3). It is useful to recall the relation between k and the order of the distribution $bv(f)$ that we denote by m; we have

$$m \leq k + 1 \quad \text{and} \quad k \leq m + \dim(\mathcal{E})$$

(cf. [7]). It is not clear at first glance if this point of view of taking bv from $\mathcal{W}$ to $\mathcal{E}$ and then restrictions to I_ξ is consistent with the previous one of taking separate bv on I_ξ along each single A^+_ξ. Here is the crucial statement which relates the two operations.

Theorem 2.8. *Let $f \in hol(\mathcal{W})$ be tempered and with $bv_{\mathcal{E}}(f) \in C^0(\mathcal{E})$; then*

$$\left(bv^{\mathcal{W}}_{\mathcal{E}}(f) \right)|_{I_\xi} = bv^{A^+_\xi}_{I_\xi}\left(f|_{A^+_\xi} \right).$$

Proof. We choose $\xi_o \in \Lambda$ and insert into a plane of parameters Λ_{ξ_o} parallel to $T_0\mathcal{E}$ and of the same dimension. We set $\mathcal{W}_{\xi_o} := \Phi(\Lambda_{\xi_o} \times \Delta^+)$; this is a manifold with boundary contained in $\mathcal{W}$. We denote by η the variable in Λ_{ξ_o}, take an approximation $\{\chi_\nu(\eta)\}_\nu$ of the Dirac measure δ_{ξ_o} at ξ_o, take a test function $\psi \in C^\infty_c(\mathcal{E})$ and set

$$F_\nu(\sigma) := \iint \chi_\nu(\eta) \left(f \circ \Phi \right)(\eta, \rho + i\sigma) \left(\psi \circ \Phi \right)(\eta, \rho)\, d\eta d\rho.$$

Since $bv(f)$ has order 0, then $|f| \lesssim \sigma^{-k}$ for $k \leq \dim(\mathcal{E})$; we then have

$$\begin{aligned}
|F^{(j)}_\nu(\sigma)| &= |(-i)^j \iint \partial^j_\rho \left(f \circ \Phi \right)(\eta, \rho + i\sigma) \chi_\nu(\eta) \left(\psi \circ \Phi \right)(\eta, \rho)\, d\eta d\rho| \\
&\lesssim \sigma^{-k} \iint |\chi_\nu \partial^j_\rho (\psi \circ \Phi)|\, d\eta d\rho \\
&\lesssim \sigma^{-k} c_j \quad \text{for } c_j = \sup |\partial^j_\rho (\psi \circ \Phi)|,
\end{aligned}$$

where the first equality follows from CR relations and the second from integration by parts. On the other hand, after k integrations, we get the bound

$$|F^{(1)}_\nu(\sigma)| \lesssim \sum_{j=0}^{k+1} c_j = ||\psi||_{C^{k+1}}.$$

It follows that $\{F_\nu\}$ is equicontinuous and therefore we can interchange $\lim_\nu \lim_\sigma$ with $\lim_\sigma \lim_\nu$. Also, by the hypothesis of continuity of $bv_{\mathcal{E}}(f)$, we have

$$(bv_{\mathcal{E}}(f))|_{I_{\xi_o}} = \lim_\nu (\chi_\nu \circ \Phi^{-1})(bv_{\mathcal{E}}(f)).$$

If $\psi_{I_{\xi_o}} \in C_c^\infty(I_{\xi_o})$ and ψ is its continuation constant in η), we then get

$$\begin{aligned}
\langle bv_{\mathcal{E}}(f)|_{I_{\xi_o}}, \psi_{I_{\xi_o}}\rangle &= \lim_\nu \langle \chi_\nu (bv_{\mathcal{E}}(f) \circ \Phi, \psi \circ \Phi\rangle \\
&= \lim_\nu \lim_\sigma \iint (f \circ \Phi)(\eta, \rho + i\sigma) \chi_\nu(\eta) (\psi \circ \Phi)(\eta, \rho)\, d\eta d\rho \\
&= \lim_\sigma \lim_\nu \iint * \\
&= \lim_\nu \lim_\sigma \iint * \\
&= \lim_\sigma \int_I (f \circ \Phi)(\xi_o, \rho + i\sigma)(\psi \circ \Phi)(\xi_o, \rho) d\rho \\
&= bv_{I_{\xi_o}}^{A_{\xi_o}^+} \langle f|_{A_{\xi_o}^+}, \psi_{I_{\xi_o}}\rangle.
\end{aligned}$$

□

In conjunction with Theorem 2.1 this yields

Theorem 2.9. *Let $\mathcal{W}$ be a wedge in a C^∞ manifold $\mathcal{M} \subset \mathbb{C}^n$ with C^∞ edge $\mathcal{E}$ and let f be a CR continuous function in $\mathcal{W}$, tempered at $\mathcal{E}$ and with continuous boundary value at $\mathcal{E}$. Then f is uniformly continuous on $\mathcal{W}' \cup \mathcal{E}$ for any $\mathcal{W}' << \mathcal{W}$.*

In case $\mathcal{W}$ is a wedge of dimension $2n$ and f is holomorphic on $\mathcal{W}$, we regain the conclusions of Rosay in [12].

Remark 2.10. We can avoid the regularization procedure used by Rosay because we know from Theorem 2.8 that $(bv_{\mathcal{E}}(f))|_{I_\xi}$ is the boundary value along A_ξ^+: thus we enter in the setting of Theorem 2.1.

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