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Abstract

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Keywords

transform, options, barrier, pricing, mellin, dynamics:, jump-diffusion, assets, approach, underlying

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Keywords: barrier options; exponential barriers; jump-diffusion dynamics; options on futures; Mellin transform; Black–Scholes kernel

MSC: 91G20; 91B25; 91G80; 35A22

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1. Introduction

A European call option is a financial contract that gives the holder the right, but not the obligation, to buy an underlying asset from the writer at an agreed strike price on a predetermined expiry date. A European put option is similar but gives the right to sell instead. Examples of the underlying asset, or simply the underlying, are stocks or futures contracts.

Options are mainly used for speculation and hedging. For example, an investor who believes that the share price for a certain stock is going to rise within the next month may invest by buying a call option on that stock. On the other hand, an investor who already owns shares of a certain stock may insure against a temporary fall in the share price by deciding to buy a put option to minimize the risk of a potential loss.

At the time the option contract is agreed upon, the holder must pay a certain amount, known as the premium or time-zero option price, to the writer. Option valuation, or option pricing, is the fundamental problem of determining a fair price for this premium. For European-style contracts, analytical expressions for the call and put premiums are given by the Nobel Prize-winning Black–Scholes formulas [1].

Call and put options can be characterized by their so-called payoff functions. We denote a payoff function by $g : \mathbb{R}_+ \to \mathbb{R}$, where $\mathbb{R}_+ = (0, \infty)$, which is typically piecewise linear. If $S(T)$ is the asset price at the expiry date T and K is the strike price, then the call and put payoffs are $g(S(T)) = (S(T) - K)^+$ and $g(S(T)) = (K - S(T))^+$, respectively, where $(z)^+ = \max(z, 0)$ for any $z \in \mathbb{R}$. Thus the respective call and put payoff functions are $g(x) = (x - K)^+$ and $g(x) = (K - x)^+$.

Options are attractive because they can be used to create a wide range of trading strategies characterised by different payoff functions. Suppose that $K_1 < K_2 < K_3 < K_4$. Denote the usual indicator function of a set A by $\mathbf{1}_A$, i.e., $\mathbf{1}_A(x) = 1$ if $x \in A$ and $\mathbf{1}_A(x) = 0$ if $x \notin A$. Some popular trading strategies with their corresponding payoff functions are given in Table 1 (see Hull [2] and Wilmott et al. [3] for more details):

Table 1. Examples of options and their payoff functions.

Type	**Payoff Function** $g(x)$
put	$(K-x)^+$
bear spread	$(K_2-x)^+ - (K_1-x)^+$
call	$(x-K)^+$
bull spread	$(x-K_1)^+ - (x-K_2)^+$
digital call	$\mathbf{1}_{[K,\infty)}(x)$
asset-or-nothing call	$x\mathbf{1}_{[K,\infty)}(x)$
butterfly spread	$(x-K_1)^+ + (x-K_3)^+ - 2(x-K_2)^+, -K_1-K_3+2K_2=0$
iron condor	$(x-K_1)^+ - (x-K_2)^+ - (x-K_3)^+ + (x-K_4)^+, -K_1+K_2+K_3-K_4=0$
straddle	$(x-K)^+ + (K-x)^+$
strip	$(x-K)^+ + 2(K-x)^+$
strap	$(K-x)^+ + 2(x-K)^+$
strangle	$(x-K_2)^+ + (K_1-x)^+$

Consider the stochastic differential equation

$$\mathrm{d}S(t) = (r-D)S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t), \tag{1}$$

where $S = \{S(t) : t \geqslant 0\}$ is the underlying asset price process and $W = \{W(t) : t \geqslant 0\}$ is a Wiener process with respect to the risk-neutral measure. Here, the risk-free rate r, the dividend yield D, and the volatility σ are assumed to be constants with $r, \sigma > 0$ and $D \geqslant 0$. Denote the generic European option price at time t by $V(t)$ and the corresponding payoff function by g. At expiry we therefore have $V(T) = g(S(T))$. It is well known that $V(t) = v(S(t), t)$, where the option pricing function $v = v(x,t)$ satisfies the Black–Scholes partial differential equation (PDE)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (r-D)x\frac{\partial v}{\partial x} - rv = 0. \tag{2}$$

The premium is obtained by setting $V(0) = v(S(0), 0)$, where $S(0)$ is today's known asset price.

While geometric Brownian motion assumed in the Black–Scholes asset price model (1) is convenient, it cannot capture many of the features of asset price returns, e.g., the skew/smile features of the implied volatility surface. In the absence of dividend payments, Merton [4] considered a jump-diffusion process that allows for the probability of the asset price to change at large magnitudes irrespective of the time interval between successive observations. The jumps in the asset price can be incorporated by introducing an additional source of uncertainty into the asset price dynamics. Empirical studies have revealed that the asset price is better described by a process with a discontinuous sample path (see, for instance, Rosenfeld [5], Jarrow and Rosenfeld [6], Ball and Torous [7], and Brown and Dybvig [8]). Cont and Tankov [9] showed that unlike standard diffusion models such as (1), jump-diffusion models produce rich structures of the distribution of asset returns and implied volatility surfaces.

To account for the possibility of instantaneous jumps in the asset price, Merton [4] proposed the following modification of (1) by assuming that the discontinuous jumps arrive as a Poisson process (here we incorporate dividend payments):

$$\mathrm{d}S(t) = [r - D - \lambda E(Y-1)]S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t) + (Y-1)S(t)\,\mathrm{d}N(t), \tag{3}$$

where Y is a nonnegative continuous random variable with $Y-1$ denoting the impulse change in the asset price from $S(t)$ to $YS(t)$ as a result of the jump, E is the expectation operator, and $N = \{N(t) : t \geqslant 0\}$ is a Poisson process with constant intensity λ and such that $\mathrm{d}N(t) = 1$ (respectively, $\mathrm{d}N(t) = 0$) with probability λdt (respectively, $1 - \lambda dt$). It is assumed that $W(t)$, $N(t)$, and Y are independent for each t, and that the asset price jumps occur independently and identically. Analogous to the Black–Scholes PDE (2), it can be shown [4] that the European option pricing function v satisfies the Black–Scholes partial integro-differential equation (PIDE)

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + (r - D)x\frac{\partial v}{\partial x} - rv + \lambda I(v; f_Y) = 0, \tag{4}$$

where $I(v; f_Y)$ is the integral operator defined by

$$I(v; f_Y)(x,t) = \int_0^\infty [v(xy,t) - v(x,t)] f_Y(y)\, dy \tag{5}$$

and f_Y is the probability density function of Y. Using Mellin transform techniques adapted from Rodrigo and Mamon [10], Li and Rodrigo [11] studied (4) and found exact pricing formulas for European options with general payoffs such as those in Table 1. A PIDE analogue of Dupire's PDE was also derived and used to find an explicit formula for the implied volatility. In the same vein, Rodrigo and Goard [12] considered (4) with a time varying D and obtained exact pricing formulas for European options on discrete dividend-paying assets.

A barrier option is an exotic path-dependent option contract where the right to buy or sell is activated (in the case of a knock-in barrier option) or extinguished (in the case of a knock-out barrier option) when the underlying reaches a certain barrier price during the lifetime of the contract. If the option expires inactive or extinguishes, then it may be worthless or there may be a cash rebate paid out. Since payoff opportunities are more limited, a barrier option is cheaper than a similar European option. Barriers are generally fixed but time-dependent barriers can be considered as well. The rationale for a barrier option is to provide a hedge at a lower premium than a conventional option.

There are many results in the academic literature on continuously monitored barrier options. One of the earliest dates back to the work of Merton [13], who gave a closed-form solution for the price of a continuously monitored down-and-out European call. One approach, mainly for fixed barriers, identifies pathwise hedging strategies for European-style derivatives that either uniquely determine or provide an admissible range for the barrier option price (see, for instance, Carr et al. [14] and Brown et al. [15]). A static hedge using calls and puts for a time-dependent single barrier option was given by Andersen et al. [16]. Their result also applies to linear diffusions with compound Poisson jumps but the hedging strategy depends on knowing the values of the barrier contract to be hedged at certain times before expiry. Geman and Yor [17] used a probabilistic approach for constant double barrier options in the Black–Scholes model. Kunitomo and Ikeda [18] introduced a method for pricing time-dependent barrier options in the Black–Scholes model with the help of the joint density of the asset and its maximum and minimum. Lattice methods were utilized by Boyle and Lau [19] and Ritchken [20], while finite difference and finite element methods were employed by Boyle and Tian [21] and Zvan et al. [22], respectively.

Recent results on time-dependent double barrier options include Fourier series expansions (Hui and Lo [23]), Green's functions (Dorleitner et al. [24]) and Laplace transforms (Pelsser [25]). Davydov and Linetsky [26] applied spectral methods to obtain constant double barrier option prices in constant elasticity of variance models. The boundary element method was used to derive an integral representation of the barrier option price in Guardasoni and Sanfelici [27] and Shen and Hsiao [28] under a Black–Scholes framework; in Guardasoni and Sanfelici [29] under stochastic volatility and jumps; and in Ballestra et al. [30] under a mixed fractional Brownian motion. Buchen and Konstandatos [31] proposed a method of images approach to price double barrier options with exponential barriers, extending the results by Buchen [32] for single fixed barriers.

Many options have as the underlying not the cash product but the corresponding futures contract, which is often more liquid and involves lower transaction costs [3]. Recall that a forward contract, made at time t^*, is a contract where the holder pays the writer the deterministic amount $F(t^*; T^*)$ (known as the forward price) at the delivery date T^* and then receives the stochastic amount $S(T^*)$ at the same time. Under the asset price dynamics (1), it is known [3] that

$$F(t^*; T^*) = S(t^*)\mathrm{e}^{(r-D)(T^*-t^*)}. \tag{6}$$

It can be shown [2,33] that when interest rates are deterministic, the price of a futures contract is the same as the price of a forward contract. One can then consider a European option, with payoff function g and expiry date T, on an underlying futures contract on an asset with price process S satisfying (1) and with delivery date T^* such that $T < T^*$. Thus the payoff of a European option on a futures contract is

$$g(F(T; T^*)) = g(S(T)\mathrm{e}^{(r-D)(T^*-T)}). \tag{7}$$

In the case of a call, where $g(x) = (x - K)^+$, then after some modification to include D the so-called "Black-76" formula [34] is obtained.

The primary objective of this article is to employ a Mellin transform approach to price barrier options with general payoffs when the underlying is modeled by the jump-diffusion dynamics (3). The use of Mellin transforms in option pricing has been developed by the present author in a series of articles [10–12,35,36]. In the absence of jumps (i.e., $\lambda = 0$ in (4)), we recover the classical Black–Scholes framework (1). In fact, the corresponding barrier option pricing problem was considered by Guardasoni, Rodrigo, and Sanfelici in [36], where the Mellin transform was used to price single and double barrier options. Although the results in [36] are applicable to general time-dependent barriers, it is necessary to solve an associated linear Volterra integral equation of the first kind (or a coupled system of two linear Volterra integral equations of the first kind for double barriers). This integral cannot be solved analytically and one must resort to a numerical approximation. In this article we incorporate jumps as in (3) but consider only exponential barriers akin to that studied in Buchen and Konstandatos [31]. Note that this includes fixed barriers as a special case. We will also use a Mellin transform technique but avoid the introduction of Volterra integral equations of the first kind.

The secondary objective of this article is to obtain, with essentially the same amount of work, exact pricing formulas for barrier options with payoffs that include those in Table 1 but when the underlying is a futures contract, with the possibility of jumps in the corresponding asset. To my knowledge, even in the absence of jumps, barrier options on futures have not been previously considered in this general framework but the proposed Mellin transform approach can handle such exotic options, with or without jumps in the asset price dynamics associated with the futures contract.

The outline of this paper is as follows. In Section 2 we recall and also derive some preliminary results involving the generalized Black–Scholes kernel and jump function with the aim of using these to obtain the European option pricing function when the underlying behaves according to jump-diffusion dynamics. We also find image function solutions in the jump-diffusion case that will be needed to price barrier options. Here the reason for choosing an exponential barrier will be evident. We formulate the barrier option pricing problem in Section 4. Knock-out barrier options are considered in Section 4 while Section 5 deals with knock-in options. Illustrative examples for both types are also given. We discuss how to price options on futures in Section 6 and give brief concluding remarks in Section 7.

2. Preliminary Results

To ensure that our results are applicable to other problems, here we will consider the PIDE

$$\mathscr{L}_\lambda v = \frac{\partial v}{\partial t} + c_2 x^2 \frac{\partial^2 v}{\partial x^2} + c_1 x \frac{\partial v}{\partial x} + c_0 v + \lambda I(v; f_Y) = 0, \tag{8}$$

where $c_0, c_1, c_2 \in \mathbb{R}$ with $c_2 > 0$, and $I(v; f_Y)$ is the integral operator defined in (5). Note that $\lambda = 0$ reduces $\mathscr{L}_0$ to a Black–Scholes-type differential operator. If the underlying asset dynamics is given by (3), then we see from (4) that $c_2 = \sigma^2/2$, $c_1 = r - D$, and $c_0 = -r$. On the other hand, for options on futures, if we make the substitution $v(x,t) = \bar{v}(\bar{x},\bar{t})$ into (4), where $\bar{x} = xe^{(r-D)(T^*-t)}$ and $\bar{t} = t$ (cf. (6)), then we obtain the PIDE

$$\frac{\partial \bar{v}}{\partial \bar{t}} + \frac{1}{2}\sigma^2 \bar{x}^2 \frac{\partial^2 \bar{v}}{\partial \bar{x}^2} - r\bar{v} + \lambda I(\bar{v}; f_Y) = 0, \tag{9}$$

which is of the form (8) with $c_2 = \sigma^2/2$, $c_1 = 0$, and $c_0 = -r$. We observe that when $\lambda = 0$, (9) simplifies to the PDE for pricing options on futures; see for instance Wilmott et al. [3]. We will return to the pricing of options on futures in Section 6.

For the convenience of the reader we summarize here some results pertaining to the Mellin transform (see, for example, ([37], pp. 362–363) or [38]). The Mellin transform of a function $f : \mathbb{R}_+ \to \mathbb{R}$ is

$$\mathscr{M}\{f(x);\xi\} = \hat{f}(\xi) = \int_0^\infty x^{\xi-1} f(x)\,\mathrm{d}x, \tag{10}$$

provided the improper integral converges at the complex number ξ. This transform is useful for Black–Scholes-type equations because of the following properties for the derivatives of f, namely,

$$\mathscr{M}\{xf'(x);\xi\} = -\xi\hat{f}(\xi), \quad \mathscr{M}\{x^2 f''(x);\xi\} = (\xi+\xi^2)\hat{f}(\xi). \tag{11}$$

The convolution of $f : \mathbb{R}_+ \to \mathbb{R}$ and $g : \mathbb{R}_+ \to \mathbb{R}$ is defined as

$$(f * g)(x) = \int_0^\infty \frac{1}{y} f\left(\frac{x}{y}\right) g(y)\,\mathrm{d}y. \tag{12}$$

Note that the convolution operator is both commutative and associative. It follows that

$$\mathscr{M}\{(f*g)(x);\xi\} = \hat{f}(\xi)\hat{g}(\xi), \tag{13}$$

also known as the convolution property.

2.1. Generalized Black–Scholes Kernel and Properties

The following lemma motivates an extension (see (18) below) of the Black–Scholes kernel originally introduced in [10,35]:

Lemma 1. *Let $a, b, c \in \mathbb{R}$ with $a > 0$. Then*

$$\mathscr{M}\left\{\frac{e^{a-b+c}}{\sqrt{2a}} x\Phi'\left(\frac{\log(x)+2a-b}{\sqrt{2a}}\right);\xi\right\} = \mathscr{M}\left\{\frac{e^{c}}{\sqrt{2a}} \Phi'\left(\frac{\log(x)-b}{\sqrt{2a}}\right);\xi\right\} = e^{a\xi^2+b\xi+c}, \tag{14}$$

where Φ is the cumulative distribution function on a standard normal random variable. Hence defining

$$\mathscr{K}(x) = \frac{e^{a-b+c}}{\sqrt{2a}} x\Phi'\left(\frac{\log(x)+2a-b}{\sqrt{2a}}\right) = \frac{e^{c}}{\sqrt{2a}} \Phi'\left(\frac{\log(x)-b}{\sqrt{2a}}\right), \tag{15}$$

it follows that

$$\mathscr{K}\left(\frac{x}{y}\right) = \frac{\partial}{\partial y}\left[-e^{a-b+c} x\Phi\left(\frac{\log(x/y)+2a-b}{\sqrt{2a}}\right)\right], \quad \frac{1}{y}\mathscr{K}\left(\frac{x}{y}\right) = \frac{\partial}{\partial y}\left[-e^{c}\Phi\left(\frac{\log(x/y)-b}{\sqrt{2a}}\right)\right]. \tag{16}$$

Proof. Let α_1, α_2, β, γ_1, and γ_2 be independent of x. Suppose that $Z \sim N(0,1)$. Using the definition of the Mellin transform (10) and a property of normal random variables, we obtain

$$\mathscr{M}\{\alpha_1 x\Phi'(\beta\log(x)+\gamma_1);\xi\} = \frac{\alpha_1}{\beta} e^{-(\xi+1)\gamma_1/\beta} E(e^{(\xi+1)/\beta Z}) = \frac{\alpha_1}{\beta} e^{-(\xi+1)\gamma_1/\beta} e^{(\xi+1)^2/(2\beta^2)}$$

and

$$\mathscr{M}\{\alpha_2\Phi'(\beta\log(x)+\gamma_2);\xi\} = \frac{\alpha_2}{\beta}\mathrm{e}^{-\gamma_2\xi/\beta}E(\mathrm{e}^{\xi Z/\beta}) = \frac{\alpha_2}{\beta}\mathrm{e}^{-\gamma_2\xi/\beta}\mathrm{e}^{\xi^2/(2\beta^2)}.$$

Choosing

$$\alpha_1 = \frac{\mathrm{e}^{a-b+c}}{\sqrt{2a}}, \quad \alpha_2 = \frac{\mathrm{e}^{c}}{\sqrt{2a}}, \quad \beta = \frac{1}{\sqrt{2a}}, \quad \gamma_1 = \frac{2a-b}{\sqrt{2a}}, \quad \gamma_2 = -\frac{b}{\sqrt{2a}}$$

and algebraic manipulations yield (14). Equation (15) is justified by continuity of the arguments of the Mellin transforms in (14). The properties in (16) can be easily shown by evaluating the derivatives on the right-hand side and comparing with (15). □

Next we set

$$a = c_2(u-t), \quad b = -(c_1-c_2)(u-t), \quad c = c_0(u-t)$$

in Lemma 1. Define the auxiliary functions

$$z_1(x,t,u) = \frac{\log(x)+(c_1+c_2)(u-t)}{\sqrt{2c_2(u-t)}}, \quad z_2(x,t,u) = \frac{\log(x)+(c_1-c_2)(u-t)}{\sqrt{2c_2(u-t)}} \tag{17}$$

and the generalized Black–Scholes kernel

$$\mathscr{K}(x,t,u) = \frac{x\mathrm{e}^{(c_0+c_1)(u-t)}}{\sqrt{2c_2(u-t)}}\Phi'(z_1(x,t,u)) = \frac{\mathrm{e}^{c_0(u-t)}}{\sqrt{2c_2(u-t)}}\Phi'(z_2(x,t,u)). \tag{18}$$

Then (16) gives the Black–Scholes kernel identities

$$\begin{aligned} \mathscr{K}\left(\frac{x}{y},t,u\right) &= \frac{\partial}{\partial y}\left[-x\mathrm{e}^{(c_0+c_1)(u-t)}\Phi\left(z_1\left(\frac{x}{y},t,u\right)\right)\right], \\ \frac{1}{y}\mathscr{K}\left(\frac{x}{y},t,u\right) &= \frac{\partial}{\partial y}\left[-\mathrm{e}^{c_0(u-t)}\Phi\left(z_2\left(\frac{x}{y},t,u\right)\right)\right]. \end{aligned} \tag{19}$$

Finally, from (14) we have

$$\hat{\mathscr{K}}(\xi,t,u) = \mathscr{M}\{\mathscr{K}(x,t,u);\xi\} = \mathrm{e}^{-p(\xi)(u-t)}, \quad p(\xi) = -c_2\xi^2+(c_1-c_2)\xi-c_0. \tag{20}$$

We see that (17)–(20) reduce to analogous results introduced in [10] in the special case when $c_2=\sigma^2/2$, $c_1=r-D$, and $c_0=-r$.

2.2. Properties of the Jump Function

We refer the reader to [11,12] for further details regarding this subsection. Construct a sequence $(h_j)_{j=0}^{\infty}$, where $h_j = h_j(x)$, by

$$h_0(x) = \delta(x-1), \quad h_1(x) = \frac{1}{x}f_Y\left(\frac{1}{x}\right), \quad h_{j+1}(x) = (h_1 * h_j)(x), \quad j \geqslant 1, \tag{21}$$

where f_Y is the probability density function of the random variable Y (see (3) and (5)) and δ is the Dirac delta function. It was shown in [11] that

$$\hat{h}_j(\xi) = \mathscr{M}\{h_j(x);\xi\} = [E(Y^{-\xi})]^j, \quad j \geqslant 0. \tag{22}$$

Defining the jump function

$$\mathscr{J}(x,t,u) = \mathrm{e}^{-\lambda(u-t)}\sum_{j=0}^{\infty}\frac{[\lambda(u-t)]^j}{j!}h_j(x), \tag{23}$$

we see from (22) that
$$\hat{\mathscr{J}}(\xi,t,u) = \mathrm{e}^{-\lambda[1-E(Y^{-\xi})](u-t)}. \tag{24}$$

Example 1. *Let Y be a lognormal random variable with parameters μ_Y and σ_Y^2. Then $Y = \mathrm{e}^X$, where $X \sim N(\mu_Y, \sigma_Y^2)$, and*
$$f_Y(y) = \frac{1}{\sigma_Y y}\Phi'\Big(\frac{\log(y)-\mu_Y}{\sigma_Y}\Big), \quad y > 0.$$
It follows that
$$E(Y^{-\xi}) = E(\mathrm{e}^{-\xi X}) = \mathrm{e}^{-\mu_Y\xi+\sigma_Y^2\xi^2/2} \tag{25}$$
using a property of normal random variables. Hence (22) gives
$$\hat{h}_j(\xi) = [E(Y^{-\xi})]^j = \mathrm{e}^{-j\mu_Y\xi+j\sigma_Y^2\xi^2/2} \tag{26}$$
and
$$h_0(x) = \delta(x-1), \quad h_j(x) = \frac{1}{\sigma_Y\sqrt{j}}\Phi'\Big(\frac{\log(x)+\mu_Y j}{\sigma_Y\sqrt{j}}\Big), \quad j \geqslant 1,$$
where we used (14) with $a = j\sigma_Y^2/2$, $b = -j\mu_Y$, and $c = 0$. Thus the jump function for a lognormal random variable Y with parameters μ_Y and σ_Y^2 is
$$\mathscr{J}(x,t,u) = \mathrm{e}^{-\lambda(u-t)}\delta(x-1) + \mathrm{e}^{-\lambda(u-t)}\sum_{j=1}^{\infty}\frac{[\lambda(u-t)]^j}{j!}\frac{1}{\sigma_Y\sqrt{j}}\Phi'\Big(\frac{\log(x)+\mu_Y j}{\sigma_Y\sqrt{j}}\Big).$$

2.3. European Option Pricing Function for Underlyings with Jump-Diffusion Dynamics

We are ready to solve the PIDE (8) together with the final condition $v(x,T) = g(x)$ using the Mellin transform. Let $v_0 = v_0(x,t)$ satisfy the final value problem for the PDE
$$\mathscr{L}_0 v_0 = \frac{\partial v_0}{\partial t} + c_2 x^2\frac{\partial^2 v_0}{\partial x^2} + c_1 x\frac{\partial v_0}{\partial x} + c_0 v_0 = 0, \quad v_0(x,T) = g(x).$$

Denote by
$$\hat{v}(\xi,t) = \int_0^\infty x^{\xi-1}v(x,t)\,\mathrm{d}x, \quad \hat{v}_0(\xi,t) = \int_0^\infty x^{\xi-1}v_0(x,t)\,\mathrm{d}x$$
the Mellin transforms of v and v_0 with respect to x, respectively. We assume that the random variable Y is such that $E(Y^{-\xi})$ is finite. Taking the Mellin transform of (8) and using the derivative properties in (11), we get
$$\hat{\mathscr{L}}_\lambda\hat{v}(\xi,t) = \frac{\partial\hat{v}}{\partial t}(\xi,t) - [p(\xi)+\lambda-\lambda E(Y^{-\xi})]\hat{v}(\xi,t) = 0, \tag{27}$$
where p is defined in (20). Moreover, $\hat{v}(\xi,T) = \hat{g}(\xi)$, where $\hat{g}$ is the Mellin transform of g. The solution of (27) is therefore
$$\hat{v}(\xi,t) = \mathrm{e}^{-p(\xi)(T-t)}\mathrm{e}^{-\lambda[1-E(Y^{-\xi})](T-t)}\hat{g}(\xi). \tag{28}$$

Since v_0 is the solution of (8) subject to $v_0(x,T) = g(x)$ when $\lambda = 0$, we see that
$$\hat{v}_0(\xi,t) = \mathrm{e}^{-p(\xi)(T-t)}\hat{g}(\xi) = \hat{\mathscr{K}}(\xi,t,T)\hat{g}(\xi) \tag{29}$$
with the help of (20). Hence substituting (29) into (28) gives
$$\hat{v}(\xi,t) = \mathrm{e}^{-\lambda[1-E(Y^{-\xi})](T-t)}\hat{v}_0(\xi,t) = \hat{\mathscr{J}}(\xi,t,T)\hat{v}_0(\xi,t), \tag{30}$$
where we used (24).

Next, we proceed to invert (28) and (29). Recalling the convolution property (13), we obtain from (29) that

$$v_0(x,t) = (\mathscr{K}(\cdot,t,T) * g)(x) = \int_0^\infty \frac{1}{y}\mathscr{K}\Big(\frac{x}{y},t,T\Big)g(y)\,\mathrm{d}y. \tag{31}$$

Similarly, (30), (13), and (23) yield

$$v(x,t) = (\mathscr{J}(\cdot,t,T) * v_0(\cdot,t))(x) = \mathrm{e}^{-\lambda(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}(v_0(\cdot,t) * h_j)(x). \tag{32}$$

Note that if $\lambda = 0$, then (32) implies that

$$v(x,t) = (v_0(\cdot,t) * h_0)(x) = \int_0^\infty \frac{1}{z}v_0\Big(\frac{x}{z},t\Big)\delta(z-1)\,\mathrm{d}z = v_0(x,t)$$

as expected since there are no jumps.

Example 2. *Continuing with Example 1 and recalling (29) and (26), we see that the Mellin transform of (32) is*

$$\hat{v}(\xi,t) = \mathrm{e}^{-\lambda(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}\mathrm{e}^{-p(\xi)(T-t)-j\mu_Y\xi+j\sigma_Y^2\xi^2/2}\hat{g}(\xi). \tag{33}$$

However, (20) gives

$$-p(\xi)(T-t) - j\mu_Y\xi + \frac{1}{2}j\sigma_Y^2\xi^2 = a\xi^2 + b\xi + c,$$

where

$$a = c_2(T-t) + \frac{1}{2}j\sigma_Y^2, \quad b = -(c_1-c_2)(T-t) - j\mu_Y, \quad c = c_0(T-t).$$

If we define $\mathscr{K}_j = \mathscr{K}_j(x,t,u)$ such that its Mellin transform is

$$\hat{\mathscr{K}_j}(\xi,t,u) = \mathrm{e}^{-p(\xi)(u-t)-j\mu_Y\xi+j\sigma_Y^2\xi^2/2}, \quad j \geqslant 0,$$

then (15) gives

$$\mathscr{K}_j(x,t,u) = \frac{x\mathrm{e}^{(c_0+c_1)(u-t)+j\mu_Y+j\sigma_Y^2/2}}{[2c_2(u-t)+j\sigma_Y^2]^{1/2}}\Phi'(z_{j,1}(x,t,u)) = \frac{\mathrm{e}^{c_0(u-t)}}{[2c_2(u-t)+j\sigma_Y^2]^{1/2}}\Phi'(z_{j,2}(x,t,u)), \tag{34}$$

where

$$\begin{aligned} z_{j,1}(x,t,u) &= \frac{\log(x) + (c_1+c_2)(u-t) + j\mu_Y + j\sigma_Y^2}{[2c_2(u-t)+j\sigma_Y^2]^{1/2}}, \\ z_{j,2}(x,t,u) &= \frac{\log(x) + (c_1-c_2)(u-t) + j\mu_Y}{[2c_2(u-t)+j\sigma_Y^2]^{1/2}}. \end{aligned} \tag{35}$$

The analogous Black–Scholes kernel identities (16) are

$$\begin{aligned} \mathscr{K}_j\Big(\frac{x}{y},t,u\Big) &= \frac{\partial}{\partial y}\Big[-x\mathrm{e}^{(c_0+c_1)(u-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{y},t,u\Big)\Big)\Big], \\ \frac{1}{y}\mathscr{K}_j\Big(\frac{x}{y},t,u\Big) &= \frac{\partial}{\partial y}\Big[-\mathrm{e}^{c_0(u-t)}\Phi\Big(z_{j,2}\Big(\frac{x}{y},t,u\Big)\Big)\Big]. \end{aligned} \tag{36}$$

Thus (33) *can be expressed as*

$$\hat{v}(\xi,t) = \mathrm{e}^{-\lambda(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}\hat{\mathscr{K}}_j(\xi,t,T)\hat{g}(\xi).$$

Therefore if Y *is a lognormal random variable with parameters* μ_Y *and* σ_Y^2, *then the exact solution of* (8) *satisfying the final condition* $v(x,T) = g(x)$ *is*

$$v(x,t) = \mathrm{e}^{-\lambda(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}\int_0^{\infty}\frac{1}{y}\mathscr{K}_j\Big(\frac{x}{y},t,T\Big)g(y)\,\mathrm{d}y, \tag{37}$$

with $\mathscr{K}_j$ *defined as in* (34).

To illustrate (37), *consider a put payoff function* $g(x) = (K-x)^+$. *Then*

$$\int_0^{\infty}\frac{1}{y}\mathscr{K}_j\Big(\frac{x}{y},t,T\Big)g(y)\,\mathrm{d}y = K\int_0^{K}\frac{1}{y}\mathscr{K}_j\Big(\frac{x}{y},t,T\Big)\,\mathrm{d}y - \int_0^{K}\mathscr{K}_j\Big(\frac{x}{y},t,T\Big)\,\mathrm{d}y.$$

The Black–Scholes kernel identities in (36) *give*

$$\begin{aligned}\int_0^{K}\frac{1}{y}\mathscr{K}_j\Big(\frac{x}{y},t,T\Big)\,\mathrm{d}y &= \int_0^{K}\frac{\partial}{\partial y}\Big[-\mathrm{e}^{c_0(T-t)}\Phi\Big(z_{j,2}\Big(\frac{x}{y},t,T\Big)\Big)\Big]\,\mathrm{d}y\\ &= \mathrm{e}^{c_0(T-t)}\Phi\Big(-z_{j,2}\Big(\frac{x}{K},t,T\Big)\Big)\end{aligned}$$

and

$$\begin{aligned}\int_0^{K}\mathscr{K}_j\Big(\frac{x}{y},t,T\Big)\,\mathrm{d}y &= \int_0^{K}\frac{\partial}{\partial y}\Big[-x\mathrm{e}^{(c_0+c_1)(T-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{y},t,T\Big)\Big)\Big]\,\mathrm{d}y\\ &= x\mathrm{e}^{(c_0+c_1)(T-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(-z_{j,1}\Big(\frac{x}{K},t,T\Big)\Big).\end{aligned}$$

Therefore (37) *becomes*

$$\begin{aligned}v(x,t) = {}&K\mathrm{e}^{(c_0-\lambda)(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}\Phi\Big(-z_{j,2}\Big(\frac{x}{K},t,T\Big)\Big)\\ &- x\mathrm{e}^{(c_0+c_1-\lambda)(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}\mathrm{e}^{j\mu_Y+j\sigma_Y^2/2}\Phi\Big(-z_{j,1}\Big(\frac{x}{K},t,T\Big)\Big),\end{aligned}$$

where $z_{j,1}$ *and* $z_{j,2}$ *are given by* (35). *Note that in the special case when* $\lambda = 0$, $c_2 = \sigma^2/2$, $c_1 = r - D$, *and* $c_0 = -r$, *this recovers the Black–Scholes formula for a European put. A general piecewise linear payoff function* g, *corresponding to any of the option strategies given in Table 1 among others, can similarly be considered since the integrals appearing in* (37) *can always be evaluated with the help of the Black–Scholes kernel identities* (16) *(or* (36) *if* Y *is lognormal).*

2.4. Image Function Solutions

Here we construct so-called image function solutions of (8) that will be used to "piece together" the barrier option pricing functions later on (cf. [31] for the special case when $\lambda = 0$, $c_2 = \sigma^2/2$, $c_1 = r$, and $c_0 = -r$).

Suppose that $B = B(t)$ is an exponential barrier, i.e.,

$$B(t) = B(T)\mathrm{e}^{-\mu(T-t)}, \tag{38}$$

where $\mu \in \mathbb{R}$. Note that (38) includes the fixed barrier case if we set $\mu = 0$, so that $B(t) = B(T)$ for all $0 \leqslant t \leqslant T$.

Let $v_1 = v_1(x,t)$ be any solution of (8). We claim that its image function

$$v_2(x,t) = \left[\frac{B(t)}{x}\right]^{\alpha} v_1\left(\frac{B(t)^2}{x}, t\right) \tag{39}$$

is also a solution of (8) for some $\alpha \in \mathbb{R}$ to be determined. Rather than use a "brute-force" verification by substituting v_2 into (8), we follow a more instructive Mellin transform route. We know that $v = v(x,t)$ is a solution of (8) if and only if its Mellin transform $\hat{v} = \hat{v}(\xi,t)$ is a solution of (27). Note that we are not imposing any final condition like $v(x,T) = g(x)$ here but are considering solutions of the PIDE and the transformed PIDE only.

Since v_1 is a solution of (8) by hypothesis, it follows from (27) that

$$\hat{\mathscr{L}}_\lambda \hat{v}_1(\xi,t) = 0. \tag{40}$$

We see that

$$\hat{v}_2(\xi,t) = \int_0^\infty x^{\xi-1} v_2(x,t)\,\mathrm{d}x = B(t)^{\alpha} \int_0^\infty x^{\xi-\alpha-1} v_1\left(\frac{B(t)^2}{x}, t\right) \mathrm{d}x.$$

Making the substitution

$$y = \frac{B(t)^2}{x}, \quad \mathrm{d}y = -B(t)^2 x^{-2}\,\mathrm{d}x = -\frac{y}{x}\,\mathrm{d}x,$$

we have

$$\hat{v}_2(\xi,t) = B(t)^{2\xi-\alpha} \int_0^\infty y^{\alpha-\xi-1} v_1(y,t)\,\mathrm{d}y = B(t)^{2\xi-\alpha} \hat{v}_1(\alpha-\xi,t),$$

so that

$$\begin{aligned}\frac{\partial \hat{v}_2}{\partial t}(\xi,t) &= (2\xi-\alpha)B(t)^{2\xi-\alpha-1}B'(t)\hat{v}_1(\alpha-\xi,t) + B(t)^{2\xi-\alpha}\frac{\partial \hat{v}_1}{\partial t}(\alpha-\xi,t) \\ &= \mu(2\xi-\alpha)B(t)^{2\xi-\alpha}\hat{v}_1(\alpha-\xi,t) + B(t)^{2\xi-\alpha}\frac{\partial \hat{v}_1}{\partial t}(\alpha-\xi,t),\end{aligned}$$

where we used the assumption (38) of an exponential barrier. Using (40), we get

$$\hat{\mathscr{L}}_\lambda \hat{v}_2(\xi,t) = B(t)^{2\xi-\alpha}\left\{\frac{\partial \hat{v}_1}{\partial t}(\alpha-\xi,t) + [\mu(2\xi-\alpha) - p(\xi) - \lambda + \lambda E(Y^{-\xi})]\hat{v}_1(\alpha-\xi,t)\right\}.$$

Our task is to introduce (40) to simplify the right-hand side.

Now let us define

$$p_\lambda(\xi) = p(\xi) - \mu\xi - \lambda E(Y^{-\xi}) \tag{41}$$

and assume that there exists α such that

$$p_\lambda(\alpha-\xi) = p_\lambda(\xi) \quad \text{for every } \xi. \tag{42}$$

Then

$$\mu(2\xi - \alpha) - p(\xi) - \lambda + \lambda E(Y^{-\xi}) = -\mu(\alpha - \xi) - p_\lambda(\xi) - \lambda = -\mu(\alpha - \xi) - p_\lambda(\alpha - \xi) - \lambda$$

upon combining (41) and (42). An application of (42) once more yields

$$\mu(2\xi - \alpha) - p(\xi) - \lambda + \lambda E(Y^{-\xi}) = -[p(\alpha - \xi) + \lambda - \lambda E(Y^{-(\alpha-\xi)})]$$

and therefore

$$\hat{\mathscr{L}}_\lambda \hat{v}_2(\xi, t) = B(t)^{2\xi-\alpha} \hat{\mathscr{L}}_\lambda \hat{v}_1(\alpha - \xi, t).$$

Recalling (40), we deduce that $\hat{v}_2$ is a solution of (27) and hence v_2 given in (39) is also a solution of (8), as was to be shown. We remark that image functions become solutions only for exponential (or fixed) barriers.

Let us now take a closer look at the assumption (42). This imposes a condition on α and/or the distribution of Y. A pair of sufficient, but not necessary, conditions for (42) to be true is

$$p(\alpha - \xi) - \mu(\alpha - \xi) = p(\xi) - \mu\xi, \quad E(Y^{-\xi}) = E(Y^{-(\alpha-\xi)}) \quad \text{for every } \xi. \tag{43}$$

Of course, if $\lambda = 0$, then the first relation above already implies (42).

Example 3. *Suppose that there are no jumps (i.e., $\lambda = 0$). The first condition in (43) holds if*

$$\alpha = \frac{c_1 - c_2 - \mu}{c_2} \tag{44}$$

from the definition of p in (20). In particular, if $c_2 = \sigma^2/2$, $c_1 = r - D$, and $c_0 = -r$, then (44) simplifies to

$$\alpha = \frac{2(r - D - \mu)}{\sigma^2} - 1.$$

In the absence of dividend yields (i.e. $D = 0$), this is precisely the value of α obtained in [31] to arrive at the mirror function solution (39) of the Black–Scholes PDE (2). Furthermore, for a fixed barrier (i.e., $\mu = 0$), this result is well known [3].

Example 4. *Now suppose that we include jumps (i.e., $\lambda \neq 0$). Then α is still given by (44) but the second condition in (43) is more restrictive as it makes further assumptions about the distribution of Y. A particular case is when Y is lognormal with parameters μ_Y and σ_Y^2. Then $E(Y^{-\xi}) = E(Y^{-(\alpha-\xi)})$ for every ξ if and only if μ_Y and σ_Y^2 are such that $\mu_Y = \alpha\sigma_Y^2/2$ (see (26)). If we assume that μ is given, and choose α in (44) and let $\mu_Y = \alpha\sigma_Y^2/2$, then (42) holds. Alternatively, if μ_Y and σ_Y^2 are arbitrary but given, then we take $\alpha = 2\mu_Y/\sigma_Y^2$ and $\mu = c_1 - c_2 - \alpha c_2$ from (44). This would also imply that (42) is also true. As pointed out above, (43) is not a necessary condition so there may exist other α and Y when $\lambda \neq 0$ such that (42) is true.*

3. Formulation of the Barrier Option Pricing Problem for Underlyings with Jump-Diffusion Dynamics

Here we formulate the barrier option pricing problem associated with the PIDE (8). The active domain is defined [32,36] to be either

$$A(t) = (0, B(t)) \quad \text{or} \quad A(t) = (B(t), \infty).$$

The active domain for down-and-out and down-and-in options is $A(t) = (B(t), \infty)$. Similarly, the active domain for up-and-out and up-and-in options is $A(t) = (0, B(t))$. Let $v_0 = v_0(x,t)$, $v_1 = v_1(x,t)$, and $v_e = v_e(x,t)$ be the solutions of the auxiliary problems

$$\begin{aligned} &\mathscr{L}_0 v_0 = 0, \quad x > 0, \quad 0 \leqslant t < T, \\ &v_0(x,T) = g(x)\mathbf{1}_{A(T)}(x), \quad x > 0, \end{aligned} \tag{45}$$

$$\begin{aligned} &\mathscr{L}_\lambda v_1 = 0, \quad x > 0, \quad 0 \leqslant t < T, \\ &v_1(x,T) = g(x)\mathbf{1}_{A(T)}(x), \quad x > 0, \end{aligned} \tag{46}$$

$$\begin{aligned} &\mathscr{L}_\lambda v_{\mathrm{e}} = 0, \quad x > 0, \quad 0 \leqslant t < T, \\ &v_{\mathrm{e}}(x,T) = g(x), \quad x > 0, \end{aligned} \tag{47}$$

respectively. We observe that v_1 and v_{e} are solutions of the same PIDE (8) but with different final conditions (we associate v_{e} with the "usual" European option pricing function). Moreover, v_0 is essentially v_1 but with $\lambda = 0$.

Equation (31) with g replaced by $g\mathbf{1}_{A(T)}$ gives

$$v_0(x,t) = \int_0^\infty \frac{1}{y}\mathscr{K}\left(\frac{x}{y}, t, T\right) g(y)\mathbf{1}_{A(T)}(y)\,\mathrm{d}y = \int_{A(T)} \frac{1}{y}\mathscr{K}\left(\frac{x}{y}, t, T\right) g(y)\,\mathrm{d}y,$$

while from (32) we have

$$v_1(x,t) = \mathrm{e}^{-\lambda(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_0^\infty \frac{1}{z} v_0\left(\frac{x}{z}, t\right) h_j(z)\,\mathrm{d}z. \tag{48}$$

An analogous expression for v_{e} can also be obtained from (31) (with $\lambda = 0$) and (32) (with $\lambda \neq 0$) by considering the payoff function g on the entire $\mathbb{R}_+$.

Example 5. *Let us continue with Examples 1 and 2 for a lognormal variable Y with parameters μ_Y and σ_Y^2. From (37) we have*

$$v_0(x,t) = \int_{A(T)} \frac{1}{y}\mathscr{K}_0\left(\frac{x}{y}, t, T\right) g(y)\,\mathrm{d}y \tag{49}$$

and

$$v_1(x,t) = \mathrm{e}^{-\lambda(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_{A(T)} \frac{1}{y}\mathscr{K}_j\left(\frac{x}{y}, t, T\right) g(y)\,\mathrm{d}y. \tag{50}$$

The barrier option pricing function $v = v(x,t)$ satisfies the PIDE

$$\mathscr{L}_\lambda v = 0, \quad x \in A(t), \quad 0 \leqslant t < T. \tag{51}$$

Note that the PIDE is only considered in the active domain. The final condition is

$$v(x,T) = g(x), \quad x \in A(T) \tag{52}$$

for knock-out options and

$$v(x,T) = 0, \quad x \in A(T) \tag{53}$$

for knock-in options. For knock-out options the condition at the barrier is

$$v(B(t),t) = 0, \quad 0 \leqslant t < T, \tag{54}$$

while for knock-in options the barrier condition is

$$v(B(t),t) = v_{\mathrm{e}}(B(t),t), \quad 0 \leqslant t < T. \tag{55}$$

Further analysis necessitates that we consider knock-out and knock-in options separately.

4. Pricing of Knock-out Barrier Options

A knock-out barrier option pricing problem is to solve

$$\begin{aligned} &\mathscr{L}_\lambda v_{\text{out}} = 0, \quad x \in A(t), \quad 0 \leqslant t < T, \\ &v_{\text{out}}(x,T) = g(x), \quad x \in A(T), \\ &v_{\text{out}}(B(t),t) = 0, \quad 0 \leqslant t < T. \end{aligned} \tag{56}$$

We claim that

$$v_{\text{out}}(x,t) = v_1(x,t) - \left[\frac{B(t)}{x}\right]^{\alpha} v_1\left(\frac{B(t)^2}{x}, t\right), \quad x \in A(t), \quad 0 \leqslant t \leqslant T \tag{57}$$

is the solution of (56). As v_{out} is a linear combination of a solution and its image function solution, linear superposition implies that v_{out} satisfies the PIDE in (56). It is clear that $v_{\text{out}}(B(t),t) = 0$ for all $0 \leqslant t < T$. Furthermore, by construction $v_1(x,T) = g(x)\mathbf{1}_{A(T)}(x)$ for $x > 0$; hence $v_1(x,T) = g(x)$ for $x \in A(T)$. If we can show that

$$v_1\left(\frac{B(T)^2}{x}, T\right) = 0, \quad x \in A(T),$$

then $v_{\text{out}}(x,T) = g(x)$ for $x \in A(T)$ in (57) and this would verify the final condition in (56). We remark that this is the reason for the choice of the final condition of v_1 in (46). Using (48), we have

$$\begin{aligned} v_1\left(\frac{B(t)^2}{x}, t\right) &= \mathrm{e}^{-\lambda(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_0^\infty \frac{1}{z} v_0\left(\frac{B(t)^2}{xz}, t\right) h_j(z)\,\mathrm{d}z \\ &= \mathrm{e}^{-\lambda(T-t)} v_0\left(\frac{B(t)^2}{x}, t\right) + \mathrm{e}^{-\lambda(T-t)} \sum_{j=1}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_0^\infty \frac{1}{z} v_0\left(\frac{B(t)^2}{xz}, t\right) h_j(z)\,\mathrm{d}z. \end{aligned}$$

Since all of the terms in the above summation are zero when $t = T$, it suffices to look at

$$\lim_{t \to T^-} v_0\left(\frac{B(t)^2}{x}, t\right) = \lim_{t \to T^-} \int_{A(T)} \frac{1}{y} \mathscr{K}\left(\frac{B(t)^2/x}{y}, t, T\right) g(y)\,\mathrm{d}y$$

and show that the limit is zero for all $x \in A(T)$. To evaluate the limit, we first investigate the bounds for

$$\left| v_0\left(\frac{B(t)^2}{x}, t\right) \right| \leqslant \int_{A(T)} \frac{1}{y} \mathscr{K}\left(\frac{B(t)^2/x}{y}, t, T\right) |g(y)|\,\mathrm{d}y.$$

Our goal here is to show that the integral on the right-hand side tends to zero as $t \to T^-$ and we would be done. As g is assumed to be piecewise linear, it is either (i) bounded or (ii) unbounded but $g(x) = O(x)$ as $x \to \infty$.

4.1. g Is Bounded

If g is bounded, then there exists $M > 0$ such that $|g(x)| \leqslant M$ for all $x > 0$, from which we deduce that

$$\int_{A(T)} \frac{1}{y} \mathscr{K}\left(\frac{B(t)^2/x}{y}, t, T\right) |g(y)|\,\mathrm{d}y \leqslant M \int_{A(T)} \frac{1}{y} \mathscr{K}\left(\frac{B(t)^2/x}{y}, t, T\right) \mathrm{d}y.$$

For a down-and-out barrier option when $A(t) = (B(t), \infty)$, Equation (19) yields

$$\int_{A(T)} \frac{1}{y} \mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)\,\mathrm{d}y = \int_{B(T)}^{\infty} \frac{\partial}{\partial y}\Big[-\mathrm{e}^{c_0(T-t)}\Phi\Big(z_2\Big(\frac{B(t)^2/x}{y}, t, T\Big)\Big)\Big]\,\mathrm{d}y$$
$$= \mathrm{e}^{c_0(T-t)}\Phi\Big(z_2\Big(\frac{B(t)^2}{B(T)x}, t, T\Big)\Big),$$

which tends to zero as $t \to T^-$ since $B(T)/x < 1$.

In the up-and-out barrier option case when $A(t) = (0, B(t))$, from (19) we obtain

$$\int_{A(T)} \frac{1}{y} \mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)\,\mathrm{d}y = \int_{0}^{B(T)} \frac{\partial}{\partial y}\Big[-\mathrm{e}^{c_0(T-t)}\Phi\Big(z_2\Big(\frac{B(t)^2/x}{y}, t, T\Big)\Big)\Big]\,\mathrm{d}y$$
$$= \mathrm{e}^{c_0(T-t)}\Phi\Big(-z_2\Big(\frac{B(t)^2}{B(T)x}, t, T\Big)\Big),$$

which also tends to zero as $t \to T^-$ since $B(T)/x > 1$ this time.

Therefore for the case when g is bounded we see that

$$v_1\Big(\frac{B(T)^2}{x}, T\Big) = \lim_{t\to T^-} v_1\Big(\frac{B(t)^2}{x}, t\Big) = 0, \quad x \in A(T).$$

4.2. *g Is Unbounded But $g(x) = O(x)$ as $x \to \infty$*

If $g(x) = O(x)$ as $x \to \infty$, then there exist $L, x_\infty > 0$ such that

$$|g(x)| \leqslant Lx \quad \text{for } x \geqslant x_\infty.$$

Without loss of generality, we may assume that $x_\infty > B(T)$; otherwise if $x_\infty \leqslant B(T)$, then $|g(x)| \leqslant Lx$ for $x \geqslant \max(x_\infty, B(T)) = B(T)$.

Consider a down-and-out barrier option, i.e. $A(t) = (B(t), \infty)$. Then

$$\int_{A(T)} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y = \int_{B(T)}^{x_\infty} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y + \int_{x_\infty}^{\infty} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y.$$

On $[B(T), x_\infty]$ we know that g is bounded, so there exists $M_1 > 0$ such that $|g(y)| \leqslant M_1$ for $B(T) \leqslant y \leqslant x_\infty$. Thus (19) implies that

$$\int_{B(T)}^{x_\infty} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y \leqslant M_1 \int_{B(T)}^{x_\infty} \frac{\partial}{\partial y}\Big[-\mathrm{e}^{c_0(T-t)}\Phi\Big(z_2\Big(\frac{B(t)^2/x}{y}, t, T\Big)\Big)\Big]\,\mathrm{d}y$$
$$= M_1\mathrm{e}^{c_0(T-t)}\Big[-\Phi\Big(z_2\Big(\frac{B(t)^2}{x_\infty x}, t, T\Big)\Big) + \Phi\Big(z_2\Big(\frac{B(t)^2}{B(T)x}, t, T\Big)\Big)\Big],$$

which tends to zero as $t \to T^-$ since $B(T)/x < 1$ and $B(T)/x_\infty < 1$. On (x_∞, ∞), we deduce from (19) that

$$\int\limits_{x_\infty}^{\infty} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y \leqslant L\int\limits_{x_\infty}^{\infty} \mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)\,\mathrm{d}y$$
$$= L\int\limits_{x_\infty}^{\infty} \frac{\partial}{\partial y}\Big[-\frac{B(t)^2}{x}\mathrm{e}^{(c_0+c_1)(T-t)}\Phi\Big(z_1\Big(\frac{B(t)^2/x}{y}, t, T\Big)\Big)\Big]\,\mathrm{d}y$$
$$= L\frac{B(t)^2}{x}\mathrm{e}^{(c_0+c_1)(T-t)}\Phi\Big(z_1\Big(\frac{B(t)^2}{x_\infty x}, t, T\Big)\Big),$$

which also tends to zero as $t \to T^-$ because $B(T)/x < 1$ and $B(T)/x_\infty < 1$. Hence

$$\lim_{t\to T^-} \int\limits_{A(T)} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y = 0$$

for the down-and-out case.

Now let us look at an up-and-out barrier option, so that $A(t) = (0, B(t))$. Since g is piecewise linear, hence bounded on $[0, B(T)]$, there exists $M_2 > 0$ such that $|g(y)| \leqslant M_2$ for $0 \leqslant y \leqslant B(T)$. Therefore

$$\int\limits_{A(T)} \frac{1}{y}\mathscr{K}\Big(\frac{B(t)^2/x}{y}, t, T\Big)|g(y)|\,\mathrm{d}y \leqslant M_2\int\limits_{0}^{B(T)} \frac{\partial}{\partial y}\Big[-\mathrm{e}^{c_0(T-t)}\Phi\Big(z_2\Big(\frac{B(t)^2/x}{y}, t, T\Big)\Big)\Big]\,\mathrm{d}y$$
$$= M_2\mathrm{e}^{c_0(T-t)}\Phi\Big(-z_2\Big(\frac{B(t)^2}{B(T)x}, t, T\Big)\Big),$$

which tends to zero as $t \to T^-$ since $B(T)/x > 1$ in this case.

Therefore for the case when g is unbounded but $g(x) = O(x)$ as $x \to \infty$, we conclude that

$$v_1\Big(\frac{B(T)^2}{x}, T\Big) = \lim_{t\to T^-} v_1\Big(\frac{B(t)^2}{x}, t\Big) = 0, \quad x \in A(T).$$

Example 6. *Consider a down-and-out asset-or-nothing call, so that $g(x) = x\mathbf{1}_{[K,\infty)}(x)$ and $A(t) = (B(t), \infty)$. For definiteness we assume that Y is lognormal with parameters μ_Y and σ_Y^2. Then (50) gives*

$$v_1(x,t) = \mathrm{e}^{-\lambda(T-t)}\sum_{j=0}^{\infty}\frac{[\lambda(T-t)]^j}{j!}\int\limits_{B(T)}^{\infty}\mathscr{K}_j\Big(\frac{x}{y}, t, T\Big)\mathbf{1}_{[K,\infty)}(y)\,\mathrm{d}y,$$

while the down-and-out barrier option pricing function from (57) is

$$v_{\mathrm{out}}(x,t) = v_1(x,t) - \Big[\frac{B(t)}{x}\Big]^{\alpha} v_1\Big(\frac{B(t)^2}{x}, t\Big), \quad x > B(t), \quad 0 \leqslant t \leqslant T. \tag{58}$$

Case (i). $B(T) < K$

Using (36), we have

$$\begin{aligned}
\int_{B(T)}^{\infty} \mathcal{K}_j\Big(\frac{x}{y}, t, T\Big) \mathbf{1}_{[K,\infty)}(y)\, \mathrm{d}y &= \int_{K}^{\infty} \mathcal{K}_j\Big(\frac{x}{y}, t, T\Big)\, \mathrm{d}y \\
&= \int_{K}^{\infty} \frac{\partial}{\partial y}\Big[-x\mathrm{e}^{(c_0+c_1)(T-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{y}, t, T\Big)\Big)\Big]\, \mathrm{d}y \\
&= x\mathrm{e}^{(c_0+c_1)(T-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{K}, t, T\Big)\Big);
\end{aligned}$$

hence

$$v_1(x,t) = x\mathrm{e}^{(c_0+c_1-\lambda)(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \mathrm{e}^{j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{K}, t, T\Big)\Big).$$

Case (ii). $B(T) \geqslant K$

Using (36) *and noting that* $y \geqslant B(T)$ *implies* $y \geqslant K$, *we have*

$$\begin{aligned}
\int_{B(T)}^{\infty} \mathcal{K}_j\Big(\frac{x}{y}, t, T\Big) \mathbf{1}_{[K,\infty)}(y)\, \mathrm{d}y &= \int_{B(T)}^{\infty} \mathcal{K}_j\Big(\frac{x}{y}, t, T\Big)\, \mathrm{d}y \\
&= \int_{B(T)}^{\infty} \frac{\partial}{\partial y}\Big[-x\mathrm{e}^{(c_0+c_1)(T-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{y}, t, T\Big)\Big)\Big]\, \mathrm{d}y \\
&= x\mathrm{e}^{(c_0+c_1)(T-t)+j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{B(T)}, t, T\Big)\Big)
\end{aligned}$$

and therefore

$$v_1(x,t) = x\mathrm{e}^{(c_0+c_1-\lambda)(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \mathrm{e}^{j\mu_Y+j\sigma_Y^2/2}\Phi\Big(z_{j,1}\Big(\frac{x}{B(T)}, t, T\Big)\Big).$$

5. Pricing of Knock-in Barrier Options

A knock-in barrier option pricing problem is to solve

$$\begin{aligned}
&\mathcal{L}_\lambda v_{\mathrm{in}} = 0, \quad x \in A(t), \quad 0 \leqslant t < T, \\
&v_{\mathrm{in}}(x,T) = 0, \quad x \in A(T), \\
&v_{\mathrm{in}}(B(t),t) = v_{\mathrm{e}}(B(t),t), \quad 0 \leqslant t < T.
\end{aligned} \tag{59}$$

The previous analysis for knock-out options can be used for knock-in options. Define $v_{\mathrm{in}} = v_{\mathrm{in}}(x,t)$ by

$$v_{\mathrm{in}}(x,t) = v_{\mathrm{e}}(x,t) - v_{\mathrm{out}}(x,t), \quad x \in A(t), \quad 0 \leqslant t \leqslant T. \tag{60}$$

We claim that v_{in} is the knock-in barrier option pricing function. By linear superposition it is clear that v_{in} satisfies the PIDE in (59). At the boundary there holds

$$v_{\mathrm{in}}(B(t),t) = v_{\mathrm{e}}(B(t),t) - v_{\mathrm{out}}(B(t),t) = v_{\mathrm{e}}(B(t),t), \quad 0 \leqslant t < T,$$

while at the expiry date we see that

$$v_{\mathrm{in}}(x,T) = v_{\mathrm{e}}(x,T) - v_{\mathrm{out}}(x,T) = g(x) - g(x) = 0, \quad x \in A(T).$$

This proves the claim.

Example 7. *Continuing with Example 6, let us price a down-and-in asset-or-nothing call. All we need is to determine* v_e*, which from (37) and (36) is*

$$\begin{aligned} v_e(x,t) &= e^{-\lambda(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_K^{\infty} \mathscr{K}_j\Big(\frac{x}{y},t,T\Big)\,dy \\ &= xe^{(c_0+c_1-\lambda)(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} e^{j\mu_Y + j\sigma_Y^2/2} \Phi\Big(z_{j,1}\Big(\frac{x}{K},t,T\Big)\Big). \end{aligned}$$

The corresponding down-and-out asset-or-nothing call option pricing function v_{out} *is given in (58). Note that there will also be two cases for* v_1 *here:* $B(T) < K$ *and* $B(T) \geqslant K$*. Therefore the down-and-in asset-or-nothing call option pricing function* v_{in} *is obtained from (60).*

6. Pricing of Barrier Options on Futures under Jump-Diffusion Dynamics

If the underlying asset pays a constant dividend and has dynamics decribed by (3), then in (57) and (60) we simply take $c_2 = \sigma^2/2$, $c_1 = r - D$, and $c_0 = -r$. In the absence of jumps, setting $\lambda = 0$ and $D = 0$ recovers the well-known results for fixed barriers [32] and exponential barriers [31].

On the other hand, if the underlying is a futures contract, as described in Section 2, we take $c_2 = \sigma^2/2$, $c_1 = 0$, $c_0 = -r$, $\bar{x} = xe^{(r-D)(T^*-t)}$, $\bar{t} = t$ in (8), and $\bar{v} = \bar{v}(\bar{x},\bar{t})$ in (32) would give the standard European option pricing function. For example, if Y is lognormal with parameters μ_Y and σ_Y^2, then (37) (which is (32) in the lognormal case) gives

$$\bar{v}(\bar{x},\bar{t}) = e^{-\lambda(T-\bar{t})} \sum_{j=0}^{\infty} \frac{[\lambda(T-\bar{t})]^j}{j!} \int_0^{\infty} \frac{1}{y} \mathscr{K}_j^{a}\Big(\frac{\bar{x}}{y},\bar{t},T\Big) g(y)\,dy, \tag{61}$$

where

$$\begin{aligned} \mathscr{K}_j^{a}(\bar{x},\bar{t},T) &= \frac{\bar{x}e^{-r(T-\bar{t})+j\mu_Y+j\sigma_Y^2/2}}{[\sigma^2(T-\bar{t})+j\sigma_Y^2]^{1/2}} \Phi'(z_{j,1}^{a}(\bar{x},\bar{t},T)) = \frac{e^{-r(T-\bar{t})}}{[\sigma^2(T-\bar{t})+j\sigma_Y^2]^{1/2}} \Phi'(z_{j,2}^{a}(\bar{x},\bar{t},T)), \\ z_{j,1}^{a}(\bar{x},\bar{t},T) &= \frac{\log(\bar{x}) + (\sigma^2/2)(T-\bar{t}) + j\mu_Y + j\sigma_Y^2}{[\sigma^2(T-\bar{t})+j\sigma_Y^2]^{1/2}}, \\ z_{j,2}^{a}(\bar{x},\bar{t},T) &= \frac{\log(\bar{x}) - (\sigma^2/2)(T-\bar{t}) + j\mu_Y}{[\sigma^2(T-\bar{t})+j\sigma_Y^2]^{1/2}}. \end{aligned} \tag{62}$$

When $\lambda = 0$, $D = 0$ and $g(x) = (x-K)^+$, (61) reduces to the "Black-76" formula [34] for a call option on a futures contract. The determination of the barrier option pricing function can then proceed as given in Sections 4 and 5.

Alternatively, instead of introducing variable transformations in (8), we can price barrier options on futures as follows. Again it suffices to consider the option pricing function of a standard European option with payoff function g on an underlying asset with jump-diffusion dynamics (3). Suppose that Y is lognormal with parameters μ_Y and σ_Y^2. Take $c_2 = \sigma^2/2$, $c_1 = r - D$, and $c_0 = -r$ in (8). The variable 'x' here is the placeholder for the asset price while '$\bar{x}$' above is for the futures price. Recall that for a forward contract, the simple contingent claim $\Pi^h(T^*)$ with contract function h (see [33] for an explanation of the terminology) is now

$$\Pi^h(T^*) = h(S(T^*)) = S(T^*) - F(t^*;T^*),$$

where $F(t^*;T^*)$ is the forward price to be determined and $h(x) = x - F(t^*;T^*)$. Note that the price process for the contingent claim at any time t can be described by $\Pi^h(t) = v(S(t),t)$, where $v = v(x,t)$

satisfies (8). As it costs nothing to enter into a forward contract at time t^*, we must have $\Pi^h(t^*) = 0$. Using the pricing formula (37), we obtain

$$\begin{aligned} v(x,t) &= \mathrm{e}^{-\lambda(T^*-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T^*-t)]^j}{j!} \int_0^{\infty} \frac{1}{y} \mathscr{K}_j\Big(\frac{x}{y}, t, T^*\Big)[y - F(t^*;T^*)]\,\mathrm{d}y \\ &= \mathrm{e}^{-\lambda(T^*-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T^*-t)]^j}{j!} [x\mathrm{e}^{-D(T^*-t)} - F(t^*;T^*)\mathrm{e}^{-r(T^*-t)}], \end{aligned}$$

where we used (36) to evaluate the integrals. The condition $\Pi^h(t^*) = v(S(t^*), t^*) = 0$ implies that the forward price is $F(t^*;T^*) = S(t^*)\mathrm{e}^{(r-D)(T^*-t^*)}$. This is the same as (6) but in the jump-diffusion case. Moreover, the prices of a futures contract and a forward contract are the same if the interest rate is deterministic since this result is model free [2]. As seen in (7), we can therefore price a European option on a futures contract using (37) by substituting $g(x\mathrm{e}^{(r-D)(T^*-T)})$, i.e., the European option pricing function $v = v(x,t)$ (with a slight abuse of notation) on a futures contract is

$$\begin{aligned} v(x,t) &= \mathrm{e}^{-\lambda(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_0^{\infty} \frac{1}{y} \mathscr{K}_j^{\mathrm{b}}\Big(\frac{x}{y}, t, T\Big) g(y\mathrm{e}^{(r-D)(T^*-T)})\,\mathrm{d}y \\ &= \mathrm{e}^{-\lambda(T-t)} \sum_{j=0}^{\infty} \frac{[\lambda(T-t)]^j}{j!} \int_0^{\infty} \frac{1}{z} \mathscr{K}_j^{\mathrm{b}}\Big(\frac{x\mathrm{e}^{(r-D)(T^*-T)}}{z}, t, T\Big) g(z)\,\mathrm{d}z, \end{aligned} \tag{63}$$

where

$$\begin{aligned} \mathscr{K}_j^{\mathrm{b}}(x,t,T) &= \frac{x\mathrm{e}^{-D(T-t)+j\mu_Y+j\sigma_Y^2/2}}{[\sigma^2(T-t)+j\sigma_Y^2]^{1/2}} \Phi'(z_{j,1}^{\mathrm{b}}(x,t,T)) = \frac{\mathrm{e}^{-r(T-t)}}{[\sigma^2(T-t)+j\sigma_Y^2]^{1/2}} \Phi'(z_{j,2}^{\mathrm{b}}(x,t,T)), \\ z_{j,1}^{\mathrm{b}}(x,t,T) &= \frac{\log(x) + (r - D + \sigma^2/2)(T-t) + j\mu_Y + j\sigma_Y^2}{[\sigma^2(T-t)+j\sigma_Y^2]^{1/2}}, \\ z_{j,2}^{\mathrm{b}}(x,t,T) &= \frac{\log(x) + (r - D - \sigma^2/2)(T-t) + j\mu_Y}{[\sigma^2(T-t)+j\sigma_Y^2]^{1/2}}. \end{aligned} \tag{64}$$

Using (62) and (64), it is straightforward to verify that

$$\mathscr{K}_j^{\mathrm{a}}(\bar{x}, \bar{t}, T) = \mathscr{K}_j^{\mathrm{b}}(x\mathrm{e}^{(r-D)(T^*-T)}, t, T).$$

Therefore the European option pricing function on a futures contract can be obtained from either (61) or (63). This is then used to price barrier options in Sections 4 and 5.

7. Discussion and Concluding Remarks

The knock-out barrier option pricing formula (57) and knock-in barrier option pricing formula (60) are solutions of the general PIDE (8). We can therefore price barrier options with general payoffs and exponential barriers when the underlyings exhibit jump-diffusion dynamics. Some of the more popular trading strategies are given in Table 1, but any barrier option with a piecewise linear payoff can be priced because the resulting integrals in (57) and (60) are evaluated explicitly with the help of the fundamental kernel identities in (19).

For more general barriers we have to use the results in [36] (with $\lambda = 0$), which also employs a Mellin transform approach. The tradeoff is that the pricing formulas in [36] are semi-analytic since they involve linear Volterra integral equations of the first kind which have to be evaluated numerically. Although we do not consider them here, double exponential barriers can also be studied. Indeed, in the absence of jumps, Equation (7) in [31] expresses the option pricing formula as a doubly infinite series of image function solutions. In principle, with the addition of jump-diffusion dynamics, one can show that an analogous series expansion is also a solution of (8) by showing that its Mellin transform is a solution of (27), just like what was done in Section 2.4

The use of the Mellin transform in the pricing of financial derivatives where the underlying dynamics are governed by geometric Brownian motion has proven to be very powerful and provides a useful tool in the quant's toolbox [10–12,35,36]. In two articles currently under review, the author has applied this tool to price perpetual American options with general payoffs, as well as a combined Mellin–Laplace transform approach to price American options with general payoffs (the latter article is joint work with Mamon). Other pricing problems that are currently being investigated by the author are lookbacks, compounds, and Parisians, among others, with the assumption of jump-diffusion dynamics. A far-reaching goal is to be able to extend the results to include stochastic volatility and early exercise features to other pricing problems. However, it should be remarked that Mellin transforms are not as useful when considering other asset price dynamics precisely because the derivative properties in (11) are not valid anymore. In these scenarios a different integral transform has to be used, although it is not clear what the appropriate transform should be and therefore the pricing problem has to be handled on a case-to-case basis.

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