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# Boundary $C^{2,\alpha}$ estimates for Monge-Ampère type equations

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## Abstract

In this paper, we obtain global second derivative estimates for solutions of the Dirichlet problem of certain Monge-Ampère type equations under some structural conditions, while the inhomogeneous term is only assumed to be Hölder continuous and bounded away from zero and infinity. These estimates correspond to those for the standard Monge-Ampère equation obtained by Trudinger and Wang (2008) and by Savin (2013), and have natural applications in optimal transportation and prescribed Jacobian equations.

## Keywords

2, c, equations, boundary, type, ampere, monge, estimates, alpha

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# BOUNDARY $C^{2,\alpha}$ ESTIMATES FOR MONGE-AMPÈRE TYPE EQUATIONS

YONG HUANG, FEIDA JIANG, AND JIAKUN LIU

ABSTRACT. In this paper, we obtain global second derivative estimates for solutions of the Dirichlet problem of certain Monge-Ampère type equations under some structural conditions, while the inhomogeneous term is only assumed to be Hölder continuous and bounded away from zero and infinity. These estimates correspond to those for the standard Monge-Ampère equation obtained by Trudinger and Wang in *Ann. of Math.* 167 (2008), 993–1028 and by Savin in *J. Amer. Math. Soc.* 26 (2013), 63–99, and have natural applications in optimal transportation and prescribed Jacobian equations.

## 1. INTRODUCTION

The Dirichlet problem of the Monge-Ampère type equations under consideration has the following general form

$$(1.1) \quad \begin{aligned} \det[D^2u - A(\cdot, Du)] &= f \quad \text{in } \Omega, \\ u &= u_0 \quad \text{on } \partial\Omega, \end{aligned}$$

where  $\Omega$  is a bounded domain in  $\mathbb{R}^n$ ,  $f$  is a positive function in  $\Omega$ ,  $A = A(\cdot, Du)$  is an  $n \times n$  symmetric matrix and  $u_0$  is a function on  $\partial\Omega$ . These equations arise in many applications, notably in optimal transportation [34], conformal geometry [26], reflector and refractor problems [9, 10, 15, 35], and have attracted significant interest in recent years [3, 29].

In a special case when the matrix  $A$  vanishes, one has the standard Monge-Ampère equation. The global  $C^{2,\alpha}$  estimates in the case when  $\partial\Omega$ ,  $u_0$  and  $f$  are sufficiently smooth and the existence of classical solutions of the Dirichlet problem

$$(1.2) \quad \begin{cases} \det D^2u = f & \text{in } \Omega, \\ u = u_0 & \text{on } \partial\Omega, \end{cases}$$

was obtained by Caffarelli, Nirenberg and Spruck [4] and Krylov [16]. When the inhomogeneous term  $f$  is assumed to be positive and only Hölder continuous in  $\Omega$ , that is  $f \in C^\alpha(\Omega)$  for some  $\alpha \in (0, 1)$ , the global  $C^{2,\alpha}$  estimates were obtained by Trudinger and Wang in [28] for  $u_0, \partial\Omega \in C^3$ . More recently, Savin [24] obtained a sharp pointwise  $C^{2,\alpha}$  estimates at boundary under appropriate local conditions on the inhomogeneous term and boundary data. However, the analysis in both [28] and [24] is highly intricate and quite complicated. In this paper, by assuming a structural condition on the matrix  $A$ , which is analogous to the Ma-Trudinger-Wang condition in the optimal transportation problem, we have some new observations, which lead to a different and simpler proof for the boundary  $C^{2,\alpha}$  estimates of the solution  $u$  of (1.1).

In optimal transportation, the potential function  $u$  satisfies the Monge-Ampère type equation (1.1), where the matrix  $A$  is determined by a given cost function  $c$ . Here, similarly we assume that the matrix  $A$  in (1.1) is given by

$$(1.3) \quad A(x, Du) = D_x^2 c(x, Tx), \quad x \in \Omega,$$

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for a smooth function  $c(x, y) : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  and an associated mapping  $T$  determined by

$$(1.4) \quad T(x) = D_x^{-1}c(x, Du(x)),$$

which is analogous to the optimal mapping in optimal transportation under some appropriate conditions [22, 33, 34]. Throughout this paper, we assume the following hypotheses:

(H1) For any  $x, p \in \mathbb{R}^n$ , there is a unique  $y \in \mathbb{R}^n$  such that  $D_x c(x, y) = p$ ; and for any  $y, q \in \mathbb{R}^n$ , there is a unique  $x \in \mathbb{R}^n$  such that  $D_y c(x, y) = q$ .

(H2) For all  $(x, p) \in \Omega \times \mathbb{R}^n$  and two vectors  $\xi, \eta \in \mathbb{R}^n$  satisfying  $\xi \perp \eta$ ,

$$(1.5) \quad D_{p_k p_l}^2 A_{ij}(x, p) \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2$$

for some constant  $c_0 > 0$ . This assumption is called (H2w) if  $c_0 = 0$ .

These conditions (H1)–(H2) essentially follow from those for the cost function in optimal transportation [22]. The condition (H1) guarantees the mapping  $T$  in (1.4) is well defined, and the condition (H2) is necessary for the regularity as shown in the Heinz-Lewy counterexample [25] and the example in optimal transportation [21]. In the above, the assumption that the function  $c$  is defined in the whole space  $\mathbb{R}^n \times \mathbb{R}^n$  is only for simplicity. In many cases it is sufficient to assume that  $c$  is defined in a proper subset  $\mathcal{U} \subset \mathbb{R}^n \times \mathbb{R}^n$  where (H1)–(H2) hold, see [20, 31] for instance. Additionally, if  $\det D_{xy}^2 c(x, y) \neq 0$  at  $(x, y) \in \Omega \times \mathbb{R}^n$  with  $y = D_x^{-1}c(x, p)$  by virtue of (H1), the inequality (1.5) can be written in terms of the function  $c$  by

$$(1.5^*) \quad \sum_{i,j,k,l,p,q,r,s} (c^{p,q} c_{ij,p} c_{q,rs} - c_{ij,rs}) c^{r,k} c^{s,l} \xi_i \xi_j \eta_k \eta_l \geq c_0 |\xi|^2 |\eta|^2,$$

where  $c_{i,j} = \frac{\partial^2 c}{\partial x_i \partial y_j}$  and  $(c^{i,j})$  is the inverse matrix of  $(c_{i,j})$ . We remark that our method and the result in this paper also extend to the more general Monge-Ampère equations (1.1) as long as the matrix  $A$  satisfies the condition (H2). The reason here to consider the equation with an optimal transportation structure is that we can directly use some established results from previous papers such as [17, 22]. In the general case, the matrix  $A$  is determined by a generating function as in [27], which plays the role of the function  $c$ , however, we need to establish the associated preliminary results which are quite formalistic and involved. These results and extensions to more general equations will be deferred to a different paper.

The Dirichlet problem (1.1) including (1.2) is the mostly studied boundary value problem for Monge-Ampère type equations due to not only the theoretical importance but also its broad applications [29]. For example, in the landmark paper [22] of Ma, Trudinger and Wang on the optimal transportation problem, a key ingredient of obtaining the interior  $C^3$  regularity of potential function is the classical solvability of Dirichlet problems (1.1) in sufficiently small balls. Moreover, to obtain the interior regularity of the reflection surface in the far-field reflector problem studied by Wang [35] and the near-field reflector problem both in the case of a point light source by Karakhanyan and Wang [15] and in the case of a parallel light source by Karakhanyan [14], the existence of smooth solutions of Dirichlet problem (1.1) in a small ball also plays a crucial role. Recently, the global regularity and the classical solvability for the Dirichlet problem (1.1) over general domains has been obtained by the second author with Trudinger and Yang [12] by assuming the existence of a subsolution and adopting the approach from [7, 8] of Guan and Guan-Spruck.

The purpose of this paper is to obtain the global  $C^{2,\alpha}$  estimate for solutions of (1.1) under the minimal hypotheses of the inhomogeneous term  $f$ , in other words, to extend the results of Trudinger and Wang in [28] and of Savin in [24] to more general Monge-Ampère type equations in (1.1). Our main result is the following, while relevant terminologies will be postponed in Section 2 below.

**Theorem 1.1.** *Let  $\Omega$  be a uniformly  $c$ -convex domain in  $\mathbb{R}^n$  with boundary  $\partial\Omega \in C^3$ . Assume that  $u_0 \in C^3(\partial\Omega)$ ,  $f \in C^\alpha(\bar{\Omega})$  for some  $\alpha \in (0, 1)$  and satisfies  $0 < \lambda \leq f \leq \Lambda$  for some positive constants*

$\lambda$  and  $\Lambda$ . Assume that the matrix  $A$  given by (1.3) satisfies the conditions (H1)–(H2). Let  $u$  be an elliptic solution of the Dirichlet problem (1.1). Then we have the a priori estimate

$$(1.6) \quad \|u\|_{C^{2,\alpha}(\bar{\Omega})} \leq C,$$

where  $C$  is a constant depending on  $n, \alpha, c, f, u_0$  and  $\partial\Omega$ .

The solution  $u$  in Theorem 1.1 is interpreted in the viscosity sense, see Definition 2.3. However, by the uniqueness it is sufficient to assume that  $u$  is a smooth solution. The proof of Theorem 1.1 relies on a local analysis of the geometry of sub-level sets of  $u$  near the boundary  $\partial\Omega$ . In fact, we obtain a more precise estimate (see Remark 4.2) that

$$(1.7) \quad |D^2u(x) - D^2u(y)| \leq C \left[ d + \int_0^d \frac{\omega(r)}{r} + d \int_d^1 \frac{\omega(r)}{r^2} \right],$$

where  $x, y \in \bar{\Omega}$ ,  $d = |x - y|$ ,  $\omega(r) = \omega_f(r) + \omega_{u_0}(r)$  is the total oscillation of  $f$  and  $u_0$  (defined in (4.12) and (4.15)), and the constant  $C > 0$  depends only on  $n, c, \lambda, \Lambda, \sup_{\Omega} |u|$ , and the modulus of continuity of  $\partial\Omega$  and  $u_0$  up to their third derivatives. The estimate (1.6) is then a consequence of (1.7).

We remark that the condition (H2), namely inequality (1.5), is not satisfied by the standard Monge-Ampère equation in (1.2). However, we have some new observations of the property of this structural condition, and by making use of them in this paper, we provide a different and simpler proof than those in [24, 28]. It would be interesting to have an estimate like (1.6) under the degenerate condition (H2w), which will be studied in our subsequent work. Besides the Dirichlet boundary condition (1.2), another interesting boundary condition for the standard Monge-Ampère equation is the prescription of the gradient image  $\Omega^* = Du(\Omega)$  which is equivalent to an oblique boundary condition for elliptic solutions. In this case the global regularity for smooth uniformly convex domain  $\Omega$  and target  $\Omega^*$  was obtained by Caffarelli [2], Delanoë [5] and Urbas [32]. For the general Monge-Ampère equation (1.1), the corresponding boundary condition is the prescription of the target image  $\Omega^* = T(\Omega)$  with  $T$  in (1.4) which is the natural boundary condition in the optimal transportation problem. The global  $C^3$  regularity was obtained by Trudinger and Wang [31] under the degenerate condition (H2w). It would also be interesting to ask for a sharp boundary estimate like (1.6) under the minimal hypotheses of the inhomogeneous term  $f$  and boundary data.

The organisation of this paper is as follows: In Section 2, we introduce some terminologies and preliminary results that will be used in subsequent context. In Section 3, we study the geometric properties of the boundary sub-level sets by a delicate local analysis, where we only assume the inhomogeneous term is positive and bounded. In Section 4, we establish a Pogorelov estimate on the boundary and use a perturbation argument to obtain the estimate (1.7), and then prove the main theorem.

## 2. PRELIMINARIES

In this section, we first introduce some notations and terminologies adopting from optimal transportation in §2.1. Then we introduce two types of transformation in §2.2 and §2.3 respectively, namely the boundary transformation flattening the boundary of domain  $\Omega$  and the coordinate transformation convexifying the sub-level sets of  $u$ , which are useful tools in our subsequent analysis. In addition, we believe that these transformations will also be valuable in investigating the boundary regularities for solutions of general Monge-Ampère equations (1.1) with other boundary conditions, for example the above mentioned  $\Omega^* = T(\Omega)$ , which includes the nature boundary condition in optimal transportation, geometric optics and more general prescribed Jacobian equations.

**2.1. Some terminologies.** The following notions of  $c$ -support,  $c$ -convexity and  $c$ -normal mapping can be found in [22] for more details.

**Definition 2.1.** Let  $u$  be a semi-convex function in  $\Omega$ , namely  $u + C|x|^2$  is convex for a large positive constant  $C$ . A  $c$ -support of  $u$  at  $x_0 \in \Omega$  is a function of the form

$$\varphi(x) = c(x, y_0) + a_0,$$

where  $a_0$  is a constant and  $y_0 \in \mathbb{R}^n$  such that

$$\begin{aligned} u(x_0) &= \varphi(x_0), \\ u(x) &\geq \varphi(x), \quad \text{for all } x \in \Omega. \end{aligned}$$

If for any point  $x_0 \in \Omega$ , there exists a  $c$ -support of  $u$ , then we say  $u$  is  $c$ -convex. If the graph of every  $c$ -support of  $u$  contacts the graph of  $u$  at one point only, we say  $u$  is *strictly*  $c$ -convex.

**Definition 2.2.** Let  $u$  be a  $c$ -convex function. The  $c$ -normal mapping of  $u$  is a set-valued mapping  $\mathcal{N}_u^c$  given by

$$\mathcal{N}_u^c(x_0) = \{y \in \mathbb{R}^n : u(x) \geq c(x, y) - c(x_0, y) + u(x_0) \quad \forall x \in \Omega\}, \quad x_0 \in \Omega.$$

For  $x_0 \in \partial\Omega$ , denote  $\mathcal{N}_u^c(x_0) = \{y \in \mathbb{R}^n : y = \lim_{k \rightarrow \infty} y_k\}$ , where  $y_k \in \mathcal{N}_u^c(x_k)$  and  $\{x_k\}$  is a sequence of interior points of  $\Omega$  such that  $x_k \rightarrow x_0$ . For any subset  $E \subset \Omega$ , denote  $\mathcal{N}_u^c(E) = \bigcup_{x \in E} \mathcal{N}_u^c(x)$ .

Note that if  $u \in C^1(\bar{\Omega})$ , by the condition (H1) and (1.4) for each  $x_0 \in \bar{\Omega}$ ,  $\mathcal{N}_u^c(x_0) = T(x_0)$  is a singleton, where  $T$  is given in (1.4). By the  $c$ -normal mapping we introduce a measure  $\mu_u$  in  $\Omega$  such that for any Borel set  $E \subset \Omega$ ,

$$(2.1) \quad \mu_u(E) = \int_{\mathcal{N}_u^c(E)} dy.$$

It was proved in [22, Section 3] that  $\mu_u$  is a Radon measure if  $u$  is a potential function in optimal transportation, where the initial and target measures can be regarded as  $fdx$  and  $dy$ , respectively. In that case, the potential  $u$  satisfies equation (1.1) with an inhomogeneous term  $|\det c_{i,j}|f$ . In this paper, we regard the inhomogeneous term as a single function  $f$ , and it is more convenient to define the generalised solution in the viscosity sense.

**Definition 2.3.** A  $c$ -convex function  $u$  is called a viscosity subsolution of (1.1), if for any  $c$ -convex function  $\varphi \in C^2(\Omega)$  such that  $u - \varphi$  has a strict local maximum at some point  $x_0 \in \Omega$ , then there holds

$$\det[D^2\varphi(x_0) - A(x_0, D\varphi(x_0))] \geq f(x_0).$$

Similarly, a  $c$ -convex function  $u$  is called a viscosity supersolution of (1.1), if for any  $c$ -convex function  $\varphi \in C^2(\Omega)$  such that  $u - \varphi$  has a strict local minimum at some point  $x_0 \in \Omega$ , then there holds

$$\det[D^2\varphi(x_0) - A(x_0, D\varphi(x_0))] \leq f(x_0).$$

Finally, a  $c$ -convex function  $u$  is a viscosity solution of (1.1) if it is both a viscosity subsolution and a viscosity supersolution of (1.1). Furthermore, we say  $u$  is a viscosity solution of (1.1) with the associated Dirichlet boundary condition if  $u \in C^0(\bar{\Omega})$  and  $u = u_0$  on  $\partial\Omega$ .

It has been shown in [18] that, even under the degenerate condition (H2w), the viscosity solution is equivalent to the generalised solution of (1.1) in the sense of Aleksandrov, which is defined by a measure preserving condition in terms of (2.1), see [22]. We remark that when a solution  $u$  is  $c$ -convex, the matrix  $Mu := \{D^2u - A(\cdot, Du)\}$  is nonnegative definite and  $u$  is also called an *elliptic* solution of (1.1).

**Definition 2.4.** A set of points  $\ell \subset \mathbb{R}^n$  is a  $c$ -segment with respect to a point  $y_0 \in \mathbb{R}^n$  if  $D_y c(\ell, y_0)$  is a line segment in  $\mathbb{R}^n$ ; and a set  $U \subset \mathbb{R}^n$  is (uniformly)  $c$ -convex with respect to another set  $V \subset \mathbb{R}^n$  if the image  $D_y c(U, y)$  is (uniformly) convex for all  $y \in V$ .

Suppose  $U$  is  $c$ -convex (or uniformly  $c$ -convex) with respect to  $y_0$ , and  $0 \in \partial U$ . In a local coordinates,  $\partial U$  is given by  $x_n = \rho(x')$  with  $D\rho(0) = 0$  such that  $e_n$  is the inner normal at 0, where  $x' = (x_1, \dots, x_{n-1})$ . Denote  $\gamma$  the inner normal of  $D_y c(U, y_0)$  at  $D_y c(0, y_0)$ . Then  $D_y c(U, y_0)$  is convex (or uniformly convex) if and only if

$$c_{ij, y_i} \gamma_i + c_{n, y_i} \gamma_i \rho_{ij} \geq 0 \quad (\text{or } \geq \kappa_0 > 0).$$

Namely,

$$(2.2) \quad c_{ij, y_k} c^{k, n} + \rho_{x_i x_j} \geq 0 \quad (\text{or } \geq \kappa_0 > 0),$$

where  $c^{i, j}$  is the inverse matrix of  $c_{i, j}$ . The formula (2.2) is the analytic expression of the  $c$ -convexity of domains [30, §2.3]. By virtue of (2.2), one can see that for any given smooth function  $c$ , a sufficiently small ball is uniformly  $c$ -convex, which was the case considered in [22]. However, for general domain  $\Omega$  in this paper, we need a further convexity assumption. In the following context, we say  $\Omega$  is (uniformly)  $c$ -convex if  $\Omega$  is (uniformly)  $c$ -convex with respect to  $\mathbb{R}^n$ .

Similarly we can define the dual notions  $c^*$ -segment,  $c^*$ -convexity of domains by exchanging  $x$  and  $y$ . We refer the reader to [22, 30] for more detailed introduction of the above notions, but note that in this paper we consider  $c$ -convex functions rather than  $c$ -concave ones therein.

Let  $u \in C^1(\bar{\Omega})$  be a  $c$ -convex function, we define the sub-level set  $S_h^0$  of  $u$  as follows.

**Definition 2.5.** Given  $h > 0$  and  $x_0 \in \Omega$ , let  $\varphi = c(\cdot, y_0) + a_0$  be the  $c$ -support of  $u$  at  $x_0$ , where the constant  $a_0 = u(x_0) - c(x_0, y_0)$  and  $y_0 = \mathcal{N}_u^c(x_0)$ , we call

$$S_{h, u}^0(x_0) = \{x \in \bar{\Omega} : u(x) < \varphi(x) + h\}$$

the sub-level set of  $u$  at  $x_0$  with height  $h$ . For simplicity, we denote it by  $S_h^0$  when  $x_0$  is a minimum point of  $u$ .

**2.2. Boundary transformation.** Let  $\Omega$  be a  $C^3$  domain. At each point  $x_0 \in \partial\Omega$ , there exists a small ball  $B = B(x_0)$  and a one-to-one mapping  $\Phi$  from  $B$  onto  $D \subset \mathbb{R}^n$  such that

$$(i) \Phi(B \cap \Omega) \subset \mathbb{R}_+^n; \quad (ii) \Phi(B \cap \partial\Omega) \subset \partial\mathbb{R}_+^n; \quad (iii) \Phi \in C^3(B), \Phi^{-1} \in C^3(D),$$

where  $\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n > 0\}$ , and  $\partial\mathbb{R}_+^n = \{x \in \mathbb{R}^n : x_n = 0\}$ . We shall say that the diffeomorphism  $\Phi$  straightens the boundary  $\partial\Omega$  near  $x_0$ .

Assume that  $e_n$  is the unit inner normal of  $\partial\Omega$  at  $0 \in \partial\Omega$ , and locally  $\partial\Omega$  is given by  $x_n = \varphi(x')$  with  $x' = (x_1, \dots, x_{n-1})$ , where  $\varphi$  is a smooth function satisfying  $\varphi(0) = 0$ ,  $D\varphi(0) = 0$ . One can see that the diffeomorphism  $\Phi$  can be given by

$$(2.3) \quad \Phi(x', x_n) = (x', x_n - \varphi(x')),$$

which straightens the boundary  $\partial\Omega$  near 0. In particular, we have

$$\Phi(S_h^0 \cap \partial\Omega) \subset \partial\mathbb{R}_+^n,$$

for  $h > 0$  sufficiently small such that  $S_h^0 \cap \Omega \subset B$ .

Under the transform (2.3), let  $\tilde{x} = \Phi(x)$  and

$$(2.4) \quad \begin{aligned} \tilde{u}(\tilde{x}) &= u(x), \\ \tilde{c}(\tilde{x}, y) &= c(x, y). \end{aligned}$$

We investigate how will this boundary transformation affect: (i) the Monge-Ampère equation (1.1); (ii) the conditions (H1)–(H2) satisfied by the function  $c$ ; and (iii) the  $c$ -convex properties.

*Claim i: Equation (1.1) is invariant under (2.4).*

By calculations, we have the Jacobian matrix

$$(2.5) \quad D\Phi^{-1} = \begin{bmatrix} I_{(n-1) \times (n-1)} & 0 \\ \nabla\varphi & 1 \end{bmatrix},$$

and furthermore

$$(2.6) \quad \frac{\partial^2 \Phi_k^{-1}}{\partial x_i \partial x_j} = \begin{cases} \delta_{nk} \varphi_{ij} & \text{if } i, j < n; \\ 0 & \text{otherwise.} \end{cases}$$

By differentiating  $\tilde{u}(\tilde{x}) = u(\Phi^{-1}(\tilde{x}))$ , we can obtain

$$(2.7) \quad D\tilde{u} = Du \cdot D\Phi^{-1},$$

and for any  $i, j = 1, \dots, n$

$$(2.8) \quad D_{ij}^2 \tilde{u} = [(D\Phi^{-1})^t \cdot D^2 u \cdot D\Phi^{-1}]_{ij} + \sum_k \frac{\partial^2 \Phi_k^{-1}}{\partial x_i \partial x_j} D_k u,$$

where  $(D\Phi^{-1})^t$  is the transpose of  $D\Phi^{-1}$ .

Similarly, for the function  $\tilde{c}(\tilde{x}, y) = c(\Phi^{-1}(\tilde{x}), y)$  we have

$$(2.9) \quad \begin{aligned} D_x \tilde{c} &= D_x c \cdot D\Phi^{-1}, \\ D_{x_i x_j}^2 \tilde{c} &= [(D\Phi^{-1})^t \cdot D_x^2 c \cdot D\Phi^{-1}]_{ij} + \sum_k \frac{\partial^2 \Phi_k^{-1}}{\partial x_i \partial x_j} D_{x_k} c, \end{aligned}$$

for any  $i, j = 1, \dots, n$ .

Therefore, we obtain that

$$(2.10) \quad \begin{aligned} D_{ij}^2 \tilde{u} - D_{x_i x_j}^2 \tilde{c} &= [(D\Phi^{-1})^t \cdot (D^2 u - D_x^2 c) \cdot D\Phi^{-1}]_{ij} \\ &+ \sum_k \frac{\partial^2 \Phi_k^{-1}}{\partial x_i \partial x_j} (D_k u - D_{x_k} c). \end{aligned}$$

From (1.4), the last term in (2.10) vanishes, and from (2.5) the determinant  $\det(D\Phi^{-1})^t = \det D\Phi^{-1} = 1$ . Hence, there holds

$$(2.11) \quad \det [D_{ij}^2 \tilde{u} - D_{x_i x_j}^2 \tilde{c}] = \tilde{f} \quad \text{in } \tilde{S}_h^0,$$

with  $\tilde{f}(\tilde{x}) = f(x)$ , where  $\tilde{S}_h^0 = \Phi(S_h^0)$ . By (1.3), this implies that the form of equation (1.1) remains invariant.

**Remark 2.1.** This invariant may seem surprising at first sight. Because for the standard Monge-Ampère equation

$$(2.12) \quad \det D^2 u = f,$$



under the nonlinear transform  $x \mapsto \Phi(x)$  in (2.3), it will become

$$(2.13) \quad \det \left[ ((D\Phi^{-1})^t \cdot D^2u \cdot D\Phi^{-1})_{ij} + \sum_k \frac{\partial^2 \Phi_k^{-1}}{\partial x_i \partial x_j} D_k u \right] = f.$$

However, in the case like optimal transportation the associated function  $c(x, y) = x \cdot y$  for (2.12) will also be changed according to the transform (2.4), namely  $c \mapsto \tilde{c}$ ,

$$\tilde{c}(x, y) = \Phi^{-1}(x) \cdot y,$$

which is not linear in general. This implies that (2.13) is actually an optimal transportation equation associated with the new cost  $\tilde{c}(x, y)$ . Indeed, by (2.9)–(2.10) the second term in the matrix of (2.13) will be eliminated from the change of the function  $c$ . Therefore, as an optimal transportation equation (2.12) remains invariant associating with the new cost function  $\tilde{c}$ .

*Claim ii: the new function  $\tilde{c}(\tilde{x}, y) = c(\Phi^{-1}(\tilde{x}), y)$  satisfies the conditions (H1)–(H2).*

For simplicity, we represent the Jacobian matrix (2.5) by  $\{g_{i,j}\}$ , i.e.

$$g_{i,j} := \frac{\partial \Phi_i^{-1}}{\partial x_j}, \quad g_{i,jk} := \frac{\partial^2 \Phi_i^{-1}}{\partial x_j \partial x_k},$$

for all  $i, j, k = 1, \dots, n$ , and  $\{g^{i,j}\} = \{g_{i,j}\}^{-1}$  is the inverse matrix. By differentiations, we have the following

$$\begin{aligned} \tilde{c}_i &= c_\alpha g_{\alpha,i}, & \tilde{c}_{,i} &= c_{,i} \\ \tilde{c}_{ij} &= c_{\alpha\beta} g_{\alpha,i} g_{\beta,j} + c_\alpha g_{\alpha,ij}, \\ \tilde{c}_{i,j} &= c_{\alpha,j} g_{\alpha,i}, & \tilde{c}^{i,j} &= c^{i,\alpha} g^{j,\alpha}, \\ \tilde{c}_{ij,k} &= c_{\alpha\beta,k} g_{\alpha,i} g_{\beta,j} + c_{\alpha,k} g_{\alpha,ij}, \\ \tilde{c}_{i,jk} &= c_{\alpha,jk} g_{\alpha,i}, \\ \tilde{c}_{ij,kl} &= c_{\alpha\beta,kl} g_{\alpha,i} g_{\beta,j} + c_{\alpha,kl} g_{\alpha,ij}, \end{aligned}$$

where the indices  $\alpha, \beta$  sum from 1 over  $n$ . Since the boundary function  $\varphi \in C^3$ , the function  $\tilde{c}$  is sufficiently smooth in the above calculations. Since  $\Phi$  is a  $C^3$  diffeomorphism and  $\det \{g_{i,j}\} = 1$ , the condition (H1) holds. It suffices to verify (1.5), the condition (H2).

By the above relations,

$$(2.14) \quad \begin{aligned} \tilde{c}^{p,q} \tilde{c}_{ij,p} \tilde{c}_{q,rs} &= c^{p,a} g^{q,a} (c_{\alpha\beta,p} g_{\alpha,i} g_{\beta,j} + c_{\alpha,p} g_{\alpha,ij}) c_{b,rs} g_{b,q} \\ &= \delta_{ab} c^{p,a} c_{\alpha\beta,p} c_{b,rs} g_{\alpha,i} g_{\beta,j} + \delta_{\alpha a} \delta_{ab} c_{b,rs} g_{\alpha,ij} \\ &= c^{p,q} c_{\alpha\beta,p} c_{q,rs} g_{\alpha,i} g_{\beta,j} + c_{\alpha,rs} g_{\alpha,ij} \\ &= c^{p,q} c_{\alpha\beta,p} c_{q,rs} g_{\alpha,i} g_{\beta,j} + \tilde{c}_{ij,rs} - c_{\alpha\beta,rs} g_{\alpha,i} g_{\beta,j}, \end{aligned}$$

thus we obtain that

$$(2.15) \quad \tilde{c}_{ij,rs} - \tilde{c}^{p,q} \tilde{c}_{ij,p} \tilde{c}_{q,rs} = (c_{\alpha\beta,rs} - c^{p,q} c_{\alpha\beta,p} c_{q,rs}) g_{\alpha,i} g_{\beta,j}.$$

Let  $\xi, \eta \in \mathbb{R}^n$  with  $\xi \perp \eta$ . Denote  $\tilde{\xi} = (\tilde{\xi}_1, \dots, \tilde{\xi}_n)$  with  $\tilde{\xi}_\alpha = \sum_i g_{\alpha,i} \xi_i$ , for  $1 \leq \alpha \leq n$ , and  $\tilde{\eta} = (\tilde{\eta}_1, \dots, \tilde{\eta}_n)$  with  $\tilde{\eta}_a = \sum_k g^{k,a} \eta_k$ , for  $1 \leq a \leq n$ . Then it is easy to see that  $\tilde{\xi} \perp \tilde{\eta}$ . From the fact that  $\det \{g_{i,j}\} = 1$  and the condition (H2) in (1.5\*), we have

$$(2.16) \quad \begin{aligned} &\sum_{i,j,k,l,p,q,r,s} (\tilde{c}_{ij,rs} - \tilde{c}^{p,q} \tilde{c}_{ij,p} \tilde{c}_{q,rs}) \tilde{c}^{r,k} \tilde{c}^{s,l} \xi_i \xi_j \eta_k \eta_l \\ &= \sum_{\alpha,\beta,a,b,p,q,r,s} (c_{\alpha\beta,rs} - c^{p,q} c_{\alpha\beta,p} c_{q,rs}) c^{r,a} c^{s,b} \tilde{\xi}_\alpha \tilde{\xi}_\beta \tilde{\eta}_a \tilde{\eta}_b \\ &\geq c_0 |\xi|^2 |\eta|^2, \end{aligned}$$

which means the new function  $\tilde{c}(x, y)$  still satisfies the condition (H2).

*Claim iii:*  $\tilde{u}$  is  $c$ -convex with respect to  $\tilde{c}$ ;  $\Phi(U)$  is (uniformly)  $\tilde{c}$ -convex with respect to  $y_0$  if  $U$  is (uniformly)  $c$ -convex with respect to  $y_0$ .

It is obvious that under the transform (2.4), the function  $\tilde{u}$  is  $c$ -convex with respect to the new function  $\tilde{c}$ , which can be seen from Definition 2.1. Alternatively, because the matrix in (2.10)–(2.11) is non-negative definite, one can also infer the  $c$ -convexity for elliptic solution  $\tilde{u}$  of (2.11) from [30, Corollary 2.1].

To see the  $c$ -convexity of domains is preserved by the transform (2.4), one can use (2.2). On the boundary  $\partial\Phi(U)$ ,  $\tilde{c}((\tilde{x}', 0), y) = c((x', \rho(x')), y)$ , by differentiation at  $(0, y_0)$  we have

$$\tilde{c}_{ij,k}\tilde{c}^{k,n} = c_{ij,k}c^{k,n} + \rho_{ij} \geq 0 \quad (\text{or } \delta > 0),$$

where  $i, j = 1, \dots, n-1$ . This implies that  $\Phi(U)$  is (uniformly)  $\tilde{c}$ -convex with respect to  $y_0$ .

From the above observations, we conclude that locally the transforms (2.3)–(2.4) give us a new setting of Dirichlet problem, where the function  $\tilde{c}$  satisfies the conditions (H1)–(H2), and the function  $\tilde{u}$  satisfies the equation (2.11) in the domain  $\Phi(S_h^0) \subset \mathbb{R}_+^n$ , which has a flat part of boundary lies on  $\{x_n = 0\}$ .

**2.3. Coordinate transformation.** In this subsection we introduce a coordinate transformation, which convexifies the sub-level sets and enables the normalisation process.

Let  $x_0 = 0$  be a fixed point on  $\partial\Omega$ , and  $y_0 = T(x_0) = 0$ , where  $T$  is given in (1.4). Make the changes

$$(2.17) \quad \begin{aligned} c(x, y) &\longrightarrow [c(x, y) - c(x, 0)] - [c(0, y) - c(0, 0)], \\ u(x) &\longrightarrow [u(x) - u(0)] - [c(x, 0) - c(0, 0)], \end{aligned}$$

such that the function  $c$  satisfies

$$(2.18) \quad c(x, 0) = c(0, y) \equiv 0 \quad \forall x, y \in \mathbb{R}^n,$$

and the function  $u$  satisfies

$$(2.19) \quad u(0) = 0, \quad Du(0) = 0, \quad u \geq 0 \quad \text{in } \Omega.$$

The sub-level set  $S_h^0$  is now given by

$$(2.20) \quad S_h^0 := S_{h,u}^0(0) = \{x \in \bar{\Omega} : u(x) < h\}$$

for a constant  $h > 0$ .

**Lemma 2.1.** *Assume that the function  $c$  satisfies conditions (H1)–(H2). The boundary of sub-level set  $\partial S_h^0 \cap \Omega$  is uniformly  $c$ -convex with respect to the origin.*

This lemma was contained in [17] and essentially follows from the monotonicity formula in [21, 30]. As remarked in [17], it is not necessary to make any smoothness assumption on  $u$ . Since the treatment therein does not depend on the bounds of  $D^2u$ , one can use the approximation argument as therein.

Since the sub-level set  $S_h^0$  is  $c$ -convex with respect to 0,  $D_y c(S_h^0, 0)$  is a convex set. We make the coordinate transform

$$(2.21) \quad x \rightarrow D_y c(x, 0)$$

so that the sub-level set  $S_h^0$  is convex, and from the  $c$ -convexity assumption,  $\Omega$  becomes uniformly convex simultaneously.

Under the transforms (2.17) and (2.21), equation (1.1) will become

$$(2.22) \quad \det [u_{ij} - A_{ij,p_k p_l} u_k u_l] = f,$$

in  $S_h^0$  and  $u = h$  on the boundary  $\partial S_h^0 \cap \Omega$ , where  $A_{ij,p_k p_l} = D_{p_k p_l}^2 A_{ij}$  satisfies (H2) and is uniformly bounded. Note that one can directly verify that (2.22) is equivalent to (1.1) by Taylor expansion, because under the above transforms there holds

$$(2.23) \quad D_{p_k} A_{ij}(\cdot, 0) \equiv 0, \quad \text{for all } i, j, k.$$

See [17, §2] for more detailed calculations. Meanwhile, the sub-level set  $S_h^0$  has now the following geometric property:

**Lemma 2.2.** *By the coordinate transform (2.21), the sub-level set  $S_h^0 \subset \mathbb{R}^n$  in (2.20) is convex and bounded.*

### 3. BOUNDARY ESTIMATE

In this section we investigate the local geometry of the boundary sub-level, which is crucial to the proof of boundary regularity in the subsequent section. First is a tangential estimate

**Lemma 3.1.** *Let  $u$  be a  $c$ -convex solution of (1.1) satisfying (2.19) and  $u = u_0$  on  $\partial\Omega$  with  $u_0 \in C^3(\partial\Omega)$ , where  $0 \in \partial\Omega$  is a boundary point. Assume that  $\partial\Omega \in C^3$  and  $\Omega$  is uniformly  $c$ -convex with respect to 0. Then, there exists a constant  $\mu > 0$  such that*

$$(3.1) \quad \mu^{-1}|x|^2 \leq u(x) \leq \mu|x|^2 \quad \text{on } \partial\Omega \text{ near } 0.$$

*Proof.* The proof of (3.1) is standard, see for example [24, Proposition 3.2] and references therein. We include it here for completeness. The second inequality is straightforward, thus it suffices to prove the first inequality. Make the coordinate transform (2.21) such that  $\Omega$  becomes uniformly convex at the origin. Note that we may assume  $D_{i,j}c(0,0) = \delta_{ij}$ , and by Taylor's expansion for each  $x$  in an  $\varepsilon$ -neighborhood of 0

$$(3.2) \quad x_i \mapsto D_{y_i}c(x, 0) = x_i + c_{jk,i}(tx, 0)x_j x_k, \quad i, j, k = 1, \dots, n$$

for some  $t \in (0, 1)$ . Hence, the transform (2.21) does not affect the inequality (3.1).

Without loss of generality, let  $e_n$  be the unit inner normal of  $\partial\Omega$  at 0 and write  $x' = (x_1, \dots, x_{n-1})$ . Since  $\partial\Omega, u_0 \in C^3$ , we have

$$u_0 = Q_0(x') + o(|x'|^3)$$

with  $Q_0$  a cubic polynomial. Note that  $u_0(0) = 0$  and  $u_0 \geq 0$  on  $\partial\Omega$ , which implies the constant and linear parts of  $Q_0$  are zero. Let its quadratic part be

$$\sum_{i < n} \mu_i x_i^2, \quad \text{with } \mu_i \geq 0.$$

It suffices to show that  $\mu_i > 0$  for all  $i = 1, \dots, n-1$ .

By contradiction, let  $\mu_1 = 0$ . For  $h > 0$  small enough, the set  $S_h^0 \cap \partial\Omega$  contains

$$\{|x_1| \leq r(h)h^{1/3}\} \cap \{|x''| \leq ch^{1/2}\}$$

for some  $c > 0$ , where  $x'' = (x_2, \dots, x_{n-1})$  and

$$r(h) \rightarrow \infty \quad \text{as } h \rightarrow 0.$$

From Lemma 2.2,  $S_h^0 \cap \Omega$  is convex. Let  $E$  be the minimum ellipsoid [13] enclosing  $S_h^0 \cap \Omega$  and since  $\Omega$  is uniformly convex, the  $x_n$ -diameter of  $E$  is larger than  $\bar{c}|r_1|^2 \geq \bar{c}(r(h)h^{1/3})^2$ , where  $r_1$  is the  $x_1$ -diameter of  $E$  and  $\bar{c}$  is a positive constant depending on  $\partial\Omega$ . Hence

$$(3.3) \quad |S_h^0 \cap \Omega| \geq c'(r(h)h^{1/3})^3 h^{(n-2)/2} \geq c'r^3(h)h^{n/2}.$$

On the other hand, let  $v$  be a solution to  $\det Mv = \inf f$  in  $S_h^0 \cap \Omega$ , where  $M$  is the matrix operator in (1.1), and  $v = h$  on  $\partial(S_h^0 \cap \Omega)$ . If  $|S_h^0| > Ch^{n/2}$  for some large  $C \gg 1$ , we have  $\inf v < 0$ . By the comparison principle [22], we have  $\inf u \leq \inf v < 0$ , which is a contradiction to  $\inf u = 0$ . Hence we obtain

$$(3.4) \quad |S_h^0 \cap \Omega| \leq Ch^{n/2}.$$

However, this is a contradiction to the inequality (3.3) as  $h \rightarrow 0$ . □

The following is an estimate of boundary sub-level sets along the normal  $x_n$ -direction.

**Lemma 3.2.** *Under the assumptions of Lemma 3.1, let  $e_n$  be the unit inner normal of  $\partial\Omega$  at 0 and locally  $\partial\Omega$  is represented by  $x_n = \rho(x')$  for a function  $\rho \in C^3$  satisfying  $\rho(0) = 0, D\rho(0) = 0$ , where  $x' = (x_1, \dots, x_{n-1})$ . For  $h > 0$  small, denote*

$$(3.5) \quad d_n(h) = \sup\{x_n : x = (x', x_n) \in S_h^0 \cap \Omega\},$$

where  $S_h^0$  is the boundary sub-level set. Then we have

- (i)  $C^{-1}h^{1/2} \leq d_n(h) \leq Ch^{1/2}$ ,
- (ii)  $C^{-1}h^{n/2} \leq |S_h^0| \leq Ch^{n/2}$ ,

where  $C$  is a positive constant depending on  $n, c, u_0, f$  and  $\partial\Omega$ .

*Proof.* Let  $x^h = \kappa(h)e_n$  for some  $\kappa = \kappa(h) > 0$ , be the intersection point  $\partial S_h^0 \cap \{x' = 0, x_n > 0\}$  of  $\partial S_h^0$  with the positive  $x_n$ -axis. From (3.5),

$$(3.6) \quad \kappa(h) \leq d_n(h).$$

Make the boundary transformation as in §2.2 such that  $S_h^0 \cap \partial\Omega$  is flat for  $h > 0$  small enough. From (3.1),  $u$  satisfies

$$(3.7) \quad \mu^{-1}|x'|^2 \leq u(x) = u(x', 0) \leq \mu|x'|^2,$$

in a neighbourhood of 0 on the boundary  $\partial\Omega$ , hence  $\partial(S_h^0 \cap \partial\Omega)$  is pinched by two cylinders  $|x'| = \mu^{-1}h^{1/2}$  and  $|x'| = \mu h^{1/2}$ .

Then, we make the coordinate transform (2.21) by setting  $x \mapsto \tilde{x} = D_y c(x, 0)$  such that  $\partial S_h^0$  becomes convex, see Figure 3.1. By (3.2), for each  $x$  in an  $\varepsilon$ -neighborhood of 0,

$$\text{dist}(\tilde{x}, x) \leq C\varepsilon^2,$$

where  $C$  depends on  $\|c\|_{C^3}$ . This implies that the image  $\tilde{x} = D_y c(x, 0)$  stays in a neighborhood of  $x$  of radius  $C|x|^2$ . Therefore,  $\tilde{x}^h = ((\tilde{x}^h)', \tilde{x}_n^h)$  satisfies  $|(\tilde{x}^h)'| \sim \kappa^2$  and  $\tilde{x}_n^h \sim \kappa + O(\kappa^2)$ , and for  $h > 0$  small the boundary  $\partial(S_h^0 \cap \partial\Omega)$  is contained in

$$(3.8) \quad \partial(S_h^0 \cap \partial\Omega) \subset \{C_1' h^{1/2} \leq |x'| \leq C_2' h^{1/2}, |x_n| \leq Ch\}.$$

Denote  $\partial S_h := \partial(S_h^0 \cap \Omega) = \partial_1 S_h \cup \partial_2 S_h$ , where

$$(3.9) \quad \partial_1 S_h = \partial S_h^0 \cap \Omega; \quad \partial_2 S_h = S_h^0 \cap \partial\Omega.$$

Let  $\pi_n$  be the projection over  $\{x_n = 0\}$ , namely  $\pi_n(x', x_n) = (x', 0)$ . From the  $c$ -convexity assumption on  $\Omega$  and (2.21), The projection  $\pi_n(\partial_2 S_h)$  is convex in  $\mathbb{R}^{n-1}$ . Let  $d_\alpha(h)$  be the radius of the minimum ellipsoid of  $\pi_n(\partial_2 S_h)$  along  $x_\alpha$ -axis, where  $\alpha = 1, \dots, n-1$ , [13]. By (3.8), we have

$$(3.10) \quad C^{-1}h^{1/2} \leq d_\alpha(h) \leq Ch^{1/2}.$$

Since  $D_{i,j}c(0, 0) = \delta_{ij}$ ,  $e_n$  remains the unit inner normal at 0. By (3.8) and the convexity of  $S_h^0 \cap \Omega$ , we have

$$(3.11) \quad |S_h^0 \cap \Omega| \geq \delta_n \Pi_{i=1}^n d_i(h) \geq \delta_n h^{\frac{n-1}{2}} d_n(h),$$

where  $\delta_n > 0$  is a constant depending on  $n$ . In fact, (3.11) is due to John's lemma [13]. Let  $E$  be the minimum ellipsoid of  $S_h^0 \cap \Omega$ ,  $|S_h^0 \cap \Omega| \geq \delta_n' |E| \geq \delta_n \Pi_{i=1}^n r_i$ , where  $r_i \geq d_i(h)$  is the  $x_i$ -diameter of  $E$ ,  $i = 1, \dots, n$ . From (3.4), we obtain the second inequalities in part (i) and in part (ii).

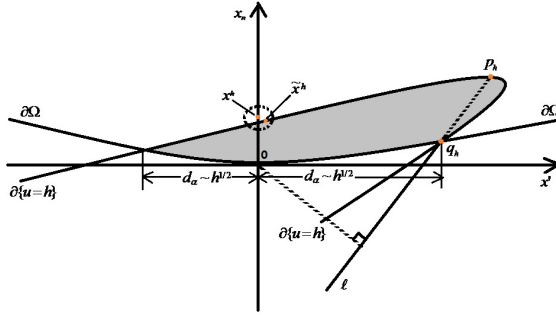


FIGURE 3.1. Convexified sub-level set on boundary

It remains to prove the first inequalities in parts (i) and (ii). In the following we replace  $u$  by its extension outside the domain  $\Omega$  such that for  $x \in \Omega^c$ , the complement of  $\Omega$ ,

$$(3.12) \quad u(x) = \sup\{c(x, y) + u(x_0) - c(x_0, y) : x_0 \in \bar{\Omega}, y \in \mathcal{N}_u^c(x_0)\}$$

where  $\mathcal{N}_u^c$  is the  $c$ -normal mapping of  $u$  in Definition 2.2, and replace the boundary sub-level set at  $0 \in \partial\Omega$  by

$$S_h^0 = \{x \in \mathbb{R}^n : u < h\}.$$

Recall that  $u(0) = 0$  and  $Du(0) = 0$ , the  $c$ -support function  $c(x, 0) + u(0) - c(0, 0) \equiv 0$  and thus the extended  $u \geq 0$  in a neighbourhood of  $\Omega$

By (3.6),  $\kappa(h) \leq Ch^{1/2}$ . From the uniform convexity of  $\partial\Omega$  and (3.1),  $\kappa(h) \geq Ch$ , for some constant  $C > 0$  depending on the constant in (3.10) and  $\partial\Omega$ . Hence, we have

$$(3.13) \quad d_n(h) \geq Ch.$$

Denote

$$(3.14) \quad d_{max} = \max_{1 \leq i \leq n} \{d_i(h)\}, \quad \text{and} \quad d_{min} = \min_{1 \leq i \leq n} \{d_i(h)\}.$$

From the above, we may assume that  $d_{max} \simeq h^{1/2}$ ,  $d_{min} = d_n(h)$ . Let the sup (3.5) attain at the point  $p_h \in \partial_1 S_h$ , by a rotation of  $x'$ -coordinates we may assume that  $p_h = (a_h, 0, \dots, 0, d_n)$ , for  $a_h > 0$ . Let  $q_h = (q_{h,1}, 0, \dots, 0, q_{h,n})$  be the point on  $\partial S_h^0 \cap \partial\Omega$  such that  $q_{h,1} > 0$ . From (3.8) and (3.10) we know  $q_{h,1} \simeq Ch^{1/2}$  and  $|q_{h,n}| \lesssim Ch$ . Let  $\ell := \ell_{p_h, q_h} \cap \Omega^c$ , where  $\ell_{p_h, q_h}$  is the straight line connecting  $p_h$  and  $q_h$ . Denote  $\partial^c S_h := \{u = h\} \cap \Omega^c$ , then by convexity we have

$$(3.15) \quad \text{dist}(0, \partial^c S_h) \leq \text{dist}(0, \ell).$$

Next we show that  $a_h < \bar{C}h^{1/2}$  for a constant  $\bar{C} > 0$ , which will be needed in the subsequent normalisation process. Otherwise, if  $a_h > Kh^{1/2}$  for a large constant  $K \gg C$ , where  $C$  is the constant in (3.10), then

$$\frac{\text{dist}(0, \ell)}{Ch^{1/2}} \leq \frac{\text{dist}(0, \ell)}{d_1(h)} \leq \frac{d_n(h)}{a_h} < CK^{-1},$$

where the last inequality is due to  $d_n(h) \leq Ch^{1/2}$  and  $a_h > Kh^{1/2}$ . Hence,

$$(3.16) \quad \text{dist}(0, \ell) < CK^{-1}h^{1/2}.$$

On the other hand, in the two dimensional  $\{x_1, x_n\}$ -space we observe that since  $d_n(h) \leq Ch^{1/2}$  and  $a_h \gg d_1(h)$ , the point  $p_h$  is sliding far along the positive  $x_1$  direction, and moreover, since  $d_1(h) \leq Ch^{1/2}$  is bounded, as  $K \rightarrow \infty$ , the sub-level set  $S_h^0$  is squeezed to almost parallel along  $x_1$ -axis due to its convexity. However, let  $x^* \in \Omega^c \cap \{u = 0\}$  close to the origin such that

$$\text{dist}(x^*, \partial^c S_h) < 2\text{dist}(0, \partial^c S_h).$$

In a neighbourhood  $B_\delta(x^*)$  of  $x^*$  the contact set  $\{u = 0\}$  contains a segment and  $x^*$  is an interior point of the segment, then by [17, Theorem 2] we have

$$0 \leq u(x) \leq C_*|x - x^*|^2 \quad \text{in } B_\delta(x^*),$$

where  $C_* > 0$  is a constant. This implies that  $\text{dist}(x^*, \partial^c S_h) \geq C_*^{-1/2} h^{1/2}$ , which can be seen as increasing  $\delta$  such that  $\partial B_\delta(x^*)$  touches  $\partial^c S_h$  at the first time. Therefore,

$$(3.17) \quad \text{dist}(0, \partial^c S_h) > \frac{1}{2} C_*^{-1/2} h^{1/2}.$$

From (3.15), (3.16) and (3.17),  $K < 2CC_*^{1/2}$  and thus we obtain  $a_h < \bar{C}h^{1/2}$  for some constant  $\bar{C} > 0$  independent of  $h$ .

Now, let's make a normalisation transform  $T = T_h$  by

$$(3.18) \quad u \mapsto u/h, \quad x_i \mapsto x_i/d_i(h)$$

for all  $i = 1, \dots, n$ . Equation (1.1) will become

$$(3.19) \quad \det \left[ D_{ij}u - hA_{ij,kl} \frac{d_i d_j}{d_k d_l} D_k u D_l u \right] = \frac{(\prod d_i)^2}{h^n} f \quad \text{in } U,$$

where  $U = T(S_h^0 \cap \Omega)$ . By (3.13), we have

$$(3.20) \quad h \frac{d_{max}^2}{d_{min}^2} \leq C,$$

and thus the coefficients  $\tilde{A}_{ij,kl} = h \frac{d_i d_j}{d_k d_l} A_{ij,kl}$  are uniformly bounded.

Since  $a_h < \bar{C}h^{1/2}$ , after the normalisation  $T$ ,  $U \subset \{|x'| \leq \max\{\bar{C}/C', 1\}\} \times [0, 1]$  is convex, where  $C'$  is the constant in (3.8), and  $\pi_n(U \cap \partial\Omega)$  is normalised (contained in a ball  $B_1$  and contains a ball  $B_r$  in  $\mathbb{R}^{n-1}$  such that  $1/r \leq C(n)$ ). Recall that  $\{u < 1\}$  is convex. Denote  $\tilde{C} = \max\{\bar{C}/C', 1\}$ , we have

$$(3.21) \quad \{u < 1\} \cap \{x_n < 0\} \subset \left\{ x \in \mathbb{R}^n : |x'| \leq 2\tilde{C}(|x_n| + \frac{1}{2}), x_n < 0 \right\}.$$

Consider equation (3.19) in the set

$$U_K = \{u < 1\} \cap \{x_n > -K\},$$

for some large constant  $K > 0$ . Set

$$(3.22) \quad w(x) = \frac{\delta}{K}|x'|^2 + \frac{\delta}{K}x_n^2 + 11\delta x_n + \delta,$$

for a small constant  $\delta > 0$ , to be determined.

By calculation

$$(3.23) \quad \begin{aligned} w(0) &= \delta, \\ w(x) &\leq \delta - \delta K < -\delta \quad \text{on } \{x_n = -K\}, \\ w(x) &\leq 1 \quad \text{on } \{u(x) = 1\}, \end{aligned}$$

by (3.21), provided  $K > 2$ . And by differentiation, we have

$$\begin{aligned} |w_i| &\leq 15\delta, \quad \text{for } 1 \leq i \leq n; \\ w_{ij} &= \frac{2\delta}{K}\delta_{ij}, \end{aligned}$$

where  $\delta_{ij}$  is the usual Kronecker delta. From (3.20) the coefficients

$$\left\{ hA_{ij,kl} \frac{d_i d_j}{d_k d_l} \right\}$$

are uniformly bounded, here the repeated indices do not indicate summation, which are fixed. It follows that the matrix

$$Mw := \left\{ w_{ij} - \sum_{k,l} hA_{ij,kl} \frac{d_i d_j}{d_k d_l} w_k w_l \right\} \geq \left( \frac{2\delta}{K} - C\delta^2 \right) I,$$

where  $I$  is the  $n \times n$  identity matrix. The matrix  $Mw$  is positive definite provided  $\delta K < 1/2C$ . Hence we have  $\det Mw \geq C'\delta^n > 0$  for some different constant  $C'$ .

Let  $U' = \{u < w\}$ , then  $0 \in U'$ . From (3.13), (3.23) and  $\kappa(h) \geq Ch$ , which implies a Lipschitz bound of  $u$ , there exists a positive constant  $r$  depending on  $\delta$  such that the ball  $B_r(0) \subset U'$ , which implies that the Lebesgue measure of  $U'$  is positive. Hence by the comparison principle we have

$$(3.24) \quad \int_{U_K} \det Mudx \geq \int_{U'} \det Mwdx \geq C\delta^n > 0.$$

On the other hand, since  $f \leq \Lambda$  and by (3.21) we have

$$(3.25) \quad \begin{aligned} \int_{U_K} \det Mudx &\leq \int_{U_K} \frac{(\prod d_i)^2}{h^n} f(x) dx \\ &\leq C \frac{(\prod d_i)^2}{h^n}. \end{aligned}$$

From (3.24) and (3.25), we then obtain the lower boundedness  $\frac{(\prod d_i)^2}{h^n} \geq C_1 > 0$ . Therefore, by (3.10) we obtain the first inequality in part (i). The first inequality in part (ii) then follows from (3.11).  $\square$

#### 4. PROOF OF THEOREM 1.1

In this section we first establish three lemmas analogous to those in the interior case [20], which will be used to prove Theorem 1.1. We start with a Pogorelov type estimate in a half-domain. A similar estimate for the standard Monge-Ampère equation was obtained in [24, 28].

**Lemma 4.1.** *Let  $u \in C^4(\bar{\Omega})$  be an elliptic solution of*

$$(4.1) \quad \det[u_{ij} - A_{ij}(x, Du)] = 1 \quad \text{in } \Omega,$$

where  $B_r(0)^+ \subset \Omega \subset B_{1/r}(0)^+$  for some constant  $r > 0$  and  $B_r(0)^+ = B_r(0) \cap \{x_n > 0\}$ . Assume that  $\Omega$  is uniformly  $c$ -convex,  $u = \frac{1}{2}|x'|^2$  on  $\partial\Omega \cap \{x_n = 0\}$  and  $u = 1$  on  $\partial\Omega \cap \{x_n > 0\}$ . Then we have the estimate

$$(4.2) \quad \|u\|_{C^4(\{u < 1/2\})} \leq C,$$

where the constant  $C$  depends on  $n, r, \|A\|_{C^2}$  and  $\sup_{\Omega} |Du|$ .

*Proof.* Applying the Pogorelov estimate [19] in the set  $F := \{u < 3/4\}$ , namely, if the maximum of

$$\tau \log \left( \frac{3}{4} - u \right) + \log w_{ii} + \frac{1}{2} \beta |Du|^2 + e^{\kappa\varphi}$$



occurs in the interior of  $F$ , where  $\varphi$  is the defining function of  $\Omega$ ,  $\beta, \kappa, \tau$  are properly-chosen positive constants, then this value is bounded by a constant depending on  $n, r, \|A\|_{C^2}$  and  $\sup_F |Du|$ . It remains to show that

$$|D^2u| \leq C \quad \text{on } E := \{x_n = 0\} \cap \{u < 3/4\}.$$

Since  $u = \frac{1}{2}|x'|^2$  on  $\partial\Omega \cap \{x_n = 0\}$ , it suffices to prove that the mixed derivatives  $|u_{\alpha n}|$  are bounded on  $E$  with  $\alpha = 1, \dots, n-1$ . Define a linearised operator

$$(4.3) \quad L := w^{ij}(D_{ij} - D_{p_k}A_{ij}(x, Du)D_k),$$

where  $(w^{ij})$  is the inverse of  $(w_{ij}) = (u_{ij} - A_{ij})$ . Then

$$(4.4) \quad Lu_\alpha = 0, \quad \text{and } u_\alpha = x_\alpha \text{ on } \{x_n = 0\}.$$

Fix  $x_0 \in E$ . Define

$$(4.5) \quad v(x) := x_\alpha - \tau x_n + \kappa\varphi - \gamma(u - \ell_{x_0}),$$

where  $\ell_{x_0}$  is the  $c$ -support of  $u$  at  $x_0$ ,  $\tau, \kappa, \gamma \geq 0$ , and the defining function  $\varphi$  satisfies  $\varphi = 0$  on  $\partial\Omega$  while  $\varphi < 0$  in  $\Omega$ . By the uniformly  $c$ -convexity of  $\Omega$ , we have

$$(4.6) \quad L\varphi \geq \delta_0 w^{ii}$$

for a constant  $\delta_0 > 0$ , [31]. By computation, we have

$$\begin{aligned} L(u - \ell_{x_0}) &= w^{ij}(w_{ij} + c_{ij} - D_{ij}\ell_{x_0} - D_{p_k}A_{ij}D_k(u - \ell_{x_0})) \\ &\leq n + Cw^{ii}, \\ L(x_\alpha - \tau x_n) &\leq C\tau w^{ii}, \end{aligned}$$

where  $C > 0$  is a constant depending on  $n, c$  and  $|Du|$ . By choosing  $\kappa$  large enough such that  $\frac{1}{2}\kappa\delta_0 > C(1 + \tau + \gamma)$ , we have

$$(4.7) \quad Lv > 0.$$

It is clear that  $v \leq u_\alpha$  on  $\{x_n = 0\}$  and  $v = u_\alpha$  at  $x_0$ . On the remaining part of  $\partial\{u < 3/4\}$ , where  $u = \frac{3}{4}$ , we consider two cases: (i),  $x_n \geq \varepsilon$  for some  $\varepsilon > 0$ . Since  $\Omega \subset B_{1/r}$  and  $\sup_\Omega |Du| \leq C$ , by choosing  $\tau$  large enough we have  $v \leq u_\alpha$ ; (ii),  $x_n < \varepsilon$ . We can restrict  $x_0$  within  $\{u < 1/2\}$ , then on the boundary part that  $\{u = 3/4\}$ ,  $u - \ell_{x_0} \geq \sigma_0$ , where  $\sigma_0 > 0$  is a constant depending on the module of convexity of  $u$ . Choosing  $\gamma$  large enough we have  $v \leq u_\alpha$  on  $\{u = 1/2\} \cap \Omega$ . Similarly, we can obtain  $v \leq u_\alpha$  on  $\{u = 3/4\} \cap \Omega$ . Hence,

$$(4.8) \quad v \leq u_\alpha \quad \text{in } \{u < 1/2\}, \quad v(x_0) = u_\alpha(x_0).$$

This gives a lower bound on  $u_{\alpha n}(x_0)$ . Changing signs of  $\tau, \kappa, \gamma$ , we can similarly obtain an upper bound.

The Monge-Ampère equation (4.1) is then uniformly elliptic in  $\{u < 1/2\}$  and by Evans-Krylov theorem and Schauder estimate [6] we obtain the desired estimate (4.2).  $\square$

**Remark 4.1.** In fact, the estimate in Lemma 4.1 holds in general uniformly  $c$ -convex domains. The proof is essentially the same except that we work with the tangential derivative  $\delta_\alpha = \partial_\alpha + \rho_\alpha \partial_n$  instead of  $\partial_\alpha$ , where  $x_n = \rho(x')$  defines the boundary of  $\Omega$  near the origin with  $\rho(0) = \rho_\alpha(0) = 0$  for all  $\alpha = 1, \dots, n-1$ . Moreover, we can remove the smooth assumption on  $u$  by an approximation process and the results in [12] so that the estimate (4.2) holds for generalised solution of (4.1).

**Lemma 4.2.** Let  $u^{(m)}$ ,  $m = 1, 2$ , be two  $c$ -convex solutions of (4.1) in  $E$ , where  $E = S_{1,u}^0(0) \cap \Omega$ . Suppose that  $\|u^{(m)}\|_{C^4} \leq C_0$ . If  $|u^{(1)} - u^{(2)}| \leq \delta$  in  $E$  for some constant  $\delta > 0$ , then we have for  $1 \leq k \leq 3$ ,

$$(4.9) \quad |D^k(u^{(1)} - u^{(2)})| \leq C\delta \quad \text{in } E',$$

where  $C$  is some positive constant and  $E' = S_{3/4,u}^0(0) \cap \Omega$ .

*Proof.* Denote  $w_{ij}^{(m)} = u_{ij}^{(m)} - A_{ij}(x, Du^{(m)})$ . We have

$$\begin{aligned} 0 &= \det w_{ij}^{(2)} - \det w_{ij}^{(1)} \\ &= \int_0^1 \frac{d}{dt} \det [w_{ij}^{(1)} + t(w_{ij}^{(2)} - w_{ij}^{(1)})] dt \\ &= a_{ij}(x) [D_{ij}(u^{(2)} - u^{(1)}) - A_{ij,p_k} D_k(u^{(2)} - u^{(1)})], \end{aligned}$$

where  $a_{ij}(x) = \int_0^1 C_{t,ij} dt$  and  $C_{t,ij}$  is the cofactor of  $(M_t)_{ij} = w_{ij}^{(1)} + t(w_{ij}^{(2)} - w_{ij}^{(1)})$ . By assumptions,  $\|w_{ij}^{(m)}\|_{C^2} \leq C_0$  for a different constant  $C_0$ , and the matrices  $M_t = \{(M_t)_{ij}\}$  are  $C^2$  smooth, positive definite for all  $t \in [0, 1]$ . Hence from the equation (4.1), the operator

$$L = a_{ij}(x)(D_{ij} - A_{ij,p_k} D_k)$$

is linear and uniformly elliptic with  $C^2$  coefficients. By Schauder estimates for linear elliptic equations [6], we obtain the estimate (4.9).  $\square$

Let  $u \in C^2(\Omega)$  be a solution of (1.1). Let  $x_0 = 0$  be a given point on  $\partial\Omega$ . By subtracting a  $c$ -support, from Section 2 we may assume that  $\varphi \equiv 0$  is the  $c$ -support of  $u$  at  $x_0$ . By making a coordinate transform (2.21) we may also assume the sub-level set

$$(4.10) \quad \hat{S}_h^0 := \{x \in \Omega : u(x) < h\}$$

is convex. By a translation and rotation of coordinates, we assume that  $E = \{\sum_i \frac{(x_i - \hat{x}_i)^2}{r_i^2} < 1\}$  is the minimum ellipsoid of  $\hat{S}_h^0$  [13], where  $\hat{x}$  is the centroid of  $E$  and  $r_1 \geq \dots \geq r_n$ . In the following we say a convex set has a *good shape* if its minimum ellipsoid satisfies  $r_1 \leq c^* r_n$  for some  $c^*$  under control. The constant  $c^*$  is called a *shape constant* [36].

**Lemma 4.3.** Let  $u$  be a  $C^2$  smooth elliptic solution of (4.1) in  $\Omega$  with  $B_r(0)^+ \subset \Omega \subset B_{1/r}(0)^+$  for some constant  $r > 0$ ,  $u = h$  on  $\partial\hat{S}_h^0 \cap \Omega$  for a constant  $h > 0$  and  $u = \varphi$  on  $\partial\hat{S}_h^0 \cap \partial\Omega$ , where  $\varphi$  is a quadratic polynomial that satisfies

$$(4.11) \quad \mu^{-1}|x|^2 \leq \varphi(x) \leq \mu|x|^2 \quad \text{on } \partial\Omega.$$

Suppose that  $D^2u(0)$  is the unit matrix (or uniformly bounded), then the domain  $\hat{S}_h^0$  is of good shape.

*Proof.* Assume by contradiction that  $\hat{S}_h^0$  does not have a good shape. Let  $E' = \{\sum_\alpha \frac{x_\alpha^2}{d_\alpha^2} < 1\}$  be the minimum ellipsoid of  $\partial\hat{S}_h^0 \cap \partial\Omega$ ,  $\alpha = 1, \dots, n-1$ . By (4.11),  $d_\alpha \simeq h^{1/2}$ . Let the superimum  $\sup\{x_n : x = (x', x_n) \in \partial\hat{S}_h^0\}$  attain at the point  $p_h \in \partial\hat{S}_h^0 \cap \Omega$ , by a rotation of  $x'$ -coordinates we may assume that  $p_h = (a_h, 0, \dots, 0, d_n)$ , for  $a_h > 0$ . From the proof of Lemma 3.2,  $a_h < \bar{C}h^{1/2}$  for a universal constant  $\bar{C} > 0$ . Make a linear transform  $T : x_i \mapsto x_i/d_i$  to normalise  $\hat{S}_h^0$  and let  $u \mapsto \bar{u} = u/h$ . Then the ratio of the largest and least eigenvalues of  $T$  is arbitrarily large and equation (4.1) transforms to

$$\det \left[ D_{ij}\bar{u} - hA_{ij,kl} \frac{d_i d_j}{d_k d_l} D_k \bar{u} D_l \bar{u} \right] = \frac{(\prod d_i)^2}{h^n}.$$

By Lemma 4.1,  $|D^2\bar{u}(0)|$  is uniformly bounded. Hence, by (4.2)  $C^{-1}I \leq D^2\bar{u}(0) \leq CI$ , where  $I$  is the unit matrix. Hence,  $D^2u(0) = T'D^2\bar{u}(0)T$  cannot be uniformly bounded, where  $T'$  is the transpose of  $T$ , and we obtain a contradiction.  $\square$

We are now ready to prove Theorem 1.1:

*Proof of Theorem 1.1.* As above we assume  $\varphi \equiv 0$  is the  $c$ -support of  $u$  at 0, so that  $u(0) = 0, Du(0) = 0$ , and the set  $\hat{S}_h^0$  in (4.10) is convex. From Section 3,  $\hat{S}_h^0 \subset B_{Ch^{1/2}}(0)$ . Choose  $h_0 > 0$  sufficiently small such that  $\hat{S}_{h_0}^0$  is compactly contained in  $B_1(0)$ .

Assume that near the origin  $\partial\Omega$  is represented by  $x_n = \rho(x')$  with  $\rho(0) = 0$  and  $D\rho(0) = 0$ . Make the boundary transformation as in §2.2 such that  $\partial\Omega \subset \{x_n = 0\}$  nearly the origin and  $\hat{S}_h^0 \subset \mathbb{R}_+^n$  for  $h > 0$  small. Denote

$$(4.12) \quad \omega_{u_0}(r) = \sup \left\{ \frac{1}{|x'|^2} |u_0(x') - \frac{1}{2} \sum_{i,j=1}^{n-1} a_{ij} x_i x_j| : |x| < r, x \in \partial\Omega \right\},$$

where  $a_{ij} = \frac{\partial^2}{\partial_i \partial_j} u_0(0)$ . If  $u_0, \partial\Omega \in C^{2+\alpha}$ , then

$$\omega_{u_0}(r) = O(r^\alpha).$$

As in Section 3, let  $d_\alpha(h)$  be the radius of the minimum ellipsoid of  $\partial\hat{S}_h^0 \cap \partial\Omega$ , where  $\alpha = 1, \dots, n-1$ , and  $d_n(h)$  is defined by (3.5). From Lemma 3.2,  $d_i(h) \simeq h^{1/2}$  for all  $i = 1, \dots, n$ . We start with  $\hat{S}_{h_0}^0$  and make a normalisation transform  $T = T_{h_0}$  by letting

$$(4.13) \quad u \mapsto u/h_0, \quad x_i \mapsto x_i/d_i(h_0), \quad i = 1, \dots, n.$$

Then equation (1.1) becomes

$$(4.14) \quad \det \left[ D_{ij}u - hA_{ij,kl} \frac{d_i d_j}{d_k d_l} D_k u D_l u \right] = \frac{(\prod d_i)^2}{h^n} f,$$

where by Lemma 3.2 the coefficients  $\tilde{A}_{ij,kl} = hA_{ij,kl} \frac{d_i d_j}{d_k d_l} = O(h)$  and the right hand side  $\tilde{f} = \frac{(\prod d_i)^2}{h^n} f$  is uniformly bounded. Hence, we may suppose  $h = 1$  and  $\hat{S}_1^0$  is normalised. Additionally, note that the proof of Lemma 3.2 implies that  $\sup_{\partial\Omega} |Du| \leq C$ , the matrix  $A$  in (4.14) satisfies the structure condition

$A(x, p) \geq -\mu_0(1 + |p|^2)I$  as in [12, (1.6)], therefore we have the global gradient bound  $\sup_{\Omega} |Du| \leq C$  thanks to [12, (4.5)].

By choosing the original  $h_0$  sufficiently small, we also have

$$(4.15) \quad \int_0^1 \frac{\omega(r)}{r} \leq \epsilon,$$

where  $\epsilon > 0$  can be as small as we want, and

$$\omega(r) := \omega_{\tilde{f}}(r) + \omega_{u_0}(r),$$

where  $\omega_{\tilde{f}}(r) = \sup\{|\tilde{f}(x) - \tilde{f}(y)| : |\bar{x} - \bar{y}| < r\}$  is the oscillation of the inhomogeneous term.

The proof is based on a perturbation argument as for the interior case in [20]. The argument was previously used by Jian-Wang [11] and Wang [37] for the standard Monge-Ampère equation and by Wang [36] for uniform elliptic and parabolic equations.

We define an approximation sequence as follows. Let  $\{u^{(k)}\}$ ,  $k = 0, 1, \dots$ , be elliptic solutions of

$$(4.16) \quad \begin{aligned} \det[u_{ij}^{(k)} - \tilde{A}_{ij}(\cdot, Du^{(k)})] &= \tilde{f}_k, & \text{in } S_k, \\ u^{(k)} &= 4^{-k}, & \text{on } \partial_1 S_k, \\ u^{(k)} &= u_0^{(k)}, & \text{on } \partial_2 S_k, \end{aligned}$$

where  $S_k := \hat{S}_{4^{-k}, u}^0$ ,  $\partial_1 S_k := \partial \hat{S}_{4^{-k}, u}^0 \cap \Omega$ ,  $\partial_2 S_k := \partial \hat{S}_{4^{-k}, u}^0 \cap \partial \Omega$ ,  $\tilde{f}_k = \inf_{S_k} \tilde{f} > 0$  is a constant, and

$$(4.17) \quad u_0^{(k)} = \sum_{i,j=1}^{n-1} \left( \frac{1}{2} a_{ij} + \omega_{u_0}(2^{-k}) \delta_{ij} \right) x_i x_j.$$

Note that by (4.12),  $|u_0^{(k)} - u_0| \leq 4^{-k} \cdot 2^{-k\alpha}$  on  $\partial_2 S_k$ . By slightly reducing the height  $4^{-k}$  on  $\partial_1 S_k$ , we may assume  $u_0^{(k)} = 4^{-k} - \varepsilon_k$  on  $\partial_1 S_k \cap \partial_2 S_k$  such that the boundary data is continuous, where  $\varepsilon_k > 0$  is a small constant and  $\varepsilon(k) \rightarrow 0$  as  $k \rightarrow \infty$ . The solvability of (4.16) has been obtained in [12]. The modification  $\varepsilon_k$  does not affect the following perturbation process.

For  $k = 0$ ,  $u^{(0)}$  is uniformly quadratic separated on the boundary  $\partial_2 S_0$ . By Lemma 4.1, we have  $\|u^{(0)}\|_{C^4(\hat{S}_{3/4, u}^0)} \leq C$ . By a transform we assume that

$$(4.18) \quad D^2 u^{(0)} = I.$$

Let  $M$  denote the matrix operator on (4.14). Since  $u^{(0)} \geq u$  on  $\partial S_0$  and  $\det M u^{(0)} \leq \det M u$  in  $S_0$ , by the comparison principle in [22] we have  $u^{(0)} \geq u$  in  $S_0$ . On the other hand,  $(u-1) \geq (u^{(0)}-1)(1+C\omega_0)$  on  $\partial S_0$ . By computation we have

$$\begin{aligned} \det^{1/n} M \left[ (1 + C\omega_0)u^{(0)} \right] &= (1 + C\omega_0) \det^{1/n} \left[ D_{ij} u^{(0)} - \frac{1}{1 + C\omega_0} \tilde{A}_{ij}(\cdot, (1 + C\omega_0)Du^{(0)}) \right] \\ &= (1 + C\omega_0) \det^{1/n} \left[ M u^{(0)} - a_{ij}^0 \right] \end{aligned}$$

where the matrix

$$a_{ij}^0 = \frac{1}{1 + C\omega_0} \tilde{A}_{ij}(\cdot, (1 + C\omega_0)Du^{(0)}) - \tilde{A}_{ij}(\cdot, Du^{(0)}),$$

from (2.23) and  $\tilde{A}_{ij,kl} = O(h)$ , for any  $\varepsilon > 0$ , by choosing  $h > 0$  sufficiently small such that

$$|a_{ij}^0| \leq Ch|Du^{(0)}|^2 < \varepsilon,$$

we have

$$\begin{aligned} \det [Mu^{(0)} - a_{ij}^0] &\geq \det Mu^{(0)} \det [I - \varepsilon|D^2u^{(0)}|\delta_{ij}] \\ &\geq \tilde{f}_0(1 - \varepsilon|D^2u^{(0)}|) \geq (1 - \tilde{C}\varepsilon)\tilde{f}_0, \end{aligned}$$

where the last estimate is due to that for classical solutions in [12]. Hence,

$$\det M [(u^{(0)} - 1)(1 + C\omega_0)] \geq \det M(u - 1) \quad \text{in } S_0.$$

By the comparison principle in [22], we obtain

$$\sup_{S_0} |u - u^{(0)}| \leq C\omega_0.$$

Similarly, we have

$$\sup_{S_1} |u - u^{(1)}| \leq C\omega_1.$$

Hence, we obtain

$$\sup_{S_1} |u^{(1)} - u^{(0)}| \leq C\omega_0.$$

Since  $S_0$  has a good shape, so does  $S_1$ , that is  $\hat{S}_{1/4, u^{(1)}}^0$ . It follows that  $\|u^{(1)}\|_{C^4(\hat{S}_{3/16, u^{(1)}}^0)} \leq C$ . By Lemma 4.2, we have

$$(4.19) \quad |D^k u^{(0)}(x) - D^k u^{(1)}(x)| \leq C\omega_0$$

for  $x \in \hat{S}_{4^{-2}, u^{(1)}}^0$ , where  $1 \leq k \leq 3$ . By Lemma 4.3, (4.19) implies that  $\hat{S}_{4^{-2}, u^{(1)}}^0$  has a good shape. In return, since  $|u - u^{(1)}| \leq C\omega_1$ , we have that  $S_2$  has a good shape.

By induction, we assume that  $S_{k+1}$  has a good shape with constant  $c^*$  independent of  $k$ . We apply the same argument to  $\hat{u}^{(0)}(x) := 4^k u^{(k)}(2^{-k}x)$  and  $\hat{u}^{(1)} := 4^k u^{(k+1)}(2^{-k}x)$ , which satisfy the equations

$$\begin{aligned} \det[\hat{u}_{ij}^{(l)} - \tilde{A}_{ij}(2^{-k}\bar{x}, 2^{-k}D\hat{u}^{(l)})] &= \tilde{f}_{k+l}, \quad \text{in } \hat{S}_{k+l}, \\ \hat{u}^{(l)} &= 4^{-l}, \quad \text{on } \partial_1 \hat{S}_{k+l}, \\ \hat{u}^{(l)} &= u_0^{(k+l)}, \quad \text{on } \partial_2 \hat{S}_{k+l}, \end{aligned}$$

where  $\hat{S}_{k+l} = 2^k S_{k+l} = \{x \in \mathbb{R}^n : 2^k x \in S_{k+l}\}$  with  $l = 1, 2$ . By the above argument we obtain, similar to (4.19)

$$(4.20) \quad |D^\gamma \hat{u}^{(0)}(x) - D^\gamma \hat{u}^{(1)}(x)| \leq C\omega_k$$

for  $x \in \hat{S}_{4^{-k-2}, u^{(k+1)}}^0$ , where  $\gamma = 1, 2, 3$ . By re-scaling back, we have that

$$|D^2 u^{(k)}(x) - D^2 u^{(k+1)}(x)| \leq C\omega_k.$$

Hence

$$(4.21) \quad |D^2 u^{(0)}(x) - D^2 u^{(k+1)}(x)| \leq C \sum_{i=0}^k \omega_i$$

for  $x \in \hat{S}_{4^{-k-2}, u^{(k+1)}}^0$ , where  $C > 0$  is independent of  $k$ . Estimates (4.21) and (4.18) imply that  $D^2u^{(k+1)}(0)$  is close to the unit matrix. Hence by Lemma 4.3,  $\hat{S}_{4^{-k-2}, u^{(k+1)}}^0$  has a good shape, and so does  $S_{k+2}$ .

Therefore, we have

$$(4.22) \quad |D^2u^{(0)}(0) - D^2u(0)| \leq C \sum_{i=0}^{\infty} \omega_i \leq C \int_0^1 \frac{\omega(r)}{r} \leq C\epsilon.$$

Hence  $|D^2u(0)| \leq C$  is uniformly bounded for a different constant  $C$  depending on  $h_0$ . Therefore we obtain, by the interior estimate in [20], that  $\sup_{\bar{\Omega}} |D^2u(x)| < C$ . This estimate implies that equation (1.1) is uniformly elliptic. Hence the desired global  $C^{2,\alpha}$  estimate (1.6) follows from [1, 16, 23], see also a renewed perturbation argument in [36, 37].  $\square$

**Remark 4.2.** *In fact, following the argument in [11, 20, 36] and Wang's lecture notes [37], we can obtain a more precise estimate (1.7). For completeness, we sketch the proof as follows: For any point  $z \in \bar{\Omega}$  near the origin,*

$$(4.23) \quad \begin{aligned} |D^2u(z) - D^2u(0)| &\leq |D^2u^{(k)}(z) - D^2u^{(k)}(0)| + |D^2u^{(k)} - D^2u(0)| + |D^2u(z) - D^2u^{(k)}(z)| \\ &=: I_1 + I_2 + I_3, \end{aligned}$$

where  $k \geq 1$  such that  $4^{-k-4} \leq u(z) \leq 4^{-k-3}$ . We may assume that the point  $z$  lies either on the  $x_n$  axis or on the boundary  $\partial\Omega$ , and will estimate  $I_i$  for each  $i = 1, 2, 3$ .

To estimate  $I_2$ , by (4.21) and recall that  $\omega(Ct) \leq C\omega(t)$ ,

$$(4.24) \quad I_2 \leq C \sum_{j=k}^{\infty} \omega_j \leq C \int_0^{|z|} \frac{\omega(r)}{r}.$$

To estimate  $I_3$ , instead of considering sub-level sets at the origin, we consider those at the point  $z$ . Let  $u_{z,j}$  be the solution of

$$(4.25) \quad \begin{aligned} \det Mu_{z,j} &= \inf_{S_{z,j}} \tilde{f}, & \text{in } S_{z,j}, \\ u_{z,j} &= u, & \text{on } \partial S_{z,j} \cap \Omega, \\ u_{z,j} &= u_0^{(z,j)}, & \text{on } \partial S_{z,j} \cap \partial\Omega, \end{aligned}$$

where  $S_{z,j} := \hat{S}_{4^{-j}, u}^0(z)$  and  $a_{ij} = \frac{\partial^2}{\partial_i \partial_j} u_0(z)$  in the definition of (4.17). Let

$$j_k := \inf\{j : \hat{S}_{4^{-j}, u}^0(z) \subset \hat{S}_{4^{-k}, u}^0\}.$$

Obviously  $j_k \geq k$ . By (4.22),  $u$  is uniformly  $c$ -convex and we also have  $j_k \leq k + l_0$  for some  $l_0 \geq 0$  independent of  $k$ . Note that  $|u_k - u_{z, k+l_0}| \leq C\omega_k$ . Applying Lemma 4.2 to  $u_k$  and  $u_{z, k+l_0}$  in  $\hat{S}_{4^{-k-l_0}, u}^0(z)$  we have

$$(4.26) \quad |D^2u_k(z) - D^2u_{z, k+l_0}(z)| \leq C\omega_k.$$

Similarly to (4.24) we have

$$(4.27) \quad |D^2u(z) - D^2u_{z,k+l_0}(z)| \leq C \sum_{j=k+l_0}^{\infty} \omega_j \leq C \int_0^{|z|} \frac{\omega(r)}{r}.$$

Combining the above two inequalities we have the estimate for  $I_3$ .

To estimate  $I_1$ , denote  $h_j = u_j - u_{j-1}$ . By re-scaling back the estimate (4.20),

$$|D^2h_j(z) - D^2h_j(0)| \leq C2^j\omega_j|z|.$$

Hence,

$$\begin{aligned} I_1 &\leq |D^2u^{(k-1)}(z) - D^2u^{(k-2)}(0)| + |D^2h_k(z) - D^2h_k(0)| \\ &\leq |D^2u^{(0)}(z) - D^2u^{(0)}(0)| + \sum_{j=1}^k |D^2h_j(z) - D^2h_j(0)| \\ &\leq C|z|(1 + \sum_{j=1}^k 2^j\omega_j) \\ &\leq C|z|(1 + \int_{|z|}^1 \frac{\omega(r)}{r^2}). \end{aligned}$$

Therefore, we obtain (1.7). We refer the reader to [36] and [11] for more detailed discussions.

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