2014

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Publication Details

Ismael, M., Nillsen, R. & Williams, G. (2014). Variation of Poincare recurrence times in discrete dynamical systems. Advances and Applications in Discrete Mathematics, 14 (2), 95-123.

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Keywords
dynamical, discrete, variation, poincare, times, systems, recurrence

Disciplines
Engineering | Science and Technology Studies

Publication Details
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VARIATION OF POINCARÉ RECURRENCE TIMES IN DISCRETE DYNAMICAL SYSTEMS

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Abstract

This paper is concerned with the variation and standard deviation of recurrence times in discrete dynamical systems. The first sections of the paper deal with dynamical systems (S, f), where *f* acts as a cyclic permutation on a finite set *S*. When *A* is a subset of *S*, an expression is derived for the standard deviation σ_A of the recurrence times of points in *A*, and the maximum and minimum of σ_A are derived over sets *A* in the family $\mathcal{A}_{r,s}$ of subsets of *S* that have *r* gaps and *s* elements. The results are applied to the case of more general discrete systems obtained as a countable "sum" of finite systems. Various aspects are discussed, and interpreted geometrically in some cases. Whereas the average value of the recurrence time is finite, the standard deviation of the recurrence time in some of these systems may be infinite.

1. Introduction

A *dynamical system* is a set S together with a transformation $f : S \to S$. If S is finite we say that the system is finite. Assume for the moment that S has a σ-algebra of subsets that has an f-invariant probability measure μ. A modern formulation of the Recurrence Theorem of Henri Poincaré says that if A is a set in the σ-algebra, then μ-almost all points of A return to A under some iterate of f, and the smallest number of iterates required is called the *recurrence time* for the point. See Brown [2] and Petersen [11] for a discussion, for example. In ergodic systems, and when $\mu(A) > 0$, Kac [7] showed that the average value of this recurrence time is $1/\mu(A)$. The question arises as to the variation, predictability and uncertainty of the recurrence times. Formulas for the moments of recurrence times in the context of stationary stochastic processes were derived by Blum and Rosenblatt [1], and in the same context a formula for the standard deviation of the recurrence time was proved by Kasteleyn [8]. A discussion of some of these matters is also in Ismael [6] and Chapter 4 of [9].

The present note is concerned with the variation in recurrence times in discrete dynamical systems. In the earlier sections, we consider when (S, f) is a finite dynamical system, and f is a cyclic permutation on S. It should possibly be noted that every finite dynamical system (T, g) contains a dynamical system of this type. For, in any such system, and with g^n denoting the nth iterate of g, we have

$$T \supseteq g(T) \supseteq g^2(T) \supseteq g^3(T) \supseteq \cdots.$$

As T is finite, there is a least value $n \in \mathbb{N}$ such that $g^n(T) = g^{n+1}(T)$. If we put $Y = g^n(T)$, then we see that $g : Y \to Y$ and that $g(Y) = g(g^n(T)) =$

$g^{n+1}(T) = Y$. But as Y is finite, this means g is a permutation on Y. But every permutation is a composition of cycles, so g acts as a cyclic permutation f on some subset S of Y, so that (S, f) is a system of the required type.

Results concerning the standard deviation of recurrence times in finite systems are derived, and these relate to how the points of A are distributed throughout S. There are results about what configurations of points in a set A maximize or minimize the standard deviation of recurrence times for A, where there are restrictions on the set A. Some of these have intuitive geometric interpretations. The results for finite systems are then applied to discrete systems that are finite or countably infinite "sums" of finite systems. Section 2 sets up the notation and concepts to be used. Section 3 derives Kac's formula for the average of the recurrence times in a finite system with some new observations. The remaining sections present new results on the variation of recurrence times in the above-mentioned families of discrete systems.

2. Preliminaries

If A is a finite set, then $|A|$ will denote the number of elements in A, and may be called the *size* of A. Let f be a cyclic permutation on a finite set S. Put $q = |S|$ and let us suppose that $S = \{u_0, u_1, ..., u_{q-1}\}$, where the elements are listed so that $u_1 = f(u_0), u_2 = f(u_1), ..., u_{q-1} = f(u_{q-2})$ and $u_0 = f(u_{q-1})$. Then S may be visualized as consisting of the qth roots of unity, with $u_j = e^{2\pi ij/q}$ for $j = 0, 1, ..., q-1$, and then f corresponds to an anticlockwise rotation through $2\pi/q$. This idea is included in Figure 1.

A subset A of S is called an *arc* if it is of the form $\{u_j, u_{j+1}, ..., u_k\}$ for some $0 \leq j \leq k \leq q-1$, or of the form $\{u_j, ..., u_{q-1}, u_0, u_1, ..., u_k\}$ for some $0 \leq k < j \leq q-1$. In each of these cases, we call u_j the *beginning point* of the arc, and u_k the *end point* of the arc. Given two arcs J and K, we say that J, K are *consecutive* if f maps the end point of J to the beginning point of K. Two arcs of S are *separated* if their union is not an arc.

The notion of an arc in S is analogous to the usual meaning of the term when S is regarded as the set of qth roots of unity in the unit circle $\mathbb{T}$ as we proceed in an anti-clockwise manner. Any non-empty subset A of S may be written uniquely as

$$A = \bigcup_{j=1}^{r} J_j, \tag{2.1}$$

where $r \in \{1, 2, ..., q\}$, and the arcs $J_1, J_2, ..., J_r$ are separated. The expression of A as in (2.1) is called the *decomposition* of A and the arcs $J_1, J_2, ..., J_r$ are called the *components* of A. Note that if $A = S$, then we have $r = 1$ and A consists of a single arc. Because the components are separated, provided that A is a proper subset of S, the complement A^c of A also has a unique decomposition into r arcs, say

$$A^c = \bigcup_{j=1}^{r} K_j. \tag{2.2}$$

The arcs J_j and K_j in (2.1) and (2.2) may be numbered so that the arcs $J_1, K_1, J_2, K_2, ..., J_r, K_r, J_1$ are consecutive. That is, in the circle interpretation, starting with J_1, as we proceed anti-clockwise around the circle $\mathbb{T}$, we encounter $K_1, J_2, K_2, J_3, ..., J_r, K_r$ and J_1 in that order. We may assume that the arcs have been numbered in this way. We call K_j the *gap* between J_j and J_{j+1} for $j = 1, 2, ..., r-1$ and K_r is called the *gap* between J_r and J_1. Note that if a subset A of S has r components, then $r \leq |A|$, $r + |A| \leq |S|$ and $r \leq |S|/2$. Also, $r + |A| = |S|$ if and only if the components of A^c are single points; while $|A| = r$ if and only if all components of A are single points.

Given a non-empty subset A of S, a cyclic permutation f of S as above, and $x \in A$, there is $n \in \mathbb{N}$ such that $f^n(x) \in A$, and we define

$$\Theta_A(x) = \min\{n : n \in \mathbb{N} \text{ and } f^n(x) \in A\}. \tag{2.3}$$

If $x \in A,$ then $\Theta_A(x)$ is called the *recurrence time* of x (relative to A). Note that for $x \in A,$ $f(x) \in f(A)$ and

$$\Theta_{f(A)}(f(x)) = \Theta_A(x).$$

Thus, A and $f(A)$ have the same recurrence properties, so that recurrence phenomena are invariant under f.

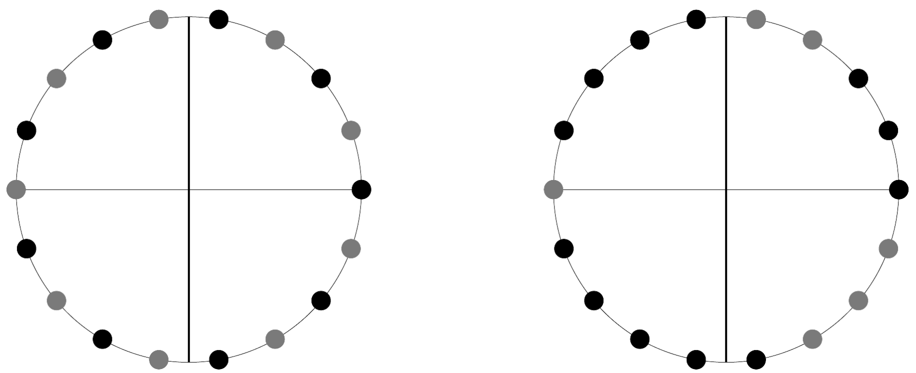

Figure 1. On the left and the right, the black and grey points together comprise the set S, in this case the 18th roots of unity. The transformation f on S is an anticlockwise rotation through 20°. On the left, the black points define a subset A of S, and the recurrence time of every point of A is 2. So, for the set A, the recurrence times are completely predictable. On the right, the black points define a subset B of S, whose points have recurrence times 1, 2, 3 or 4. So, if all we know is that a point is in B, its recurrence time is not predictable. The question is: how much variation can occur and what determines it?

3. Recurrence Times and Kac's Formula

We continue to consider a finite system (S, f), where f is a cyclic permutation on S. The recurrence times of points in a subset A of S are determined by the size of the arcs in the decomposition of S, and by the size of the gaps between the arcs. Thus, if $A = \bigcup_{j=1}^{r} J_j$ is the decomposition of

A as in (2.1), and if $x \in J_j$ but x is not the endpoint of A, then $\Theta_A(x) = 1$; but if x is the endpoint of J_j, $\Theta_A(x)$ is 1 plus the size of the gap between x and the beginning point of the next arc in the decomposition. Specifically, if x is the endpoint of J_j and if $\Theta_A(x) = k$, then $K_j = f(x), f^2(x), \ldots, f^{k-1}(x)$. We see that $\Theta_A(x) \leq |S| - |A| + 1$ for all $x \in A$. The following gives us an idea of the possible distribution of the recurrence times.

Theorem 3.1. *Let S be a finite set with* $|S| \geq 2$, *let* $A \subseteq S$ *and let f be a cyclic permutation on S, as above. Also, as above,* $\Theta_A(x)$ *denotes the recurrence time of* $x \in A$ *relative to A. Then the following statements hold*:

(i) *For all* $x \in A$, $\Theta_A(x) \in \{1, 2, \ldots, |S| - |A| + 1\}$.

(ii) *For all* $k \in \{2, 3, \ldots, |S|\}$,

$$\frac{1}{|A|}|\{x : x \in A \text{ and } \Theta_A(x) = k\}| \leq \frac{1}{k-1} \cdot \frac{|A^c|}{|A|}. \tag{3.1}$$

(iii) *Given* $k \in \{2, 3, \ldots, |S|\}$, *equality holds in* (3.1) *if and only if every point of A has recurrence time* 1 *or k, in which case* $k - 1$ *divides* $|S| - |A|$ *and A has* $(|S| - |A|)/(k-1)$ *components.*

(iv) *If there is a point of A with recurrence time* $|S|$, *then* $|A| = 1$ *and equality holds in* (3.1) *with both sides being equal to* 1.

(v) *If the subset A has a decomposition with r components, we have*

$$\frac{1}{|A|}|\{x : x \in A \text{ and } \Theta_A(x) = 1\}| = 1 - \frac{r}{|A|}, \tag{3.2}$$

and the maximum possible recurrence time for a point in A is $|S| - |A| - r + 2$. *This is attained when there is a gap of size* 1 *between* $r - 1$ *components of the decomposition of A and the size of the remaining gap is* $|S| - |A| - r + 1$.

Proof. (i) It was noted above that $\Theta_A(x) \in \{1, 2, \ldots, |S| - |A| + 1\}$.

(ii) Let $k \in \{2, 3, \ldots, |S|\}$ and suppose there are ℓ elements in A such that $\Theta_A(x) = k$. Then, by the above remarks we have $|A^c| \geq \ell(k-1)$, from which (3.1) follows.

(iii) Suppose $k \in \{2, 3, \ldots, |S|\}$ is given. If equality holds in (3.1), we have

$$(k-1)|\{x : x \in A \text{ and } \Theta_A(x) = k\}| = |S| - |A|, \tag{3.3}$$

and so $k-1$ divides $|S| - |A|$. But each $x \in A$ with $\Theta_A(x) = k$ produces an arc in the decomposition of A^c having $k-1$ elements, so we deduce from (3.3) that the union of such arcs is the whole of A^c. This means that all points x of A with $\Theta_A(x) \neq k$ must have $\Theta_A(x) = 1$. It then follows that A has $(|S| - |A|)/(k-1)$ components.

On the other hand, if every point of A has recurrence time 1 or k, we see that A^c will have $|\{x : x \in A \text{ and } \Theta_A(x) = k\}|$ components, each having $k-1$ elements. So,

$$|A| + (k-1)|\{x : x \in A \text{ and } \Theta_A(x) = k\}| = |S|,$$

and equality holds in (3.1).

(iv) If there is a point of A with recurrence time $|S|$, then we see that $|A| = 1$ and equality holds for $k = |S|$ in (3.1) with both sides of the inequality being equal to 1.

(v) When A has r components, $|\{x : x \in A \text{ and } \Theta_A(x) = 1\}| = |A| - r$, and (3.2) follows. The final statement is immediate on realising that the maximum possible recurrence time comes from having a gap of maximum possible size in the decomposition, and then calculating the size of the gap and the corresponding recurrence time. □

The *average* of Θ over A is defined to be $\sum_{x \in A} \Theta_A(x)/|A|$. In a general ergodic dynamical system, Kac [7] derived a formula for the average recurrence time for a given measurable subset of positive measure. Here, because the system is finite, we can give a very quick derivation based on counting the recurrence times of the elements of the subset.

Theorem 3.2 (Kac's formula, finite version). *Let* (S, f) *be a finite dynamical system, let f act as a cyclic permutation on S, and let A be a non-empty subset of S. Then the average of the recurrence time* Θ_A *over A equals* $|S|/|A|$.

Proof. Write the decompositions of A and A^c as in (2.1) and (2.2). Then, using the remarks at the beginning of this section, we see that the average of Θ over A equals

$$\frac{1}{|A|}\sum_{x\in A}\Theta(x) = \frac{1}{|A|}\left[\sum_{j=1}^{r}|K_j| + 1 + |J_j| - 1\right]$$

$$= \frac{1}{|A|}\left[\sum_{j=1}^{r}|K_j| + |J_j|\right] = \frac{|S|}{|A|}. \qquad \square$$

Note that Kac's formula shows that the average recurrence time depends only upon the *number* of points in the set *A*, and not upon the *distribution* of the points of *A* throughout *S*. However, we shall see that for the variation of recurrence times, the distribution of the points of *A* in *S* affects the variation.

4. The Recurrence Formula and the Standard Deviation

We continue to consider a finite dynamical system (S, f), where *f* is acting as a cyclic permutation. If *A* is a non-empty subset of *S* and $\Theta_A(x)$ denotes the recurrence time of points $x \in A$, the *standard deviation* of Θ_A over *A* is defined to be σ_A, where

$$\sigma_A = \sqrt{\frac{1}{|A|}\sum_{x\in A}\left(\Theta_A(x) - \frac{|S|}{|A|}\right)^2}. \tag{4.1}$$

The standard deviation is a measure of how much the recurrence times $\Theta_A(x)$ vary from their average value $|S|/|A|$, as given by Kac's formula in Theorem 3.2. The *variation* of Θ_A is defined to be σ_A^2. As $\Theta_A(x) \leq |S|$, the rough estimate

$$\sigma_A \leq \max\left\{|S| - \frac{|S|}{|A|}, \frac{|S|}{|A|} - 1\right\}$$

follows from the definition. Note that if $A = S$, then $\Theta_A = 1$ and $\sigma_A = 0$. So, our discussion assumes that A is a non-empty but proper subset of S and we assume that its decomposition consists of r arcs as in (2.1). Then the decomposition of A^c also consists of r arcs, say $K_1, K_2, \ldots, K_r$. Using (4.1) and the discussion on recurrence times at the beginning of Section 3, we see that

$$\sigma_A^2 = \frac{1}{|A|}\left[\sum_{j=1}^{|A|-r}\left(1 - \frac{|S|}{|A|}\right)^2 + \sum_{j=1}^{r}\left(|K_j| + 1 - \frac{|S|}{|A|}\right)^2\right], \text{ so that}$$

$$\sigma_A = \sqrt{\left(1 - \frac{r}{|A|}\right)\left(1 - \frac{|S|}{|A|}\right)^2 + \frac{1}{|A|}\sum_{j=1}^{r}\left(|K_j| + 1 - \frac{|S|}{|A|}\right)^2}. \tag{4.2}$$

Note that σ_A depends upon the sizes $|K_j|$ of the gaps arising from the decomposition of A^c, but it does not depend upon the order in which the gaps appear.

The following result is a special case of well-known results concerning extreme values of convex functions, see [12, pp. 122-126], for example. It is included here to make the exposition as complete and as accessible as possible.

Lemma 4.1. *Let* $r \in \mathbb{N}$, $b \in [r, \infty)$ *and Z be the subset of* $\mathbb{R}^r$ *given by*

$$Z = \left\{(x_1, x_2, \ldots, x_r) : x_1, \ldots, x_r \geq 1 \text{ and } \sum_{j=1}^{r} x_j = b\right\}.$$

Let $a \in \mathbb{R}$. *Let* $h : Z \to [0, \infty)$ *be the function given by*

$$h(x_1, x_2, \ldots, x_r) = \sum_{j=1}^{r} (x_j - a)^2. \tag{4.3}$$

Then h attains a maximum and a minimum over Z. The maximum occurs at the r points of the form $(1, \ldots, 1, b - r + 1, 1, \ldots, 1)$ *and equals* $(b - a - r + 1)^2 + (r - 1)(a - 1)^2$. *The minimum occurs at a unique point which is* $(b/r, b/r, \ldots, b/r)$ *and it equals* $r(b/r - a)^2$.

Proof. Clearly, h is continuous on Z and so attains a maximum and a minimum over Z, since Z is closed and bounded in $\mathbb{R}^r$ (see [10, p. 114] or [13, pp. 115 and 119], for example). Note the *symmetry property* of h, which is that h has the same value at $(x_1, \ldots, x_r)$ and at any point obtained by rearranging the x_j.

Now, let us suppose h attains a maximum at $x = (x_1, x_2, \ldots, x_r) \in Z$ with, say, $x_1 > 1$ and $x_2 > 1$. Put $x' = (x_1 + x_2 - 1, 1, x_3, \ldots, x_r) \in Z$. Then $x' \in Z$ and we have

$$\begin{aligned} h(x') - h(x) &= (x_1 + x_2 - 1 - a)^2 + (a - 1)^2 - (x_1 - a)^2 - (x_2 - a)^2 \\ &= 2(x_1 x_2 - x_1 - x_2 + 1) \\ &= 2(x_1 - 1)(x_2 - 1) \\ &> 0, \end{aligned}$$

which contradicts the assumption that h has a maximum over Z at x. Thus, by the symmetry property of h, we see that the points in Z where h attains its maximum are precisely those of the form $(1, \ldots, 1, b - r + 1, 1, \ldots, 1)$, where $b - r + 1$ is in position j for some $j \in \{1, 2, \ldots, r\}$. Also, we see that the maximum of h over Z is the value of h at each of these points, which is $(b - a - r + 1)^2 + (r - 1)(a - 1)^2$, as stated.

In considering the minimum of h over Z, we show first that this occurs at a unique point in Z. Then the symmetry property of h implies that all coordinates of the point where the minimum occurs must be equal. So, let us assume that h has a minimum value m at points $u, v \in Z$. Putting $u = (u_1, ..., u_r)$ and $v = (v_1, ..., v_r)$, and noting that $h(u) = h(v) = m$, a straightforward calculation gives

$$m = \frac{1}{2}h(u) + \frac{1}{2}h(v) = h\left(\frac{u+v}{2}\right) + \sum_{j=1}^{r}\left(\frac{u_j - v_j}{2}\right)^2 \geq m + \sum_{j=1}^{r}\left(\frac{u_j - v_j}{2}\right)^2,$$

which implies that $\sum_{j=1}^{r}(u_j - v_j)^2 = 0$, so that $u = v$. Thus, h assumes its minimum at a unique point, all coordinates of this point must be equal by symmetry, so the minimum is assumed at $(b/r, b/r, ..., b/r)$. We also see that the minimum value is $r(b/r - a)^2$. □

We will use the preceding result to show that under the given conditions, the maximum of the standard deviation of the recurrence times, taken over all subsets A with a given number of points and a given number of components, is attained when the components are as close together as possible, but with two components having a larger gap, in general. The corresponding minimum is attained when the components have the same number of points and are equally spaced.

Now, let (S, f) be a finite dynamical system with $|S| \geq 2$ and assume that f acts as a cyclic permutation on S. Then, if $r, s \in \{1, 2, ..., |S| - 1\}$ with $r \leq s$, make the definition that $\mathcal{A}_{r,s}$ consists of the family of all subsets A of S such that $|A| = s$ and the decomposition of A consists of r arcs. The following result gives the maximum and the minimum of the standard deviation over a set $\mathcal{A}_{r,s}$.

Theorem 4.2. *Let (S, f) be a finite dynamical system where f acts as a cyclic permutation on S and $|S| \geq 2$. Let $r, s \in \{1, 2, ..., |S| - 1\}$ be given with $r \leq s$. Then*

$$\max_{A\in\mathcal{A}_{r,s}} \sigma_A$$

$$= \sqrt{\left(1-\frac{r}{s}\right)\left(1-\frac{|S|}{s}\right)^2 + \frac{1}{s}\left(|S| - s - r - \frac{|S|}{s} + 2\right)^2 + \frac{r-1}{s}\left(-\frac{|S|}{s} + 2\right)^2}. \tag{4.4}$$

Now, assume further that r divides $|S| - s$. *Then*

$$\min_{A\in\mathcal{A}_{r,s}} \sigma_A = \left(\frac{|S|}{s} - 1\right)\sqrt{\frac{s}{r} - 1}. \tag{4.5}$$

The maximum in (4.4) *is attained for any set A in* $\mathcal{A}_{r,s}$ *such that in the decomposition of A the size of the gap between* $r-1$ *consecutive arcs is* 1, *while the size of the remaining gap is* $|S| - r - s + 1$. *The minimum in* (4.5) *is attained for any set A in* $\mathcal{A}_{r,s}$ *such that in the decomposition of A, the size of all gaps between consecutive arcs is* $(|S| - s)/r$.

Proof. If $A \in \mathcal{A}_{r,s}$ observe that we have $|S| = |A| + |A^c| = s + \sum_{j=1}^{r} |K_j|$, so that

$$\sum_{j=1}^{r} |K_j| = |S| - s. \tag{4.6}$$

The idea is to use Lemma 4.1. In view of (4.2) and (4.6), observe that the problem is equivalent to finding the maxima and minima of the function $h : \mathbb{R}^r \to [0, \infty)$ given by

$$h(x_1, x_2, \ldots, x_r) = \sum_{j=1}^{r}\left(x_j + 1 - \frac{|S|}{s}\right)^2, \tag{4.7}$$

subject to the conditions that

$$x_j \geq 1 \text{ for } j = 1, 2, \ldots, r \text{ and } \sum_{j=1}^{r} x_j = |S| - s. \tag{4.8}$$

However, note that for the maxima and minima of the standard deviation over $\mathcal{A}_{r,s}$ we need to have them occurring for positive integer values of $x_1, x_2, \ldots, x_r$.

Now, the conditions on $(x_1, x_2, \ldots, x_r)$ in (4.8) define a subset Z of $\mathbb{R}^r$ and, in fact, we see that Lemma 4.1 now applies to h over Z with h as given in (4.7) and with $a = |S|/s - 1$ and $b = |S| - s$. By Lemma 4.1, the maximum of h over Z occurs at the points $e_1, e_2, \ldots, e_r$, where

$$e_j = (1, \ldots, 1, |S| - s - r + 1, 1, \ldots, 1),$$

and $|S| - s - r + 1$ is in position j. Note that the coordinates of e_j are all positive integers. Also, this maximum value is

$$(b - a - r + 1)^2 + (r - 1)(a - 1)^2$$

$$= \left(|S| - s - r + 2 - \frac{|S|}{s}\right)^2 + (r - 1)\left(\frac{|S|}{s} - 2\right)^2.$$

Consequently, using (4.2) and (4.7), we see that the maximum value of σ_A^2 over $\mathcal{A}_{r,s}$ is

$$\left(1 - \frac{r}{s}\right)\left(1 - \frac{|S|}{s}\right)^2 + \frac{1}{s}\left(|S| - s - r + 2 - \frac{|S|}{s}\right)^2 + \frac{r-1}{s}\left(\frac{|S|}{s} - 2\right)^2,$$

and so the conclusion (4.4) follows.

Lemma 4.1 applies also to considering the minimum, and we see that h has a minimum over Z at the point

$$\left(\frac{|S| - s}{r}, \frac{|S| - s}{r}, \ldots, \frac{|S| - s}{r}\right),$$

and note that because we are assuming that r divides $|S| - s$, the coordinates of this point are positive integers and so will give a minimum for the standard deviation, not just a minimum for h. The value of this minimum is

$$r\left(\frac{b}{r} - a\right)^2 = r\left(\frac{|S| - s}{r} + 1 - \frac{|S|}{s}\right)^2.$$

Thus, again using (4.2) and (4.7), we see that the minimum value of σ_A^2 over $\mathcal{A}_{r,s}$ is

$$\left(1-\frac{r}{s}\right)\left(1-\frac{|S|}{s}\right)^2+\frac{r}{s}\left(\frac{|S|-s}{r}+1-\frac{|S|}{s}\right)^2=\left(\frac{|S|}{s}-1\right)^2\left(\frac{s}{r}-1\right), \quad (4.9)$$

and the statement about the minimum of σ_A^2 follows. □

The following result derives from Theorem 4.2 and gives an estimate on the size of the standard deviation of the recurrence time over A in terms of the number of elements of A only.

Corollary 4.3. *If A is any non-empty subset of S and σ_A is the standard deviation as in Theorem* 4.2 *above, we have*

$$\sigma_A \le \sqrt{6}|S||A|^{-1/2}.$$

Proof. We put $s = |A|$ as in Theorem 4.2 and we have $r \le s \le |S|$, $r + s \le |S|$ and $|S| \ge 2$. So, $0 \le 1 - r/s \le 1$ and $(r-1)/s \le 1$. By the first estimate in Theorem 4.2, we have

$$\begin{aligned}
\sigma_A^2 &\le \left(\frac{|S|}{s}-1\right)^2+\frac{|S|^2}{s}\left(1-\frac{r+s}{|S|}-\frac{1}{s}+\frac{2}{|S|}\right)^2+\left(\frac{|S|}{s}-2\right)^2 \\
&\le \frac{|S|^2}{s^2}\left(1-\frac{|s|}{|S|}\right)^2+\frac{|S|^2}{s}\left(\max\left\{\left(1+\frac{2}{|S|}\right)^2,\left(\frac{r+s}{|S|}+\frac{1}{s}\right)^2\right\}\right) \\
&\quad+\frac{|S|^2}{s^2}\left(1-\frac{2s}{|S|}\right)^2 \\
&\le \frac{|S|^2}{s}+\frac{4|S|^2}{s}+\frac{|S|^2}{s} \\
&= 6\cdot\frac{|S|^2}{s}.
\end{aligned}$$

□

A special case of Theorem 4.2 is when $r = |S| - s,$ which is when the sizes of the gaps arising from the decomposition of *A* all equal 1. In this case, by (4.2), the standard deviation is the same for all elements of $\mathcal{A}_{r,s},$ so the maximum and minimum in (4.4) and (4.5) are equal and a calculation using (4.2) or (4.4) or (4.5) shows that they are equal to

$$\sqrt{\left(\frac{|S|}{s} - 1\right)\left(2 - \frac{|S|}{s}\right)}.$$

Note also that in Theorem 4.2 the expression for the maximum value looks complicated compared with the one for the minimum value. This is not too surprising, as the minimum is attained when the points in the minimising set are equally spaced, but for the maximising set the points are irregularly spaced (see Figure 2).

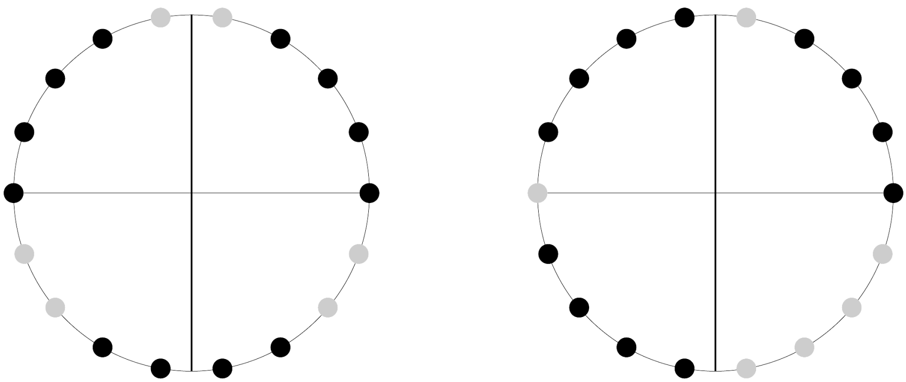

Figure 2. On the left and the right the set *S* consists of the 18th roots of unity, as in Figure 1. On the left, the set *A* given by the black dots is in $\mathcal{A}_{3,12}$. Note that 3 divides into $18 - 12 = 6$. The components of *A* are equally spaced and so the set *A* minimizes the standard deviation of the recurrence time over the sets in $\mathcal{A}_{3,12}$. On the right, the set *B* indicated by the black dots is also in $\mathcal{A}_{3,12}$. This time the components are placed as close together as possible, but of necessity leaving a larger gap between 2 of the components. So, the set *B* maximizes the standard deviation of the recurrence time over the sets in $\mathcal{A}_{3,12}$.

Another special case of interest in Theorem 4.2 is when the components of A are single points, a condition equivalent to having $r = s$. In this case, if s divides $|S| - s$, the minimum in (4.5) is 0, and arises from having the s points in $A \in \mathcal{A}_{s,s}$ equally spaced around the unit circle, if we interpret the system as in Figure 1. Now, for the maximum, (4.4) gives that the maximum of σ_A with $A \in \mathcal{A}_{s,s}$ is

$$\sqrt{s-1}\left(\frac{|S|}{s} - 2\right). \tag{4.10}$$

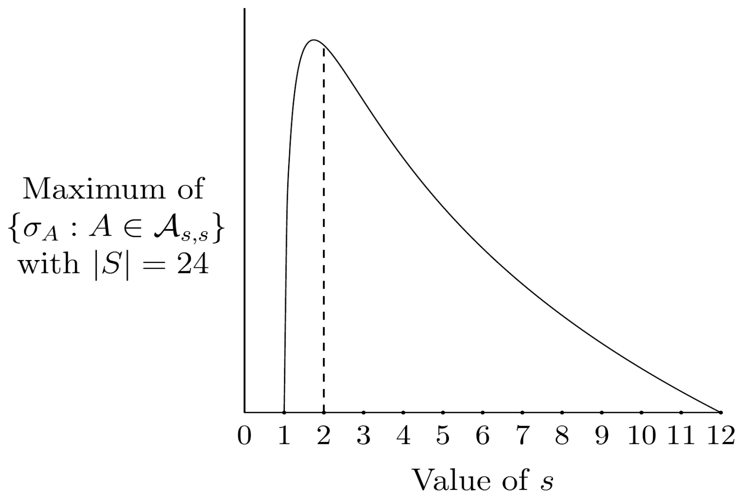

Figure 3. The graph illustrates how the maximum of the standard deviation varies when $|S| = 24$ and $r = s$. The latter condition is equivalent to requiring that the components of any set $A \in \mathcal{A}_{s,s}$ consist of single points. In accordance with (4.10), the graph is of the function $s \mapsto \sqrt{s-1}(24/s - 2)$. If we restrict ourselves to discrete values only, as needed in a discrete context, the function maximum is at $s = 2$.

Figure 3 illustrates the maximum of the standard deviation when $r = s$ and $|S| = 24$. In the general case, still with $r = s$ but where $|S|$ is given but arbitrary, the maximum occurs again for $s = 2$. An intuitive interpretation is as follows. When $r = s = 1$, the recurrence time is constant and so the standard deviation is 0. When $r = s = 2$, the maximum of σ_A is when one point has recurrence time 2, the other has recurrence time $|S| - 2$. As more points are added the maximum recurrence time needed for a maximum of σ_A decreases, and there is "less room" to deviate from the average value. More precisely, the maximum of the function $s \mapsto \sqrt{s-1}(|S|/s - 2)$ occurs at $(-|S| + \sqrt{|S|^2 + 16|S|})/4$, which is less than 2 and approaches 2 as $|S| \to \infty$. So, when considering only discrete values, the maximum must occur at $s = 2$ regardless of the value of $|S|$.

Now, let S be a given finite set, and let $s \in \{2, 3, \ldots, |S|\}$ be given. Considering (4.4), the maximum of σ_A over the sets $A \in \mathcal{A}_{r,s}$ is given by the function f where

$$f(r) = \sqrt{\left(1 - \frac{r}{s}\right)\left(1 - \frac{|S|}{s}\right)^2 + \frac{1}{s}\left(|S| - s - r - \frac{|S|}{s} + 2\right)^2 + \frac{r-1}{s}\left(-\frac{|S|}{s} + 2\right)^2}.$$

Allowing r for the moment to take on real values, a calculation of the derivative shows that

$$f'(r) = -\frac{1}{2sf(r)}(1 + 2(|S| - s - r)).$$

This implies that f is decreasing as a function of r, because $s + r \leq |S|$ and so $f'(r) < 0$. Intuitively, we can interpret this as follows. As r increases, there are more gaps in the decomposition of sets in $\mathcal{A}_{r,s}$. This generally causes the recurrence times to decrease, and they have a narrower range of values which causes the standard deviation to decrease. Even so, it may happen that $\sigma_A < \sigma_B$ even if $A \in \mathcal{A}_{r,s}$ and $B \in \mathcal{A}_{r+1,s}$.

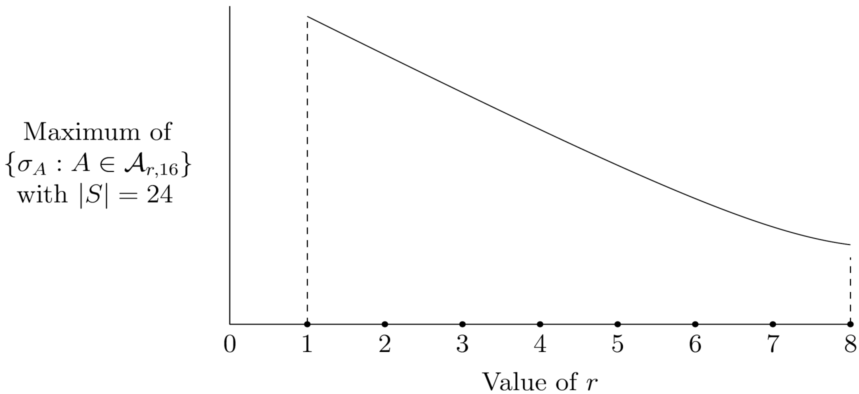

Figure 4. The graph is of the function

$$r \mapsto \sqrt{\frac{1}{4}\left(1-\frac{r}{16}\right)+\frac{1}{16}\left(\frac{17}{2}-r\right)^2+\frac{1}{4}\left(\frac{r-1}{16}\right)},$$

which comes from (4.4) taking $|S| = 24$ and $|A| = 16$. It shows how the maximum of σ_A over subsets A with $|A| = 16$ and a given number r of gaps decreases as more gaps are allowed in the decomposition of the sets A.

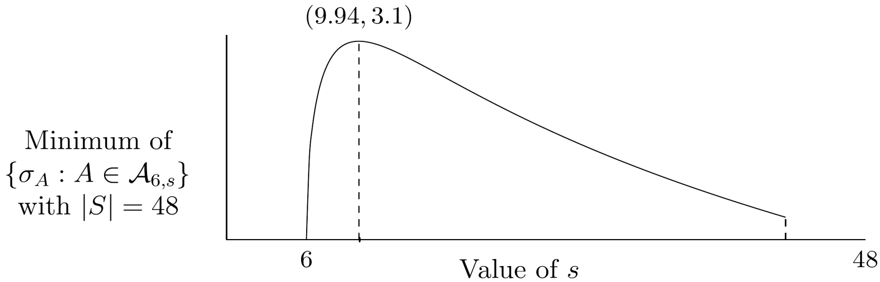

Figure 5. The graph is of the function $s \mapsto \left(\frac{48}{s}-1\right)\sqrt{\frac{s}{6}-1}$, which comes from (4.5) taking $|S| = 48$ and $r = 6$. For $s \in \{6, 12, 18, 24, 30, 36, 42\}$ the function value is the minimum of σ_A, over subsets $A \in \mathcal{A}_{6,s}$. So, the graph shows how the minimum of the standard deviation σ_A, taken over $A \in \mathcal{A}_{6,s}$, varies with s. Note that because we only consider subsets having 6 components, we must have $s \leq 42$.

Now, consider what happens when the number r of gaps is kept fixed, but the number s of elements in a subset A of S varies. (See Figure 5 for a special case.) In general, if r is given, by (4.5) the minimum of σ_A taken over sets $A \in \mathcal{A}_{r,s}$, is the minimum of the function $s \to (|S|/s - 1)\sqrt{s/r - 1}$, taken over $r \leq s \leq |S| - r$. A calculation shows that this occurs at

$$s_0(r) = \frac{4r|S|}{|S| + \sqrt{|S|^2 + 8|S|r}}. \tag{4.11}$$

A way of thinking about this is that if s is small and $A \in \mathcal{A}_{r,s}$ has a small standard deviation, as points are added one-by-one to the A, the minimum possible standard deviation may increase for a while - but, as points continue to be added, the set becomes more “crowded”, the recurrence times are reduced regardless of how the points are added, and so beyond the point $s_0(r)$ in (4.11) the minimum standard deviation decreases.

5. Recurrence Phenomena in Countable Sums of Finite Systems

Let Z denote either the set $\mathbb{N}$ or a finite set. Then, for each $j \in Z$, let S_j be a non-empty finite set and let f_j be a cyclic permutation on S_j. Thus, (S_j, f_j) is a finite dynamical system of the type considered in the previous sections. Assume that the sets $(S_j)_{j \in Z}$ are disjoint. Define a set S and the transformation f on S by putting

$$S = \bigcup_{j \in Z} S_j \text{ and } f(x) = f_j(x), \text{ for } x \in S_j.$$

Then f is a one-to-one and onto transformation on S. The dynamical system (S, f) is called the *sum* of the dynamical systems $(S_j, f_j)_{j \in Z}$.

Furthermore, for each $j \in Z$, let $w_j \in (0, 1]$ be such that $\sum_{j \in Z} w_j = 1$. The numbers w_j are called *weights*. Then, when $A \subseteq S$, we put

$$P(A) = \sum_{j \in Z} w_j \frac{|A \cap S_j|}{|S_j|}, \tag{5.1}$$

and it may be called the *probability* of *A*. Note that if $x \in S_j$, then $P(\{x\}) = w_j / |S_j|$. It is significant that *f* is *P*-invariant, in the sense that if $A \subseteq S$, $P(f^{-1}(A)) = P(A)$. In fact, the following result shows that if (S, f) is any dynamical system where *S* is countable and where *f* is *P*-invariant, then it is a sum of systems of the type above.

Theorem 5.1. *Let S be a finite or countable set and for each* $x \in S$, *let* $v_x > 0$ *and suppose that* $\sum_{x \in S} v_x = 1$. *When* $A \subseteq S$, *put* $P(A) = \sum_{x \in A} v_x$. *Let f be a transformation on S that is P-invariant, meaning that* $P(f^{-1}(A)) = P(A)$ *for all* $A \subseteq S$. *Then there is a subset Z of S such that for each* $z \in Z$ *there is a finite set* S_z *such that f acts as a cyclic permutation* f_z *on* S_z, *the sets* $(S_z)_{z \in Z}$ *are pairwise disjoint,* $P(\{x\})$ *has the same value for all* $x \in S_z$, (S, f) *is the sum of the systems* $(S_z, f_z)_{z \in Z}$, *and f is a one-to-one and onto transformation on S.*

Proof. Just within this proof, we will write $P(y)$ for $P(\{y\})$ when $y \in S$. Now, let $x \in S$. Then $x \in f^{-1}(f(x))$. So, $P(x) \le P(f^{-1}(f(x))) = P(f(x))$. Repeating this step gives

$$0 < P(x) \le P(f(x)) \le P(f^2(x)) \le \cdots.$$

If all the points $x, f(x), f^2(x), \ldots$ are distinct, then we would deduce that

$$1 = P(S) \ge \sum_{n=1}^{\infty} P(f^{n-1}(x)) = \infty,$$

a contradiction. So, there is a minimum value of $n \in \mathbb{N}$ such that $f^n(x) = f^m(x)$ for some $0 \le m \le n-1$. Then *f* acts as a cyclic permutation on the set $T_x = \{f^m(x), f^{m+1}(x), \ldots, f^{n-1}(x)\}$. Now we have $f^{j-1}(x) \in f^{-1}(f^j(x))$

so $P(f^{j-1}(x)) \leq P(f^{-1}(f^{j}(x))) = P(f^{j}(x))$. Also, $f^{n-1}(x) \in f^{-1}(f^{m}(x))$, and we deduce similarly that $P(f^{n-1}(x)) \leq P(f^{m}(x))$. Thus,

$$P(f^{m}(x)) \leq P(f^{m+1}(x)) \leq P(f^{m+2}(x)) \leq \cdots \leq P(f^{n-1}(x)) \leq P(f^{m}(x)),$$

and it follows that for all points y in T_x, $P(y)$ has the same value. So, if f_x denotes the restriction of f to T_x, we see that T_x is finite, that f_x acts as a cyclic permutation on T_x, and that for each point w in T_x, $P(w)$ has the same value. We put $T = \bigcup_{x \in S} T_x$, and note that $f : T \to T$.

Now it is clear that as f_x acts as a cyclic permutation on T_x, given $x, y \in S$, either $T_x = T_y$ or $T_x \bigcap T_y = \emptyset$. Thus, there is a subset Z of S such that for any $z, z' \in Z$, T_z and $T_{z'}$ are disjoint and

$$T = \bigcup_{x \in S} T_x = \bigcup_{z \in Z} T_z.$$

We claim that $T = S$, and it will then follow that (S, f) is the sum of the dynamical systems $(S_z, f_z)_{z \in Z}$. Let us assume that $T^c \neq \emptyset$. Then $f : T^c \to T^c$. For, if not, $T^c \bigcap f^{-1}(T) \neq \emptyset$. We would then have

$$P(T) = P(f^{-1}(T)) = P(T \bigcup (T^c \bigcap f^{-1}(T)))$$

$$= P(T) + P(T^c \bigcap f^{-1}(T)) > P(T),$$

a contradiction. We deduce that $f : T^c \to T^c$. Now as $T^c \neq \emptyset$, the above analysis can be applied to the dynamical system (T^c, f) in place of (S, f), and so T^c must contain a finite set upon which f acts as a cyclic permutation. However, this is impossible, since the construction of the system (T, f) ensures that it contains every finite subset of S upon which f acts as a cyclic permutation. We deduce that $T^c = \emptyset$, and then we see that (S, f) is the sum of the systems $(T_z,\ f_z)_{z\in Z}$. The fact that f is one-to-one and onto is immediate from this, and from the fact that each cyclic permutation f_z is one-to-one and onto on the set T_z. □

Now, let $(S,\ f)$ be the sum of the finite dynamical systems $(S_j)_{j\in Z}$ as described above, and let $A \subseteq S$ be given. For $x \in A$, we define the recurrence time $\Theta_A(x)$ as in (2.3). Note for $x \in A \cap S_j$, $\Theta_A(x)$ in the system $(S,\ f)$ equals $\Theta_{A\cap S_j}(x)$ in the system $(S_j,\ f_j)$. The *expectation* or *average* of Θ_A over A in $(S,\ f)$ is

$$E_A(\Theta_A) = \frac{1}{P(A)} \sum_{x\in A} \Theta_A(x) P(\{x\}). \tag{5.2}$$

Now, observe that the earlier results apply to each finite dynamical system $(S_j,\ f_j)$. In particular, if $j \in Z$ and $A \cap S_j \neq \varnothing$, then we have from Theorem 3.2 that

$$\sum_{x\in A\cap S_j} \Theta_A(x) = |\, S_j \,|. \tag{5.3}$$

The following result is a form of Kac's theorem for a system that is a direct sum.

Theorem 5.2. *Let Z denote either the set $\mathbb{N}$ or a finite set. For each $j \in Z$, let S_j be a non-empty finite set and let f_j be a cyclic permutation on S_j, and assume that the sets $(S_j)_{j\in Z}$ are disjoint. Let $(S,\ f)$ be the sum of the dynamical systems $(S_j)_{j\in Z}$. Let weights $(w_j)_{j\in Z}$ be given as above, and let the probability of a subset A of S be given by* (5.1). *Then, if $A \subseteq S$ is such that $A \cap S_j \neq \varnothing$ for all $j \in Z$, the average $E_A(\Theta_A)$ of Θ_A over A is given by*

$$E_A(\Theta_A) = \frac{1}{P(A)}.$$

If $A \subseteq S$ *is such that* $A \bigcap S_k = \varnothing$ *for some* $k \in Z$, *then*

$$E_A(\Theta_A) < \frac{1}{P(A)}.$$

Proof. If $A \subseteq S$ and $A \neq \varnothing$, put $Z' = \{j : j \in Z \text{ and } A \bigcap S_j \neq \varnothing\}$. Then

$$\begin{aligned} E_A(\Theta_A) &= \frac{1}{P(A)} \sum_{x \in A} \Theta_A(x) P(\{x\}) \\ &= \frac{1}{P(A)} \sum_{j \in Z'} \sum_{x \in A \bigcap S_j} \Theta_A(x) \frac{w_j}{|S_j|} \\ &= \frac{1}{P(A)} \sum_{j \in Z'} w_j, \end{aligned} \tag{5.4}$$

by (5.3). Here, there are two possibilities. If $A \bigcap S_j \neq \varnothing$ for all $j \in Z$, then $Z = Z'$, and as $\sum_{j \in Z} w_j = 1$ (5.4) becomes $E_A(\Theta_A) = 1/P(A)$. That is, in this case, Kac's formula remains true. But if $A \bigcap S_k = \varnothing$ for some k, then $Z' \subsetneq Z$ and so, as $\sum_{j \in Z'} w_j < 1$, (5.4) becomes

$$E_A(\Theta_A) = \frac{1}{P(A)} \sum_{j \in Z'} w_j < \frac{1}{P(A)},$$

and Kac's formula fails. □

We now consider the standard deviation in a system which is a sum of finite systems, as above. When $A \subseteq S$ and A is non-empty, the standard deviation of Θ_A over A in the system (S, f) is taken to be σ_A, where

$$\sigma_A^2 = \frac{1}{P(A)} \sum_{x \in A} |\Theta_A(x) - E_A(\Theta_A)|^2 P(\{x\}). \tag{5.5}$$

Note that $P(\{x\}) = w_j / S_j$ for $x \in A \bigcap S_j$, by (5.1). However, when the system (S, f) is a sum of the systems (S_j, f_j), the probability of a single point within any one system (S_j, f_j) considered by itself is $1/|S_j|$. Now, if $A \subseteq S$ is such that $A \cap S_j \neq \emptyset$, then a straightforward calculation using (5.2) shows that

$$E_{A \cap S_j}(\Theta_A) = \frac{|S_j|}{|A \cap S_j|}, \tag{5.6}$$

so that from (5.5) and (5.6), we see that in (S, f),

$$\begin{aligned} \sigma^2_{A \cap S_j} &= \frac{1}{P(A \cap S_j)} \sum_{x \in A \cap S_j} \left| \Theta_A(x) - \frac{|S_j|}{|A \cap S_j|} \right|^2 P(\{x\}) \\ &= \frac{1}{|A \cap S_j|} \sum_{x \in A \cap S_j} \left| \Theta_A(x) - \frac{|S_j|}{|A \cap S_j|} \right|^2. \end{aligned} \tag{5.7}$$

We see from (5.6) and (5.7) that the expectation of Θ_A over $A \cap S_j$, whether taken in the system (S, f) or (S_j, f_j), is the same, and that the same is true for the standard deviation. Thus, the expressions $E_{A \cap S_j}(\Theta_A)$ and $\sigma_{A \cap S_j}$ have no ambiguity. The following result expresses the relationship between the standard deviation σ_A of recurrence times in a system (S, f) that is a sum of systems (S_j, f_j) in terms of the standard deviations of the corresponding component systems (S_j, f_j).

Theorem 5.3. *Let Z denote either the set* $\mathbb{N}$ *or an interval in* $\mathbb{N}$ *of the form* $\{1, 2, ..., r\}$. *For each* $j \in Z$, *let* S_j *be a non-empty finite set and let* f_j *be a cyclic permutation on* S_j, *and assume that the sets* $(S_j)_{j \in Z}$ *are disjoint. Let* (S, f) *be the sum of the dynamical systems* $(S_j, f_j)_{j \in Z}$. *Let weights* $(w_j)_{j \in Z}$ *be given as above, and let the probability of a subset A of S be given by* (5.1). *Then, if* $A \subseteq S$ *is such that* $A \cap S_j \neq \emptyset$ *for all* $j \in Z$, *the standard deviation* σ_A *of* Θ_A *over A in* (S, f) *is given by*

$$\sigma_A^2 = \frac{1}{P(A)}\left(\sum_{j\in Z} w_j \frac{|A\cap S_j|}{|S_j|}\sigma_{A\cap S_j}^2\right) + \frac{1}{P(A)}\sum_{j\in Z} w_j \frac{|S_j|}{|A\cap S_j|} - \frac{1}{P(A)^2}. \tag{5.8}$$

Proof. As $A \cap S_j \neq \varnothing$ for all $j \in Z$, Theorem 5.2 gives $E_A(\Theta_A) = 1/P(A)$. Then, using (5.5), we have

$$\sigma_A^2 = \frac{1}{P(A)}\sum_{x\in A}\Theta_A(x)^2 P(\{x\}) - \frac{2}{P(A)^2}\sum_{x\in A}\Theta_A(x)P(\{x\}) + \frac{1}{P(A)^3}\sum_{x\in A}P(\{x\})$$

$$= \frac{1}{P(A)}\sum_{j\in Z}\frac{w_j}{|S_j|}\left(\sum_{x\in A\cap S_j}\Theta_A(x)^2\right) - \frac{1}{P(A)^2}$$

$$= \frac{1}{P(A)}\sum_{j\in Z}\frac{w_j}{|S_j|}\left(\sum_{x\in A\cap S_j}\left(\Theta_A(x) - \frac{|S_j|}{|A\cap S_j|}\right)^2 + \sum_{x\in A\cap S_j}\frac{|S_j|^2}{|A\cap S_j|}\right)$$

$$- \frac{1}{P(A)^2}, \text{ where we have used (5.3),}$$

$$= \frac{1}{P(A)}\left(\sum_{j\in Z} w_j \frac{|A\cap S_j|}{|S_j|}\sigma_{A\cap S_j}^2\right) + \frac{1}{P(A)}\sum_{j\in Z} w_j \frac{|S_j|}{|A\cap S_j|} - \frac{1}{P(A)^2},$$

where we have used (5.7). □

Theorem 5.4. *For each* $j \in \mathbb{N}$ *let* S_j *be a finite set and let* f_j *be a cyclic permutation on* S_j. *Assume that the sets in* $(S_j)_{j\in\mathbb{N}}$ *are disjoint, and let* (S, f) *be the sum of the dynamical systems* $(S_j, f_j)_{j\in\mathbb{N}}$. *Let weights* $(w_j)_{j\in\mathbb{N}}$ *be given such that* $w_j > 0$ *for all j and* $\sum_{j=1}^{\infty} w_j = 1$, *and let P be the probability on the family of all subsets of S as given in* (5.1). *Let A be a subset of S such that* $A \cap S_j \neq \varnothing$ *for all* $j \in \mathbb{N}$. *Let* σ_A *denote the standard deviation of the recurrence time for* Θ_A *in the system* (S, f). *Then the* *following hold*:

(i) *If* $\sum_{j=1}^{\infty} w_j |S_j| < \infty$, *then* $\sigma_A < \infty$.

(ii) *If there is* $M > 0$ *such that* $|S_j| \le M |A \cap S_j|^{1/2}$ *for all* $j \in \mathbb{N}$, *then* $\sigma_A < \infty$.

Proof. (i) By (5.8) and by applying Corollary 4.3 to the set $A \cap S_j$ in S_j for each $j \in \mathbb{N}$, we obtain

$$\sigma_A^2 \le \frac{6}{P(A)}\left(\sum_{j=1}^{\infty} w_j \frac{|A \cap S_j|}{|S_j|}\frac{|S_j|^2}{|A \cap S_j|}\right)$$

$$+ \frac{1}{P(A)}\sum_{j\in\mathbb{N}} w_j \frac{|S_j|}{|A \cap S_j|} - \frac{1}{P(A)^2}$$

$$\le \frac{6}{P(A)}\sum_{j=1}^{\infty} w_j |S_j| + \frac{1}{P(A)}\sum_{j=1}^{\infty} w_j |S_j| - \frac{1}{P(A)^2}$$

$$< \infty,$$

as we are assuming $\sum_{j=1}^{\infty} w_j |S_j| < \infty$.

(ii) In this case, we again use Corollary 4.3 and apply (5.8) to obtain

$$\sigma_A^2 \le \frac{6M}{P(A)}\sum_{j=1}^{\infty} w_j + \frac{M}{P(A)}\sum_{j=1}^{\infty} w_j - \frac{1}{P(A)^2} = \frac{7M}{P(A)} - \frac{1}{P(A)^2} < \infty. \quad \square$$

If it happens that $\sum_{j=1}^{\infty} w_j |S_j| = \infty$ in the theorem above, then the standard deviation σ_A may or may not be finite, but part (ii) of Theorem 5.4 gives a sufficient condition for finiteness of σ_A in this case. However, the following example shows it may happen that $\sigma_A = \infty$.

Example. Consider the case in Theorem 5.3 when $Z = \mathbb{N}$ and $|A \cap S_j| = 1$ for all $j \in \mathbb{N}$. Then $\Theta_A(x) = |S_j|$ for all $x \in A \cap S_j$, so that $\sigma_{A \cap S_j} = 0$ for all $j \in \mathbb{N}$. We see from (5.8) that

$$\sigma_A^2 = \frac{1}{P(A)} \sum_{j \in \mathbb{N}} w_j |S_j| - \frac{1}{P(A)^2}.$$

Now once the weights w_j are given, it is clear that we can choose the sets S_j in such a way that $\sum_{j \in Z} w_j |S_j| = \infty$ and we then have $\sigma_A = \infty$. However, as we have seen in Theorem 5.2, $E_A(\Theta_A) < \infty$. So, the average of Θ_A is always finite but the standard deviation may be infinite.

The preceding example is relevant to the calculation of the standard deviation of recurrence times in a different context [9, p. 275]. There, a formula for the standard deviation gave a finite value when

$$\sum_{k=1}^{\infty} P\left(\bigcap_{j=0}^{k-1} f^{-j}(A^c) \right) < \infty,$$

and there was no discussion of what happened if this condition was not satisfied. Now in the example put $B_k = \bigcap_{j=0}^{k-1} f^{-j}(A^c)$ for $k \in \mathbb{N}$. As $A \cap S_n$ is a single point, and as *f* is the cyclic permutation f_n on S_n, we see from the definitions that

$$|B_k \cap S_n| = \begin{cases} |S_n| - k, & \text{for } 1 \le k \le |S_n|, \\ 0, & \text{for } k \ge |S_n| + 1. \end{cases}$$

It follows that

$$\sum_{k=1}^{\infty} P\left(\bigcap_{j=0}^{k-1} f^{-j}(A^c) \right) = \sum_{k=1}^{\infty} P(B_k)$$

$$= \sum_{k=1}^{\infty} \left(\sum_{n=1}^{\infty} w_n \frac{|B_k \cap S_n|}{|S_n|} \right)$$

$$= \sum_{n=1}^{\infty} w_n \left(\sum_{k=1}^{|S_n|} \frac{|S_n| - k}{|S_n|} \right)$$

$$= \frac{1}{2} \sum_{n=1}^{\infty} w_n (|S_n| - 1).$$

Again, we see that if the weights w_n are given, we can choose the sets S_j such that $\sum_{n=1}^{\infty} w_n (|S_n| - 1) = \infty,$ which means that the condition

$$\sum_{k=1}^{\infty} P\left(\bigcap_{j=0}^{k-1} f^{-j}(A^c) \right) < \infty,$$

appearing in [9, p. 275] for the finiteness of the standard deviation, is not satisfied in this case.

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