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Semigroup actions on higher-rank graphs and their graph C*- algebras

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I hereby certify that the work embodied in this thesis is the result of original research and has not been submitted for a higher degree to any other University or Institution.

(Signed) ______________________________

Ben Maloney

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Contents

Abstract

Higher-rank graphs and their C^*-algebras were introduced by Kumjian and Pask as a graphical means to provide combinatorial models of the Cuntz-Krieger families of Robertson and Steger. If a group G acts on a directed graph E, the universal property of $C^*(E)$ shows that this action on the graph induces an action of G on $C^*(E)$, and hence the crossed product $C^*(E) \rtimes G$ can be formed. A theorem of Kumjian and Pask says that if a group G acts freely on a directed graph E, then the associated crossed product $C^*(E) \rtimes G$ of the graph algebra is stably isomorphic to the graph algebra $C^*(G\backslash E)$ of the quotient graph. A similar result was established by Pask, Raeburn and Yeend for certain actions of semigroups on directed graphs.

The main purpose of this thesis is to prove this relationship in the most general case: that is, for an Ore semigroup S and a free action $\alpha : S \to \operatorname{End} \Sigma$ on a higher-rank graph Σ that admits a fundamental domain, the crossed product $C^*(\Sigma) \times_{\alpha_*} S$ is stably isomorphic to $C^*(S\backslash\Sigma)$. We will show that there exists an isomorphism from $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$. That is, the associated crossed product $C^*(\Sigma) \times_{\alpha_*} S$ of the graph algebra is stably isomorphic to the graph algebra $C^*(S\backslash\Sigma)$ of the quotient k-graph.

The isomorphism is realised in three stages: the method that we use is an improvement of the method used by Pask, Raeburn and Yeend using the full weight of a dilation result by Laca. The extra generality means that new proofs at each stage were required. We first prove a version of the Gross-Tucker theorem for semigroup actions on higher-rank graphs. Second, we give a new formulation of Laca's result involving endomorphisms of Ore semigroups actions on C^*-algebras that is specially tailored to the higher-rank graph case. By doing so, we also have managed to shorten the proof for the directed graph case. We also generalise the work of Kaliszewski, Quigg and Raeburn to recognise Cuntz-Krieger families of crossed products in associated tensor products, and provide an explicit isomorphism for the relationship. For each step we have managed to give an explicit isomorphism by breaking down the steps used in the argument of Pask, Raeburn and Yeend. We apply multiplier algebra techniques to produce an isomorphism between the semigroup crossed product and the dilated group crossed product. In order to do this, we provide new results on convergence in the multiplier algebra of a higher-rank graph C^*-algebra.

Finally, we give some applications of our results, such as criteria for simplicity and pure infiniteness of skew-product graph C^*-algebras. We consider a dual k-graph construction that allows us to consider non-free actions on a k-graph that are free on some associated dual graph. We extend the idea of primitivity, and give some equivalent conditions to aperiodicity and cofinality that are more straightforward to check. We also give simplicity criteria for the fixed-point algebra of the gauge action. As an alternative to our efficient use of Laca's dilation theory for endomorphic actions of Ore semigroups on C^*-algebras we could have performed the complicated direct limit construction. We feel these calculations are of independent interest, and so they are also provided.

CHAPTER 1

Introduction

1.1. History

In [**11, 12**], Cuntz and Krieger studied a class of C^*-algebras generated by nonzero partial isometries, which satisfy relations determined by a $\{0, 1\}$–matrix A. These relations are now known as the Cuntz-Krieger relations. In [**21**], by viewing A as the vertex matrix of a directed graph, Enomoto and Watatani showed that the C^*-algebras introduced by Cuntz and Krieger can be interpreted as the C^*-algebras associated to finite directed graphs. In [**39**], Kumjian, Pask, Raeburn and Renault generalised these ideas to the infinite case using groupoid techniques.

In [**38**], Kumjian, Pask and Raeburn showed using a more direct approach that given a row-finite directed graph E, there exists a universal C^*-algebra $C^*(E)$. The graph algebra $C^*(E)$ is generated by a family of partial isometries satisfying relations determined by E, and these relations reduce to those introduced by Cuntz and Krieger for a finite graph with no sinks or sources. Since then there has been much interest in these C^*-algebras and their applications. The literature has progressed in several directions. We are primarily interested in the generalisation to higher-rank graphs.

In [**65, 66**], Robertson and Steger were looking at the crossed product of groups acting on the boundaries of buildings and found two sets of interacting Cuntz-Krieger families, which led them to introduce and study higher-rank versions of the Cuntz-Krieger algebras. In [**36**], Kumjian and Pask introduced row-finite higher-rank graphs with no sources as a graphical means to provide combinatorial models for the Cuntz-Krieger algebras of Robertson and Steger. A *higher-rank graph* or *k-graph* Λ is a combinatorial structure, and is a k-dimensional analogue of a directed graph. Using groupoid techniques, they showed how to construct a C^*-algebra $C^*(\Lambda)$ that is associated to a row-finite k-graph.

In [**57, 58**], Raeburn, Sims and Yeend introduced general methods for associating a C^*-algebra to a wider class of higher-rank graphs; however, we shall focus exclusively on row-finite k-graphs with no sources.

One of the reasons for the great interest in graph algebras is that important properties of the associated graph C^*-algebra can be read off from the combinatorial properties of a directed graph E or a higher-rank graph Λ. We mention briefly four examples of this phenomenon.

(1) *Ideal Structure.* In [**11**], Cuntz introduced the notion of hereditary subsets, and gave a classification of the ideal structure for finite directed graphs that satisfied a condition known as Condition (II). In [**28**], an Huef and Raeburn described the ideal structure when this condition was not necessarily satisfied. In [**39**, Theorem 6.6], Kumjian, Pask, Raeburn and Renault extended this to infinite graphs and showed that the ideals of a row-finite directed graph C^*-algebra are indexed by certain subsets of vertices that are associated to the connectivity of the graph. These results were extended in [**6, 7**] to the general case. Analogous results for the gauge-invariant ideals of k-graph C^*-algebras were provided by Raeburn, Sims and Yeend [**57**], Robertson and Sims [**63**], and Sims [**69**].

(2) *Entropy.* In [**29, 30**], Jeong, and in [**8**], Boca and Goldstein showed that for a directed graph E whose graph algebra $C^*(E)$ is simple, the entropy of the canonical endomorphism is equal to the entropy of the shift map on the shift space associated to E. The entropy is the logarithm of the spectral radius of the connectivity matrix of the graph. Analogous results for k-graphs whose C^*-algebras are simple were provided by Skalski and Zacharias [**70, 71**].

(3) *Classifiable C^*-algebras.* Graph C^*-algebras provide a wealth of tractable examples of C^*-algebras that can be classified by their K-theory:

In [**15, 75**], it is shown that up to Morita equivalence, every AF algebra can be realised by a graph algebra whose graph has no cycles. It was shown in [**19**] by Elliott that every AF-algebra was classifiable by its K-theory.

Necessary and sufficient conditions for a directed graph C^*-algebra to be simple and purely infinite were given in [**7, 27, 38, 39**]. These conditions, namely cofinality and aperiodicity, can be readily checked as they appear in terms of the distribution of cycles within the graph. These results were generalised to k-graphs by Kumjian and Pask in [**36**], and with further generalisation in [**63, 64**] by Robertson and Sims, and in [**42**] by Lewin and Sims with the generalisations of aperiodicity and cofinality conditions given. The classification by K-theory was done by Kirchberg and Phillips in [**52**].

In [**48**], Pask, Raeburn, Rørdam, and Sims gave examples of 2-graphs whose C^*-algebras are simple and real rank zero. That these algebras are classified by their K-theory, together with the dimension data, was shown by Elliott in [**20**].

(4) *K-theory.* The K-theory of a directed graph C^*-algebra may be computed in terms of its edge connectivity matrix. This has been shown with increasing generality in [**11, 46, 59, 73**]. It was generalised in [**4, 22, 65**] for k-graph C^*-algebras.

The above list shows that many results within the theory relating to C^*-algebras of directed graphs has been generalised to the k-graph C^*-algebra analogue, though rarely has this process been straightforward. There have been many technical issues due to the extra complexity of the relations satisfied by the generators of k-graph C^*-algebras.

The purpose of this thesis is to generalise the results of Pask, Raeburn, Yeend [**50**] about crossed products of graph algebras by semigroup actions from directed graphs to k-graphs.

1.2. Overview

In [**35**], it was shown that for the action α of a group G on a directed graph, E, the universal property of $C^*(E)$ can be used to induce an action of G on the graph C^*-algebra, $C^*(E)$, thereby giving rise to a crossed product $C^*(E) \rtimes G$. An elegant theorem of Kumjian and Pask says that if a group G acts freely on a directed graph E, then the associated crossed product $C^*(E) \rtimes G$ of the graph algebra is stably isomorphic to the graph algebra $C^*(E/G)$ of the quotient graph. The proof of this result relies on a theorem by Gross and Tucker; for a free action of a group G on a directed graph E, we can recover, up to equivariant isomorphism, the original graph and action of G from the quotient graph E/G by a skew-product construction. The argument proving this was first outlined in [**25**, Theorem 2.2.2] stating that for a directed graph E and a group G there exists a function $c : E/G \to G$ such that $E/G \times_c G$ is isomorphic to E.

A skew-product graph $F \times_c G$ is formed from a function $c : F^1 \to G$. In [**32**, Theorem 2.4] it was shown that $C^*(F \times_c G)$ is the crossed product of $C^*(F)$ by a coaction δ_c of G. Combining this with the Gross-Tucker theorem, we have $C^*(E \times_c G) \rtimes G$ is isomorphic to $C^*(E) \otimes \mathcal{K}(\ell^2(G))$; see [**32, 35**]. The Gross-Tucker theorem [**25**, Theorem 2.2.2] can be applied, as in [**35**, Corollary 3.10] and it was shown using groupoid techniques that $C^*(E) \rtimes G$ is isomorphic to $C^*(E/G) \otimes \mathcal{K}(\ell^2(G))$.

This result of Kumjian and Pask is reminiscent of results of Green and Rieffel [**62**, Situation 10] for proper actions on locally compact spaces. This indicates that $C^*(E)$ is behaving like the functions on a non-commutative space. The result was revisited by [**32, 46**], each time with extra generality. In [**32**, Theorem 2.4], Kaliszewski, Quigg and Raeburn, using techniques from non-abelian duality, showed that the graph algebra $C^*(E \times_c G)$ is isomorphic to $C^*(E) \rtimes_{\delta_c} G$. They also showed that the reduced crossed product $C^*(E \times_c G) \rtimes_{\alpha,r} G$ is isomorphic to $C^*(E) \otimes \mathcal{K}(\ell^2(G))$ [**32**, Corollary 2.5]. Pask and Raeburn [**47**, Theorem 1.6, Corollary 3.1] showed that $C^*(E) \rtimes_\alpha G$ is isomorphic to $C^*(E) \rtimes_{\alpha,r} G$, which is Morita equivalent to $C^*(E/G)$.

For group actions on k-graphs, analogous theorems were first proved in [**36**] by Kumjian and Pask, and then with greater generality in [**45**] by Pask, Quigg and Raeburn.

In [**50**] Pask, Raeburn, and Yeend considered semigroup actions on directed graphs, proving the result that if an Ore semigroup acts on a directed graph E, then $C^*(E) \times_{\alpha_*} S$ is isomorphic to $C^*(E/S) \otimes \mathcal{K}(\ell^2(S))$. This was proved by providing a version of the Gross-Tucker theorem for semigroup actions, direct limit techniques, and dilation results for endomorphic actions of Ore semigroups on C^*-algebras by Laca [**41**].

In this thesis we are looking to bring these efforts together to consider semigroup actions on k-graphs. We shall be proving an analogue of the Gross-Tucker theorem [**25**, Theorem 2.2.2] for semigroup actions, and be using the dilation results of Laca [**41**], as well as generalising the work of [**32**] regarding groups acting on the graph algebras of skew-product graphs.

Our main result is Theorem 6.1, which states that given an Ore semigroup acting on a higher-rank graph with no sources, the crossed product of the k-graph C^*-algebra by the induced action of the Ore semigroup is stably isomorphic to the graph algebra of the quotient k-graph. This directly extends the work of Pask, Raeburn and Yeend in [**50**] to higher-rank graphs, but our proof has some interesting new features. First of these is our more efficient use of Laca's dilation theory for endomorphic actions [**41**]: by exploiting his uniqueness theorem, we have been able to avoid the complicated direct-limit constructions used in [**50**]. Second, we have found an explicit isomorphism (and have thus been able to avoid using the symbol $\cong$!). Searching for explicit formulas has led us to revisit the case of group actions, and we think a third feature of general interest is our direct approach to crossed products of the C^*-algebras of skew-product graphs, which is based on the treatment of skew-products of directed graphs in [**32**, §3].

Some technical difficulties were encountered along the way: our generalisation of the Gross-Tucker theorem (Theorem 3.11) needed to be carefully formulated so that it can efficiently interface with Laca's results (Corollary 4.4). In the search for an explicit isomorphism, we needed a deep understanding of the multiplier algebra of the C^*-algebra of a higher-rank graph to prove new results involving strict convergence in the multiplier algebra (Proposition 2.8).

1.3. Outline of the thesis

Chapter 2. In this chapter, after a brief discussion of notation and background material, we discuss higher-rank graphs and their C^*-algebras. We consider the multiplier algebra of a graph C*-algebra, and the strict topology. We consider conditions for extendibility: we need to understand when a k-graph induces a map between the associated k-graph C^*-algebras extends to a morphism between their multiplier algebras. We prove two general results about extendibility to the multiplier algebras of C^*-algebras of higher-rank graphs.

Chapter 3. In this chapter we prove that the skew-product graph construction produces a k-graph, and that the action of an Ore semigroup can be used to construct a quotient k-graph. We prove results about actions of semigroups, including a version of the Gross-Tucker theorem that says given a free action of an Ore semigroup S on a higher-rank graph Σ we can produce a functor $\eta : \Sigma \to S$ that allows us to recognise the higher-rank graph as a skew-product graph, $S\backslash\Sigma \times_\eta S$. We demonstrate that an application of the Gross-Tucker theorem allows us to recover from a quotient k-graph and an action, up to equivariant isomorphism, the original k-graph and action of an Ore semigroup. The isomorphism produced by our generalisation of the Gross-Tucker theorem will later be used to induce the isomorphism ψ_1 of $C^*(\Sigma) \times_{\alpha_*} S$ onto a corner of $C^*(S\backslash\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$. The requirement of a fundamental domain for the action is considered, and we provide a classification of fundamental domains.

Chapter 4. In this chapter we apply Laca's dilation theory to higher-rank graph algebras. Using the language of k-graphs and restating the results of Laca allow us to avoid the direct limit construction that has previously been used to prove the result. We therefore obtain an explicit formula for an isomorphism ψ_2 of $C^*(S\backslash\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$ onto a corner of $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$, where Γ is the enveloping group of the Ore semigroup S.

Chapter 5. In this chapter we discuss group actions on skew-products. For directed graph, it was shown in [**32**] that there exists an isomorphism from the crossed product of a directed skew-product graph algebra, $C^*(E \times_c G) \rtimes_\gamma G$ onto $C^*(E) \otimes \mathcal{K}(\ell^2(G))$. We extend the result to higher-rank graphs, and prove an analogous result, giving an explicit formula for the isomorphism. This requires us being able to recognise a Cuntz-Krieger $(S\backslash\Sigma)$-family in $M(C^*(S\backslash\Sigma \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$. We combine this with our generalisation of the Gross-Tucker theorem to give an isomorphism ϕ of $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$ onto $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(\Gamma))$. We then restrict to a subspace of $\ell^2(\Gamma)$, and the restriction of ϕ gives an isomorphism ψ_3 from a corner of $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$ onto $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$.

Chapter 6. In this we pull the pieces together and prove our main theorem. Each isomorphism from Chapters 3, 4 and 5 are carefully considered, and shown to be compatible.

The following diagram is intended to served as an indication of the big picture, and an easy reference for the isomorphisms and spaces we are considering.

$$\begin{array}{ccc} C^*(\Sigma)\times_{\alpha_*} S & \xrightarrow{\psi_1 \text{ (Theorem 3.11)}} & C^*(S\backslash\Sigma\times_\eta S)\times_{\mathrm{lt}_*} S \\ \Big\downarrow{\scriptstyle\text{(Theorem 6.1)}} & & \Big\downarrow{\scriptstyle\psi_2\text{ (Corollary 4.4)}} \\ C^*(S\backslash\Sigma)\otimes\mathcal{K}(\ell^2(S)) & \xleftarrow{\psi_3 \text{ (Theorem 5.1)}} & p(C^*(S\backslash\Sigma\times_\eta\Gamma)\times_{\mathrm{lt}_*}\Gamma)p \end{array} \tag{1.1}$$

Chapter 7. In this chapter we consider two applications of the main result, Theorem 6.1. First, we consider the dual graph $p\Sigma$ of a k-graph Σ, and consider actions that are non-free on Σ, but are free on some dual graph of Σ. Second, by applying [**63**, Theorem 3.1], we reduce the problem of determining simplicity of $C^*(\Sigma)$ to considering the aperiodicity and cofinality of the quotient k-graph, $S\backslash\Sigma$.

Chapter 8. In this chapter we take a further look at the simplicity of skew-product k-graph C^*-algebras, and prove some results about their aperiodicity and cofinality. We extend the idea of primitivity from [**51**], and attempt to find some equivalent conditions to aperiodicity and cofinality that are easier to check than attempting to apply the original definitions.

Chapter 9. In this chapter we provide the direct limit arguments relating the direct systems of k-graphs that would have been necessary if not for our efficient use of the results of [**41**]. We feel that these results are of independent interest.

1.4. Background and notation

We remind the reader of some basic notions that we will use throughout the thesis.

A *directed graph* $E = (E^0, E^1, r, s)$ is a quadruple, consisting of countable sets E^0 and E^1 and functions $r, s : E^1 \to E^0$. We say that elements of E^0 are the vertices of E, elements of E^1 are the edges of E, and for each edge $e \in E^1$, $s(e) \in E^0$ is the source of e and $r(e) \in E^0$ is the range of e.

We take $\{e_i : 1 \le i \le k\}$ to be the canonical basis for $\mathbb{N}^k$. For $m, n \in \mathbb{N}^k$, we have $m \le n$ if and only if $m_i \le n_i$ for $1 \le i \le k$.

On categories

A *category* $\mathcal{C}$ consists of two classes: $\mathcal{C}^0$ and $\mathcal{C}^*$, two functions $r, s : \mathcal{C}^* \to \mathcal{C}^0$, an associative partially defined product $(f, g) \mapsto (fg)$ from $\{(f, g) \in \mathcal{C}^* \times \mathcal{C}^* : s(f) = r(g)\}$ to $\mathcal{C}^*$, and for each object $x \in \mathcal{C}^0$, there exists an identity morphism $\iota_x \in \mathcal{C}^*$ such that the maps r, s satisfy the following relations: $r(fg) = r(f)$ and $s(fg) = s(g)$, $r(\iota_x) = x = s(\iota_x)$, $\iota_x f = f$ when $r(f) = x$, and $g\iota_x = g$ when $s(g) = x$.

The elements of $\mathcal{C}^0$ are called the objects of the category $\mathcal{C}$, the elements of $\mathcal{C}^*$ are the morphisms of the category, for $f \in \mathcal{C}^*$, $s(f)$ is called the domain, and $r(f)$ is called the codomain of f. If $\mathcal{C}^0$ and $\mathcal{C}^*$ are sets rather than classes, $\mathcal{C}$ is a *small category.*

A *functor* $F : \mathcal{C} \to \mathcal{D}$ associates to each object $x \in \mathcal{C}^0$ an object $F(x) \in \mathcal{D}^0$ and to each morphism $\lambda \in \mathcal{C}^*$ a morphism $F(\lambda) \in \mathcal{D}^*$, in such a way that $F(\iota_v) = \iota_{F(v)}$, and $F(\lambda \circ \mu) = F(\lambda) \circ F(\mu)$ when λ is composable with μ.

On actions

An *action* of a semigroup with identity on a C^*-algebra A is an identity-preserving homomorphism α of S into the semigroup $\operatorname{End}(A)$ of endomorphisms of A, which is itself a semigroup under composition. This data gives a semigroup dynamical systems, which we denote (A, S, α). A pair (π, V) consisting of a nondegenerate homomorphism π of A into a C^*-algebra B and a homomorphism V of S into the semigroup of isometries in $M(B)$ is a *covariant representation* of (A, S, α) in B if $\pi(\alpha_t(a)) = V_t \pi(a) V_t^*$ for $a \in A$ and $t \in S$. The *crossed product* $A \times_\alpha S$ is generated by a universal covariant representation (i_A, i_S) in $A \times_\alpha S$. (In the recent literature, this is called the "Stacey crossed product" [**72**].) When $S = \Gamma$, the endomorphisms are automorphisms, and we recover the usual crossed product $A \rtimes_\alpha \Gamma$.

The *gauge invariant uniqueness theorem* for directed graphs was originally proved as [**28**, Theorem 2.3]. The following version for k-graphs is from [**36**, Theorem 3.4]: suppose $\alpha : \mathbb{T}^k \to \operatorname{Aut} C^*(\Lambda)$ is the gauge action, B is a C^*-algebra, $\pi : C^*(\Lambda) \to B$ is a homomorphism and let $\beta : \mathbb{T}^k \to \operatorname{Aut}(B)$ be an action such that $\pi \circ \alpha_t = \beta_t \circ \pi$ for all $t \in \mathbb{T}^k$. Then π is faithful if and only if $\pi(p_v) \neq 0$ for all $v \in \Lambda^0$.

On stable isomorphisms

Two C^*-algebras A, B are stably isomorphic if there exists an isomorphism $A \to B \otimes \mathcal{K}$. Hence, to talk about stable isomorphisms, we need to consider tensor products with the algebra $\mathcal{K}(\mathcal{H})$ of compact operators. Since $\mathcal{K}(\mathcal{H})$ is nuclear, there is no ambiguity in writing $A \otimes \mathcal{K}(\mathcal{H})$. However, we are interested in C^*-algebras which have universal properties, and we view $A \otimes \mathcal{K}(\mathcal{H})$ as the maximal tensor product $A \otimes_{\max} \mathcal{K}(\mathcal{H})$ which is universal for pairs of commuting representations of A and $\mathcal{K}(\mathcal{H})$ (see [**60**, Theorem B.27]).

CHAPTER 2

Higher-rank graphs and their C^*-algebras

The purpose of this chapter is to give some background about higher-rank graphs, and their associated k-graph C^*-algebras. We give new results about the multipliers and multiplier algebra of a k-graph C^*-algebra. In order to this, we review the topology on multiplier algebras, and give a necessary condition for a k-graph morphism to induce a homomorphism between k-graph C^*-algebras. We also require an understanding of extendability of k-graph morphisms in the strict topology. Our main results include Proposition 2.8, which will allow us to make sense of infinite sums of partial isometries in a k-graph C^*-algebra, so that we can determine when these maps can be extended into the multiplier algebra. Our second main result is Proposition 2.14, which gives a condition of a k-graph morphism that ensures that the induced maps between k-graph C^*-algebras are extendible.

2.1. Background on higher-rank graphs

Suppose $k \in \mathbb{N}$ and $k \geq 1$. A *graph of rank* k, or k*-graph*, is a countable category Λ with domain and codomain maps r and s, together with a functor $d : \Lambda \to \mathbb{N}^k$ satisfying the k*-graph factorisation property*: for every $\lambda \in \Lambda$ and decomposition $d(\lambda) = m + n$ with $m, n \in \mathbb{N}^k$, there is a unique pair (μ, ν) in $\Lambda \times \Lambda$ such that $s(\mu) = r(\nu)$, $d(\mu) = m$, $d(\nu) = n$ and $\lambda = \mu\nu$. We define $\Lambda^n := d^{-1}(n)$ for $n \in \mathbb{N}^k$. By an application of the factorisation property, we can identify the objects of Λ with $d^{-1}(0)$, and write Λ^0 for the set of objects. For $\lambda \in \Lambda$ and $0 \leq m \leq n \leq d(\lambda)$, we define $\lambda(m, n)$ to be the unique morphism in Λ^{n-m} obtained from the k-graph factorisation property such that $\lambda = \lambda'(\lambda(m, n))\lambda''$ for some $\lambda' \in \Lambda^m$ and $\lambda'' \in \Lambda^{d(\lambda)-n}$. Given $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, $v\Lambda^n := \{\lambda \in \Lambda^n : r(\lambda) = v\}$. Similarly, $\Lambda^n v := \{\lambda \in \Lambda^n : s(\lambda) = v\}$. As in [**36**], we assume throughout that our k-graphs are *row-finite* and *have no sources*, in the sense that $r^{-1}(v) \cap \Lambda^n$ is finite and nonempty for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$. A k*-graph morphism* is a structure-preserving map: a degree-preserving functor, from Λ, viewed as a category, into another k-graph, Σ say, also viewed as a category. An *endomorphism* of a k-graph is a k-graph morphism of Λ into itself, and the set $\operatorname{End}(\Lambda)$ is a semigroup under composition.

EXAMPLES 2.1. (Some k-graphs)

(1) Take a 1-graph Λ. Then with $E^0 = \Lambda^0$ and $E^1 = \Lambda^1$ with $s_E(\lambda) = r(\lambda)$ and $r_E(\lambda) = s(\lambda)$; then the quadruple (E^0, E^1, r_E, s_E) is a directed graph.

Conversely, given a directed graph E, $E^* = \bigcup_{n\geq 0} E^n$ can be viewed as a small category with $s = r_E$ and $r = s_E$. Then with $d : E^* \to \mathbb{N}$ as the length function, $\Lambda_E := (E^*, d)$ is a 1-graph. Example 1.3 in [**36**] shows how the path category of a directed graph is a 1-graph, and vice versa. Two examples of directed graphs giving rise to 1-graphs: for $n \geq 1$ we denote by B_n the path category of the directed graph consisting of a single vertex and n edges. We can also have L_r, which is the path category of a directed graph consisting of a loop with r edges. This 1-graph is considered in [**40**, §6.3].

(2) The category T_k consists of a single object and is generated by k commuting morphisms $\{f_1, \ldots, f_k\}$. Define $d : T_k \to \mathbb{N}^k$ by $d(f_1^{n_1} \ldots f_k^{n_k}) = (n_1, \ldots, n_k)$. It is then straightforward to check that T_k is a row-finite k-graph.

(3) This k-graph was defined in [**57**, Example 2.2(ii)]. Let $(\Omega_k)^* = \{(m, n) \in \mathbb{N}^k \times \mathbb{N}^k : m \leq n\}$ and $(\Omega_k)^0 = (m, m) : m \in \mathbb{N}^k$. Then gifted with the structure maps $r(m, n) = m$, $s(m, n) = n$, composition $(m, n)(n, p) = (m, p)$ and degree map $d(m, n) = n - m$, one checks that (Ω_k, d) is a row-finite k-graph.

Closely associated to Ω_k is the k-graph Δ_k. It is the $\mathbb{Z}^k$-lattice with $(\Delta_k)^0 = \{p \in \mathbb{Z}^k\}$, $(\Delta_k)^* = \{(m, n) : m \leq n \text{ for } m, n \in \mathbb{Z}^k\}$, $r(m, n) = m$, $s(m, n) = n$, composition $(m, n)(n, p) = (m, p)$, and $d_{\Delta_k} = n - m$.

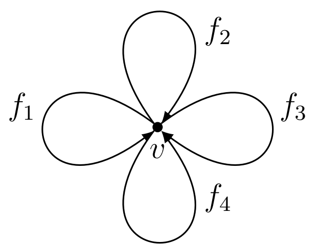
(4) For $n \geq 1$ let $\underline{n} = \{1, \ldots, n\}$. For $m, n \geq 1$ let $\theta : \underline{m} \times \underline{n} \to \underline{m} \times \underline{n}$ be a bijection. Let $\mathbb{F}_\theta^2$ be the category with single vertex v and edges $f_1, \ldots, f_m, g_1, \ldots, g_n$. Define a map $d(f_i) = e_1$ for $i \in \underline{m}$, $d(g_j) = e_2$ for $j \in \underline{n}$ and factorisation rules $f_i g_j = g_{j'} f_{i'}$ where $\theta(i, j) = (i', j')$ for $(i, j) \in \underline{m} \times \underline{n}$. For further details, See [**13, 14, 53**]. By [**24, 26**] we assert that with this degree map and factorisation rules, $\mathbb{F}_\theta^2$ is a 2-graph.

(5) In [**36**], it was shown that given two higher-rank graphs Λ_1, Λ_2 of rank k_1, k_2, we can form the Cartesian product $\Lambda_1 \times \Lambda_2$, which is a $(k_1 + k_2)$-graph.

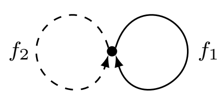
We visualise a k-graph in terms of its 1-skeleton: a k-coloured directed graph $(\Lambda^0, \bigcup_{i=1}^k \Lambda^{e_i}, r, s)$ with the edges of each Λ^{e_i} utilising a distinct colour. Most of our examples are 2-graphs, and we will use a convention of blue and red edges, or solid and dashed edges. Blue or solid edges represent morphisms of degree e_1, red or dashed edges represent morphisms of degree e_2. The 1-skeleton does not uniquely determine Λ as it does not encode the k-graph factorisation property. To do this, we use the factorisation property to determine all factorisations of morphisms of degree $e_i + e_j$, $ef = gh$ such that $d(e) = d(h) = e_j$ and $d(f) = d(g) = e_i$ for $i \neq j$ to specify the factorisation rules. If there is a unique path of degree $e_i + e_j$ from vertices v to w, then we do not need to specify a factorisation rule. These visualisations of k-graphs are discussed in [**57**] and [**55**, Chapter 10]. Indeed, [**36**] asserted that for $k = 2$, a 1-skeleton and a suitable set of factorisation rules uniquely specify a k-graph. It was proved as [**24**, Remark 3.4] for $k > 2$. For a complete description of the relationship between k-graphs and 1-skeletons, see [**26**]. The 1-skeleton only allows one set of factorisation rules when there is a unique blue-red path and red-blue path between any pair of vertices.

Examples 2.2. (Some k-graph 1-skeletons)

(1) The 1-skeleton of B_4 (the bouquet of four edges with a single vertex) is



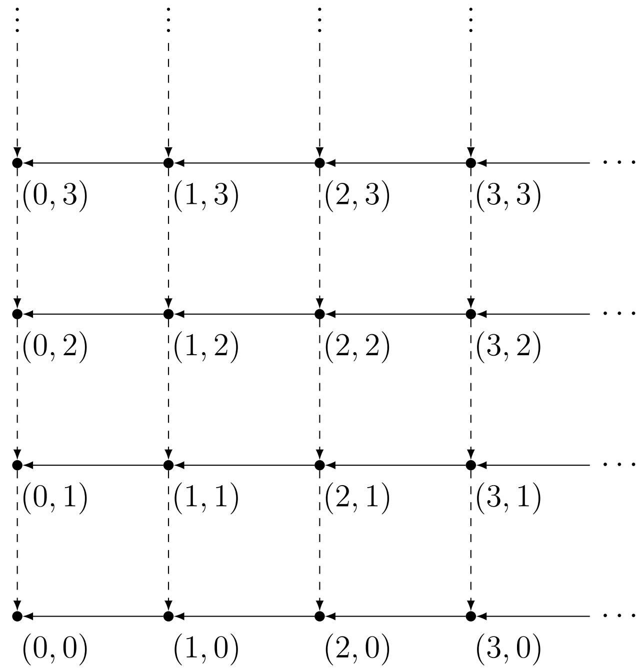
(2) The 1-skeleton of the 2-graph T_2 is a single vertex with two edges of different colours:



We are using the convention that lines of different style indicate edges of different degree. Note that no factorisation rule need to be given, as there is no ambiguity of a path of degree $e_1 + e_2$.

(3) The 1-skeleton of the 2-graph Ω_2 is related to the $\mathbb{N}^2$-lattice. The 1-skeleton is usually drawn in colour, and the edges would be referred as the blue edges

and red edges.



There is no need for explicit factorisation rules for Ω_k, as for a path of edges of degree e_i followed by e_j, there is a unique path of degree e_j followed by e_i that form a commuting square. Using the convention that different colours represent edges of different degree, in Ω_2, this is equivalent to for each blue-red path, there is a unique red-blue path that forms a commuting square.

For a row-finite k-graph with no sources, a *Cuntz-Krieger* Λ*-family* in a C^*-algebra B consists of partial isometries $\{S_\lambda : \lambda \in \Lambda\}$ in B satisfying the *Cuntz-Krieger relations*:

(CK1) $\{S_v : v \in \Lambda^0\}$ are mutually orthogonal projections;
(CK2) $S_\lambda S_\mu = S_{\lambda\mu}$ whenever $s(\lambda) = r(\mu)$;
(CK3) $S_\lambda^* S_\lambda = S_{s(\lambda)}$ for every $\lambda \in \Lambda$;
(CK4) $S_v = \sum_{\{\lambda \in v\Lambda^n\}} S_\lambda S_\lambda^*$ for every $v \in \Lambda^0$ and $n \in \mathbb{N}^k$.

The k-graph C^*-algebra $C^*(\Lambda)$ is generated by a universal Cuntz-Krieger Λ-family $\{s_\lambda\}$. When there is more than one graph around, we may write $\{s_\lambda^\Lambda\}$ for emphasis. Each vertex projection s_v (and hence by (CK3) each s_λ) is non-zero [**36**, Proposition 2.11], and so there exists a universal k-graph C^*-algebra for a Cuntz-Krieger Λ-family. Moreover,

$$C^*(\Lambda) = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \lambda, \mu \in \Lambda, s(\lambda) = s(\mu)\} \quad \text{(see [36, Lemma 3.1]).}$$

Examples 2.3. The k-graph C^*-algebras of some of our previous examples are well-known (see [**36**] for details):

(1) Recall the k-graph T_k defined in Examples 2.1. It can be shown that the Cuntz-Krieger relations imply that the set $\{T^{e_i}\}$ is a set of commuting unitaries: the set $(s_1, \dots, s_k)$ satisfy (CK1) and (CK2), hence $s_v = \mathrm{id}$. From (CK2), $s_i s_j = s_j s_i$. For (CK3), $s_i^* s_i = s_v = \mathrm{id}$. For (CK4), $s_v = \sum s_i s_i^* = \mathrm{id}$. Hence we have k commuting unitaries. The universal algebra of k commuting unitaries is the group C^*-algebras $C^*(\mathbb{Z}^k)$, which is isomorphic to $C(\mathbb{T}^k)$.

(2) Recall the k-graph Ω_k defined in Examples 2.1. We claim the set $\{s_{(m,m+e_i)} : m \in \mathbb{N}^k, i = 1 \dots k\}$ forms a system of matrix units, so that $s_{(m,m+e_i)}$ can be identified with the rank-one operators $\theta : \delta_m \to \delta_{m+e_i}$ in $\ell^2(\mathbb{N}^k)$, and we can see that $C^*(\Omega_k)$ is isomorphic to $\mathcal{K}(\ell^2(\mathbb{N}^k))$.

(3) If Λ is a 1-graph, then there is some directed graph E such that $\Lambda = \Lambda_E$ where Λ_E is the path category of E. Take a Cuntz-Krieger Λ_E-family $\{s_\lambda\}$. Then $s_e^* s_e = s_{s(e)}$ for $e \in \Lambda_E^{e_i}$ and for $f \in \Lambda_E^{e_i}$, $s_v = \sum_{r(f)=v} s_f s_f^*$. We then have a Cuntz-Krieger family for $C^*(E)$, giving a map $C^*(E) \to C^*(\Lambda_E)$ by the universal property. We can also take a Cuntz-Krieger E-family $\{f_k, p_v\}$ with $s_v = p_v$ and $s_\lambda = s_{\lambda_1} \dots s_{\lambda_k}$, and check that $\{s_\lambda : \lambda \in \Lambda_E\}$ is a Cuntz-Krieger E-family, and get a map $C^*(\Lambda_E) \to C^*(E)$.

2.2. Multipliers, and the multiplier algebra of a k-graph C^*-algebra.

We will in later chapters be proving results involving crossed products, and so will require an understanding of multiplier algebras. In particular, we are interested in the multiplier algebras of k-graph C^*-algebras as we are going to need to be able to make sense of infinite sums of projections and partial isometries of a k-graph C^*-algebra. For instance, in the diagram (1.1), the isomorphism ψ_3 from the crossed product $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$ to $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$ in particular requires a detailed understanding of the multiplier algebra.

We first consider some general facts. Every C^*-algebra has a multiplier algebra. Most of the ideas contained in this section regarding multiplier algebras can be attributed to Busby, [**9**]. The format of the following results follows a set of informal lecture notes by Raeburn [**56**].

A *multiplier* of a C^*-algebra A is a pair of bounded linear maps (L, R) of A into A such that $L(a)b = L(ab), bR(a) = R(ba)$ and $R(a)b = aL(b)$ for $a, b \in A$. For an operator $T \in B(A)$, define $T^\sharp \in B(A)$ by $T^\sharp(a) := T(a^*)^*$.

A simple example is to multiply by elements of A: for $c \in A$ define the multiplier (L_c, R_c) by $L_c(a) = ca$ and $R_c(a) = ac$, and then the required relations are immediate by associativity of multiplication. In a unital C^*-algebra all multipliers can be characterised in this way.

LEMMA 2.4. *Let A be a C^*-algebra. Suppose $a \in A$, and take (L_a, R_a) to be the multiplier of A defined by $L_a(b) = ab$ and $R_a(b) = ba$. Then for scalars λ, μ and $a, b \in A$,*

(1) $(L_{\lambda a+\mu b}, R_{\lambda a+\mu b}) = (\lambda L_a + \mu L_b, \lambda R_a + \mu R_b)$
(2) $(L_{ab}, R_{ab}) = (L_a \circ L_b, R_b \circ R_a)$
(3) $(L_{a^*}, R_{a^*}) = (R_a^\sharp, L_a^\sharp)$
(4) $\|a\| = \|L_a\| = \|R_a\|$.

If the C^*-algebra A has an identity, then every multiplier of A has the form (L_a, R_a).

PROOF. Utilising the axioms of an algebra, we can demonstrate the first assertion. Take $a, b, c \in A$:

$$L_{\lambda a+\mu b}(c) = (\lambda a + \mu b)c = \lambda L_a(c) + \mu L_b(c)$$
$$R_{\lambda a+\mu b}(c) = c(\lambda a + \mu b) = \lambda R_a(c) + \mu R_b(c)$$
$$dL_{\lambda a+\mu b}(c) = d(\lambda a + \mu b)c = R_{\lambda a+\mu b}(d)c.$$

The second assertion is also straightforward using the axioms of an algebra:

$$L_{ab}(c) = abc = aL_b(c) = L_a(L_b(c))$$
$$R_{ab}(c) = cab = R_a(c)b = R_b(R_a(c)).$$

For the third assertion, we calculate:

$$L_{a^*}(b) = a^*b = (b^*a)^* = R_a(b^*)^* = R_a^\sharp(b)$$
$$R_{a^*}(b) = ba^* = (ab^*)^* = L_a(b^*)^* = L_a^\sharp(b).$$

The final assertion takes a little more effort: first note that L is a bounded linear operator since $\|L_a(b)\| = \|ab\| \leq \|a\|\|b\|$; a similar calculation show that R is a bounded linear operator. Hence $\|L_a\| \leq \|a\|$. Choose $b := \|a\|^{-1}a^*$, and use the C^*-algebraic identity $\|aa^*\| = \|a\|^2$ and that $L_a(\|a\|^{-1}a^*) = a\|a\|^{-1}a^*$. So

$$\|L_a(\|a\|^{-1}a^*)\| = \|a\|a\|^{-1}a^*\| = \|a\|^{-1}\|aa^*\| = \|a\|^{-1}\|a\|^2 = \|a\|,$$

so $\|a\| = \|L_a\|$. A similar argument show that $\|Ra\| = \|a\|$.

Suppose A has an identity, 1, and (L, R) is a multiplier of A. Then for some $a \in A$, $a = L(1) = 1L(1) = R(1)1 = R(1)$. Hence for all $b \in A$, $L(b) = L(1b) = L(1)b = ab = L_a(b)$. A similar argument shows $R = R_a$. □

We can now define the multiplier algebra for a C^*-algebra.

PROPOSITION 2.5. *Suppose A is a C^*-algebra. Define the following operations on $M(A)$:*

(1) $\lambda(L_1, R_1) + \mu(L_2, R_2) := (\lambda L_1 + \mu L_2, \lambda R_1 + \mu R_2)$,
(2) $(L_1, R_1)(L_2, R_2) := (L_1 \circ L_2, R_2 \circ R_1)$,
(3) $(L, R)^* := (R^\sharp, L^\sharp)$,

(4) $\|(L,R)\| := \|L\| = \|R\|$.

Then the set $M(A)$ of multipliers of A is a C^*-algebra with identity $1_{M(A)} := (\mathrm{id}, \mathrm{id})$, and is called the multiplier algebra of A. The map $\iota_A : A \to M(A) : a \mapsto (L_a, R_a)$ is an isometric $*$-isomorphism onto a closed ideal in $M(A)$, and we then have

(5) $(L,R)(L_a, R_a) = (L_{L(a)}, R_{L(a)})$.

If A is a C^*-algebra with identity, then the map ι_A is an isomorphism of A onto all of $M(A)$.

Proof. The operations in (1) make $M(A)$ into a vector space. The right hand side of (2) can be seen to be a multiplier: for $a, b \in A$,

$$a(L_1 \circ L_2)(b) = aL_1(L_2(b)) = R_1(a)L_2(b) = R_2(R_1(a))b = (R_2 \circ R_1)(a)b.$$

Using operations (1) and (2), $M(A)$ is an algebra. We next verify that the right hand side of operation (3) is a multiplier: for $a, b \in A$,

$$aR^\sharp(b) = aR(b^*)^* = (R(b^*)a^*)^* = (b^*L(a^*))^* = L(a^*)^*b = L^\sharp(a)b.$$

Hence $(R^\sharp, L^\sharp)$ is a multiplier. Using the definition of the mapping $T \mapsto T^\sharp$, the assignment is conjugate linear, and satisfies $(T^\sharp)^\sharp = T$. Hence, the assignment $(L,R) \mapsto (L,R)^\sharp$ is conjugate linear and satisfies $(L,R)^{**} = (L,R)$. A short calculation shows $(ST)^\sharp = S^\sharp T^\sharp$:

$$(ST)^\sharp(a) = ((ST)(a^*))^* = (S(T(a^*)))^* = (S(T(a^*)^*)^*)^* = S^\sharp(T(a^*)^*) = S^\sharp T^\sharp(a).$$

Using this identity, as well as operations (1) and (2), we can prove:

$$\begin{aligned}((L_1,R_1),(L_2,R_2))^* &= (L_1L_2, R_2R_1)^* = ((R_2R_1)^\sharp, (L_1L_2)^\sharp) \\ &= (R_2^\sharp R_1^\sharp, L_1^\sharp L_2^\sharp) = (R_2^\sharp, L_2^\sharp)(R_1^\sharp, L_1^\sharp) \\ &= (L_2, R_2)^*(L_1, R_1)^*.\end{aligned}$$

The operations $(1), (2), (3)$ are sufficient to show $M(A)$ is a $*$-algebra.

Moving from algebraic concerns to analytic, we use that the operator norm in $B(A)$ is submultiplicative to conclude that $\|(L,R)\|$ is also a submultiplicative norm. We wish for the norm to satisfy the C^*-identity. A short calculation

$$\|R^\sharp(a)\| = \|R(a^*)^*\| = \|R(a^*)\| \le \|R\|\|a^*\| = \|R\|\|a\|$$

shows that $\|R^\sharp\| \leq \|R\|$. The following calculation uses this along with the definition of the norm, operation (4) and that A is a C^*-algebra:

$$\begin{aligned}\|(L,R)\|^2 = \|L\|^2 &= \sup\{\|L(a)\|^2 : \|a\| \leq 1\} = \sup\{\|L(a)^*L(a)\| : \|a\| \leq 1\} \\ &= \sup\{\|L^\sharp(a^*)L(a)\| : \|a\| \leq 1\} = \sup\{\|R(L^\sharp(a^*))a\| : \|a\| \leq 1\} \\ &\leq \|(L,R)(L,R)^*\| = \|(L,R)^*(L,R)\| \\ &\leq \|(L,R)^*\|\|(L,R)\| = \|(R^\sharp, L^\sharp)\|\|(L,R)\| \\ &= \|R^\sharp\|\|L\| \leq \|R\|\|L\| = \|(L,R)\|^2.\end{aligned}$$

Therefore we have equality throughout, and so have the C^*-identity for $M(A)$.

Finally, we show that $M(A)$ is a complete space: suppose $\{(L_n, R_n)\}$ is a Cauchy sequence of multipliers in $M(A)$, then the two sequences $\{L_n\}$ and $\{R_n\}$ are themselves Cauchy sequences in $B(A)$. The set of bounded linear operators is complete, therefore $\{L_n\}$ and $\{R_n\}$ converge: there exist $L, R \in B(A)$ such that $L_n \to L$ and $R_n \to R$. The relation $aL_n b = R_n(a)b$ for each n implies that $aL(b) = R(a)b$, and so the limit (L,R) is a multiplier. Using operation (4) $\|(L_n, R_n) - (L,R)\| = \|L_n - L\|$, and so we have $(L_n, R_n) \to (L,R)$. Thus $M(A)$ is complete. Therefore, with these operations, $M(A)$ is a Banach algebra with a submultiplicative norm that satisfies the C^*-identity, and hence is a C^*-algebra.

Using Lemma 2.4, the map ι_A is isometric, preserves the adjoint as

$$\iota_A(a)^* = (L_a, R_a)^* = (R_a^\sharp, L_a^\sharp) = (L_{a^*}, R_{a^*}) = \iota_A(a^*),$$

and is by definition an isomorphism onto its range. The range is closed and is a sub-$*$-algebra of $M(A)$. To demonstrate (5) it is sufficient to demonstrate the range is an two-sided ideal: take $(L,R) \in M(A)$ and $i_A(a) = (L_a, R_a) \in \iota_A(A)$. Then $(L,R)(L_a,R_a) = (L \circ L_a, R_a \circ R)$ and $(L_a, R_a)(L,R) = (L_a \circ L, R \circ R_a)$.

$$\begin{aligned}(L \circ L_a)(b) &= L(L_a(b)) = L(ab) = L(a)b = L_{L(a)}(b) \\ (R_a \circ R)(b) &= R_a(R(b)) = R(b)a = bL(a) = R_{L(a)}(b) \\ (L_a \circ L)(b) &= L_a(L(b)) = aL(b) = R(a)b = L_{R(a)}(b) \\ (R \circ R_a)(b) &= R(R_a)(b) = R(ba) = bR(a) = R_{R(a)}(b).\end{aligned}$$

Hence, the range of ι_A is a two-sided ideal.

If A is has an identity, 1, then applying Lemma 2.4 again, we can conclude that ι_A is a surjective map, and is therefore an isomorphism onto all of $M(A)$. □

2.3. The strict topology

For a C^*-algebra A, the strict topology of $M(A)$ is generated by the seminorms $\lambda_a(x) = \|ax\|$ and $\rho_a(x) = \|xa\|$ for each $a \in A$ and $x \in M(A)$. In Proposition 2.8 we prove an important result about the convergence in this topology of multipliers, which will be critical in later chapters. By convention, we write $m \in M(A)$ for the element (L, R). Hence the topology on $M(A)$ may be described as follows: $\lambda_a(m) = \|am\| = \|R(a)\|$ and $\rho_a(m) = \|ma\| = \|L(a)\|$ where $m = (L, R) \in M(A)$.

DEFINITION 2.6. Suppose A is a separable C^*-algebra and $\{m_j : j \in \mathbb{N}\}$ is a sequence in $M(A)$. The sequence $\{m_j\}$ *converges strictly* to m means for every $a \in A$, $m_j a \to ma$ and $am_j \to am$ in norm in A.

The following lemma was presented without proof as [**50**, Lemma 5.1] and [**46**, Lemma 3.3.1].

LEMMA 2.7. *Suppose A is a separable C^*-algebra, that B is a dense subset of A, that $\{m_j\}$ is a norm-bounded sequence in $M(A)$, and that there exists $m \in M(A)$ such that $m_j b \to mb$ and $bm_j \to bm$ in norm for $b \in B$. Then $\{m_j\}$ converges strictly to m.*

PROOF. Fix $a \in A$ and $\varepsilon > 0$. To show that $\{m_j\}$ converges strictly to m, we find J such that for all $j > J$, $\|m_j a - ma\| < \varepsilon$ and $\|am_j - am\| < \varepsilon$.

Since $\{m_j\}$ is a norm-bounded sequence, there exists M such that $\|m_j\| \leq M$ for all j and $\|m\| \leq M$. Hence $\|am_j - bm_j\| \leq \|a - b\|\|m_j\| \leq \|a - b\|M$. Since B is dense in A, there exists $b \in B$ such that $\|a - b\| < \frac{\varepsilon}{3M}$. Since bm_j converges to bm in norm, there exists J_1 such that for all $j > J_1$, $\|bm_j - bm\| < \frac{\varepsilon}{3}$.

Then for $j > J_1$,

$$\begin{aligned} \|am_j - am\| &\leq \|am_j - bm_j\| + \|bm_j - bm\| + \|bm - am\| \\ &\leq \|a - b\|\|m_j\| + \|bm_j - bm\| + \|b - a\|\|m\| \\ &\leq \frac{\varepsilon}{3M} M + \frac{\varepsilon}{3} + \frac{\varepsilon}{3M} M = \varepsilon. \end{aligned}$$

Hence $\{am_j\}$ converges to am in norm.

A similar argument finds J_2 such that for $j \geq J$, we have $\|m_j a - ma\| < \varepsilon$. Then $J = \max\{J_1, J_2\}$ has the required property. □

In particular, we now consider the multiplier algebra of a k-graph C^*-algebra. The following result is going to allow us to consider infinite sums of projections and partial isometries in the multiplier algebra of a k-graph C^*-algebra.

PROPOSITION 2.8. *Suppose that Λ is a row-finite k-graph with no sources, $m \in \mathbb{N}^k$, V is a subset of Λ^m such that the paths in V all have different sources and $\{s_\lambda : \lambda \in \Lambda\}$ is the universal Cuntz-Krieger Λ-family. Let $\{F_n\}$ be an increasing sequence of finite subsets of V such that $V = \bigcup_n F_n$. Then each $s_n := \sum_{\mu \in F_n} s_\mu$ is a partial isometry in $C^*(\Lambda)$, and there is a partial isometry $s_V \in M(C^*(\Lambda))$ such that $s_n \to s_V$ strictly. The limit s_V is independent of the choice of F_n, and satisfies*

$$s_V s_\alpha s_\beta^* = \begin{cases} s_{\mu\alpha} s_\beta^* & \textit{if } r(\alpha) = s(\mu) \textit{ for some } \mu \in V \\ 0 & \textit{otherwise,} \end{cases} \tag{2.1}$$

and, for paths β with $d(\beta) \geq m$,

$$s_\alpha s_\beta^* s_V = \begin{cases} s_\alpha s_{\beta'}^* & \textit{if } \beta = \mu\beta' \textit{ for some } \mu \in V \\ 0 & \textit{otherwise.} \end{cases} \tag{2.2}$$

If $V \subset \Lambda^m$ and $W \subset \Lambda^p$ are two such sets, then $s_V s_W$ is the partial isometry s_{VW} associated to the set $VW := \{\mu\nu : \mu \in V,\ \nu \in W \text{ and } s(\mu) = r(\nu)\}$.

PROOF. Since all the μ have the same degree, (CK3) and (CK4) imply that

$$s_n^* s_n = \sum_{\mu,\nu \in F_n} s_\mu^* s_\nu = \sum_{\mu \in F_n} s_\mu^* s_\mu = \sum_{\mu \in F_n} s_{s(\mu)};$$

since $s(\mu) \neq s(\nu)$ for $\mu \neq \nu$ in V, this is a sum of mutually orthogonal projections, and hence is a projection. Thus s_n is a partial isometry. For $\alpha, \beta \in \Lambda$, we have

$$s_n s_\alpha s_\beta^* = \begin{cases} s_{\mu\alpha} s_\beta^* & \text{if } r(\alpha) = s(\mu) \text{ for some } \mu \in F_n \\ 0 & \text{otherwise.} \end{cases} \tag{2.3}$$

If $r(\alpha) = s(\mu)$ for some $\mu \in V$, then $\mu \in F_n$ for large n, and hence the right-hand side of (2.3) is eventually constant for every $s_\alpha s_\beta^*$. Take $A_0 := \operatorname{span}\{s_\alpha s_\beta^*\}$, which is a dense sub-$*$-algebra of $C^*(\Lambda)$. We give a standard $\varepsilon/3$ argument to show that $\{s_n a\}$ is Cauchy for every $a \in C^*(\Lambda)$: fix $\varepsilon > 0$ and $a \in C^*(\Lambda)$. Take $a_0 \in A_0$ such that $\|a - a_0\| < \varepsilon/3$. Since $\{s_n\}$ is a sequence of partial isometries, it is norm-bounded by 1, and so $\|s_n a - s_n a_0\| < \varepsilon/3$ and $\|s_m a_0 - s_m a\| < \varepsilon/3$. Since the right-hand side of (2.3) is eventually constant, there exists $N \in \mathbb{N}$ such that $\|s_n a_0 - s_m a_0\| < \varepsilon/3$ for all $n, m > N$. Hence:

$$\begin{aligned} \|s_n a - s_m a\| &< \|s_n a - s_n a_0\| + \|s_n a_0 - s_m a_0\| + \|s_m a_0 - s_m a\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \end{aligned}$$

A similar calculation shows that $s_\alpha s_\beta^* s_n$ is eventually constant whenever $d(\beta) \geq m$. However, (CK4) and (CK2) imply that

$$\operatorname{span}\left\{s_\alpha s_\beta^* : \alpha, \beta \in \Lambda\right\} = \operatorname{span}\left\{s_\alpha s_\beta^* : \alpha, \beta \in \Lambda,\ d(\beta) \geq m\right\},$$

so $s_\alpha s_\beta^* s_n$ is eventually constant for all α, β, and we deduce as before that $\{a s_n\}$ is Cauchy for all $a \in C^*(\Lambda)$. Since $M(C^*(\Lambda))$ is complete in the strict topology [**9**, Proposition 3.6], we have that there exists a multiplier such that $a s_n \to a s_V$ and $s_n a \to s_V a$ in norm for $a \in C^*(\Lambda)$. We can use Lemma 2.7 to see that s_n converges strictly to a multiplier s_V. Then (2.3) implies (2.1), and similarly for (2.2).

The formula (2.1) implies that s_V is independent of the choice of sequence $\{F_n\}$. For $\alpha, \beta \in \Lambda$, (2.1) and the adjoint of (2.2) show that $s_V s_V^* s_V s_\alpha^* s_\beta = 0 = s_V s_\alpha s_\beta^*$ unless $r(\alpha) = s(\mu)$ for some $\mu \in V$, and in that case

$$s_V s_V^* s_V s_\alpha^* s_\beta = s_V s_V^* s_{\mu\alpha} s_\beta^* = s_V s_\alpha s_\beta^*;$$

either way, we have $s_V s_V^* s_V s_\alpha s_\beta^* = s_V s_\alpha s_\beta^*$. Thus $s_V s_V^* s_V = s_V$, and s_V is a partial isometry. The final assertion follows from two applications of (2.1). □

REMARK 2.9. Proposition 2.8 applies when $m = 0$, in which case the summands are projections and so is the limit s_V. To emphasise this, we write p_V for s_V when $m = 0$.

2.4. Extendibility

When using the dilation results of Laca [**41**] in later chapters, we will need to know when a homomorphism from a C^*-algebra A into the multiplier algebra of another C^*-algebra B induces a homomorphism from $M(A)$ to $M(B)$. When we are working with non-unital algebras, we will require our maps to be extendible.

A *approximate identity* for a separable C^*-algebra A is a sequence $\{a_j\}$ such that $ma_j \to m$ and $a_j m \to m$ for all $m \in A$; equivalently, $a_j \to 1$ strictly in $M(A)$. For C^*-algebras A and B, a homomorphism ϕ from A to the multiplier algebra $M(B)$ is *extendible* if there is an approximate identity $\{a_j\}$ for A and a projection p_ϕ in $M(B)$ such that $\phi(a_j) \to p_\phi$ strictly in $M(B)$. An action $\alpha : S \to \mathrm{End}(A)$ of a semigroup on a C^*-algebra is extendible if each α_s is extendible.

First, we give some background about nondegenerate representations. This is material that is generally known: the following treatment is referenced to Adji, [**2**].

A representation π of a C^*-algebra A on a Hilbert space H is nondegenerate if

$$\overline{\mathrm{span}}\{\pi(a)h : a \in A, h \in H\} = H.$$

This is equivalent to: for any approximate identity $\{a_j\}$ in A, $\pi(a_j)$ converges strongly to the identity operator 1_H on H. Suppose the C^*-algebra A has a unit, then if $\pi(1) = 1_H$, a representation π of A is nondegenerate.

Every non-degenerate representation π of A on H extends to a unital representation $\overline{\pi}$ of the multiplier algebra $M(A)$ onto the set

$$\{T \in B(H) : T\pi(a), \pi(a)T \in \pi(A) \text{ for all } a \in A\}. \tag{2.4}$$

Every nondegenerate homomorphism is extendible with $p = 1_{M(B)}$.

For an arbitrary representation π of A on H, the subspace

$$K = \overline{\mathrm{span}}\{\pi(a)h : a \in A, h \in H\}$$

is invariant for $\pi(A)$, as is its complement, $K^\perp$. The representation π is unitarily equivalent to $\pi|_K \oplus 0$ by the unitary $U : (h, k) \mapsto h + k$ of $K \oplus K^\perp$ onto H. The restriction of π to K, π_K is itself a non-degenerate representation.

The composition $\overline{\pi} \circ \phi$ of a nondegenerate homomorphism $\phi : A \to B$ and nondegenerate representation $\pi : B \to B(H)$ is a nondegenerate representation of B on H. This is because for each $h \in H$, there exists $b \in B$ such that $h \sim \pi(b)h$, since π is a non-degenerate representation, and $\pi(b)h \sim \pi(\phi(a_j)b)h$, since ϕ is a non-degenerate homomorphism, for some j.

A k-graph morphism doesn't necessarily induce a homomorphism between k-graph C^*-algebras. In Proposition 2.14, we will given a necessary condition for such an homomorphism to be induced. We introduce the idea of extendibility. The following proposition and proof is from Adji, [**2**].

PROPOSITION 2.10. *A homomorphism $\phi : A \to M(B)$ is extendible if and only if there exists a strictly continuous homomorphism $\overline{\phi}$ of $M(A)$ into $M(B)$ such that $\overline{\phi}|_A = \phi$. If so, then $\overline{\phi}(1) = p_\phi$.*

PROOF. The 'if' direction is straightforward: if ϕ has a strictly continuous extension $\overline{\phi}$, let $p = \overline{\phi}(1)$, and then for any approximate identity $\{a_i\}$ in A, $\phi(a_i)$ will converge strictly to $\overline{\phi}(1)$. Hence ϕ is extendible.

The converse is much more complicated. Suppose we have an extendible homomorphism ϕ and a faithful nondegenerate representation π of B on H. Then π extends to a faithful unital representation $\overline{\pi}$ of $M(B)$ onto

$$\overline{\pi}(M(B)) = \{T \in B(H) : \pi(b)T, T\pi(b) \in \pi(B) \text{ for } b \in B\}.$$

For an approximate identity $\{a_j\}$, and $a \in A$,

$$aa_j \to a \Rightarrow \phi(aa_j) \to \phi(a) \Rightarrow \phi(a)\phi(a_j) \to \phi(a),$$

but also, as ϕ is extendible, $\phi(a)\phi(a_j) \to \phi(a)p$ strictly, so we have $\phi(a)p = \phi(a)$. A similar argument shows $p\phi(a) = \phi(a)$, and we conclude $p\phi(a) = \phi(a) = \phi(a)p$.

Consider the composition of the extension of the faithful nondegenerate representation π of B on H to a unital representation $\overline{\pi}$ of $M(B)$ with the extendible homomorphism ϕ, so we have $\overline{\pi} \circ \phi : A \to B(H)$. Take $k = (\overline{\pi} \circ \phi)(a)h$, so

$$k = \overline{\pi}(p\phi(a))h = \overline{\pi}(p)\overline{\pi}(\phi(a))h = \overline{\pi}(p)k,$$

hence the range $\overline{\pi} \circ \phi(a)$ for each $a \in A$ is contained in $\overline{\pi}(p)H$. The equation $\phi(a)p = p\phi(a)$ also implies that $\overline{\pi}(p)H$ is invariant under $\overline{\pi}(\phi(a))$ for all $a \in A$. This implies that $(\overline{\pi}(p)H)^\perp$ is also invariant. A short calculation: if $l \in (\overline{\pi}(p)H)^\perp$, then $\overline{\pi}(p)l = 0$, and $\overline{\pi} \circ \phi(a)l = \overline{\pi}(\phi(a)pl) = 0$. So $\overline{\pi} \circ \phi|_{(\overline{\pi}(p)H)^\perp} = 0$, and therefore $\overline{\pi} \circ \phi$ is unitarily equivalent to $\overline{\pi} \circ \phi|_{\overline{\pi}(p)H} \oplus 0$.

If we identify the Hilbert space H with $K \oplus K^\perp$, we then have $\overline{\pi} \circ \phi = \overline{\pi} \circ \phi|_{\overline{\pi}(p)H} \oplus 0$. For the sake of simplicity, $\overline{\pi} \circ \phi|_{\overline{\pi}(p)H}$ is denoted π_p, and so we have $\overline{\pi} \circ \phi(a) = \pi_p(a) \oplus 0$ for $a \in A$. We stated previously that the composition of a non-degeneration homomorphism π and a non-degenerate representation π was non-degenerate, hence π_p is a nondegenerate representation of A on $\overline{\pi}(p)H$. Therefore π_p extends to a unital representation $\overline{\pi_p}$ of $M(A)$ onto

$$\overline{\pi_p}(M(A)) = \{S \in B(\overline{\pi}(p)H) : S\pi_p(a), \pi_p S \in \pi(A), \text{ for } a \in A\}.$$

Claim: $\overline{\pi_p}(M(A)) \oplus 0$ is contained in $\overline{\pi}(M(B))$, so we can define:

$$\overline{\phi} : M(A) \to M(B) \text{ by } \overline{\phi}(m) = (\overline{\pi})^{-1} \circ (\overline{\pi_p}(m) \oplus 0). \tag{2.5}$$

If such a map exists,

$$\overline{\phi}(1) = (\overline{\pi})^{-1} \circ (\overline{\pi_p}(1) \oplus 0) = (\overline{\pi})^{-1}(1_{\overline{\pi}(p)H} \oplus 0) = (\overline{\pi})^{-1} \circ \overline{\pi}(p) = p.$$

Proving the claim: suppose $m \in M(A)$ and $b \in B$. Then a long calculation, where the following limits are for strict convergence, and these limits exist as $\overline{\pi_p}$ is extendible:

$$\begin{aligned}(\overline{\pi_p}(m) \oplus 0)\pi(b) &= (\overline{\pi_p}(m) \oplus 0)(\overline{\pi_p}(1) \oplus 0)\pi(b) = (\overline{\pi_p}(m) \oplus 0)(1_{\overline{\pi}(p)H} \oplus 0)\pi(b) \\ &= (\overline{\pi_p}(m) \oplus 0)\overline{\pi}(p)\pi(b) = (\overline{\pi_p}(m) \oplus 0)\pi(pb) \\ &= \lim_j (\overline{\pi_p}(m) \oplus 0)\pi(\phi(a_j)b) = \lim_j (\overline{\pi_p}(m) \oplus 0))(\pi_p(a_j) \oplus 0)\pi(b) \\ &= \lim_j (\pi_p(ma_j) \oplus 0)\pi(b) = \lim_j (\overline{\pi} \circ \phi)(ma_j)\pi(b) \\ &= \lim_j \pi(\phi(ma_j)b)\end{aligned}$$

implies that $(\overline{\pi_p}(m) \oplus 0)\pi(b) \in \pi(B)$ since each $\pi(\phi(ma_j)b) \in \pi(B)$, and $\pi(B)$ is closed as it is the image by a homomorphism on an algebra. A similar argument with multiplication of the left rather than the right shows that $\pi(b)(\overline{\pi_p}(m) \oplus 0) \in \pi(B)$ too, thus proving the claim. The map $\overline{\phi}$ can now be defined by Equation (2.5). Since $\overline{\phi}(m^*) = (\overline{\pi})^{-1} \circ (\overline{\pi_p}(m)^* \oplus 0)$, we have, for $m, n \in M(A)$,

$$\begin{aligned}\overline{\phi}(mn) &= (\overline{\pi})^{-1} \circ (\overline{\pi_p}(mn) \oplus 0) \\ &= (\overline{\pi})^{-1} \circ ((\overline{\pi_p}(m) \oplus 0)(\overline{\pi_p}(n) \oplus 0)) \\ &= ((\overline{\pi})^{-1} \circ (\overline{\pi_p}(m) \oplus 0))((\overline{\pi})^{-1} \circ (\overline{\pi_p}(n) \oplus 0)).\end{aligned}$$

Hence, $\overline{\phi}$ is a $*$-homomorphism. We required it to be strictly continuous: let $\{m_j\}$ be a sequence in $M(A)$ converging to $m \in M(A)$ in the strict topology; hence, $\{m_j\}$ is norm-bounded. We require $\overline{\phi}(m_j) \to \overline{\phi}(m)$ strictly in $M(B)$. Suppose $b \in B$, then

$$\overline{\phi}(m_j)b = \overline{\phi}(m_j 1)b = \overline{\phi}(m_j)\overline{\phi}(1)b = \overline{\phi}(m_j)pb.$$

If we approximate pb by $\phi(a_{i_0})b$ for some i_0 in the sequence, then

$$\begin{aligned}\overline{\phi}(m_j)b = \overline{\phi}(m_j)pb &\sim \overline{\phi}(m_j)\phi(a_{i_0})b = \phi(m_j a_{i_0})b \to \phi(m a_{i_0})b \\ &= \overline{\phi}(m)\phi(a_{i_0})b \sim \overline{\phi}(m)pb = \overline{\phi}(m)b.\end{aligned}$$

A similar argument, multiplying on the left instead of the right, show that $b\overline{\phi}(m_j)$ converges to $b\overline{\phi}(m)$. Therefore, $\overline{\phi}$ is strictly continuous. □

LEMMA 2.11. *Suppose $\phi : A \to M(B)$ is an extendible homomorphism. There is a unique strictly continuous homomorphism $\overline{\phi} : M(A) \to M(B)$ such that $\overline{\phi}|_A = \phi$.*

PROOF. Proposition 2.10 demonstrates the existence $\overline{\phi}$. Suppose $\psi : M(A) \to M(B)$ is another strictly continuous homomorphism extending ϕ. Every $m \in M(A)$ is the strict limit of $\{a_i m\}$ where $\{a_i\}$ is an approximate identity for A. Then

$$\overline{\phi}(m) = \lim_i \overline{\phi}(a_i m) = \lim_i \psi(a_i m) = \psi(m).$$

Hence, $\overline{\phi} = \psi$. □

The following lemma is from [**50**, Lemma 4.2]. It is key to proving our result, as it provides a method of checking when maps are extendible. We follow the proof of [**50**], adding the argument that $\{\phi(a_j)b\}$ is Cauchy.

LEMMA 2.12. *Suppose $\phi : A \to M(B)$ is a homomorphism, and there is a dense sub-$*$-algebra B_0 of B and an approximate identity $\{a_j\}_{j\in J}$ in A such that $\{\phi(a_j)b\}_{j\in J}$ is Cauchy for every $b \in B_0$. Then ϕ is an extendible homomorphism.*

PROOF. Fix $\varepsilon > 0$. Take $b \in B$ and $\{a_j\}$ to be an approximate identity; so in particular $\|a_j\| \leq 1$. As B_0 is a dense sub-$*$-algebra of B, there exists $b_0 \in B_0$ such that $\max\{\|\phi(a_j)\|, \|\phi(a_k)\|\}\|b - b_0\| < \varepsilon/3$. Since $\{\phi(a_j)b_0\}$ is Cauchy, there exists K such that for all $j, k > K$, $\|\phi(a_j)b_0 - \phi(a_k)b_0\| < \varepsilon/3$. So

$$\begin{aligned}\|\phi(a_j)b - \phi(a_k)b\| &\leq \|\phi(a_j)b - \phi(a_j)b_0\| + \|\phi(a_j)b_0 - \phi(a_k)b_0\| + \|\phi(a_k)b_0 - \phi(a_k)b\| \\ &< \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon.\end{aligned}$$

Hence $\{\phi(a_j)b\}$ is a Cauchy sequence for all $b \in B$.

Our aim is to demonstrate the existence of a projection $p_\phi \in M(B)$ such that $\phi(a_j) \to p_\phi$ strictly in $M(B)$. Define maps $L_p, R_p : B \to B$ by $L_p(b) := \lim_j \phi(a_j)b$ and $R_p(b) := \lim_j b\phi(a_j)$. The second map exists because the adjoint of the limit $(\lim_j \phi(a_j)b)^*$ is equal to the limit of the adjoint $\lim_j \phi(a_j)b^*$. Calculating for $b, c \in B$,

$$R_p(b)c = \lim_j b\phi(a_j)c = b \lim_j \phi(a_j)c = bL_p(c).$$

Hence the pair (L_p, R_p) is a multiplier of B and there exists a multiplier $p \in M(B)$ such that $pb = L_p(b)$ and $bp = R_p(b)$. By our definition of p, $\phi(a_j) \to p$ strictly. The set $\{a_j\}$ is an approximate identity, and so each a_j is self-adjoint. Using this, p is also self-adjoint. Using some properties of the limit, for any $a \in A$, we can calculate

$$\phi(a)pb = \lim_j \phi(aa_j)b = \phi(\lim_j aa_j) = \phi(a)b.$$

Using that $\phi(a_j) \to p$ strictly and the previous calculation,

$$\|p^2b - pb\| \leq \|p(pb) - \phi(a_j)(pb)\| + \|\phi(a_j)pb - pb\| \to 0.$$

Hence $p^2 = p$, and $p \in M(B)$ is a projection. Take $p_\phi = p$ and ϕ is extendible. □

Recall that a k-graph morphism is a structure-preserving map; that is, a degree-preserving functor, from Λ, viewed as a category, into another k-graph Σ. An endomorphism of a k-graph is a k-graph morphism of Λ into itself, and the set $\mathrm{End}(\Lambda)$ is a semigroup under composition.

DEFINITION 2.13. Allen [**3**] gave the following definition of saturation of a k-graph morphism: given two k-graphs Λ_1, Λ_2, with $X \subseteq \Lambda_1^0$, a k-graph morphism $\phi : \Lambda_1 \to \Lambda_2$ is *saturated* with respect to X if $\phi : X\Lambda_1 \to \phi(X)\Lambda_2$ is a bijection. If $X = \Lambda_1^0$ and Λ_1 is a sub-k-graph of Λ_2, then the condition is that $\phi : \Lambda_1 \to \phi(\Lambda_1^0)\Lambda_2$ is a bijection. This is equivalent to $r(\sigma) \in \pi(\Lambda_1^0) \Rightarrow \sigma \in \pi(\Lambda_1)$.

In [**50**], Pask, Raeburn and Yeend defined saturation of subgraphs of directed graphs instead: a subgraph E is saturated in F if for each vertex $v \in E^0$, either all edges $e \in F^1$ with $s(e) = v$ lie in E^1 or none do; that is, if $s^{-1}(s(E^1)) = E^1$. There is an immediate connection between the two definitions. Both definitions explain when a map between directed graphs (or k-graphs) can be extended to a map between their associated k-graph C^*-algebras.

PROPOSITION 2.14. *Suppose that $\pi : \Lambda \to \Sigma$ is an injective saturated k-graph morphism between row-finite graphs with no sources. Then there is a homomorphism $\pi_* : C^*(\Lambda) \to C^*(\Sigma)$ such that $\pi_*(s_\lambda^\Lambda) = s_{\pi(\lambda)}^\Sigma$, and π_* is injective and extendible with $\overline{\pi_*}(1) = p_{\pi(\Lambda^0)}$. The assignment $\pi \mapsto \pi_*$ is functorial in the sense that $(\pi \circ \tau)_* = \pi_* \circ \tau_*$.*

PROOF. The saturation of $\pi : \Lambda \to \Sigma$ means that $\{\sigma \in \Sigma : r(\sigma) = \pi(v)\} = \{\pi(\lambda) : r(\lambda) = v\}$ for every $v \in \Lambda^0$, so the Cuntz-Krieger relation (CK4) in Σ implies the analogous relation for the family $\{s_{\pi(\lambda)}^\Sigma : \lambda \in \Lambda\}$. Thus $\{s_{\pi(\lambda)}^\Sigma\}$ is a Cuntz-Krieger Λ-family, and there is a homomorphism π_* satisfying $\pi_*(s_\lambda^\Lambda) = s_{\pi(\lambda)}^\Sigma$. Since π is injective and every projection $s_w^\Sigma \neq 0$, the gauge-invariant uniqueness theorem [**36**, Theorem 3.4] implies that π_* is faithful.

To see that π_* is extendible, write $\Lambda^0 = \bigcup_n F_n$ as an increasing union of finite sets. Then $p_n := \sum_{v \in F_n} s_v^\Lambda$ is an approximate identity for $C^*(\Lambda)$. The images $\pi(F_n)$ satisfy $\bigcup_n \pi(F_n) = \pi(\Lambda^0)$, and since π is injective,

$$\pi(p_n) = \sum_{v \in F_n} p_{\pi(v)}^\Sigma = \sum_{w \in \pi(F_n)} p_w^\Sigma,$$

which by Proposition 2.8 converge strictly to $p_{\pi(\Lambda^0)}$. Thus π_* is extendible with $\overline{\pi_*}(1) = p_{\pi(\Lambda^0)}$. The functoriality follows from the formula $\pi_*(s_\lambda^\Lambda) = s_{\pi(\lambda)}^\Sigma$: suppose $\tau : \Delta \to \Lambda$ is an injective saturated k-graph morphism. Then

$$\pi_* \circ \tau_*(s_\delta^\Delta) = \pi_*(s_{\tau(\delta)}^\Lambda) = s_{\pi(\tau(\delta))}^\Sigma \text{ and } (\pi \circ \tau)_*(s_\delta^\Delta) = s_{(\pi\circ\tau)(\delta)}^\Sigma. \qquad \square$$

CHAPTER 3

A Gross-Tucker theorem

We consider free actions of an Ore semigroup S on a k-graph Σ and show in Proposition 3.5 that the Ore condition allows us to define a quotient object $S\backslash\Sigma$ that is also a k-graph, generalising remarking from [**50**]. We define a skew-product k-graph using a functor η from a k-graph into an Ore semigroup. We show that there is an action of the semigroup by left-translation on $\Sigma \times_\eta S$. Moreover, this is how all free actions of Ore semigroups on skew-product graphs occur. In particular, in Theorem 3.11 we construct an isomorphism from Σ to $S\backslash\Sigma \times_\eta S$, thereby providing a generalisation of the Gross-Tucker theorem. By doing so, we prove that every free left action of an Ore semigroup S on a k-graph Σ is equivariantly isomorphic to an action by left-translation on $S\backslash\Sigma \times_\eta S$.

As in [**50**], not every graph can be realised as a skew-product graph: we need to insist on a technical condition that the action admits a fundamental domain. We then in Corollary 3.12 induce an isomorphism between the associated k-graph C^*-algebras of Σ and $S\backslash\Sigma \times_\eta S$. We can then define the isomorphism ψ_1 in the diagram 1.1 from $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*(S\backslash\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$.

We consider some examples before providing a classification of fundamental domains for Ore semigroup actions on k-graphs in Theorem 3.14. We show that cohomology class of the functor $\eta : S\backslash\Sigma \to S$ from Theorem 3.11 is uniquely determined by the action α, and we can classify the free actions of S on a k-graph up to equivariant isomorphism.

3.1. Ore semigroups and actions

A *semigroup* $(S, \cdot_S)$ consists of a set S, and an associative, closed, binary operation $\cdot_S : S \times S \to S$. Every semigroup is assumed to be countable, with an identity 1_S. We say that S is *cancellative* if for all $a, b \in S$, $a \cdot_S c = b \cdot_S c \Rightarrow a = b$ and $c \cdot_S a = c \cdot_S b \Rightarrow a = b$. A cancellative semigroup is *Ore* if it is right-reversible, in the sense that $S \cdot_S s \cap S \cdot_S t \neq \emptyset$ for all $s, t \in S$. There is a corresponding notion of left-reversibility as well: if $s \cdot_S S \cap t \cdot_S S \neq \emptyset$ for all $s, t \in S$. If the context is clear, we shall omit the symbol $\cdot_S$. Ore [**44**], and later Dubreil [**16, 17**] (see [**41**], for example, for a proof) demonstrated that a semigroup is Ore if and only if it can be embedded in a group Γ such that $\Gamma = S^{-1}S$; the group Γ is unique up to isomorphism, and we call it the *enveloping group* of S.

EXAMPLES 3.1. (Ore semigroups)

(1) Any abelian semigroup S is automatically left- and right-reversible since $st = ts \in Ss \cap St$ and in $sS \cap tS$ for all $s, t \in S$. Every group G is a left- and right-reversible semigroup since for all $s \in G$ we have $Gs = G = sG$.
(2) Let $\mathbb{N}$ denote the semigroup of natural numbers under addition and $\mathbb{N}^{\times}$ denote the semigroup of nonzero natural numbers under multiplication. Let $S = \mathbb{N} \times \mathbb{N}^{\times}$ be gifted with the associative binary operation $\star$ given by

$$(m_1, n_1) \star (m_2, n_2) = (m_1 + n_1 m_2, n_1 n_2).$$

One can easily check that S is a non-abelian right-reversible semigroup. It is, however, not left-reversible: for example, $(m, n)S \cap (p, q)S = \emptyset$ when $n = q = 0$ and $m \neq p$.
(3) The free semigroup $\mathbb{F}_n^+$ on $n \geq 2$ generators is not an Ore semigroup since for all $s, t \in \mathbb{F}_n^+$ with $s \neq t$ we have $\mathbb{F}_n^+ s \cap \mathbb{F}_n^+ t = \emptyset$ as there is no cancelation, and so the right-reversibility of the Ore condition is not satisfied. Similarly, the lack of cancelation means $\mathbb{F}_n^+$ cannot be left-reversible either.

An *action* of a semigroup S on a k-graph Λ is a homomorphism $\alpha : S \to \mathrm{End}(\Lambda)$ such that $\alpha(1_S)$ is the identity map. Suppose that α is a left action of an Ore semigroup S on a k-graph Σ, and that α is *free* in the sense that $\alpha_t(\lambda) = \alpha_u(\lambda)$ implies $t = u$ for all $\lambda \in \Sigma$. We note that since S acts by k-graph morphisms, it suffices to check freeness on vertices.

Examples 3.2. (S-actions on k-graphs)

(1) Recall the k-graph Ω_k from Examples 2.1. There is a free action of $\mathbb{N}^k$ on Ω_k by translation. For $m \in \mathbb{N}^k$ and $(p, q) \in \Omega_k$, define $\alpha_m(p, q) = (p+m, q+m)$. This action is free. We note that this action is not invertible, as $\mathbb{N}^k \backslash \{0\}$ has no additive inverses.
(2) Recall the k-graph Δ_k from Examples 2.1. There is a free action of $\mathbb{N}^k$ on Δ_k by translation as in (1) above. However, this action can be extended to an action β of $\mathbb{Z}^k$ in an obvious way.
(3) Consider a $\mathbb{Z}^k \times \mathbb{N}^l$-action by translation on either the $(k+l)$-graph Δ_{k+l} or on the $(k+l)$-graph $\Delta_k \times \Omega_l$ which is defined for $\lambda \in \Delta_k$, $\mu \in \Omega_k$, $m \in \mathbb{Z}^k$, $n \in \mathbb{N}^k$ by $\delta_{(m,n)}(\lambda, \mu) = (\beta_m(\lambda), \alpha_n(\mu))$. This action is free. On Δ_{k+l} the action is extendible to an action of $\mathbb{Z}^{k+l}$, but not on $\Delta_k \times \Omega_l$.
(4) We considered L_r, the path category consisting of a loop with r vertices, in Examples 2.1. There is a natural action of $\mathbb{N}$ that permutes the edges. Defining the action $\alpha_j(e_i) = e_{i+j(mod\, r)}$, one can see that such an action is not free. This example can be extended to an action of $\mathbb{N}^k$ on $L_r \times \ldots \times L_r$.
(5) We considered B_r, the path category consisting of a single vertex with r loops, in Examples 2.1. There is a natural action of the cyclic group

of order r, C_r, on B_r that permutes the edges. Defining this action by $\alpha_j(f_i) = f_{i+j (mod\, r)}$, one can see that the action is not free.

(6) Any action of an infinite semigroup on a finite k-graph must be non-free.

3.2. Skew-product k-graphs and quotient k-graphs

Suppose that Λ is a k-graph and $\eta : \Lambda \to S$ is a functor into a semigroup S. The semigroup S can be viewed as a category with one object, 1_S, and with morphisms $s \in S$. As in [**36**, Definition 5.1], we can make the set-theoretic product $\Lambda \times S$ into a k-graph $\Lambda \times_\eta S$. When S is a group, that $\Lambda \times_\eta S$ is the skew-product k-graph has been presented previously in the literature, but not with extensive details, or at this level of generalisation.

Proposition 3.3. *Suppose (Λ, d) is a k-graph, S is a semigroup, and $\eta : \Lambda \to S$ is a functor. Then $((\Lambda \times_\eta S)^0, (\Lambda \times_\eta S)^*, r, s, d_{\Lambda \times_\eta S})$ is a k-graph with $(\Lambda \times_\eta S)^0 = \Lambda^0 \times S$, $(\Lambda \times_\eta S)^* = \Lambda^* \times S$, range and source maps $r, s : \Lambda \times_\eta S \to (\Lambda \times_\eta S)^0$ defined by*

$$r(\lambda, t) = (r(\lambda), t) \quad \textit{and} \quad s(\lambda, t) = (s(\lambda), t\eta(\lambda)),$$

defining the composition by

$$(\lambda, t)(\mu, u) = (\lambda\mu, t) \quad \textit{when } s(\lambda, t) = r(\mu, u) \textit{ (which is equivalent to } u = t\eta(\lambda) \textit{)},$$

and defining $d_{\Lambda \times_\eta S} : \Lambda \times_\eta S \to \mathbb{N}^k$ by $d_{\Lambda \times_\eta S}(\lambda, t) = d_\Lambda(\lambda)$.

Proof. We verify that $\Lambda \times_\eta S$ satisfies the axioms of a category: take $(\lambda, t), (\mu, u) \in \Lambda \times_\eta S$ to be composable pairs such that $s(\lambda, t) = r(\mu, u)$. By our definitions of r, s this is equivalent to $s(\lambda) = r(\mu)$ and $u = t\eta(\lambda)$. If so:

$$\begin{aligned} r((\lambda, t)(\mu, t\eta(\lambda))) &= r(\lambda\mu, t) = (r(\lambda\mu), t) = (r(\lambda), t) = r(\lambda, t) \\ s((\lambda, t)(\mu, t\eta(\lambda))) &= s(\lambda\mu, t) = (s(\lambda\mu), t\eta(\lambda\mu)) \\ &= (s(\mu), t\eta(\lambda)\eta(\mu)) = s(\mu, t\eta(\lambda)). \end{aligned}$$

Take $(\lambda, t), (\mu, u), (\nu, v) \in \Lambda \times_\eta S$ such that $s(\lambda, t) = r(\mu, u)$ and $s(\mu, u) = r(\nu, v)$. Therefore $s(\lambda) = r(\mu)$ and $s(\mu) = r(\nu)$, and $u = t\eta(\lambda)$ and $v = u\eta(\mu) = t\eta(\lambda)\eta(\mu)$. Then:

$$\begin{aligned} [(\lambda, t)(\mu, t\eta(\lambda))](\nu, t\eta(\lambda)\eta(\mu)) &= (\lambda\mu, t)(\nu, t\eta(\lambda\mu)) = (\lambda\mu\nu, t), \\ (\lambda, t)[(\mu, t\eta(\lambda))(\nu, t\eta(\lambda)\eta(\mu))] &= (\lambda, t)(\mu\nu, t\eta(\lambda)) = (\lambda\mu\nu, t); \end{aligned}$$

hence, composition is associative.

The identity morphisms of $\Lambda \times_\eta S$ are $\{\iota_{v,t} := (\iota_v, t) : \iota_v \in \Lambda, t \in S\}$. Therefore

$$\begin{aligned} &r(\iota_v, t) = (r(\iota_v), t) = (v, t), \text{ and} \\ &s(\iota_v, t) = (s(\iota_v), t\eta(\iota_v)) = (v, t), \text{ since } \eta(\iota_v) = 1_S. \end{aligned}$$

Suppose $s(\iota_v, t) = r(\lambda, u)$ and $s(\mu, x) = r(\iota_w, y)$. Then $u = t\eta(\iota_v) = t$ and $y = x\eta(\mu)$, and we have that

$$(\iota_v, t)(\lambda, t\eta(\iota_v)) = (\iota_v\lambda, t) = (\lambda, t)$$
$$(\mu, x)(\iota_w, x\eta(\mu)) = (\mu\iota_v, x) = (\mu, x).$$

Hence, $\Lambda \times_\eta S$ satisfies the axioms of a category.

Showing that $\Lambda \times_\eta S$ is a k-graph: it is straightforward to show $d_{\Lambda\times_\eta S}$ is a functor: take (λ, t) composable with (μ, u), then as the degree map d_Λ of Λ is a functor,

$$d_{\Lambda\times_\eta S}((\lambda, t)(\mu, u)) = d_{\Lambda\times_\eta S}(\lambda\mu, t) = d_\Lambda(\lambda\mu), \text{ and}$$
$$d_{\Lambda\times_\eta S}(\lambda, t)d_{\Lambda\times_\eta S}(\mu, u)) = d_\Lambda(\lambda)d_\Lambda(\mu) = d_\Lambda(\lambda\mu).$$

Checking the k-graph factorisation property: suppose $d_{\Lambda\times_\eta S}(\lambda, t) = m + n$, then by definition $d_\Lambda(\lambda) = m+n$ as well. Since Λ is a k-graph, by the factorisation property of Λ there exist unique $\mu, \nu \in \Lambda$ such that $\lambda = \mu\nu$, $d(\mu) = m$ and $d(\nu) = n$. Then $d_{\Lambda\times_\eta S}(\mu, t) = m$ and $d_{\Lambda\times_\eta S}(\nu, t\eta(\mu)) = n$, and (μ, t) and $(\nu, t\eta(\mu))$ are morphisms in $\Lambda \times_\eta S$ such that $(\lambda, t) = (\mu, t)(\nu, t\eta(\mu))$. To see that they are unique, we need only consider the S-coordinate, as μ and ν are uniquely determined. However, the definition of composition in $\Lambda \times_\eta S$ requires the S-coordinates to be t and $t\eta(\mu)$ respectively. Therefore the category $\Lambda \times_\eta S$ has the k-graph factorisation property, and we can conclude that $(\Lambda \times_\eta S, d_{\Lambda\times_\eta S})$ is a k-graph. □

LEMMA 3.4. *There exists a natural free left action of an Ore semigroup S on the skew-product k-graph $\Lambda \times_\eta S$ by left-translation, defined for $\lambda \in \Lambda$, $s, t \in S$ by* $\mathrm{lt}_t(\lambda, s) = (\lambda, ts)$.

PROOF. It is straightforward to verify that lt is a free action: take $t, u \in S$ and suppose $\mathrm{lt}_t(\lambda, s) = \mathrm{lt}_u(\lambda, s)$. Then $(\lambda, ts) = (\lambda, us)$, and as S is assumed to be cancellative, we have $t = u$, showing that lt is a free action. □

Our goal is to show that if a free left action α of an Ore semigroup S on a k-graph Σ admits a fundamental domain (to be defined in Definition 3.8), then there exists an isomorphism of Σ onto a skew-product k-graph that carries α into a canonical action of S by left translation. Such results were first proved for actions of groups on directed graphs by Gross and Tucker (see, for example, [**25**, Theorem 2.2.2]). When S is a group, a version of Theorem 3.11 below was proved by Kumjian and Pask [**36**, Remark 5.6].

Even the first step, which is the construction of the quotient graph, relies on the Ore property of S. Given an action of an Ore semigroup S on a k-graph Σ, we define a relation $\sim$ of S on Σ by

$$\lambda \sim \mu \iff \text{ if there exist } t, u \in S \text{ such that } \alpha_t(\lambda) = \alpha_u(\mu).$$

The relation $\sim$ is trivially reflexive and symmetric. To see that it is transitive, suppose $\lambda \sim \mu$ and $\mu \sim \nu$, so that there exist $s, t, u, v \in S$ such that $\alpha_s(\lambda) = \alpha_t(\mu)$ and $\alpha_u(\mu) = \alpha_v(\nu)$. Since S is Ore it is right-reversible, and so there exist $x, y \in S$ such that $xt = yu$. Then $\alpha_{xs}(\lambda) = \alpha_{xt}(\mu) = \alpha_{yu}(\mu) = \alpha_{yv}(\nu)$, which implies that $\lambda \sim \nu$. Thus $\sim$ is an equivalence relation on Σ. Since equivalent elements have the same degree, it makes sense to write $(S\backslash\Sigma)^0$ for the set of equivalence classes of vertices, $S\backslash\Sigma$ for the set of all equivalence classes, and to define $d : S\backslash\Sigma \to \mathbb{N}^k$ by $d_{S\backslash\Sigma}([\lambda]) = d_\Sigma(\lambda)$. We show that it is straightforward to check that there are well-defined maps $r, s : S\backslash\Sigma \to (S\backslash\Sigma)^0$ such that $r([\lambda]) = [r(\lambda)]$ and $s([\lambda]) = [s(\lambda)]$: suppose $[\lambda] = [\mu]$, so there exist $t, u \in S$ such that $\alpha_t(\lambda) = \alpha_u(\mu)$. Since $\alpha : S \to \mathrm{End}(\Lambda)$, each α_t is an endomorphism on Σ, so:

$$r(\alpha_t(\lambda)) = r(\alpha_u(\mu)) \Longrightarrow \alpha_t(r(\lambda)) = \alpha_u(r(\mu)) \Longrightarrow [r(\lambda)] = [r(\mu)].$$

A similar argument shows $s([\lambda]) = [s(\lambda)]$ to be well-defined.

For the following result, when S is a group, that $S\backslash\Sigma$ is the quotient k-graph has been presented previously in the literature, but not with extensive details, or at this level of generalisation.

PROPOSITION 3.5. *Suppose Σ is a k-graph, S is an Ore semigroup, and α is a free left-action of S of Σ. Then $((S\backslash\Sigma)^0, S\backslash\Sigma, r, s, d_{S\backslash\Sigma})$ is a k-graph, with composition defined by*

$$[\lambda][\mu] = [\alpha_t(\lambda)\alpha_u(\mu)] \text{ where } t, u \in S \text{ satisfy } \alpha_t(s(\lambda)) = \alpha_u(r(\mu)), \tag{3.1}$$

and $q : \lambda \mapsto [\lambda]$ is a k-graph morphism.

PROOF. To verify that $S\backslash\Sigma$ is a k-graph, we first check that the right-hand side of equation (3.1) is independent of the choice of t and u: suppose $[s(\lambda)] = [r(\mu)]$, and there exist $p, q, u, v \in S$ such that $\alpha_p(s(\lambda)) = \alpha_q(r(\mu))$ and $\alpha_u(s(\lambda)) = \alpha_v(r(\mu))$. The Ore semigroup S is right-reversible, so there exist $s, t \in S$ such that $sp = tu$. So $\alpha_{sp}(s(\lambda)) = \alpha_{sq}(r(\mu))$ and $\alpha_{tu}(s(\lambda)) = \alpha_{tv}(r(\mu))$, and since α is a free action, this forces $sq = tv$. Therefore:

$$\alpha_s(\alpha_p(\lambda)\alpha_q(\mu)) = \alpha_{sp}(\lambda)\alpha_{sq}(\mu) = \alpha_{tu}(\lambda)\alpha_{tv}(\mu) = \alpha_t(\alpha_u(\lambda)\alpha_v(\mu)).$$

Therefore $\alpha_p(\lambda)\alpha_q(\mu) \sim \alpha_u(\lambda)\alpha_v(\mu)$ as required.

We check the right-hand side of (3.1) is independent of the choice of coset representatives: if $[\lambda] = [\nu]$, then we want $[\lambda][\mu] = [\nu][\mu]$. Then there exist $c, d \in S$ such that $\alpha_c(\nu) = \alpha_d(\lambda)$. That λ is composable with μ implies $s([\lambda]) = r([\mu])$, so there exist $p, q \in S$ such that $\alpha_p(s(\lambda)) = \alpha_q(r(\mu))$. The Ore semigroup S is right-reversible so there exist $t, u \in S$ such that $td = up$, then

$$\alpha_{tc}(s(\nu)) = \alpha_{td}(s(\lambda)) = \alpha_{up}(s(\lambda)) = \alpha_{uq}(r(\mu)).$$

Therefore $[\nu]$ is composable with $[\mu]$ and

$$\alpha_{tc}(\nu)\alpha_{uq}(\mu) = \alpha_{td}(\lambda)\alpha_{uq}(\mu) = \alpha_{up}(\lambda)\alpha_{uq}(\mu) = \alpha_u(\alpha_p(\lambda)\alpha_q(\mu)),$$

which implies $\alpha_{tc}(\nu)\alpha_{uq}(\mu) \sim \alpha_p(\lambda)\alpha_q(\mu)$, as required.

Checking that our construction is a category: we first check that $r([\lambda][\mu]) = r([\lambda])$ and $s([\lambda][\mu]) = s([\mu])$:

$$r([\lambda][\mu]) = r([\alpha_s(\lambda)\alpha_t(\mu)]) = [r(\alpha_s(\lambda)\alpha_t(\mu))] = [r(\alpha_s(\lambda))] = r([\lambda])$$
$$s([\lambda][\mu]) = s([\alpha_t(\lambda)\alpha_u(\mu)]) = [s(\alpha_t(\lambda)\alpha_u(\mu))] = [s(\alpha_u(\mu))] = s([\mu]).$$

We check associativity of composition: suppose $[s(\lambda)] = [r(\mu)]$ and $[s(\mu)] = [r(\nu)]$, so there exist $t, u, v, w \in S$ such that $\alpha_t(s(\lambda)) = \alpha_u(r(\mu))$ and $\alpha_v(s(\mu)) = \alpha_w(r(\nu))$. The right-reversibility of the Ore semigroup S implies there exist $a, b \in S$ such that $au = bv$. Then $\alpha_{au}(s(\mu)) = \alpha_{bv}(s(\mu)) = \alpha_{bw}(r(\nu))$ and $\alpha_{bv}(r(\mu)) = \alpha_{au}(r(\mu)) = \alpha_{at}(s(\lambda))$, and so:

$$\begin{aligned}([\lambda][\mu])[\nu] &= [\alpha_t(\lambda)\alpha_u(\mu)][\nu] = [\alpha_{at}(\lambda)\alpha_{au}(\mu)][\nu]\\ &= [\alpha_{at}(\lambda)\alpha_{bv}(\mu)\alpha_{bw}(\nu)]\\ [\lambda]([\mu][\nu]) &= [\lambda][\alpha_v(\mu)\alpha_w(\nu)] = [\lambda][\alpha_{bv}(\mu)\alpha_{bw}(\nu)]\\ &= [\alpha_{at}(\lambda)\alpha_{au}(\mu)\alpha_{bw}(\nu)].\end{aligned}$$

We check that the map from the objects of $S\backslash\Sigma$ to the identity morphisms $\iota : [v] \mapsto [\iota_v]$ is well-defined: suppose $[v] = [w]$, then there exist $s, t \in S$ such that $\alpha_s(v) = \alpha_t(w)$. The action α is a homomorphism onto $\mathrm{End}(\Sigma)$, hence $\iota(\alpha_s(v)) = \iota(\alpha_t(w)) \Rightarrow [\iota_v] = [\iota_w]$. By the definition of the range and source maps, $r([\iota_v]) = [r(\iota_v)] = [v]$ and $s([\iota_v]) = [s(\iota_v)] = v$. Suppose $r([\lambda]) = [v]$: for any $s, t \in S$ such that $\alpha_s(s(\iota_v)) = \alpha_s(v) = [v] = \alpha_t(r(\lambda))$,

$$[\iota_v][\lambda] = [\alpha_s(\iota_v)\alpha_t(\lambda)] = [\iota_v\alpha_t(\lambda)] = [\alpha_t(\lambda)] = [\lambda].$$

Similarly, if $s([\lambda]) = [v]$, $[\lambda][\iota_v] = [\lambda]$. We can conclude that $S\backslash\Sigma$ is a category.

We check that the degree map is well-defined: suppose $[\lambda] = [\mu]$, so there exist $s, t \in S$ such that $\alpha_s(\lambda) = \alpha_t(\mu)$; α_s is an endomorphism of a k-graph, so it preserves degree and so $d(\alpha_s(\lambda)) = d(\alpha_t(\mu))$ implies $d(\lambda) = d(\mu)$.

We check the factorisation property for k-graphs: suppose $[\lambda] \in S\backslash\Sigma$ and $m, n \in \mathbb{N}^k$ such that $d([\lambda]) = m + n$. Then by definition $d(\lambda) = m + n$. Since Σ is a k-graph there exist unique $\mu \in \Sigma^m$ and $\nu \in \Sigma^n$ such that $\lambda = \mu\nu$. Then $d([\mu]) = m$, $d([\nu]) = n$ and $[\lambda] = [\mu][\nu]$. To show uniqueness of $[\mu]$ and $[\nu]$, suppose $d([\xi]) = m$, $d([\zeta]) = n$ and $[\lambda] = [\xi][\zeta]$. Then $d(\xi) = m$ and $d(\zeta) = n$, and there exist $\xi' \in [\xi]$ and $\zeta' \in [\zeta]$ such that $s(\xi') = r(\zeta')$. Then $[\xi][\zeta] = [\xi'\zeta'] = [\lambda] = [\mu\nu]$. So there exists $t, u \in S$ such that $\alpha_t(\xi'\zeta') = \alpha_u(\lambda)$. Then $\alpha_t(\xi')\alpha_t(\zeta') = \alpha_u(\mu)\alpha_u(\nu)$ and the uniqueness of the factorisation implies that $\alpha_t(\xi') = \alpha_u(\mu)$ and $\alpha_t(\zeta') = \alpha_u(\nu)$.

Therefore $[\xi] = [\xi'] = [\mu]$ and $[\zeta] = [\zeta'] = [\nu]$. Hence $((S\backslash\Sigma)^0, S\backslash\Sigma, r, s, d_{S\backslash\Sigma})$ is a k-graph.

Finally, if λ and μ are composable, we can take $t = u = 1$ in (3.1), and deduce that $q(\lambda\mu) = q(\lambda)q(\mu)$. $\square$

EXAMPLES 3.6. (Quotient k-graphs)

(1) In Examples 3.2(1), we saw there was an action of $\mathbb{N}^k$ on Ω_k by left-translation. We can form the quotient k-graph $\mathbb{N}^k\backslash\Omega_k$. It can be shown that $\mathbb{N}^k\backslash\Omega_k$ is isomorphic to T^k.
(2) In Examples 3.2(3), there was defined an action of $\mathbb{Z}^k \times \mathbb{N}^l$ by translation on either the $(k+l)$-graph Δ_{k+l} or on the $(k+l)$-graph $\Delta_k \times \Omega_l$. In both cases, we get a quotient k-graph $T^k \times T^l \cong T^{k+l}$.
(3) In Examples 3.2(4), for the path category L_r, we defined an action of $\mathbb{N}$ that permuted the edges. Even though this action is not free, we can still find the quotient k-graph: $\mathbb{N}\backslash L_r = T^1$. For the path B_r, we defined an action of C_r that permuted the edges. The quotient k-graph by this action will also be T^1. Since neither action is free, we will not be able to apply our generalisation of the Gross-Tucker theorem, and hence these graphs cannot be written as a skew-product of the quotient k-graph.

3.3. A generalisation of the Gross-Tucker theorem

We claim that the quotient of a skew-product k-graph $\Lambda \times_\eta S$ by the action by left-translation of S is isomorphic to Λ.

PROPOSITION 3.7. *Suppose Λ is a k-graph, S is an Ore semigroup, and $\eta : \Lambda \to S$ is a functor. Then $\tau : S\backslash(\Lambda \times_\eta S) \to \Lambda$ defined $\tau([(\lambda, s)]) = \lambda$ is an isomorphism.*

PROOF. Take two elements of $\Lambda \times_\eta S$, $(\lambda, s), (\mu, t)$. They are in the same equivalence class if there exists $u, v \in S$ such that $\mathrm{lt}_u(\lambda, s) = \mathrm{lt}_v(\mu, t)$; that is, $\lambda = \mu$ and $us = vt$. Since S is right-reversible, $Ss \cap St \neq \emptyset$ for all $s, t \in S$. Hence $(\lambda, s) \in [(\lambda, t)]$ for all $s, t \in S$. Therefore $\tau : [(\lambda, s)] \mapsto \lambda$ is well-defined.

The map τ is injective since $\tau([(\lambda, s)]) = \tau([(\mu, t)])$ only if $\lambda = \mu$, in which case $[(\lambda, s)] = [(\mu, t)]$, as the equivalence classes are independent of the S-coordinate. The map τ is clearly surjective as for all $\lambda \in \Lambda$, there exists $[(\lambda, s)] \in S\backslash(\Lambda \times_\eta S)$ such that $\tau([(\lambda, s)]) = \lambda$. $\square$

The converse of this result is a generalisation of the Gross-Tucker theorem, [**25**, Theorem 2.2.2]. The version of Gross-Tucker theorem implicit in [**36**, Remark 5.6] says that every free action of a group Γ on a k-graph Σ is isomorphic to the action by left-translation of Γ, lt, on a skew-product $(\Gamma\backslash\Sigma) \times_\eta \Gamma$. We wish to generalise the Gross-Tucker theorem to actions of Ore semigroups. To do so, as in [**50**], one has to insist that the action admits a fundamental domain.

DEFINITION 3.8. A *fundamental domain* for a free action α of S is a subset F of Σ such that for every $\sigma \in \Sigma$ there are exactly one $\mu \in F$ and one $t \in S$ such that $\alpha_t(\mu) = \sigma$, and such that $r(\mu) \in F$ for every $\mu \in F$.

REMARK 3.9.
(1) For a k-graph Λ, an Ore semigroup S, and a functor $\eta : \Lambda \to S$, the skew-product graph $\Lambda \times_\eta S$ will always have a fundamental domain for lt; namely $\{(\lambda, 1_S) : \lambda \in \Lambda\}$. It will be shown later as Lemma 3.15 that $\Lambda \times_\eta S$ will have a fundamental domain for each invertible element of S.
(2) Suppose F is a fundamental domain for a free action α of an Ore semigroup S on a k-graph Σ, and $s, t \in S$. Then for $\lambda, \mu \in F$, if $\alpha_s(\lambda) = \alpha_t(\mu)$, then $s = t$ and $\lambda = \mu$.
(3) The action of a group Γ on a k-graph automatically admits a fundamental domain. We will return to this example later in the chapter.

The definition of a fundamental domain here is tailored for use in higher-rank graph examples. It gives a fundamental domain in a 1-graph that is compatible with the definition in [**50**].

EXAMPLES 3.10 (Fundamental Domains). We consider the k-graphs and actions from Examples 3.2, and give descriptions of some of the fundamental domains if they exist.

(1) For the $\mathbb{N}^k$-action by translation on Ω_k, the fundamental domain F has $F^0 = \{0 \in \mathbb{N}^k\}$ and $F^* = \{\sigma \in \Omega_k : r(\sigma) = 0\}$. Then for $\mu \in \Omega_k$ with $d(\mu) = n$, $\sigma = (0, n) \in F^*$ such that $\alpha_{d(\mu)}(0, n) = \mu$. Alternately, the fundamental domain can be described as $F = \{(0, m) : m \in \mathbb{N}^k\}$.
(2) The action of $\mathbb{N}^k$ by translation on Δ_k does not have a fundamental domain. We prove this by contradiction: suppose there exists a fundamental domain F, and so F^0 is non-empty. Take $a \in \mathbb{N}^k$ to be the element such that for all $b \in F^0$, $a \le b$. Take $\sigma \in \Delta_k$ such that $r(\sigma) < a$. Then since the action by $\mathbb{N}^k$ has no invertibles, there does not exist $t \in \mathbb{N}^k$ such that $\alpha_t(\mu) = \sigma$ for any $\mu \in F^*$.
(3) Recall Example 3.2(3). There is an action of $\mathbb{Z} \times \mathbb{N}$ by translation acting on $\Delta_1 \times \Omega_1$. For all $x \in \mathbb{Z}$, we can define a fundamental domain F_x such that $F_x^0 = \{(x, n) \in \mathbb{Z} \times \mathbb{N}\}$ and $F_x^0 = \{\sigma \in \Delta_1 \times \Omega_1 : r(\sigma) \in F_x^0\}$.

An immediate question is whether it is possible to take the entire k-graph to be a fundamental domain for some free left action of an Ore semigroup. The answer is no: for contradiction suppose so, and let $\mu = \alpha_s(\lambda)$ for $s \neq 1_S$. Then $\mu = \alpha_{1_S}(\mu)$, contradicting either that α is free or that Σ is a fundamental domain, as there is no uniqueness. Hence if S has more than a single element, Σ cannot be its own fundamental domain.

We can now state and prove our generalisation of [**25**, Theorem 2.2.2]:

THEOREM 3.11. *Suppose that* Σ *is a row-finite* k*-graph with no sources, and* α *is a free left action of an Ore semigroup* S *on* Σ *that admits a fundamental domain* F*. Let* $q : \Sigma \to S\backslash\Sigma$ *be the quotient map, and define* $c : S\backslash\Sigma \to F$*,* $\eta : S\backslash\Sigma \to S$ *and* $\xi : \Sigma \to S$ *by*

$$q(c(\lambda)) = \lambda, \quad s(c(\lambda)) = \alpha_{\eta(\lambda)}(c(s(\lambda))) \quad \text{and} \quad \sigma = \alpha_{\xi(\sigma)}(c(q(\sigma))). \tag{3.2}$$

Then $\eta : S\backslash\Sigma \to S$ *is a functor, and the map* $\psi(\lambda, t) = \alpha_t(c(\lambda))$ *is an isomorphism of the skew-product* $(S\backslash\Sigma) \times_\eta S$ *onto* Σ*, with inverse given by* $\phi := \psi^{-1}$ *defined* $\phi(\sigma) := (q(\sigma), \xi(\sigma))$*. The isomorphism* ψ *satisfies* $\psi \circ \mathrm{lt}_t = \alpha_t \circ \psi$ *for all* $t \in S$*.*

PROOF. Take $\lambda \in S\backslash\Sigma$. Since the action α of S on Σ admits a fundamental domain, we can choose the unique $\mu \in F$ such that $q(\mu) = \lambda$. Then $\mu = c(\lambda)$ with $r(\mu) \in F$ and $q(s(\mu)) = s(\lambda)$. For $s(\lambda)$, there exists a unique $v \in F$ such that $q(v) = s(\lambda)$. The definition of $\eta : S\backslash\Sigma \to S : \lambda \mapsto \eta(\lambda)$ such that $s(\lambda) = \alpha_{\eta(\lambda)}(v)$, is uniquely specified by the freeness of the action.

Suppose $s(\lambda_1) = r(\lambda_2)$. Take $\mu_i := c(\lambda_i)$ such that $q(\mu_i) = \lambda_i$. Then using that $c(r(\lambda_i)) = r(c(\lambda_i))$,

$$\begin{aligned} s(\lambda_1) = r(\lambda_2) \Rightarrow c(s(\lambda_1)) = c(r(\lambda_2)) &\Rightarrow \alpha_{\eta(\lambda_1)}(c(s(\lambda_1))) = \alpha_{\eta(\lambda_1)}(c(r(\lambda_2))) \\ &\Rightarrow s(\mu_1) = r(\alpha_{\eta(\lambda_1)}(\mu_2)). \end{aligned}$$

Therefore μ_1 is composable with $\alpha_{\eta(\lambda_1)}(\mu_2)$ in Σ.

Take $q(\mu_1\alpha_{\eta(\lambda)}(\mu_2))$; since it has range in F, $\mu_1\alpha_{\eta(\lambda_1)}(\mu_2)$ is the unique image by c of $\lambda_1\lambda_2$. So $c(\lambda_1\lambda_2) = c(\lambda_1)\alpha_{\eta(\lambda_1)}(c(\lambda_2))$. The map η can be viewed as measuring by how much the map c fails to be a functor.

We show that η is multiplicative:

$$\begin{aligned} s(c(\lambda_1\lambda_2)) = s(c(\lambda_1)\alpha_{\eta(\lambda_1)}(c(\lambda_2))) &= s(\alpha_{\eta(\lambda_1)}(c(\lambda_2))) = \alpha_{\eta(\lambda_1)}(s(c(\lambda_2))) \\ &= \alpha_{\eta(\lambda_1)}\alpha_{\eta(\lambda_2)}(c(s(\lambda_2))) = \alpha_{\eta(\lambda_1)\eta(\lambda_2)}(c(s(\lambda_1\lambda_2))). \end{aligned}$$

But $s(c(\lambda_1\lambda_2)) = \alpha_{\eta(\lambda_1\lambda_2)}(c(s(\lambda_1\lambda_2)))$ by definition of η, so $\eta(\lambda_1\lambda_2) = \eta(\lambda_1)\eta(\lambda_2)$ by freeness of the action. As any multiplication-preserving map into a semigroup is a functor, $\eta : S\backslash\Sigma \to S$ is a functor.

We claim that the map $\psi : S\backslash\Sigma \times_\eta S \to \Sigma$ defined $\psi : (\lambda, t) \mapsto \alpha_t(c(\lambda))$ is an equivariant isomorphism. We first verify that ψ respects the range map and the source map:

$$\begin{aligned} &r(\psi(\lambda, t)) = r(\alpha_t(c(\lambda))) = \alpha_t(r(c(\lambda))) \\ &\psi(r(\lambda, t)) = \psi(r(\lambda), t) = \alpha_t(c(r(\lambda))) = \alpha_t(r(c(\lambda))) \text{ because } c(\lambda) \in F. \\ &s(\psi(\lambda, t)) = s(\alpha_t(c(\lambda))) = \alpha_t(s(c(\lambda))) \\ &\psi(s(\lambda, t)) = \psi(s(\lambda), t\eta(\lambda)) = \alpha_t\alpha_{\eta(\lambda)}(c(s(\lambda))) = \alpha_t(s(c(\lambda))). \end{aligned} \tag{3.3}$$

From the properties of the semigroup skew-product k-graph (Proposition 3.3), and the quotient k-graph (Proposition 3.5),

$$d_\Sigma(\psi(\lambda,t)) = d_\Sigma(\alpha_t(c(\lambda))) = d_\Sigma(c(\lambda)) = d_{S\backslash\Sigma}(\lambda) = d_{S\backslash\Sigma\times_\eta S}(\lambda,t).$$

An immediate corollary to this calculation is that ψ maps the objects of $S\backslash\Sigma\times_\eta S$ to the objects of Σ.

We verify ψ is multiplicative: suppose (λ,t) and $(\mu,t\eta(\lambda))$ are composable in $S\backslash\Sigma\times_\eta S$. Then equation (3.3) implies that $\psi(\lambda,t)$ and $\psi(\mu,t\eta(\lambda))$ are composable. Using that $c(\lambda)\alpha_{\eta(\lambda)}(c(\mu)) = c(\lambda\mu)$ implies:

$$\begin{aligned}\psi((\lambda,t)(\mu,t\eta(\lambda))) &= \psi(\lambda\mu,t) = \alpha_t(c(\lambda\mu))\\ \psi(\lambda,t)\psi(\mu,t\eta(\lambda)) &= \alpha_t(c(\lambda))\alpha_t\alpha_{\eta(\lambda)}(c(\mu))\\ &= \alpha_t(c(\lambda)\alpha_{\eta(\lambda)}(c(\mu))) = \alpha_t(c(\lambda\mu)).\end{aligned}$$

Hence ψ is a k-graph morphism.

Suppose $\psi(\lambda,t) = \psi(\mu,u)$, then $\alpha_t(c(\lambda)) = \alpha_u(c(\mu))$. Taking the quotient shows $\lambda = \mu$, and freeness of the action forces $t = u$. Hence ψ is injective.

Suppose $\gamma \in \Sigma$, then there exists $t \in S$ and $c(\lambda) \in c(\Lambda)$ such that $\gamma = \alpha_t(c(\lambda))$. Then $\psi(\lambda,t) = \alpha_t(c(\lambda)) = \gamma$. Hence ψ is surjective. Therefore ψ is an isomorphism of $S\backslash\Sigma\times_\eta S$ onto Σ.

Suppose $\beta \in \Sigma$. Then $\beta = \alpha_{\xi(\beta)}(c(q(\beta)))$. We claim the map $\phi := \psi^{-1} : \Sigma \to S\backslash\Sigma\times_\eta S$ defined $\phi(\beta) = (q(\beta),\xi(\beta))$ is the inverse of ψ.

$$\begin{aligned}&(\phi\circ\psi)(\lambda,t) = \phi(\alpha_t(c(\lambda))) = (\lambda,t),\\ &(\psi\circ\phi)(\beta) = \psi\circ\phi(\alpha_{\xi(\beta)}(c(q(\beta)))) = \psi(q(\beta),\xi(\beta)) = \alpha_{\xi(\beta)}(c(q(\beta))) = \beta.\end{aligned}$$

We require $\psi\circ\mathrm{lt}_u = \alpha_u\circ\psi$, where lt is left-translation on the skew-product k-graph:

$$(\phi\circ\mathrm{lt}_u)(\lambda,t) = \phi(\lambda,ut) = \alpha_{ut}(c(\lambda)) = \alpha_u\alpha_t(c(\lambda)) = (\alpha_u\circ\phi)(\lambda,t).$$

Hence ψ is equivariant. □

COROLLARY 3.12. *With Σ,α,S as defined in Theorem 3.11, there exists an isomorphism $\phi_* : C^*(\Sigma) \to C^*(S\backslash\Sigma\times_\eta S)$ such that $\phi_*\circ\alpha_* = \mathrm{lt}_*\circ\phi_*$.*

PROOF. The isomorphism $\phi : \Sigma \to S\backslash\Sigma\times_\eta S$ produced by Theorem 3.11 satisfied the hypotheses of Proposition 2.14. Hence there exists an induced isomorphism $\phi_* : C^*(\Sigma) \to C^*(S\backslash\Sigma\times_\eta S)$ such that $\phi_*(s^\Sigma_\sigma) = s^{S\backslash\Sigma\times_\eta S}_{(q(\sigma),\xi(\sigma))}$. That the isomorphism ϕ_* intertwines the induced actions follows from the equivariance of ϕ.

□

REMARK 3.13. If the action were the lt action, then the k-graph would be a skew-product graph, and it is straightforward to write down what the maps η and c from Theorem 3.11 are. Given a k-graph Λ, an Ore semigroup S, and a functor $\xi : \Lambda \to S$, we can apply Theorem 3.11 to the k-graph $\Sigma := \Lambda \times_\xi S$ together with the fundamental domain $\Lambda \times_\xi \{1_S\}$. The map $c : S\backslash\Sigma \to \Sigma$ is defined $c([\lambda, s]) = (\lambda, 1_S)$ and $\eta : S\backslash\Sigma \to S$ is characterised by $\alpha_{\eta([\lambda,s])}c([\lambda, s]) = (\lambda, s)$. We claim the map $q : S\backslash\Sigma \to \Lambda$ defined by $q([\lambda, s]) = \lambda$ is a well-defined isomorphism and that $\eta = \xi \circ q$. So $q \times \mathrm{id} : S\backslash\Sigma \times_\eta S \to \Lambda \times_\xi S$ is an equivariant isomorphism for the left action.

If we have an action by a group rather than a semigroup, taking any element of the equivalence class, the group action will fill out the class: for each $[\sigma] \in G\backslash\Sigma$, we can choose $\lambda_{[\sigma]} \in \Sigma$ such that $\lambda_{[\sigma]} \in [\sigma]$. Then $\{\lambda_{[\sigma]} : [\sigma] \in G\backslash\Sigma\}$ is a fundamental domain. Any map $c : G\backslash\Sigma \to \Sigma$ with the property that $q \circ c = \mathrm{id}_{G\backslash\Sigma}$ (such a map is called a section) will satisfy the fundamental domain condition. Hence, if S is a group, every free action of S on a k-graph Λ admits a fundamental domain as a group action will fill the orbit, and we recover the result of [**36**, Remark 5.6]. Indeed, the proof of [**36**, Remark 5.6] starts by constructing a suitable fundamental domain, and the rest of the proof then carries over to show that the formula we give for ϕ^{-1} defines an isomorphism of $(S\backslash\Sigma) \times_\eta S$ onto Σ.

3.4. Classification of fundamental domains and free actions

We saw earlier in Example 3.10 that some actions admit multiple fundamental domains, whereas some admit only one, or even none at all. We are interested in classifying the possible fundamental domains for an Ore semigroup acting on a k-graph. The main result of this section is Theorem 3.14, which says that when an Ore semigroup is acting freely on a k-graph, the fundamental domains are in one-to-one correspondence with the invertible elements of the Ore semigroup. Consequently, if S has no invertible elements, other than 1_S, the fundamental domain is unique.

THEOREM 3.14. *Suppose Σ is a k-graph, i is an invertible element in an Ore semigroup S and $F \subseteq \Sigma$ is a fundamental domain for a free left-action α of S on a k-graph Σ. Then $\alpha_i(F) \neq F$ is a fundamental domain for α. Conversely, if $\alpha_i(F)$ is a fundamental domain, then $i \in S$ is invertible. Hence, there is a one-to-one correspondence between invertible elements of S and fundamental domains.*

PROOF. For $\sigma \in \Sigma$, since F is a fundamental domain, there exists $s \in S$ and $\mu \in F$ such that $\sigma = \alpha_s(\mu)$. Since i is invertible in S, $si^{-1} \in S$, and hence $\sigma = \alpha_{si^{-1}}\alpha_i(\mu)$ with $si^{-1} \in S$ and $\alpha_i(\mu) \in \alpha_i(F)$. Since $r(\mu) \in F$, $r(\alpha_i(\mu)) = \alpha_i(r(\mu)) \in \alpha_i(F)$. Suppose $\alpha_i(F) = F$, then given $\mu \in F$, there exists $\lambda \in F$, $s \in S$ with $\alpha_s(\lambda) = \mu$. But $\alpha_{si^{-1}}\alpha_i(\lambda) = \mu \in F$, contradicting the uniqueness condition of the fundamental domain definition.

Conversely, suppose α is a free action of S on Σ and $\alpha_i(F)$ is a fundamental domain. For $\mu \in F$, there exists $\nu \in \alpha_i(F)$ and $j \in S$ such that $\mu = \alpha_j(\nu)$. By definition, there exists $\nu' \in F$ such that $\nu = \alpha_i(\nu')$. Therefore $\nu = \alpha_j\alpha_i(\nu')$. Hence $r(\mu) = \alpha_j\alpha_i r(\nu')$. Therefore $r(\mu) = r(\nu')$ and $1_S = ji$, and i is invertible in S.

The final statement follows from these two arguments. □

Using Theorem 3.14, we can categorise the fundamental domains of the action by left translation of an Ore semigroup on the skew-product $\Lambda \times_\eta S$. Suppose $I(S)$ is the group of invertible elements of S.

Lemma 3.15. *Suppose Λ is a k-graph, S is an Ore semigroup, and $\eta : \Lambda \to S$ is a functor. Then for all $i \in I(S)$, $F_i = \{(\lambda, i) : \lambda \in \Lambda, i \in I(S)\}$ is a fundamental domain for the action of S by left-translation on $\Lambda \times_\eta S$.*

Proof. Since $r(\lambda, 1_S) = (r(\lambda), 1_S)$, it is immediate that $\{(\lambda, 1_S) : \lambda \in \Lambda\}$ is a fundamental domain for $\Lambda \times_\eta S$. Take $(\lambda, s) \in \Lambda \times_\eta S$. Then apply Theorem 3.14 with lt the action of S on $\Lambda \times_\eta S$ by left-translation: since $i \in I$, $si^{-1} \in S$ and $\mathrm{lt}_{si^{-1}}(\lambda, i) = (\lambda, si^{-1}i) = (\lambda, s)$ as required. □

Recall from Remark 3.9(2) that a skew-product k-graph $\Lambda \times_\eta S$ will always have at least one fundamental domain, $F = \{(\lambda, 1_S) : \lambda \in \Lambda\}$. The Gross-Tucker theorem can now be used.

Corollary 3.16. *Suppose α is a free left action of an Ore semigroup S on a k-graph Σ and $c : S\backslash\Sigma \to F$, $\eta : S\backslash\Sigma \to S$, and $\psi : S\backslash\Sigma \times_\eta S \to \Sigma$ are defined as in Theorem 3.11. Then $F := \{\psi(\lambda, 1_S) : \lambda \in S\backslash\Sigma\} = \{c(\lambda) : \lambda \in S\backslash\Sigma\}$ is a fundamental domain for α in Σ.*

Proof. We claim $F := \{\psi(\lambda, 1_S) : \lambda \in S\backslash\Sigma\}$ is a fundamental domain in Σ. We need to prove that for each $\beta \in \Sigma$ there exist unique $t \in S$ and $\gamma \in F$ such that $\beta = \alpha_t(\gamma)$, and that for all $\delta \in F$, $r(\delta) \in F$. Take $\beta \in \Sigma$. The isomorphism ψ is onto, so there exists unique $u \in S$ and $\mu \in S\backslash\Sigma$ such that, since ψ is an equivariant isomorphism,

$$\beta = \psi(\mu, u) = \psi(\mathrm{lt}_u(\lambda, 1_S)) = \alpha_u(\psi(\mu, 1_S)).$$

Hence $\gamma = \psi(\mu, 1_S) \in F$.

For each $(\lambda, 1_S) \in S\backslash\Sigma \times_\eta S$, $r(\psi(\lambda, 1_S)) = \psi(r(\lambda, 1_S)) = \psi(r(\lambda), 1_S)$, so for each $\delta = \psi(\lambda, 1_S)$, $r(\delta) \in F$. Hence F is a fundamental domain for α. □

Example 3.17. If we consider the actions from Examples 3.2, we can see that only some of the actions have invertible elements. The action of $\mathbb{N}^k$ on Ω_k by translation has no invertible elements, as $\mathbb{N}^k\backslash\{0\}$ has no additive inverses, hence there exists a unique fundamental domain for the action of $\mathbb{N}^k$ on Ω_k. If we compare this to the action of $\mathbb{Z}^k \times \mathbb{N}^l$ on either the $(k+l)$-graph Δ_{k+l} or the $(k+l)$-graph

$\Delta_k \times \Omega_l$ defined in Examples 3.2(3), there are inverses in the 'x'-direction, but not the 'y' direction, and so there are fundamental domains for the action of $\mathbb{Z}^k \times \mathbb{N}^l$ indexed by the invertible elements of $\mathbb{Z}^k$.

Definition 3.18. Let Λ be a k graph and $\eta_1, \eta_2 : \Lambda \to S$ be two functors. Then η_1 and η_2 are *cohomologous* if there exists $b : \Lambda^0 \to S$ and an automorphism $\psi : \Lambda \to \Lambda$ such that

$$\eta_1(\lambda)b(s(\lambda)) = b(r(\lambda))\eta_2(\psi(\lambda)) \text{ for all } \lambda \in \Lambda. \tag{3.4}$$

If so, we write $\eta_1 \sim \eta_2$. This relation is reflexive with $\psi = \mathrm{id}$, and symmetric as ψ is an automorphism and hence its inverse is also an automorphism $\Lambda \to \Lambda$.

The necessity of introducing the automorphism ψ in the definition of cohomology of functors (Definition 3.18) can be seen by considering the following example from Rho [**61**, Remark 11.5(ii)]. Suppose Λ is a k-graph, ψ is an automorphism of Λ, $\eta : \Lambda \to S$ is a functor into an Ore semigroup. Let $\eta' = \eta \circ \psi$. Then $\Lambda \times_\eta S$ is equivariantly isomorphic to $\Lambda \times_{\eta'} S$. A simple example is to take an asymmetric k-graph and a functor $\eta : \Lambda \to \mathbb{Z}_2$. We define the automorphism ψ on $\Lambda \times_\eta \mathbb{Z}_2$ as the map that flips the graph over.

For the following result, recall $I(S)$ is the set of invertible elements in S.

Lemma 3.19. *Let Λ be a row-finite k-graph, $\eta : \Lambda \to I(S)$ a functor, and $\Lambda \times_\eta S$ the associated skew-product graph. There exists an* lt*-equivariant isomorphism $\phi : \Lambda \times_{\eta_1} S \to \Lambda \times_{\eta_2} S$ if and only if $\eta_1, \eta_2 : \Lambda \to S$ are cohomologous functors.*

Proof. Suppose $\eta_1 \sim \eta_2$. Hence there exists a map $b : \Lambda^0 \to S$ and automorphism $\psi : \Lambda \to \Lambda$ that satisfy equation (3.4). Define $\phi : \Lambda \times_{\eta_1} S \to \Lambda \times_{\eta_2} S$ by $(\lambda, t) \mapsto (\psi(\lambda), tb(r(\lambda)))$. It is straightforward to see that ϕ respects the range and source maps, composition, and degree of the first coordinate, and hence is an equivariant k-graph morphism.

To check injectivity: suppose $\phi(\lambda, s) = \phi(\mu, t)$. Then $\psi(\lambda) = \psi(\mu)$ and $sb(r(\lambda)) = tb(r(\mu))$. Hence, $\lambda = \mu$ since ψ is an automorphism, and $s = t$ since S is cancellative. Surjectivity follows from observing that for $(\lambda, t) \in \Lambda \times_{\eta_2} S$, $\phi(\psi^{-1}(\lambda), t(b(r(\lambda)))^{-1}) = (\lambda, t)$.

Let $\mathrm{lt}_1, \mathrm{lt}_2$, denote the left actions of S by left-translation on $\Lambda \times_{\eta_1} S$ and $\Lambda \times_{\eta_2} S$. Suppose $\phi : \Lambda \times_{\eta_1} S \to \Lambda \times_{\eta_2} S$ is an equivariant isomorphism. Then $\phi(\lambda, 1) = (\lambda', t)$ for some $\lambda' \in \Lambda$ and $t \in S$. The equivariance of ϕ implies $\phi((\mathrm{lt}_1)_s(\lambda, 1)) = (\mathrm{lt}_2)_s(\phi(\lambda, 1))$, which implies $\phi(\lambda, s) = (\lambda', st)$. Define $\psi : \Lambda \to \Lambda$ by $\psi(\lambda) = \lambda'$, so $\phi(\lambda, s) = (\psi(\lambda), st)$. That ϕ is an isomorphism is sufficient to demonstrate $\psi : \Lambda \to \Lambda$ is an automorphism.

For $v \in \Lambda^0$, define $b : \Lambda^0 \to S$ by $\phi(v, 1_S) = (\psi(v), b(v))$. The equivariance of ϕ implies that $\phi(r(\lambda, 1_S)) = (\psi(r(\lambda)), b(r(\lambda)))$ and $r(\phi(\lambda, 1_S)) = (r(\psi(\lambda)), t)$, and we can conclude $\phi(\lambda, s) = (\psi(\lambda), s(b(r(\lambda))))$.

To see that $\eta_1 \sim \eta_2$, consider

$$\begin{aligned}\phi(s(\lambda, 1_S)) &= (\psi(s(\lambda)), \eta_1(\lambda) b(s(\lambda)))\\ s(\phi(\lambda, 1_S)) &= (s(\psi(\lambda)), b(r(\lambda))\eta_2(\psi(\lambda))).\end{aligned}$$ □

The following proposition considers what happens when we have more than one fundamental domain for an action on a k-graph. When we apply Theorem 3.11 to two different fundamental domains for a free left action, we get two different functors on the same k-graph. We claim that such functors must be cohomologous.

Proposition 3.20. *Suppose that α is a free action of an Ore semigroup S on a k-graph Σ. Let F_1, F_2 be fundamental domains for α, and $c_1, c_2 : S\backslash\Sigma \to \Sigma$ are the maps such that $c_i(S\backslash\Sigma) = F_i$ for $i = 1, 2$ defined by $q(c_i(\lambda)) = \lambda$.*

(1) *There is a function $b : (S\backslash\Sigma)^0 \to I(S)$ such that $\alpha_{b(r(\lambda))}(c_2(\lambda)) = c_1(\lambda)$ for all $\lambda \in S\backslash\Sigma$.*
(2) *Let $\eta_1, \eta_2 : S\backslash\Sigma \to S$ be functors such that $s(c_i(\lambda)) = \alpha_{\eta_i(\lambda)}(c_i(s(\lambda)))$ for all $\lambda \in S\backslash\Sigma$. Then η_1 is cohomologous to η_2.*

Proof. By Theorem 3.11, the fundamental domains F_i induce maps c_i such that for $\lambda \in S\backslash\Sigma$, $r(c_i(\lambda)) = c_i(r(\lambda))$. We can define a map $b : \Lambda^0 \to S$ by $\alpha_{b(v)}(c_2(v)) = c_1(v)$. Then

$$\alpha_{b(r(\lambda))}(c_2(r(\lambda))) = c_1(r(\lambda)) \Longrightarrow r(\alpha_{b(r(\lambda))}(c_2(\lambda))) = r(c_1(\lambda)).$$

Since we are in a fundamental domain, $\alpha_{b(r(\lambda))}(c_2(\lambda)) = c_1(\lambda)$ is the unique image by the section c of λ in the fundamental domain, as by the previous calculation it must have range in the fundamental domain.

That $b(v)$ is invertible is straightforward. By applying the construction again, by freeness of the action, $c_2(\lambda) = \alpha_{(b(r(\lambda)))^{-1}}(c_1(\lambda))$.

For (2), applying Theorem 3.11 shows that there exist equivariant isomorphisms $\psi_1 : S\backslash\Sigma \times_{\eta_1} S \to \Sigma$ and $\phi_2 : S\backslash\Sigma \times_{\eta_2} S \to \Sigma$ such that

$$\psi_1((\lambda, s)) = \alpha_s(c_1(\lambda)) = \alpha_s \alpha_{b(r(\lambda))}(c_2(\lambda)) = \psi_2^{-1}((\lambda, sb(r(\lambda)))).$$

Hence $\psi_2^{-1} \circ \psi_1 : S\backslash\Sigma \times_{\eta_1} S \to S\backslash\Sigma \times_{\eta_2} S : (\lambda, s) \mapsto (\lambda, sb(r(\lambda)))$ is an equivariant isomorphism. In particular, when restricted to the first coordinate, this is the identity map $\mathrm{id} : \lambda \mapsto \lambda$, and so η_1 is cohomologous to η_2. □

Proposition 3.20 says that if we choose different fundamental domains, Theorem 3.11 will give two functors, which will be cohomologous. Therefore, the cohomology class of the functor from Theorem 3.11 is uniquely determined by the action α. Hence, Proposition 3.20 can be used to classify the free actions on a k-graph up to equivariance.

CHAPTER 4

Dilating semigroup actions

For a free action α of an Ore semigroup S on a k-graph Σ, Theorem 3.11 produces a functor $\eta : S\backslash\Sigma \to S$ such that $S\backslash\Sigma \times_\eta S$ is isomorphic to Σ. This naturally gives rise to a crossed product $C^*(S\backslash\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$. Consider the enveloping group Γ of S: We can then consider $\eta : S\backslash\Sigma \to \Gamma$, and can form $S\backslash\Sigma \times_\eta \Gamma$, with an action by left-translation. This gives a crossed product $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$. The isomorphism ψ_2 of diagram 1.1 maps the crossed product $C^*(S\backslash\Sigma\times_\eta S)\times_{\mathrm{lt}_*} S$ to $C^*(S\backslash\Sigma\times_\eta\Gamma)\times_{\mathrm{lt}_*}\Gamma$ in such a way that the crossed product by S sits in a full corner of the crossed product by Γ. This isomorphism dilates a non-invertible action of S to an invertible action of Γ. In Corollary 4.4 we will give an explicit formula of the isomorphism. This will involve a reformulation of a result of Laca [**41**] about the dilation of an action of an Ore semigroup on a C^*-algebra. One of the hypotheses of the main result of this chapter, Corollary 4.4, is that the action is by injective extendible endomorphisms: since Σ is not necessarily a finite k-graph, $C^*(\Sigma)$ is not necessarily a unital C^*-algebra, and hence we must work with extendible endomorphisms. We determine what properties the action α must satisfy to induce such extendible endomorphisms on a C^*-algebra. We see in Theorem 4.2 that the condition required for extendible endomorphisms is the existence of a fundamental domain.

Previously the construction of this isomorphism required two results, Theorems 2.1 and 2.4 of [**41**]: a direct limit construction was provided, followed by a projection onto a corner. By our reformulation, we are able to do it with one result.

Recall from Definition 2.13 the definition of a saturated k-graph morphism: $\phi : \Lambda_1 \to \Lambda_2$ is saturated with respect to X if $\phi : X\Lambda_1 \to \phi(X)\Lambda_2$ is a bijection. Recall that an *action* of a semigroup on a k-graph Λ is an identity-preserving homomorphism α of S into the semigroup $\mathrm{End}(\Lambda)$ of endomorphisms of Λ, which is a semigroup under composition. We first need to know when an action of a k-graph extends to an endomorphism of the k-graph C^*-algebra. This is done with the following two results.

Lemma 4.1. *Suppose S is an Ore semigroup, Λ a row-finite k-graph, and $\alpha : S \to \mathrm{End}(\Lambda)$ is a free left action which admits a fundamental domain. Then $\alpha_t : \Lambda \to \Lambda$ is saturated for each $t \in S$.*

Proof. Fix $t \in S$. Take F to be the fundamental domain for α. We claim that $\alpha_t : \Lambda^0\Lambda \to \alpha_t(\Lambda^0)\Lambda$ is a bijection. For injectivity: suppose $\alpha_t(\lambda) = \alpha_t(\mu)$ for

$\lambda, \mu \in \Lambda$. As F is a fundamental domain, there exist unique $r, s \in S$ and $\lambda', \mu' \in F$ such that $\lambda = \alpha_r(\lambda')$ and $\mu = \alpha_s(\mu')$. Then $\alpha_{tr}(\lambda') = \alpha_{ts}(\mu')$. Hence $tr = ts$ and $\lambda' = \mu'$ as F is a fundamental domain. Cancellativity in S implies $r = s$. We can conclude $\lambda = \mu$ demonstrating injectivity.

To demonstrate surjectivity: fix $t \in S$ and $\mu \in \Lambda$ with $r(\mu) \in \alpha_t(\Lambda^0)$. Then the injectivity of α_t implies there exists a unique $v \in \Lambda^0$ such that $r(\mu) = \alpha_t(v)$. Since α has a fundamental domain there exist unique $u \in S$ and $\mu' \in F$ such that $\mu = \alpha_u(\mu')$, with $r(\mu') \in F$. Then $\alpha_u(r(\mu')) = r(\mu) = \alpha_t(v)$. For $v \in \Lambda^0$, as F is a fundamental domain there exists unique $u' \in S$ and $v' \in F^0$ such that $v = \alpha_{u'}(v')$. Hence $\alpha_u(r(\mu')) = \alpha_t(\alpha_{u'}(v'))$, implying $r(\mu') = v'$ and $\alpha_u(r(\mu')) = \alpha_t(\alpha_{u'}(r(\mu')))$. Since $r(\mu') \in F^0$, we have that $u = tu'$. Therefore $\mu = \alpha_u(\mu') = \alpha_t(\alpha_{u'}(\mu'))$. Therefore α_t is surjective onto $\alpha_t(\Lambda^0)\Lambda$. Hence, for each $t \in S$, $\alpha_t : \Lambda \to \Lambda$ is saturated. □

If the action was by a group, rather than a semigroup, then it would be an action by automorphisms, and since a fundamental domain would always exist, saturation would be automatic.

The following theorem was proved for directed graphs and their graph algebras as [**50**, Proposition 3.7]. We generalise the result to semigroups acting on k-graphs, but note that the argument is quite similar to the argument that was used for directed graphs.

THEOREM 4.2. *Suppose S is an Ore semigroup, Λ is a row-finite k-graph, and $\alpha : S \to \mathrm{End}(\Lambda)$ is a free action that admits a fundamental domain. Then each α_t is saturated, and $\alpha_* : t \mapsto (\alpha_*)_t := (\alpha_t)_*$ is an action of S by injective extendible endomorphisms of $C^*(\Lambda)$.*

PROOF. That α has a fundamental domain, together with Lemma 4.1 implies that $\alpha_t : \Lambda \to \Lambda$ is saturated for every $t \in S$. We are then able to apply Proposition 2.14 to $\alpha_t : \Lambda \to \Lambda$, and hence there exists an injective extendible homomorphism $(\alpha_t)_* : C^*(\Lambda) \to C^*(\Lambda)$. Again by Proposition 2.14, the assignment $\alpha_t \mapsto (\alpha_t)_*$ is functorial, and so the action α_* of S on $C^*(\Lambda)$ defined $t \mapsto (\alpha_*)_t$ is a semigroup homomorphism. □

We now turn to the results of Laca [**41**]: We require a short summary of semigroup dynamical systems. We use notation and terminology consistent with [**41, 50, 55**]. A *semigroup dynamical system* (A, S, α) is a triple consisting of a C^*-algebra A (not necessarily unital), a semigroup S, and a homomorphism $\alpha : S \to \mathrm{End}(A)$. We denote by α_s the endomorphism $\alpha(s)$.

A *covariant representation* is a pair (π, V), where π is a nondegenerate representation of A on a Hilbert space $\mathcal{H}$, and $V : s \mapsto V_s$ is a homomorphism of S into the semigroup of isometries on $\mathcal{H}$ such that:

$$V_s \pi(a) V_s^* = \pi(\alpha_s(a)) \text{ for all } a \in A, s \in S. \tag{4.1}$$

A *crossed product* for a semigroup dynamical system (A, S, α) is a triple (B, i_A, i_S) such that B is a C^*-algebra, $i_A : A \to M(B)$ is a nondegenerate homomorphism, $i_S : S \to \operatorname{Isom}(M(B))$ is a semigroup homomorphism satisfying the following conditions:

(1) $i_A(\alpha_s(a)) = i_S(s) i_A(a) i_S(s)^*$, for all $a \in A, s \in S$,
(2) for any covariant representation (π, V) of (A, S, α) on a Hilbert space $\mathcal{H}$, there exists a nondegenerate representation $\pi \times V$ of B on $\mathcal{H}$ such that $(\pi \times V) \circ i_A = \pi$ and $\overline{(\pi \times V)} \circ i_S = V$;
(3) B is generated as a C^*-algebra by $\{i_A(a) i_S(s) : a \in A, s \in S\}$.

We usually denote the semigroup crossed product by $(A \times_\alpha S, i_A, i_S)$.

If we had a group crossed product with Γ instead of a semigroup S, then the C^*-algebra B would be closed spanned, rather than generated, by the set $\{i_A(a) i_\Gamma(g) : a \in A, g \in \Gamma\}$.

The following is a formulation of two of Laca's dilation arguments, [**41**, Theorem 2.1, Theorem 2.4] that were used by Pask, Raeburn and Yeend in [**50**], which are useful in our particular context of k-graph C^*-algebras. These give necessary conditions for an action of an Ore semigroup of a C^*-algebra to be dilated to an action of the enveloping group, and describe a map from a crossed product by endomorphisms to a corner of a crossed product by automorphisms of a group.

THEOREM 4.3 (Laca). *Suppose that S is an Ore semigroup with enveloping group $\Gamma = S^{-1}S$, and $\alpha : S \to \operatorname{End}(A)$ is an action of S by injective extendible endomorphisms of a C^*-algebra A.*

(a) *There is an action β of Γ on a C^*-algebra B and an injective extendible homomorphism $j : A \to B$ such that*

(L1) *$j \circ \alpha_u = \beta_u \circ j$ for $u \in S$, and*

(L2) *$\bigcup_{u \in S} \beta_u^{-1}(j(A))$ is dense in B;*

the triple (B, β, j) with these properties is unique up to isomorphism.

(b) *Suppose (B, β, j) has properties* (L1) *and* (L2), *write $p := \overline{i_B \circ j}(1)$, and define $v_s := i_\Gamma(s)p$. Then $(i_B \circ j, v)$ is a covariant representation of (A, S, α) and $(i_B \circ j) \times v$ is an isomorphism of $A \times_\alpha S$ onto $p(B \rtimes_\beta \Gamma)p$.*

For the unital case, part (a) is Theorem 2.1 of [**41**]. Laca proves the existence of (B, Γ, β) using a direct limit construction, and j is the canonical embedding α^1 of the first copy A_1 of A in the direct limit A_∞. Lemma 4.3 of [**50**] says that if the endomorphisms are all extendible, then so is $j := \alpha^1$. Laca's proof of uniqueness carries over verbatim. Part (b) is proved for the unital case in [**41**, Theorem 2.4], and again the proof carries over: the crucial step, which is Lemma 2.3 of [**41**], is purely representation-theoretic.

In the context of k-graph C^*-algebras, Laca's theorem takes the following form. Recall the definition of a saturated subgraph from Definition 2.13. In the following result, we have $\{s^\Omega_\omega\}$ to be the universal Cuntz-Krieger Ω-family for $C^*(\Omega)$, and p_{Ω^0} to be the projection from Proposition 2.8.

COROLLARY 4.4. *Suppose that S is an Ore semigroup with enveloping group $\Gamma = S^{-1}S$, and β is a free action of Γ on a row-finite k-graph Λ. Suppose that Ω is a saturated subgraph of Λ such that $\beta_u(\Omega) \subset \Omega$ for all $u \in S$ and $\bigcup_{u\in S} \beta_u^{-1}(\Omega) = \Lambda$. Write $\alpha_u := \beta_u|_\Omega$, and set $p := \overline{i_{C^*(\Lambda)}}(p_{\Omega^0})$. Then there is an isomorphism ψ of $C^*(\Omega) \times_{\alpha_*} S$ onto $p(C^*(\Lambda) \rtimes_{\beta_*} \Gamma)p$ such that*

$$\psi(i_{C^*(\Omega)}(s^\Omega_\omega)) = i_{C^*(\Lambda)}(s^\Lambda_\omega) \quad \textit{and} \quad \overline{\psi}(i_S(u)) = i_\Gamma(u)p.$$

PROOF. We first ensure that we can apply Theorem 4.3. Let $\pi : \Omega \to \Lambda$ be the inclusion map. Since Ω is saturated in Λ, Proposition 2.14 implies that π induces an injective extendible homomorphism $\pi_* : C^*(\Omega) \to p_{\Omega^0} C^*(\Lambda) p_{\Omega^0}$ such that $\pi_*(s^\Omega_\omega) = s^\Lambda_\omega$ for $\omega \in \Omega$ and $\overline{\pi_*}(1) = p_{\Omega^0}$. Since each β_u is an automorphism, it is saturated, and we claim that the restriction α_u is saturated as a graph morphism from Ω to Ω. Indeed, if $\omega \in \Omega$ has $r(\omega) \in \alpha_u(\Omega^0)$, say $r(\omega) = \alpha_u(v)$, then

$$r(\beta_u^{-1}(\omega)) = \beta_u^{-1}(r(\omega)) = \beta_u^{-1}(\alpha_u(v)) = \beta_u^{-1}(\beta_u(v)) = v$$

belongs to Ω^0, $\beta_u^{-1}(\omega)$ belongs to Ω because Ω is saturated in Λ, and $\omega = \alpha_u(\beta_u^{-1}(\omega))$ belongs to $\alpha_u(\Omega)$. Now Theorem 4.2 implies that α induces an action α_* of S on $C^*(\Omega)$ by injective extendible endomorphisms. Hence the hypotheses of Theorem 4.3 are satisfied, and we can apply the result.

We will show that the system $(C^*(\Lambda), \Gamma, \beta_*)$ and $j := \pi_*$ have the properties (L1) and (L2) of Theorem 4.3 relative to the semigroup dynamical system $(C^*(\Omega), S, \alpha_*)$. Homomorphisms are determined by how they act on generators, so for $\omega \in \Omega$ and $u \in S$, the calculation

$$\pi_*((\alpha_*)_u(s^\Omega_\omega)) = \pi_*(s^\Omega_{\alpha_u(\omega)}) = s^\Lambda_{\alpha_u(\omega)} = s^\Lambda_{\beta_u(\omega)} = (\beta_*)_u(s^\Lambda_\omega) = (\beta_*)_u(\pi_*(s^\Omega_\omega))$$

implies that $\pi_* \circ (\alpha_*)_u = (\beta_*)_u \circ \pi_*$, which is (L1). Next, note that for $u \in S$ we have

$$(\beta_*)_u^{-1}(\pi_*(C^*(\Omega))) \supset \{(\beta_*)_u^{-1}(s^\Lambda_\omega) : \omega \in \Omega\} = \{s^\Lambda_{\beta_u^{-1}(\omega)} : \omega \in \Omega\},$$

which by the hypothesis that $\bigcup_{u\in S}\beta_u^{-1}(\Omega) = \Lambda$ implies that the set A_0 defined $\bigcup_{u\in S}(\beta_*)_u^{-1}(\pi_*(C^*(\Omega)))$ contains all the generators of $C^*(\Lambda)$. Thus to check (L2), it is enough to prove that A_0 is a $*$-algebra, and the only non-obvious point is whether A_0 is closed under multiplication. Let $a \in (\beta_*)_u^{-1}(\pi_*(C^*(\Omega)))$ and $b \in (\beta_*)_t^{-1}(\pi_*(C^*(\Omega)))$ for $u, t \in S$. Since S is Ore, there exist $r, w \in S$ such that $ru = wt = x$, say. Since $(\beta_*)_r \circ \pi_* = \pi_* \circ (\alpha_*)_r$, we have $\operatorname{range}(\beta_*)_r \circ \pi_* \subset \operatorname{range}\pi_*$, and

$$(\beta_*)_u^{-1}(\pi_*(C^*(\Omega))) = (\beta_*)_{ru}^{-1} \circ (\beta_*)_r(\pi_*(C^*(\Omega))) \subset (\beta_*)_x^{-1}(\pi_*(C^*(\Omega))).$$

Similarly,

$$(\beta_*)_t^{-1}(\pi_*(C^*(\Omega))) \subset (\beta_*)_x^{-1}(\pi_*(C^*(\Omega))).$$

Since $(\beta_*)_x^{-1}(\pi_*(C^*(\Omega)))$ is an algebra, we have $ab \in (\beta_*)_x^{-1}(\pi_*(C^*(\Omega))) \subset A_0$, as required.

We can now set $v_s := i_\Gamma(s)\overline{i_{C^*(\Lambda)} \circ \pi_*}(1) = i_\Gamma(s)p$, and deduce from Theorem 4.3 that $\psi := (i_{C^*(\Lambda)} \circ \pi_*) \times v$ is an isomorphism of $C^*(\Omega) \times_{\alpha_*} S$ onto $p(C^*(\Lambda) \rtimes_{\beta_*} \Gamma)p$. This isomorphism has the required properties. $\square$

CHAPTER 5

Crossed products of the C^*-algebras of skew-product graphs

For this chapter, we wish to construct the isomorphism ψ_3 of diagram 1.1 from $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$ to $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$. This result was originally stated for higher-rank graphs by Kumjian and Pask [**36**] using groupoid techniques. It was revisited for directed graphs by Kaliszewski, Quigg and Raeburn in [**32**, Theorem 5.7] whose treatment we will follow. We will adapt their methods, and in doing so, produce an explicit isomorphism.

In Theorem 5.1 we will prove that for a row-finite k-graph with no sources, Λ, an Ore semigroup S, and a functor η, there exists an isomorphism ϕ from $C^*(\Lambda \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$ onto $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$.

Since the isomorphism we will construct maps a crossed product into a tensor product, we give a brief background of tensor products and how they can be represented. We are able to give an explicit isomorphism in terms of generators of the crossed product.

We prove our result in three steps: Lemma 5.2 gives a homomorphism π_S from $C^*(\Lambda \times_\eta \Gamma)$ to $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$ by finding a Cuntz-Krieger $\Lambda \times_\eta \Gamma$-family in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$. Lemmas 5.2 and 5.3 together allow us to construct a covariant representation of $(C^*(\Lambda \times_\eta \Gamma), \Gamma, \mathrm{lt}_*)$ in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$. Lemmas 5.5 and 5.6 allow us to construct an inverse for π_S.

The following discussion of tensor products is based on notes from [**60**]. The tensor product of two vector spaces V, W can be viewed as a vector space spanned by elementary tensors $v \otimes w$ with rules that make $(v, w) \mapsto v \otimes w$ bilinear, or as an abstract vector space on which linear maps are given by bilinear maps on $V \times W$. When $\mathcal{H}$ and $\mathcal{K}$ are Hilbert spaces, there is a natural inner product on the vector space tensor product that can be completed to give the Hilbert space tensor product $\mathcal{H} \otimes \mathcal{K}$ ([**60**, Lemma 2.59]). If A and B are C*-algebras, there is a natural way to make the vector space tensor product into a $*$-algebra ([**60**, Lemma B.1]), but a variety of ways to normalise the vector space tensor product and complete it to give a C^*-algebra.

We can get the *spatial* (or *minimal*) tensor product $A \otimes_{\min} B$, which represents A and B faithfully on Hilbert spaces $\mathcal{H}$ and $\mathcal{K}$, and then represents the vector space tensor product faithfully on $\mathcal{H} \otimes \mathcal{K}$. The norm is pulled back from $B(\mathcal{H} \otimes \mathcal{K})$ to the vector space tensor product and completed.

The other tensor product we are interested in is the *maximal* tensor product $A \otimes_{\max} B$, which is universal for commuting pairs of representation of A and B ([**60**, Theorem B.27]).

By [**68**, Definition 2.1.1], a C^*-algebra is *nuclear* if the map $A \otimes_{\max} B \to A \otimes_{\min} B$ is a bijection for every C^*-algebra B. That is, A is nuclear if and only if for every C^*-algebra B, there exists a unique C^*-norm on the algebraic tensor product [**5**, II.9.4.1]. Hence, if A or B is nuclear, we may write $A \otimes B$ for their (unique) tensor product.

For a k-graph Λ, the higher-rank graph C^*-algebra $C^*(\Lambda)$ is nuclear ([**36**, Theorem 5.5]).

Suppose Γ is a group. We use the standard notation that λ and ρ are the left and right regular representations of Γ on $\ell^2(\Gamma)$ respectively such that $\lambda_g \xi(h) = \xi(g^{-1}h)$ and $\rho_g \xi(h) = \xi(hg)$. For $F \subset \Gamma$, χ_F is the operator on $\ell^2(\Gamma)$ which multiplies by the characteristic function of F, and $\chi_g := \chi_{\{g\}}$. When S is a subsemigroup of Γ, we identify $\ell^2(S)$ with the subspace $\overline{\text{span}}\{\delta_t : t \in S\}$ of $\ell^2(\Gamma)$, and then $t \mapsto \lambda_t^S := \lambda_t|_{\ell^2(S)}$ is the usual Toeplitz representation of S on $\ell^2(S)$.

We will require the commutation relation for λ and ρ with χ_G, the operator on $\ell^2(\Gamma)$ that multiplies by the characteristic function of G where G is a subset of Γ. For singleton sets $\{g\}$, we will write χ_g. For $g, h, k \in \Gamma$ and $\xi \in \mathcal{B}(\ell^2(\Gamma))$,

$$\begin{aligned}(\lambda_h \chi_g)\xi(k) &= \lambda_h(\chi_g(\xi))(k) = \chi_g(\xi)(h^{-1}k) = \chi_g(h^{-1}k)\xi(h^{-1}k) \\ &= \begin{cases} \xi(h^{-1}k) & \text{if } h^{-1}k = g \text{ (or } k = hg) \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{hg}(k)\xi(h^{-1}k) = \chi_{hg}(\lambda_h(\xi))(k)\end{aligned} \tag{5.1}$$

$$\begin{aligned}(\rho_h \chi_g)\xi(k) &= \rho_h(\chi_g(\xi))(k) = \chi_g(\xi)(kh) = \chi_g(kh)\xi(kh) \\ &= \begin{cases} \xi(kh) & \text{if } g = kh \text{ (or } k = gh^{-1}) \\ 0 & \text{otherwise} \end{cases} \\ &= \chi_{gh^{-1}}(k)\xi(kh) = \chi_{gh^{-1}}(\rho_h(\xi))(k).\end{aligned}$$

So we have the relations $\lambda_h \chi_g = \chi_{hg} \lambda_h$ and $\rho_h \chi_g = \chi_{gh^{-1}} \rho_h$.

For the space $\ell^2(\Gamma)$, we consider $\{e_g : g \in \Gamma\}$ to be the usual orthonormal basis of point mass functions. For h, k in some Hilbert Space $\mathcal{H}$, denote $h \otimes \overline{k}$ as the rank-one operator defined $h \otimes \overline{k}(g) = \langle g|k \rangle h$. Then $\{e_{ij} := e_i \otimes e_j\}$ is a family of non-zero matrix units in the C^*-algebra $\mathcal{K}(\ell^2(\Gamma))$.

Recognising that a group is also an Ore semigroup, we can adapt an earlier discussion (Lemma 3.4) to conclude that if Γ is a group, then a skew-product graph $\Lambda \times_\eta \Gamma$ carries a natural action lt of Γ. This action on $\Lambda \times_\eta \Gamma$ induces an action lt_*

of Γ on the graph algebra $C^*(\Lambda \times_\eta \Gamma)$. It was first proved by Kumjian and Pask [**35**] that the crossed product by this action is stably isomorphic to $C^*(\Lambda)$. Their proof used a groupoid model for the graph algebra and results of Renault about skew-product groupoids. Kumjian and Pask themselves generalised their result to k-graphs, again using their groupoid model [**36**, Theorem 5.7]. In the following generalisation of the directed graph case of [**32**, Theorem 3.1], that the algebras are isomorphic follows from [**36**, Theorem 5.7] or [**45**, Corollary 5.1] by taking $H = G$, but we seek an explicit isomorphism.

THEOREM 5.1. *Suppose that Λ is a row-finite k graph with no sources, and $\eta : \Lambda \to \Gamma$ is a functor into a group Γ. Then there is an isomorphism ϕ of $C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma$ onto $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$ such that*

$$\phi(i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})) = s_\lambda \otimes \chi_g \rho_{\eta(\lambda)} \quad \textit{and} \quad \overline{\phi}(i_\Gamma(h)) = 1 \otimes \lambda_h. \tag{5.2}$$

Our proof of Theorem 5.1 will require several preliminary results. We begin by proving the following two lemmas about $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$. The first shows how we can form a Cuntz-Krieger $(\Lambda \times_\eta \Gamma)$-family in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$ that is covariant for the action of Γ on $\mathcal{K}(\ell^2(\Gamma))$ by the left regular representation.

LEMMA 5.2. *Suppose that Λ is a row-finite k-graph with no sources, and $\eta : \Lambda \to \Gamma$ is a functor into a group Γ.*

(1) *For $\lambda \in \Lambda$, $\{S_{(\lambda,g)} := s_\lambda \otimes \chi_g \rho_{\eta(\lambda)}\}$ is a Cuntz-Krieger $(\Lambda \times_\eta \Gamma)$-family in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$.*
(2) *Moreover, $(1 \otimes \lambda_h) S_{(\lambda,g)} = S_{(\lambda,hg)}(1 \otimes \lambda_h)$ for $\lambda \in \Lambda$ and $g, h \in \Gamma$.*

PROOF. To check (CK1), we take (v, g) and (w, h) in $(\Lambda \times_\eta \Gamma)^0 = \Lambda^0 \times \Gamma$: then $\eta(v) = \eta(w) = 1$, and $S_{(v,g)} S_{(w,h)} = s_v s_w \otimes \chi_g \chi_h$, which gives (CK1). Next, suppose that (λ, g) and (μ, h) are composable, so that $s(\lambda) = r(\mu)$ and $g\eta(\lambda) = h$. Then as $\rho_k \chi_{gk} = \chi_g \rho_k$ by equation (5.1), we have that

$$\begin{aligned} S_{(\lambda,g)} S_{(\mu,h)} &= (s_\lambda \otimes \chi_g \rho_{\eta(\lambda)})(s_\mu \otimes \chi_h \rho_{\eta(\mu)}) = (s_\lambda s_\mu) \otimes (\chi_g \rho_{\eta(\lambda)} \chi_{g\eta(\lambda)} \rho_{\eta(\mu)}) \\ &= s_{\lambda\mu} \otimes (\chi_g \chi_g \rho_{\eta(\lambda)} \rho_{\eta(\mu)}) = s_{\lambda\mu} \otimes (\chi_g \rho_{\eta(\lambda\mu)}) = S_{(\lambda\mu,g)}, \end{aligned}$$

which is (CK2). For (CK3), suppose $(\lambda, g) \in \Lambda \times_\eta \Gamma$, and we calculate:

$$\begin{aligned} S_{(\lambda,g)}^* S_{(\lambda,g)} &= (s_\lambda^* \otimes \rho_{\eta(\lambda)}^* \chi_g^*)(s_\lambda \otimes \chi_g \rho_{\eta(\lambda)}) = s_\lambda^* s_\lambda \otimes \rho_{\eta(\lambda)}^* \chi_g^* \chi_g \rho_{\eta(\lambda)} \\ &= s_{s(\lambda)} \otimes \chi_{g\eta(\lambda)}^* \rho_{\eta(\lambda)}^{-1} \rho_{\eta(\lambda)} \chi_{g\eta(\lambda)} = s_{s(\lambda)} \otimes \chi_{g\eta(\lambda)} = S_{s(\lambda,g)}. \end{aligned}$$

For (CK4), suppose $(v, g) \in (\Lambda \times_\eta \Gamma)^0$, and we calculate:

$$\begin{aligned} S_{(v,g)} = s_v \otimes \chi_g &= \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^* \otimes \chi_g = \sum_{\lambda \in v\Lambda^n} s_\lambda s_\lambda^* \otimes \chi_g \rho_{\eta(\lambda)} \rho_{\eta(\lambda)}^* \chi_g^* \\ &= \sum_{\lambda \in v\Lambda^n} (s_\lambda \otimes \chi_g \rho_{\eta(\lambda)})(s_\lambda \otimes \chi_g \rho_{\eta(\lambda)})^* = \sum_{\{(\lambda,g): r(\lambda,g)=(v,g)\}} S_{(\lambda,g)} S_{(\lambda,g)}^*. \end{aligned}$$

Hence $\{S_{(\lambda,g)} := s_\lambda \otimes \chi_g \rho_{\eta(\lambda)}\}$ is a Cuntz-Krieger $\Lambda \times_\eta \Gamma$ family in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$, and we have proved item (1).

Using that for $g, h \in \Gamma$, $\lambda_h \chi_g = \chi_{hg} \lambda_h$ and $\rho_g \lambda_h = \lambda_h \rho_g$, we can calculate:

$$\begin{aligned}(1 \otimes \lambda_h) S_{(\lambda,g)} &= (1 \otimes \lambda_h)(s_\lambda \otimes \chi_g \rho_{\eta(\lambda)}) = s_\lambda \otimes \lambda_h \chi_g \rho_{\eta(\lambda)} \\ &= s_\lambda \otimes \chi_{hg} \lambda_h \rho_{\eta(\lambda)} = s_\lambda \otimes \chi_{hg} \rho_{\eta(\lambda)} \lambda_h \\ &= (s_\lambda \otimes \chi_{hg} \rho_{\eta(\lambda)})(1 \otimes \lambda_h) = S_{(\lambda,hg)}(1 \otimes \lambda_h),\end{aligned}$$

giving item (2). □

The following lemma shows that the map π_S from Lemma 5.2 is non-degenerate.

LEMMA 5.3. *Suppose that Λ is a row-finite k-graph with no sources, and $\eta : \Lambda \to \Gamma$ is a functor into a group Γ. Then if F_n and G_n are increasing sequences of finite subsets of Λ^0 and Γ respectively such that $\Lambda^0 = \bigcup_n F_n$ and $\Gamma = \bigcup_n G_n$, then $\sum_{(v,g) \in F_n \times G_n} S_{(v,g)}$ converges strictly to the identity in $M(C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma)))$.*

PROOF. Observe that for $v \in \Lambda^0$, $\eta(v) = 1$, and so $S_{(v,g)} = s_v \otimes \chi_g$. Then

$$\sum_{(v,g) \in F_n \times G_n} S_{(v,g)} = \Big(\sum_{v \in F_n} s_v \Big) \otimes \Big(\sum_{g \in G_n} \chi_g \Big) = a_n \otimes b_n,$$

say, and then the result follows because $\{a_n\}$ and $\{b_n\}$ are approximate identities for $C^*(\Lambda)$ and $\mathcal{K}(\ell^2(\Gamma))$ respectively. □

REMARK 5.4. Item (1) of Lemma 5.2 implies that there is a homomorphism π_S from $C^*(\Lambda \times_\eta \Gamma)$ to $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$ taking $s_{(\lambda,g)}$ to $S_{(\lambda,g)}$, and Lemma 5.3 then says that π_S is nondegenerate. As $(\mathrm{lt}_*)_h S_{(\lambda,g)} = S_{(\lambda,hg)}$, item (2) of Lemma 5.2 implies that $(\pi_S, 1 \otimes \lambda)$ is a covariant representation of $(C^*(\Lambda \times_\eta \Gamma), \Gamma, \mathrm{lt}_*)$ in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$, and $\phi := \pi_S \rtimes (1 \otimes \lambda)$ satisfies (5.2). The image of each spanning element $s_{(\lambda,g)} s^*_{(\mu,k)} i_\Gamma(h)$ belongs to $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$, and hence ϕ has range in $C^*(\Lambda) \otimes \mathcal{K}(\ell^2(\Gamma))$.

So now we have a homomorphism ϕ satisfying (5.2). To prove Theorem 5.1, we will build an inverse for ϕ. To do this, we must find a Cuntz-Krieger Λ-family in $M(C^*(\Lambda \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma)$. Before we do this, we need a technical lemma.

LEMMA 5.5. *Suppose that $\{y_g : g \in \Gamma\}$ is a set of mutually orthogonal projections in a C^*-algebra D, and $u : \Gamma \to UM(D)$ is a homomorphism such that for $g, h \in \Gamma$,*

$$u_h y_g = y_{hg} u_h. \tag{5.3}$$

Then there is a homomorphism $y \times u : \mathcal{K}(\ell^2(\Gamma)) \to D$ such that $y \times u(\lambda_h \chi_g) = u_h y_g$.

PROOF. Using that y_1 is a projection, we observe that for $e_{g,h} := u_g y_1 u_h^*$ for $g, h \in \Gamma$,

$$e_{g,h}^* = u_h y_1^* u_g^* = u_h y_1 u_g^* = e_{h,g} \text{ and}$$
$$e_{g,h}e_{k,l} = u_g y_1 u_h^* u_k y_1 u_l^* = \begin{cases} u_g y_1^2 u_l^* = u_g y_1 u_l^* & \text{if } h = k \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\{e_{g,h}\}$ is a set of matrix units in D, and thus Corollary A.9 of [**55**] gives a homomorphism $y \times u : \mathcal{K}(\ell^2(\Gamma)) \to D$ such that for the canonical generator $e_g \otimes \overline{e}_h \in \mathcal{K}(\ell^2(\Gamma))$, $(y \times u)(e_g \otimes \overline{e}_h) = u_g y_1 u_h^*$. We claim that $\lambda_h \chi_g = e_{hg} \otimes \overline{e}_g$. First,

$$\begin{aligned}(\lambda_h \chi_g)\xi(k) &= \lambda_h(\chi_g(\xi))(k) = \chi_g(\xi)(h^{-1}k) = \chi_g(h^{-1}k)\xi(h^{-1}k) \\ &= \begin{cases} \xi(h^{-1}k) & \text{if } h^{-1}k = g \text{ (or } k = hg) \\ 0 & \text{otherwise} \end{cases}\end{aligned}$$

Recall that the rank one operator $h \otimes \overline{k}$ is defined by $g \mapsto \langle g|k\rangle h$ for g in some Hilbert space:

$$(e_{hg} \otimes \overline{e_g})\xi(k) = \langle \xi | e_g \rangle e_{hg}(k) = \xi(g) e_{hg}(k) = \begin{cases} \xi(g) & \text{if } k = hg \\ 0 & \text{otherwise.} \end{cases}$$

Hence $\lambda_h \chi_g = e_{hg} \otimes \overline{e}_g$ as they have the same effect on $\xi \in \ell^2(\Gamma)$. We then have $y \times u(\lambda_h \chi_g) = u_{hg} y_1 u_g^* = u_{hg} y_{gg^{-1}} u_g^* = u_{hg} u_{g^{-1}} y_g = u_h y_g$. □

The purpose of the following lemma is to construct a Cuntz-Krieger Λ-family in $M(C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$. It will take several discrete steps, and we have set out the result to reflect this.

LEMMA 5.6. *Suppose that Λ, Γ and η are as in Theorem 5.1.*

(a) *For $g, h \in \Gamma$, the elements*

$$y_g := \overline{i_{C^*(\Lambda \times_\eta \Gamma)}}(p_{\Lambda^0 \times \{g\}}) \quad \text{and} \quad u_h := i_\Gamma(h)$$

of $M(C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$ satisfy (5.3).

(b) *The homomorphism $y \times u$ from Lemma 5.5 is nondegenerate; for $k \in \Gamma$, the elements $w_k := \overline{y \times u}(\rho_k)$ commute with u_h and satisfy*

$$w_k y_g = y_{gk^{-1}} w_k. \tag{5.4}$$

(c) *The partial isometries*

$$T_\lambda := \overline{i_{C^*(\Lambda \times_\eta \Gamma)}}(s_{\{\lambda\} \times \Gamma}) w_{\eta(\lambda)}^{-1} \in M(C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$$

commute with every y_g, u_h and w_k.

(d) *$\{T_\lambda : \lambda \in \Lambda\}$ is a Cuntz-Krieger Λ-family in $M(C^*(\Lambda \times_\eta \Gamma) \rtimes_{\mathrm{lt}_*} \Gamma)$.*

PROOF. We choose increasing sequences of finite subsets G_n of Λ^0 and H_n of Γ such that $\Lambda^0 = \bigcup_n G_n$ and $\Gamma = \bigcup_n H_n$. Then the strict continuity of $\overline{i_{C^*(\Lambda\times_\eta\Gamma)}}$ implies that

$$i_{C^*(\Lambda\times_\eta\Gamma)}\Big(\sum_{v\in G_n} s_{(v,g)}\Big) \to y_g \ \text{ strictly.}$$

For each n, the covariance of $(i_{C^*(\Lambda\times_\eta\Gamma)}, i_\Gamma)$ implies that

$$u_h i_{C^*(\Lambda\times_\eta\Gamma)}\Big(\sum_{v\in G_n} s_{(v,g)}\Big) = i_{C^*(\Lambda\times_\eta\Gamma)}\Big(\sum_{v\in G_n} s_{(v,hg)}\Big)u_h,$$

and since the right-hand side converges strictly to $y_{hg}u_h$, (5.3) follows. We have now proved (a).

Since $r(\alpha,g) = (r(\alpha),g)$ belongs to $\Lambda^0\times\{g\}$, the formula (2.1) of Proposition 2.8 shows that $s_{(\alpha,g)}s^*_{(\beta,h)} = y_g s_{(\alpha,g)}s^*_{(\beta,h)}$, and this implies that $y\times u$ is nondegenerate. So the formula for w_k makes sense. It has the described properties because ρ_k commutes with λ_h and satisfies $\rho_k\chi_g = \chi_{gk^{-1}}\rho_k$. We have now proved (b).

Since elements of $\Lambda^0\times\{g\}$ and $\{\lambda\}\times\Gamma$ have different sources, the last assertion in Proposition 2.8 implies that

$$y_g T_\lambda = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(p_{\Lambda^0\times\{g\}}s_{\{\lambda\}\times\Gamma})w^{-1}_{\eta(\lambda)} = i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})w^{-1}_{\eta(\lambda)}. \tag{5.5}$$

On the other hand, (5.4) implies that $w^{-1}_{\eta(\lambda)}y_g = y_{g\eta(\lambda)}w^{-1}_{\eta(\lambda)}$, and thus

$$T_\lambda y_g = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma}p_{\Lambda^0\times\{g\eta(\lambda)\}})w^{-1}_{\eta(\lambda)},$$

which since $s(\lambda,g) = (s(\lambda), g\eta(\lambda))$ is the same as the right-hand side of (5.5). Thus y_g commutes with T_λ.

To see that u_h commutes with T_λ, we realise $s_{\{\lambda\}\times\Gamma}$ as the strict limit of the finite sums $s_{\{\lambda\}\times H_n} := \sum_{g\in H_n} s_{(\lambda,g)}$. Then T_λ is the strict limit of $t_n := i_{C^*(\Lambda\times_\eta\Gamma)}(s_{\{\lambda\}\times H_n})$, and u_hT_λ is the strict limit of u_ht_n. Covariance of $(i_{C^*(\Lambda\times_\eta\Gamma)}, i_\Gamma)$ implies that

$$u_h t_n = i_\Gamma(h) i_{C^*(\Lambda\times_\eta\Gamma)}(s_{\{\lambda\}\times H_n}) = i_{C^*(\Lambda\times_\eta\Gamma)}(s_{\{\lambda\}\times hH_n})u_h, \tag{5.6}$$

and since the limit $s_{\{\lambda\}\times\Gamma}$ is independent of the choice of increasing subsets, the right-hand side of (5.6) converges strictly to $T_\lambda u_h$. Thus $u_hT_\lambda = T_\lambda u_h$. Since T_λ commutes with everything in the ranges of $y\times u$ and $\overline{y\times u}$, including w_k, we have proved (c).

Since $\eta(v) = 1$ for every vertex v, the relation (CK1) for $\{T_\lambda\}$ follows from the assertion $s_Vs_W = s_{VW}$ in Proposition 2.8. Take $V = \{v\}\times\Gamma$ and $W = \{w\}\times\Gamma$, then $T_VT_W = 0$ when $v \neq w$. For (CK2), we suppose λ and μ are composable in Λ. Then because $w^{-1}_{\eta(\lambda)}$ and T_μ commute, we have

$$T_\lambda T_\mu = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})T_\mu w^{-1}_{\eta(\lambda)} = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma}s_{\{\mu\}\times\Gamma})(w_{\eta(\lambda)\eta(\mu)})^{-1}.$$

The right-hand side then reduces to $T_{\lambda\mu}$ because $s_V s_W = s_{VW}$ as $\{\lambda\}\times\Gamma$ and $\{\mu\}\times\Gamma$ have different sources, $k \mapsto w_k$ is a homomorphism, and η is a functor. For (CK3), we need to compute

$$T_\lambda^* T_\lambda = w_{\eta(\lambda)} \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s^*_{\{\lambda\}\times\Gamma} s_{\{\lambda\}\times\Gamma}) w^{-1}_{\eta(\lambda)}.$$

We claim that $s^*_{\{\lambda\}\times\Gamma} s_{\{\lambda\}\times\Gamma} = s_{\{s(\lambda)\}\times\Gamma}$. From (2.1) and the adjoint of (2.2), we invoke Proposition 2.8 to deduce that $(s^*_{\{\lambda\}\times\Gamma} s_{\{\lambda\}\times\Gamma})(s_{(\alpha,g)} s^*_{(\beta,h)})$ vanishes unless $r(\alpha) = s(\lambda)$, and then is $s_{(\alpha,g)} s^*_{(\beta,h)}$; thus left multiplication on a spanning set $\{s_{(\alpha,g)} s^*_{(\beta,h)}\}$ by $s^*_{\{\lambda\}\times\Gamma} s_{\{\lambda\}\times\Gamma}$ is the same as left multiplication by $s_{\{s(\lambda)\}\times\Gamma}$. Then $s^*_{\{\lambda\}\times\Gamma} s_{\{\lambda\}\times\Gamma} = s_{\{s(\lambda)\}\times\Gamma}$ as $\{s_{(\alpha,g)} s^*_{(\beta,h)}\}$ spans a dense sub-$*$-algebra of $M(C^*(\Lambda\times_\eta\Gamma))$, and the claim is established. Thus $T_\lambda^* T_\lambda = w_{\eta(\lambda)} T_{s(\lambda)} w^{-1}_{\eta(\lambda)}$, and since $w_{\eta(\lambda)}$ commutes with $T_{s(\lambda)}$, we recover (CK3). For (CK4) we fix $v \in \Lambda^0$ and $n \in \mathbb{N}^k$, and then

$$\sum_{\lambda\in v\Lambda^n} T_\lambda T_\lambda^* = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}\Big(\sum_{\lambda\in v\Lambda^n} s_{\{\lambda\}\times\Gamma} s^*_{\{\lambda\}\times\Gamma}\Big). \tag{5.7}$$

A calculation similar to that used in the previous claim using the formulas in Proposition 2.8 shows that left multiplication by the inside sum is the same as left multiplication by $s_{\{v\}\times\Gamma}$. Then the right-hand side of (5.7) equals T_v, by definition. This gives (CK4), and we have proved (d). □

Proof of Theorem 5.1. Using Lemmas 5.2 and 5.3, we constructed a homomorphism in Remark 5.4

$$\phi := \pi_S \times (1\otimes\lambda) : C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma \to C^*(\Lambda)\otimes\mathcal{K}(\ell^2(\Gamma))$$

that satisfied (5.2). We claim that ϕ is surjective. Since (5.2) is satisfied, we note that the range of ϕ contains every element $s_\lambda \otimes \chi_g\rho_{\eta(\lambda)}\lambda_h$. As part of Lemma 5.5, we showed that $\chi_{hg}\lambda_h = \lambda_h\chi_g = e_{hg}\otimes\overline{e}_g$. A similar argument shows that $\rho_h\chi_g\chi_{gh^{-1}}\rho_h = e_{gh^{-1}}\otimes\overline{e}_g$. Hence $\chi_g\rho_{\eta(\lambda)} = e_g\otimes\overline{e}_{g\eta(\lambda)}$ and $\lambda_h = \chi_{hh^{-1}}\lambda_h = e_1\otimes\overline{e}_{h^{-1}}$. Therefore the operator $\chi_g\rho_{\eta(\lambda)}\lambda_h$ is the rank-one operator $e_g\otimes\overline{e}_{h^{-1}g\eta(\lambda)}$. For each λ, each matrix unit $e_p\otimes\overline{e}_q$ arises for a suitable choice of g and h. Thus the range of ϕ contains every $s_\lambda\otimes(e_p\otimes\overline{e}_q)$, and every

$$s_\lambda s_\mu^* \otimes (e_p\otimes\overline{e}_q) = \big(s_\lambda\otimes(e_p\otimes\overline{e}_q)\big)\big(s_\mu\otimes(e_q\otimes\overline{e}_q)\big)^*;$$

since these elements span a dense sub-$*$-algebra of $C^*(\Lambda)\otimes\mathcal{K}(\ell^2(\Gamma))$, and homomorphisms of C^*-algebras have closed range, we deduce that ϕ is surjective.

For injectivity, we consider the homomorphism $y\times u : \mathcal{K}(\ell^2(\Gamma)) \to M(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma)$ associated to the elements y_g and u_h described in Lemma 5.6(a), and the homomorphism π_T of $C^*(\Lambda)$ into $M(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma)$ given by the Cuntz-Krieger family $\{T_\lambda\}$ of Lemma 5.6. Lemma 5.6(b) implies that π_T and $y\times u$ have commuting ranges, and hence give a homomorphism $\theta := \pi_T\otimes(y\times u)$ of $C^*(\Lambda)\otimes\mathcal{K}(\ell^2(\Gamma))$ into $M(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma)$ such that $\theta(a\otimes k) = \pi_T(a)(y\times u)(k)$ (by [**60**, Theorem B.27]).

Finally we compute, using in particular the formula (5.4):

$$\begin{aligned}\theta \circ \phi\big(i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})i_\Gamma(h)\big) &= \theta\big(s_\lambda \otimes \chi_g\rho_{\eta(\lambda)}\lambda_h\big)\\ &= \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})w_{\eta(\lambda)}^{-1}y_g w_{\eta(\lambda)}u_h = \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}(s_{\{\lambda\}\times\Gamma})y_{g\eta(\lambda)}w_{\eta(\lambda)}^{-1}w_{\eta(\lambda)}u_h\\ &= \overline{i_{C^*(\Lambda\times_\eta\Gamma)}}\big(s_{\{\lambda\}\times\Gamma}p_{\Lambda^0\times\{g\eta(\lambda)\}}\big)i_\Gamma(h) = i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})i_\Gamma(h).\end{aligned}$$

Since the elements $i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})i_\Gamma(h)$ generate the crossed product, this proves that $\theta\circ\phi$ is the identity, and in particular that ϕ is injective. □

COROLLARY 5.7. *Suppose that Λ is a row-finite k-graph with no sources, S is an Ore semigroup with enveloping group Γ, and $\eta : \Lambda \to \Gamma$ is a functor. Then there is an isomorphism φ of $p\big(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p$ onto $C^*(\Lambda)\otimes\mathcal{K}(\ell^2(S))$ where $p := \overline{i_{C^*(\Lambda\times_\eta S)}}(p_{\Lambda^0\times S})$.*

PROOF. Theorem 5.1 gives an isomorphism ϕ of $C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma$ onto $C^*(\Lambda)\otimes\mathcal{K}(\ell^2(\Gamma))$ such that

$$\phi\big(i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,g)})\big) = s_\lambda\otimes\chi_g\rho_{\eta(\lambda)} \text{ and } \overline{\phi}(i_\Gamma(h)) = 1\otimes\lambda_h.$$

Since ϕ is an isomorphism, it extends to the multiplier algebra, and restricts an isomorphism of $p\big(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p$ onto $\overline{\phi}(p)\big(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)\overline{\phi}(p)$.

Write Λ^0 as an increasing union of finite sets, $\Lambda^0 = \bigcup_n G_n$ and S as an increasing union of finite sets, $S = \bigcup_n H_n$. Then by Proposition 2.8, $p_{\Lambda^0\times S}$ is by definition the strict limit of $p_n := \sum_{(w,u)\in G_n\times H_n} p_{(w,u)}$. Thus, since $\eta(w) = 1_S$ for every vertex w, $\overline{\phi}(p)$ is the strict limit of

$$\begin{aligned}\phi\big(i_{C^*(\Lambda\times_\eta\Gamma)}(p_n)\big) &= \sum_{(w,u)\in G_n\times H_n}\phi\big(i_{C^*(\Lambda\times_\eta\Gamma)}(p_{(w,u)})\big)\\ &= \sum_{(w,u)\in G_n\times H_n} p_w\otimes\chi_u = \Big(\sum_{w\in G_n}p_w\Big)\otimes\Big(\sum_{u\in H_n}\chi_u\Big).\end{aligned}$$

Since $\sum_{w\in G_n}p_w$ converges strictly to $1_{M(C^*(\Lambda))}$ and $\sum_{u\in H_n}\chi_u$ converges strictly to χ_S, a multiplier in $\mathcal{K}(\ell^2(\Gamma))$, the assertion about strict continuity in Lemma 6.2 implies that $\overline{\phi}(p) = 1_{M(C^*(\Lambda))}\otimes\chi_S$. A calculation on elementary tensors shows that

$$(1\otimes\chi_S)(C^*(\Lambda)\otimes\mathcal{K}(\ell^2(\Gamma)))(1\otimes\chi_S) = C^*(\Lambda)\otimes\chi_S\mathcal{K}(\ell^2(\Gamma))\chi_S.$$

Since we are identifying $\ell^2(S)$ with a subspace of $\ell^2(\Gamma)$ and χ_S is then the orthogonal projection of $\ell^2(\Gamma)$ onto $\ell^2(S)$, $\chi_S\mathcal{K}(\ell^2(\Gamma))\chi_S$ is naturally identified with $\mathcal{K}(\ell^2(S))$. When we make this identification, $\chi_S\lambda_u\chi_S$ is the generator $\lambda_u^S := \lambda_u\chi_S$ of the Toeplitz representation of S on $\ell^2(S)$. Thus restricting ϕ gives an isomorphism

$$\varphi : p\big(C^*(\Lambda\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p \to C^*(\Lambda)\otimes\mathcal{K}(\ell^2(S))$$

such that for t and u in S,

$$\varphi\big(p\, i_{C^*(\Lambda\times_\eta\Gamma)}(s_{(\lambda,t)})p\big) = s_\lambda\otimes(\chi_t\rho_{\eta(\lambda)})|_{\ell^2(S)} \text{ and } \overline{\varphi}(i_\Gamma(u)) = \lambda_u^S.$$

We note that although $\rho_{\eta(\lambda)}$ does not leave $\ell^2(S)$ invariant, the product $\chi_t \rho_{\eta(\lambda)}$ does. $\square$

CHAPTER 6

The main theorem

We are now in a position to prove the main result of the thesis: the crossed product of the k-graph C^*-algebra by the action of an Ore semigroup is stably isomorphic to the k-graph C^*-algebra of the quotient k-graph. We will use the isomorphisms from each of our three main steps. Theorem 3.11 allows us to recognise a row-finite k-graph with no sources Σ and free left-action α as a skew-product graph being acted on by left-translation. In Corollary 3.12 we showed that this induces an isomorphism between $C^*(\Sigma)$ and $C^*(S\backslash\Sigma \times_\eta S)$, which can be used to induce an isomorphism ψ_1 from $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*(S\backslash\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$. Using Corollary 4.4, we dilate the semigroup action to obtain the crossed product by the enveloping group, and give an isomorphism ψ_2 from $C^*(S\backslash\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$ onto a corner of $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$. Theorem 5.1 allows to recognise the crossed product in a tensor product. In Corollary 5.7, we show that a restriction to an appropriate full corner gives an isomorphism ψ_3 from a corner of $C^*(S\backslash\Sigma \times_\eta \Gamma) \times_{\mathrm{lt}_*} \Gamma$ onto $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$. Each of these maps needs to be shown to be compatible in terms of hypotheses with the next isomorphism. In particular, isomorphisms ψ_2 and ψ_3 maps onto corners, and we need to ensure these corners are full, and that the ranges are surjective onto the domain of the following isomorphism. Having explicit isomorphisms allows us to do this far more easily. We will consider some of the consequences of this result in Chapter 7.

Recall the notation for the maps $c : S\backslash\Sigma \to F$, $\eta : S\backslash\Sigma \to S$, $\xi : \Sigma \to S$ from Theorem 3.11. We wish to compute the image of $\psi_3 \circ \psi_2 \circ \psi_1$ from diagram 1.1.

THEOREM 6.1. *Suppose that Σ is a row-finite k-graph with no sources, and α is a free action of an Ore semigroup S on Σ which admits a fundamental domain F. Let $q : \Sigma \to S\backslash\Sigma$ be the quotient map, and define for $\lambda \in S\backslash\Sigma$ and $\sigma \in \Sigma$, $c : S\backslash\Sigma \to F$, $\eta : S\backslash\Sigma \to S$, $\xi : \Sigma \to S$ by*

$$q(c(\lambda)) = \lambda, \quad s(c(\lambda)) = \alpha_{\eta(\lambda)}(c(s(\lambda))) \quad \textit{and} \quad \sigma = \alpha_{\xi(\sigma)}(c(q(\sigma))). \tag{6.1}$$

Then there is an isomorphism ψ of $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$ such that for $\sigma \in \Sigma$ and $u \in S$

$$\psi(i_{C^*(\Sigma)}(s^\Sigma_\sigma)) = s_{q(\sigma)} \otimes (\chi_{\xi(\sigma)}\rho_{\eta(q(\sigma))}|_{\ell^2(S)}) \quad \textit{and} \quad \overline{\psi}(i_S(u)) = 1 \otimes \lambda^S_u.$$

We need a general lemma about tensor products of multipliers, as we will need to be rigorous about the images of infinite sums of projections in multiplier algebras.

Lemma 6.2. *Suppose that A and B are C^*-algebras. For each $m \in M(A)$ and $n \in M(B)$ there is a multiplier $m \otimes_{\max} n$ of $A \otimes_{\max} B$ such that*

$$(m \otimes_{\max} n)(a \otimes b) = ma \otimes nb \quad \text{and} \quad (a \otimes b)(m \otimes_{\max} n) = am \otimes bn. \tag{6.2}$$

The map $\iota : (m, n) \mapsto m \otimes_{\max} n$ is strictly continuous in the following weak sense: if $m_i \to m$ strictly in $M(A)$, $n_i \to n$ strictly in $M(B)$, and both $\{m_i\}$ and $\{n_i\}$ are bounded, then $m_i \otimes_{\max} n_i \to m \otimes_{\max} n$.

Proof. Consider the canonical maps $j_A : A \to M(A \otimes_{\max} B)$ and $j_B : B \to M(A \otimes_{\max} B)$, as in, for example, [**60**, Theorem B.27]. Then j_A and j_B are nondegenerate homomorphisms with commuting ranges such that $j_A(a)j_B(b) = a \otimes b$ [**60**, Theorem B.27(a)]. The extensions $\overline{j_A}$ to $M(A)$ and $\overline{j_B}$ to $M(B)$ also have commuting ranges, and hence there is a homomorphism $\overline{j_A} \otimes_{\max} \overline{j_B}$ of $M(A) \otimes_{\max} M(B)$ into $M(A \otimes_{\max} B)$ such that $\overline{j_A} \otimes_{\max} \overline{j_B}(m \otimes n) = \overline{j_A}(m)\overline{j_B}(n)$. We define $m \otimes_{\max} n := \overline{j_A} \otimes_{\max} \overline{j_B}(m \otimes n)$ Then

$$\begin{aligned}(m \otimes_{\max} n)(a \otimes b) &= (\overline{j_A}(m)\overline{j_B}(n))(j_A(a)j_B(b)) = (\overline{j_A}(m)j_A(a))(\overline{j_B}(n)j_B(b))\\ &= j_A(ma)j_B(nb) = ma \otimes nb,\end{aligned}$$

and similarly on the other side

$$\begin{aligned}(a \otimes b)(m \otimes_{\max} n) &= (j_A(a)j_B(b))(\overline{j_A}(m)\overline{j_B}(n)) = (j_A(a)\overline{j_A}(m))(j_B(b)\overline{j_B}(n))\\ &= j_A(am)j_B(bn) = am \otimes bn.\end{aligned}$$

Since $\overline{j_A}$ and $\overline{j_B}$ are strictly continuous, $\overline{j_A}(m_i) \to \overline{j_A}(m)$ and $\overline{j_B}(n_i) \to \overline{j_B}(n)$, and the strict continuity of multiplication on bounded sets implies that $m_i \otimes_{\max} n_i = \overline{j_A}(m_i)\overline{j_B}(n_i)$ converges to $\overline{j_A}(m)\overline{j_B}(n) = m \otimes_{\max} n$. □

When we apply Lemma 6.2, at least one of A or B is nuclear, and $A \otimes_{\max} B$ coincides with the usual spatial tensor product; then, since there is at most one multiplier satisfying (6.2), $m \otimes_{\max} n$ coincides with the usual spatially defined $m \otimes n$. However, $M(A)$ and $M(B)$ need not be nuclear (even for $B = \mathcal{K}(\mathcal{H})$!), so this observation merely says that $\overline{j_A} \otimes_{\max} \overline{j_B}$ on $M(A) \otimes_{\max} M(B)$ factors through the spatial tensor product.

Proof of Theorem 6.1. In our version of the Gross-Tucker theorem, Theorem 3.11, we describe an isomorphism ϕ of Σ onto the skew-product $(S\backslash\Sigma) \times_\eta S$ such that $\phi \circ \alpha_t = \mathrm{lt}_t \circ \phi$. By Corollary 3.12, the induced isomorphism ϕ_* of $C^*(\Sigma)$ onto $C^*((S\backslash\Sigma) \times_\eta S)$ satisfies $\phi_* \circ \alpha_* = \mathrm{lt}_* \circ \phi_*$, and hence induces an isomorphism ψ_1 of $C^*(\Sigma) \times_{\alpha_*} S$ onto $C^*((S\backslash\Sigma) \times_\eta S) \times_{\mathrm{lt}_*} S$ satisfying

$$\psi_1(i_{C^*(\Sigma)}(s_\sigma^\Sigma)) = i_{C^*((S\backslash\Sigma)\times_\eta S)}(s_{(q(\sigma),\xi(\sigma))}) \quad \text{and} \quad \overline{\psi_1}(i_S(u)) = i_S(u).$$

We want to apply Corollary 4.4 with $\Lambda = S\backslash\Sigma \times_\eta \Gamma$, $\Omega = S\backslash\Sigma \times_\eta S$ and $\beta = \mathrm{lt}$. The subgraph Ω is saturated, because $r(\lambda, g) = (r(\lambda), g)$ belongs to Ω^0 precisely when $g \in S$, in which case (λ, g) belongs to Ω. We trivially have $\mathrm{lt}_t(\Omega) \subset \Omega$ for $t \in S$, and because $\Gamma = S^{-1}S$, every $g \in \Gamma$ can be written as $t^{-1}u$ for $t, u \in S$, and then every $(\lambda, g) = \mathrm{lt}_t^{-1}(\lambda, u)$ belongs to $\bigcup_{t\in S} \mathrm{lt}_t^{-1}(\Omega)$. The restriction of lt_u to Ω is the action by left translation on $S\backslash\Sigma\times_\eta S$, lt_u, from the previous paragraph. So with $p := \overline{i_{C^*(S\backslash\Sigma\times_\eta S)}}(p_{(S\backslash\Sigma)^0\times S})$, Corollary 4.4 gives an isomorphism ψ_2 of $C^*(S\backslash\Sigma\times_\eta S)\times_{\mathrm{lt}_*} S$ onto $p\big(C^*(S\backslash\Sigma\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p$ such that

$$\psi_2\big(i_{C^*(S\backslash\Sigma\times_\eta S)}(s_{(\lambda,t)})\big) = i_{C^*(S\backslash\Sigma\times_\eta\Gamma)}(s_{(\lambda,t)}) \quad\text{and}\quad \overline{\psi_2}(i_S(u)) = i_\Gamma(u)p. \tag{6.3}$$

Apply Corollary 5.7 with $\Lambda := S\backslash\Sigma$ to obtain an isomorphism ψ_3 from $p\big(C^*(S\backslash\Sigma\times_\eta\Gamma)\rtimes_{\mathrm{lt}_*}\Gamma\big)p$ onto $C^*(S\backslash\Sigma)\otimes\mathcal{K}(\ell^2(S))$ such that for t and u in S,

$$\psi_3\big(p\, i_{C^*(S\backslash\Sigma\times_\eta\Gamma)}(s_{(\lambda,t)})p\big) = s_\lambda\otimes(\chi_t\rho_{\eta(\lambda)})|_{\ell^2(S)} \quad\text{and}\quad \overline{\psi_3}(i_\Gamma(u)) = \lambda_u^S.$$

Now $\psi := \psi_3\circ\psi_2\circ\psi_1$ has the required properties. □

Corollary 6.3. *Suppose Λ is a row-finite k-graph with no sources, S is an Ore semigroup and $\eta : \Lambda\to S$ is a functor. Then $C^*(\Lambda\times_\eta S)\times_{\mathrm{lt}_*} S$ is isomorphic to $C^*(\Lambda)\otimes\mathcal{K}(\ell^2(S))$.*

Proof. We wish to apply Theorem 6.1 with $\Sigma = \Lambda\times_\eta S$ and $\alpha = \mathrm{lt}$. By Lemma 3.15, the action lt on $\Lambda\times_\eta S$ admits a fundamental domain. Hence, the hypotheses of Theorem 6.1 are satisfied, and we have the desired result. □

CHAPTER 7

Applications

We consider two interesting applications of Theorem 6.1. The first involves taking a non-free action that will be free on some dual graph, and showing that the resultant crossed products are isomorphic. The second application involves finding necessary and sufficient conditions for a crossed product to be simple.

7.1. An example involving the dual graph

The following is an interesting example using concepts introduced by Allen, Pask and Sims in [**4**]. Summarising [**4**, Definition 3.1,Proposition 3.2]: for a k-graph (Σ, d) and $p \in \mathbb{N}^k$, we can define a dual k-graph $(p\Sigma, d_p)$ by taking $p\Sigma := \{\sigma \in \Sigma : d(\sigma) \geq p\}$, range and source maps $r_p(\sigma) = \sigma(0,p)$, $s_p(\sigma) = \sigma(d(\sigma) - p, d(\sigma))$ for $\sigma \in p\Sigma$, and composition by $\sigma \circ_p \mu = \sigma\mu(p, d(\mu)) = \sigma(0, d(\sigma) - p)\mu$ whenever $s_p(\sigma) = r_p(\mu)$. The degree map d_p on $p\Sigma$ is defined $d_p(\sigma) = d(\sigma) - p$.

Taking $\{s_\sigma\}$ to be the universal Cuntz-Krieger Σ-family in $C^*(\Sigma)$ and $\{t_\sigma\}$ to be the universal Cuntz-Krieger $p\Sigma$-family in $C^*(p\Sigma)$, [**4**, Theorem 3.5] shows there exists an isomorphism $\phi : C^*(p\Sigma) \to C^*(\Sigma)$ such that $\phi(t_\sigma) = s_\sigma s^*_{s_p(\sigma)}$.

Suppose $\alpha : S \to \mathrm{End}(\Sigma)$ is a left action. (Note that we do not necessarily require freeness.) We check that the induced map, $\widehat{\alpha}$, is also an action on $p\Sigma$: we require $\alpha_s(s_p(\sigma)) = s_p(\alpha_s(\sigma))$ for σ with $d(\sigma) \geq p$ (and similarly for r_p). We observe that $\sigma = \sigma' s_p(\sigma)$. Then as α is a functor, $\alpha_s(\sigma) = \alpha_s(\sigma')\alpha_s(s_p(\sigma))$. The term $s_p(\sigma)$ has degree p, and as α is degree-preserving, so does $\alpha_s(s_p(\sigma))$. Hence by the k-graph factorisation property, $s_p(\alpha_s(\sigma)) = \alpha_s(s_p(\sigma))$. Similarly $r_p(\alpha_s(\sigma)) = \alpha_s(r_p(\sigma))$.

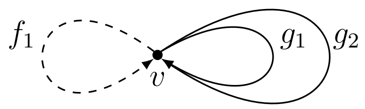
We are now able to prove that the isomorphism ϕ from [**4**, Theorem 3.5] is equivariant. Noting that $\widehat{\alpha}_s(t_\sigma) = t_{\alpha_s(\sigma)}$, we calculate:

$$(\alpha_s \circ \phi)(t_\sigma) = \alpha_s(s_\sigma s^*_{s_p(\sigma)}) = s_{\alpha_s(\sigma)} s^*_{\alpha_s(s_p(\sigma))}$$
$$(\phi \circ \widehat{\alpha}_s)(t_\sigma) = \phi(t_{\alpha_s(\sigma)}) = s_{\alpha_s(\sigma)} s^*_{s_p(\alpha_s(\sigma))}.$$

Proposition 7.1. *Suppose Σ is a row-finite k-graph with no sources and α is a left action of an Ore semigroup on Σ. Then $C^*(\Sigma) \times_{\alpha_*} S$ is isomorphic to $C^*(p\Sigma) \times_{\widehat{\alpha}_*} S$. Furthermore, if the action of $\widehat{\alpha}$ of S on $p\Sigma$ is free and admits a fundamental domain, $C^*(\Sigma) \times_{\alpha^*} S$ is a stably isomorphic to $C^*(S\backslash p\Sigma)$.*

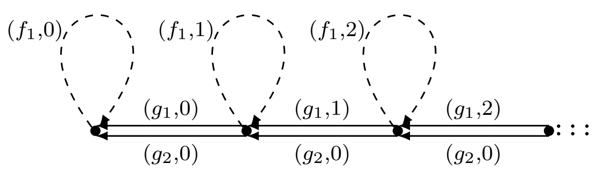
Proof. With notation as above, take $\phi : C^*(p\Sigma) \to C^*(\Sigma)$ such that $\phi(t_\sigma) = s_\sigma s^*_{s_p(\sigma)}$. A previous calculation showed that $\alpha_s \circ \phi = \phi \circ \widehat{\alpha}_s$, and so ϕ is an equivariant isomorphism. Therefore, $C^*(\Sigma) \times_{\alpha_*} S$ is isomorphic to $C^*(p\Sigma) \times_{\widehat{\alpha}_*} S$. If the action of S on $p\Sigma$ is free, we can apply Theorem 6.1 and conclude $C^*(\Sigma) \times_{\alpha_*} S$ is isomorphic to $C^*(S \backslash p\Sigma) \otimes \mathcal{K}(\ell^2(S))$, using $C^*(p\Sigma) \times_{\widehat{\alpha}_*} S$ as an intermediate step. □

EXAMPLE 7.2. This example uses the 2-graph $\mathbb{F}^2_\theta$ of [**14**]. Suppose we have a 2-graph with the following 1-skeleton:



with factorisation rules: $f_1 g_1 = g_2 f_1$ and $f_1 g_2 = g_1 f_1$. Define a map $\theta : \underline{1} \times \underline{2} \to \underline{1} \times \underline{2}$ by $\theta(1,1) = (1,2)$ and $\theta(1,2) = (1,1)$. Hence we have a 2-graph $\mathbb{F}^2_\theta$ as defined in Example 2.1.

Define a functor $\eta : \mathbb{F}^2_\theta \to \mathbb{N}$ by $\eta(f_1) = 0$ and $\eta(g_1) = \eta(g_2) = 1$. We can then form the skew-product graph $\mathbb{F}^2_\theta \times_\eta \mathbb{N}$, which has the following 1-skeleton:

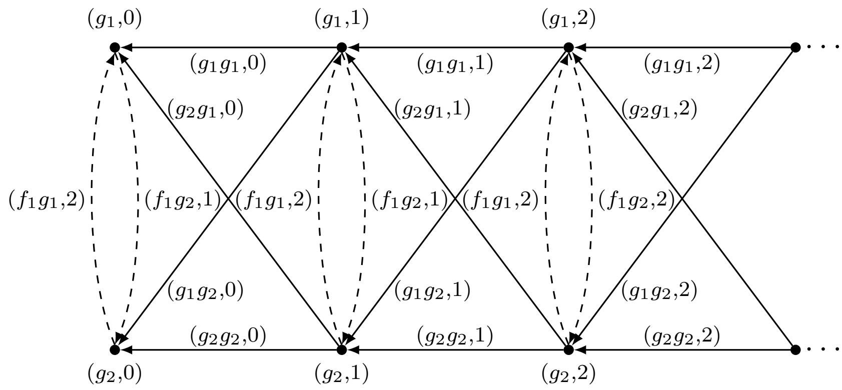


We claim that there is a non-free action α of $\mathbb{N} \times \mathbb{Z}_2$ on the skew-product graph: define for $m, n \in \mathbb{N}$, $i \in \{1,2\}$, $j \in \{0,1\}$,

$$\alpha_{(n,j)}(f_1, m) = (f_1, m+n),\ \alpha_{(n,0)}(g_i, m) = (g_i, m+n),$$
$$\alpha_{(n,1)}(g_1, m) = (g_2, m+n),\ \alpha_{(n,1)}(g_2, m) = (g_1, m+n).$$

Note that $\alpha_{(0,j)}$ fixes (v, n) and (f_1, n) for all $n \in \mathbb{N}$, and hence the action is not free.

If we instead take the e_1-dual of $\mathbb{F}_\theta^2 \times_\eta \mathbb{N}$, then $(e_1(\mathbb{F}_\theta^2 \times_\eta \mathbb{N}))^0 = \Sigma^{e_1}$ and $(e_1(\mathbb{F}_\theta^2 \times_\eta \mathbb{N}))^n = \Sigma^{n+e_1}$. The dual graph $e_1(\mathbb{F}_\theta^2 \times_\eta \mathbb{N})$ has 1-skeleton:



The induced action of $\mathbb{N} \times \mathbb{Z}_2$ switches the upper and lower labels, and $\mathbb{N}$-shifts to the right. This action is free.

There is a fundamental domain for this 2-graph: $F = \{\mu \in e_2(\mathbb{F}_\theta^2 \times_\eta \mathbb{N}) : r(\mu) = (g_i, 0)\}$. In which case the quotient 2-graph $\mathbb{N}\backslash e_2(\mathbb{F}_\theta^2 \times_\eta \mathbb{N})$ is $\mathbb{F}_\theta^2$, which is not unexpected, as $e_2(\mathbb{F}_\theta^2 \times_\eta \mathbb{N})$ is a skew-product of $\mathbb{F}_\theta^2$. This, however, will not be true in general.

By Proposition 7.1 we have that $C^*(e_2(\mathbb{F}_\theta^2 \times_\eta \mathbb{N})) \times_{\widehat{\alpha_*}} (\mathbb{N} \times \mathbb{Z}_2)$ is isomorphic to $C^*(\mathbb{F}_\theta^2) \otimes \mathcal{K}(\ell^2(\mathbb{N}))$, and using our main result, Theorem 6.1, $C^*(\mathbb{F}_\theta^2) \otimes \mathcal{K}(\ell^2(\mathbb{N}))$ is isomorphic to $C^*(\mathbb{F}_\theta^2 \times_\eta \mathbb{N}) \times_{\alpha_*} (\mathbb{N} \times \mathbb{Z}_2)$.

7.2. Applications to simplicity

Let α be a free action of an Ore semigroup S on a row-finite k-graph Σ. Theorem 6.1 says that the crossed product $C^*(\Sigma) \times_{\alpha_*} S$ is isomorphic to $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$. Since $\mathcal{K}(\ell^2(S))$ is simple, the crossed product is simple if and only if $C^*(S\backslash\Sigma)$ is simple. We are interested in the simplicity of the semigroup crossed product. We give necessary and sufficient conditions for a skew-product k-graph to be cofinal, and necessary and sufficient conditions for a quotient k-graph to be aperiodic. In Corollary 7.10 we prove that these conditions used together give necessary and sufficient conditions for the crossed product to be simple.

We use the Robertson-Sims aperiodicity condition [**63**, Theorem 3.2]:

Definition 7.3. A row-finite k-graph Λ with no sources has *no local periodicity* at $v \in \Lambda^0$ if for all $m \neq n \in \mathbb{N}^k$ there exists a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

Λ is called *aperiodic* if every $v \in \Lambda^0$ has no local periodicity.

Examples 7.4. (Aperiodicity)

(1) The k-graph Ω_k is aperiodic for all $k \geq 1$. First observe that there is no local periodicity at $v = (0,0)$. Given $m \neq n \in \mathbb{N}^k$, let $N = m \vee n$; then $\lambda = (0, N)$ is the only element of $v\Omega_k$. Then $\lambda(m,m) = (m,m) \neq (n,n) = \lambda(n,n)$. For any other vertex $w = (n,n)$ in Ω_k, we have $v\Omega_k w \neq \emptyset$ and so there is no local periodicity at w by [**33**, Lemma 3.6].

(2) The k-graph T_k is not aperiodic for all $k \geq 1$. For all $n \in \mathbb{N}^k$ one checks that $f_1^{n_1} \cdots f_k^{n_k}$ is the only element of vT_k^n. Hence given $m \neq n \in \mathbb{N}^k$ it follows that for all $\lambda \in v\Lambda^{m \vee n}$ we have

$$\lambda(m, m + (m \vee n)) = \lambda(n, n + (m \vee n)).$$

We use the Lewin-Sims [**42**, Remark A.3] definition of cofinality for a row-finite k-graph:

Definition 7.5. A row-finite, k-graph Λ with no sources is *cofinal* if for all pairs $v, w \in \Lambda^0$ there exists $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$.

We will use the following result from Robertson and Sims [**63**, Theorem 3.1] to assist us when we are considering the simplicity of a graph algebra.

Theorem 7.6. *[Robertson-Sims] Let Γ be a row–finite k-graph with no sources, then $C^*(\Gamma)$ is simple if and only if Γ is cofinal and aperiodic.*

Definition 7.7. Let Λ be a row-finite k-graph with no sources and S an Ore semigroup acting on Λ. We say that (Λ, S, α) is *S–aperiodic* if for all $v \in \Lambda^0$, $m \neq n \in \mathbb{N}^k$ and $s, t \in S$ there is $\lambda \in v\Lambda^N$ where $N \geq m \vee n$ such that

$$\alpha_s \lambda(m, m + d(\lambda) - m \vee n) \neq \alpha_t \lambda(n, n + d(\lambda) - m \vee n).$$

It is clear that S–aperiodicity is preserved for equivariant isomorphisms.

Theorem 7.8. *Suppose Σ is a row-finite k-graph with no sources, S is an Ore semigroup, and α is a free left action of S on Σ. Then (Σ, S, α) is S–aperiodic if and only if $S\backslash\Sigma$ is aperiodic.*

Proof. Fix $[v] \in (S\backslash\Sigma)^0$, $m \neq n \in \mathbb{N}^k$, and suppose (Σ, S, α) is S–aperiodic. Given $v \in \Sigma^0$, $m \neq n \in \mathbb{N}^k$, $s, t \in S$, there exists $\sigma \in v\Sigma^N$ where $N > m \vee n$ such that

$$\alpha_s \sigma(m, m + d(\sigma) - (m \vee n)) \neq \alpha_t \sigma(n, n + d(\sigma) - (m \vee n)).$$

Then by Proposition 3.5, using definition of the quotient graph, and using that $d_\Sigma(\sigma) = d_{S\backslash\Sigma}(S\backslash\sigma)$ we have,

$$[\sigma](m, m + d([\sigma]) - (m \vee n)) \neq [\sigma](n, n + d([\sigma]) - (m \vee n))$$

and so $S\backslash\Sigma$ is aperiodic.

Suppose that $S\backslash\Sigma$ is aperiodic. Fix $v \in \Sigma^0$, $m \neq n \in \mathbb{N}^k$ and $s, t \in S$, then for $[v] \in (S\backslash\Sigma)^0$ there is $[\sigma] \in [v](S\backslash\Sigma)$ and $N > m \vee n$ such that

$$[\sigma](m, m + d([\sigma]) - (m \vee n)) \neq [\sigma](n, n + d([\sigma]) - (m \vee n)).$$

We claim that $\alpha_s\sigma(m, m + d(\sigma) - (m \vee n)) \neq \alpha_t\sigma(n, n + d(\sigma) - (m \vee n))$. For contradiction, suppose not: then since $\alpha_s\sigma(m, m + d(\sigma) - (m \vee n)) = \alpha_t\sigma(n, n + d(\sigma) - (m \vee n))$ we have $[\sigma](m, m + d([\sigma]) - (m \vee n)) = [\sigma](n, n + d([\sigma]) - (m \vee n))$ which contradicts $S\backslash\Sigma$ being aperiodic. Hence $\alpha_s\sigma(m, m + d([\sigma]) - (m \vee n)) \neq \alpha_t\sigma(n, n + d([\sigma]) - (m \vee n))$ and so (Σ, S, α) is S–aperiodic. ☐

This completes the picture between properties of the dynamical system and properties of the quotients. However, in practice, it will probably be simpler to check that the quotient is aperiodic rather than the dynamical system being S–aperiodic.

COROLLARY 7.9. *Suppose Σ is a row-finite k-graph with no sources, S is an Ore semigroup, $\eta : \Sigma \to S$ is a functor, and α is a free left action of S on Σ that admits a fundamental domain. Then $(\Sigma \times_\eta S, S, \mathrm{lt})$ is S–aperiodic if and only if Σ is aperiodic.*

PROOF. Using Proposition 3.7, we observe that the quotient of $\Sigma \times_\eta S$ by the lt-action is Σ, and the result follows by applying Theorem 7.8. ☐

COROLLARY 7.10. *Suppose Σ be a row-finite k-graph with no sources, α is a free action of an Ore semigroup S that admits a fundamental domain. Then $C^*(\Sigma) \times_{\alpha_*} S$ is simple if and only if $S\backslash\Sigma$ is cofinal and (Σ, S, α) is S–aperiodic.*

PROOF. By Theorem 6.1 we have $C^*(\Sigma) \times_{\alpha_*} S$ is isomorphic to $C^*(S\backslash\Sigma) \otimes \mathcal{K}(\ell^2(S))$. Since Morita equivalence preserves simplicity, it follows that the crossed product $C^*(\Sigma) \times_{\alpha_*} S$ is simple if and only if $C^*(S\backslash\Sigma)$ is simple. By Theorem 7.6, this is equivalent to $S\backslash\Sigma$ being aperiodic and cofinal. By Theorem 7.8, $S\backslash\Sigma$ is aperiodic if and only if (Σ, S, α) is aperiodic, which gives the result. ☐

COROLLARY 7.11. *Suppose Σ be a row-finite k-graph with no sources, α is a free action of an Ore semigroup S that admits a fundamental domain. Then $C^*(\Sigma \times_\eta S) \times_{\mathrm{lt}_*} S$ is simple if and only if Σ is cofinal and aperiodic.*

PROOF. Applying Corollary 7.9 and Corollary 7.10 to the skew-product k-graph $\Sigma \times_\eta S$ gives the required result. ☐

We can use Theorem 6.1 to show that the crossed product $C^*(\Sigma) \times_{\alpha_*} S$ arising from a free action α of an Ore semigroup S on a row-finite k-graph Σ is purely infinite and simple.

Let Σ be a row-finite k-graph with no sources. Following [**69**, Definition 8.7] we say that $\sigma \in \Sigma$ is a *loop with an entrance* if $s(\sigma) = r(\sigma)$ and there exists $\alpha \in s(\sigma)\Sigma$ with $d(\sigma) \geq d(\alpha)$ and $\sigma(0, d(\alpha)) \neq \alpha$. We say that a vertex $v \in \Sigma^0$ can be *reached from a loop with an entrance* if there is a loop with an entrance $\sigma \in \Sigma$ such that $v\Sigma s(\sigma) \neq \emptyset$.

Lemma 7.12. *Let Σ be an aperiodic row-finite k-graph. Then every loop σ with $d(\sigma) > 0$ has an entrance.*

Proof. Let Σ be aperiodic. Suppose, for contradiction, that σ is a loop with $d(\sigma) > 0$ that does not have an entrance. Then for all $\alpha \in s(\sigma)\Sigma$ with $d(\sigma) \geq d(\alpha)$ we have $\sigma(0, d(\alpha)) = \alpha$. Therefore $|s(\sigma)\Sigma^n| = 1$ for all $0 \leq n < d(\sigma)$.

By the factorisation property one sees that $\sigma\sigma$ is also a loop that does not have an entrance. By the above argument it then follows that $|s(\sigma)\Sigma^n| = 1$ for all $0 \leq n < 2d(\sigma)$. Since $d(\sigma) > 0$ it follows that $|s(\sigma)\Sigma^n| = 1$ for all $n \in \mathbb{N}^k$.

Fix $m = d(\sigma)$ and $n = 2d(\sigma)$ then $m \neq n$ and for all $N \geq m \vee n = 2d(\sigma)$ and $\mu \in s(\sigma)\Sigma^N$ we have

$$\begin{aligned}\mu(d(\sigma), d(\sigma) + d(\mu) - 2d(\sigma)) &= \sigma(d(\sigma), d(\mu) - d(\sigma)) \\ &= \sigma^p\sigma(0, q) \text{ where } N - 2d(\sigma) = pd(\sigma) + q \\ &= \mu(2d(\sigma), d(\mu)) \\ &= \mu(2d(\sigma), 2d(\sigma) + d(\mu) - 2d(\sigma)).\end{aligned}$$

But this contradicts the hypothesis that Σ is aperiodic. □

Proposition 7.13. *Let Σ be an aperiodic, cofinal, row-finite k-graph with no sources. If there is a loop σ with $d(\sigma) > 0$ then $C^*(\Sigma)$ is simple and purely infinite.*

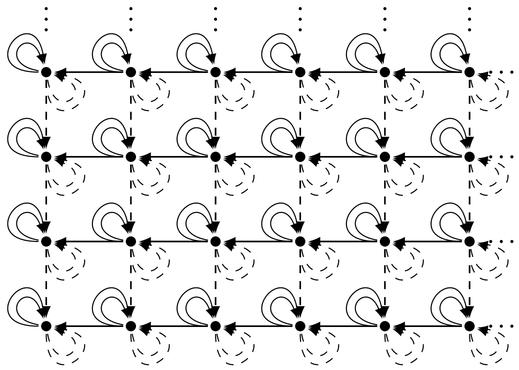
Proof. By Theorem 7.6 it follows that $C^*(\Sigma)$ is simple since Σ is aperiodic and cofinal. From [**69**, Proposition 8.8] it suffices to show that every vertex $v \in \Sigma^0$ can be reached from a loop with an entrance. Fix $v \in \Sigma^0$ and $w = s(\sigma)$. Since Σ is cofinal there is N such that for all $\alpha \in w\Sigma^N$ there is $\beta \in v\Sigma s(\alpha)$. Since $d(\sigma) > 0$ there is n such that $d(\sigma^n) > N$. Hence there is $\beta \in v\Sigma w$ and since σ has an entrance by Lemma 7.12 the proof is complete. □

Example 7.14. Recall the definition of $\mathbb{F}^2_\theta$ given in Examples 2.1. Let $\Gamma = \mathbb{F}^2_\theta$ where $\theta : \underline{3} \times \underline{3} \to \underline{3} \times \underline{3}$ is given by

$$\begin{aligned}&\theta(2, j) = (1, j), \quad \theta(1, j) = (2, j), \quad \theta(3, j) = (3, j) \quad \text{for } j = 1, 3 \text{ and} \\ &\theta(i, 2) = (i, 2) \ \text{ for } i = 1, 2, 3.\end{aligned}$$

Define a map $c : \Gamma \to \mathbb{N}^2$ by $c(g_3) = (0, 1)$, $c(f_3) = (1, 0)$, $c(f_i) = (0, 0)$, $c(g_i) = (0, 0)$, for $i = 1, 2$ then since c preserves the factorization rules in Γ, it extends to a functor $c : \Gamma \to \mathbb{N}^2$.

Then the 1-skeleton of $\Sigma = \Gamma \times_c \mathbb{N}^2$ is given as follows.



and factorization rules induced from those in Γ. There is a natural free action of $\mathbb{N}^2$ on Σ given by left translation. As in [**33**, Example 5.7] we see that Γ is aperiodic.

Since Γ has only one vertex, it is straightforward to see that it is cofinal, and so by Theorem 7.6 it follows that $C^*(\Gamma)$ is simple. Moreover, one checks that $\sigma = f_1 g_1$ is a loop in Γ with $d(\sigma) = (1,1) > 0$ and so by Proposition 7.13 $C^*(\Gamma)$ is purely infinite. Then by Theorem 6.1 and [**68**, Proposition 4.1.8] it follows that $C^*(\Sigma) \times_{\mathrm{lt}_*} \mathbb{N}^2$ is purely infinite and simple.

CHAPTER 8

Semigroups and coactions

The motivation for this chapter is to find conditions for which the fixed point algebra is simple. In Section 8.1 we consider some conditions for when the skew-product graph is aperiodic. In Section 8.2 we consider the conditions for when the skew-product graph is cofinal, and give necessary and sufficient conditions for the simplicity of the C^*-algebra of the skew-product graph. The main result is Theorem 8.19. In Section 8.3, we state a generalised form of primitivity in the context of skew-product graphs. We are able to restate Theorem 8.19 using these new results as Theorem 8.27. In Section 8.4, we consider group skew-product graphs. We know that an action by a group gives rise to a natural coaction. In [**54**], Quigg gives necessary and sufficient conditions for the fixed point algebra of the coaction crossed product to be simple. In Theorem 8.31, we apply these conditions for a group skew-product graph. In Section 8.5 we apply Theorem 8.31 to the gauge action to give the result Theorem 8.33, which says that the fixed point algebra of a k-graph C^*-algebra is simple if and only if the k-graph is primitive. This is of interest because of its potential applications to results of Takehana and Katayama [**34**] regarding outer automorphisms.

We have previously taken a dynamical system and have been interested in the simplicity of the associated crossed product. Suppose that we start with a skew-product graph and a functor, and consider the simplicity of the skew-product graph algebra.

Given a k-graph Λ, an Ore semigroup S, and functor $\eta : \Lambda \to S$, we are interested in the simplicity of $C^*(\Lambda \times_\eta S)$. We aim to apply a simplicity result of Robertson-Sims, Theorem 7.6, and so we need to know about more about aperiodicity and cofinality of skew-product k-graphs.

In Proposition 3.3, we saw that functor η multiplies on the right in the definition of the source map in a skew product graph $\Lambda \times_\eta S$. This forces us to consider left-reversible semigroups here. A countable, cancellative semigroup with identity S is said to be *left-reversible* if for all $s, t \in S$ we have $sS \cap tS \neq \emptyset$. It is more common to work with right-reversible semigroups, which are then called Ore semigroups. In analogy with the results of Dubreil it can be shown that a left-reversible semigroup has an enveloping group Γ such that $\Gamma = SS^{-1}$. This is a counterpoint to our previous work, where we started with an action of a semigroup, whereas here we are starting with a functor into a semigroup.

EXAMPLES 8.1. (Left-reversible semigroups)

(1) Every group is right- and left-reversible.
(2) Any abelian semigroup is automatically right- and left-reversible.
(3) Let $\mathbb{N}$ denote the semigroup of natural numbers under addition and $\mathbb{N}^\times$ denote the semigroup of nonzero natural numbers under multiplication. Let $S = \mathbb{N} \times \mathbb{N}^\times$ be gifted with the associative binary operation $\star$ given by

$$(m_1, n_1) \star (m_2, n_2) = (m_1 n_2 + m_2, n_1 n_2),$$

then one checks that S is a non-abelian left-reversible semigroup. It is not right-reversible; for example, $S(m,n) \cap S(p,q) = \emptyset$ when $n = q = 0$ and $m \neq p$.
(4) The free semigroup $\mathbb{F}_n^+$ on $n \geq 2$ generators is not an left-reversible semigroup since for all $s, t \in \mathbb{F}_n^+$ with $s \neq t$ we have $s\mathbb{F}_n^+ \cap t\mathbb{F}_n^+ = \emptyset$ as there is no cancelation, and so not only the left-reversibility but also the right-reversibility conditions cannot be satisfied.

We recall some basic facts about preordered sets: A *preorder* is a reflexive, transitive relation $\leq$ on a set X. A preordered set $(X, \leq)$ is *directed* if the following condition holds: for every $x, y \in X$, there exists $z \in X$ such that $x \leq z$ and $y \leq z$. A subset Y of X is *cofinal* if for each $x \in X$ there exists $y \in Y$ such that $x \leq y$. We say that $t \in S$ is *strictly positive* if $\{t^n\}$ is a cofinal set in S. For $X \subseteq Y$, we say that $X \leq Y$ if $x \leq y$ for all $x \in X$ and for all $y \in Y$.

The following result is an analogue of [**50**, Lemma 2.2], but has been adapted for left-reversible semigroups. We will see the version for Ore semigroups in Chapter 9.

LEMMA 8.2. *Let S be a left-reversible semigroup with enveloping group Γ, and define $\leq_l$ on Γ by $g \leq_l h$ if and only if $g^{-1}h \in S$. Then $\leq_l$ is a left-invariant preorder that directs Γ, and for any $t \in S$, tS is cofinal in S.*

PROOF. To see that $\leq_l$ is reflexive: $g \leq_l g \Leftrightarrow g^{-1}g \in S \Leftrightarrow \mathrm{id} \in S$; transitive: suppose $g \leq_l h$, $h \leq_l k$, then $g^{-1}h \in S$, $h^{-1}k \in S$, so $g^{-1}hh^{-1}k \in S \Leftrightarrow g \leq_l k$; left-invariant:

$$kg \leq_l kh \Leftrightarrow (kg)^{-1}kh \in S \Leftrightarrow g^{-1}k^{-1}kh \in S \Leftrightarrow g^{-1}h \in S \Leftrightarrow g \leq_l h.$$

So $\leq_l$ is a left-invariant preorder.

We claim that the left-reversibility of the semigroup S implies $\leq_l$ directs Γ. Take $x, y \in S$. The left-reversibility of S implies $xS \cap yS \neq \emptyset$. Then

$$z \in xS, z \in yS \Leftrightarrow x^{-1}z \in S, y^{-1}z \in S \Leftrightarrow x \leq_l z, y \leq_l z.$$

We verify that tS is cofinal in S for each $t \in S$. Fix $s \in S$. We want to find $x \in tS$ such that $s \leq_l x$. The left-reversibility of S implies there exist $u, v \in S$ such that $tu = sv$. Now $s \leq_l sv = tu \in tS$. Take $x = tu$. ∎

8.1. Aperiodicity

We show that if a row-finite k-graph with no sources is aperiodic, then a skew-product graph of it is also aperiodic (Corollary 8.6). The converse is not true in general. We first recall from Definition 7.3 that a k-graph Λ is aperiodic if every $v \in \Lambda^0$ has no local periodicity: that is, for all $m \neq n \in \mathbb{N}^k$ there exists a path $\lambda \in v\Lambda$ such that $d(\lambda) \geq m \vee n$ and

$$\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n)).$$

DEFINITION 8.3. Let Λ and Γ be row-finite k-graphs. A surjective k-graph morphism $p : \Lambda \to \Gamma$ has *r-path lifting* if for all $v \in \Lambda$ and $\lambda \in p(v)\Gamma$ there is $\lambda' \in v\Lambda$ such that $p(\lambda') = \lambda$.

EXAMPLE 8.4. Let Λ be a row-finite k-graph and $\eta : \Lambda \to S$ a functor where S is a semigroup, and $\Lambda \times_\eta S$ the associated skew product graph. The map $\pi : \Lambda \times_\eta S \to \Lambda$ defined $\pi(\lambda, s) = \lambda$ is a surjective k-graph morphism with the r-path lifting property.

THEOREM 8.5. *Suppose Λ and Γ are row-finite k-graphs and $p : \Lambda \to \Gamma$ have r-path lifting. If Γ is aperiodic, then Λ is aperiodic.*

PROOF. Suppose that Γ is aperiodic. Let $v \in \Lambda^0$ and $m \neq n \in \mathbb{N}^k$. Since Γ is aperiodic, there exists $\lambda \in p(v)\Gamma$ with $d(\lambda) \geq m \vee n$ such that $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$. Hence, $\lambda' \in v\Lambda$ with $p(\lambda') = \lambda$ is such that $d(\lambda') \geq m \vee n$,

$$\lambda'(m, m + d(\lambda) - (m \vee n)) \neq \lambda'(n, n + d(\lambda) - (m \vee n)),$$

since p is a k-graph morphism, and so Λ is aperiodic. $\square$

COROLLARY 8.6. *Let Λ be a row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor where S is a semigroup and $\Lambda \times_\eta S$ the associated skew-product graph. If Λ is aperiodic then $\Lambda \times_\eta S$ is aperiodic.*

PROOF. The map $\pi : \Lambda \times_\eta S \to \Lambda$ has the unique path lifting property by Example 8.4, so we can apply Theorem 8.5 to obtain the result. $\square$

Unfortunately, the converse to Theorem 8.5 is not true, as demonstrated by the following example.

EXAMPLE 8.7. The surjective k-graph morphism $p : \Omega_1 \to T_1$ given by $p(m, m + 1) = f_1$ for all $m \geq 0$ has r-path lifting. However by Examples 7.4 we see that Ω_1 is aperiodic but T_1 is not.

The following result gives a condition for the aperiodicity of a skew-product graph.

Lemma 8.8. *Suppose S is an left-reversible semigroup, Λ is a row-finite k-graph, $\eta : \Lambda \to S$ is a functor, and there exists a map $\phi : S \to \mathbb{Z}^k$ such that $d = \phi \circ \eta$. Then $\Lambda \times_\eta S$ is aperiodic.*

Proof. Fix $(v, s) \in (\Lambda \times_\eta S)^0$ and $m \neq n \in \mathbb{N}^k$. Let $\lambda \in (v, s)(\Lambda \times_\eta S)$ be such that $d(\lambda) \geq m \vee n$. We claim $\Lambda(m, m) \neq \lambda(n, n)$. Since $\lambda(m, m) = s(\lambda(0, m))$, $\Lambda(m, m)$ is of the form $(w, s\eta(\lambda(0, m)))$ for some $w \in \Lambda^0$. Similarly, $\lambda(n, n)$ is of the form $(w', s\eta(\lambda(0, n)))$ for some $w' \in \Lambda^0$.

Suppose $\eta(\lambda(0, n)) = \eta(\lambda(0, m))$. Then $n = \phi \circ \eta(\lambda(0, n)) = \phi \circ \eta(\lambda(0, m)) = m$, which provides a contradiction. Then $\eta(\lambda(0, m)) \neq \eta(\lambda(0, n))$, and so $\lambda(m, m) \neq \lambda(n, n)$, and hence we have the aperiodicity condition: $\lambda(m, m + d(\lambda) - (m \vee n)) \neq \lambda(n, n + d(\lambda) - (m \vee n))$. □

Corollary 8.9. *Suppose Λ is a row-finite k-graph. Then $\Lambda \times_d \mathbb{Z}^k$ is aperiodic.*

Proof. Apply Lemma 8.8 with $\eta = d$. Then the map $\phi : \mathbb{Z}^k \to \mathbb{Z}^k$ is the identity map, and the result is immediate. □

8.2. Cofinality

We examine the connection between strong-connectivity and cofinality. In Corollary 8.14 we show that for a row-finite k-graph with no sources, if the skew-product graph is cofinal, then the k-graph itself is cofinal. We first recall from Definition 7.5 that a row-finite k-graph Λ with no sources is cofinal if for all pairs $v, w \in \Lambda^0$ there exists $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$.

We say that a k-graph Λ is *connected* if the equivalence relation on Λ^0 generated by the relation $\{(u, v) : u\Lambda v \neq \emptyset\}$ is $\Lambda^0 \times \Lambda^0$. A k-graph Λ is *strongly connected* if for all $u, v \in \Lambda^0$ there is $\lambda \in u\Lambda v$ such that $d(\lambda) > 0$, in the sense that $d(\lambda)e_i \neq 0$ for all $1 \leq i \leq k$.

Lemma 8.10. *Let Λ be a row-finite k-graph with no sources.*

(1) *If Λ is cofinal then Λ is connected.*
(2) *If Λ is cofinal then for all $v, w \in \Lambda^0$ there is $N \in \mathbb{N}^k$ such that for all $n \geq N$ we have $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^n$.*

Proof. Fix $v, w \in \Lambda^0$. If Λ is cofinal it follows that there is $\alpha \in w\Lambda$ such that $w\Lambda s(\alpha)$ and $v\Lambda s(\alpha)$ are non-empty. It follows that (v, w) belongs to the equivalence relation described in the definition above of connected. Since v, w were arbitrary it follows that the equivalence relation is $\Lambda^0 \times \Lambda^0$ and so Λ is connected.

Fix $v, w \in \Lambda^0$, then as Λ is cofinal there is $N \in \mathbb{N}^k$ such that $v\Lambda s(\alpha) \neq \emptyset$ for every $\alpha \in w\Lambda^N$. Let $n \geq N$ and consider $\beta \in w\Lambda^n$ then $\beta' = \beta(0, N) \in w\Lambda^N$ and so by hypothesis there is $\lambda \in v\Lambda s(\beta')$. Then $\lambda\beta(N, n) \in v\Lambda s(\beta)$ and the result follows. □

We require a short technical lemma in preparation to showing the connection between strong connectivity and cofinality of k-graphs.

LEMMA 8.11. *Let Λ be a k-graph with no sinks, and Λ^0 finite. Then for all $v \in \Lambda^0$, there exists $w \in \Lambda^0$ and $\alpha \in w\Lambda w$ such that $d(\alpha) > 0$ and $w\Lambda v \neq \emptyset$.*

PROOF. Let $p = (1, \ldots, 1) \in \mathbb{N}^k$. Since v is not a sink, there exists $\beta_1 \in \Lambda^p v$. Since $r(\beta_1)$ is not a sink, there exists $\beta_2 \in \Lambda^p r(\beta_1)$. Inductively, there exist infinitely many β_i such that $d(\beta_i) = p$ and $r(\beta_i) = s(\beta_{i+1})$. Since Λ^0 is finite, there exists $w \in \Lambda^0$ such that $r(\beta_i) = w$ for infinitely many i. Suppose $r(\beta_n) = w = r(\beta_m)$ with $m > n$. Then $\alpha = \beta_m \ldots \beta_{n+1}$ has the requisite properties, and $w\Lambda v \neq \emptyset$, since $\beta_n \ldots \beta_1 \in w\Lambda v$. □

Connectivity is not enough to give cofinality. The following proposition establishes a link between cofinality and strongly connectivity for a row-finite k-graph.

PROPOSITION 8.12. *Suppose Λ is a row-finite k-graph with no sources.*

(1) *If Λ is strongly connected then Λ is cofinal.*
(2) *If Λ is cofinal, has no sinks and Λ^0 finite then Λ is strongly connected.*

PROOF. Suppose Λ is strongly connected. Fix $v, w \in \Lambda^0$ then for $N = e_1$ we have $v\Lambda s(\alpha) \neq \emptyset$ for all $\alpha \in w\Lambda^N$ since Λ is strongly connected, and so Λ is cofinal.

Suppose Λ is cofinal. Fix $u, v \in \Lambda^0$. Then by Lemma 8.11, there exists $w \in \Lambda^0$ and $\alpha \in w\Lambda w$ such that $d(\alpha) > 0$ and $w\Lambda v \neq \emptyset$. Let $\alpha' \in w\Lambda v$. Given $u, w \in \Lambda^0$, since Λ is cofinal and has no sources, by Lemma 8.10(ii) there exists $N \in \mathbb{N}^k$ such that for all $n \geq N$ and all $\alpha'' \in w\Lambda^n$, there exists $\beta \in u\Lambda s(\alpha'')$. Since $d(\alpha) > 0$ we may choose $t \in \mathbb{N}$ such that $td(\alpha) > N$. Then $\alpha^t \in w\Lambda^n$ where $n > N$, and so by cofinality of Λ exists $\beta \in u\Lambda s(\alpha^t) = u\Lambda w$. Hence $\beta\alpha\alpha' \in u\Lambda v$ with $d(\beta\alpha\alpha') > d(\alpha) > 0$ and so Λ is strongly connected. □

THEOREM 8.13. *Suppose Λ and Γ are row-finite k-graphs and $p : \Lambda \to \Gamma$ have r-path lifting. If Λ is cofinal then Γ is cofinal.*

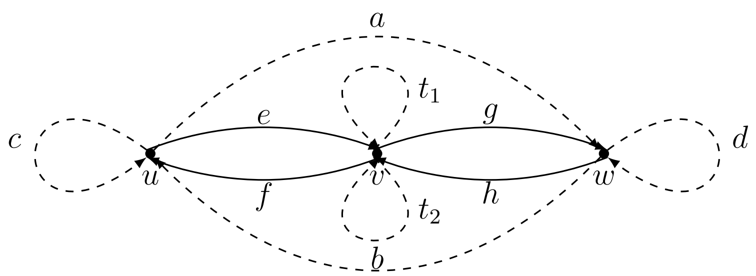
PROOF. Suppose that Λ is cofinal. Fix $v, w \in \Gamma^0$. Let $v', w' \in \Lambda^0$ be such that $p(v') = v$ and $p(w') = w$. Since Λ is cofinal there is an N such that for all $\alpha' \in w'\Lambda^N$ there is $\beta' \in v'\Lambda s(\alpha')$. Then for $\alpha \in v\Gamma^N$ there is $\alpha' \in v'\Lambda^N$ with $p(\alpha') = \alpha$. By hypothesis there is $\beta' \in v'\Lambda s(\alpha')$, and so $\beta = p(\beta')$ is such that $s(\beta) = s(\alpha)$ and $r(\beta) = v$ which implies that $v\Lambda(s(\alpha)) \neq \emptyset$ as required. □

COROLLARY 8.14. *Let Λ be a row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor where S is a semigroup and $\Lambda \times_\eta S$ the associated skew-product graph. If $\Lambda \times_\eta S$ is cofinal then Λ is cofinal.*

Proof. The map $\pi : \Lambda \times_\eta S \to \Lambda$ has the unique path lifting property by Example 8.4, so we can apply Theorem 8.13 to obtain the result. □

Unfortunately, the converse to Theorem 8.13 is not true, as demonstrated by Example 8.15, which follows. We first recall the following standard notation: given a k-graph Λ, for $1 \le i \le k$ and $u, v \in \Lambda^0$, we define k non-negative matrices $M_i : 1 \le i \le k$ with entries $M_i(u,v) = |v\Lambda^{e_i}u|$. Using the k-graph factorisation property, we have that $|u\Lambda^{e_i+e_j}v| = |u\Lambda^{e_j+e_i}v|$, implying $M_iM_j = M_jM_i$. For $m = (m_1, m_2, \ldots, m_k) \in \mathbb{N}^k$, we have $|v\Lambda^m u| = (M_1^{m_1}M_2^{m_2}\ldots M_k^{m_k})(u,v) = M^m(u,v)$, using multiindex notation.

Example 8.15. Consider the following strongly connected 2-graph Λ with 1-skeleton:



and factorisation rules: $ec = t_1e$ and $ha = t_2e$ for paths from u to v, $cf = ft_1$ and $bg = ft_2$ for paths from v to u, $hd = t_1h$ and $eb = t_2h$ for paths from w to v and $dg = gt_1$ and $af = gt_2$ for paths from v to w. Note there are no paths of degree $e_1 + e_2$ from a vertex to itself.

In this example, $M_1 = \begin{pmatrix} 0&1&0\\1&0&1\\0&1&0 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 1&0&1\\0&2&0\\1&0&1 \end{pmatrix}$. We calculate that $M^{(2j_1,j_2)} = 2^{j_1+j_2-1}M_2$ and $M^{(2j_1+1,j_2)} = 2^{j_1+j_2+1}M_1$. From these matrices, we can calculate that $M^{(2j-1)(e_1+e_2)} = \begin{pmatrix} 0&4j&0\\4j&0&4j\\0&4j&0 \end{pmatrix}$ and $M^{(2j)(e_1+e_2)} = \begin{pmatrix} 4j&0&4j\\0&8j&0\\4j&0&4j \end{pmatrix}$.

We claim that the skew product graph $\Lambda \times_d \mathbb{Z}^2$ is not cofinal. Consider $v_1 = (v,(m,n))$ and $v_2 = (v,(m+1,n))$ in $(\Lambda \times_d \mathbb{Z}^2)^0$. We claim that for all $N \ge 0$, for all $\alpha \in v_1(\Lambda \times_d \mathbb{Z}^2)^N$, we have $v_2(\Lambda \times_d \mathbb{Z}^2)s(\alpha) \ne \emptyset$. Let $N = (N_1, N_2)$. Suppose N_1 is even. Then for all $\alpha \in v_1(\Lambda \times_d \mathbb{Z}^2)^N$, $s(\alpha) = (v,(m+N_1, n+N_2))$. In order for this vertex to connect to $(v,(m+1,n))$, we must have $M^{(N_1-1,N_2)}(v,v) \ne 0$. But $N_1 - 1$ is odd, and this matrix entry is zero. If N_1 is odd, then $s(\alpha) = (u,(m+N_1,n+N_2))$ or $s(\alpha) = (w,(m+N_1,n+N_2))$. In order for either of these vertices to connect to $(v,(m+1,n))$, we must have $M^{(N_1-1,N_2)}(u,v) \ne 0$, or $M^{(N_1-1,N_2)}(w,v) \ne 0$. But $N_1 - 1$ is even, and so both of these matrix entries are zero. Hence $\Lambda \times_d \mathbb{Z}^k$ is not cofinal.

Example 8.15 shows that cofinality of a k-graph Λ is not sufficient to guarantee cofinality of the skew-product k-graph $\Lambda \times_\eta S$. We need the functor η to satisfy the following technical condition.

DEFINITION 8.16. Let Λ be a k-graph and $\eta : \Lambda \to S$ a functor, where S is a semigroup. The system (Λ, S, η) is *cofinal* if for all $v, w \in \Lambda^0$, $a, b \in S$, there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$, there exists $\beta \in v\Lambda s(\alpha)$ such that $a\eta(\beta) = b\eta(\alpha)$.

PROPOSITION 8.17. *Let Λ be a k-graph and $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew-product graph. Then the system (Λ, S, η) is cofinal if and only if $\Lambda \times_\eta S$ is cofinal.*

PROOF. Suppose $\Lambda \times_\eta S$ is cofinal. Fix $a, b \in S$ and $v, w \in \Lambda^0$. By hypothesis there exists $N \in \mathbb{N}^k$ such that $(v, a)(\Lambda \times_\eta S)s(\alpha, b)$ is non-empty for every $(\alpha, b) \in (w, b)(\Lambda \times_\eta S)^N$. In particular for all $\alpha \in w\Lambda^N$ there exists $\beta \in w\Lambda^N$ such that $a\eta(\beta) = b\eta(\alpha)$, and so (Λ, S, η) is cofinal.

Now suppose (Λ, S, η) is cofinal. Fix $(v, a), (w, b) \in \Lambda \times_\eta S$. By hypothesis there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$, there exists $\beta \in v\Lambda s(\alpha)$ with $a\eta(\beta) = b\eta(\alpha)$. In particular for all $(\alpha, b) \in (w, b)(\Lambda \times_\eta S)^N$ there is $(\beta, a) \in (v, a)\Lambda s(\alpha, b)$, and so $\Lambda \times_\eta S$ is cofinal. □

If we apply Corollary 8.14 and Proposition 8.17, we can conclude that if (Λ, S, η) is cofinal, then Λ is cofinal. In particular, if$(\Lambda, \mathbb{N}^k, d)$ is cofinal, then Λ is cofinal. The converse is not true: In Example 8.15 we saw a 2-graph Λ which is cofinal, however $\Lambda \times_d \mathbb{N}^2$ is not cofinal and so by Proposition 8.17, $(\Lambda, \mathbb{N}^2, d)$ is not cofinal.

REMARK 8.18. If $\eta, \eta' : \Lambda \to S$, are cohomologous functors (Definition 3.18) then by Lemma 3.19 $\Lambda \times_\eta S$ is isomorphic to $\Lambda \times_{\eta'} S$. Applying Proposition 8.17 we conclude (Λ, S, η) is cofinal if and only if (Λ, S, η') is cofinal.

THEOREM 8.19. *Let Λ be an aperiodic row-finite k-graph with no sources, $\eta : \Lambda \to S$ a functor, where S is a semigroup and $\Lambda \times_\eta S$ the associated skew product graph. Then $C^*(\Lambda \times_\eta S)$ is simple if and only if the system (Λ, S, η) is cofinal.*

PROOF. Suppose that the system (Λ, S, η) is cofinal. Then by Proposition 8.17, $\Lambda \times_\eta S$ is cofinal. By Corollary 8.6, $\Lambda \times_\eta S$ is aperiodic, and so by Theorem 7.6, $C^*(\Lambda \times_\eta S)$ is simple.

Now suppose that $C^*(\Lambda \times_\eta S)$ is simple. Then by Theorem 7.6, $\Lambda \times_\eta S$ is cofinal. By Proposition 8.17 this implies that (Λ, S, η) is cofinal. □

8.3. S-primitivity

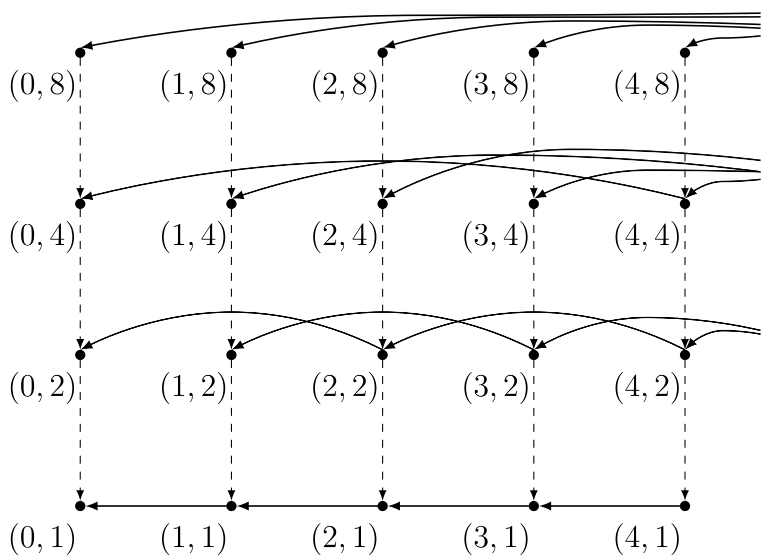
Our motivation is the results of [**51**]. We are generalising the idea of primitivity from [**37**, Definition 2.2]: a k-graph Λ is primitive if there exists strictly positive $p \in \mathbb{N}^k$ such that $v\Lambda^p w \neq \emptyset$ for all $v, w \in \Lambda^0$. The condition will be a key hypothesis for the simplicity of the core of the graph algebra. We aim to generalise from the group case to the semigroup $\mathbb{N}^k$.

Definition 8.20. Let Λ be a k-graph and $\eta : \Lambda \to S$ be a functor where S is an left-reversible semigroup. We say that η is *S–primitive* if there is a strictly positive $t \in S$ such that for all $v, w \in \Lambda^0$ we have $v\eta^{-1}(s)w \neq \emptyset$ for all s such that $t \leq_l s$.

Remark 8.21. If $\eta : \Lambda \to S$ is a S–primitive where S is a left-reversible semigroup, then if we extend $\eta : \Lambda \to \Gamma = SS^{-1}$ then η is Γ–primitive.

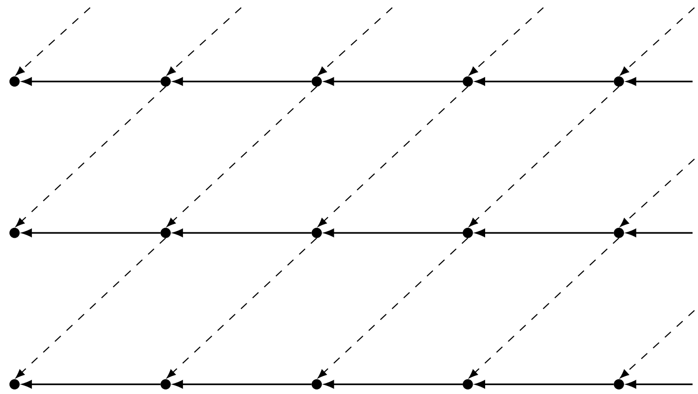
Examples 8.22. (S–primitive functors)

(1) Let Λ be a k-graph. Then d is $\mathbb{N}^k$–primitive if and only if Λ is primitive.
(2) Let $X = \{2^n : n \in \mathbb{N}\} \subseteq \mathbb{N}^\times$, which is a semigroup under multiplication. Define a functor η from T_2 to the left-reversible semigroup $\mathbb{N} \times X$ such that $\eta(f_1) = (1,1)$, and $\eta(f_2) = (0,2)$. Note that $(1,1)$ and $(0,2)$ are generators of $\mathbb{N} \times X$. With the calculations: $(n, 2^k)(1,1) = (n + 2^k, 2^k)$ and $(n, 2^k)(0,2) = (n, 2^{k+1})$, we may form the skew-product $T_2 \times_\eta (\mathbb{N} \times X)$ with the following 1-skeleton:



We claim that the functor η is $(\mathbb{N} \times X)$-primitive: take $t = (1,2) \in \mathbb{N} \times X$. Since there is one vertex it suffices to show that $v\eta^{-1}(s)v \neq \emptyset$ for all $s \geq_l t$. Since $s = (m, n) \geq_l (1,2)$, $m = i \geq 1$, and $n = 2^j \geq 2$, then $\eta(f_1^i f_2^j) = s$, and $v\eta^{-1}(s)v \neq \emptyset$.

(3) Taking T_2 again, we define a functor $\eta : T_2 \to \mathbb{N}^2$ by $\eta(f_1) = (1,1)$ and $\eta(f_2) = (1,0)$. The skew-product graph has 1-skeleton:



For the skew-product graph $T_2 \times_\eta \mathbb{N}^k$, $v\eta^{-1}(m,n)v = \emptyset$ whenever $n > m$, as the range of $\eta = \{(m,n) : m < n\}$, hence the functor is not $\mathbb{N}^2$-primitive.

It is clear from these last two examples that S–primitivity of η is not sufficient to guarantee cofinality of $\Lambda \times_\eta S$. This is a counterpoint to the situation when $S = \mathbb{N}^k$ and $\eta = d$, where such a condition would be sufficient.

The following result gives the required extra condition. We note that it is a necessary and sufficient condition, allowing us the best possible classification.

Definition 8.23. Let Λ be a k-graph and $\eta : \Lambda \to S$ be a functor where S is a left-reversible semigroup. We will say that η is *upper dense* if for all $w \in \Lambda^0$ and $u, b \in S$ there exists $N \in \mathbb{N}^k$ such that $b\eta(w\Lambda^N) \geq_l u$.

Proposition 8.24. *Let Λ be a row-finite k-graph and $\eta : \Lambda \to S$ be a S–primitive functor where S is a left-reversible semigroup. Then (Λ, S, η) is cofinal if and only if η is upper dense.*

Proof. Suppose that (Λ, S, η) is cofinal. Fix $w \in \Lambda^0$ and $u, b \in S$ and let v be any vertex of Λ. By cofinality of (Λ, S, η) there exists $N \in \mathbb{N}^k$ such that for all $\alpha \in w\Lambda^N$ there is $\beta \in v\Lambda s(\alpha)$ such that $u\eta(\beta) = b\eta(\alpha)$. Then any element of $b\eta(w\Lambda^N)$ is of the form

$$b\eta(\alpha) = u\eta(\beta) \geq_l u.$$

Conversely, suppose that for all $w \in \Lambda^0$ and $u, b \in S$ there exists $N \in \mathbb{N}^k$ such that $b\eta(w\Lambda^N) \geq_l u$. Since η is S–primitive there exists $t \in S$ such that for all $v, w \in \Lambda^0$ we have $v\eta^{-1}(s)w \neq \emptyset$ for all $s \geq_l t$. Fix $v, w \in \Lambda^0$ and $a, b \in S$. Since η is upper dense there exists $N \in \mathbb{N}^k$ such that $b\eta(\alpha) \geq_l at$ for all $\alpha \in w\Lambda^N$. Since S is left-reversible, it is directed, and so by definition $b\eta(\alpha) = atu$ for some $u \in S$. But $tu \geq_l t$ and so since η is S–primitive there exists $\beta \in v\Lambda s(\alpha)$ such that $\eta(\beta) = tu$ and hence $b\eta(\alpha) = a\eta(\beta)$. $\square$

When $\eta = d$, the upper dense condition is automatically satisfied since $d(w\Lambda^N) = N$, and we obtain the following corollary.

Corollary 8.25. *Let Λ be a row-finite k-graph such that d is $\mathbb{N}^k$–primitive, then d is upper dense and hence $(\Lambda, \mathbb{N}^k, d)$ is cofinal.*

Proof. Suppose that Λ is d–primitive. For any $b, u \in \mathbb{N}^k$ we have $b + d(w\Lambda^N) = b + N \geq u$ provided $N \geq u - b$. Hence $(\Lambda, \mathbb{N}^k, d)$ is cofinal by Proposition 8.24. □

Example 8.26. Recall Examples 8.22(ii) and (iii). We use Proposition 8.24 to prove that these examples are not cofinal as the upper dense condition can not be satisfied. Examples 8.22(ii) is not cofinal because no vertex connects to both $(0, 2)$ and $(1, 2)$. For Example 8.22(iii), we consider any vertex above the main diagonal; there does not exist a path to the origin.

These examples show that upper density is not automatic.

Theorem 8.27. *Let Λ be an aperiodic row-finite k-graph, $\eta : \Lambda \to S$ be a functor into a left-reversible semigroup, and η be S–primitive. Then $C^*(\Lambda \times_\eta S)$ is simple if and only if η is upper dense.*

Proof. This is essentially a restatement of Theorem 8.19. We use Proposition 8.24 to replace the condition of cofinality (Λ, S, η) with S-primitivity of η and upper density of η. □

The following result gives a sufficient condition for primitivity of particular k-graphs.

Theorem 8.28. *Let Λ be a row-finite k-graph with no sinks and no sources and Λ^0 finite. If $(\Lambda, \mathbb{Z}^k, d)$ is cofinal then Λ is primitive.*

Proof. We claim that for $v \in \Lambda^0$ there is $N(v) \in \mathbb{N}^k$ such that for all $n \geq N(v)$ we have $v\Lambda^n v \neq \emptyset$. Fix $(v, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ then for each $w \in \Lambda^0$, when we apply the cofinality condition to $(w, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$ we obtain $N_w \in \mathbb{N}^k$ such that $(v, 0)(\Lambda \times_d \mathbb{Z}^k)s(\alpha, 0) \neq \emptyset$ for all $(\alpha, 0) \in (w, 0)(\Lambda \times_d \mathbb{Z}^k)^{N_w}$. Define $N = \max_{w \in \Lambda^0}\{N_w\}$, which is finite since Λ^0 is finite.

By Proposition 8.12 it follows that Λ is strongly connected, hence there exists $\alpha \in v\Lambda v$ with $d(\alpha) = r > 0$. Hence, there exists $t \geq 1$ such that $tr \geq N$. Let $N(v) = tr$.

Let $m = n - tr \geq 0$. Since Λ has no sources, $v\Lambda^m \neq \emptyset$; hence there exists $\gamma \in v\Lambda^m$. Let $w = s(\gamma)$. For $(v, 0), (w, 0) \in (\Lambda \times_d \mathbb{Z}^k)^0$, we have $(\alpha^t, 0) \in (v, 0)(\Lambda \times_d \mathbb{Z}^k)^{tr}$ where $tr \geq N \geq N_w$. Hence by cofinality and Lemma 8.10 (b), there exists $(\beta, 0) \in (w, 0)(\Lambda \times_d \mathbb{Z}^k)(v, tr)$ as $s(\alpha^t, 0) = (v, tr)$. As $\beta \in w\Lambda^{tr}v$ it follows that $\gamma\beta \in v\Lambda^n v$, which proves the claim. □

8.4. Skew-products by groups

Let Λ be a row-finite k-graph. Take $\{s_\lambda\}$ to be a Cuntz-Krieger Λ-family, and G to be a group. A functor $\eta : \Lambda \to G$ defines a coaction δ_η on $C^*(\Lambda)$ determined by $\delta_\eta(s_\lambda) = s_\lambda \otimes \eta(\lambda)$; see [**45**]. For more details on coactions, see Section 3 of [**18**]. Following [**45**, Lemma 7.9], for $g \in G$ the spectral subspace $C^*(\Lambda)_g$ of the coaction δ_η is given by

$$C^*(\Lambda)_g = \overline{\operatorname{span}}\{s_\lambda s_\mu^* : \eta(\lambda)\eta(\mu)^{-1} = g\}.$$

We define $\operatorname{sp}(\delta_\eta) = \{g \in G : C^*(\Lambda)_g \neq \emptyset\}$, to be the collection of non-empty spectral subspaces. The *fixed point algebra*, $C^*(\Lambda)^{\delta_\eta}$ of the coaction is defined to be $C^*(\Lambda)_{1_G}$. For more details on the coactions of discrete groups on k-graph algebras, see [**45**, §7] and [**54**]. The following definition was inspired by [**54**, Definition 2.9].

Definition 8.29. Let Λ be a row-finite k-graph, G be a discrete group and $\eta : \Lambda \to G$ a functor, then we define

$$\Gamma(\eta) = \{g \in G : g = \eta(\lambda)\eta(\mu)^{-1} \text{ for some } \lambda, \mu \in \Lambda \text{ with } s(\lambda) = s(\mu)\}.$$

Lemma 8.30. *Let Λ be a row-finite graph with no sources and $\eta : \Lambda \to G$ a functor, where G is a discrete group.*

(1) *If (Λ, G, η) is cofinal then $\Gamma(\eta) = G$.*
(2) *$\operatorname{sp}(\delta_\eta) = G$ if and only if $\Gamma(\eta) = G$.*

Proof. Fix $g \in G$ and write $g = b^{-1}a$ for some $a, b \in G$. Now fix $v, w \in \Lambda^0$; since (Λ, G, η) is cofinal there exist $\lambda, \mu \in \Lambda$ with $s(\lambda) = s(\mu)$ such that $a\eta(\mu) = b\eta(\lambda)$. Hence $b^{-1}a = \eta(\lambda)\eta(\mu)^{-1}$ and so $g \in \Gamma(\eta)$. Since g was arbitrary the result follows.

The second statement follows by definition. □

Theorem 8.31. *Let Λ be an aperiodic row-finite k-graph with no sources, $\eta : \Lambda \to G$ a functor and δ_η the associated coaction of G on $C^*(\Lambda)$. Then $C^*(\Lambda \times_\eta G)$ is simple if and only if $C^*(\Lambda)^{\delta_\eta}$ is simple and $\Gamma(\eta) = G$.*

Proof. By [**45**, Theorem 7.1] it follows that $C^*(\Lambda \times_\eta G)$ is isomorphic to $C^*(\Lambda) \times_{\delta_\eta} G$. Then by [**54**, Theorem 2.10] $C^*(\Lambda) \times_{\delta_\eta} G$ is simple if and only if $C^*(\Lambda)^{\delta_\eta}$ is simple and $\operatorname{sp}(\delta_\eta) = G$. The result now follows from Lemma 8.30(2). □

Example 8.32. Let Λ be a row-finite k-graph with no sources and $d : \Lambda \to \mathbb{N}^k$ be the degree functor. We claim that $\Gamma(d) = \mathbb{Z}^k$. Fix $p \in \mathbb{Z}^k$, and write $p = m - n$ where $m, n \in \mathbb{N}^k$. Since Λ has no sources, for every $v \in \Lambda^0$ there is $\lambda \in \Lambda^m v$ and $\mu \in \Lambda^n v$. Then

$$d(\lambda) - d(\mu) = m - n = p \in \Gamma(d),$$

and so $\Gamma(d) = \mathbb{Z}^k$. Since $\Gamma(d) = \mathbb{Z}^k$, and $(\Lambda, \mathbb{Z}^k, d)$ is $\mathbb{Z}^k$–aperiodic, we have that $C^*(\Lambda)^{\delta_d}$ is simple if and only $(\Lambda, \mathbb{N}^k, d)$ is cofinal.

8.5. The gauge coaction

In Section 8.4, we introduced a coaction δ_d of $\mathbb{Z}^k$ on $C^*(\Lambda)$. A coaction of an Abelian group G corresponds to an action of its Pontryagin dual $\widehat{G}$, and so we can consider a coaction of $\mathbb{Z}^k$ on $C^*(\Lambda)$ to be an action of $\mathbb{T}^k$ on $C^*(\Lambda)^{\delta_d}$. A comment preceding [**10**, Corollary 4.9] says that since $\mathbb{Z}^k$ is abelian, the coactions of $\mathbb{Z}^k$ on $C^*(\Lambda)$ defined using the degree functor correspond via the Fourier transform to the canonical gauge action γ of $\mathbb{T}^k$ of $C^*(\Lambda)$ (see [**10**, Corollary 4.9]), and hence the fixed point algebra $C^*(\Lambda)^{\delta_d}$ is identical to the fixed point algebra $C^*(\Lambda)^{\gamma}$.

The AF core of a k-graph algebra plays a significant role in the development of crossed products by endomorphisms. Results of Takehana and Katayama [**34**] show that when Λ is a finite 1-graph such that the AF core of $C^*(\Lambda)$ is simple, then every nontrivial automorphism of $C^*(\Lambda)$ is outer (see [**49**, Proposition 3.4]).

THEOREM 8.33. *Let Λ be a row-finite k-graph with no sinks or sources, and Λ^0 finite. Then $C^*(\Lambda)^{\delta_d}$ is simple if and only if Λ is primitive.*

PROOF. We aim to use Theorem 8.31 to conclude that $C^*(\Lambda)^{\delta_d}$ is simple. To do this, we must show that $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple. Suppose Λ is primitive. By Example 8.22 we have that the degree map $d : \Lambda \to \mathbb{N}^k$ is $\mathbb{N}^k$-primitive. By Corollary 8.25, we have that the degree maps is upper dense. Also by Corollary 8.25, $(\Lambda, \mathbb{N}^k, d)$ is cofinal. By Remark 8.21, d is $\mathbb{Z}^k$-primitive, and by Proposition 8.24, $(\Lambda, \mathbb{Z}^k, d)$ is cofinal, and so by Theorem 8.27, $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple. Hence by Theorem 8.31, we may conclude $C^*(\Lambda)^{\delta_d}$ is simple.

For the converse, we will aim to show that $(\Lambda, \mathbb{Z}^k, d)$ is cofinal and apply Theorem 8.28. Suppose that $C^*(\Lambda)^{\delta_d}$ is simple. By Example 8.32, we have that $\Gamma(d) = \mathbb{Z}^k$. Hence by Theorem 8.31, $C^*(\Lambda \times_d \mathbb{Z}^k)$ is simple, which by Theorem 7.6 implies that $\Lambda \times_d \mathbb{Z}^k$ is aperiodic and cofinal. By Proposition 8.17, we then have $(\Lambda, \mathbb{Z}^k, d)$ is cofinal. We then apply Theorem 8.28 and conclude Λ is primitive, as required. □

EXAMPLE 8.34. Since it has one vertex $\mathbb{F}_\theta^2$ is a primitive graph for all θ. Therefore, $C^*(\mathbb{F}_\theta^2)^{\gamma}$ is simple. By [**14**], this graph algebra is the $(mn)^\infty$-UHF algebra.

8.6. AF algebras

We are interested in generalising a result of Kumjian and Pask from [**36**] that gives a condition for a particular skew-product graph C^*-algebra to be AF. We revisit the result and the methods used to prove it. We attempt to find other skew-product graphs for which this result applies.

Take Λ to be a row-finite k-graph, and consider $(\Lambda, \mathbb{Z}^k, d)$. We claim the k-graph C^*-algebra of the skew-product graph, $C^*(\Lambda \times_d \mathbb{Z}^k)$ is always AF. To see this we first define $b : \Lambda \times_d \mathbb{Z}^k \to \mathbb{Z}^k$ by $b(\lambda, n) = n$. Then since $d(\lambda, n) = d(\lambda)$, and

$$b(s(\lambda, n)) - b(r(\lambda, n)) = b(s(\lambda), n + d(\lambda)) - b(r(\lambda, n)) = n + d(\lambda) - n = d(\lambda).$$

Hence, by a result of [**35**], the skew-product graph algebra is AF.

We attempt to generalise this result. First, a technical lemma.

LEMMA 8.35. *Let Λ be a row-finite k-graph with no sources and let $\eta : \Lambda \to S$ be a functor. Suppose there is a functor $\phi : S \to \mathbb{Z}^k$ such that $d = \phi \circ \eta$ and a map $c : \Lambda^0 \to S$ such that $c(r(\lambda))\eta(\lambda) = c(s(\lambda))$ for all $\lambda \in \Lambda$. Then $C^*(\Lambda)$ is AF.*

PROOF. Define $b : \Lambda^0 \to S$ by $b = \phi \circ c$, then for $\lambda \in \Lambda$ we have

$$b(r(\lambda)) + d(\lambda) = \phi(\eta(\lambda)) + \phi(c(r(\lambda))) = \phi(c(r(\lambda)\eta(\lambda))) = \phi(c(s(\lambda))) = b(s(\lambda))$$

and so $d(\lambda) = b(s(\lambda)) - b(r(\lambda))$. It then follows from [**35**, Lemma 5.4] that $C^*(\Lambda)$ is AF. □

THEOREM 8.36. *Let Λ be a row-finite k-graph with no sources and let $\eta : \Lambda \to S$ be a functor. Suppose there is a functor $\phi : S \to \mathbb{Z}^k$ such that $d = \phi \circ \eta$ then $C^*(\Lambda \times_\eta S)$ is AF.*

PROOF. Define $\eta' : \Lambda \times_\eta S \to S$ by $\eta'(\lambda, s) = \eta(\lambda)$ then one checks that η' is a functor. Define $c : (\Lambda \times_\eta S)^0 \to S$ by $b(v,t) = t$, then for $(\lambda, t) \in \Lambda \times_\eta S$ we have

$$c(r(\lambda, t))\eta'(\lambda, t) = c(r(\lambda), t)\eta'(\lambda, t) = t\eta(\lambda) = c(s(\lambda), t\eta(\lambda)).$$

The result now follows by Lemma 8.35. □

EXAMPLES 8.37. (AF algebras)

(1) Let $\Lambda = B_2$ and define $\eta : \Lambda \to \mathbb{F}_2^+ = \langle a, b \rangle$ by $\eta(a_1) = a$ and $\eta(a_2) = b$. Let $\phi : \mathbb{F}_2^+ \to \mathbb{N}$ be defined by $\phi(w) = |w|$, the length of $w \in \mathbb{F}_2^+$, then one checks that ϕ is a functor such that $\phi \circ \eta = d$. Hence $C^*(\Lambda \times_\eta \mathbb{F}_2^+)$ is AF by Theorem 8.36.
(2) Take η to be a map from the 2-graph $\mathbb{F}_\theta^2$ to the semigroup $\mathbb{F}_\theta^2$, defined $\eta(\lambda) = \lambda$. Then $C^*(\mathbb{F}_\theta^2 \times_\eta \mathbb{F}_\theta^2)$ is an AF graph algebra, since we can define $\phi : \mathbb{F}_\theta^2 \to \mathbb{N}^2$ by $\phi(\lambda) = \{|\{f_1, f_2\}|, |\{g_1, g_2\}|\}$ and see that $\phi(\eta(\lambda)) = \phi(\lambda) = d(\lambda)$.
(3) Recall Example 8.22(ii): define $\eta : \Lambda \to \mathbb{N}^2$ by $\eta(a_1) = (1,1)$ and $\eta(a_2) = (1,0)$. Let $\phi : \mathbb{N}^2 \to \mathbb{Z}^2$ be defined by $\phi(m,n) = (n, m-n)$ then one checks that ϕ is a functor. For $\lambda \in \Lambda^N$ consisting of N_1 occurrences of a_1 and N_2 occurrences of a_2 we have $\eta(\lambda) = (N_1 + N_2, N_1)$ and

$$\phi(\eta(\lambda)) = \phi(N_1 + N_2, N_1) = (N_1, N_2) = N = d(\lambda).$$

Hence $C^*(\Lambda \times_\eta \mathbb{N}^2)$ is AF by Theorem 8.36.

(4) Let $X = \{2^n : n \in \mathbb{N}^\times\} \subseteq \mathbb{N}^\times$, a semigroup. Let $\Lambda = T_2$ and define $\eta : \Lambda \to \mathbb{N} \times X$ by $\eta(a_1) = (1, 1)$ and $\eta(a_2) = (0, 2)$. Then there is no functor $\phi : \mathbb{N} \times X \to \mathbb{Z}$ such that $d = \phi \circ \eta$. To see this observe that $\eta(f_1 f_1 f_2) = (2, 2)$ and $\eta(f_2 f_1) = (2, 2)$ and so if there were such a functor ϕ we would have

$$(1, 1) = d(f_2 f_1) = \phi(\eta(f_2 f_1)) = \phi(\eta(f_1 f_1 f_2)) = d(f_1 f_1 f_2) = (2, 1),$$

which is nonsense.

(5) If we proceed as per the last example, but take $\Lambda = B_2$ instead, and define $\eta : \Lambda \to \mathbb{N} \times X$ by $\eta(a_1) = (1, 1)$ and $\eta(a_2) = (0, 2)$. Then there is no functor $\phi : \mathbb{N} \times \{2^n : n \geq 0\} \to \mathbb{N}$ such that $d = \phi \circ \eta$. To see this observe that $\eta(a_1 a_1 a_2) = (2, 2)$ and $\eta(a_2 a_1) = (2, 2)$ and so if there were such a functor ϕ we would have

$$2 = d(a_2 a_1) = \phi(\eta(a_2 a_1)) = \phi(\eta(a_1 a_1 a_2)) = d(a_1 a_1 a_2) = 3,$$

which as with the previous example, is nonsense. However $\Lambda \times_\eta (\mathbb{N} \times X)$ is AF since it is a 1-graph with no loops (see [**38**]).

This final example shows that Theorem 8.36 cannot be an if and only if statement.

CHAPTER 9

Direct Limits

This chapter is inspired by the original method of proof of Theorem 6.1 for directed graphs by Pask, Raeburn and Yeend in [**50**]. It is included here for completeness. The dilation was shown to be a direct limit, and results regarding direct limit C^*-algebras were proved.

Turning our attention back to semigroup actions, we show the action of an Ore semigroup S on a k-graph Σ naturally gives rise to an inductive limit. In particular, when the k-graph is the skew-product graph $S\backslash\Sigma \times_\eta S$, we may dilate the action of S on the inductive limit to get an action of the enveloping group Γ on the inductive limit graph $S\backslash\Sigma \times_\eta \Gamma$. In Remark 9.12 we show that this justifies the choice of $\Omega = \Lambda \times_\eta S$ in Corollary 4.4.

We set up the notion of a direct limit of a system of k-graphs. In Theorem 9.8, we show that this notion can be used for the related system of graph algebras: for a directed system of k-graphs, the direct limit exists and is functorial in the sense that $C^*(\varinjlim \Lambda_x) = \varinjlim C^*(\Lambda_x)$.

9.1. A direct system of k-graphs

Recall from Chapter 8 the definitions of a preorder, a directed set, and a cofinal set.

Definition 9.1. Suppose $\{\Lambda_x : x \in X\}$ is a family of k-graphs with the index set X directed by $\leq$, and that for every $x \leq y$ there exists a graph morphism $\pi^x_y : \Lambda_x \to \Lambda_y$ such that $\pi^x_x = \mathrm{id}$ and $\pi^y_z \circ \pi^x_y = \pi^x_z$ whenever $x \leq y \leq z$. Then $(\{\Lambda_x : x \in X\}, \{\pi^x_y\})$ is a *direct system* of k-graphs with *linking maps* $\{\pi^x_y\}$.

The following proposition is a generalisation of [**50**, Proposition 2.1].

Proposition 9.2. *Suppose $(\{\Lambda_x : x \in X\}, \{\pi^x_y\})$ is a direct system of k-graphs directed by $\leq$. There exists a k-graph F and k-graph morphisms $\pi^x : \Lambda \to F$ satisfying $\pi^x = \pi^y \circ \pi^x_y$ whenever $x \leq y$, with the following property: whenever G is a k-graph and $\psi^x : \Lambda_x \to G$ are k-graph morphisms satisfying $\psi^x = \psi^y \circ \pi^x_y$, there is a unique k-graph morphism $\psi : F \to G$ such that $\psi \circ \pi^x = \psi^x$.*

If $\bigcup_{x\in X} \psi^x(\Lambda_x) = G$ then ψ is surjective. If each ψ^x is injective, then ψ is injective.

Proof. Define $\sim$ on the disjoint union $\bigsqcup_{x\in X}\Lambda_x$ by $\Lambda_x \ni \lambda \sim \mu \in \Lambda_y$ if and only if there exists $z \geq x, y$ with $\pi_z^x(\lambda) = \pi_z^y(\mu)$. We claim that $\sim$ is an equivalence relation: take $y = x$ and any $z \geq x$ to see that $\sim$ is reflexive; from the definition of $\sim$, the same choice of z demonstrates that $\sim$ is symmetric; if $\lambda \sim \mu$ and $\mu \sim \nu$ for $\lambda \in \Lambda_x$, $\mu \in \Lambda_y$, $\nu \in \Lambda_z$, then as X is directed, there exists $v \geq x, y$ with $\pi_v^x(\lambda) = \pi_v^y(\mu)$ and $w \geq y, z$ with $\pi_w^y(\mu) = \pi_w^z(\nu)$. Since X is directed, there exists $u \geq v, w$ with $\pi_u^v(\pi_v^x(\lambda)) = \pi_u^w(\pi_w^z(\nu)) \Rightarrow \pi_u^x(\lambda) = \pi_u^z(\nu)$, so $\lambda \sim \nu$, and so $\sim$ is transitive. Hence $\sim$ is an equivalence relation.

Define $F := \bigsqcup_{x\in X}\Lambda_x/\sim$. We claim with range and source maps $r, s : F^* \to F^0$ defined $r([\lambda]) = [r(\lambda)]$ and $s([\lambda]) = [s(\lambda)]$ respectively, composition $F \times F \to F$ defined for $\lambda \in \Lambda_x, \mu \in \Lambda_y : s([\lambda]) = r([\mu])$ by $[\lambda][\mu] = [\pi_z^x(\lambda)\pi_z^y(\mu)]$, and degree map $d_F : F \to \mathbb{N}^k$ for $\lambda \in \Lambda_x$ defined $d_F([\lambda]) = d_{\Lambda_x}(\lambda)$ satisfying the k-graph unique factorisation property, F is a k-graph. We use our definition of the quotient k-graph, Proposition 3.5.

We verify $r, s : F \to F^0$ are well-defined maps: suppose $\lambda \in \Lambda_x$ and take $\lambda' \in \Lambda_y$ such that $\lambda' \in [\lambda]$, then there exists $z \geq x, y$ with

$$\begin{aligned}\pi_z^x(\lambda) = \pi_z^y(\lambda') &\Rightarrow r(\pi_z^x(\lambda)) = r(\pi_z^y(\lambda'))\\ &\Rightarrow \pi_z^x(r(\lambda)) = \pi_z^y(r(\lambda')), \text{ using that } \{\pi_y^x\} \text{ are } k\text{-graph morphisms},\\ &\Rightarrow [r(\lambda)] = [r(\lambda')].\end{aligned}$$

A similar argument demonstrates $s : F \to F^0$ to be well-defined.

Composition $F \times F \to F$ is defined by taking $\lambda \in \Lambda_x, \mu \in \Lambda_y$ such that $s([\lambda]) = r([\mu])$, by $[\lambda][\mu] = [\pi_z^x(\lambda)\pi_z^y(\mu)]$. Suppose $\lambda \in \Lambda_x, \mu \in \Lambda_y$ with $s([\lambda]) = r([\mu])$, equivalently $[s(\lambda)] = [r(\mu)]$. Therefore, there exists $z \geq x, y$ with $\pi_z^x(s(\lambda)) = \pi_z^y(r(\mu)) \Leftrightarrow s(\pi_z^x(\lambda)) = r(\pi_z^y(\mu))$. Then $\pi_z^x(\lambda)$ is composable with $\pi_z^y(\mu)$ in Λ_z. Demonstrating well-defined: suppose $\pi_v^x(s(\lambda)) = \pi_v^y(r(\mu))$. Then $\pi_v^x(\lambda)$ is composable with $\pi_v^y(\mu)$ in Λ_v. There exists $w \geq z, v$ such that,

$$\pi_w^z(\pi_z^x(\lambda)\pi_z^y(\mu)) = \pi_w^x(\lambda)\pi_w^y(\mu) = \pi_w^v(\pi_v^x(\lambda)\pi_v^y(\mu)),$$

and so $\pi_z^x(\lambda)\pi_z^y(\mu) \sim \pi_v^x(\lambda)\pi_v^y(\mu)$.

Next we want if $\lambda \sim \lambda'$ then $[\lambda][\mu] = [\lambda'][\mu]$; ie. $[\pi_z^x(\lambda)\pi_z^y(\mu)] = [\pi_v^u(\lambda')\pi_v^y(\mu)]$. Suppose $\lambda \in \Lambda_x, \lambda' \in \Lambda_u, \mu \in \Lambda_y$. Since $\lambda \sim \lambda'$, $[\lambda] = [\lambda']$, then $s([\lambda]) = s([\lambda'])$; take $v \geq u, y$, so $[\lambda']$ is composable with $[\mu]$ in Λ_v. Choose $t \geq z, v$, then

$$\pi_t^z(\pi_z^x(\lambda)\pi_z^y(\mu)) = \pi_t^v(\pi_v^u(\lambda')\pi_v^y(\mu)).$$

Then $[\pi_z^x(\lambda)\pi_z^y(\mu)] = [\pi_v^u(\lambda')\pi_v^y(\mu)]$ and $[\lambda][\mu] = [\lambda'][\mu]$. A similar argument proves that if $\mu \sim \mu'$, $[\lambda][\mu] = [\lambda][\mu']$. Hence composition is well-defined.

We claim that F is a category. First, suppose $\lambda \in \Lambda_x$, $\mu \in \Lambda_y$, $\nu \in \Lambda_z$. Then

$$\begin{aligned} r([\mu][\nu]) &= r([\pi_t^x(\mu)\pi_t^y(\nu)]) = [r(\pi_t^x(\mu)\pi_t^y(\nu))] \\ &= [r(\pi_t^x(\mu))] = r([\pi_t^x(\mu)]) = r([\mu]). \end{aligned}$$

Similarly $s([\lambda][\nu]) = s([\nu])$.

Next, suppose $s([\lambda]) = r([\mu])$ and $s([\mu]) = r([\nu])$, then there exist $t \geq x, y$, $t' \geq y, z$ and $u \geq t, t'$ such that

$$\begin{aligned} ([\lambda][\mu])[\nu] &= [\pi_t^x(\lambda)\pi_t^y(\mu)][\nu] \\ &= [\pi_u^t(\pi_t^x(\lambda)\pi_t^y(\mu))\pi_u^z(\nu)] = [\pi_u^x(\lambda)\pi_u^y(\mu)\pi_u^z(\nu)] \\ [\lambda]([\mu][\nu]) &= [\lambda][\pi_{t'}^y(\mu)\pi_{t'}^z(\nu)] \\ &= [\pi_u^x(\lambda)\pi_u^{t'}(\pi_{t'}^y(\mu)\pi_{t'}^z(\nu))] = [\pi_u^x(\lambda)\pi_u^y(\mu)\pi_u^z(\nu)]. \end{aligned}$$

The linking maps $\pi_y^x : \Lambda_x \to \Lambda_y$ are functors, so $\pi_y^x(\iota_v) = \iota_{\pi_y^x(v)}$, hence $[\iota_v] = [\iota_{\pi_y^x(v)}]$.

$$r([\iota_v]) = [r(\iota_v)] = [v] = [s(\iota_v)] = s([\iota_v]).$$

Suppose $r([\lambda]) = [v]$, so $[r(\lambda)] = [v]$; for some $x, y \in X$, $\iota_v \in \Lambda_x$, $\lambda \in \Lambda_y$, then

$$\begin{aligned} [\iota_v][\lambda] &= [\pi_z^x(\iota_v)\iota_z^y(\lambda)] = [\iota_{\pi_z^x}\pi_z^y(\lambda)] \\ &= [\pi_z^y(\lambda)] = [\lambda], \end{aligned}$$

since Λ_z is a k-graph. Similarly, given $s([\lambda]) = [v]$, $[\lambda][\iota_v] = [\lambda]$. Hence F satisfies the axioms of a category.

We verify that the degree map d, defined $d : F \to \mathbb{N}^k$ by $d_F([\lambda]) = d_{\Lambda_x}(\lambda)$ if and only if $\lambda \in \Lambda_x$, is a well-defined functor that satisfies the unique factorisation property for k-graphs: suppose $\lambda \in \Lambda_x$ and $\lambda' \in \Lambda_y$ such that $[\lambda] = [\lambda']$, so there exists $z \geq x, y$ with $\pi_z^x(\lambda) = \pi_z^y(\lambda')$. Then

$$d_F([\lambda]) = d_F([\lambda']) \Longleftrightarrow d_{\Lambda_x}(\lambda) = d_{\Lambda_y}(\lambda') \Longrightarrow d_{\Lambda_z}(\pi_z^x(\lambda)) = d_{\Lambda_z}(\pi_z^y(\lambda')),$$

so d is well-defined.

We verify that d is a functor: suppose $\lambda \in \Lambda_x$, $\mu \in \Lambda_y$ with $[\lambda], [\mu] \in F$ composable, then

$$\begin{aligned} d_F([\lambda][\mu]) &= d_F(\pi_z^x(\lambda)\pi_z^y(\mu)) = d_{\Lambda_z}(\pi_z^x(\lambda)\pi_z^y(\mu)) \\ &= d_{\Lambda_z}(\pi_z^x(\lambda)) + d_{\Lambda_z}(\pi_z^y(\mu)) = d_{\Lambda_x}(\lambda) + d_{\Lambda_y}(\mu) \\ &= d_F([\lambda]) + d_F([\mu]). \end{aligned}$$

Finally, the unique factorisation property for k-graphs: suppose $\lambda \in \Lambda_x$, such that $[\lambda] \in F$ with $d_F([\lambda]) = m + n$. Since Λ_x is a k-graph there exist unique $\mu, \nu \in \Lambda_x$ such that $d_{\Lambda_x}(\mu) = d_F([\mu]) = m$, $d_{\Lambda_x}(\nu) = d_F([\nu]) = n$, and $\lambda = \mu\nu$. Therefore $[\mu]$ and $[\nu]$ are composable as $s([\mu]) = [s(\mu)] = [r(\nu)] = r([\nu])$, and $[\mu\nu] = [\pi_x^x(\mu)\pi_x^x(\nu)] = [\mu][\nu]$. Showing uniqueness: suppose there exists $\mu' \in \Lambda_w$,

$\nu' \in \Lambda_y$ with $d([\mu']) = m$, $d([\nu']) = n$ and $[\lambda] = [\mu'][\nu']$. Since $[\mu']$ composable with $[\nu']$, there exists $z \geq w, y$ and maps π_z^w, π_z^y such that $[\mu'][\nu'] = [\pi_z^w(\mu')\pi_z^y(\nu')]$. So

$$[\mu'][\nu'] = [\lambda] = [\mu][\nu] \Rightarrow [\pi_z^w(\mu')\pi_z^y(\nu')] = [\pi_x^x(\mu)\pi_x^x(\nu)] = [\mu\nu].$$

Hence there exists $t \geq z, x$ such that

$$\pi_t^z(\pi_z^w(\mu')\pi_z^y(\nu')) = \pi_t^x(\mu\nu) = \pi_t^x(\lambda) \Rightarrow \pi_t^w(\mu')\pi_t^y(\nu') = \pi_t^x(\mu)\pi_t^x(\nu) = \pi_t^x(\lambda).$$

Λ_t is a k-graph, and hence has the unique factorisation property, and since

$$d_{\Lambda_t}(\pi_t^w(\mu')) = m = d_{\Lambda_t}(\pi_t^x(\mu)) \text{ and } d_{\Lambda_t}(\pi_t^y(\nu')) = n = d_{\Lambda_t}(\pi_t^x(\nu)),$$

we have $\pi_t^w(\mu') = \pi_t^x(\mu)$ and $\pi_t^y(\nu') = \pi_t^x(\nu)$, so $[\mu'] = [\mu]$ and $[\nu'] = [\nu]$. Hence (F, d) is a k-graph.

Each map $\pi^x : \Lambda_x \to F : \lambda \mapsto [\lambda]$ is a k-graph morphism. Take $\lambda \in \Lambda_x$: then $\pi^x(\lambda) = [\lambda]$ and $\pi^y(\pi_y^x(\lambda)) = [\pi_y^x(\lambda)] = [\lambda]$. Therefore, for $y \geq x$, $\pi^x = \pi^y \circ \pi_y^x$.

Next, suppose there exists a k-graph G and k-graph morphisms $\psi^x : \Lambda_x \to G$ satisfying $\psi^x = \psi^y \circ \pi_y^x$. We define a map $\psi : F \to G$ as follows: take $u \in F$, choose $x \in X$ and $p \in \Lambda_x$ such that $\pi^x(p) = u$ and define $\psi(u) = \psi^x(p)$. Demonstrating ψ well-defined: suppose $\pi^x(p) = u = \pi^y(q)$ for some $q \in \Lambda_y$, then there exists $z \in X$ such that $\pi_z^x(p) = \pi_z^y(q)$, so $\psi^x(p) = \psi^z(\pi_z^x(p)) = \psi^z(\pi_z^y(q)) = \psi^y(q)$. Verifying ψ is a k-graph morphism: first, that it is degree-preserving:

$$\begin{aligned} d_G(\psi(u)) &= d_G(\psi^x(p)) = d_{\Lambda_x}(p), \text{ since } \psi^x \text{ is a } k\text{-graph morphism,} \\ &= d_G(\pi^x(p)) = d_F(u), \text{ since } \pi^x \text{ is a } k\text{-graph morphism.} \end{aligned}$$

That ψ is degree-preserving means it maps the degree-zero morphisms of F to the degree-zero morphisms of G; that is, it takes the objects of F to the objects of G. For $u = \pi^x(p)$, $\psi(\pi^x(p)) = \psi^x(p)$, hence

$$\begin{aligned} r(\psi(u)) &= r(\psi(\pi^x(p))) = r(\psi^x(p)) = \psi^x(r(p)), \text{ since } \psi^x \text{ is a } k\text{-graph morphism,} \\ &= \psi(\pi^x(r(p))) = \psi(r(\pi^x(p))) = \psi(r(u)). \end{aligned}$$

A similar argument demonstrates $\psi(s(u)) = s(\psi(u))$.

Lastly, demonstrating that ψ respects composition: take $u = \pi^x(p)$, $v = \pi^y(q)$ such that $s(u) = r(v)$. Either $x \geq y$ or $y \geq x$; suppose $y \geq x$, then $\pi^x = \pi^y \circ \pi_y^x$. Using that ψ^y is a k-graph morphism:

$$\begin{aligned} \psi(uv) &= \psi(\pi^x(p)\pi^y(q)) = \psi(\pi^y(\pi_y^x(p)q)) = \psi^y(\pi_y^x(p)q) \\ &= \psi^y(\pi_y^x(p))\psi^y(q) = \psi^x(p)\psi^y(q) = \psi(\pi^x(p))\psi(\pi^y(q)) = \psi(u)\psi(v). \end{aligned}$$

Hence $\psi : F \to G$ is a k-graph morphism.

To see that $\psi : F \to G$ is unique, suppose $\phi : F \to G$ satisfies $\phi \circ \pi^x = \psi^x = \psi \circ \pi^x$. Since every $p \in F$ can be written in the form $\pi^x(\lambda) = p$,

$$\phi(p) = \phi(\pi^x(\lambda)) = \psi(\pi^x(\lambda)) = \psi(p).$$

Hence, given the hypotheses, there exists a unique k-graph morphism $\psi : F \to G$ such that $\psi \circ \pi^x = \psi^x$.

Suppose $\bigcup_{x\in X} \psi^x(\Lambda_x) = G$. Then for each $g \in G$, there exists $x \in X$ and $\lambda \in \Lambda_x$ such that $\psi^x(\lambda) = g$. Combined with the identity $\psi \circ \pi^x = \psi^x$, this shows ψ is surjective.

Suppose ψ^x are injective for all $x \in X$. Suppose $\psi([\lambda]) = \psi([\mu])$, then there exist $x, y \in X$, $\lambda \in \Lambda_x$ and $\mu \in \Lambda_y$ such that $\pi^x(\lambda) = [\lambda]$ and $\pi^y(\mu) = [\mu]$. Therefore,

$$\psi([\lambda]) = \psi([\mu]) \Longrightarrow \psi(\pi^x(\lambda)) = \psi(\pi^y(\mu)) \Longrightarrow \psi^x(\lambda) = \psi^y(\mu).$$

The set X is directed, so there exists $z \in X$ such that $z \geq x, y$, and using the relation $\psi^x = \psi^z \circ \pi^x_z$, this implies $\psi^z(\pi^x_z(\lambda)) = \psi^z(\pi^y_z(\mu))$. The k-graph morphism ψ^z is injective by assumption, implying

$$\pi^x_z(\lambda) = \pi^y_z(\mu) \Rightarrow \pi^z(\pi^x_z(\lambda)) = \pi^z(\pi^y_z(\mu)) \Rightarrow \pi^x(\lambda) = \pi^y(\mu) \Rightarrow [\lambda] = [\mu].$$

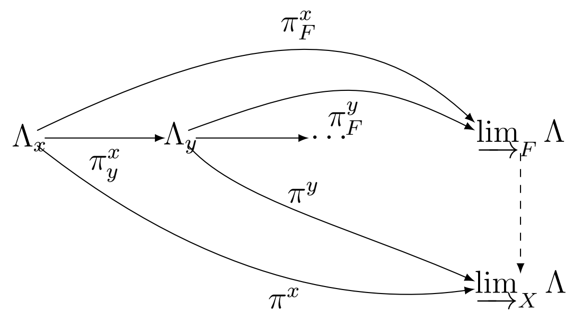
Hence ψ is injective. □

The universal property of the inductive limit is unique up to isomorphism, and so it makes sense to define $(F, \{\pi^x : x \in X\})$ to be the direct limit of the direct system $(\{\Lambda_x : x \in X\}, \{\pi^x_y\})$. We denote the limit by $\varinjlim \Lambda_x$.

This next lemma demonstrates that the direct limit of a cofinal subset of a directed set gives rise to the same direct limit, up to isomorphism.

LEMMA 9.3. *Suppose X is a directed set, F is a cofinal subset of X, and $(\{\Lambda_x : x \in x\}, \{\pi^x_y\})$ is a direct system of k-graphs directed by $\geq$. Then there exists an isomorphism $\pi^\infty_F : \varinjlim_F \Lambda_{f\in F} \to \varinjlim_X \Lambda_{x\in X}$.*

PROOF. For every $x, y \in X$, and in particular for every $x, y \in F$ we have commuting maps such that:



So applying Proposition 9.2, there exists a k-graph morphism $\pi^\infty_F : \varinjlim_F \Lambda \to \varinjlim_X \Lambda$ such that $\pi^\infty_F \circ \pi^x_F = \pi^x$ for all $x \in F$. We wish for π^∞_F to be an isomorphism. We verify injectivity: the direct limit over the cofinal set F can be written as $\varinjlim_F \Lambda = \bigcup_{x\in F} \pi^x_F(\lambda)$. Suppose $\pi^\infty_F(\pi^x_F(\lambda)) = \pi^\infty_F(\pi^y_F(\mu))$. This is true if and only if $\pi^x(\lambda) = \pi^y(\mu)$ in $\varinjlim_X \Lambda$. Using the definition of the direct limit, and that the set

X is directed, there exists $z \in X$ such that $z \geq x, y$ such that $\pi_z^x(\lambda) = \pi_z^y(\mu)$. Now as F is a cofinal subset of X, there exists $z' \in F$ such that $z' \geq z$, and

$$\pi_{z'}^x(\lambda) = \pi_{z'}^z(\pi_z^x(\lambda)) = \pi_{z'}^z(\pi_z^y(\mu)) = \pi_{z'}^y(\mu).$$

Therefore $\pi_F^{z'}(\pi_{z'}^x(\lambda)) = \pi_F^{z'}(\pi_{z'}^x(\mu))$, and so $\pi_F^x(\lambda) = \pi_F^y(\mu)$, hence $\pi_F^\infty : \varinjlim_F \Lambda \to \varinjlim_X \Lambda$ is injective.

We verify surjectivity: for all $x \in X$ and $\lambda \in \Lambda_x$, $\pi^x(\lambda) \in \varinjlim_X \Lambda$. F is a cofinal subset of X, so there exists $z \in F$ such that $z \geq x$, then $\pi^x(\lambda) = \pi_F^\infty \circ \pi_F^z \circ \pi_z^x(\lambda)$. As there exists $z \in F$ and $\pi_F^z(\pi_z^x(\lambda)) \in \varinjlim_F \Lambda$, π_F^∞ is surjective, and is therefore an isomorphism from $\varinjlim_F \Lambda$ onto $\varinjlim_X \Lambda$. □

The definition of a saturated k-graph morphism was given as Definition 2.13. The next lemma demonstrates the connection between injectivity of the linking maps in a direct system, and the injectivity of the embedding maps into the direct limit.

LEMMA 9.4. *Suppose* $(\{\Lambda_x : x \in X\}, \{\pi_y^x\})$ *is a direct system of* k*-graphs. If the linking maps* $\{\pi_y^x\}$ *are injective for all* $x \leq y$*, then the embedding maps* $\{\pi^x : \Lambda_x \to \varinjlim_X \Lambda\}$ *are injective.*

PROOF. Suppose $\lambda, \mu \in \Lambda_x$ such that $\pi^x(\lambda) = \pi^x(\mu)$. Then the classes of λ and μ in the direct limit are equal, $[\lambda] = [\mu]$. By our definition of the direct limit and a direct system in Proposition 9.2, there exists $z \geq x$ such that $\pi_z^x(\lambda) = \pi_z^x(\mu)$. We have assumed $\{\pi_y^x\}$ to be injective, so $\lambda = \mu$. □

The following lemma draws together the concept of a direct system with the concept of saturation. It was originally proved for directed graphs by Pask, Raeburn and Yeend as [**50**, Theorem 3.5].

LEMMA 9.5. *Let* $(\{\Lambda_x : x \in X\}, \{\pi_y^x\})$ *be a direct system of* k*-graphs with injective linking maps. If for* $x \leq y$*,* π_y^x *is a saturated* k*-graph morphism of* Λ_x *to* Λ_y*, then* π^y *is a saturated* k*-graph morphism of* Λ_y *to* $\varinjlim_X \Lambda$ *for all* $y \in X$*.*

PROOF. We claim that $\pi^x : \Lambda_x \to \pi^x(\Lambda_x)^0 \varinjlim_X \Lambda$ is bijective as this will demonstrate that π^x is a saturated morphism. To demonstrate surjectivity: suppose $\lambda \in \varinjlim_X \Lambda$ with $r(\lambda) \in \pi^x(\Lambda_x)^0$. The direct limit is the union of the images of $\{\Lambda_x\}$ by the embedding maps, modulo an appropriate equivalence relation, so there exists $y \in X$ such that $y \geq x$ and $\lambda_y \in \Lambda_y$ such that $\pi^y(\lambda_y) = \lambda$ and $r(\lambda_y) \in \pi_y^x(\Lambda_x)^0$. However, because the maps $\{\pi_y^x\}$ are assumed saturated, there exists a unique $\lambda_x \in \Lambda_x$ such that $\pi_y^x(\lambda_x) = \lambda_y$. Then $\pi^x(\lambda_x) = \pi^y(\pi_y^x(\lambda_x)) = \lambda$, and so $\pi^x : \Lambda_x \to \pi^x(\Lambda_x^0) \varinjlim_X \Lambda$ is surjective.

By Lemma 9.4, $\pi^x : \Lambda_x \to \varinjlim_X \Lambda$ is injective, hence $\pi^x : \Lambda \to \pi^x(\Lambda_x)^0 \varinjlim \Lambda_x$ is also injective. □

From a direct system of k-graphs, we move to C^*-algebras, and demonstrate that our results have analogues in a C^*-algebraic sense. The concept of a direct system can be applied to C^*-algebras:

REMARK 9.6. A direct system of C^*-algebras consists of a set of C^*-algebras $\{A_x\}_{x\in X}$ indexed by a directed set X along with homomorphisms $\{\pi^x_y : A_x \to A_y : x \leq y\}$ satisfying $\pi^y_z \circ \pi^x_y = \pi^x_z$ for all $x, y, z \in X$ such that $x \leq y \leq z$. The linking maps will be assumed to be injective to prevent trivial cases from arising.

We assert the existence of a direct limit C^*-algebra for the directed system $(\{A_x : x \in X\}, \{\pi^x_y\})$. We will denote such an algebra in familiar notation as $\varinjlim A_x$, and the canonical embeddings of $A_x \hookrightarrow \varinjlim A_x$ will be written π^x. For a C^*-algebra B and homomorphisms $\psi^x : A_x \to B$ satisfying $\psi^x = \psi^y \circ \pi^x_y$ for all $x \leq y$, the direct limit homomorphism will be denoted $\varinjlim \psi^x : \varinjlim A_x \to B$. By [**31**, Proposition 11.4.1(i)], such a direct limit C^*-algebra exists. Laca [**41**, Proposition 2.1] asserts that such a C^*-algebra is unique.

Using this notation, we give the following lemma from Pask, Raeburn and Yeend, [**50**, Lemma 3.1].

LEMMA 9.7. *Suppose $(\{A_x : x \in X\}, \{\pi^x_y\})$ is a direct system of C^*-algebras, B is a C^*-algebra, and there exists homomorphism $\psi^x : A_x \to B$ such that $\psi^x = \psi^y \circ \pi^x_y$ for all $x \leq y$. Denote $\varinjlim \psi^x$ to be the induced map from $\varinjlim A_x$ to B. If $\bigcup_{x\in X} \psi^x(A_x)$ is dense in B, then the homomorphism $\varinjlim \psi^x$ is surjective. If each $\psi^x : A_x \to B$ is injective, then $\varinjlim \psi^x$ is injective.*

The following theorem has an analogue that was originally proved for directed graphs as [**50**, Theorem 3.5]: we show that the C^*-algebra of the inductive limit of a directed system of k-graphs is the inductive limit of a directed system of k-graph C^*-algebras. Despite the fact that we are here dealing with a direct system of C^*-algebras encoded by k-graphs rather than directed graphs, the argument is similar.

THEOREM 9.8. *Let $(\{\Lambda_x : x \in X\}, \{\pi^x_y\})$ be a direct system of row-finite k-graphs with injective linking maps, and suppose that $\pi^x_y : \Lambda_x \to \Lambda_y$ is a saturated k-graph morphism for each $x \leq y$. Then $(\{C^*(\Lambda_x)\}, \{(\pi^x_y)_*\})$ is a direct system of C^*-algebras, and the induced map $\varinjlim(\pi^x)_*$ is an isomorphism of $\varinjlim C^*(\Lambda_x)$ onto $C^*(\varinjlim \Lambda_x)$.*

PROOF. The proof of this theorem relies very heavily on Proposition 2.14, Lemma 9.5 and the properties of a direct system, Definition 9.1.

Proposition 2.14 implies the existence of homomorphisms $(\pi^x_y)_* : C^*(\Lambda_x) \to C^*(\Lambda_y)$. Also by Proposition 2.14, and by the relations on the linking maps, for

$x, y, z \in X$ with $x \leq y \leq z$, $(\pi^x_y)_* \circ (\pi^y_z)_* = (\pi^x_y \circ \pi^y_z)_* = (\pi^x_z)_*$. Hence $(\{C^*(\Lambda_x)\}, \{(\pi^x_y)_*\})$ is a direct system of C^*-algebras. Lemmas 9.4 and 9.5 imply that the k-graph morphisms $\pi^x : \Lambda_x \to \varinjlim \Lambda_x$ are injective and saturated, and hence by Proposition 2.14 and the gauge-invariant uniqueness theorem, see for example [**36**, Theorem 3.4], induce injective homomorphisms $(\pi^x)_* : C^*(\Lambda_x) \to C^*(\varinjlim \Lambda_x)$. Proposition 2.14 and the relations on the limit maps imply $(\pi^y)_* \circ (\pi^x_y)_* = (\pi^y \circ \pi^x_y)_* = (\pi^x)_*$. Consequently, for each $x \in X$, there exists a homomorphism $\varinjlim (\pi^x)_* : \varinjlim C^*(\Lambda_x) \to C^*(\varinjlim \Lambda_x)$, which is injective because each $(\pi^x)_*$ being injective by Lemma 9.7. If we consider the direct limit as defined to be $\bigcup_{x \in X} \pi^x(\Lambda_x)$, for each $\lambda \in \Lambda_x$, including degree zero representations of objects, for every $x \in X$ there exist $\mu \in \varinjlim \Lambda_x$ such that $(\pi^x)_*(s_\lambda) = t_\mu$. We then have

$$\bigcup_{x \in X} \{(\pi^x)_*(s_\lambda) : \lambda \in \Lambda\} = \{t_\mu : \mu \in (\varinjlim \Lambda_x)\}.$$

Therefore, $\bigcup_{x \in X} \{(\pi^x)_*(C^*(\Lambda_x))\}$ is dense in $C^*(\varinjlim \Lambda_x)$, and Lemma 9.7 implies that the map $\varinjlim (\pi^x)_*$ is an isomorphism onto $C^*(\varinjlim \Lambda_x)$. □

Returning to idea of the thesis, namely semigroup actions; we need to know when the action induces a saturated map. We first require the following lemma, which appeared as Lemma 2.2 of [**50**]. We proved an analogue for left-reversible semigroups as Lemma 8.2. The proof for an Ore (right-reversible) semigroup is quite similar.

LEMMA 9.9. *Let S be an Ore semigroup with enveloping group Γ, and define $\preceq_r$ on Γ by $g \preceq_r h$ if and only if $hg^{-1} \in S$. Then $\preceq_r$ is a right-invariant preorder that directs Γ, and for any $t \in S$, St is cofinal in S.*

9.2. The direct limit of a directed system

Next we consider a result from Laca [**41**, Theorem 2.1] related to a direct system of C^*-algebras rather than k-graphs. In Pask, Raeburn and Yeend, [**50**, Theorem 2.3], the situation with directed graphs is proved by following Laca's argument. We shall follow both Laca [**41**] and Pask, Raeburn and Yeend [**50**] to the analogous theorem involving k-graphs. Such an inductive limit occurs naturally when we have an action of an Ore semigroup on system of k-graphs.

THEOREM 9.10. *Let $\alpha : S \to \mathrm{End}(\Lambda)$ be an action of an Ore semigroup on a k-graph. Then the family of k-graphs $\{\Lambda_t := \Lambda : t \in S\}$, together with graph morphisms $\alpha^s_t = \alpha_{ts^{-1}} : \Lambda = \Lambda_s \to \Lambda_t = \Lambda$, form a direct system. Moreover, there is an action α^∞ of Γ on $\Lambda_\infty := \varinjlim \Lambda_x$ such that:*

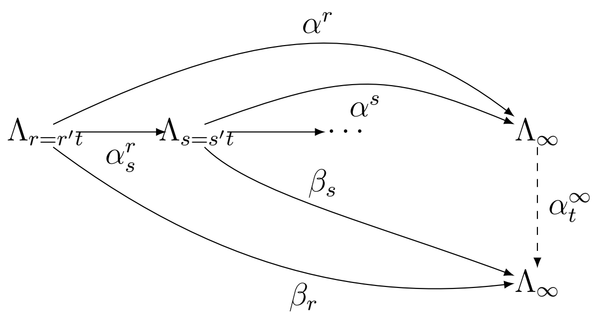
(1) *if $s \in St$, then $\alpha^\infty_t \circ \alpha^s = \alpha^{st^{-1}}$ on $\Lambda_s = \Lambda = \Lambda_{st^{-1}}$*
(2) *α^∞ dilates α in the sense that, for any $t \in S$, $\alpha^\infty_t \circ \alpha^{\mathrm{id}} = \alpha^{\mathrm{id}} \circ \alpha_t$ on $\Lambda = \Lambda_{\mathrm{id}}$.*

PROOF. The argument follows [**50**, Theorem 2.3]: that $(\{\Lambda_t\}, \{\alpha_t^s\} : s, t \in S)$ is a direct system is straightforward: $\preceq_r$ directs S by Lemma 9.9. For all $s \in S$, $\alpha_s^s = \alpha_{ss^{-1}} = \alpha_{\mathrm{id}} = \mathrm{id}$, and $\alpha_t^s \circ \alpha_s^r = \alpha_{ts^{-1}} \circ \alpha_{sr^{-1}} = \alpha_{tr^{-1}} = \alpha_t^r$.

For each $t \in S$, consider the sub-direct system $\{\Lambda_s : s \in St\}$ together with the same morphisms $\{\alpha_t^s\}$. By Lemma 9.9, St is cofinal in S, and hence by 9.3 the sub-direct system has the same direct limit, Λ_∞. We define graph morphisms $\beta^s : \Lambda_s \to \Lambda_\infty$ for $s \in St$ by $\beta^s = \alpha^{st^{-1}} : \Lambda_{st^{-1}} \to \Lambda_\infty$. By Proposition 9.2, the set of k-graph morphisms $\{\beta^s\}$ will generate a k-graph morphism, $\alpha_t^\infty : \Lambda_\infty \to \Lambda_\infty$ if and only if for all $r, s \in St$ with $r \preceq_r s$, $\beta^r = \beta^s \circ \alpha_s^r$. We calculate:

$$\beta^s \circ \alpha_s^r = \alpha^{st^{-1}} \circ \alpha_{sr^{-1}} = \alpha^{st^{-1}} \circ \alpha_{st^{-1}tr^{-1}} = \alpha^{st^{-1}} \circ \alpha_{st^{-1}}^{rt^{-1}} = \alpha^{rt^{-1}} = \beta^r.$$

Therefore, again by Proposition 9.2, there exists a graph morphism $\alpha_t^\infty : \Lambda_\infty \to \Lambda_\infty$ such that for $s \in St$, $\alpha_t^\infty \circ \alpha^s = \beta^s$ demonstrating (1). Our situation can be viewed in part as represented here:



We can now apply Lemma 9.3 to the systems $\{\Lambda_s := \Lambda\}$, indexed by the set S and by the cofinal subset St, and conclude α_t^∞ is an automorphism $\Lambda_\infty \to \Lambda_\infty$.

A short calculation utilising the relations between the k-graph morphisms will demonstrate (2):

$$\alpha_t^\infty \circ \alpha^{\mathrm{id}} = \beta^{\mathrm{id}} = \beta^t \circ \alpha_t^{\mathrm{id}} = \alpha^{tt^{-1}} \circ \alpha_{t\,\mathrm{id}^{-1}} = \alpha^{\mathrm{id}} \circ \alpha_t. \qquad \square$$

The next proposition demonstrates the idea that when a k-graph Σ can be identified as a skew-product with an free action α by left translation, which by Proposition 3.3 is equivalent to the existence of a free action and a fundamental domain, the direct limit can be identified by the skew-product over the enveloping group, and the action α^∞ is left translation on the resultant group skew-product k-graph. This proposition was proposed as [**50**, Proposition 2.4] by Pask, Raeburn and Yeend for directed graphs. We propose a new result for k-graphs. Whilst the idea of the argument from [**50**, Proposition 2.4] is retained, we utilise our new definitions of the skew-product, Proposition 3.3 and the emphasis on the role of the skew-product functor η. To simplify the statement of the following result, we have defined $\Lambda := S\backslash\Sigma$.

Proposition 9.11. *Suppose the system* $(\Sigma, S, \alpha) = (\Lambda \times_\eta S, S, \mathrm{lt})$ *is a skew-product with* lt *the action of* S *by left translation. For* $\mathrm{lt}^t : (\Lambda \times_\eta S)_t \to (\Lambda \times_\eta S)_\infty$ *defined* $\mathrm{lt}^t(\lambda, s) \mapsto (\lambda, ts)$ *and* $\psi^t : (\Lambda \times_\eta S)_t \to (\Lambda \times_\eta \Gamma)$ *defined* $\psi^t(\lambda, s) = (\lambda, t^{-1}s)$*, there is an isomorphism* ψ *from* $(\Lambda \times_\eta S)_\infty$ *onto* $\Lambda \times_\eta \Gamma$ *characterised by* $\psi^t = \psi \circ \mathrm{lt}^t$ *and* $\psi \circ \mathrm{lt}^\infty = \sigma \circ \psi$*, where* σ *is action of* Γ *by left translation on* $\Lambda \times_\eta \Gamma$*.*

Proof. For each $t \in S$, define a map $\psi^t : (\Lambda \times_\eta S)_t \to \Lambda \times_\eta \Gamma : (\lambda, s) \mapsto (\lambda, t^{-1}s)$. For $t \in S$ and $(\lambda, a) \in (\Lambda \times_\eta S)_t$, using Proposition 9.2 and our definition of the semigroup skew-product:

$$\psi^t(s(\lambda, a)) = \psi^t(s(\lambda), a\eta(\lambda)) = (s(\lambda), t^{-1}a\eta(\lambda)) = s(\lambda, t^{-1}a) = s(\psi^t(\lambda, a))$$
$$\psi^t(r(\lambda, a)) = \psi^t(r(\lambda), a) = (r(\lambda), t^{-1}a) = r(\lambda, t^{-1}a) = r(\psi^t(\lambda, a)).$$

The map ψ^t preserves degree: $d_{\Lambda \times_\eta \Gamma}(\psi^t(\lambda, a)) = d_{\Lambda \times_\eta \Gamma}(\lambda, t^{-1}a) = d_\Lambda(\lambda) = d_{\Lambda \times_\eta \Gamma}(\lambda, a)$ using the analogous definition for the group skew-product. That ψ^t maps the objects of $(\Lambda \times_\eta S)_t$ to the objects of $\Lambda \times_\eta \Gamma$ is a consequences of ψ^t preserving degree.

Take $(\lambda, a), (\mu, b) \in (\Lambda \times_\eta S)_t$ such that $s(\lambda, a) = r(\mu, b)$, hence $b = a\eta(\lambda)$:

$$\begin{aligned}\psi^t((\lambda, a)(\mu, b)) &= \psi^t(\lambda\mu, a) = (\lambda\mu, t^{-1}a) = (\lambda, t^{-1}a)(\mu, t^{-1}a\eta(\lambda)) \\ &= (\lambda, t^{-1}a)(\mu, t^{-1}b) = \psi^t(\lambda, a)\psi^t(\mu, b).\end{aligned}$$

Hence for all $t \in S$, $\psi^t : (\Lambda \times_\eta S)_t \to \Lambda \times_\eta \Gamma$ is a k-graph morphism.

For appropriate $s, t \in S$, take $\mathrm{lt}^s_t : (\Lambda \times_\eta S)_s \to (\Lambda \times_\eta S)_t : (\lambda, a) \mapsto (\lambda, ts^{-1}a)$. Then

$$\psi^t(\mathrm{lt}^s_t(\lambda, a)) = \psi^t(\lambda, ts^{-1}a) = (\lambda, t^{-1}ts^{-1}ta) = \psi^s(\lambda, a),$$

so $\psi^t \circ \mathrm{lt}^s_t = \psi^s$.

We claim $(\{(\Lambda \times_\eta S)_t : t \in S\}, \{\mathrm{lt}^s_t\})$ is a direct system of k-graphs with linking maps $\{\mathrm{lt}^s_t\}$. Taking $s = t$, $\mathrm{lt}^s_s(\lambda, a) = (\lambda, ss^{-1}a) = (\lambda, \mathrm{id}\, a)$. Suppose $r \leq s \leq t$:

$$\mathrm{lt}^s_t(\mathrm{lt}^r_s(\lambda, a)) = \mathrm{lt}^s_t(\lambda, sr^{-1}a) = (\lambda, ts^{-1}sr^{-1}a) = \mathrm{lt}^r_t(\lambda, a).$$

For the direct system we have linking maps $\{\mathrm{lt}^s_t\}$, k-graph morphisms $\mathrm{lt}^t : (\Lambda \times_\eta S)_t \to (\Lambda \times_\eta S)_\infty$ such that $\mathrm{lt}^t \circ \mathrm{lt}^s_t = \mathrm{lt}^s$ and k-graph morphisms $\psi^t : (\Lambda \times_\eta S)_t \to \Lambda \times_\eta \Gamma$ such that $\psi^t \circ \mathrm{lt}^s_t = \psi^s$. Therefore we can utilise Proposition 9.2, and hence there exists a unique k-graph morphism $\psi := \varinjlim \psi^t : (\Lambda \times_\eta S)_\infty \to \Lambda \times_\eta \Gamma$ satisfying $\psi \circ \mathrm{lt}^t = \psi^t$ for all $t \in S$.

Take $(\lambda, g) \in \Lambda \times_\eta \Gamma$. Γ is the enveloping group of an Ore semigroup S such that $\Gamma = S^{-1}S$. So for $g \in \Gamma$, there exist $s, t \in S$ such that $g = s^{-1}t$. Then

$$\psi(\mathrm{lt}^s(\lambda, t)) = \psi^s(\lambda, t) = (\lambda, s^{-1}t) = (\lambda, g),$$

and so ψ is a surjective k-graph morphism of $(\Lambda \times_\eta S)_\infty \to \Lambda \times_\eta \Gamma$.

If for all $t \in S$ we can verify ψ^t to be injective, then by Proposition 9.2, ψ is injective: suppose $\psi^t(\lambda, a) = \psi^t(\mu, b)$. Then $(\lambda, t^{-1}a) = (\mu, t^{-1}b)$. So $\lambda = \mu$ and the cancellative property of S implies $a = b$. So all ψ^t are injective, hence ψ is injective. Therefore ψ is an isomorphism of $(\Lambda \times_\eta S)_\infty$ onto $\Lambda \times_\eta \Gamma$.

Demonstrating ψ to be equivariant: we require $\psi \circ \mathrm{lt}_g^\infty = \sigma_g \circ \psi$ for all $g \in \Gamma$. Fixing $t \in S$, we know St is a cofinal subset of S and that the set $\{(\Lambda \times_\eta S)_s := (\Lambda \times_\eta S)\}$ indexed by $s \in St$ has the same direct limit, $(\Lambda \times_\eta S)_\infty$, as when indexed by $s \in S$. For each $(\lambda, a) \in \Lambda_s$ and $s \in St$, using relations Theorem 9.10 (1), (2), we have:

$$\begin{aligned}\psi \circ \mathrm{lt}_t^\infty(\mathrm{lt}^s(\lambda, a)) &= \psi \circ \mathrm{lt}^{st^{-1}}(\lambda, a) = \psi^{st^{-1}}(\lambda, a) = (\lambda, ts^{-1}a)\\ &= \sigma_t \circ \psi^s(\lambda, a) = \sigma_t \circ \psi(\mathrm{lt}^s(\lambda, a)).\end{aligned}$$

For $g \in \Gamma$, there exist $s, t \in S$ such that $g = s^{-1}t$, and

$$\begin{aligned}\psi \circ \mathrm{lt}_g^\infty &= \psi \circ \mathrm{lt}_{s^{-1}t}^\infty = \psi \circ (\mathrm{lt}_s^\infty)^{-1} \circ \mathrm{lt}_t^\infty = \psi \circ (\psi \circ \mathrm{lt}_s^\infty)^{-1} \circ (\psi \circ \mathrm{lt}_t^\infty)\\ &= \psi \circ (\sigma_s \circ \psi)^{-1} \circ (\sigma_t \circ \psi) = (\sigma_s)^{-1} \circ \sigma_t \circ \psi = \sigma_g \circ \psi,\end{aligned}$$

so ψ is an equivariant isomorphism of $(\Lambda \times_\eta S)_\infty$ onto $\Lambda \times_\eta \Gamma$. □

Remark 9.12. Consider the system in Corollary 4.4, we can see that $C^*(\Lambda)$ is the inductive limit of $C^*(\Omega)$. We have that $\Lambda \times_\eta \Gamma$ is the direct limit of $\Lambda \times_\eta S$. Moreover,

$$C^*(\Lambda \times_\eta \Gamma) = C^*(\varinjlim \Lambda \times_\eta S) = \varinjlim C^*(\Lambda \times_\eta S),$$

which is equivariant for lt_*. We observe that Ω appearing in Corollary 4.4 is a copy of $\Lambda \times_\eta S \subseteq \Lambda \times_\eta \Gamma$.

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