Exact travelling wave solutions of a variety of Boussinesq-like equations

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Abstract
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Keywords
wave, solutions, variety, boussinesq, exact, like, travelling, equations

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Exact Travelling Wave Solutions of a Variety of Boussinesq-Like Equations

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In this paper, we obtain exact traveling wave solutions of a variety of Boussinesq-like equations by using two distinct methods with symbolic computation. The Boussinesq equations play an important role in physical applications, such as in nonlinear lattice waves, acoustic waves, iron sound waves in a plasma, and vibrations in a nonlinear string. More precisely, the modified tanh-coth method is employed to obtain single soliton solutions, and the extended Jacobi elliptic function method is applied to derive doubly periodic wave solutions. Further, it is shown that soliton solutions and triangular solutions can be established as the limits of the Jacobi doubly periodic wave solutions. The employed approaches are quite efficient for the determination of the solutions, and are practically well suited for solving nonlinear evolution equations arising in physics.

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I. INTRODUCTION

Nonlinear partial differential equations are important in many fields of science and engineering, because these equations are used to describe the real features in a wide variety of scientific models, including the propagation of shallow water waves, hydrodynamics, plasma physics, solid state physics, chemical kinematics, optical fibers, solid state physics, fluid dynamics, chemical kinetics, and so on. Further, determination of the wave solutions for nonlinear evolution equations has been a subject of great interest, due its significant applications in real world problems [1–3]. For this reason, a variety of powerful methods have been developed to construct exact wave solutions for nonlinear evolution equations, such as the homotopy perturbation method [4], moving least square method [5], $G/G'$-expansion method [6], homotopy analysis method [7], Hirota’s method [8], tanh-coth function method [9, 10], mapping methods [11–13], Lie symmetry analysis [14], and so on. However, practically there is no unified method that can be used to handle all types of nonlinearity.

For the past decades, studies of wave propagation on the surface of water has become an important research area. The Boussinesq equation is completely integrable and is used to describe the propagation of long waves in shallow water and in other physical applications,
such as nonlinear lattice waves, acoustic waves, iron sound waves in a plasma, and vibrations in a nonlinear string. Therefore, the search for exact solutions of the Boussinesq equation is of great importance and interest. Shokri and Dehghan [15] obtained numerical solutions of the improved Boussinesq equation using collocation and approximating the solution by radial basis functions based on the third-order time discretization. Wazwaz [8] obtained the approximate solution of the Boussinesq-like \( B(m, n) \) equations by replacing some integer-order time derivatives by fractional derivatives. In this paper, we consider the following variety of Boussinesq-like equations as in [8]:

\[
\begin{align*}
    u_{tt} - u_{xx} - (6u^2u_x + u_{xxx})_x &= 0, \\
    u_{tt} - u_{xx} - (6u^2u_x + u_{xt})_x &= 0, \\
    u_{tt} - u_{xt} - (6u^2u_x + u_{xxt})_x &= 0, \\
    u_{tt} - (6u^2u_x + u_{xxx})_x &= 0.
\end{align*}
\]

One of the most effective and direct methods for constructing soliton solutions for nonlinear evolution equations is the extended Jacobi elliptic function method, which was proposed in [16] to obtain doubly periodic wave solutions. It is noted that soliton and triangular solutions can be established as the limits of the Jacobi doubly periodic wave solutions. More recently, Bhrawy et al. [17] obtained cnoidal and snoidal wave solutions to coupled nonlinear wave equations by using the extended Jacobi elliptic function method. Another important method developed in [18, 19] is rather heuristic and useful for computing physically relevant solitary wave and shock-wave solutions for a large class of nonlinear PDEs [20, 21]. Gomez [22] used the improved tanh-coth method to obtain wave solutions to a Korteweg-de Vries equation with higher-order nonlinearity, from which the solutions of the standard KdV and the modified Korteweg-de Vries equations with variable coefficients can be derived as particular cases. More recently, Ghany [23] obtained some white noise functional solutions for stochastic generalized Hirota-Satsuma coupled KdV equations by using the modified tanh-coth method together with Hermite transforms. The aim of this paper is to obtain more exact traveling wave solutions of the above Boussinesq-like equations by applying the extended Jacobi elliptic function and modified tanh-coth methods. These two methods possess powerful features that make them practical for the determination of exact wave solutions for a wide class of nonlinear evolution equations.

II. WAVE SOLUTIONS VIA JACOBI ELLIPTIC FUNCTION METHOD

First, we present a brief description about the extended Jacobi elliptic function method for a given general partial differential equation (PDE). Suppose that a nonlinear PDE is given by

\[
P(u, u_t, u_x, u_{xx}, u_{tt}, \ldots) = 0,
\]

where \( u = u(x, t) \) is an unknown function, \( P \) is a polynomial in \( u \) and its various partial derivatives. A transformation \( u = u(\eta), \eta = \alpha x - \omega t \) converts the PDE (1) to an ODE

\[
Q(u, -\omega u', \alpha u', \alpha^2 u'', \ldots) = 0,
\]

where \( \alpha \) and \( \omega \) are constants.
where $\alpha$ and $\omega$ are constants to be determined; $\omega$ denotes the wave speed. Using the extended Jacobi elliptic function method, $u(\eta)$ can be expressed as a finite series in the Jacobi elliptic function $\text{sn}(\eta)$, i.e., the ansatz

$$u(x, t) = u(\eta) = \sum_{i=0}^{n} a_i \text{sn}^i(\eta) + \sum_{j=1}^{n} b_j \text{sn}^{-j}(\eta),$$  

(3)

where $a_i (i = 0, 1, \ldots, n)$, $b_j (j = 1, 2, \ldots, n)$ are constants to be determined. Further,

$$\text{cn}^2(\eta) = 1 - \text{sn}^2(\eta), \quad \text{dn}^2(\eta) = 1 - m^2 \text{sn}^2(\eta),$$  

(4)

$$\frac{d}{d\eta} \text{sn}(\eta) = \text{cn}(\eta)\text{dn}(\eta), \quad \frac{d}{d\eta} \text{cn}(\eta) = -\text{sn}(\eta)\text{dn}(\eta), \quad \frac{d}{d\eta} \text{dn}(\eta) = -m^2 \text{sn}(\eta)\text{cn}(\eta),$$  

(5)

where $\text{cn}(\eta)$ and $\text{dn}(\eta)$ are Jacobi elliptic functions with modulus $m$ ($0 < m < 1$). We define a polynomial degree function as $D(u(\eta)) = n$, thus we have

$$D \left( \frac{d^p u(\eta)}{d\eta^p} \right)^q = q(n + s), \quad p, q = 0, 1, 2, \ldots.$$  

(6)

Also, the parameter $n$ is determined by balancing the highest order linear term with the nonlinear term in Eq. (2). If $n$ is a positive integer, an analytic solution in closed form may be obtained. Substitute (3)-(6) in (2) and equating to zero the coefficients of all powers of $\text{sn}^i(\eta)\text{cn}^j(\eta)\text{dn}^k(\eta)$, $(i = 0, 1, 2, \ldots, j, k = 0, 1)$ yields a set of algebraic equations with respect to $\omega, \alpha, a_i (i = 0, 1, 2, \ldots, n)$ and $b_j (j = 1, 2, \ldots, n)$. Moreover, by substituting each solution of this set of algebraic equations in (3) we can get Jacobi doubly periodic wave solutions for the given equation. Also, when $m \to 1$ and $m \to 0$, Jacobi elliptic function solutions degenerate to the hyperbolic and triangular function solutions, respectively.

In this section, the extended Jacobi elliptic function method will be implemented to handle the first Boussinesq-like equation. The first Boussinesq-like equation is given by [8]

$$u_{tt} - u_{xx} - (6u^2u_x + u_{xxx})_x = 0.$$  

(7)

To look for the traveling wave solutions of Eq. (7), we make the transformations $u(x, t) = u(\eta), \eta = \alpha x - \mu t$, where $\alpha \neq 0$ and $\mu \neq 0$ are constants. Now, Eq. (7) can be written as the following nonlinear ordinary differential equation of the form

$$(\mu^2 - \alpha^2)u'' - 12\alpha^2 u(u')^2 - 6\alpha^2 u^2 u'' - \alpha^4 u^{(4)} = 0,$$  

(8)

where the prime denotes derivative with respect to $\eta$. To determine the parameter $M$, we balance the linear terms of highest order in Eq. (8) with the highest order nonlinear terms. By balancing $u^{(4)}$ and $u(u')^2$, we get $M + 4 = M + 2(M + 1)$, this in turn gives $M = 1$. As a result, the extended Jacobi elliptic function method admits the use of the finite expansion

$$u(\eta) = a_0 + a_1 \text{sn}(\eta) + b_1 \text{sn}^{-1}(\eta).$$  

(9)

Substituting Eq. (9) in Eq. (8) along with Eqs. (4) and (5), and setting each coefficient of $\text{sn}(\eta)$, $\text{cn}(\eta)$, $\text{dn}(\eta)$, $i = 0, 1, \ldots, j$, $k = 0, 1$ to zero yields a system of algebraic equations.
for \(a_0, a_1, b_1, \alpha,\) and \(\mu\). Solving this system by using Maple, the following sets of solutions are obtained:

\[
\begin{align*}
&\{ a_0 = 0, \ a_1 = \pm m\alpha i, \ b_1 = 0, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - \alpha^2 - m^2 \alpha^2} \}, \\
&\{ a_0 = 0, \ a_1 = 0, \ b_1 = \pm \alpha i, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - \alpha^2 - m^2 \alpha^2} \}, \\
&\{ a_0 = 0, \ a_1 = \pm m\alpha i, \ b_1 = \mp \alpha i, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - \alpha^2 - m^2 \alpha^2 + 6m^2 \alpha^2} \}, \\
&\{ a_0 = 0, \ a_1 = \pm m\alpha i, \ b_1 = \pm \alpha i, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - \alpha^2 - m^2 \alpha^2 - 6m^2 \alpha^2} \}.
\end{align*}
\]

From Eqs. (9) and (10)–(13), we obtain the following the exact Jacobi elliptic function solutions:

\[
\begin{align*}
u_1(x, t) &= \pm m\alpha \sin \left\{ \alpha \left( x \pm \sqrt{1 - \alpha^2 - m^2 \alpha^2} t \right) \right\}, \\
u_2(x, t) &= \pm \alpha \sin^{-1} \left\{ \alpha \left( x \pm \sqrt{1 - \alpha^2 - m^2 \alpha^2} t \right) \right\}, \\
u_3(x, t) &= \pm \alpha i \left[ \text{sn}(\eta) - \text{sn}^{-1}(\eta) \right], \\
u_4(x, t) &= \pm \alpha i \left[ \text{sn}(\eta) + \text{sn}^{-1}(\eta) \right],
\end{align*}
\]

where \(\eta = \alpha \left( x \pm \sqrt{1 - \alpha^2 - m^2 \alpha^2} + 6m^2 \alpha^2 \right)\).

Further, as \(m \to 1\), \(\text{sn}(\eta) = \tanh(\eta)\), we obtain the solitary wave solutions in the following form:

\[
\begin{align*}
\tilde{u}_1(x, t) &= \pm \alpha \alpha \tanh \left\{ \alpha \left( x \pm \sqrt{1 - 2\alpha^2} t \right) \right\}, \\
\tilde{u}_2(x, t) &= \pm \alpha \alpha \coth \left\{ \alpha \left( x \pm \sqrt{1 - 2\alpha^2} t \right) \right\}, \\
\tilde{u}_3(x, t) &= \pm \alpha i \left[ \tanh \left\{ \alpha \left( x \pm \sqrt{1 + 4\alpha^2} t \right) \right\} - \coth \left\{ \alpha \left( x \pm \sqrt{1 + 4\alpha^2} t \right) \right\} \right], \\
\tilde{u}_4(x, t) &= \pm \alpha i \left[ \tanh \left\{ \alpha \left( x \pm \sqrt{1 - 8\alpha^2} t \right) \right\} + \coth \left\{ \alpha \left( x \pm \sqrt{1 - 8\alpha^2} t \right) \right\} \right].
\end{align*}
\]
Next, we consider the second Boussinesq-like equation in the form \[8\]

\[ u_{tt} - u_{xx} - (6u^2 u_x + u_{xxt})_x = 0. \] (22)

To look for the traveling wave solutions of Eq. (22), we use the transformation \( u(x,t) = u(\eta), \eta = \alpha x - \mu t \). Then, Eq. (22) becomes

\[(\mu^2 - \alpha^2)u'' - 12\alpha^2 u(u')^2 - 6\alpha^2 u^2 u'' - \alpha^2 \mu^2 u^{(4)} = 0,\] (23)

where the prime denote differentiations. By considering the homogeneous balance between \( u^{(4)} \) and \( u(u')^2 \), we find \( M = 1 \). Then, the solution of Eq. (23) can be written in the form

\[ u(\eta) = a_0 + a_1 \text{sn}(\eta) + b_1 \text{sn}^{-1}(\eta), \] (24)

where \( a_0, a_1 \) and \( b_1 \) are to be determined later. Substituting Eq. (24) in Eq. (23) and making use of Eqs. (4) and (5), and setting each coefficients of \( \text{sn}^i(\eta), \text{cn}^j(\eta), \text{dn}^k(\eta), i = 0, 1, \ldots, j; k = 0, 1 \) to zero, yields a system of algebraic equations for \( a_0, a_1, b_1, \alpha, \) and \( \mu \). Solving the resulting system with the aid of Maple, we find four kinds of solutions, namely

\[
\begin{align*}
\{ a_0 &= 0, \ a_1 = \pm m \mu i, \ b_1 = 0, \ \alpha = \pm \mu \sqrt{\frac{1}{1 - m^2 \mu^2 - \mu^2}}, \ \mu = \mu \}, \\
\{ a_0 &= 0, \ a_1 = 0, \ b_1 = \pm \mu i, \ \alpha = \pm \mu \sqrt{\frac{1}{1 - m^2 \mu^2 - \mu^2}}, \ \mu = \mu \}, \\
\{ a_0 &= 0, \ a_1 = \pm m \mu i, \ b_1 = \pm \mu i, \ \alpha = \pm \mu \sqrt{\frac{1}{1 - m^2 \mu^2 - 6 m \mu^2 - \mu^2}}, \ \mu = \mu \}, \\
\{ a_0 &= 0, \ a_1 = \pm m \mu i, \ b_1 = \mp \mu i, \ \alpha = \pm \mu \sqrt{\frac{1}{1 - m^2 \mu^2 + 6 m \mu^2 - \mu^2}}, \ \mu = \mu \}.
\end{align*}
\]

(25) \hspace{1cm} (26) \hspace{1cm} (27) \hspace{1cm} (28)

Then, the corresponding traveling solution to the equation Eq. (22) is given by

\[
\begin{align*}
u_1(x,t) &= \pm m \mu \text{sn} \left\{ \mu \left( \pm \sqrt{\frac{1}{1 - m^2 \mu^2 - \mu^2}} x - t \right) \right\}, \\
u_2(x,t) &= \pm \mu \text{sn}^{-1} \left\{ \mu \left( \pm \sqrt{\frac{1}{1 - m^2 \mu^2 - \mu^2}} x - t \right) \right\}, \\
u_3(x,t) &= \pm \mu i \left[ \text{sn}(\eta) + \text{sn}^{-1}(\eta) \right], \\
u_4(x,t) &= \pm \mu i \left[ \text{sn}(\eta) - \text{sn}^{-1}(\eta) \right],
\end{align*}
\]

where \( \eta = \mu \left( \pm \sqrt{\frac{1}{1 - m^2 \mu^2 - 6 m \mu^2 - \mu^2}} x - t \right). \) (29) \hspace{1cm} (30) \hspace{1cm} (31) \hspace{1cm} (32)
where \( \eta = \mu \left( \pm \sqrt{\frac{1}{1-m^2\mu^2+6m\mu^2-\mu^2}} x - t \right) \). Further, as \( m \to 1 \), \( \text{sn}(\eta) = \tanh(\eta) \), we obtain the solitary wave solutions in the following form:

\[
\tilde{u}_1(x,t) = \pm \mu i \tanh \left\{ \mu \left( \pm \sqrt{\frac{1}{1-2\mu^2}} x - t \right) \right\},
\]

(33)

\[
\tilde{u}_2(x,t) = \pm \mu i \coth \left\{ \mu \left( \pm \sqrt{\frac{1}{1-2\mu^2}} x - t \right) \right\},
\]

(34)

\[
\tilde{u}_3(x,t) = \pm \mu i \left[ \tanh \left\{ \mu \left( \pm \sqrt{\frac{1}{1-8\mu^2}} x - t \right) \right\} + \coth \left\{ \mu \left( \pm \sqrt{\frac{1}{1-8\mu^2}} x - t \right) \right\} \right],
\]

(35)

\[
\tilde{u}_4(x,t) = \pm \mu i \left[ \tanh \left\{ \mu \left( \pm \sqrt{\frac{1}{1+4\mu^2}} x - t \right) \right\} - \coth \left\{ \mu \left( \pm \sqrt{\frac{1}{1+4\mu^2}} x - t \right) \right\} \right].
\]

(36)

As \( m \to 0 \), these solutions degenerate to the soliton solutions

\[
\hat{u}(x,t) = \pm \mu i \csc \left\{ \mu \left( \pm \sqrt{\frac{1}{1-\mu^2}} x - t \right) \right\}.
\]

(37)

Now, we consider the third Boussinesq equation [8]:

\[
u_{tt} - u_{xt} - (6u^2 u_x + u_{xx} t) = 0.
\]

(38)

To look for the traveling wave solutions of Eq. (38), we make the transformations \( u(x,t) = u(\eta) \), where \( \eta = \alpha x - \mu t \). Now Eq. (38) can be written as

\[
\mu(\mu + \alpha)u'' - 12\alpha^2 u(u')^2 - 6\alpha^2 u^2 u'' + \alpha^3 u^{(4)} = 0.
\]

(39)

With balancing \( u^{(4)} \) and \( u(u')^2 \), we get \( M + 4 = M + 2(M + 1) \), hence \( M = 1 \).

Then, the solution of Eq. (39) can be written in the following form:

\[
\begin{align*}
u(\eta) = a_0 + a_1 \text{sn}(\eta) + b_1 \text{sn}^{-1}(\eta),
\end{align*}
\]

(40)

Substituting Eq. (40) in Eq. (39) and setting each of the coefficients of \( \text{sn}^{(i)}(\eta) \), \( \text{cn}^{(i)}(\eta) \), \( \text{dn}^{(i)}(\eta) \), \( i = 0,1,\ldots,j \), \( k = 0,1 \) to zero gives an over-determined system of nonlinear algebraic equations with respect to \( a_0 \), \( a_1 \), \( b_1 \), \( \alpha \), and \( \mu \). Solving the resulting system of nonlinear algebraic equations with the aid of Maple, we obtain the following solutions:

\[
\begin{align*}
&\begin{align*}
&\left\{ a_0 = 0, \ a_1 = \pm m\alpha \sqrt{\alpha^2 + m^2\alpha^2 - 1}, \ b_1 = 0, \ \alpha = \alpha, \ \mu = \alpha(\alpha^2 + m^2\alpha^2 - 1) \right\},
\end{align*}
&\left\{ a_0 = 0, \ a_1 = 0, \ b_1 = \pm \alpha \sqrt{\alpha^2 + m^2\alpha^2 - 1}, \ \alpha = \alpha, \ \mu = \alpha(\alpha^2 + m^2\alpha^2 - 1) \right\},
\end{align*}
\]

(41)
\[
\begin{align*}
\{ a_0 &= 0, a_1 = \pm m \sqrt{\alpha^2 + m^2 \alpha^2 + 6m^2} - 1, \\
\{ b_1 &= \mp \alpha \sqrt{\alpha^2 + m^2 \alpha^2 + 6m^2} - 1, \\
\alpha &= \alpha, \quad \mu = \alpha (\alpha^2 + m^2 \alpha^2 + 6m^2 - 1) \}, \\
\alpha &= \alpha, \quad \mu = \alpha (\alpha^2 + m^2 \alpha^2 - 6m^2 - 1) \}.
\end{align*}
\]

(43)

Further, with the above set of values, we find the following wave solutions to Eq. (38):

\[
u_1(x, t) = \pm m \alpha \sqrt{\alpha^2 + m^2 \alpha^2 - 1} \text{sn} \left\{ \alpha \{ x - (\alpha^2 + m^2 \alpha^2 - 1)t \} \right\},
\]

(45)

\[
u_2(x, t) = \pm \alpha \sqrt{\alpha^2 + m^2 \alpha^2 - 1} \text{sn}^{-1} \left\{ \alpha \{ x - (\alpha^2 + m^2 \alpha^2 - 1)t \} \right\},
\]

(46)

\[
u_3(x, t) = \pm \alpha \sqrt{\alpha^2 + m^2 \alpha^2 + 6m^2} - 1 \left[ \text{msn}(\eta) + \text{sn}^{-1}(\eta) \right],
\]

(47)

where \( \eta = \alpha \{ x - (\alpha^2 + m^2 \alpha^2 + 6m^2 - 1)t \} \).

\[
u_4(x, t) = \pm \alpha \sqrt{\alpha^2 + m^2 \alpha^2 - 6m^2 - 1} \left[ \text{msn}(\eta) - \text{sn}^{-1}(\eta) \right],
\]

(48)

where \( \eta = \alpha \{ x - (\alpha^2 + m^2 \alpha^2 - 6m^2 - 1)t \} \). Also, some solitary wave solutions can be obtained if the modulus \( m \to 1 \) in Eqs. (45)–(48):

\[
\tilde{\nu}_1(x, t) = \pm \alpha \sqrt{2\alpha^2 - 1} \text{tanh} \left\{ \alpha \{ x - (2\alpha^2 - 1)t \} \right\},
\]

(49)

\[
\tilde{\nu}_2(x, t) = \pm \alpha \sqrt{2\alpha^2 - 1} \text{coth} \left\{ \alpha \{ x - (2\alpha^2 - 1)t \} \right\},
\]

(50)

\[
\tilde{\nu}_3(x, t) = \pm \alpha \sqrt{8\alpha^2 - 1} \left[ \text{tanh} \left( \alpha \{ x - (8\alpha^2 - 1)t \} \right) + \text{coth} \left( \alpha \{ x - (8\alpha^2 - 1)t \} \right) \right],
\]

(51)

\[
\tilde{\nu}_4(x, t) = \pm \alpha \sqrt{-4\alpha^2 - 1} \left[ \text{tanh} \left( \alpha \{ x + (4\alpha^2 + 1)t \} \right) - \text{coth} \left( \alpha \{ x + (4\alpha^2 + 1)t \} \right) \right].
\]

(52)

Note that if the modulus \( m \) approaches to zero, a trigonometric function solution can be obtained for Eq. (38):

\[
\tilde{\nu}(x, t) = \pm \alpha \sqrt{\alpha^2 - 1} \text{csc} \left\{ \alpha \{ x - (\alpha^2 - 1)t \} \right\}.
\]

(53)

Finally, we consider the fourth Boussinesq-like equation as in [8]:

\[
u_{tt} - (6u^2 u_x + u_{xxx})_x = 0,
\]

(54)

where the dissipation term \( u_{xx} \) does not exist. This equation also arises in other physical applications, for example, iron sound waves in plasma, nonlinear lattice waves, and in vibrations in a nonlinear string. To obtain the traveling wave solutions of Eq. (54), we consider
the transformation $u(x, t) = u(\eta), \eta = \alpha x - \mu t$. With the help of above transformation, Eq. (54) can be written as

$$\mu^2 u'' - 12\alpha^2 u(u')^2 - 6\alpha^2 u^2 u'' - \alpha^4 u^{(4)} = 0. \tag{55}$$

By taking the balance between $u^{(4)}$ and $u(u')^2$, we obtain $M = 1$. As a result, the Jacobi elliptic function method admits a solution of Eq. (55) in the form

$$u(\eta) = a_0 + a_1 sn(\eta) + b_1 sn^{-1}(\eta).$$

By the same calculation as above, we obtain a system of algebraic equations for $a_0, a_1, b_1, \alpha,$ and $\mu$. Solving the system of resulting algebraic equations with the help of Maple software, we obtain the following four solutions:

$$\begin{align*}
\{a_0 &= 0, \ a_1 = \pm m \alpha i, \ b_1 = 0, \alpha = \alpha, \ \mu = \pm \alpha^2 i \sqrt{m^2 + 1}\}, \\
\{a_0 &= 0, \ a_1 = 0, \ b_1 = \pm \alpha i, \ \alpha = \alpha, \ \mu = \pm \alpha^2 i \sqrt{m^2 + 1}\},
\end{align*} \tag{56}$$

$$\begin{align*}
\{a_0 &= 0, \ a_1 = \pm m \alpha i, \ b_1 = \pm \alpha i, \ \alpha = \alpha, \ \mu = \pm \alpha^2 i \sqrt{m^2 + 6m + 1}\}, \\
\{a_0 &= 0, \ a_1 = \pm m \alpha i, \ b_1 = \mp \alpha i, \ \alpha = \alpha, \ \mu = \pm \alpha^2 i \sqrt{m^2 - 6m + 1}\}. \tag{57}
\end{align*}$$

From the above set of values, we obtain the following traveling wave solutions:

$$u_1(x, t) = \pm m \alpha i sn\left\{\alpha \left( x \pm \alpha i \sqrt{m^2 + 1} t \right) \right\}, \tag{60}$$

$$u_2(x, t) = \pm m \alpha i sn^{-1}\left\{\alpha \left( x \pm \alpha i \sqrt{m^2 + 1} t \right) \right\}, \tag{61}$$

$$u_3(x, t) = \pm \alpha i \left[ msn\left\{\alpha \left( x \pm \alpha i \sqrt{m^2 + 6m + 1} t \right) \right\} \right] + sn^{-1}\left\{\alpha \left( x \pm \alpha i \sqrt{m^2 + 6m + 1} t \right) \right\}, \tag{62}$$

$$u_4(x, t) = \pm \alpha i \left[ msn\left\{\alpha \left( x \pm \alpha i \sqrt{m^2 - 6m + 1} t \right) \right\} \right] - sn^{-1}\left\{\alpha \left( x \pm \alpha i \sqrt{m^2 - 6m + 1} t \right) \right\}. \tag{63}$$

Also, if the modulus $m$ approaches 1, from the Eqs. (60)–(63) we obtain the following solitary wave solutions:

$$\tilde{u}_1(x, t) = \pm \alpha i \tanh\left\{\alpha \left( x \pm \sqrt{2} \alpha i t \right) \right\}, \tag{64}$$

$$\tilde{u}_2(x, t) = \pm \alpha i \coth\left\{\alpha \left( x \pm \sqrt{2} \alpha i t \right) \right\}. \tag{65}$$
FIG. 1: Real part of the solution (62) when $\alpha = 1$ and $m = 0.3$.

FIG. 2: Imaginary part of the solution (62) when $\alpha = 1$ and $m = 0.3$.

$$\tilde{u}_3(x, t) = \pm\alpha i \left[ \tanh \left\{ \alpha \left( x \pm 2\sqrt{2}it \right) \right\} + \coth \left\{ \alpha \left( x \pm 2\sqrt{2}it \right) \right\} \right] ,$$  \hspace{1cm} (66)

$$\tilde{u}_4(x, t) = \pm\alpha i \left[ \tanh \left\{ \alpha (x \pm 2\alpha t) \right\} + \coth \left\{ \alpha (x \pm 2\alpha t) \right\} \right] .$$  \hspace{1cm} (67)

Further, if the modulus $m$ approaches to zero then a trigonometric function solution can be obtained:

$$\tilde{u}(x, t) = \pm\alpha i \csc \left\{ \alpha (x \pm \alpha it) \right\} .$$  \hspace{1cm} (68)

The solution $u_2(x, t)$ of Eq. (62) such as the real part, imaginary part, and modulus are provided in Figs. 1, 2, and 3, respectively, with values of the parameters given in their captions.
III. WAVE SOLUTIONS VIA THE MODIFIED TANH-COTH METHOD

The hyperbolic tangent (tanh) method is a powerful technique developed by Malfliet [24] for computing traveling waves solutions of nonlinear evolution equations. Further, Fan [25] proposed an extended tanh method and obtained new traveling wave solutions for nonlinear equations that cannot be obtained by the tanh method. Subsequently, Wazwan [18] modified the extended tanh method (modified tanh-coth method) and obtained new solutions for some nonlinear PDEs. Now, we briefly discuss the modified tanh-coth function method in its systematized form [19, 23] and apply it to the variety of Boussineq equations. The nonlinear evolution equations we want to investigate are commonly written in the form of a partial differential equation (PDE) for a function $u(x, t)$. The first step is to combine the independent variables, $x$ and $t$, into a new variable, $\eta = \alpha x - \mu t$, which defines the traveling frame of reference; here $\alpha$ and $\mu$ represent the wave number and velocity of the traveling wave and both are undetermined parameters. Substitution into the PDE yields an ordinary differential equation (ODE) for $u(\eta)$. Hence, in what follows we deal with ODEs rather than with PDEs. Our aim is to find exact solutions for those ODEs in tanh-coth form. The ordinary differential equation is then integrated as long as all terms contain derivatives, where the integration constants are considered as zero. Then, the resulting ODE is solved by the tanh-coth method which admits the use of a finite series of functions of the form

$$u(\eta) = a_0 + \sum_{n=1}^{M} a_n Y^n(\eta) + \sum_{n=1}^{M} b_n Y^{-n}(\eta),$$  \hspace{1cm} (69)

where $Y$ satisfies the Riccati equation,

$$Y' = A + CY^2,$$  \hspace{1cm} (70)

with $A$, $B$, and $C$ being constants to be determined later. Further, $M$ and $N$ are positive integers that will be obtained by balancing the linear terms of highest order in the
resulting equation with the highest order nonlinear terms. Substituting (69) in the ODE and using (70) results in an algebraic system of equations in powers of $Y$ that will lead to the determination of the parameters $a_n$, $b_n$ ($n = 0, \ldots, M$), $\alpha$, and $\mu$. Having determined these parameters, we obtain an analytic solution $u(x, t)$ in a closed form.

**Remark 1** More precisely, we will consider the following special solutions of the Riccati equation (70)

(i) $A = \frac{1}{2}$, $C = -\frac{1}{2}$, Eq. (70) has solutions $Y = \tanh \eta \pm i \sech \eta$ and $Y = \coth \eta \pm \csch \eta$.

(ii) $A = 1$, $C = -4$, Eq. (70) has solutions $Y = \frac{1}{2} \tanh 2\eta$ and $Y = \frac{1}{4} (\tanh \eta + \coth \eta)$.

(iii) $A = 1$, $C = 4$, Eq. (70) has solutions $Y = \frac{1}{2} \tanh 2\eta$ and $Y = \frac{1}{4} (\tanh \eta - \cot \eta)$.

In order to obtain more traveling wave solutions for Eq. (7), we consider the same transformation $u(x, t) = u(\eta)$, $\eta = \alpha x - \mu t$, as in the previous section. To determine the parameter $M$, we balance the linear terms of highest order in Eq. (8) with the highest order nonlinear terms. This in turn gives $M = 1$. As a result, the modified tanh-coth method (69) admits the use of the finite expansion

$$u(\eta) = a_0 + a_1 Y + \frac{b_1}{Y}.$$  

(71)

Substituting Eq. (71) in the reduced ODE (8) and using Eq. (70), collecting the coefficients of $Y$, yields a system of algebraic equations for $a_0, a_1, b_1, A, C, \alpha,$ and $\mu$.

**Case (I):** If we set $A = \frac{1}{2}, C = -\frac{1}{2}$ in Equation (70), and solve the system of algebraic equations using Maple, we obtain the following four sets of solutions:

$$\begin{align*}
\{ a_0 &= 0, \ a_1 = \pm \frac{\alpha}{2} i, \ b_1 = 0, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - \frac{\alpha^2}{2}} \}, \\
\{ a_0 &= 0, \ a_1 = 0, \ b_1 = \pm \frac{\alpha}{2} i, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - \frac{\alpha^2}{2}} \}, \\
\{ a_0 &= 0, \ a_1 = \pm \frac{\alpha}{2} i, \ b_1 = \pm \frac{\alpha}{2} i, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 - 2\alpha^2} \}, \\
\{ a_0 &= 0, \ a_1 = \pm \frac{\alpha}{2} i, \ b_1 = \mp \frac{\alpha}{2} i, \ \alpha = \alpha, \ \mu = \pm \alpha \sqrt{1 + \alpha^2} \}.
\end{align*}$$

(72) (73) (74) (75)

Substituting $Y = \tanh \eta \pm i \sech \eta$ and $Y = \coth \eta \pm \csch \eta$ in Equation (71), the first and second set gives the soliton solutions

$$u_{1,1}(x, t) = \pm \frac{\alpha}{2} i \left[ \tanh \eta \pm i \sech \eta + \frac{1}{\tanh \eta \pm i \sech \eta} \right].$$

(76)
and

\[ u_{2,1}(x,t) = \pm \frac{\alpha}{2} i \left[ \coth \eta \pm \text{csch} \eta + \frac{1}{\coth \eta \pm \text{csch} \eta} \right], \tag{77} \]

where \( \eta = \alpha \left( x \pm \sqrt{1-2\alpha^2 t} \right) \). Further, the third and fourth set gives the soliton solutions

\[ u_{3,1}(x,t) = \pm \frac{\alpha}{2} i \left[ \tanh \eta \pm \text{sech} \eta - \frac{1}{\tanh \eta \pm \text{sech} \eta} \right] \tag{78} \]

and

\[ u_{4,1}(x,t) = \pm \frac{\alpha}{2} i \left[ \coth \eta \pm \text{csch} \eta - \frac{1}{\coth \eta \pm \text{csch} \eta} \right], \tag{79} \]

where \( \eta = \alpha \left( x \pm \sqrt{1+\alpha^2 t} \right) \).

**Case (II):** If we set \( A = 1, C = -4 \) in Equation (70) and by the same calculation as above, the following sets of solutions are obtained:

\[ u_{1,2}(x,t) = \pm 2\alpha i \left[ \tanh \left\{ 2\alpha \left( x \pm \sqrt{1-32\alpha^2 t} \right) \right\} + \coth \left\{ 2\alpha \left( x \pm \sqrt{1-32\alpha^2 t} \right) \right\} \right], \tag{80} \]

\[ u_{2,2}(x,t) = \pm \alpha \left[ \frac{1}{\tanh \eta + \coth \eta} \right], \tag{81} \]

where \( \eta = \alpha \left( x \pm \sqrt{1-32\alpha^2 t} \right) \).

\[ u_{3,2}(x,t) = \pm 2\alpha i \left[ \tanh \left\{ 2\alpha \left( x \pm \sqrt{1+32\alpha^2 t} \right) \right\} - \coth \left\{ 2\alpha \left( x \pm \sqrt{1+32\alpha^2 t} \right) \right\} \right], \tag{82} \]

\[ u_{4,2}(x,t) = \pm \alpha \left[ \frac{1}{\tanh \eta + \coth \eta} \right], \tag{83} \]

where \( \eta = \alpha \left( x \pm \sqrt{1+32\alpha^2 t} \right) \).

**Case (III):** If we set \( A = 1, C = 4 \) in Eq. (70), and solve the resulting system of algebraic equations using Maple, we get the following sets of solutions:

\[ u_{1,3}(x,t) = \pm 2\mu i \left[ \tan \left\{ 2\alpha \left( x \pm \sqrt{1-16\alpha^2 t} \right) \right\} \right], \tag{84} \]

\[ u_{2,3}(x,t) = \pm \alpha i \left[ \frac{1}{\tan \eta - \cot \eta} \right], \tag{85} \]

where \( \eta = \alpha \left( x \pm \sqrt{1-16\alpha^2 t} \right) \).

\[ u_{3,3}(x,t) = \pm 2\alpha i \left[ \tan \left\{ 2\alpha \left( x \pm \sqrt{1+32\alpha^2 t} \right) \right\} - \cot \left\{ 2\alpha \left( x \pm \sqrt{1+32\alpha^2 t} \right) \right\} \right], \tag{86} \]
\[ u_{4,3}(x,t) = \pm \alpha i \left[ \tan \eta - \cot \eta - \frac{4}{\tan \eta + \cot \eta} \right], \]  
\[ \eta = \alpha \left( x \pm \sqrt{1 + 32\alpha^2 t} \right). \]  

Further, the real and imaginary parts and the modulus of the solution \( u_{1,1}(x,t) \) of Eq. (76) are plotted in Figs. 4, 5, and 6 with the values of parameters given in their captions.

Now, we find the traveling wave solutions of (22). To determine the parameter \( M \), we balance the linear terms of highest order in Eq. (23) with the highest order nonlinear terms. This in turn gives \( M = 1 \). Moreover, the modified tanh-coth method (69) admits the use of the finite expansion

\[ u(\eta) = a_0 + a_1 Y + \frac{b_1}{Y}, \]  

Substituting Eq. (88) in the reduced ODE (23) and using Eq. (70) then collecting the coefficients of \( Y \) yields a system of algebraic equations for \( a_0, a_1, b_1, A, C, \alpha, \) and \( \mu \).
FIG. 6: Modulus of the solution (76) when $\alpha = 0.5$.

**Case (I):** If we set $A = \frac{1}{2}$ and $C = -\frac{1}{2}$ in Equation (70), and solve the system of algebraic equations using Maple, we obtain the following four sets of solutions:

\[
\begin{align*}
    u_{1,1}(x,t) &= \pm \frac{\mu}{2} i \left[ \tanh \eta \pm \text{sech} \eta \pm \frac{1}{\tanh \eta \pm \text{sech} \eta} \right], \quad (89) \\
    u_{1,2}(x,t) &= \pm \frac{\mu}{2} i \left[ \coth \eta \pm \text{csch} \eta \pm \frac{1}{\coth \eta \pm \text{csch} \eta} \right], \quad (90) \\
    u_{1,3}(x,t) &= \pm \frac{\mu}{2} i \left[ \tanh \eta \pm \text{sech} \eta - \frac{1}{\tanh \eta \pm \text{sech} \eta} \right], \quad (91) \\
    u_{1,4}(x,t) &= \pm \frac{\mu}{2} i \left[ \coth \eta \pm \text{csch} \eta - \frac{1}{\coth \eta \pm \text{csch} \eta} \right], \quad (92)
\end{align*}
\]

where $\eta = \mu \left( \pm \sqrt{\frac{1}{1-2\mu^2}} x - t \right)$.

**Case (II):** If we set $A = 1$ and $C = -4$ in Equation (70), and by the same calculation as above the following sets of solutions are obtained:

\[
\begin{align*}
    u_{2,1}(x,t) &= \pm 2\mu i \left[ \tanh \left( 2\mu \left( \pm \sqrt{\frac{1}{1-32\mu^2}} x - t \right) \right) \right] + \coth \left( 2\mu \left( \pm \sqrt{\frac{1}{1-32\mu^2}} x - t \right) \right), \quad (93) \\
    u_{2,2}(x,t) &= \pm \mu i \left[ \tanh \eta + \coth \eta + \frac{4}{\tanh \eta + \coth \eta} \right], \quad (94)
\end{align*}
\]
where \( \eta = \mu \left( \pm \sqrt{\frac{1}{1 - 32\mu^2} x - t} \right) \).

\[
u_{2,3}(x, t) = \pm 2\mu \left[ \tanh \left( 2\mu \left( \pm \sqrt{\frac{1}{1 + 16\mu^2} x - t} \right) \right) - \coth \left( 2\mu \left( \pm \sqrt{\frac{1}{1 + 16\mu^2} x - t} \right) \right) \right], (95)
\]

\[
u_{2,4}(x, t) = \pm \mu \left[ \tanh \eta + \coth \eta - \frac{4}{\tanh \eta + \coth \eta} \right],
\] (96)

where \( \eta = \mu \left( \pm \sqrt{\frac{1}{1 + 16\mu^2} x - t} \right) \).

**Case (III):** If we set \( A = 1 \) and \( C = 4 \) in Eq. (70), and solve the system of algebraic equations using Maple, we get the following sets of solutions:

\[
u_{3,1}(x, t) = \pm 2\mu \left[ \tan \left( 2\mu \left( \pm \sqrt{\frac{1}{1 - 32\mu^2} x - t} \right) \right) + \cot \left( 2\mu \left( \pm \sqrt{\frac{1}{1 - 32\mu^2} x - t} \right) \right) \right], (97)
\]

\[
u_{3,2}(x, t) = \pm \mu \left[ \tan \eta - \cot \eta + \frac{4}{\tan \eta - \cot \eta} \right],
\] (98)

where \( \eta = \mu \left( \pm \sqrt{\frac{1}{1 - 32\mu^2} x - t} \right) \).

\[
u_{3,3}(x, t) = \pm 2\mu \left[ \tan \left( 2\mu \left( \pm \sqrt{\frac{1}{1 + 32\mu^2} x - t} \right) \right) - \cot \left( 2\mu \left( \pm \sqrt{\frac{1}{1 + 32\mu^2} x - t} \right) \right) \right], (99)
\]

\[
u_{3,4}(x, t) = \pm \mu \left[ \tan \eta - \cot \eta - \frac{4}{\tan \eta - \cot \eta} \right],
\] (100)

where \( \eta = \mu \left( \pm \sqrt{\frac{1}{1 + 32\mu^2} x - t} \right) \).

Next, we look for the traveling wave solutions to the third and fourth Boussinesq-like Eqs. (38) and (54). To determine the parameter \( M \), we balance the linear terms of highest order in Eq. (39) and (55) with the highest order nonlinear terms. This in turn gives \( M = 1 \).

As a result, the modified tanh-coth method (69) admits the use of the finite expansion

\[
u(\eta) = a_0 + a_1 Y + \frac{b_1}{Y}, \tag{101}\]

Substituting Eq. (101) in the reduced ODE (39) and (55) and taking into account relations Eq. (70), we can obtain a system of algebraic equations for \( a_0, a_1, b_1, A, C, \alpha \) and \( \mu \). Solving the resulting system of equations by using Maple, we find that a solution of (38) and (54) exists only in the following cases:

**Case (I):** If we set \( A = \frac{1}{2} \) and \( C = -\frac{1}{2} \) in Equation (70), and solve the system of algebraic equations using Maple, we obtain the following sets of solutions for the Eqs. (38):

\[
u_{1,1}(x, t) = \pm \frac{\alpha}{2} \sqrt{2\alpha^2 - 1} \left[ \tanh \eta \pm \text{sech} \eta + \frac{1}{\tanh \eta \pm \text{sech} \eta} \right], \tag{102}\]
\begin{align}
    u_{1,2}(x, t) &= \pm \frac{\alpha}{2} \sqrt{2\alpha^2 - 1} \left[ \coth \eta \pm \csch \eta + \frac{1}{\coth \eta \pm \csch \eta} \right], \\
    u_{1,3}(x, t) &= \pm \frac{\alpha}{2} \sqrt{-\alpha^2 - 1} \left[ \tanh \eta \pm \sech \eta - \frac{1}{\tanh \eta \pm \sech \eta} \right], \\
    u_{1,4}(x, t) &= \pm \frac{\alpha}{2} \sqrt{-\alpha^2 - 1} \left[ \coth \eta \pm \csch \eta - \frac{1}{\coth \eta \pm \csch \eta} \right],
\end{align}

where \( \eta = \alpha \{ x - (2\alpha^2 - 1)t \} \).

Further, we obtain the following set of wave solutions for Eq. (54):

\begin{align}
    u_{1,5}(x, t) &= \pm \frac{\alpha}{2} i \left[ \tanh \eta \pm \sech \eta + \frac{1}{\tanh \eta \pm \sech \eta} \right], \\
    u_{1,6}(x, t) &= \pm \frac{\alpha}{2} i \left[ \coth \eta \pm \csch \eta + \frac{1}{\coth \eta \pm \csch \eta} \right],
\end{align}

where \( \eta = \alpha \{ x \pm \sqrt{2\alpha t} \} \).

\begin{align}
    u_{1,7}(x, t) &= \pm \frac{\alpha}{2} i \left[ \tanh \eta \pm \sech \eta - \frac{1}{\tanh \eta \pm \sech \eta} \right], \\
    u_{1,8}(x, t) &= \pm \frac{\alpha}{2} i \left[ \coth \eta \pm \csch \eta - \frac{1}{\coth \eta \pm \csch \eta} \right],
\end{align}

where \( \eta = \alpha \{ x \pm \alpha t \} \).

**Case (II):** If we set \( A = 1 \) and \( C = -4 \) in Equation (70), by the same calculation as above, the following sets of solutions are obtained for Eq. (38):

\begin{align}
    u_{2,1}(x, t) &= \pm 2\alpha \sqrt{32\alpha^2 - 1} \left[ \tanh \{2\alpha\{ x - (32\alpha^2 - 1)t \} \} \right] + \coth \{2\alpha\{ x - (32\alpha^2 - 1)t \} \}], \\
    u_{2,2}(x, t) &= \pm \alpha \sqrt{32\alpha^2 - 1} \left[ \tanh \eta + \coth \eta + \frac{4}{\tanh \eta + \coth \eta} \right],
\end{align}

where \( \eta = \alpha \{ x - (32\alpha^2 - 1)t \} \).

\begin{align}
    u_{2,3}(x, t) &= \pm 2\alpha \sqrt{-16\alpha^2 - 1} \left[ \tanh \{2\alpha\{ x + (16\alpha^2 + 1)t \} \} \right] - \coth \{2\alpha\{ x + (16\alpha^2 + 1)t \} \}], \\
    u_{2,4}(x, t) &= \pm 2\alpha \sqrt{-16\alpha^2 - 1} \left[ \tanh \eta + \coth \eta - \frac{4}{\tanh \eta + \coth \eta} \right].
\end{align}
where \( \eta = \alpha \left\{ x + (16\alpha^2 + 1)t \right\} \).

Also, with the same calculation as above, we get the following wave solutions for Eq. (54):

\[
u_{2,5}(x, t) = \pm 2\alpha i \left[ \tanh \left\{ 2\alpha \left( x \pm 4\sqrt{2}\alpha i t \right) \right\} + \coth \left\{ 2\alpha \left( x \pm 4\sqrt{2}\alpha i t \right) \right\} \right],
\]

\[
u_{2,6}(x, t) = \pm \alpha i \left[ \tanh \eta + \coth \eta + \frac{4}{\tanh \eta + \coth \eta} \right],
\]

where \( \eta = \alpha \left( x \pm 4\sqrt{2}\alpha i t \right) \).

\[
u_{2,7}(x, t) = \pm 2\alpha i \left[ \tanh \left\{ 2\alpha \left( x \pm 4\alpha t \right) \right\} \right],
\]

\[
u_{2,8}(x, t) = \pm \alpha i \left[ \tanh \eta + \coth \eta \right].
\]

\[\text{(114)}\]

\[\text{(115)}\]

\[\text{(116)}\]

\[\text{(117)}\]

\[\text{Case (III): If we set } A = 1 \text{ and } C = 4 \text{ in Eq. (70), and solve the system of algebraic equations using Maple, we obtain the following set of wave solutions for (38):}\]

\[
u_{3,1}(x, t) = \pm 2\alpha \sqrt{16\alpha^2 - 1} \left[ \tan \left\{ 2\alpha \left( x - (16\alpha^2 - 1)t \right) \right\} \right],
\]

\[
u_{3,2}(x, t) = \pm \alpha \sqrt{16\alpha^2 - 1} \left[ \tan \eta - \cot \eta + \frac{4}{\tan \eta - \cot \eta} \right],
\]

where \( \eta = \alpha \left( x - (16\alpha^2 - 1)t \right) \).

\[
u_{3,3}(x, t) = \pm 2\alpha \sqrt{-32\alpha^2 - 1} \left[ \tan \left\{ 2\alpha \left( x + (32\alpha^2 + 1)t \right) \right\} \right],
\]

\[
u_{3,4}(x, t) = \pm 2\alpha \sqrt{-32\alpha^2 - 1} \left[ \tan \eta - \cot \eta \right],
\]

where \( \eta = \alpha \left( x + (32\alpha^2 + 1)t \right) \).

Moreover, with similar calculations as above, the wave solutions to Eq. (54) are given by

\[
u_{3,5}(x, t) = \pm 2\alpha i \left[ \tan \left\{ 2\alpha \left( x \pm 4\alpha i t \right) \right\} \right],
\]

\[
u_{3,6}(x, t) = \pm \alpha i \left[ \tanh \eta - \coth \eta + \frac{4}{\tanh \eta - \coth \eta} \right]
\]

where \( \eta = \alpha \left( x \pm 4\alpha i t \right) \).

\[
u_{3,7}(x, t) = \pm 2\alpha i \left[ \tan \left\{ 2\alpha \left( x \pm 4\sqrt{2}\alpha t \right) \right\} \right],
\]

\[
u_{3,8}(x, t) = \pm \alpha i \left[ \tanh \eta - \coth \eta \right]
\]

where \( \eta = \alpha \left( x \pm 4\sqrt{2}\alpha t \right) \).
IV. CONCLUSION

In this paper, an extended Jacobi elliptic function and modified tanh-coth function technique have been implemented to obtain a series of exact doubly periodic solutions to a variety of Boussinesq-like equations. Further, the solitary wave solutions and triangular periodic solutions can be established as the limits of Jacobi doubly periodic wave solutions. When $m$ approaches 1, the Jacobi functions degenerate to the hyperbolic function solutions, whereas when $m$ approaches 0 the Jacobi functions degenerate to the triangular function solutions. Also, graphs of some solution structure are provided in order to understand those new solutions and understand the physical phenomena of the Boussinesq-like equations. The result reveals that the implemented techniques are quite efficient and practically well suited for obtaining traveling wave solutions. The correctness of these solutions are ensured by testing them with the aid of the computation software Maple.

References