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Abstract

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A RIGIDITY THEOREM FOR IDEAL SURFACES WITH FLAT BOUNDARY

JAMES MCCOY AND GLEN WHEELER

ABSTRACT. We consider surfaces with boundary satisfying a sixth order nonlinear elliptic partial differential equation corresponding to extremising the L^2 -norm of the gradient of the mean curvature. We show that such surfaces with small L^2 -norm of the second fundamental form and satisfying so-called ‘flat boundary conditions’ are necessarily planar.

1. INTRODUCTION

We are interested in extremal surfaces of the geometric energy

$$(1) \quad \mathcal{F}[f] = \int_{\Sigma} |\nabla H|^2 d\mu$$

under the hypothesis

$$(2) \quad \int_{\Sigma} |A|^2 d\mu \leq \varepsilon_0$$

where $\varepsilon_0 > 0$ is a small, universal constant. Here $f : \Sigma \rightarrow \mathbb{R}^3$ is a smooth immersion of surface Σ with boundary; $d\mu$ is the induced area element; $H = \kappa_1 + \kappa_2$ and $|A|^2 = \kappa_1^2 + \kappa_2^2$ are respectively the mean curvature and the norm of the second fundamental form of $f(\Sigma)$ and ∇ is the covariant derivative on $f(\Sigma)$. We set up our notation in more detail in Section 2. We will call extremal surfaces for (1) ‘ideal surfaces’. Clearly minimal surfaces and surfaces of constant mean curvature are ideal surfaces that are absolute minimisers of (1).

Previous work related to the type of result of this article includes rigidity of manifolds with Ricci curvature bounds whose volume is close to that of the sphere [A], work on rigidity and classification of minimal submanifolds [CdCK, FC, L, S, R] and hypersurfaces of constant mean curvature [NS]. More recently and closer in spirit to our result here is work on higher-order geometric gap lemmas. The first of these, for Willmore surfaces,

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appeared in [KS] (Theorem 2.7) as part of a larger investigation of the gradient flow for the Willmore functional

$$\int_{\Sigma} |\mathbf{H}|^2 d\mu$$

for surfaces Σ without boundary immersed in \mathbb{R}^n , $n \geq 3$, satisfying the smallness condition

$$(3) \quad \int_{\Sigma} |A^0|^2 d\mu \leq \varepsilon_0.$$

Above \mathbf{H} is the mean curvature vector while $|A^0|^2 = |A|^2 - \frac{1}{2}H^2 = \frac{1}{2}(\kappa_1 - \kappa_2)^2$, the norm of the trace-free second fundamental form, is a pointwise measure of closeness to each other of the two principal curvatures κ_1 and κ_2 . The gap lemma of [KS] gives that the resulting surface without boundary is either an embedded plane or sphere. Later the second author of this article obtained a gap lemma for stationary solutions of the surface diffusion flow [W1] without boundary and under the smallness condition

$$\int_{\Sigma} |A^0|^n d\mu \leq \varepsilon_0$$

for surfaces of dimension $n = 2$ and $n = 3$; again such a surface is either an embedded plane or sphere. The second author further obtained a gap lemma for biharmonic surfaces in [W2] and, together with the first author, for some Helfrich surfaces [MW]. This result was extended to some other Helfrich surfaces in [BWW]. With Parkins the authors obtained a gap lemma for triharmonic surfaces [MPW]; Parkins additionally obtained the corresponding result for polyharmonic surfaces in [P].

For several of the abovementioned results there are also versions for surfaces with boundary, with either of two boundary conditions:

- (1) *umbilic boundary conditions* $|\nabla A^0| = |A^0| = 0$; or
- (2) *flat boundary conditions* $|\nabla A| = |A| = 0$.

With suitable smallness conditions, umbilic boundary conditions lead to parts of spheres and planes, while flat boundary conditions allow parts of planes only [W3].

In many cases above results hold for arbitrary codimension.

The main result of this article is as follows:

Theorem 1.1. Suppose $f : \Sigma \rightarrow \mathbb{R}^3$ satisfies

$$(4) \quad \mathcal{J}[f] := \Delta^2 H + |A|^2 \Delta H - (A^0)^{ij} \nabla_i H \nabla_j H = 0$$

with boundary conditions

$$(5) \quad |A| = 0 \text{ and } \nabla_{\eta} H = \nabla_{\eta} \Delta H = 0.$$

If f also satisfies (2) for $\varepsilon_0 > 0$ sufficiently small, then $f(\Sigma)$ is part of a flat plane.

Above we have used η to denote the unit conormal to the boundary and Δ the Laplace-Beltrami operator. We also have $\Delta^2 H := \Delta \Delta H$ and use the standard Einstein summation convention of summing over repeated indices. Notation is further clarified in Section 2

Remarks:

1. The boundary conditions (5) are understood in the sense of limits approaching the boundary within the surface.
2. We may replace (4) by the slightly more general condition

$$H \mathcal{J}[f] = 0.$$

3. In the case of high codimension, where $f : \Sigma \rightarrow \mathbb{R}^n$, $n > 3$, (4) may be replaced by the relatively weak orthogonality condition

$$\langle \mathcal{J}[f], \mathbf{H} \rangle_{\mathbb{R}^n} = 0.$$

With analogous boundary conditions (5) and smallness condition (2) we obtain the same result as Theorem 1.1. In this article we restrict to the case $n = 3$ for notational simplicity; the workings in the general case are essentially the same. Note that in high codimension the variational equation for (1) is not the same as $\mathcal{J}[f] = 0$, so this ‘easy’ high-codimension result is not particularly natural.

4. Theorem 1.1 includes a nonexistence result: there are no surfaces $f(\Sigma)$ satisfying (4) and (2) with boundary conditions (5) whose boundaries do not lie within a plane in \mathbb{R}^3 .

The structure of this article is as follows. In Section 2 we set up notation and state some fundamental results that are needed in the proof of Theorem 1.1. In Section 3 we compute the normal variation of (1) showing how (4) and the boundary conditions 5 arise. In Section 4 we establish various estimates that culminate in the proof of Theorem 1.1. The authors would like to thank the anonymous referee whose careful reading and comments have led to improvements in the article.

2. PRELIMINARIES

We begin this section by setting up our notation in more detail. For a local orthonormal frame $\{\partial_1, \partial_2\}$ on Σ , that is, sections into $T\Sigma \times T\Sigma$ corresponding to an arbitrarily chosen chart on Σ , we define for any smooth function $\varphi : \Sigma \rightarrow \mathbb{R}$ the directional derivative

$$\nabla_i(\varphi) := \partial_i(\varphi)$$

in each of the directions corresponding to $i = 1$ and $i = 2$. From this we define the tangential gradient of φ by

$$\nabla(\varphi) := (\nabla_1\varphi)\partial_1 + (\nabla_2\varphi)\partial_2,$$

a locally-defined, smooth tangential vector field along Σ , that is, a locally-defined smooth section into $T\Sigma$.

For an immersion $f : \Sigma \rightarrow \mathbb{R}^3$ we have the induced metric

$$(g_f)_{ij} := \langle \partial_i f, \partial_j f \rangle_{\mathbb{R}^3}$$

with inverse metric tensor denoted (g_f^{ij}) . For later reference we also have the induced area element on Σ

$$d\mu_f := d\text{vol}_f = \sqrt{\det g_f} dx.$$

We now define the divergence operator

$$\text{div}_f X := \nabla^j X_j := g_f^{ij} \nabla_i X_j$$

for smooth vector fields $X = X_1 \partial_1 + X_2 \partial_2$ on Σ . Above we have also implicitly defined the standard convention where a raised index indicates contraction over the metric. Finally we define the Laplace-Beltrami operator corresponding to f , and its square, respectively by

$$\Delta_f(\varphi) := (\text{div}_f \circ \nabla)(\varphi) \equiv g_f^{ij} \nabla_i \nabla_j(\varphi)$$

and

$$(\Delta_f)^2(\varphi) := (\Delta_f \circ \Delta_f)(\varphi).$$

As there is no cause for confusion we will now drop the subscript f notation where quantities are naturally associated to the immersion f without ambiguity.

Throughout this work we will employ cut-off functions defined as follows. We take $\tilde{\gamma} \in C_c^2(\mathbb{R}^3)$ of the form $\tilde{\gamma}(x) = \hat{\gamma}\left(\frac{1}{\rho}|x|\right)$, for any $\rho > 0$, where $\hat{\gamma} : \mathbb{R}_+ \cup \{0\} \rightarrow [0, 1]$ satisfies

$$\hat{\gamma}(s) = \begin{cases} 1 & 0 \leq s \leq \frac{1}{2} \\ 0 & s \geq 1. \end{cases}$$

Then $\gamma = \tilde{\gamma} \circ f : \Sigma \rightarrow [0, 1]$ satisfies

$$\|\nabla \gamma\|_\infty \leq c_\gamma, \quad \text{and} \quad \|\nabla^2 \gamma\|_\infty \leq c_\gamma(c_\gamma + |A|),$$

for $c_\gamma = \frac{c}{\rho}$ where $c > 0$ is an absolute constant.

We will also need the Michael-Simon Sobolev inequality [MS] for surfaces with boundary. A proof in this setting appears for example in [W3].

Theorem 2.1. For $f : M^m \rightarrow \mathbb{R}^n$ a smooth immersion of M with boundary ∂M into \mathbb{R}^n and any $u \in C^1(\overline{M})$,

$$(6) \quad \left[\int_M |u|^{\frac{m}{m-1}} d\mu \right]^{\frac{m-1}{2}} \leq \frac{4^{m+1}}{\omega_m^{1/m}} \left[\int_M (|\nabla u| + |H||u|) d\mu + \int_{\partial M} |u| d\sigma \right]$$

where ω_m is the volume of the unit ball in \mathbb{R}^m and $d\sigma := d\text{Vol}_{f \circ \iota}$ is the area element on ∂M , where $\iota : \partial \Sigma \hookrightarrow \Sigma$ is the boundary-injection.

Remark: We only need to apply the above with $m = 2$ and $n = 3$ (or general n in view of the earlier Remark 3). Moreover our boundary conditions (5) always ensure in our applications of the above that the boundary term is identically equal to zero. With these settings (6) gives

$$\int_{\Sigma} u^2 d\mu \leq c \left[\int_{\Sigma} (|\nabla u| + |H| |u|) d\mu \right]^2,$$

where $c = \frac{32\sqrt{3}}{\sqrt{\pi}}$.

Let us finally mention that from the Codazzi equations

$$\nabla_i A_{jk} = \nabla_j A_{ki} = \nabla_k A_{ij}$$

one can show (see, eg [P]) that the k -derivatives of the full A tensor are controlled by those of A^0 :

$$(7) \quad |\nabla^{(k)} H| \leq |\nabla^{(k)} A| \leq c(n) |\nabla^{(k)} A^0|.$$

3. EXTREMAL SURFACES WITH BOUNDARY FOR ENERGY (1)

We calculate the normal variation of energy 1 as follows.

Lemma 3.1. Given a smooth normal variation $\phi : \Sigma \rightarrow \mathbb{R}^3$ of $f : \Sigma \rightarrow \mathbb{R}^3$,

$$(8) \quad \frac{d}{d\epsilon} \mathcal{F}[f + \epsilon\phi] \Big|_{\epsilon=0} = -2 \int_{\Sigma} [\Delta^2 H + |A|^2 \Delta H - (A^0)^{ij} \nabla_i H \nabla_j H] \varphi d\mu \\ + 2 \int_{\partial\Sigma} \langle (\Delta\varphi + |A|^2 \varphi) \nabla H + \nabla \Delta H \varphi - \Delta H \nabla \varphi, \eta \rangle d\sigma.$$

Here ν denotes a smooth choice of unit normal to $f(\Sigma)$ and $\varphi := \langle \phi, \nu \rangle$ is the signed length of the normal section ϕ . We have used $d\sigma$ for the measure induced on $\partial\Sigma$ and η is the outer unit co-normal to $\partial\Sigma$. The boundary integral can be considered intrinsically, in which case η is tangent to Σ , normal to $\partial\Sigma$ and the inner product is with respect to the metric on Σ induced via the isometric immersion f ; alternatively we may consider the boundary integral extrinsically via the immersion f , in which case the inner product is the usual one in Euclidean space.

Proof: Equation (8) follows from the variations

$$\frac{\partial}{\partial \epsilon} g_{ij}^\epsilon \Big|_{\epsilon=0} = -2\varphi A_{ij}, \quad \frac{\partial}{\partial \epsilon} g_\epsilon^{ij} \Big|_{\epsilon=0} = 2\varphi A^{ij}, \\ \frac{\partial}{\partial \epsilon} \sqrt{\det(g_{ij}^\epsilon)} \Big|_{\epsilon=0} = -H\varphi \sqrt{\det(g_{ij})}, \quad \frac{\partial}{\partial \epsilon} H_\epsilon \Big|_{\epsilon=0} = \Delta\varphi + \varphi |A|^2.$$

Calculations of these may be found in [E] and [MW], for example.

Using the above we calculate, with slight abuse of notation and suppressing ε where there is no chance of confusion

$$(9) \quad \frac{d}{d\varepsilon} \int_{\Sigma} |\nabla H|^2 d\mu \Big|_{\varepsilon=0} = \frac{d}{d\varepsilon} \int_U g^{ij} \frac{\partial}{\partial x_i} H \frac{\partial}{\partial x_j} H \sqrt{\det(g_{ij}^\varepsilon)} dx \Big|_{\varepsilon=0}$$

$$= \int_{\Sigma} (2\varphi A^{ij}) \nabla_i H \nabla_j H d\mu + 2 \int_{\Sigma} \nabla^i (\Delta\varphi + |A|^2 \varphi) \nabla_j H d\mu + \int_{\Sigma} |\nabla H|^2 (-H\varphi) d\mu$$

The first and last terms on the right hand side of (9) combine to give the third term in 8. On the second term of (9) we ‘integration by parts’ on Σ with boundary, that is, we apply Stokes’ Theorem on smooth manifold Σ with boundary,

$$\int_{\Sigma} \operatorname{div}_f X d\mu = \int_{\partial\Sigma} \langle X, \eta \rangle d\sigma,$$

where X is any smooth tangent vector field in $\Gamma(T\Sigma)$, and div_f is the divergence on $f(\Sigma)$.

We have by Stokes’ Theorem

$$\int_{\Sigma} \operatorname{div} [(\Delta\varphi + \varphi|A|^2) \nabla H] d\mu = \int_{\partial\Sigma} (\Delta\varphi + \varphi|A|^2) \langle \nabla H, \eta \rangle d\sigma;$$

expanding out the left hand side by the product rule therefore yields

$$\int_{\Sigma} \nabla^i (\Delta\varphi + \varphi|A|^2) \nabla_i H d\mu = - \int_{\Sigma} (\Delta\varphi + \varphi|A|^2) \Delta H d\mu + \int_{\partial\Sigma} (\Delta\varphi + \varphi|A|^2) \langle \nabla H, \eta \rangle d\sigma.$$

The second term of (8) and the first boundary term are now clearly visible. We integrate by parts twice more on the first term above:

$$\begin{aligned} \int_{\Sigma} \Delta H \Delta\varphi d\mu &= - \int_{\Sigma} \nabla^i \Delta H \nabla_i \varphi d\mu + \int_{\partial\Sigma} \Delta H \langle \nabla \varphi, \eta \rangle d\mu \\ &= \int_{\Sigma} \Delta^2 H \varphi d\mu - \int_{\partial\Sigma} \langle \nabla \Delta H, \eta \rangle \varphi d\sigma + \int_{\partial\Sigma} \Delta H \langle \nabla \varphi, \eta \rangle d\sigma \end{aligned}$$

revealing the remaining terms in (8). \square

If $f(\Sigma)$ were closed without boundary, there would be no boundary terms in the expression (8) and critical points of $\mathcal{F}[f]$ would satisfy (4).

In our setting we impose *flat boundary conditions* (5) on $\partial\Sigma$. Then the boundary terms in (8) disappear (on the last term we integrate by parts on $\partial\Sigma$ that itself has no boundary) and we are left with (4) for critical points of the energy. We study smooth solutions (4) with boundary conditions (5) and smallness condition (2).

4. PROOF OF THEOREM 1.1

We establish in turn estimates that facilitate the proof of Theorem 1.1. The first four Lemmas below do not require the smallness condition (2).

Using Stokes’ Theorem on Σ (integration by parts) we begin with

LEMMA 4.1. *Surfaces satisfying (5) also satisfy*

$$\begin{aligned} \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu &= \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + \int_{\Sigma} |A|^2 |\nabla H|^2 \gamma^p d\mu \\ &\quad + \int_{\Sigma} H \nabla^i H \nabla_i |A|^2 \gamma^p d\mu + \int_{\Sigma} H (A^0)^{ij} \nabla_i H \nabla_j H \gamma^p d\mu \\ &\quad + p \int_{\Sigma} [H \nabla^i \Delta H + (H |A|^2 - \Delta H) \nabla^i H] \nabla_i \gamma \cdot \gamma^{p-1} d\mu \end{aligned}$$

Proof: We multiply (4) by $H \gamma^p$ for some constant $p > 0$ to be chosen later:

$$(10) \quad \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu = \int_{\Sigma} H \Delta^2 H \gamma^p d\mu + \int_{\Sigma} |A|^2 H \Delta H \gamma^p d\mu - \int_{\Sigma} H (A^0)^{ij} \nabla_i H \nabla_j H \gamma^p d\mu.$$

Noting that

$$\begin{aligned} \int_{\Sigma} \operatorname{div} (H \nabla \Delta H \gamma^p) d\mu &= \int_{\Sigma} H \Delta^2 H \gamma^p d\mu + \int_{\Sigma} \nabla^i H \nabla_i \Delta H \gamma^p d\mu \\ &\quad + p \int_{\Sigma} H \nabla^i \Delta H \nabla_i \gamma \cdot \gamma^{p-1} d\mu = \int_{\partial \Sigma} H \nabla_{\eta} \Delta H \gamma^p d\sigma = 0 \end{aligned}$$

where the boundary integral is equal to zero in view of (5), we have

$$(11) \quad \int_{\Sigma} H \Delta^2 H \gamma^p d\mu = - \int_{\Sigma} \nabla^i H \nabla_i \Delta H \gamma^p d\mu - p \int_{\Sigma} H \nabla^i \Delta H \nabla_i \gamma \cdot \gamma^{p-1} d\mu.$$

Next we compute

$$\begin{aligned} \int_{\Sigma} \operatorname{div} (\nabla H \Delta H \gamma^p) d\mu &= \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + \int_{\Sigma} \nabla^i H \nabla_i \Delta H \gamma^p d\mu \\ &\quad + p \int_{\Sigma} \gamma^{p-1} \Delta H \nabla^i H \nabla_i \gamma d\mu = \int_{\partial \Sigma} \Delta H \nabla_{\eta} H \gamma^p d\sigma = 0, \end{aligned}$$

where similarly (5) implies the boundary integral is equal to zero. Hence

$$(12) \quad - \int_{\Sigma} \nabla^i H \nabla_i \Delta H \gamma^p d\mu = \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + p \int_{\Sigma} \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu.$$

Combining (11) and (12) we obtain

$$(13) \quad \begin{aligned} \int_{\Sigma} H \Delta^2 H \gamma^p d\mu \\ = \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + p \int_{\Sigma} \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu - p \int_{\Sigma} H \nabla^i \Delta H \nabla_i \gamma \cdot \gamma^{p-1} d\mu. \end{aligned}$$

Now

$$\begin{aligned} \int_{\Sigma} \operatorname{div} (H|A|^2 \nabla H \gamma^p) d\mu &= \int_{\Sigma} \nabla^i (|A|^2 H) \nabla_i H \gamma^p d\mu + \int_{\Sigma} |A|^2 H \Delta H \gamma^p d\mu \\ &\quad + p \int_{\Sigma} \gamma^{p-1} H |A|^2 \nabla^i H \nabla_i \gamma d\mu = \int_{\partial \Sigma} H |A|^2 \nabla_{\eta} H \gamma^p d\sigma = 0 \end{aligned}$$

where we have again used (5). Thus

$$(14) \quad \int_{\Sigma} |A|^2 H \Delta H \gamma^p d\mu = - \int_{\Sigma} \nabla^i (|A|^2 H) \nabla_i H \gamma^p d\mu - p \int_{\Sigma} \gamma^{p-1} H |A|^2 \nabla^i H \nabla_i \gamma d\mu.$$

Substituting (13) and (14) into (10) we obtain

$$\begin{aligned} &\int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu \\ &= \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + p \int_{\Sigma} \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu - p \int_{\Sigma} H \nabla^i \Delta H \nabla_i \gamma \cdot \gamma^{p-1} d\mu \\ &\quad - \int_{\Sigma} \nabla^i (|A|^2 H) \nabla_i H \gamma^p d\mu - p \int_{\Sigma} \gamma^{p-1} H |A|^2 \nabla^i H \nabla_i \gamma d\mu - \int_{\Sigma} H (A^0)^{ij} \nabla_i H \nabla_j H \gamma^p d\mu. \end{aligned}$$

The result follows. \square

LEMMA 4.2. *Surfaces satisfying (5) also satisfy*

$$\begin{aligned} \int_{\Sigma} |\nabla^2 H|^2 \gamma^p d\mu &\leq c \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu \\ &\quad + p \int_{\Sigma} [H \nabla^i \Delta H + (H |A|^2 - \Delta H) \nabla^i H] \nabla_i \gamma \cdot \gamma^{p-1} d\mu \end{aligned}$$

Proof: Interchanging covariant derivatives, using the Codazzi equations and Stokes Theorem we may obtain exactly as in [W3] that there is a universal constant c such that

$$\begin{aligned} \frac{1}{c} \int_{\Sigma} (|\nabla^2 H|^2 + H^2 |\nabla H|^2) \gamma^p d\mu \\ \leq \int_{\Sigma} [(\Delta H)^2 + |A^0|^2 |\nabla H|^2] \gamma^p d\mu + c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu. \end{aligned}$$

Further, we estimate

$$\int_{\Sigma} H (A^0)^{ij} \nabla_i H \nabla_j H \gamma^p d\mu \leq \frac{1}{2c} \int_{\Sigma} H^2 |\nabla H|^2 \gamma^p d\mu + \frac{c}{2} \int_{\Sigma} |A^0|^2 |\nabla H|^2 \gamma^p d\mu$$

and

$$\int_{\Sigma} H \nabla_i |A|^2 \nabla^i H \gamma^p d\mu \leq \tilde{c} \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu.$$

Combining these with the Lemma 4.1 yields the result. \square

LEMMA 4.3. *Surfaces satisfying (5) also satisfy*

$$\begin{aligned} & \int_{\Sigma} \left(|\nabla^2 H|^2 + |A|^4 |A^0|^2 \right) \gamma^p d\mu \\ & \leq c \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu + c \int_{\Sigma} |A^0|^6 \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu \\ & \quad + c c_{\gamma}^4 \int_{\Sigma} |A^0|^2 \gamma^{p-4} d\mu + p \int_{\Sigma} [H \nabla^i \Delta H + (H |A|^2 - \Delta H) \nabla^i H] \nabla_i \gamma \cdot \gamma^{p-1} d\mu \end{aligned}$$

Proof: By the same argument as for [W3, Lemma 5] we have

$$\begin{aligned} & \int_{\Sigma} \left(H^4 |A^0|^2 + H^2 |\nabla A^0|^2 \right) \gamma^p d\mu \\ & \leq c \int_{\Sigma} \left(H^2 |\nabla H|^2 + |A^0|^2 |\nabla A^0|^2 + |A^0|^6 \right) \gamma^p d\mu + c c_{\gamma}^4 \int_{\Sigma} |A^0|^2 \gamma^{p-4} d\mu. \end{aligned}$$

Moreover for $\varepsilon > 0$,

$$\begin{aligned} \int_{\Sigma} |A|^4 |A^0|^2 \gamma^p d\mu &= \int_{\Sigma} \left(H^2 + \frac{1}{2} |A^0|^2 \right)^2 |A^0|^2 d\mu \\ &\leq (1 + \varepsilon) \int_{\Sigma} H^4 |A^0|^2 \gamma^p d\mu + (1 + c(\varepsilon)) \int_{\Sigma} |A^0|^6 \gamma^p d\mu. \end{aligned}$$

Thus from Lemma 4.2 we obtain

$$\begin{aligned} & \int_{\Sigma} \left(|\nabla^2 H|^2 + |A|^4 |A^0|^2 \right) \gamma^p d\mu \\ & \leq c \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu \\ & \quad + p \int_{\Sigma} [H \nabla^i \Delta H + (H |A|^2 - \Delta H) \nabla^i H] \nabla_i \gamma \cdot \gamma^{p-1} d\mu \\ & \quad + (1 + \varepsilon) \int_{\Sigma} H^4 |A^0|^2 \gamma^p d\mu + (1 + c(\varepsilon)) \int_{\Sigma} |A^0|^6 \gamma^p d\mu \\ & \leq c \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu \\ & \quad + (1 + \varepsilon) \left[c \int_{\Sigma} \left(H^2 |\nabla H|^2 + |A^0|^2 |\nabla A^0|^2 + |A^0|^6 \right) \gamma^p d\mu + c c_{\gamma}^4 \int_{\Sigma} |A^0|^2 \gamma^{p-4} d\mu \right] \\ & \quad + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu + p \int_{\Sigma} [H \nabla^i \Delta H + (H |A|^2 - \Delta H) \nabla^i H] \nabla_i \gamma \cdot \gamma^{p-1} d\mu \end{aligned}$$

which gives the statement of the Lemma. \square

LEMMA 4.4. *Surfaces satisfying (5) also satisfy*

$$\begin{aligned} & \int_{\Sigma} \left(|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^0|^2 \right) \gamma^p d\mu \\ & \leq c \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu + c \int_{\Sigma} |A^0|^6 \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu \\ & \quad + c c_{\gamma}^4 \int_{\Sigma} |A^0|^2 \gamma^{p-4} d\mu + p \int_{\Sigma} [H \nabla^i \Delta H + (H |A|^2 - \Delta H) \nabla^i H] \nabla_i \gamma \cdot \gamma^{p-1} d\mu \end{aligned}$$

Proof: Using Simons' identity [S]

$$\Delta A_{ij} = \nabla_i \nabla_j H + H A_{im} A_j^m - |A|^2 A_{ij}$$

we may write

$$\Delta A_{ij}^0 = \Delta A_{ij} - \frac{1}{2} g_{ij} \Delta H = \nabla_i \nabla_j H - \frac{1}{2} g_{ij} \Delta H + H A_{im} A_j^m - |A|^2 A_{ij}.$$

Using $A_{ij} = A_{ij}^0 + \frac{1}{2} H g_{ij}$ again, this becomes

$$\begin{aligned} \Delta A_{ij}^0 &= \nabla_i \nabla_j H - \frac{1}{2} g_{ij} \Delta H + H \left(A_{im}^0 + \frac{1}{2} H g_{im} \right) \left((A^0)^m_j + \frac{1}{2} H g_j^m \right) \\ &\quad - \left(|A^0|^2 + \frac{1}{2} H^2 \right) \left(A_{ij}^0 + \frac{1}{2} H^2 g_{ij} \right) \\ &= \nabla_i \nabla_j H - \frac{1}{2} g_{ij} \Delta H + H A_{im}^0 (A^0)^m_j + \frac{1}{2} H^2 A_{ij}^0 - |A^0|^2 A_{ij}^0 - \frac{1}{2} H |A^0|^2 g_{ij}. \end{aligned}$$

Hence for an absolute constant c we have

$$|\Delta A^0| \leq |\nabla^2 H| + c |H| |A^0|^2 + c H^2 |A^0| + c |A^0|^3$$

and so

$$|\Delta A^0|^2 \leq 2 |\nabla^2 H|^2 + c H^4 |A^0|^2 + c |A^0|^6.$$

Interchange of second covariant derivatives and application of Stokes' Theorem then shows

$$\begin{aligned} & \int_{\Sigma} |\nabla^2 A^0|^2 \gamma^p d\mu \\ & \leq 2 \int_{\Sigma} |\Delta A^0|^2 \gamma^p d\mu + c \int_{\Sigma} \left(|A|^2 |\nabla A|^2 + |A^0|^6 \right) \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu. \end{aligned}$$

Bearing in mind (7) and using also [W3] inequality (31) we estimate

$$\begin{aligned}
& \int_{\Sigma} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^0|^2) \gamma^p d\mu \\
& \leq 2 \int_{\Sigma} |\Delta A|^2 \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu + \int_{\Sigma} |A|^4 |A^0|^2 \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu \\
& \leq c \int_{\Sigma} |\nabla^2 H|^2 \gamma^p d\mu + c \int_{\Sigma} H^4 |A^0|^2 \gamma^p d\mu + \int_{\Sigma} |A^0|^6 \gamma^p d\mu + c \int_{\Sigma} |A|^2 |\nabla A|^2 \gamma^p d\mu \\
& \quad + \int_{\Sigma} |A|^4 |A^0|^2 \gamma^p d\mu + c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu
\end{aligned}$$

The result then follows using Lemma 4.3. \square

LEMMA 4.5. *Surfaces satisfying (5) and (2) also satisfy*

$$\int_{\Sigma} (|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^0|^2) \gamma^p d\mu \leq c \int_{\Sigma} \mathcal{J}[f] H \gamma^p d\mu + c c_{\gamma}^4 \int_{\Sigma} |A|^2 \gamma^{p-4} d\mu$$

Proof: Write $\|A\|_{2, [\gamma > 0]}^2 = \int_{[\gamma > 0]} |A|^2 d\mu$. The idea is to use the smallness condition (2) to estimate the terms on the right hand side of Lemma 4.4. In [W3] it was shown using the Michael-Simon Sobolev inequality

$$\begin{aligned}
& \int_{\Sigma} (|A^0|^2 |A|^4 + |A|^2 |\nabla A|^2) \gamma^p d\mu \\
& \leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} (|\nabla^2 A^0|^2 + |A|^2 |\nabla A^0|^2 + |A|^4 |A^0|^2) \gamma^p d\mu + c c_{\gamma}^4 \|A\|_{2, [\gamma > 0]}^4;
\end{aligned}$$

this result applies in the case of our boundary conditions (5). Thus we can absorb the non- c_{γ} terms on the right hand side of Lemma 4.4.

We estimate the c_{γ} terms from Lemma 4.4 as follows:

$$c c_{\gamma}^4 \int_{\Sigma} |A^0|^2 \gamma^{p-4} d\mu \leq c c_{\gamma}^4 \int_{\Sigma} |A|^2 \gamma^{p-4} d\mu;$$

via Stokes' Theorem and the Cauchy-Schwarz and Peter-Paul inequalities we have for $\varepsilon > 0$

$$c c_{\gamma}^2 \int_{\Sigma} |\nabla A^0|^2 \gamma^{p-2} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^2 A|^2 \gamma^p d\mu + c(\varepsilon) c_{\gamma}^4 \int_{\Sigma} |A|^2 \gamma^{p-4} d\mu;$$

with this in turn we estimate

$$\begin{aligned}
& p \int_{\Sigma} \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu \\
& \leq c c_{\gamma} \int_{\Sigma} |\nabla^2 A| |\nabla H| \gamma^{p-1} d\mu \leq \varepsilon \int_{\Sigma} |\nabla^2 A|^2 \gamma^p d\mu + c(\varepsilon) c_{\gamma}^4 \int_{\Sigma} |A|^2 \gamma^{p-4} d\mu
\end{aligned}$$

and

$$p \int_{\Sigma} H |A|^2 \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu \leq \varepsilon \int_{\Sigma} |A|^2 |\nabla A^0|^2 \gamma^p d\mu + c(\varepsilon) c_{\gamma}^2 \int_{\Sigma} H^2 |A|^2 \gamma^{p-2} d\mu.$$

Now by the Michael-Simon Sobolev inequality ($|A| = 0$ on $\partial\Sigma$ from (5))

$$\begin{aligned} \int_{\Sigma} H^2 |A|^2 \gamma^{p-2} d\mu &\leq \int_{\Sigma} |A|^4 \gamma^{p-2} d\mu \\ &\leq c \left(\int_{\Sigma} |\nabla |A|^2| \gamma^{\frac{p-2}{2}} d\mu \right)^2 + \left(\int_{\Sigma} |A|^3 \gamma^{\frac{p-2}{2}} d\mu \right)^2 \\ &\leq c \left(\int_{\Sigma} |\nabla A| |A| \gamma^{\frac{p-2}{2}} d\mu \right)^2 + c c_{\gamma}^2 \left(\int_{\Sigma} |A|^2 \gamma^{\frac{p-4}{2}} d\mu \right)^2 \\ &\quad + c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} |A|^4 \gamma^{p-2} d\mu. \end{aligned}$$

Absorbing on the left and using the Cauchy-Schwarz inequality we obtain

$$\int_{\Sigma} |A|^4 \gamma^{p-2} d\mu \leq c \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} |\nabla A|^2 \gamma^{p-2} d\mu + c c_{\gamma}^2 \|A\|_{2, [\gamma > 0]}^4$$

and so

$$c c_{\gamma}^2 \int_{\Sigma} H^2 |A|^2 \gamma^{p-2} d\mu \leq c c_{\gamma}^2 \|A\|_{2, [\gamma > 0]}^2 \int_{\Sigma} |\nabla A|^2 \gamma^{p-2} d\mu + c c_{\gamma}^4 \|A\|_{2, [\gamma > 0]}^4.$$

For the remaining term from Lemma 4.4 we use Stokes' Theorem ($H = 0$ on $\partial\Sigma$ in view of (5))

$$\begin{aligned} \int_{\Sigma} H \nabla^i \Delta H \nabla_i \gamma \cdot \gamma^{p-1} d\mu &= - \int_{\Sigma} \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu - \int_{\Sigma} H \Delta H \Delta \gamma \cdot \gamma^{p-1} d\mu \\ &\quad - (p-1) \int_{\Sigma} H \Delta H |\nabla \gamma|^2 \gamma^{p-2} d\mu. \end{aligned}$$

We now estimate for $\varepsilon > 0$,

$$- \int_{\Sigma} \Delta H \nabla^i H \nabla_i \gamma \cdot \gamma^{p-1} d\mu \leq \varepsilon \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + c(\varepsilon) c_{\gamma}^2 \int_{\Sigma} |\nabla H|^2 \gamma^{p-2} d\mu$$

and

$$\begin{aligned} &- \int_{\Sigma} H \Delta H \Delta \gamma \cdot \gamma^{p-1} d\mu \\ &\leq c c_{\gamma} \int_{\Sigma} |H| |\Delta H| (c_{\gamma} + |A|) \gamma^{p-1} d\mu \\ &\leq \varepsilon \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + c c_{\gamma}^4 \int_{\Sigma} |A|^2 \gamma^{p-2} d\mu + c c_{\gamma}^2 \int_{\Sigma} H^2 |A|^2 \gamma^{p-2} d\mu \end{aligned}$$

and

$$-(p-1) \int_{\Sigma} H \Delta H |\nabla \gamma|^2 \gamma^{p-2} d\mu \leq \varepsilon \int_{\Sigma} (\Delta H)^2 \gamma^p d\mu + c c_{\gamma}^4 \int_{\Sigma} H^2 \gamma^{p-4} d\mu$$

Inserting all these estimates and absorbing on the left yields the result. \square

Completion of the proof of Theorem 1.1: Using Lemma 4.5, surfaces satisfying (5), (2) and (4) also satisfy

$$\int_{\Sigma} \left(|\nabla^2 A|^2 + |A|^2 |\nabla A|^2 + |A|^4 |A^0|^2 \right) \gamma^p d\mu \leq \frac{c}{\rho^4} \int_{\Sigma} |A|^2 \gamma^{p-4} d\mu \leq \frac{c}{\rho^4} \varepsilon_0$$

for an absolute constant c . With $p = 4$, taking $\rho \rightarrow \infty$ we see that $f(\Sigma)$ must have

$$|A|^4 |A^0|^2 \equiv 0.$$

Since

$$|A^0|^6 \leq |A|^4 |A^0|^2$$

we have that

$$|A^0|^2 \equiv 0$$

implying $f(\Sigma)$ is either part of a sphere or part of a plane. The boundary condition (5) implies $f(\Sigma)$ is part of a plane. \square

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