2019

Factorisation of equivariant spectral triples in unbounded KK-theory

Iain G. Forsyth
University of Wollongong, Leibniz University of Hannover, Australian National University

Adam C. Rennie
University of Wollongong, renniea@uow.edu.au

Factorisation of equivariant spectral triples in unbounded KK-theory

Abstract

We provide sufficient conditions to factorise an equivariant spectral triple as a Kasparov product of unbounded classes constructed from the group action on the algebra and from the fixed point spectral triple. We show that if factorisation occurs, then the equivariant index of the spectral triple vanishes. Our results are for the action of compact abelian Lie groups, and we demonstrate them with examples from manifolds and θ-deformations. In particular, we show that equivariant Dirac-type spectral triples on the total space of a torus principal bundle always factorise. Combining this with our index result yields a special case of the Atiyah-Hirzebruch theorem. We also present an example that shows what goes wrong in the absence of our sufficient conditions (and how we get around it for this example).

Disciplines

Engineering | Science and Technology Studies

Publication Details

Forsyth, I. & Rennie, A. (2019). Factorisation of equivariant spectral triples in unbounded KK-theory. Journal of the Australian Mathematical Society, 107 (2), 145-180.

FACTORISATION OF EQUIVARIANT SPECTRAL TRIPLES IN UNBOUNDED KK-THEORY.

IAIN FORSYTH and ADAM RENNIE

(October 28, 2018)

Abstract

We provide sufficient conditions to factorise an equivariant spectral triple as a Kasparov product of unbounded classes constructed from the group action on the algebra and from the fixed point spectral triple. We show that if factorisation occurs, then the equivariant index of the spectral triple vanishes. Our results are for the action of compact abelian Lie groups, and we demonstrate them with examples from manifolds and θ-deformations. In particular we show that equivariant Dirac-type spectral triples on the total space of a torus principal bundle always factorise. Combining this with our index result yields a special case of the Atiyah-Hirzebruch theorem. We also present an example that shows what goes wrong in the absence of our sufficient conditions (and how we get around it for this example).

1. Introduction

This paper is motivated by recent applications of the Kasparov product to gauge theory, [4, 7]. In particular, it becomes important to know to what degree an equivariant spectral triple, regarded as encoding the geometry of the total space of a noncommutative principal bundle, can be factored over the base space.

We provide sufficient conditions to factorise a G-equivariant spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, for G compact abelian, as a Kasparov product of a 'fixed point' spectral triple for the base space and a Kasparov module constructed solely from the action of the group on the algebra. These two components of the product represent respectively the 'horizontal' and 'vertical' parts of the noncommuative principal bundle. More precisely, given our sufficient conditions, we construct unbounded cycles representing classes in

$KK_G^{\dim G}(A, A^G)$ and $KK_G^{j+\dim G}(A^G, \mathbb{C})$, with A the norm completion of $\mathcal{A}$, such that the Kasparov product of these classes

$$KK_G^{\dim G}(A, A^G) \times KK_G^{j+\dim G}(A^G, \mathbb{C}) \to KK_G^j(A, \mathbb{C})$$

recovers the class of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ in $KK^j(A, \mathbb{C})$. The construction of these unbounded Kasparov modules is new, although in the case $G = \mathbb{T}$ the cycle for $KK_G^{\dim G}(A, A^G)$ agrees with the cycle constructed in [6, Section 2] up to a sign.

In order to define the Kasparov module with class in $KK_G^{\dim G}(A, A^G)$, we require that the action of G on A satisfies the spectral subspace assumption of [6]. To define the unbounded Kasparov module with class in $KK_G^{j+\dim G}(A^G, \mathbb{C})$, we need a Clifford action

$$\eta : \mathbb{C}\mathrm{l}(T_eG) \cong \mathbb{C}\mathrm{l}_{\dim G} \to B(\mathcal{H})$$

satisfying a few compatibility conditions. Finally, the product of these classes represents the class of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, provided that one positivity constraint is satisfied: this constraint arises from Kucerovsky's criteria [16].

Our factorisation results show that the class of our equivariant spectral triple is the product of classes with unbounded representatives, which are defined in terms of the original spectral triple subject to some geometric constraints. As a consequence, we show that if our conditions are satisfied and factorisation occurs, then the equivariant index of the spectral triple vanishes, when this is defined.

The constructive approach to the Kasparov product, [4, 14, 20, 21], seeks to construct a spectral triple from unbounded representatives of composable KK-classes. Having obtained a factorisation, say,

$$[(\mathcal{A}, \mathcal{H}, \mathcal{D})] = [(\mathcal{A}', E_{A^G}, \mathcal{D}_1)] \widehat{\otimes}_{A^G} [(\mathcal{A}^G, \mathcal{H}_2, \mathcal{D}_2)]$$

it is natural to ask whether the constructive product of $(\mathcal{A}', E_{A^G}, \mathcal{D}_1)$ and $(\mathcal{A}^G, \mathcal{H}_2, \mathcal{D}_2)$ makes sense and recovers the original triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. We examine equivariant Dirac-type spectral triples $(C^\infty(M), L^2(S), \mathcal{D})$ on a compact Riemannian manifold with a free isometric torus action, where we show that factorisation holds in our sense. As an easy corollary we derive a particular case of the Atiyah-Hirzebruch theorem, [1]. In this special case, we show that the constructive method produces a spectral triple $(C^\infty(M), L^2(S), T)$ whose KK-class is the same as that of $(C^\infty(M), L^2(S), \mathcal{D})$. The operator T is a self-adjoint elliptic first order differential operator, but the difference $\mathcal{D} - T$ is typically unbounded. If each orbit in M is an isometrically embedded copy of $\mathbb{T}^n$, we find that $\mathcal{D} - T$ is bounded. Thus we see evidence in these examples that the constructive product is sensitive to metric data.

Factorisation of circle-equivariant spectral triples has also been studied in [4], [10, 11] and the Ph.D. thesis of A. Zucca, [32]. The last three of these works study such factorisations under the condition of "fibres of constant length", a condition which is also satisfied in the examples studied in [4]. Such a condition appears in Corollary 6.9, and corresponds to the isometric embedding of orbits (up to a constant multiple).

Finally, we consider in detail the factorisation of the Dirac operator over the 2-sphere, for rotation by the circle. In this case, the circle action is not free and factorisation for $C(S^2)$ is not possible, but we show that factorisation is nevertheless possible if one restricts to the C^*-algebra of continuous functions vanishing at the poles.

Acknowledgements. Both authors were supported by the Australian Research Council, and the authors thank the University of Wollongong, the Australian National University and the Hausdorff Institute for Mathematics for hospitality. We would also like to thank Simon Brain, Magnus Goffeng and Walter van Suijlekom for useful discussions.

2. The construction of the unbounded KK-cycles.

Definition 2.1. Let A and B be $\mathbb{Z}_2$-graded C^*-algebras carrying respective actions α and β by a compact group G. An *unbounded equivariant Kasparov A-B-module* $(\mathcal{A}, E_B, \mathcal{D})$ consists of an invariant dense sub-$*$-algebra $\mathcal{A} \subset A$, a countably generated $\mathbb{Z}_2$-graded right Hilbert B-module E with a homomorphism V from G into the invertible degree zero bounded linear (not necessarily adjointable) operators on E, a $\mathbb{Z}_2$-graded $*$-homomorphism $\phi : A \to \mathrm{End}_B(E)$, and an odd, self-adjoint, regular operator $\mathcal{D} : \mathrm{dom}(\mathcal{D}) \subset E \to E$ such that:
(1) $V_g(\phi(a)eb) = \phi(\alpha_g(a))V_g(e)\beta_g(b)$ and $(V_g e|V_g f)_B = \beta_g((e|f)_B)$ for all $g \in G$, $a \in A$, $e \in E$ and $b \in B$;
(2) $\phi(a) \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, and the graded commutator $[\mathcal{D}, \phi(a)]_\pm$ is bounded for all $a \in \mathcal{A}$;
(3) $\phi(a)(1 + \mathcal{D}^2)^{-1/2}$ is a compact endomorphism for all $a \in \mathcal{A}$;
(4) $V_g \cdot \mathrm{dom}(\mathcal{D}) \subset \mathrm{dom}(\mathcal{D})$, and $[\mathcal{D}, V_g] = 0$.

Remark. We normally suppress the notation ϕ. The unbounded Kasparov module $(\mathcal{A}, E_B, \mathcal{D})$ defines a class in the abelian group $KK_G(A, B)$, [2].

Remark. We will only employ unbounded equivariant Kasparov A-B-modules for which the action of G on B is trivial. Then for all $g \in G$, V_g is adjointable with adjoint $V_g^* = V_{g^{-1}}$.

Definition 2.2. Let A be a $\mathbb{Z}_2$-graded C^*-algebra with an action by a compact group G. An *even equivariant spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ for A is an unbounded equivariant Kasparov A-$\mathbb{C}$-module. If A is trivially $\mathbb{Z}_2$-graded, then one can also define an *odd equivariant spectral triple* $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, which has the same definition, except that $\mathcal{H}^1 = \{0\}$ and $\mathcal{D}$ need not be odd.

Throughout this section, G is a compact abelian Lie group, equipped with the normalised Haar measure, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded separable C^*-algebra A carrying an action α by G. (The case that the spectral triple is odd is considered later.)

There are some differences between the cases of G even dimensional and G odd dimensional. We introduce the following notation so that we may handle both cases simultaneously.

Definition 2.3. Let $\mathbb{C}\mathrm{l}_1$ be the Clifford algebra generated by a self-adjoint unitary c, which is $\mathbb{Z}_2$-graded by

$$\mathbb{C}\mathrm{l}_1^j = \operatorname{span}\{c^j\}, \quad j \in \mathbb{Z}_2.$$

We denote by $\mathfrak{C}$ the $\mathbb{Z}_2$-graded C^*-algebra

$$\mathfrak{C} = \begin{cases} \mathbb{C} & \text{if } G \text{ is even dimensional} \\ \mathbb{C}\mathrm{l}_1 & \text{if } G \text{ is odd dimensional.} \end{cases}$$

We also denote by $\mathfrak{c}$ the generator of $\mathfrak{C}$; i.e.

$$\mathfrak{c} = \begin{cases} 1 & \text{if } G \text{ is even dimensional} \\ c & \text{if } G \text{ is odd dimensional.} \end{cases}$$

We will construct three unbounded KK-cycles. The first cycle (referred to as the left-hand module), is constructed using the spin Dirac operator over G, and defines a class in $KK_G(A, A^G \widehat{\otimes} \mathfrak{C})$. The second cycle, which we call the middle module, represents a class in $KK_G(A^G \widehat{\otimes} \mathfrak{C}, A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG))$. The module is simply the Morita equivalence between $A^G \widehat{\otimes} \mathfrak{C}$ and $A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG) \cong A^G \widehat{\otimes} \mathbb{C}\mathrm{l}_n$, and so contains no homological information. The third cycle (the right-hand module) is constructed by restricting the spectral triple to a spectral subspace of $\mathcal{H}$, and adding a representation of $\mathbb{C}\mathrm{l}(T_eG)$, so that it defines a class in $KK_G(A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG), \mathbb{C})$.

2.1. The left-hand module. Let $\mathrm{Char}(G)$ be the characters of G, which is the set of smooth homomorphisms $\chi : G \to U(1)$. Since G is abelian, the characters form a group under multiplication. For each $\chi \in \mathrm{Char}(G)$, let

$$A_\chi = \{a \in A : \alpha_g(a) = \chi(g)a\}$$

be the *spectral subspace* of A associated with the character χ. Note that $\bigoplus_{\chi \in \mathrm{Char}(G)} A_\chi$ is dense in A. For each $\chi \in \mathrm{Char}(G)$, define $\Phi_\chi : A \to A$ by

$$\Phi_\chi(a) = \int_G \chi^{-1}(g)\alpha_g(a)\, dg.$$

Each Φ_χ is a continuous idempotent with range $\Phi_\chi = A_\chi$.

Definition 2.4. The action of G on A is said to satisfy the *spectral subspace assumption* (SSA) if the norm closure $\overline{A_\chi A_\chi^*}$ is a complemented ideal in the fixed point algebra A^G for each $\chi \in \mathrm{Char}(G)$.

Remark. A particular case of the spectral subspace assumption is if $\overline{A_\chi A_\chi^*} = A^G$ for all $\chi \in \mathrm{Char}(G)$. In this case we say that A has **full spectral subspaces**. This is equivalent to the action of G on A being **free** or **saturated**, [25, 29]. If $A = C_0(X)$ for a locally compact Hausdorff G-space X, then $C_0(X)$ has full spectral subspaces if and only if the action of G on X is free, [25, Proposition 7.1.12 and Theorem 7.2.6].

We define an A^G-valued inner product on A by

$$(a|b)_{A^G} := \Phi_1(a^*b) = \int_G \alpha_g(a^*b)\, dg.$$

With this inner product, A is a right pre-Hilbert A^G-module. Hence the completion of A with respect to $(\cdot|\cdot)_{A^G}$ is a right Hilbert A^G-module, which we denote by X. The $\mathbb{Z}_2$-grading of A defines a $\mathbb{Z}_2$-grading of X, which makes X into a $\mathbb{Z}_2$-graded right Hilbert A^G-module. The action of G on A extends to a unitary action $\alpha : G \to \mathrm{End}_{A^G}(X)$.

Remark. Let $\chi \in \mathrm{Char}(G)$, and let $a, b \in A_\chi$. Then $a^*b \in A^G$, so $(a|b)_{A^G} = a^*b$. Hence A_χ is closed in X, and so

$$X_\chi := \{x \in X : \alpha_g(x) = \chi(g)x\} = A_\chi.$$

The following is a more general version of [23, Lemma 4.2] or [6, Lemma 2.4]. The result there is for the case $G = \mathbb{T}$, but the proof is much the same as in the general case.

Lemma 2.5. *For each $\chi \in \mathrm{Char}(G)$, the map $\Phi_\chi : A \to A$ extends to an adjointable projection $\Phi_\chi : X \to X$ with range A_χ. Moreover,*

$$(x|y)_{A^G} = \sum_{\chi \in \mathrm{Char}(G)} \Phi_\chi(x)^* \Phi_\chi(y)$$

for all $x, y \in X$, and the sum $\sum_{\chi \in \mathrm{Char}(G)} \Phi_\chi$ converges strictly to the identity on X.

Let $\$_G$ be the trivial flat complex spinor bundle over G, with Dirac operator $\mathcal{D}_G$. The left multiplication of G on itself lifts to a strongly continuous unitary representation V on $L^2(\$_G)$ which makes $(C^\infty(G), L^2(\$_G), \mathcal{D}_G)$ into a G-equivariant spectral triple, which is even if and only if $\dim G$ is even, [30]. Then $(C^\infty(G), (L^2(\$_G)\widehat{\otimes}\mathfrak{C})_{\mathfrak{C}}, \mathcal{D}_G\widehat{\otimes}\mathfrak{c})$ is a G-equivariant unbounded Kasparov $C(G)$-$\mathfrak{C}$-module for G either even or odd dimensional.

Definition 2.6. Let $X\widehat{\otimes}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ be the external tensor product of X and $L^2(\$_G)\widehat{\otimes}\mathfrak{C}$, which is a $\mathbb{Z}_2$-graded right Hilbert $A^G\widehat{\otimes}\mathfrak{C}$-module. Let E_1 be the invariant submodule of $X\widehat{\otimes}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ under the diagonal action $g\cdot(x\widehat{\otimes}(s\widehat{\otimes}z)) = \alpha_g(x)\widehat{\otimes}(V_g s\widehat{\otimes}z)$. Let V_1 be the homomorphism from G into the unitaries of E_1 defined by

$$V_{1,g}(x\widehat{\otimes}(s\widehat{\otimes}z)) = \alpha_g(x)\widehat{\otimes}(s\widehat{\otimes}z).$$

For each $\chi \in \mathrm{Char}(G)$, let $p'_\chi \in B(L^2(\$_G))$ be the orthogonal projection onto

$$L^2(\$_G)_\chi = \{s \in L^2(\$_G) : V_g(s) = \chi(g)s\},$$

and define $p_\chi \in \mathrm{End}_{\mathfrak{C}}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ by $p_\chi(s\widehat{\otimes}z) = p'_\chi s\widehat{\otimes}z$.

The following result is elementary, but will be quite useful in later calculations.

Lemma 2.7. *For elements of homogeneous degree, the $A^G\widehat{\otimes}\mathfrak{C}$-valued inner product on E_1 can be expressed (for x_1, $x_2 \in X$ and s_1, $s_2 \in L^2(\$_G)\widehat{\otimes}\mathfrak{C}$) as*

$$\begin{aligned}(x_1\widehat{\otimes}s_1|x_2\widehat{\otimes}s_2)_{A^G\widehat{\otimes}\mathfrak{C}} =& (-1)^{\deg s_1\cdot(\deg x_1+\deg x_2)}\times \\ &\times \sum_{\chi\in\mathrm{Char}(G)} \Phi_\chi(x_1)^*\Phi_\chi(x_2)\widehat{\otimes}(p_{\chi^{-1}}s_1|p_{\chi^{-1}}s_2)_{\mathfrak{C}}.\end{aligned}$$

Proposition 2.8. *Define an action of $\bigoplus_{\chi\in\mathrm{Char}(G)} A_\chi$ on E_1 by*

$$\sum_{\chi\in\mathrm{Char}(G)} a_\chi\cdot(x\widehat{\otimes}s) := \sum_{\chi\in\mathrm{Char}(G)} a_\chi x\widehat{\otimes}\chi s,$$
$$\text{for } \sum a_\chi \in \bigoplus_{\chi\in\mathrm{Char}(G)} A_\chi, \quad x\widehat{\otimes}s \in E_1.$$

This action extends to a $\mathbb{Z}_2$-graded $*$-homomorphism $\phi : A \to \mathrm{End}_{A^G\widehat{\otimes}\mathfrak{C}}(E_1)$ satisfying

$$V_{1,g}(\phi(a)e) = \phi(\alpha_g(a))V_{1,g}(e), \qquad a \in A,\ e \in E_1.$$

Proof. Suppose $a_\chi \in A_\chi$ and $x = \sum_{\nu\in\mathrm{Char}(G)} x_\nu \in X$, where $x_\nu \in A_\nu$ for all $\nu \in \mathrm{Char}(G)$. Then

$$\|a_\chi x\|^2 = \sum_{\phi\in\mathrm{Char}(G)} \|a_\chi x_\nu\|^2 \leq \|a_\chi\|^2\|x\|^2$$

by Lemma 2.5, so $a_\chi x$ is a well-defined element of x.

Since $\alpha_g(a_\chi^*) = \alpha_g(a_\chi)^* = \overline{\chi(g)}a_\chi^* = \chi^{-1}(g)a_\chi^*$, it follows that $a_\chi^* \in A_{\chi^{-1}}$. Hence if $a_\chi \in A_\chi$ and $x_i\widehat{\otimes}s_i \in E_1$, $i = 1, 2$, each of homogeneous degree, then

$$\begin{aligned}
&\big(x_1\widehat{\otimes}s_1\big|a_\chi \cdot (x_2\widehat{\otimes}s_2)\big)_{A^G\widehat{\otimes}\mathfrak{C}} \\
&\quad= \big(x_1\widehat{\otimes}s_1\big|a_\chi x_2\widehat{\otimes}\chi s_2\big)_{A^G\widehat{\otimes}\mathfrak{C}} \\
&\quad= (-1)^{\deg s_1\cdot(\deg x_1+\deg a_\chi+\deg x_2)}(x_1|a_\chi x_2)_{A^G}\widehat{\otimes}(s_1|\chi s_2)_{\mathfrak{C}} \\
&\quad= (-1)^{\deg s_1\cdot(\deg x_1+\deg a_\chi+\deg x_2)}(a_\chi^* x_1|x_2)_{A^G}\widehat{\otimes}(\chi^{-1}s_1|s_2)_{\mathfrak{C}} \\
&\quad= \big(a_\chi^* x_1\widehat{\otimes}\chi^{-1}s_1\big|x_2\widehat{\otimes}s_2\big)_{A^G\widehat{\otimes}\mathfrak{C}} = \big(a_\chi^*\cdot(x_1\widehat{\otimes}s_1)\big|x_2\widehat{\otimes}s_2\big)_{A^G\widehat{\otimes}\mathfrak{C}}.
\end{aligned}$$

So the action of $\bigoplus_\chi A_\chi$ on E_1 defines a $*$-homomorphism on the direct sum $\bigoplus_\chi A_\chi \to \mathrm{End}_{A^G\widehat{\otimes}\mathfrak{C}}(E_1)$, which extends to a $*$-homomorphism $\phi : A \to \mathrm{End}_{A^G\widehat{\otimes}\mathfrak{C}}(E_1)$. That ϕ is $\mathbb{Z}_2$-graded and equivariant is obvious. □

Definition 2.9. Let $\mathcal{D}_G : \mathrm{dom}(\mathcal{D}_G) \subset L^2(\$_G) \to L^2(\$_G)$ be the spin Dirac operator on G, and let $\mathfrak{c}$ be the generator of $\mathfrak{C}$. Define a closed operator $\mathcal{D}_1 : \mathrm{dom}(\mathcal{D}_1) \subset E_1 \to E_1$ initially on the linear span of elements of the form $x\widehat{\otimes}(s\widehat{\otimes}z)$, where $x \in X$, $s \in \mathrm{dom}(\mathcal{D}_G)$ and $z \in \mathfrak{C}$ are of homogeneous degree, by

$$\mathcal{D}_1(x\widehat{\otimes}(s\widehat{\otimes}z)) := (-1)^{\deg x}x\widehat{\otimes}(\mathcal{D}_G s\widehat{\otimes}\mathfrak{c}z),$$

and then take the operator closure. Since $\mathcal{D}_G$ is equivariant, this is well-defined.

Proposition 2.10. *The triple $(\oplus_\chi A_\chi, (E_1)_{A^G\widehat{\otimes}\mathfrak{C}}, \mathcal{D}_1)$ is an unbounded equivariant Kasparov A-$A^G\widehat{\otimes}\mathfrak{C}$-module if and only if the action of G on A satisfies the spectral subspace assumption. When the action of G on A satisfies the spectral subspace condition, we call the Kasparov module $(\oplus_\chi A_\chi, (E_1)_{A^G\widehat{\otimes}\mathfrak{C}}, \mathcal{D}_1)$ the* left-hand module*.*

Proof. See [6, Proposition 2.9] and the preceding lemmas for a proof when $G = \mathbb{T}$. The general case requires only minor modifications, as in [5, Chapter 5]. □

We henceforth assume that the action of G on A satisfies the spectral subspace assumption.

2.2. The middle module.

Recall that G is a compact abelian Lie group, equipped with the trivial spinor bundle $\$_G$, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded separable C^*-algebra A. We will now construct the middle module, whose job is to correct for the spinor bundle dimensions between the left hand module and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

Let $W := (\$_G)_e$, and let $\rho : \mathrm{Cl}(T_eG) \to B(W)$ be the Clifford representation, which is a $*$-homomorphism. When G is even dimensional, ρ

is a $\mathbb{Z}_2$-graded $*$-homomorphism, but this is not the case when G is odd dimensional. Note that we have the $*$-isomorphisms $\mathbb{C}\mathrm{l}(T_eG) \cong \mathbb{C}\mathrm{l}_n$ and

$$\mathbb{C}\mathrm{l}_n \cong \begin{cases} M_{2^{\dim G/2}}(\mathbb{C}) & \text{if } G \text{ is even dimensional} \\ M_{2^{(\dim G-1)/2}}(\mathbb{C}) \oplus M_{2^{(\dim G-1)/2}}(\mathbb{C}) & \text{if } G \text{ is odd dimensional.} \end{cases}$$

On the other hand,

$$W = (\$_G)_e) \cong \begin{cases} \mathbb{C}^{2^{\dim G/2}} & \text{if } G \text{ is even dimensional} \\ \mathbb{C}^{2^{(\dim G-1)/2}} & \text{if } G \text{ is odd dimensional.} \end{cases}$$

Let $\mathfrak{c}$ be the generator of the C^*-algebra $\mathfrak{C}$, as in Definition 2.3. The $\mathbb{Z}_2$-graded $*$-homomorphism $\widetilde{\rho} : \mathbb{C}\mathrm{l}(T_eG) \to \mathrm{End}_{\mathfrak{C}}(W \widehat{\otimes} \mathfrak{C})$ defined on elements of homogeneous degree by

$$\widetilde{\rho}(s)(w \widehat{\otimes} z) = \rho(s) w \widehat{\otimes} \mathfrak{c}^{\deg s} z, \tag{1}$$

is an isomorphism.

The isomorphism (1) implies that $W \widehat{\otimes} \mathfrak{C}$ is a $\mathbb{Z}_2$-graded Morita equivalence bimodule between $\mathbb{C}\mathrm{l}(T_eG)$ and $\mathfrak{C}$, where the left inner product is defined by

$$\widetilde{\rho}\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w_1|w_2)\big) w_3 = w_1 (w_2|w_3)_{\mathfrak{C}}.$$

Hence the conjugate module $(W \widehat{\otimes} \mathfrak{C})^*$, [26, p. 49] is a $\mathbb{Z}_2$-graded Morita equivalence bimodule between $\mathfrak{C}$ and $\mathbb{C}\mathrm{l}(T_eG)$.

The fixed point algebra A^G is a $\mathbb{Z}_2$-graded right Hilbert module over itself, and left multiplication on itself defines a $\mathbb{Z}_2$-graded $*$-homomorphism $A^G \to \mathrm{End}_{A^G}(A^G)$.

The external tensor product $A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*$ is a $\mathbb{Z}_2$-graded right Hilbert $A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)$-module, which carries a $\mathbb{Z}_2$-graded representation $A^G \widehat{\otimes} \mathfrak{C} \to \mathrm{End}_{A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)}(A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)$. Since $A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*$ is a Morita equivalence bimodule, the triple

$$(A^G \widehat{\otimes} \mathfrak{C}, (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)_{A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)}, 0)$$

is an (unbounded) equivariant Kasparov $A^G \widehat{\otimes} \mathfrak{C}$-$A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)$-module. The C^*-algebras and the Hilbert module carry the trivial action by G. We call this module the *middle module*.

2.3. The right-hand module. To define the right-hand module we require greater compatibility between the action α of G on A and $\mathcal{A} \subset A$ than we have assumed so far. We say that $\mathcal{A}$ is *α-compatible* if

$$\mathcal{A}_\chi := \mathcal{A} \cap A_\chi \text{ is dense in } A_\chi \text{ for all } \chi \in \mathrm{Char}(G).$$

Compatibility is implied by α restricting to a continuous action on $\mathcal{A}$ for some finer complete topology on $\mathcal{A}$.

Definition 2.11. For each $\chi \in \mathrm{Char}(G)$, let $\mathcal{H}_\chi = \{\xi \in \mathcal{H} : V_g\xi = \chi(g)\xi\}$ be the spectral subspace corresponding to χ, and define an operator $\mathcal{D}_\chi : \mathrm{dom}(\mathcal{D}) \cap \mathcal{H}_\chi \subset \mathcal{H}_\chi \to \mathcal{H}_\chi$ by $\mathcal{D}_\chi\xi := \mathcal{D}\xi$. The Hilbert space $\mathcal{H}_\chi$ inherits the $\mathbb{Z}_2$-grading of $\mathcal{H}$.

LEMMA 2.12. *Suppose that $\mathcal{A}$ is α-compatible. Let $\mathcal{A}^G$ be the fixed point algebra of $\mathcal{A}$. Then for each $\chi \in \mathrm{Char}(G)$, $(\mathcal{A}^G, \mathcal{H}_\chi, \mathcal{D}_\chi)$ is an even equivariant spectral triple for A^G, where $\mathcal{H}_\chi$ inherits the action of G on $\mathcal{H}$.*

PROOF. Since G acts on $\mathcal{H}$ unitarily, there is an orthogonal decomposition $\mathcal{H} = \bigoplus_{\chi\in\mathrm{Char}(G)} \mathcal{H}_\chi$. The density of $\mathrm{dom}(\mathcal{D})$ in $\mathcal{H}$ thus implies that $\mathrm{dom}(\mathcal{D}_\chi)$ is dense in $\mathcal{H}_\chi$ for all $\chi \in \mathrm{Char}(G)$.

The operator $(1+\mathcal{D}^2)^{-1/2} \in B(\mathcal{H})$ is self-adjoint, and since $\mathcal{D}$ commutes with the action of G, so too does $(1+\mathcal{D}^2)^{-1/2}$. Hence $(1+\mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi}$ is a bounded self-adjoint operator on $\mathcal{H}_\chi$, and $(1+\mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi} = (1+\mathcal{D}_\chi^2)^{-1/2}$ for all $\chi \in \mathrm{Char}(G)$. Hence

$$F_\chi := \mathcal{D}(1+\mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi} = \mathcal{D}_\chi(1+\mathcal{D}_\chi^2)^{-1/2}$$

is also a bounded self-adjoint operator on $\mathcal{H}_\chi$. Since $\mathcal{D}_\chi = F_\chi(1-F_\chi^2)^{-1/2}$, it follows from [18, Theorem 10.4] that $\mathcal{D}_\chi$ is a self-adjoint operator on $\mathcal{H}_\chi$.

Since $[\mathcal{D}_\chi, a] = [\mathcal{D}, a]|_{\mathcal{H}_\chi}$ and $a(1+\mathcal{D}_\chi^2)^{-1/2} = a(1+\mathcal{D}^2)^{-1/2}|_{\mathcal{H}_\chi}$ for all $a \in \mathcal{A}^G$, it follows that $(\mathcal{A}^G, \mathcal{H}_\chi, \mathcal{D}_\chi)$ satisfies the conditions of Definition 2.1, and hence $(\mathcal{A}^G, \mathcal{H}_\chi, \mathcal{D}_\chi)$ is an even equivariant spectral triple. □

We wish to use the operator $\mathcal{D}_\zeta$ to construct our final Kasparov module, for some fixed $\zeta \in \mathrm{Char}(G)$. However, the middle module is an unbounded Kasparov A^G-$A^G\hat{\otimes}\mathbb{C}\mathrm{l}(T_eG)$-module, whereas $(\mathcal{A}^G, \mathcal{H}_\zeta, \mathcal{D}_\zeta)$ is an unbounded Kasparov A^G-$\mathbb{C}$-module. Hence we need a representation of $\mathbb{C}\mathrm{l}(T_eG)$ on $\mathcal{H}_\zeta$, which will define an action of $A^G\hat{\otimes}\mathbb{C}\mathrm{l}(T_eG)$ on $\mathcal{H}_\zeta$. The conditions we impose below on the action and the character ζ ensure that we obtain an even spectral triple for $A^G\hat{\otimes}\mathbb{C}\mathrm{l}(T_eG)$, and in addition that Kucerovsky's connection criteria is satisfied (Proposition 3.5).

Simple examples show that $\mathcal{H}_\chi$ may be trivial for any given $\chi \in \mathrm{Char}(G)$, including the trivial character $\chi(g) = 1$. We therefore impose the condition $\overline{A\mathcal{H}_\zeta} = \mathcal{H}$ on the character ζ in order to construct the right-hand module. Choosing ζ in this way allows us to recover the original Hilbert space $\mathcal{H}$ from the three modules.

REMARK. Even if $\overline{A\mathcal{H}_\chi} = \mathcal{H}$ for all $\chi \in \mathrm{Char}(G)$, the positivity criterion may be satisfied for some choices of ζ but not for others. For an example see Section 8.

Definition 2.13. Suppose that $\mathcal{A}$ is α-compatible. Let $\zeta \in \mathrm{Char}(G)$ be such that $\overline{A\mathcal{H}_\zeta} = \mathcal{H}$, and let $\eta : \mathbb{C}\mathrm{l}(T_eG) \to B(\mathcal{H})$ be a unital, equivariant

$\mathbb{Z}_2$-graded $*$-homomorphism such that
1) $[\eta(s), a]_\pm = 0$ for all $s \in \mathbb{C}\mathrm{l}(T_eG)$ and $a \in A^G$, and
2) $a\eta(s) \cdot \mathrm{dom}(\mathcal{D}_\zeta) \subset \mathrm{dom}(\mathcal{D})$ and $[\mathcal{D}, \eta(s)]_\pm aP_\zeta$ is bounded on $\mathcal{H}$ for all $a \in \oplus_\chi \mathcal{A}_\chi$ and
$s \in \mathbb{C}\mathrm{l}(T_eG)$, where $P_\zeta \in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_\zeta$.

We call η the *Clifford representation* when it exists.

We define a $\mathbb{Z}_2$-graded $*$-homomorphism $A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG) \to B(\mathcal{H}_\zeta)$ by $(a \widehat{\otimes} s) \cdot \xi := a\eta(s)\xi$. If $\mathcal{A}$ is α-compatible, the conditions on η and Lemma 2.12 ensure that $(\mathcal{A}^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG), \mathcal{H}_\zeta, \mathcal{D}_\zeta)$ is an even equivariant spectral triple for A^G, which we call the *right-hand module.*

Remark. Condition 2) of Definition 2.13 is stronger than necessary to ensure that we obtain an equivariant spectral triple for $A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)$, but this stronger condition is sufficient to prove that Kucerovsky's connection criteria is satisfied.

Remark. The conditions of Definition 2.13 are quite restrictive. For instance, if the group acts trivially on both the algebra and Hilbert space of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, then Definition 2.13 requires that we have a spectral triple $(\mathcal{A} \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG), \mathcal{H}, \mathcal{D})$, whence the class of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is zero. So, as an example, the class of $(\mathbb{C}, \mathbb{C}_\mathbb{C}, 0)$ in $KK^{\mathbb{T}}(\mathbb{C}, \mathbb{C})$ does not satisfy the conditions of Definition 2.13.

3. The Kasparov product of the left-hand, middle and right-hand modules.

Recall that G is a compact abelian Lie group, equipped with the normalised Haar measure and a trivial spinor bundle $\$_G$, and $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ is an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded separable C^*-algebra A. Let $\zeta \in \mathrm{Char}(G)$ and $\eta : \mathbb{C}\mathrm{l}(T_eG) \to B(\mathcal{H})$ satisfy the conditions of Definition 2.13, so in particular $\mathcal{A}$ is α-compatible.

The next result can be proved with a straightforward application of Kucerovsky's criteria, [16, Theorem 13].

Proposition 3.1. *The product of the left-hand and middle modules is represented by* $(\oplus_\chi A_\chi, (E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*))_{A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)}, \mathcal{D}_1 \widehat{\otimes} 1)$.

To determine whether the Kasparov product of the left-hand, middle and right-hand modules is represented by $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, we first construct an isomorphism

$$\Psi : (E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)) \widehat{\otimes}_{A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_eG)} \mathcal{H}_\zeta \to \mathcal{H},$$

which will allow us to use Kucerovsky's criteria, [16, Theorem 13]. We would like to define the map Ψ on elements of homogeneous degree by

$$\Psi\Big(\big((y\widehat{\otimes}u)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi\Big) := (-1)^{\deg u\cdot\deg a}\sum_{\chi\in\mathrm{Char}(G)}\Phi_\chi(y)a\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}u|w)\big)\xi, \quad (2)$$

where $p_\chi\in\mathrm{End}_{\mathfrak{C}}(L^2(\$_G)\widehat{\otimes}\mathfrak{C})$ and $\Phi_\chi\in\mathrm{End}_{A^G}(X)$ are the spectral subspace projections of Definition 2.6 and Lemma 2.5 respectively.

To see that Ψ is well-defined, even on homogeneous elements, we need to know that the sum over characters converges. This is established by the following lemma.

Lemma 3.2. *For $i=1,2$ let $\big((y_i\widehat{\otimes}u_i)\widehat{\otimes}(a_i\widehat{\otimes}\overline{w_i})\big)\widehat{\otimes}\xi_i$ be an element of $(E_1\widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}}(A^G\widehat{\otimes}(W\widehat{\otimes}\mathfrak{C})^*))\widehat{\otimes}_{A^G\widehat{\otimes}\mathbb{C}\mathrm{l}(T_eG)}\mathcal{H}_\zeta$. Then*

$$\Big\langle\Psi\Big(\big((y_1\widehat{\otimes}u_1)\widehat{\otimes}(a_1\widehat{\otimes}\overline{w_1})\big)\widehat{\otimes}\xi_1\Big),\Psi\Big(\big((y_2\widehat{\otimes}u_2)\widehat{\otimes}(a_2\widehat{\otimes}\overline{w_2})\big)\widehat{\otimes}\xi_2\Big)\Big\rangle = \big\langle\big((y_1\widehat{\otimes}u_1)\widehat{\otimes}(a_1\widehat{\otimes}\overline{w_1})\big)\widehat{\otimes}\xi_1,\big((y_2\widehat{\otimes}u_2)\widehat{\otimes}(a_2\widehat{\otimes}\overline{w_2})\big)\widehat{\otimes}\xi_2\big\rangle$$

and hence Ψ is a well-defined isometry.

Proof. Suppose that both elements are of homogeneous degree. Then using Lemma 2.7,

$$\begin{aligned}
&\big\langle\big((y_1\widehat{\otimes}u_1)\widehat{\otimes}(a_1\widehat{\otimes}\overline{w_1})\big)\widehat{\otimes}\xi_1,\big((y_2\widehat{\otimes}u_2)\widehat{\otimes}(a_2\widehat{\otimes}\overline{w_2})\big)\widehat{\otimes}\xi_2\big\rangle\\
&=(-1)^{\deg u_1\cdot(\deg y_1+\deg y_2)+(\deg u_1+\deg u_2)\cdot\deg a_2+\deg w_1\cdot(\deg a_1+\deg y_1+\deg y_2+\deg a_2)}\\
&\quad\times\sum_{\chi\in\mathrm{Char}(G)}\big\langle\xi_1,a_1^*\Phi_\chi(y_1)^*\Phi_\chi(y_2)a_2\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w_1|w_2(p_{\chi^{-1}}u_2|p_{\chi^{-1}}u_1)_{\mathfrak{C}})\big)\xi_2\big\rangle\\
&=(-1)^{\deg u_1\cdot\deg a_1+\deg u_2\cdot\deg a_2}\sum_{\chi\in\mathrm{Char}(G)}\\
&\big\langle\Phi_\chi(y_1)a_1\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}u_1|w_1)\big)\xi_1,\Phi_\chi(y_2)a_2\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}u_2|w_2)\big)\xi_2\big\rangle\\
&=\Big\langle\Psi\Big(\big((y_1\widehat{\otimes}u_1)\widehat{\otimes}(a_1\widehat{\otimes}\overline{w_1})\big)\widehat{\otimes}\xi_1\Big),\Psi\Big(\big((y_2\widehat{\otimes}u_2)\widehat{\otimes}(a_2\widehat{\otimes}\overline{w_2})\big)\widehat{\otimes}\xi_2\Big)\Big\rangle.
\end{aligned}$$

The penultimate line follows from

$$ {}_{\mathbb{C}\mathrm{l}(T_eG)}(w_1|\chi^{-1}p_{\chi^{-1}}u_1)_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}u_2|w_2) = {}_{\mathbb{C}\mathrm{l}(T_eG)}(w_1|w_2(p_{\chi^{-1}}u_2|p_{\chi^{-1}}u_1)_{\mathfrak{C}}), \quad (3)$$

which in turn follows from $(\chi^{-1}p_{\chi^{-1}}u_2|\chi^{-1}p_{\chi^{-1}}u_1)_{\mathfrak{C}}=(p_{\chi^{-1}}u_2|p_{\chi^{-1}}u_1)_{\mathfrak{C}}$.

We have already established that the sum over characters

$$\sum_{\chi\in\mathrm{Char}(G)}\Phi_\chi(y)a\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}u|w)\big)\xi$$

converges. It only remains to check that Ψ is well-defined with respect to the balanced tensor products, which is a straightforward exercise. $\square$

PROPOSITION 3.3. *The map Ψ is a unitary, equivariant, $\mathbb{Z}_2$-graded, A-linear isomorphism. The inverse*

$$\Psi^{-1} : \mathcal{H} \to (E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)) \widehat{\otimes}_{A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_e G)} \mathcal{H}_\zeta$$

is defined as follows. Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\$_G$, and let $(\phi_\ell)_{\ell=1}^\infty$ be an approximate identity for A^G of homogeneous degree zero. For $\xi \in \mathcal{H}$, choose sequences $(a_k)_{k=1}^\infty \subset A$ and $(\xi_k)_{k=1}^\infty \subset \mathcal{H}_\zeta$ such that $a_k \xi_k \to \xi$ as $k \to \infty$. Then

$$\Psi^{-1}(\xi) := \sum_{\chi \in \mathrm{Char}(G)} \sum_{j=1}^n \lim_{k\to\infty} \lim_{\ell\to\infty} \big((\Phi_\chi(a_k) \widehat{\otimes} (\chi x_j \widehat{\otimes} 1)) \widehat{\otimes} (\phi_\ell \widehat{\otimes} \overline{x_j \widehat{\otimes} 1}) \big) \widehat{\otimes} \xi_k.$$

PROOF. It is immediate that Ψ is equivariant and $\mathbb{Z}_2$-graded, and Ψ is an isometry by Lemma 3.2. So it remains to show that (i) Ψ is A-linear, and (ii) Ψ^{-1} is an inverse for Ψ.

(i) Let $b \in A$. Then

$$\begin{aligned}
\Psi\Big(b \cdot \big((y \widehat{\otimes} u) \widehat{\otimes} (a \widehat{\otimes} \overline{w})\big) \widehat{\otimes} \xi\Big) &= \sum_{\mu \in \mathrm{Char}(G)} \Psi\Big(\big((\Phi_\mu(b) y \widehat{\otimes} \mu u) \widehat{\otimes} (a \widehat{\otimes} \overline{w})\big) \widehat{\otimes} \xi\Big) \\
&= (-1)^{\deg u \cdot \deg a} \sum_{\chi, \mu \in \mathrm{Char}(G)} \Phi_\chi(\Phi_\mu(b) y) a \eta\big({}_{\mathbb{C}\mathrm{l}(T_e G)}(\chi^{-1} p_{\chi^{-1}} \mu u | w)\big) \xi \\
&= (-1)^{\deg u \cdot \deg a} \sum_{\chi, \mu} \Phi_\mu(b) \Phi_\chi(y) a \eta\big({}_{\mathbb{C}\mathrm{l}(T_e G)}(\chi^{-1} p_{\chi^{-1}} u | w)\big) \xi \\
&= b \Psi\Big(\big((y \widehat{\otimes} u) \widehat{\otimes} (a \widehat{\otimes} \overline{w})\big) \widehat{\otimes} \xi\Big),
\end{aligned}$$

so Ψ is A-linear.

(ii) We first check that Ψ^{-1} is well-defined, which means checking that the limits exist and that the sum converges. Suppose $\xi \in \mathcal{H}$, and choose sequences $(a_k)_{k=1}^\infty \subset A$ and $(\xi_k)_{k=1}^\infty \subset \mathcal{H}_\zeta$ such that $a_k \xi_k \to \xi$ as $k \to \infty$, which exist since $\overline{A \mathcal{H}_\zeta} = \mathcal{H}$. Since $\sum_{j=1}^n {}_{\mathbb{C}\mathrm{l}(T_e G)}(x_j \widehat{\otimes} 1 | x_j \widehat{\otimes} 1) = 1$,

$$\begin{aligned}
&\Psi\left(\sum_{j=1}^n \big((\Phi_\chi(a_k) \widehat{\otimes} (\chi x_j \widehat{\otimes} 1)) \widehat{\otimes} (\phi_\ell \widehat{\otimes} \overline{x_j \widehat{\otimes} 1}) \big) \widehat{\otimes} \xi_k \right) \\
&= \sum_{j=1}^n \Phi_\chi(a_k) \phi_\ell \eta\big({}_{\mathbb{C}\mathrm{l}(T_e G)}(x_j \widehat{\otimes} 1 | x_j \widehat{\otimes} 1)\big) \xi_k \\
&= \Phi_\chi(a_k) \phi_\ell \xi_k = P_{\chi\zeta}(a_k \phi_\ell \xi_k),
\end{aligned}$$

where $P_{\chi\zeta} \in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_{\chi\zeta}$, and

$$\lim_{k\to\infty}\lim_{\ell\to\infty} P_{\chi\zeta}(a_k\phi_\ell\xi_k) = \lim_{k\to\infty} P_{\chi\zeta}(a_k\xi_k) = P_{\chi\zeta}\xi.$$

Since Ψ is an isometry, this establishes that the limits exist. Moreover,

$$\sum_{\chi\in\mathrm{Char}(G)} P_{\chi\zeta}\xi = \sum_{\chi\in\mathrm{Char}(G)} P_{\chi}\xi = \xi,$$

so the sum converges. This calculation also shows that Ψ^{-1} is a right-inverse for Ψ, so that Ψ is surjective. Since Ψ is injective, it follows that Ψ is invertible with inverse Ψ^{-1}. □

Now that we have the isomorphism Ψ, we can use Kucerovsky's criteria, [16, Theorem 13], to determine if $(\mathcal{A},\mathcal{H},\mathcal{D})$ represents the Kasparov product of the left-hand, middle and right-hand modules. More precisely, $(\mathcal{A},\mathcal{H},\mathcal{D})$ is unitarily equivalent as an unbounded equivariant Kasparov module to $(\mathcal{A},(E_1\widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}}(A^G\widehat{\otimes}(W\widehat{\otimes}\mathfrak{C})^*))\widehat{\otimes}_{A^G\widehat{\otimes}\mathbb{C}\mathrm{l}(T_eG)}\mathcal{H}_\zeta, \Psi^{-1}\circ\mathcal{D}\circ\Psi)$, and Kucerovsky's criteria may now be applied to determine whether factorisation has been achieved.

Theorem 3.4 (The criterion for factorisation). *Let $\zeta\in\mathrm{Char}(G)$ and $\eta : \mathbb{C}\mathrm{l}(T_eG)\to B(\mathcal{H})$ satisfy the conditions of Definition 2.13, so in particular $\mathcal{A}$ is α-compatible. Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\$_G$, and for each $\chi\in\mathrm{Char}(G)$, let $P_\chi\in B(\mathcal{H})$ be the orthogonal projection onto $\mathcal{H}_\chi$. If there is some $R\in\mathbb{R}$ such that*

$$\sum_{j=1}^n \Big(\big\langle \mathcal{D}\xi, \eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi\big\rangle \\ + \big\langle \eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi, \mathcal{D}\xi\big\rangle \Big) \geq R\|\xi\|^2 \qquad (4)$$

for all $\chi\in\mathrm{Char}(G)$, $\xi\in\mathrm{dom}(\mathcal{D})$, then $(\mathcal{A},\mathcal{H},\mathcal{D})$ represents the Kasparov product of left-hand, middle and right-hand modules.

Remark. Although [16, Theorem 13] is stated for the non-equivariant case, it requires no modification in the equivariant case, [17].

Theorem 3.4 is proved by showing that Kucerovsky's domain and connection conditions hold under the existing assumptions. The remaining positivity condition is precisely condition (4).

Condition (4) is essentially about whether the part of the operator $\mathcal{D}$ which acts in the direction of the group G is (more-or-less) proportional to the operator $\mathcal{D}_G$. The "more-or-less" is quantified by condition (4), as is the fact that the remaining part of the operator $\mathcal{D}$ should (more-or-less) anticommute with $\mathcal{D}_G$.

In practice this condition is checkable once we have identified the Clifford representation, as all the operators are given and computable. In the final two sections we will see that this condition is indeed checkable when we apply it in examples.

Proposition 3.5 (The connection criterion). *For each*

$$e \in E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*),$$

let $T_e : \mathcal{H}_\zeta \to (E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)) \widehat{\otimes}_{A^G \widehat{\otimes} \mathbb{C}\mathrm{l}(T_e G)} \mathcal{H}_\zeta$ *be the creation operator. The graded commutators*

$$\left[\begin{pmatrix} \Psi^{-1} \circ \mathcal{D} \circ \Psi & 0 \\ 0 & \mathcal{D}_\zeta \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix} \right]_\pm$$

are bounded for all $e \in Y$, *where* $Y \subset E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)$ *is the dense subspace*

$$Y := \operatorname{span}\{(z \widehat{\otimes} s) \widehat{\otimes} (a \widehat{\otimes} \overline{w}) \in E_1 \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} (A^G \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*) : z \in \oplus_\chi \mathcal{A}_\chi,\ a \in \mathcal{A}^G\}.$$

Proof. Consider vectors $e = (z \widehat{\otimes} s) \widehat{\otimes} (a \widehat{\otimes} \overline{w}) \in Y$, $\psi \in \operatorname{dom}(\mathcal{D}_\zeta)$, and $\big((y \widehat{\otimes} t) \widehat{\otimes} (b \widehat{\otimes} \overline{v})\big) \widehat{\otimes} \xi \in \operatorname{dom}(\Psi^{-1} \circ \mathcal{D} \circ \Psi)$, each of homogeneous degree. Then the upper entry of the column vector

$$\left[\begin{pmatrix} \Psi^{-1} \circ \mathcal{D} \circ \Psi & 0 \\ 0 & \mathcal{D}_\zeta \end{pmatrix}, \begin{pmatrix} 0 & T_e \\ T_e^* & 0 \end{pmatrix} \right]_\pm \begin{pmatrix} \big((y \widehat{\otimes} t) \widehat{\otimes} (b \widehat{\otimes} \overline{v})\big) \widehat{\otimes} \xi \\ \psi \end{pmatrix}$$

is

$$\begin{aligned}
&\Psi^{-1} \circ \mathcal{D} \circ \Psi \circ T_e \psi - (-1)^{\deg z + \deg s + \deg a + \deg w} T_e \circ \mathcal{D}_\zeta \psi \\
&= \Psi^{-1} \circ \mathcal{D} \circ \Psi \Big(\big((z \widehat{\otimes} s) \widehat{\otimes} (a \widehat{\otimes} \overline{w})\big) \widehat{\otimes} \psi \Big) \\
&- (-1)^{\deg z + \deg s + \deg a + \deg w} \big((z \widehat{\otimes} s) \widehat{\otimes} (a \widehat{\otimes} \overline{w})\big) \widehat{\otimes} \mathcal{D}_\zeta \psi \\
&= (-1)^{\deg s \cdot \deg a} \Psi^{-1} \circ \mathcal{D} \sum_{\chi \in \mathrm{Char}(G)} \Phi_\chi(z) a \eta \big({}_{\mathbb{C}\mathrm{l}(T_e G)} (\chi^{-1} p_{\chi^{-1}} s | w) \big) \psi \\
&- (-1)^{\deg z + \deg s + \deg a + \deg w + \deg s \cdot \deg a} \times \\
&\qquad \times \Psi^{-1} \sum_{\chi \in \mathrm{Char}(G)} \Phi_\chi(y) a \eta \big({}_{\mathbb{C}\mathrm{l}(T_e G)} (\chi^{-1} p_{\chi^{-1}} s | w) \big) \mathcal{D}_\zeta \psi \\
&= (-1)^{\deg s \cdot \deg a} \Psi^{-1} \sum_{\chi \in \mathrm{Char}(G)} [\mathcal{D}, \Phi_\chi(z) a \eta \big({}_{\mathbb{C}\mathrm{l}(T_e G)} (\chi^{-1} p_{\chi^{-1}} s | w) \big)]_\pm \psi,
\end{aligned}$$

and we estimate

$$\left\|(-1)^{\deg s\cdot\deg a}\Psi^{-1}\sum_{\chi\in\mathrm{Char}(G)}[\mathcal{D},\Phi_\chi(z)a\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm\psi\right\|^2$$
$$=\sum_{\chi\in\mathrm{Char}(G)}\big\|[\mathcal{D},\Phi_\chi(z)a\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm P_\zeta\psi\big\|^2$$
$$\leq\|\psi\|^2\sum_{\chi\in\mathrm{Char}(G)}\big\|[\mathcal{D},\Phi_\chi(z)a\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}s|w)\big)]_\pm P_\zeta\big\|^2,$$

where the sum converges since $z\in\oplus_\chi\mathcal{A}_\chi$. Hence the upper entry is a bounded function of ψ. For the lower entry we have

$$\begin{aligned}&\mathcal{D}_\zeta\circ T_e^*\big(((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v))\widehat{\otimes}\xi\big)\\ &=\mathcal{D}_\zeta\big(((z\widehat{\otimes}s)\widehat{\otimes}(a\widehat{\otimes}\overline{w})\big|(y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v))_{A^G\widehat{\otimes}\mathbb{C}\mathrm{l}(T_eG)}\xi\big)\\ &=(-1)^{\deg s\cdot(\deg z+\deg y)+\deg w\cdot(\deg a+\deg z+\deg y+\deg b)+\deg b\cdot(\deg s+\deg t)}\\ &\times\sum_{\chi\in\mathrm{Char}(G)}\mathcal{D}_\zeta\Big(a^*\Phi_\chi(z)^*\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}s|w)_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big),\end{aligned}$$

using Lemma 2.7 and Equation (3). Let $(x_j)_{j=1}^n$ be a G-invariant, global orthonormal frame for $\$_G$, and let $(\phi_\ell)_{\ell=1}^\infty$ be an approximate identity for A^G of homogeneous degree zero. For each $\chi\in\mathrm{Char}(G)$, let $(c_k^\chi)_{k=1}^\infty\subset A$ and $(\sigma_k^\chi)_{k=1}^\infty\subset\mathcal{H}_\zeta$ be sequences such that

$$\lim_{k\to\infty}c_k^\chi\sigma_k^\chi=\mathcal{D}\big(\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\big).$$

Then

$$\begin{aligned}&T_e^*\circ\Psi^{-1}\circ\mathcal{D}\circ\Psi\big(((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v))\widehat{\otimes}\xi\big)\\ &=(-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)}\\ &\times\sum_{\chi,\nu\in\mathrm{Char}(G)}\sum_{j=1}^n\lim_{k\to\infty}a^*\Phi_\nu(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|(x_j\widehat{\otimes}1)\cdot(\nu x_j\widehat{\otimes}1|p_{\nu^{-1}}s)_{\mathfrak{C}})\big)\Phi_\nu(c_k^\chi)\sigma_k^\chi\\ &=(-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)}\times\\ &\qquad\times\sum_\chi\sum_{j=1}^n a^*\Phi_\chi(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|(x_j\widehat{\otimes}1)\cdot(\chi x_j\widehat{\otimes}1|p_{\chi^{-1}}s)_{\mathfrak{C}})\big)\times\\ &\qquad\qquad\times\mathcal{D}\Big(\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big)\end{aligned}$$

where we have used

$$\begin{aligned}\lim_{k\to\infty}\Phi_\nu(c_k^\chi)\sigma_k^\chi&=\lim_{k\to\infty}P_{\nu\zeta}c_k^\chi\sigma_k^\chi=P_{\nu\zeta}\mathcal{D}\big(\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\big)\\ &=\delta_{\nu,\chi}\mathcal{D}\big(\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\big).\end{aligned}$$

Since $\chi^{-1}p_{\chi^{-1}}s = \sum_{j=1}^{n}(x_j\widehat{\otimes}1)\cdot(\chi x_j\widehat{\otimes}1|p_{\chi^{-1}}s)_{\mathfrak{C}}$,

$$\begin{aligned}
&T_e^* \circ \Psi^{-1} \circ \mathcal{D} \circ \Psi\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) \\
&\ = (-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)} \sum_{\chi\in\mathrm{Char}(G)} \\
&a^*\Phi_\chi(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\chi^{-1}p_{\chi^{-1}}s)\big)\mathcal{D}\Big(\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big).
\end{aligned}$$

Hence the lower entry is

$$\begin{aligned}
&\mathcal{D}_\zeta \circ T_e^*\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) \\
&- (-1)^{\deg z+\deg s+\deg a+\deg w} T_e^* \circ \Psi^{-1} \circ \mathcal{D} \circ \Psi\big(\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}v)\big)\widehat{\otimes}\xi\big) \\
&= (-1)^{\deg s\cdot(\deg z+\deg y)+\deg w\cdot(\deg a+\deg z+\deg y+\deg b)+\deg b\cdot(\deg s+\deg t)} \sum_{\chi\in\mathrm{Char}(G)} \\
&\times \mathcal{D}_\zeta\Big(a^*\Phi_\chi(z)^*\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}s|w)_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big) \\
&- (-1)^{\deg z+\deg s+\deg a+\deg w+\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)} \sum_{\chi\in\mathrm{Char}(G)} \\
&\times a^*\Phi_\chi(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\chi^{-1}p_{\chi^{-1}}s)\big)\mathcal{D}\Big(\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big) \\
&= (-1)^{\deg t\cdot\deg b+\deg s\cdot\deg z+\deg w\cdot(\deg a+\deg z)} \\
&\times \sum_{\chi\in\mathrm{Char}(G)} [\mathcal{D}, a^*\Phi_\chi(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\chi p_\chi s)\big)]_\pm \Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi p_\chi t|v)\big)\xi.
\end{aligned}$$

The sum $\sum_{\chi\in\mathrm{Char}(G)}\Phi_\chi(y)a\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|w)\big)\xi$ converges since Ψ is an isometry, so

$$\begin{aligned}
\sum_{\chi\in\mathrm{Char}(G)} &[\mathcal{D}, a^*\Phi_\chi(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\chi^{-1}p_{\chi^{-1}}s)\big)]_\pm \times \\
&\times \Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi \\
= &\left(\sum_\nu P_\zeta[\mathcal{D}, a^*\Phi_\nu(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm\right) \times \\
&\times \left(\sum_\chi \Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\right).
\end{aligned}$$

Thus we can estimate the lower entry by

$$\begin{aligned}
&\Big\| \sum_{\chi\in\mathrm{Char}(G)} [\mathcal{D}, a^*\Phi_\chi(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\chi^{-1}p_{\chi^{-1}}s)\big)]_\pm \times \\
&\times \Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\Big\|^2 \\
&\leq \Big\| \sum_{\nu} P_\zeta[\mathcal{D}, a^*\Phi_\nu(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm\Big\|^2 \times \\
&\times \sum_\chi \big\|\Phi_\chi(y)b\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}p_{\chi^{-1}}t|v)\big)\xi\big\|^2 \\
&= \Big\| \sum_{\nu\in\mathrm{Char}(G)} P_\zeta[\mathcal{D}, a^*\Phi_\nu(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm\Big\|^2 \times \\
&\times \big\|\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}\overline{v})\big)\widehat{\otimes}\xi\big\|^2,
\end{aligned}$$

since Ψ is an isometry. We note that

$$\sum_{\nu\in\mathrm{Char}(G)} P_\zeta[\mathcal{D}, a^*\Phi_\nu(z)^*\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(w|\nu^{-1}p_{\nu^{-1}}s)\big)]_\pm$$

is a finite sum of bounded operators and hence is bounded. Therefore the lower entry is a bounded function of $\big((y\widehat{\otimes}t)\widehat{\otimes}(b\widehat{\otimes}\overline{v})\big)\widehat{\otimes}\xi$. $\square$

Lemma 3.6. *Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\$_G$, let $\mathcal{D}_G$ be the Dirac operator on $\$_G$, and let $P_\chi \in B(\mathcal{H})$ be the projection onto $\mathcal{H}_\chi$ for $\chi \in \mathrm{Char}(G)$. Then*

$$\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1} = \sum_{\chi\in\mathrm{Char}(G)}\sum_{j=1}^n \eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}.$$

Proof. Let $\mathfrak{c}$ be the generator of $\mathfrak{C}$, let $\xi \in \mathrm{dom}(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})$, and choose sequences $(a_k)_{k=1}^\infty \subset A$ and $(\xi_k)_{k=1}^\infty \subset \mathcal{H}_\zeta$ such that $a_k\xi_k \to \xi$ as $k \to \infty$. Then

$$\begin{aligned}
&\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1}\xi \\
&= \Psi\sum_{\chi\in\mathrm{Char}(G)}\sum_{j=1}^n \lim_{k,\ell\to\infty}(-1)^{\deg a_k}\Big(\big(\Phi_\chi(a_k)\widehat{\otimes}(\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c})\big)\widehat{\otimes}(\phi_\ell\widehat{\otimes}\overline{x_j\widehat{\otimes}1})\Big)\widehat{\otimes}\xi_k \\
&= \sum_{\chi\in\mathrm{Char}(G)}\sum_{j=1}^n \lim_{k\to\infty}\eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)\Phi_\chi(a_k)\xi_k \\
&= \sum_{\chi\in\mathrm{Char}(G)}\sum_{j=1}^n \eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi^{-1}x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi.
\end{aligned}$$

$\square$

Proposition 3.7 (The domain criterion). *For all $\mu \in \mathbb{R}\setminus\{0\}$, the resolvent $(i\mu+\mathcal{D})^{-1}$ maps the submodule $C_c^\infty(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})\mathcal{H}$ into* $\mathrm{dom}(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})$.

Proof. By Lemma 3.6 and the compactness of $(1+\mathcal{D}_G)^{-1/2}$, if $\xi\in C_c^\infty(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})\mathcal{H}$, then $P_\chi\xi=0$ for all but finitely many $\chi\in\mathrm{Char}(G)$. Since $(i\mu+\mathcal{D})^{-1}$ commutes with the action of G, it preserves $\mathcal{H}_\chi$ for all $\chi\in\mathrm{Char}(G)$. Hence if $\xi\in C_c^\infty(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})\mathcal{H}$, then $P_\chi(i\mu+\mathcal{D})^{-1}\xi=0$ for all but finitely many $\chi\in\mathrm{Char}(G)$. Lemma 3.6 then implies that $(i\mu+\mathcal{D})^{-1}\in\mathrm{dom}(\Psi\circ(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1\circ\Psi^{-1})$. □

Since the connection and domain criteria of [16, Theorem 13] are satisfied (Propositions 3.5 and 3.7 respectively), Theorem 3.4 is proved by combining the remaining positivity condition with Lemma 3.6.

4. Factorisation for an odd spectral triple.

Recall that G is a compact abelian Lie group, equipped with the normalised Haar measure and a trivial spinor bundle $\$_G$. However, suppose that rather than an *even* G-equivariant spectral triple, we instead have an *odd* G-equivariant spectral triple $(\mathcal{A},\mathcal{H},\mathcal{D})$.

The K-homology class of an odd spectral triple is defined by associating to it an even spectral triple. Let $\gamma=\left(\begin{smallmatrix}1&0\\0&-1\end{smallmatrix}\right)\in B(\mathbb{C}^2)$, and equip $\mathbb{C}^2$ with the $\mathbb{Z}_2$-grading defined by γ. Let c be the generator of the Clifford algebra $\mathbb{C}\mathrm{l}_1$, and define a $\mathbb{Z}_2$-graded $*$-homomorphism $\mathbb{C}\mathrm{l}_1\to B(\mathbb{C}^2)$ by $c\mapsto\left(\begin{smallmatrix}0&1\\1&0\end{smallmatrix}\right)$. Equip $A\widehat{\otimes}\mathbb{C}\mathrm{l}_1$ and $\mathcal{H}\widehat{\otimes}\mathbb{C}^2$ with the obvious actions by G. Let $\omega=\left(\begin{smallmatrix}0&-i\\i&0\end{smallmatrix}\right)\in B(\mathbb{C}^2)$. Then $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathcal{H}\widehat{\otimes}\mathbb{C}^2,\mathcal{D}\widehat{\otimes}\omega)$ is an even G-equivariant spectral triple. The class of $(\mathcal{A},\mathcal{H},\mathcal{D})$ in odd K-homology is defined to be $[(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathcal{H}\widehat{\otimes}\mathbb{C}^2,\mathcal{D}\widehat{\otimes}\omega)]\in KK_G(A\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathbb{C})=KK^1_G(A,\mathbb{C})$, [8, Prop. IV.A.13].

We make the following definition analogously to Definition 2.13.

Definition 4.1. Let $(\mathcal{A},\mathcal{H},\mathcal{D})$ be an odd, G-equivariant spectral triple for a trivially $\mathbb{Z}_2$-graded separable C^*-algebra A, and suppose that $\mathcal{A}$ is α-compatible. Let $\zeta\in\mathrm{Char}(G)$ satisfy $\overline{A\mathcal{H}_\zeta}=\mathcal{H}$, and let $\eta:\mathbb{C}\mathrm{l}(T_eG)\to B(\mathcal{H})$ be a unital, equivariant $*$-homomorphism such that
1) $[\eta(s),a]=0$ for all $s\in\mathbb{C}\mathrm{l}(T_eG)$ and $a\in A^G$, and
2) $a\eta(s)\cdot\mathrm{dom}(\mathcal{D}_\zeta)\subset\mathrm{dom}(\mathcal{D})$ and $(\mathcal{D}\eta(s)-(-1)^{\deg s}\eta(s)\mathcal{D})aP_\zeta$ is bounded on $\mathcal{H}$ for every $a\in\oplus_\chi\mathcal{A}_\chi$, $s\in\mathbb{C}\mathrm{l}(T_eG)$, where $P_\zeta\in B(\mathcal{H})$ is the orthogonal projection onto $\mathcal{H}_\zeta$.

We define a $\mathbb{Z}_2$-graded $*$-homomorphism $\widetilde{\eta}:\mathbb{C}\mathrm{l}(T_eG)\to B(\mathcal{H}\widehat{\otimes}\mathbb{C}^2)$ by $\widetilde{\eta}(s)=\eta(s)\widehat{\otimes}\omega^{\deg s}$, where $(\eta(s)\widehat{\otimes}\omega^{\deg s})(\xi\widehat{\otimes}v)=\eta(s)\xi\widehat{\otimes}\omega^{\deg s}v$.

It is easy to see that the pair $(\zeta,\widetilde{\eta})$ satisfy the conditions of Definition 2.13 for the even G-equivariant spectral triple $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1,\mathcal{H}\widehat{\otimes}\mathbb{C}^2,\mathcal{D}\widehat{\otimes}\omega)$.

The next result follows easily from Theorem 3.4 applied to the even G-equivariant spectral triple $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}\widehat{\otimes}\mathbb{C}^2, \mathcal{D}\widehat{\otimes}\omega)$.

THEOREM 4.2. *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd, G-equivariant spectral triple for a trivially $\mathbb{Z}_2$-graded C^*-algebra A, and let $\zeta \in \mathrm{Char}(G)$ and $\eta : \mathbb{C}\mathrm{l}(T_eG) \to B(\mathcal{H})$ be as in Definition 4.1, so in particular $\mathcal{A}$ is α-compatible. Let $(x_j)_{j=1}^n$ be a G-invariant global orthonormal frame for $\$_G$. If there is some $R \in \mathbb{R}$ such that*

$$\sum_{j=1}^{n}\Big(\big\langle \mathcal{D}\xi, \eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi\big\rangle \\ + \big\langle \eta\big({}_{\mathbb{C}\mathrm{l}(T_eG)}(\chi^{-1}\mathcal{D}_G(\chi x_j)\widehat{\otimes}\mathfrak{c}|x_j\widehat{\otimes}1)\big)P_{\chi\zeta}\xi, \mathcal{D}\xi\big\rangle\Big) \geq R\|\xi\|^2$$

for all $\chi \in \mathrm{Char}(G)$, $\xi \in \mathrm{dom}(\mathcal{D})$, then the odd spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ represents the Kasparov product of the left-hand, middle and right-hand modules for $(\mathcal{A}\widehat{\otimes}\mathbb{C}\mathrm{l}_1, \mathcal{H}\widehat{\otimes}\mathbb{C}^2, \mathcal{D}\widehat{\otimes}\omega)$.

5. The θ-deformation of a $\mathbb{T}^n$-equivariant spectral triple and factorisation.

Given a $\mathbb{T}^n$-equivariant spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ and a skew-symmetric matrix $\theta \in M_n(\mathbb{R})$, one can construct the θ-deformed $\mathbb{T}^n$-equivariant spectral triple $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$. We show that if factorisation is achieved for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$, then it is also achieved for $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$.

We first recall the construction of a θ-deformed $\mathbb{T}^n$-equivariant spectral triple, [9, 28].

Definition 5.1. Let $\theta \in M_n(\mathbb{R})$ be a skew-symmetric matrix. The *noncommutative torus* $C(\mathbb{T}^n)_\theta$ is the universal C^*-algebra generated by n unitaries $U_1, \dots, U_n$ subject to the commutation relations $U_jU_k = e^{2\pi i\theta_{jk}}U_kU_j$ for $j, k = 1, \dots, n$.

The noncommutative torus $C(\mathbb{T}^n)_\theta$ carries an action by the n-torus $\mathbb{T}^n$, which is given by $t \cdot U_j = e^{2\pi i t^j}U_j$, where $t = (t^1, \dots, t^n) \in \mathbb{T}^n$ are the standard torus coordinates.

Definition 5.2. Let A be a $\mathbb{Z}_2$-graded C^*-algebra with an action α by $\mathbb{T}^n$. Let $\theta \in M_n(\mathbb{R})$ be a skew-symmetric matrix, and equip the tensor product $A\widehat{\otimes}C(\mathbb{T}^n)_\theta$ with the diagonal action $t \cdot (a\widehat{\otimes}b) = \alpha_t(a)\widehat{\otimes}(t \cdot b)$ by $\mathbb{T}^n$. The *θ-deformation* of A is the invariant sub-C^*-algebra $A_\theta := (A\widehat{\otimes}C(\mathbb{T}^n)_\theta)^{\mathbb{T}^n}$.

The θ-deformation A_θ carries an action $\alpha^{(\theta)}$ by $\mathbb{T}^n$, given by $\alpha_t^{(\theta)}(a\widehat{\otimes}b) = \alpha_t(a)\widehat{\otimes}b$.

Definition 5.3. Let $\mathcal{H} = \mathcal{H}^0 \oplus \mathcal{H}^1$ be a $\mathbb{Z}_2$-graded Hilbert space with a strongly continuous unitary representation $V : \mathbb{T}^n \to U(\mathcal{H})$ such that $V_t \cdot \mathcal{H}^j \subset \mathcal{H}^j$ for $t \in \mathbb{T}^n$, $j \in \mathbb{Z}_2$. Let $\theta \in M_n(\mathbb{R})$ be a skew-symmetric matrix. Viewing $C(\mathbb{T}^n)_\theta$ as a right Hilbert module over itself, form the $\mathbb{Z}_2$-graded right Hilbert $C(\mathbb{T}^n)_\theta$-module $\mathcal{H}\widehat{\otimes}C(\mathbb{T}^n)_\theta$. This module carries an action by $\mathbb{T}^n$, given by $t \cdot (\xi\widehat{\otimes}b) = V_t\xi\widehat{\otimes}(t \cdot b)$. The *$\theta$-deformation* of $\mathcal{H}$ is the $\mathbb{Z}_2$-graded Hilbert space $\mathcal{H}_\theta := (\mathcal{H}\widehat{\otimes}C(\mathbb{T}^n)_\theta)^{\mathbb{T}^n}$. We define a unitary represenation $V^{(\theta)} : \mathbb{T}^n \to U(\mathcal{H}_\theta)$ by $V_t^{(\theta)}(\xi\widehat{\otimes}b) = V_t\xi\widehat{\otimes}b$.

We now define the θ-deformed $\mathbb{T}^n$-equivariant spectral triple $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$.

Definition 5.4. Suppose that $\mathcal{A}$ is α-compatible. Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be a $\mathbb{T}^n$-equivariant spectral triple, and let $\theta \in M_n(\mathbb{R})$ be skew-symmetric. Represent A_θ on $\mathcal{H}_\theta$ by $(a\widehat{\otimes}b)(\xi\widehat{\otimes}c) = a\xi\widehat{\otimes}bc$ (for $a \in A$, $b \in C(\mathbb{T}^n)_\theta$), and setting $U^k := U_1^{k_1} \cdots U_n^{k_n}$ for $k \in \mathbb{Z}^n$, let

$$\mathcal{A}_\theta = \mathrm{span}\{a_k\widehat{\otimes}U^{-k} \in A_\theta : a_k \in \mathcal{A} \cap A_k,\ k \in \mathbb{Z}^n\}$$

which is a dense sub-$*$-algebra of A_θ compatible with $\alpha^{(\theta)}$, and define an operator $\mathcal{D}_\theta$ on $\mathcal{H}_\theta$ by $\mathcal{D}_\theta(\xi\widehat{\otimes}b) = \mathcal{D}\xi\widehat{\otimes}b$ for $\xi \in \mathrm{dom}(\mathcal{D})$. Then $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$ is a $\mathbb{T}^n$-equivariant spectral triple for A_θ, which we call the *θ-deformation* of $(\mathcal{A}, \mathcal{H}, \mathcal{D})$.

PROPOSITION 5.5. *Let A be a C^*-algebra with an action by $\mathbb{T}^n$, and let $\theta \in M_n(\mathbb{R})$ be skew-symmetric. Then A_θ satisfies the spectral subspace assumption if and only if A does.*

PROOF. Let $\psi : A^{\mathbb{T}^n} \to A_\theta^{\mathbb{T}^n}$ be the $*$-isomorphism $\psi(a) = a\widehat{\otimes}1$. Then $\psi(\overline{A_k A_k^*}) = \overline{(A_\theta)_k(A_\theta)_k^*}$ for all $k \in \mathbb{Z}^n$. □

Definition 5.6. Define a unitary isomorphism $u : \mathcal{H} \to \mathcal{H}_\theta$ by

$$u\left(\sum_{k\in\mathbb{Z}^n} \xi_k\right) = \sum_{k\in\mathbb{Z}^n} \xi_k\widehat{\otimes}U^{-k}.$$

This isomorphism intertwines the actions of $\mathbb{T}^n$, so that $u : \mathcal{H}_\ell \to (\mathcal{H}_\theta)_\ell$ for all $\ell \in \mathbb{Z}^n$.

Given $\eta : \mathrm{Cl}(\mathbb{T}^n_e) \to B(\mathcal{H})$, define $\eta_\theta : \mathrm{Cl}(\mathbb{T}^n_e) \to B(\mathcal{H}_\theta)$ by $\eta_\theta(s) = u \circ \eta(s) \circ u^*$.

PROPOSITION 5.7. *The pair (ℓ, η_θ) satisfies the conditions of Definition 2.13 for $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$ if and only if (ℓ, η) satisfies those conditions for $(\mathcal{A}, \mathcal{H}, \mathcal{D})$. Consequently $(\mathcal{A}_\theta, \mathcal{H}_\theta, \mathcal{D}_\theta)$ factorises if and only if $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ does.*

Proof. If $\xi\widehat{\otimes}U^{-\ell} \in (\mathcal{H}_\theta)_\ell$ and $a\widehat{\otimes}U^{-k} \in (A_\theta)_k$ are homogenous, then $(a\widehat{\otimes}U^{-k})(\xi\widehat{\otimes}U^{-\ell}) = \lambda a\xi\widehat{\otimes}U^{-k-\ell}$ for some $\lambda \in U(1)$. Hence $\overline{A_\theta(\mathcal{H}_\theta)_\ell} = \mathcal{H}$ if and only if $\overline{A\mathcal{H}_\ell} = \mathcal{H}$.

Recall the $*$-isomorphism $\psi : A^{\mathbb{T}^n} \to A_\theta^{\mathbb{T}^n}$, $\psi(a) = a\widehat{\otimes}1$. Then $u(a\xi) = \psi(a)u(\xi)$ for all $a \in A^{\mathbb{T}^n}$, $\xi \in \mathcal{H}$. Hence $u \circ [\eta(s), a]_\pm \circ u^* = [\eta_\theta(s), \psi(a)]_\pm$ for all $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$, $a \in A^{\mathbb{T}^n}$, so Condition (1) is satisfied for the θ-deformation if and only if it is satisfied for the original spectral triple.

By construction, $\oplus_k(\mathcal{A}_\theta)_k = \mathcal{A}_\theta$. Let $a\widehat{\otimes}U^{-k} \in (\mathcal{A}_\theta)_k$ and let $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$. If $\xi\widehat{\otimes}U^{-\ell} \in (\mathcal{H}_\theta)_\ell$ then $u^*\big((a\widehat{\otimes}U^{-k})\eta_\theta(s)(\xi\widehat{\otimes}U^{-\ell})\big) = \lambda a\eta(s)\xi$ for some $\lambda \in U(1)$. Since $\mathcal{D}_\theta = u \circ \mathcal{D} \circ u^*$, it follows that $a\eta(s) \cdot \mathrm{dom}(\mathcal{D}_\ell) \subset \mathrm{dom}(\mathcal{D})$ for all $a \in \oplus_k\mathcal{A}_k$, $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$ if and only if $b\eta_\theta(s) \cdot \mathrm{dom}((\mathcal{D}_\theta)_\ell) \subset \mathrm{dom}(\mathcal{D}_\theta)$ for all $b \in \mathcal{A}_\theta$, $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$.

Let $a\widehat{\otimes}U^{-k} \in (\mathcal{A}_\theta)_k$, and let $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$. Then

$$u^* \circ [\mathcal{D}_\theta, \eta_\theta(s)]_\pm (a\widehat{\otimes}U^{-k})P_\ell \circ u = \lambda[\mathcal{D}, \eta(s)]_\pm aP_\ell$$

for some $\lambda \in U(1)$ depending on k, ℓ and θ. Therefore (ℓ, η) satisfies Condition (2) if and only if (ℓ, η_θ) satisfies Condition (2).

Since $\mathcal{D}_\theta = u\circ\mathcal{D}\circ u^*$ and $\eta_\theta = u\circ\eta\circ u^*$, clearly the factorisation criterion (Theorems 3.4, 4.2) is satisfied for (ℓ, η_θ) and the θ-deformed spectral triple if and only if it is satisfied for (ℓ, η) and the original spectral triple. $\square$

6. Factorisation of a torus-equivariant Dirac-type operator over a compact manifold.

Throughout this section, let (M, g) be a compact Riemannian manifold with a smooth, free, isometric left action by the n-torus $\mathbb{T}^n$, and let S be a (possibly $\mathbb{Z}_2$-graded) Clifford module over M. We suppose that that either (i) the action of $\mathbb{T}^n$ lifts to an action of S, or (ii) the action of $\mathbb{T}^n$ on M by the double covering $\mathbb{T}^n \to \mathbb{T}^n$ lifts to an action of S, so that in either case S is a $\mathbb{T}^n$-equivariant Clifford module, [3, p.186]. The reason we include both these cases is that if S is the spinor bundle over M, then either the action of $\mathbb{T}^n$ or that of its double cover lifts to an action on the spinor bundle, making the spin Dirac operator into a equivariant Dirac operator, [1, Proposition on p. 22]. This more general setting will be necessary to deduce a particular case of the Atiyah-Hirzebruch theorem in Corollary 7.2.

Remark. The results generalise easily to an action via any finite covering of $\mathbb{T}^n$, but in the interest of readability we restrict ourselves to the single or double cover.

We suppose that the equivariant Clifford module S is equipped with a $\mathbb{T}^n$-equivariant Clifford connection ∇^S. Then $(C^\infty(M), L^2(S), \mathcal{D})$ is a $\mathbb{T}^n$-equivariant spectral triple, where $\mathcal{D}$ is the associated Dirac operator on S. The spectral triple is even if S is $\mathbb{Z}_2$-graded; otherwise it is odd.

We will show that $(C^\infty(M), L^2(S), \mathcal{D})$ can always be factorised. If the torus action is free, $C(M)$ has full spectral subspaces (a special case of the spectral subspace assumption) by [25, Thm. 7.2.6]. If the torus action is via the double cover, then it is no longer free, but the spectral subspace assumption is still satisfied. This is because the non-zero spectral subspaces of $C(M)$ are precisely the spectral subspaces of $C(M)$ under the original torus action. Hence $\overline{C(M)_k C(M)_k^*} = C(M)^{\mathbb{T}^n}$ if $k/2 \in \mathbb{Z}^n$, and $C(M)_k = \{0\}$ otherwise. We show that the remaining two conditions for factorisation (that is, the existence of the Clifford representation $\eta : \mathbb{C}\mathrm{l}(\mathbb{T}^n_e) \to B(L^2(S))$ and the positivity criterion) are satisfied in turn. Compatibility of $C^\infty(M)$ with the action is satisfied since we assume the action to be smooth.

Remark. A particular case of this situation is when the Dirac operator $\mathcal{D}$ on the total space M is constructed from a spin structure on the base space $M/\mathbb{T}^n$, as in [3, p. 335]. In this case $\mathcal{D}$ is constructed as a Kasparov product and so it is not difficult to see that factorisation occurs.

6.1. The Clifford representation. We require a character $\ell \in \mathbb{Z}^n$ and a map $\eta : \mathbb{C}\mathrm{l}(\mathbb{T}^n_e) \to B(L^2(S))$ satisfying the conditions of Definition 2.13 (or Definition 4.1 if S is trivially graded). The following lemma shows that any $\ell \in \mathbb{Z}^n$ satisfies the condition if the action is free (resp. $\ell \in 2\mathbb{Z}^n$ if the action is via the double cover), and indeed factorisation is achieved for any choice of ℓ (resp. $\ell \in 2\mathbb{Z}^n$).

Lemma 6.1. *Let N be a Riemannian manifold with a smooth free left action by the n-torus $\mathbb{T}^n$, and let F be an equivariant Hermitian vector bundle over N. Then $\overline{C_0(N)L^2(F)_\ell} = L^2(F)$ for all $\ell \in \mathbb{Z}^n$.*

Proof. Since $L^2(F) = \bigoplus_{k\in\mathbb{Z}^n} L^2(F)_k$, it is enough to show that the subspace $C_0(N)_{k-\ell}L^2(F)_\ell$ is dense in $L^2(F)_k$ for all $k \in \mathbb{Z}^n$. We show that $C_0(N)_{k-\ell}\Gamma_c(F)_\ell = \Gamma_c(F)_k$ for all $k \in \mathbb{Z}$, which since $\Gamma_c(F)$ is dense in $L^2(F)$ proves the result.

Let $\xi \in \Gamma_c(F)_k$. Since ξ has compact support, there is a finite collection of open sets $(U_i)_{i=1}^N$ which cover the support of ξ, such that $U_i \cong \pi(U_i) \times \mathbb{T}^n$ as $\mathbb{T}^n$-spaces, recalling the quotient map $\pi : N \to N/\mathbb{T}^n$. Let $(\phi_n)_{n=1}^N$ be an invariant paritition of unity for $\bigcup_{i=1}^N U_i$ subordinate to $(U_i)_{i=1}^N$. For each $i = 1, \ldots, N$, let $f_i \in C_0(\pi(U_i))$ be a function such that $(f_i \circ \pi)\phi_i = f_i \circ \pi$, and let $a_i, b_i \in C_0(U_i)$ be the functions corresponding to $f_i \otimes \chi_{k-\ell}$ and $f_i \otimes \chi_{\ell-k}$ respectively under the equivariant $*$-isomorphism $C_0(U_i) \cong C_0(\pi(U_i)) \otimes C(\mathbb{T}^n)$. Note that $b_i a_i \phi_i = \phi_i$ and $a_i \xi \in \Gamma_c(F)_\ell$, so $\xi = \sum_{i=1}^N \phi_i \xi = \sum_{i=1}^N b_i a_i \phi_i \xi_i \in C_0(N)_{k-\ell}\Gamma_c(F)_\ell$. □

We will assume that $\ell \in \mathbb{Z}^n$ (resp. $\ell \in 2\mathbb{Z}^n$) is fixed from now on. This choice does not affect the factorisation. This means we could choose

$\ell = 0$ for convenience, but we will leave ℓ arbitrary in order to show that factorisation is achieved for all choices of ℓ.

Next we define the map $\eta : \mathbb{C}\mathrm{l}(\mathbb{T}^n_e) \to B(L^2(S))$. First recall that the fundamental vector field $X^{(v)} \in \Gamma^\infty(TM)$ associated to $v \in T_e\mathbb{T}^n$ is $X^{(v)}_x = \frac{d}{dt}\exp(tv)\cdot x\big|_{t=0}$. Since the *original* action of the n-torus $\mathbb{T}^n$ on M is free, the fundamental vector field of a non-zero vector in $T_e\mathbb{T}^n$ is non-vanishing. The fundamental vector field map and the canonical isomorphism $TM \cong T^*M$ define an equivariant, $\mathbb{Z}_2$-graded map $T_e\mathbb{T}^n \to \Gamma^\infty(T^*M)$. However, this map need not be an isometry and hence need not extend to a $*$-homomorphism $\mathbb{C}\mathrm{l}(\mathbb{T}^n_e) \to \Gamma^\infty(\mathbb{C}\mathrm{l}(M))$. We will modify this map to obtain a $*$-homomorphism. For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(TM)^{\mathbb{T}^n}$ be the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$. Observe that $\{X_1(x), \ldots, X_n(x)\}$ is a linearly independent set for every $x \in M$. For each $x \in M$, let $W(x) = (W^{jk}(x))^n_{j,k=1} \in M_n(\mathbb{R})$ be the inverse square root of the positive-definite matrix $(g(X_j(x), X_k(x)))^n_{j,k=1}$. Letting x vary, we obtain functions $W^{jk} \in C^\infty(M)^{\mathbb{T}^n}$ for $j, k = 1, \ldots, n$. Let

$$v_k = \sum_{j=1}^n X_j^\flat W^{jk} \in \Gamma^\infty(T^*M)^{\mathbb{T}^n}, \quad k = 1, \ldots, n, \tag{5}$$

where $TM \to T^*M$, $X \mapsto X^\flat$ is the canonical isomorphism. Then the set $\{v_1(x), \ldots, v_n(x)\}$ is orthonormal for all $x \in M$. We call the functions $W^{jk} \in C^\infty(M)^{\mathbb{T}^n}$, $j, k = 1, \ldots, n$ the *normalisation functions*.

Definition 6.2. The map

$$T_e\mathbb{T}^n \ni dt^k \mapsto -v_k = -\sum_{j=1}^n X_j^\flat W^{jk} \in \Gamma^\infty(T^*M)^{\mathbb{T}^n}$$

is now not only equivariant and $\mathbb{Z}_2$-graded (when S is $\mathbb{Z}_2$-graded), but is also an isometry. It therefore extends to a unital $*$-homomorphism $\eta : \mathbb{C}\mathrm{l}(\mathbb{T}^n_e) \to \Gamma^\infty(\mathbb{C}\mathrm{l}(M)) \subset B(L^2(S))$.

Remark. The action of the torus on sections of the Clifford module S is $V_{\exp(tv)}u(x) = \exp(tv)\cdot u(\exp(-tv)\cdot x)$, explaining the appearance of a minus sign in the definition of η. So the more natural convention to define η is to use the vector field $Y^{(v)}_x = \frac{d}{dt}\exp(-tv)\cdot x\big|_{t=0} = -X^{(v)}_x$.

As functions are central in the endomorphisms, η satisfies Condition 1) of Definition 2.13, so it remains to check Condition 2). Since the image of η consists of smooth sections of $\mathbb{C}\mathrm{l}(M)$, $\eta(s)\cdot \mathrm{dom}(\mathcal{D}) \cap L^2(S)_\ell \subset \mathrm{dom}(\mathcal{D})$ for all $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$. Before showing that $[\mathcal{D}, \eta(s)]_\pm P_\ell$ is bounded for all $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$, we prove a lemma.

Lemma 6.3. *Let N be a Riemannian manifold, and let G be a Lie group acting smoothly by isometries on N. Let F be an equivariant Hermitian vector bundle over N. This defines a unitary representation $V : G \to U(L^2(F))$.*

Let $v \in \mathfrak{g}$, and let $X^{(v)} \in \Gamma^\infty(TN)$ be the fundamental vector field associated to v. Define a one-parameter unitary group on $L^2(F)$ by $\gamma_v(t) = V_{\exp(tv)}$. Let A be the infinitesimal generator of γ_v, characterised by $\gamma_v(t) = e^{itA}$. Then
1) $A : \Gamma^\infty(F) \to \Gamma^\infty(F)$, and
2) $iA + \nabla_{X^{(v)}} \in \Gamma^\infty(\mathrm{End}(F))$ for any connection ∇ on F.

In particular, if N is compact, then $iA + \nabla_{X^{(v)}} \in B(L^2(F))$ for any connection ∇.

Proof. Let $u \in \Gamma^\infty(F)$. Working on a local trivialisation of F, we can view u as a $\mathbb{C}^k$-valued function on N. Since $\gamma_v(t)u(x) = \exp(tv)\cdot u(\exp(-tv)\cdot x)$, in this trivialisation,

$$iAu(x) = \frac{d}{dt}\gamma_v(t)u(x)\Big|_{t=0} = Bu(x) - X_x^{(v)}(u),$$

where $B \in M_k(\mathbb{C})$ is the derivative at $t = 0$ of the curve $t \mapsto \exp(tv) \in M_k(\mathbb{C})$. This shows 1) and 2), since if ∇ is a connection then locally $\nabla_{X^{(v)}} = X^{(v)} + \omega$, where ω is a locally-defined $M_k(\mathbb{C})$-valued function on N. □

The next result shows that the pair (ℓ, η) satisfy the remaining condition (2) of Definition 2.13.

Proposition 6.4. *Let η be as in Definition 6.2 and $\ell \in 2\mathbb{Z}^n$ (or $\ell \in 2\mathbb{Z}^n$ if the action is via the double cover of $\mathbb{T}^n$). Then the graded commutator $[\mathcal{D}, \eta(s)]_\pm P_\ell$ is bounded for all $s \in \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$.*

Proof. For $j = 1, \ldots, n$, let X_j be the fundamental vector field associated to $\frac{\partial}{\partial t^j}$, and let $v_j = \sum_{k=1}^n X_k W^{kj}$ be the normalised vector field as in Equation (5). Let $U \subset M$ be an open set such that $M|_U$ is parallelisable, and choose vector fields $(w_1, \ldots, w_{m-n}) \subset \Gamma^\infty(TU)$ (where $m := \dim M$) such that $(v_1, \ldots, v_n, w_1, \ldots, w_{m-n})$ is an orthonormal frame for TU. We can locally express the Dirac operator $\mathcal{D}$ as

$$\mathcal{D}|_U = \sum_{j=1}^n c(v_j^\flat)\nabla^S_{v_j} + \sum_{i=1}^{m-n} c(w_i^\flat)\nabla^S_{w_i},$$

where $v \mapsto v^\flat$ is the isomorphism $TM \to T^*M$ determined by the Riemannian metric, and c denotes Clifford multiplication.

Since the C^*-algebra $\mathbb{C}\mathrm{l}(\mathbb{T}^n_e)$ is generated by $(c(dt^k))_{k=1}^n$, we need only show that the anticommutator $\{\mathcal{D}, c(v_j^\flat)\}P_\ell$ is bounded for $j = 1, \ldots, n$. Letting ∇^{LC} be the Levi-Civita connection on T^*M and using the compatibility between ∇^S and ∇^{LC}, we have

$$\begin{aligned}\{\mathcal{D}, c(v_j^\flat)\}|_U &= \sum_{i=1}^n c(v_i^\flat)c(v_j^\flat)\nabla^S_{v_i} + \sum_{i=1}^m c(w_i^\flat)c(v_j^\flat)\nabla^S_{w_i} + \sum_{i=1}^n c(v_i^\flat)c(\nabla^{LC}_{v_i} v_j^\flat)\\ &\quad + \sum_{i=1}^m c(w_i^\flat)c(\nabla^{LC}_{w_i} v_j^\flat) + \sum_{i=1}^n c(v_j^\flat)c(v_i^\flat)\nabla^S_{v_i} + \sum_{i=1}^m c(v_j^\flat)c(w_i^\flat)\nabla^S_{w_i}\\ &= -2\nabla^S_{v_j} + \sum_{i=1}^n c(v_i^\flat)c(\nabla^{LC}_{v_i} v_j^\flat) + \sum_{i=1}^m c(w_i^\flat)c(\nabla^{LC}_{w_i} v_j^\flat).\end{aligned}$$

The second and third terms are smooth endomorphisms which are independent of the choice of $(f_1, \ldots, f_{m-n})$, and so globally

$$\begin{aligned}\{\mathcal{D}, c(v_j^\flat)\} &= -2\nabla^S_{v_j} + \text{bundle endomorphism}\\ &= -2\sum_{k=1}^n W^{kj}\nabla^S_{X_k} + \text{bundle endomorphism}.\end{aligned}$$

Since M is compact, every endomorphism is bounded, and so it is enough to show that $\nabla^S_{X_j} P_\ell$ is bounded. By Lemma 6.3, $\nabla^S_{X_j} = -iA_j + \omega$ for some $\omega \in \Gamma^\infty(\mathrm{End}(S))$, where A_j is the infinitesimal generator of the one-parameter unitary group $s \mapsto V_{\exp(s\frac{\partial}{\partial t^j})} \in U(L^2(S))$. Since

$$\exp(s\frac{\partial}{\partial t^j}) = (0, \ldots, 0, \underbrace{s}_{j^{\text{th}}}, 0, \ldots, 0), \qquad s \in \mathbb{R},$$

$V_{\exp(s\frac{\partial}{\partial t^j})} = \sum_{k \in \mathbb{Z}^n} e^{2\pi i s k_j} P_k$, where we note that $P_k = 0$ if the action is by the double cover of $\mathbb{T}^n$ and $k \notin 2\mathbb{Z}^n$. Hence $A_j = \sum_{k\in\mathbb{Z}^n} 2\pi k_j P_k$, and thus

$$\nabla^S_{X_j} P_\ell = -iA_j P_\ell + \omega P_\ell = -2\pi i \ell_j P_\ell + \omega P_\ell$$

is bounded, and so we have shown that $\{\mathcal{D}, c(v_j^\flat)\}P_\ell$ is bounded. $\square$

6.2. The positivity criterion. Now that we have a pair (ℓ, η) satisfying the conditions of Definition 2.13, it remains to check the positivity criterion. To this end we derive an explicit formula for $\Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}$, recalling from Equation (2) the isomorphism

$$\Psi : (E_1 \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} (C(M)^{\mathbb{T}^n} \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*)) \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathrm{Cl}(\mathbb{T}^n_e)} L^2(S)_\ell \to L^2(S),$$

where we recall $W = (\$_{\mathbb{T}^n})_e$.

Lemma 6.5. *For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(TM)$ be the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$, with corresponding covector field $X_j^\flat$, and let A_j be the infinitesimal generator of the one-parameter unitary group*

$t \mapsto V_{\exp(t\frac{\partial}{\partial t^j})} \in U(L^2(S))$. *Let* $W^{jk} \in C^\infty(M)^{\mathbb{T}^n}$ *be the normalisation functions. Then*

$$\Psi \circ (\mathcal{D}_1 \hat{\otimes} 1)\hat{\otimes} 1 \circ \Psi^{-1} = -i \sum_{j,r=1}^{n} W^{rj} c(X_r^\flat)(A_j - 2\pi \ell_j)$$

PROOF. Let $(x_r)_{r=1}^{2^{\lfloor n/2 \rfloor}}$ be an invariant, global orthonormal frame for $\$_{\mathbb{T}^n}$, corresponding to some orthonormal basis for $(\$_{\mathbb{T}^n})_e$. By Lemma 3.6,

$$\Psi \circ (\mathcal{D}_1 \hat{\otimes} 1)\hat{\otimes} 1 \circ \Psi^{-1} = \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(\chi_k^{-1}\mathcal{D}_{\mathbb{T}^n}(\chi_k x_r)\hat{\otimes}\mathfrak{c}|x_r \hat{\otimes} 1)\big) P_{k+\ell}.$$

Since we are using the trivial flat spinor bundle over $\mathbb{T}^n$, $\mathcal{D}_{\mathbb{T}^n} x_r = 0$ for all r, and

$$[\mathcal{D}_{\mathbb{T}^n}, \chi_k] = 2\pi i \sum_{j=1}^{n} k_j \chi_k c(dt^j).$$

Recall $\eta : \mathbb{C}l(\mathbb{T}^n_e) \to B(L^2(S))$ is defined by $c(dt^j) \mapsto -\sum_{r=1}^n c(X_r^\flat) W^{rj}$. Hence

$$\begin{aligned}
\Psi \circ (\mathcal{D}_1 \hat{\otimes} 1)\hat{\otimes} 1 \circ \Psi^{-1} &= 2\pi i \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \sum_{j=1}^{n} k_j \eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(c(dt^j)x_r \hat{\otimes}\mathfrak{c}|x_r \hat{\otimes} 1)\big) P_{k+\ell} \\
&= 2\pi i \sum_{k \in \mathbb{Z}^n} \sum_{r=1}^{2^{\lfloor n/2 \rfloor}} \sum_{j=1}^{n} k_j \eta(c(dt^j))\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r \hat{\otimes} 1|x_r \hat{\otimes} 1)\big) P_{k+\ell} \\
&= -2\pi i \sum_{k} \sum_{j,p=1}^{n} k_j W^{pj} c(X_p^\flat) P_{k+\ell} = -i \sum_{j,r=1}^{n} W^{rj} c(X_r^\flat)(A_j - 2\pi \ell_j).
\end{aligned}$$

□

THEOREM 6.6. *The positivity criterion is satisfied; that is there is some* $R \in \mathbb{R}$ *such that*

$$\big\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \hat{\otimes} 1)\hat{\otimes} 1 \circ \Psi^{-1}\xi \big\rangle + \big\langle \Psi \circ (\mathcal{D}_1 \hat{\otimes} 1)\hat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \big\rangle \geq R\|\xi\|^2$$

for all $\xi \in \operatorname{dom}(\mathcal{D}) \cap \Psi(\operatorname{dom}((\mathcal{D}_1 \hat{\otimes} 1)\hat{\otimes} 1))$. *Thus* $(C^\infty(M), L^2(S), \mathcal{D})$ *factorises.*

PROOF. For $j = 1, \ldots, n$, let $X_j \in \Gamma^\infty(TM)$ be the fundamental vector field corresponding to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$, and let $v_j = \sum_{p=1}^n X_p W^{pj}$ be the normalised vector field as in Equation (5). Let $U \subset M$ be an open set such that $M|_U$ is parallelisable, and choose vector fields $(w_1, \ldots, w_{m-n}) \subset \Gamma^\infty(TU)$

(where $m := \dim M$) such that $(v_1, \ldots, v_n, w_1, \ldots, w_{m-n})$ is an orthonormal frame for TU. Recall that we can locally express the Dirac operator $\mathcal{D}$ as

$$\mathcal{D}|_U = \sum_{j=1}^{n} c(v_j^\flat)\nabla^S_{v_j} + \sum_{i=1}^{m-n} c(w_i^\flat)\nabla^S_{w_i}.$$

Since M is compact, by using a partition of unity it is enough to prove the positivity for sections with support in an open set V with $\overline{V} \subset U$.

Let A_j generate the one-parameter unitary group $s \mapsto V_{\exp(s\frac{\partial}{\partial t^j})} \in U(L^2(S))$ for $j = 1, \ldots, n$. Then for $\xi \in \operatorname{dom}(\mathcal{D}) \cap \Psi(\operatorname{dom}((\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1))$ with support in V,

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&= \sum_{j,p} \left\langle c(v_j^\flat)\nabla^S_{v_j}\xi, -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi \right\rangle \\
&\qquad + \sum_{j,p} \left\langle c(w_j^\flat)\nabla^S_{w_j}\xi, -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi \right\rangle \\
&+ \sum_{j,p} \left\langle -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi, c(v_j^\flat)\nabla^S_{v_j}\xi \right\rangle \\
&\qquad + \sum_{j,p} \left\langle -ic(v_p^\flat)(A_p - 2\pi\ell_p)\xi, c(w_j^\flat)\nabla^S_{w_j}\xi \right\rangle.
\end{aligned}$$

Given $X \in \Gamma^\infty(TM)$, the (formal) adjoint of ∇_X is $(\nabla^S_X)^* = -\nabla^S_X - \operatorname{div} X$. Using the compatibility between ∇^S and the Levi-Civita connection ∇^{LC} on T^*M, we compute

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&= 4\pi i \sum (k_j - \ell_j) \left\langle \xi, \nabla^S_{v_j} P_k \xi \right\rangle - 2\pi i \sum (k_p - \ell_p) \times \\
&\times \Bigg(\left\langle \xi, \big(c(\nabla^{LC}_{v_j} v_j^\flat)c(v_p^\flat) + c(v_j^\flat)c(\nabla^{LC}_{v_j} v_p^\flat) + (\operatorname{div} v_j)c(v_j^\flat)c(v_p^\flat)\big)P_k\xi \right\rangle \\
&+ \left\langle \xi, \big(c(\nabla^{LC}_{w_j} w_j^\flat)c(v_p^\flat) + c(w_j^\flat)c(\nabla^{LC}_{w_j} v_p^\flat) + (\operatorname{div} w_j)c(w_j^\flat)c(v_p^\flat)\big)P_k\xi \right\rangle \Bigg).
\end{aligned}$$

Let $\omega_j = \nabla^S_{X_j} + iA_j \in \Gamma^\infty(\mathrm{End}(S))$, as in Lemma 6.3. Since $A_j P_k = 2\pi k_j P_k$,

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&= 8\pi^2 \sum k_p (k_j - \ell_j) \left\langle \xi, W^{jp} P_k \xi \right\rangle + 4\pi i \sum (k_j - \ell_j) \left\langle \xi, W^{jp} \omega_p P_k \xi \right\rangle \\
&\quad - 2\pi i \sum (k_p - \ell_p) \times \\
&\times \Bigg(\left\langle \xi, \left(c(\nabla^{LC}_{v_j} v_j^\flat) c(v_p^\flat) + c(v_j^\flat) c(\nabla^{LC}_{v_j} v_p^\flat) + (\operatorname{div} v_j) c(v_j^\flat) c(v_p^\flat) \right) P_k \xi \right\rangle \\
&+ \left\langle \xi, \left(c(\nabla^{LC}_{w_j} w_j^\flat) c(v_p^\flat) + c(w_j^\flat) c(\nabla^{LC}_{w_j} v_p^\flat) + (\operatorname{div} w_j) c(w_j^\flat) c(v_p^\flat) \right) P_k \xi \right\rangle \Bigg).
\end{aligned}$$

We estimate:

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&\geq 8\pi^2 \sum_{j,p,k} k_p (k_j - \ell_j) \left\langle P_k \xi, W^{jp} P_k \xi \right\rangle - \sum_{p,k} |k_p - \ell_p| C_p \left\langle P_k \xi, P_k \xi \right\rangle,
\end{aligned}$$

for some constants $C_p \in [0, \infty)$, $p = 1, \dots, n$, which are based on the norms of the endomorphisms such as $W^{jp}\omega_p$ and $(\operatorname{div} w_j) c(w_j^\flat) c(v_p^\flat)$ on the compact set $\overline{V}$.

For $x \in M$, let $\lambda(x) > 0$ be the smallest eigenvalue of the positive-definite real matrix $(W^{jp}(x))_{p,q=1}^n$. Then $\sum_{j,p,q=1}^n k_j k_p W^{jp}(x) \geq \lambda(x) \sum_{j=1}^n k_j^2$, and so we can estimate

$$\begin{aligned}
&\left\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1}\xi \right\rangle + \left\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1) \widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \right\rangle \\
&\geq 8\pi^2 \inf_{x \in M} \{\lambda(x)\} \sum_{j,k} k_j^2 \|P_k \xi\|^2 \\
&\qquad - 8\pi^2 n \sup_{j,p} \left\{ |\ell_p| \sup_{x \in M} \{|W^{jp}(x)|\} \right\} \sum_{r,k} |k_r| \|P_k \xi\|^2 \\
&\qquad\quad - \sum_{p,k} |k_p - \ell_p| C_p \|P_k \xi\|^2 \\
&\geq \sum_{k \in \mathbb{Z}^n} \left(a \sum_j k_j^2 - b \sum_j |k_j| - d \sum_j |k_j - \ell_j| \right) \|P_k \xi\|^2,
\end{aligned}$$

where we have relabelled some constants and set $d := \sup_p \{C_p\}$. Since M is compact, the constant $a = 8\pi^2 \inf_{x \in M}\{\lambda(x)\}$ is strictly positive, and so the function

$$Q : \mathbb{Z}^n \to \mathbb{R}, \qquad Q(k) = a \sum_j k_j^2 - b \sum_j |k_j| - d \sum_j |k_j - \ell_j|$$

is bounded from below by some $R \in \mathbb{R}$. Hence

$$\begin{aligned}\big\langle \mathcal{D}\xi, \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi \big\rangle + \big\langle \Psi \circ (\mathcal{D}_1 \widehat{\otimes} 1)\widehat{\otimes} 1 \circ \Psi^{-1}\xi, \mathcal{D}\xi \big\rangle \\ \geq R \sum_{k \in \mathbb{Z}^n} \|P_k \xi\|^2 = R\|\xi\|^2.\end{aligned}$$

□

6.3. The constructive Kasparov product for manifolds. Recall that (M, g) is a compact Riemannian manifold with a free, isometric left action by $\mathbb{T}^n$, (S, ∇^S) is an equivariant Clifford module over M with Dirac operator $\mathcal{D}$, for either the free action of $\mathbb{T}^n$ or via the double cover $\mathbb{T}^n \to \mathbb{T}^n$, and $\ell \in \mathbb{Z}^n$ (resp. $\ell \in 2\mathbb{Z}^n$) is fixed.

We have seen that $(C^\infty(M), L^2(S), \mathcal{D})$ represents the product of the unbounded Kasparov modules

$$(\oplus_k C(M)_k, (E_1 \widehat{\otimes}_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathfrak{C}} (C(M)^{\mathbb{T}^n} \widehat{\otimes} (W \widehat{\otimes} \mathfrak{C})^*))_{C(M)^{\mathbb{T}^n} \widehat{\otimes} \mathbb{C}\mathrm{l}(\mathbb{T}^n_e)}, \mathcal{D}_1 \widehat{\otimes} 1)$$

(the product of left and middle modules) and $(C^\infty(M)^{\mathbb{T}^n} \widehat{\otimes} \mathbb{C}\mathrm{l}(\mathbb{T}^n_e), \mathcal{H}_\ell, \mathcal{D}_\ell)$ (the right-hand module). We now show that the constructive Kasparov product [4, 14, 21] can be used to produce a representative of the product of these two cycles. The representative thus obtained is unitarily equivalent to[1] $(C^\infty(M), L^2(S), T)$ for some self-adjoint, first order elliptic differential operator T on S. If the orbits of $\mathbb{T}^n$ are embedded isometrically into M, then T is a bounded perturbation of the original operator $\mathcal{D}$.

Definition 6.7. Let G be a compact group, and let A and B be $\mathbb{Z}_2$-graded C^*-algebras carrying respective actions α and β by G. Let E_A be a $\mathbb{Z}_2$-graded right Hilbert A-module with a homomorphism V from G into the invertible degree zero bounded operators on E such that $V_g(ea) = V_g(e)\alpha_g(a)$ for all $g \in G$, $a \in A$ and $e \in E$, and let $(\mathcal{A}, F_B, T)$ be an unbounded equivariant Kasparov A-B-module. There is a natural action of G on $E \widehat{\otimes}_A \operatorname{End}_B(F_B)$ given by $g \cdot (e \widehat{\otimes} B) = V_g(e) \widehat{\otimes} U_g B U_g^{-1}$, where U is the action of G on F_B. A *T-connection* on E_A is a linear map ∇ from a dense subspace $\mathcal{E} \subset E_A$ which is a right $\mathcal{A}$-module into $E \widehat{\otimes}_A \operatorname{End}_B(F_B)$, such that $g \cdot \nabla(e) = \nabla(V_g(e))$ for all $g \in G$, $e \in E$, and

$$\nabla(ea) = \nabla(e)a + (-1)^{\deg e} e \widehat{\otimes} [T, a]_\pm, \qquad e \in \mathcal{E},\ a \in \mathcal{A}. \tag{6}$$

---

[1] Here we replace the algebra $\oplus_k C(M)_k$ by $\oplus_k C^\infty(M)_k$, and even by $C^\infty(M)$. The distinction between these algebras is unimportant for KK-classes, but may produce differences for (unitary equivalence classes of) spectral triples, where the choice of smooth algebra enters. We will ignore numerous subtleties involved in the choice of smooth algebra, which is harmless in the context of first order differential operators on compact manifolds.

We define a closed operator $1\widehat{\otimes}_\nabla T$ initially on $\mathrm{span}\{e\widehat{\otimes} f : e \in \mathcal{E}, f \in \mathrm{dom}(T)\} \subset E\widehat{\otimes}_A F$ by

$$(1\widehat{\otimes}_\nabla T)(e\widehat{\otimes} f) = (-1)^{\deg e} e\widehat{\otimes} Tf + \nabla(e)f.$$

The equivariance of ∇ ensures that $1\widehat{\otimes}_\nabla T$ is equivariant. We say that ∇ is *Hermitian* if

$$(e_1|\nabla e_2)_{\mathrm{End}_B(F_B)} - (\nabla e_1|e_2)_{\mathrm{End}(F_B)} = (-1)^{\deg e_1}[T,(e_1|e_2)_A]_\pm, \quad e_1, e_2 \in \mathcal{E}.$$

If ∇ is Hermitian, then the operator $1\widehat{\otimes}_\nabla T$ is symmetric.

Let $x \in M$. Choose tangent vectors $(v_1, \ldots, v_{m-n})$ spanning the subspace $\mathrm{span}\{X_1(x), \ldots, X_n(x)\}^\perp \subset T_x M$, where we recall that X_j is the fundamental vector field associated to $\frac{\partial}{\partial t^j} \in T_e\mathbb{T}^n$. Let $(x^1, \ldots, x^n, y^1, \ldots, y^{m-n})$ be the geodesic normal coordinates around x corresponding to the tangent vectors $(X_1(x), \ldots, X_n(x), v_1, \ldots, v_{m-n})$. There is a neighbourhood U of x and V of $e \in \mathbb{T}^n$ such that $U \cong \pi(U) \times V$ as $\mathbb{T}^n$-spaces, where $\pi : M \to M/\mathbb{T}^n$ is the quotient map (if the action is free we may take $V = \mathbb{T}^n$), so the standard coordinates $(t^1, \ldots, t^n) \in (0,1)^n$ on $\mathbb{T}^n$ give us coordinates $(t^1, \ldots, t^n, y^1, \ldots, y^{m-n})$ in a neighbourhood of x. Since $g(X_j(x), v_p) = 0$ and $X_j = \frac{\partial}{\partial t^j}$, it follows from the fact that a geodesic is orthogonal to one orbit of $\mathbb{T}^n$ if and only if it is orthogonal to every orbit of $\mathbb{T}^n$ that it intersects, [27, Prop. 2], that $g(\frac{\partial}{\partial t^j}, \frac{\partial}{\partial y^p}) = 0$ on the coordinate chart for $j = 1, \ldots, n$, $p = 1, \ldots, m-n$.

Let $(U_i)_{i=1}^N$ be a finite cover of M by such coordinate neighbourhoods, and for each $k \in \mathbb{Z}^n$, $i = 1, \ldots, N$, define $\chi_{i,k} \in C^\infty(U_i)$ by

$$\chi_{i,k}(t^1, \ldots, t^n, y^1, \ldots, y^{m-n}) = e^{-2\pi i \sum_{j=1}^n k_j t^j}.$$

Observe that if $g \in C(M)_k$ has support in U_i, then $g\chi_{i,k}^{-1} \in C(M)^{\mathbb{T}^n}$. Let $(\phi_i)_{i=1}^N$ be an invariant partition of unity subordinate to $(U_i)_{i=1}^N$, and for each $i = 1, \ldots, N$, let $\psi_i \in C^\infty(M)$ be an invariant function with support in U_i, such that ψ_i is 1 in a neighbourhood of $\mathrm{supp}\,\phi_i$.

Then for $f \in C(M)$,

$$\Phi_k(f) = \sum_i \phi_i\psi_i\Phi_k(f) = \sum_i \phi_i\chi_{i,k}(\Phi_k(f)\psi_i\chi_{i,k}^{-1}).$$

Let $(x_r)_{r=1}^{2^{\lfloor n/2\rfloor}}$ be an invariant orthonormal frame for $\$_{\mathbb{T}^n}$ of homogeneous degree, such that x_1 is of even degree (in the case $\$_{\mathbb{T}^n}$ is $\mathbb{Z}_2$-graded). Then given $\big((f\widehat{\otimes} u)\widehat{\otimes}(h\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi$ in

$$(E_1\widehat{\otimes}_{C(M)^{\mathbb{T}^n}\widehat{\otimes}\mathfrak{C}}(C(M)^{\mathbb{T}^n}\widehat{\otimes}(W\widehat{\otimes}\mathfrak{C})^*))\widehat{\otimes}_{C(M)^{\mathbb{T}^n}\widehat{\otimes}\mathrm{Cl}(\mathbb{T}^n_e)}L^2(S)_\ell,$$

we may write

$$\big((f\widehat{\otimes}u)\widehat{\otimes}(h\widehat{\otimes}\overline{w})\big)\widehat{\otimes}\xi = \sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{i=1}^{N}\big(\phi_i\chi_{i,k}\widehat{\otimes}(\chi_k x_r\widehat{\otimes}1)\big)\widehat{\otimes}(1\widehat{\otimes}\overline{x_1\widehat{\otimes}1})\big) \tag{7}$$
$$\widehat{\otimes}\Phi_k(f)\psi_i\chi_{i,k}^{-1}h\eta\big({}_{\mathbb{C}\mathrm{l}(\mathbb{T}^n_e)}(x_1\widehat{\otimes}(x_r\widehat{\otimes}1|\chi_k^{-1}p_{\chi_k^{-1}}u)_{\mathfrak{C}}|w)\big)\xi.$$

Define a $\mathcal{D}_\ell$-connection on $E_1\widehat{\otimes}_{C(M)^{\mathbb{T}^n}\widehat{\otimes}\mathfrak{C}}(C(M)^{\mathbb{T}^n}\widehat{\otimes}(W\widehat{\otimes}\mathfrak{C})^*)$ by

$$\begin{aligned}&\nabla\big((f\widehat{\otimes}u)\widehat{\otimes}(h\widehat{\otimes}\overline{w})\big)\\&:=\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{i=1}^{N}(-1)^{\deg x_r}((\phi_i\chi_{i,k}\widehat{\otimes}(\chi_k x_r\widehat{\otimes}1))\widehat{\otimes}(1\widehat{\otimes}\overline{x_1\widehat{\otimes}1}))\\&\qquad\widehat{\otimes}\big[\mathcal{D},\Phi_k(f)\psi_i\chi_{i,k}^{-1}h\eta\big({}_{\mathbb{C}\mathrm{l}(\mathbb{T}^n_e)}(x_1\widehat{\otimes}(x_r\widehat{\otimes}1|\chi_k^{-1}p_{\chi_k^{-1}}u)_{\mathfrak{C}}|w)\big)\big]_{\pm}.\end{aligned}$$

That ∇ is equivariant and satisfies Equation (6) follows from Equation (7). Since ∇ is built from a frame, [21], it is also Hermitian.

Writing $1\widehat{\otimes}_\nabla\mathcal{D}_\ell=(1\widehat{\otimes}1)\widehat{\otimes}_\nabla\mathcal{D}_\ell$ and $\mathcal{D}_1\widehat{\otimes}1=(\mathcal{D}_1\widehat{\otimes}1)\widehat{\otimes}1$ for short, the following result shows that the constructive Kasparov product yields a spectral triple.

THEOREM 6.8. *For $j=1,\dots,n$, let $X_j\in\Gamma^\infty(M)$ be the fundamental vector field associated to $\frac{\partial}{\partial t^j}\in T_e\mathbb{T}^n$. Let $(h_{jk})_{j,k=1}^n=(g(X_j,X_k))_{j,k=1}^n$, $(h^{jk})=(h_{jk})^{-1}$, and let $(W^{jk})_{j,k=1}^n$ be the normalisation functions. Then*

$$\Psi\circ\Big(1\widehat{\otimes}_\nabla\mathcal{D}_\ell+\mathcal{D}_1\widehat{\otimes}1\Big)\circ\Psi^{-1}=\mathcal{D}+\sum_{j,r=1}^{n}(W^{rj}-h^{rj})c(X_r^\flat)\nabla^S_{X_j}+B,$$

where $B\in\Gamma^\infty(\mathrm{End}(S))$. Thus $\Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell+\mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1}$ is a first order, self-adjoint, equivariant, elliptic differential operator. Hence the data $(C^\infty(M),L^2(S),\Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell+\mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1})$ defines an equivariant spectral triple representing the Kasparov product (which is also represented by $(C^\infty(M),L^2(S),\mathcal{D})$).

PROOF. Given $\xi\in L^2(S)$,

$$\Psi^{-1}(\xi)=\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\big((\psi_i\chi_{i,k-\ell}\widehat{\otimes}(\chi_{k-\ell}^{-1}x_r\widehat{\otimes}1))\widehat{\otimes}(1\widehat{\otimes}\overline{x_r\widehat{\otimes}1})\big)\widehat{\otimes}\chi_{i,\ell-k}\phi_iP_k\xi.$$

Using this we can compute

$$
\begin{aligned}
\Psi \circ 1\widehat{\otimes}_\nabla \mathcal{D}_\ell \circ \Psi^{-1} &= \sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{i,j=1}^{N}(-1)^{\deg x_r}\phi_i\chi_{i,k-\ell}\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\\
&\quad\times\big[\mathcal{D},\psi_j\chi_{j,k-\ell}\psi_i\chi_{i,k-\ell}^{-1}\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_\pm\chi_{j,\ell-k}\phi_jP_k\\
&\quad+\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\psi_i\chi_{i,k-\ell}\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\mathcal{D}\chi_{i,\ell-k}\phi_iP_k\\
&= \sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{i,j=1}^{N}(-1)^{\deg x_r}\phi_i\chi_{i,k-\ell}\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\\
&\quad\times\big[\mathcal{D},\psi_j\chi_{j,k-\ell}\psi_i\chi_{i,k-\ell}^{-1}\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_\pm\chi_{j,\ell-k}\phi_jP_k\\
&\quad+\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n}\chi_{i,k-\ell}[\mathcal{D},\psi_i\chi_{i,\ell-k}]\phi_iP_k+\mathcal{D},
\end{aligned}
\tag{8}
$$

where we have used $\sum_{r=1}^{2^{\lfloor n/2\rfloor}}{}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)=1$ and $\sum_{i=1}^N\phi_i=1$. Let I denote the first term of Equation (8). By several applications of the graded commutator relation $[a,bc]_\pm=(-1)^{\deg b}b[a,c]_\pm+[a,b]_\pm c$, the first term of Equation (8) can be simplified to

$$
\begin{aligned}
I &= \sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{N}[\mathcal{D},\psi_j\chi_{j,k-\ell}]\chi_{j,\ell-k}\phi_jP_k+\sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{i=1}^{N}(-1)^{\deg x_r}\phi_i\chi_{i,k-\ell}\\
&\quad\times\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)[\mathcal{D},\psi_i\chi_{i,k-\ell}^{-1}]\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)P_k\\
&\quad+\sum_{r=1}^{2^{\lfloor n/2\rfloor}}(-1)^{\deg x_r}\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\big[\mathcal{D},\eta\big({}_{\mathrm{Cl}(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_\pm.
\end{aligned}
$$

With respect to the $(t^1,\dots,t^n,y^1,\dots,y^{m-n})$ coordinates on U_i, $\chi_{i,k}=e^{-2\pi i\sum_{j=1}^n t^jk_j}$, and so

$$
\chi_{i,k}^{-1}[\mathcal{D},\psi_i\chi_{i,k}]=\chi_{i,k}^{-1}c(d\chi_{i,k})=-2\pi i\sum_{j=1}^{n}k_jc(dt^j).
$$

Write $\mathcal{D}=\sum_{j=1}^n c(dt^j)\nabla^S_{X_j}+\sum_{s=1}^{m-n}c(dy^s)\nabla^S_{\partial_{y^s}}$. Since $g(\partial_{t^j},\partial_{y^p})=0$ and $X_j^\flat=\sum_{p=1}^n h_{jk}dt^k$, the Clifford vector $c(dy^p)$ anticommutes with $c(X_j^\flat)$ and hence graded commutes with the image of $\mathrm{Cl}(\mathbb{T}^n_e)$ under η for each $p=1,\dots,m-n$. Using this fact as well as the compatibility of ∇^S with the Levi-Civita connection, the first term of Equation (8) is locally

$$
\begin{aligned}
I &= -2\pi i \sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{n} c(dt^j)(k_j-\ell_j)P_k \\
&+ 2\pi i \sum_{k\in\mathbb{Z}^n}\sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{j=1}^{n}(-1)^{\deg x_r}\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big) \\
&\qquad\qquad\qquad \times c(dt^j)\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)(k_j-\ell_j)P_k \\
&+ \sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{j=1}^{n}(-1)^{\deg x_r}\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\times \\
&\qquad\qquad\qquad \times \big[c(dt^j)\nabla^S_{X_j},\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_{\pm} \\
&+ \sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{p=1}^{m-n}\sum_{j=1}^{n}(-1)^{\deg x_r}\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\times \\
&\qquad\qquad\qquad \times \big[c(dy^p)\nabla^S_{\partial_{y^p}},\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_{\pm} \\
&= \sum_{j=1}^{n} c(dt^j)(\nabla^S_{X_j}+\omega_j-2\pi\ell_j) - \sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{j=1}^{n}(-1)^{\deg x_r}\Big(\\
&\quad \eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)c(dt^j)\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)(\nabla^S_{X_j}+\omega_j-2\pi\ell_j) \\
&\quad - \eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)c(dt^j)\nabla^{LC}_{X_j}\big(\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big) \\
&\quad - \eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)\big[c(dt^j),\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big]_{\pm}\nabla^S_{X_j}\Big) \\
&+ \sum_{r=1}^{2^{\lfloor n/2\rfloor}}\sum_{p=1}^{m-n}(-1)^{\deg x_r}\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)c(dy^p)\times \\
&\qquad\qquad\qquad \times \nabla^{LC}_{\partial_{y^p}}\big(\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big)
\end{aligned}
$$

for $\omega_j \in \Gamma^\infty(\mathrm{End}(S))$ for $j = 1,\dots,n$, using $A_j = 2\pi\sum_{k\in\mathbb{Z}^n} k_jP_k$ and Lemma 6.3. Here ∇^{LC} denotes the extension of the Levi-Civita connection on the cotangent bundle to the Clifford bundle. Using

$$\sum_{r=1}^{2^{\lfloor n/2\rfloor}} {}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1) = 1$$

and the fact that $c(dy^p)$ graded commutes with the image of η, we can make some cancellations and, working locally, simplify the first term of Equation

(8) to

$$\begin{aligned}
I = \sum_{j=1}^{n} c(dt^j)(\omega_j - 2\pi\ell_j) - \sum_{r=1}^{2^{\lfloor n/2\rfloor}} \sum_{j=1}^{n} (-1)^{\deg x_r}\Big(\\
\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)c(dt^j)\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)(\omega_j - 2\pi\ell_j) \\
- \eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)c(dt^j)\nabla^{LC}_{X_j}\big(\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big)\Big) \\
+ \sum_{r=1}^{2^{\lfloor n/2\rfloor}} \sum_{p=1}^{m-n} (-1)^{\deg x_r}\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_r\widehat{\otimes}1|x_1\widehat{\otimes}1)\big)c(dy^p)\times \\
\times \nabla^{LC}_{\partial_{y^p}}\big(\eta\big({}_{\mathbb{C}l(\mathbb{T}^n_e)}(x_1\widehat{\otimes}1|x_r\widehat{\otimes}1)\big)\big) \\
\in \Gamma^\infty(\mathrm{End}(S)).
\end{aligned}$$

The second term of Equation (8) is

$$\begin{aligned}
\sum_{i=1}^{N}\sum_{k\in\mathbb{Z}^n} \chi_{i,k-\ell}[\mathcal{D},\psi_i\chi_{i,\ell-k}]\phi_i P_k = 2\pi i \sum_{k\in\mathbb{Z}^n}\sum_{j=1}^{n} c(dt^j)(k_j-\ell_j)P_k \\
= -\sum_{j=1}^{n} c(dt^j)(\nabla^S_{X_j} + \omega_j - 2\pi\ell_j) = -\sum_{j,q=1}^{n} h^{jq}c(X_q^\flat)(\nabla^S_{X_j} + \omega_j - 2\pi\ell_j)
\end{aligned}$$

for some $\omega_j \in \Gamma^\infty(\mathrm{End}(S))$ by Lemma 6.3. Putting the expressions for Equation (8) together with Lemma 6.5 and Lemma 6.3 yields

$$\begin{aligned}
\Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1} = \mathcal{D} + \sum_{j,r=1}^{n}(W^{rj}-h^{rj})c(X_r^\flat)\nabla^S_{X_j} + B \\
= \sum_{p=1}^{m-n} c(dy^p)\nabla^S_{\partial_{y^p}} + \sum_{j,r,q=1}^{n} W^{rj}h_{rq}c(dt^q)\nabla^S_{X_j} + B
\end{aligned}$$

for some $B \in \Gamma^\infty(\mathrm{End}(S))$, which establishes that $\Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell+\mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1}$ is a first order differential operator. Since $(W^{rj})^n_{rj=1}$ and $(h_{rq})^n_{r,q=1}$ are invertible, this also shows that the operator $\Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1}$ is elliptic. Since ∇ is Hermitian, $1\widehat{\otimes}_\nabla\mathcal{D}_\ell$ is symmetric, and so $1\widehat{\otimes}_\nabla\mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1$ is the sum of a symmetric operator with a self-adjoint operator, which is symmetric. Elliptic operator theory, [13, 19], implies that $\Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1}$ is essentially self-adjoint with compact resolvent, and hence $(C^\infty(M), L^2(S), \Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell+\mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1})$ is an equivariant spectral triple. That $(C^\infty(M), L^2(S), \Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell+\mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1})$ represents the product is now a straightforward application of Kucerovsky's criteria. □

Corollary 6.9. *Suppose that each orbit is an isometric embedding of $\mathbb{T}^n$ in M. That is, the fundamental vector fields $T_e\mathbb{T}^n \ni v \mapsto X^{(v)} \in \Gamma^\infty(TM)$ satisfy $(X^{(v)}|X^{(v)})_{C(M)} = \|v\|^2$. Then*

$$\mathcal{D} - \Psi\circ(1\widehat{\otimes}_\nabla\mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1)\circ\Psi^{-1} \in \Gamma^\infty(\mathrm{End}(S)).$$

PROOF. In this case, the normalisation functions are $W^{jk} = \delta^{jk}$, and so Lemma 6.8 becomes $\Psi \circ (1\widehat{\otimes}_\nabla \mathcal{D}_\ell + \mathcal{D}_1\widehat{\otimes}1) \circ \Psi^{-1} = \mathcal{D} + B$ where $B \in \Gamma^\infty(\mathrm{End}(S))$. $\square$

7. Applications to index theory.

As an easy application of factorisation, we can say that when an even equivariant unital spectral triple factorises by our method, then its equivariant index is zero.

PROPOSITION 7.1. *Let G be a compact abelian Lie group of positive dimension, and let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an even G-equivariant spectral triple for a $\mathbb{Z}_2$-graded unital G-algebra A. Suppose that the spectral subspace assumption and the conditions of Theorem 3.4 are satisfied, so that factorisation occurs. Then the equivariant index of $\mathcal{D}$ is zero; i.e.*

$$\mathrm{index}_G(\mathcal{D}) := [(\mathbb{C}, \mathcal{H}, \mathcal{D})] = 0 \in KK_G(\mathbb{C}, \mathbb{C}) \cong R(G).$$

PROOF. Since the spectral triple $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ factorises, the index is given by

$$\mathrm{index}_G(\mathcal{D}) = [(\mathbb{C}, (E_1)_{A^G\widehat{\otimes}\mathfrak{C}}, \mathcal{D}_1)]\widehat{\otimes}_{A^G\widehat{\otimes}\mathfrak{C}}\, y,$$

where y is the Kasparov product of the middle and right-hand modules. The class of the module $(\mathbb{C}, (E_1)_{A^G\widehat{\otimes}\mathfrak{C}}, \mathcal{D}_1)$ is represented by

$$(\mathbb{C}, (\ker \mathcal{D}_1)_{A^G\widehat{\otimes}\mathfrak{C}}, 0) = (\mathbb{C}, (A^G\widehat{\otimes}(\ker \mathcal{D}_G\widehat{\otimes}\mathfrak{C}))_{A^G\widehat{\otimes}\mathfrak{C}}, 0).$$

It is clear that $(\mathbb{C}, (A^G\widehat{\otimes}(\ker \mathcal{D}_G\widehat{\otimes}\mathfrak{C}))_{A^G\widehat{\otimes}\mathfrak{C}}, 0)$ is the external Kasparov product of $(\mathbb{C}, A^G_{A^G}, 0)$ and $(\mathbb{C}, (\ker \mathcal{D}_G\widehat{\otimes}\mathfrak{C})_{\mathfrak{C}}, 0)$. The spinor bundle over G is $\$_G = G \times W$, where $W = (\$_G)_e$. Since

$$\mathcal{D}_G = \sum_{j=1}^{\dim G} c(X_j^\flat) X_j$$

for an orthonormal invariant frame $\{X_1, \ldots, X_{\dim G}\}$ for TG, $\ker \mathcal{D}_G$ is precisely the sections of $G \times W$ which are constant on each connected component of G, and so $\ker \mathcal{D}_G \cong \mathbb{C}^N\widehat{\otimes}W$, where N is the number of connected components of G. Because G is abelian, its action on W is trivial, [30], and hence its action is also trivial on $\ker \mathcal{D}_G\widehat{\otimes}\mathfrak{C}$. It follows that the even and odd parts of $\ker \mathcal{D}_G\widehat{\otimes}\mathfrak{C}$ are equivariantly isomorphic as Hilbert $\mathfrak{C}$-modules (for $\dim G > 0$). For $\dim G$ even, this is because Clifford multiplication by a vector in T_eG of norm 1 defines a unitary isomorphism between the even and odd parts of W, and hence the even and odd parts of $\ker \mathcal{D}_G$ are isomorphic. For $\dim G$ odd, $\ker \mathcal{D}_G$ is trivially graded, and the Hilbert module isomorphism between the even and odd parts of $\ker \mathcal{D}_G \widehat{\otimes} \mathbb{C}\mathrm{l}_1$ is implemented by the generator c of $\mathbb{C}\mathrm{l}_1$. Hence $(\mathbb{C}, (\ker \mathcal{D}_G \widehat{\otimes} \mathfrak{C})_{\mathfrak{C}}, 0)$ defines a trivial class in $KK_G(\mathbb{C}, \mathfrak{C})$. Thus $\mathrm{index}_G(\mathcal{D}) = \big([(\mathbb{C}, A^G_{A^G}, 0)] \widehat{\otimes} 0\big) \widehat{\otimes}_{A^G \widehat{\otimes} \mathfrak{C}} y = 0$. $\square$

Combining this result with Theorem 6.6 recovers a special case of a theorem of Atiyah and Hirzebruch, [1].

Corollary 7.2 (Atiyah-Hirzebruch). *Let M be a compact spin manifold which admits a free isometric action by a torus. Then the spin Dirac operator has zero equivariant index, and hence $\widehat{A}(M) = 0$.*

Proof. Either the torus action or the action of its double cover lifts to the spinor bundle, and in either case the spin Dirac operator is equivariant. The spectral triple defined by the Dirac operator factorises by Theorem 6.6, and the spin Dirac operator has trivial equivariant index by Proposition 7.1. $\square$

8. Example: the Dirac operator on the 2-sphere.

The spinor Dirac operator $\mathcal{D}$ on the sphere S^2 defines an even spectral triple $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$. The circle acts on S^2 by rotation about the north-south axis, and there are countably infinitely many lifts of this action to $L^2(\$_{S^2})$, such that $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ is an equivariant spectral triple. One can then ask whether any of these spectral triples can be factorised, but since the action of $\mathbb{T}$ on S^2 is not free we cannot apply the earlier theory.

In fact, we cannot factorise $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$, since the spectral subspace assumption is not satisfied, and, more seriously, $K^1(C(S^2)^{\mathbb{T}}) = K^1([0,1]) = \{0\}$. Since the class of the triple $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ in $K^0(C(S^2))$ is non-zero, it is impossible to recover this class under the Kasparov product between $KK^1(C(S^2), C(S^2)^{\mathbb{T}})$ and $KK^1(C(S^2)^{\mathbb{T}}, \mathbb{C}) = \{0\}$.

Instead, we remove the poles, and restrict the spectral triple to the complement and consider $(C_c^\infty(S^2 \setminus \{N, S\}), L^2(\$_{S^2}), \mathcal{D})$ and ask whether this equivariant spectral triple can be factorised. The circle now acts freely, and hence the spectral subspace assumption is satisfied.

We show that factorisation is achieved for $(C_c^\infty(S^2 \setminus \{N, S\}), L^2(\$_{S^2}), \mathcal{D})$ for every possible lift of the circle action. Unlike for a free action on a compact manifold, the positivity criterion is satisfied for precisely two choices of the character $\ell \in \mathbb{Z}$ of Defintion 2.13 used to define the right-hand module.

We will describe the Dirac operator $\mathcal{D}$ on the spinor bundle $\$_{S^2}$ over S^2, [12, 31].

Let N be the North pole of S^2, and let U_N be $S^2 \setminus \{N\}$. A chart for U_N is given by stereographic projection onto $\mathbb{C}$. This chart defines a trivialisation of the spinor bundle $\$_{S^2}$. All work will be done in the U_N trivialisation unless explicitly stated otherwise. We will work in the standard polar coordinates $(\theta, \phi) \in (0, \pi) \times (0, 2\pi)$.

The spinor Dirac operator is given by

$$\mathcal{D} = \begin{pmatrix} 0 & e^{i\phi}\big(i\partial_\theta + \csc(\theta)\partial_\phi + i\cot(\theta/2)/2\big) \\ e^{-i\phi}\big(i\partial_\theta - \csc(\theta)\partial_\phi + i\cot(\theta/2)/2\big) & 0 \end{pmatrix}. \tag{9}$$

The Hilbert space $L^2(\$_{S^2})$ is graded by $\gamma = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. The action of the circle $\mathbb{T}$ on S^2 is $t \cdot (\theta, \phi) = (\theta, \phi + 2\pi t)$. There are countably infinitely many lifts of this action which make $(C^\infty(S^2), L^2(\$_{S^2}), \mathcal{D})$ into a $\mathbb{T}$-equivariant spectral triple.

PROPOSITION 8.1. *Any even unitary action of* $\mathbb{T}$ *on* $L^2(\$_{S^2})$ *which commutes with* $\mathcal{D}$ *and which is compatible with the action on* $C(S^2)$ *is equal to* $V_k : \mathbb{T} \to U(L^2(\$_{S^2}))$ *for some* $k \in \mathbb{Z}$*, where*

$$V_{k,t}\begin{pmatrix} f(\theta,\phi) \\ g(\theta,\phi) \end{pmatrix} := \begin{pmatrix} e^{2\pi i k t} f(\theta, \phi - 2\pi t) \\ e^{2\pi i (k-1) t} g(\theta, \phi - 2\pi t) \end{pmatrix}.$$

PROOF. We require the action of $\mathbb{T}$ on $L^2(\$_{S^2})$ to be compatible with the action α of $\mathbb{T}$ on $C(S^2)$, which is $\alpha_t(f)(\theta, \phi) = f(\theta, \phi - 2\pi t)$. Hence the action on spinors is of the form

$$V_t\begin{pmatrix} f(\theta,\phi) \\ g(\theta,\phi) \end{pmatrix} = \begin{pmatrix} a & b \\ d & h \end{pmatrix}\begin{pmatrix} f(\theta, \phi - 2\pi t) \\ g(\theta, \phi - 2\pi t) \end{pmatrix},$$

where a, b, d and h can *a priori* depend on θ, ϕ and t. Since the action of $\mathbb{T}$ should commute with the grading, we require $b = d = 0$. Requiring that the action is unitary, that it commutes with $\mathcal{D}$ and that it is a group homomorphism determines that $a = e^{2\pi i k t}$ and $h = e^{2\pi i (k-1) t}$ for some $k \in \mathbb{Z}$. □

REMARK. None of these actions preserve the real structure on $\$_{S^2}$, so they are spin^c but not spin actions. There is however a unique lift of the action of $\mathbb{T}$ via the double covering $\mathbb{T} \to \mathbb{T}$, $t \cdot (\theta, \phi) = (\theta, \phi + 4\pi t)$, to a spin action given by setting $k = 1/2$ and replacing t by $2t$ in Proposition 8.1.

We fix $k \in \mathbb{Z}$ for the remainder of the section, fixing a representation $V_k : \mathbb{T} \to U(L^2(\$_{S^2}))$. The spectral subspaces of $C(S^2)$ are

$$C(S^2)_j = \begin{cases} \{f(\theta) : f \in C([0,1])\} & \text{if } j = 0 \\ \{f(\theta)e^{-ij\phi} : f \in C_0((0,1))\} & \text{if } j \neq 0. \end{cases}$$

Hence

$$\overline{C(S^2)_j C(S^2)_j^*} \cong \begin{cases} C([0,1]) & \text{if } j = 0 \\ C_0((0,1)) & \text{if } j \neq 0. \end{cases}$$

Since $C_0((0,1))$ is not a complemented ideal in $C(S^2)^{\mathbb{T}} \cong C([0,1])$, $C(S^2)$ does not satisfy the spectral subspace assumption, and so we cannot define the left-hand module if we use the C^*-algebra $C(S^2)$. However, the SSA is satisfied for $C_0(S^2 \setminus \{N,S\})$, since the action on $S^2 \setminus \{N,S\}$ is free, by [25, Thm. 7.2.6].

By taking the fundamental vector field map and normalising as in Section 6, we define the map $\eta : \mathbb{C}l(\mathbb{T}_e) \to B(L^2(\$_{S^2}))$ by

$$\eta(c(dt)) = -\frac{1}{\sqrt{g(d\phi, d\phi)}}c(d\phi) = \begin{pmatrix} 0 & -e^{i\phi} \\ e^{-i\phi} & 0 \end{pmatrix}$$

We check that η satisfies the conditions of Definition 2.13. Clearly $\eta(c(dt))$ commutes with the algebra, so Condition (1) is satisfied. Since $a\eta(c(dt))$ is a smooth bundle endomorphism for all $a \in C_c^\infty(S^2 \setminus \{S,N\})$, $a\eta(c(dt))$ preserves $\mathrm{dom}(\mathcal{D})$. It remains to check the commutation condition. We compute:

$$\{\mathcal{D}, \eta(c(dt))\} = \begin{pmatrix} 2\csc(\theta)\partial_\phi - i\csc(\theta) & 0 \\ 0 & 2\csc(\theta)\partial_\phi + i\csc(\theta) \end{pmatrix}.$$

Hence if $f(\theta)e^{-ij\phi} \in C_c^\infty(S^2 \setminus \{S,N\})_j$, then

$$\begin{aligned}
&\{\mathcal{D}, \eta(c(dt))\}f(\theta)e^{-ij\phi}P_\ell \\
&= \begin{pmatrix} 2i\csc(\theta)(k-\ell-j) - i\csc(\theta) & 0 \\ 0 & 2i\csc(\theta)(k-\ell-j-1) + i\csc(\theta) \end{pmatrix} f(\theta)e^{-ij\phi}P_\ell \\
&= -i\csc(\theta)(2j+2\ell-2k+1)f(\theta)e^{-ij\phi}P_\ell.
\end{aligned}$$

Since $f \in C_c((0,\pi))$, this is bounded, and so Condition (2) of Definition 2.13 is satisfied. Therefore (ℓ, η) satisfy the conditions of Definition 2.13 for any $\ell \in \mathbb{Z}$.

Let $n, \ell \in \mathbb{Z}$, and let $\xi = \begin{pmatrix} f(\theta)e^{i(k-n-\ell)\phi} \\ g(\theta)e^{i(k-n-\ell-1)\phi} \end{pmatrix} \in \mathrm{dom}(\mathcal{D}) \cap L^2(\$_{S^2})_{n+\ell}$. Then the positivity criterion reduces to

$$\begin{aligned}
&\langle \mathcal{D}\xi, in\eta(c(dt))P_{n+\ell}\xi\rangle + \langle in\eta(c(dt))P_{n+\ell}\xi, \mathcal{D}\xi\rangle \\
&= 4\pi n(n-k+\ell+1/2)\int_0^\pi d\theta\Big(|f(\theta)|^2 + |g(\theta)|^2\Big).
\end{aligned}$$

If $p(n) = 2n(n-k+\ell+1/2)$ is non-negative for all $n \in \mathbb{Z}$, then the factorisation condition is satisfied. Conversely, since $\int_0^\pi d\theta\Big(|f(\theta)|^2 + |g(\theta)|^2\Big)$ is not bounded by $\|\xi\|^2$, if $p(n) < 0$ for some $n \in \mathbb{Z}$, then $\langle \mathcal{D}\xi, -in\eta(c(dt))P_{n+\ell}\xi\rangle + \langle -in\eta(c(dt))P_{n+\ell}\xi, \mathcal{D}\xi\rangle$ is not bounded from below and the factorisation condition is not satisfied.

Since $\ell \in \mathbb{Z}$ has thus far not been fixed, we will determine for which values of ℓ the polynomial $p : \mathbb{Z} \to \mathbb{R}$ is non-negative. As a real-valued polynomial, p has a minimum at $x = (k-\ell)/2 - 1/4$.

Suppose $k-\ell$ is even. Then the integer values of n either side of this minimum are $n=(k-\ell)/2-1$ and $n=(k-\ell)/2$, at which $p(n)$ has respective values $-(\ell-k+2)(\ell-k-1)/2$ and $-(\ell-k+1)(\ell-k)/2$. The smallest of these two values is $p((k-\ell)/2)=-(\ell-k+1)(\ell-k)/2$. As a function of ℓ, $q(\ell)=-(\ell-k+1)(\ell-k)/2$ has a maximum at $\ell=k-1/2$. The integer values on either side of this with $k-\ell$ even are $\ell=k$ and $\ell=k-2$, at which $q(\ell)$ has respective values 0 and -1. Therefore if $k-\ell$ is even, then $p(n)$ is non-negative if and only if $\ell=k$.

Suppose now that $k-\ell$ is odd. Then the integer values of n either side of the minimum $n=(k-\ell)/2-1/4$ are $n=(k-\ell)/2-1/2$ and $n=(k-\ell)/2+1/2$, at which $p(n)$ has respective values $-(\ell-k+1)(\ell-k)/2$ and $-(\ell-k+2)(\ell-k-1)/2$, the smallest of which is $p((k-\ell)/2-1/2)=-(\ell-k+1)(\ell-k)/2$. As a function of ℓ, $r(\ell)=-(\ell-k+1)(\ell-k)/2$ has a maximum at $\ell=k-1/2$. The values on either side such that $k-\ell$ is odd are $\ell=k-1$ and $\ell=k+1$, at which $r(\ell)$ has respective values 0 and -1. Therefore if $k-\ell$ is odd, then $p(n)$ is non-negative if and only if $\ell=k-1$.

Thus factorisation is achieved for the nonunital equivariant spectral triple $(C_c^\infty(S^2\setminus\{N,S\}),L^2(\$_{S^2}),\mathcal{D})$ for any lift V_k of the circle action to $L^2(\$_{S^2})$, by choosing the characters $\ell=k$ or $\ell=k-1$ when constructing the right-hand module.

We conclude the 2-sphere example by examining the operator on the right-hand module, which, upon identifying $C_0(S^2\setminus\{N,S\})^{\mathbb{T}}$ with $C_0((0,\pi))$ and $\mathbb{C}\mathrm{l}(\mathbb{T}_e)$ with $\mathbb{C}\mathrm{l}_1$, defines a spectral triple for $C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1$. One might wonder whether it can be obtained from an odd spectral triple for $C_0((0,\pi))$, such as that defined by (some self-adjoint extension of) the Dirac operator on $(0,\pi)$. We show that this is not the case; for each $\ell\in\mathbb{Z}$ there is no odd spectral triple $(C_c^\infty((0,\pi)),\mathcal{H}',\mathcal{D}')$ such that the right-hand module is the even spectral triple corresponding to $(C_c^\infty((0,\pi)),\mathcal{H}',\mathcal{D}')$.

Let $k,\ell\in\mathbb{Z}$ be fixed, where $V_k:\mathbb{T}\to U(L^2(\$_{S^2}))$ is the representation and (ℓ,η) is the pair of Definition 2.13. Define $F:\mathcal{H}_\ell\to L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2$ by

$$F\left(\begin{pmatrix} f(\theta)e^{i(k-\ell)\phi} \\ g(\theta)e^{i(k-\ell-1)\phi}\end{pmatrix}\right)=\sqrt{\sin\theta}\begin{pmatrix} if(\theta) \\ g(\theta)\end{pmatrix}.$$

The map F is a $C_0((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1$-linear $\mathbb{Z}_2$-graded unitary isomorphism between $L^2(\$_{S^2})_\ell$ and $L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2$, where the latter space is graded by $1\widehat{\otimes}\begin{pmatrix}1&0\\0&-1\end{pmatrix}$ and the action of $\mathbb{C}\mathrm{l}_1$ is given by $c\mapsto 1\widehat{\otimes}\begin{pmatrix}0&1\\1&0\end{pmatrix}$. We can compute

$$F\circ\mathcal{D}_\ell\circ F^{-1}=-i\partial_\theta\widehat{\otimes}\omega-(k-\ell-1/2)\csc(\theta)\widehat{\otimes}c,$$

where $\omega=\begin{pmatrix}0&-i\\i&0\end{pmatrix}$. Hence the right-hand module is unitarily equivalent to the spectral triple

$$\Big(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2,\ -i\partial_\theta\widehat{\otimes}\omega-(k-\ell-1/2)\csc(\theta)\widehat{\otimes}c\Big).$$

If $(C_c^\infty((0,\pi)), L^2([0,\pi]), \mathcal{D}')$ is an odd spectral triple, then the corresponding even spectral triple is $(C_c^\infty((0,\pi))\widehat{\otimes}\mathbb{C}\mathrm{l}_1, L^2([0,\pi])\widehat{\otimes}\mathbb{C}^2, \mathcal{D}'\widehat{\otimes}\omega)$. The presence of the $(k-\ell-1/2)\csc(\theta)\widehat{\otimes}c$ factor means that the right-hand module is not the even spectral triple corresponding to any odd spectral triple.

References

[1] M. Atiyah and F. Hirzebruch, *Spin-manifolds and group actions*, Essays on Topology and Related Topics (Mémoires dédiés à Georges de Rham), Springer-Verlag, Berlin, 1970, pp. 18–28.

[2] S. Baaj and P. Julg, *Théorie bivariante de Kasparov et opérateurs non bornées dans les C^*-modules hilbertiens*, C. R. Acad. Sci. Paris **296** (1983), 875–878.

[3] N. Berline, E. Getzler, and M. Vergne, *Heat Kernels and Dirac Operators*, Springer, Berlin, 2004.

[4] S. Brain, B. Mesland, and W. D. van Suijlekom, *Gauge theory for spectral triples and the unbounded Kasparov product*, J. Noncommut. Geom. **10** (2016), no. 1, 135–206.

[5] A. L. Carey, V. Gayral, A. Rennie, and F. A. Sukochev, *Index theory for locally compact noncommutative geometries*, Mem. Amer. Math. Soc. **231** (2014), no. 1085.

[6] A. L. Carey, S. Neshveyev, R. Nest, and A. Rennie, *Twisted cyclic theory, equivariant KK-theory and KMS states*, J. reine angew. Math. **650** (2011), 161–191.

[7] A. Chamseddine, A. Connes, and W. van Suijlekom, *Inner fluctuations in noncommutative geometry without the first order condition*, J. Geom. Phys. **73** (2013), 222,234.

[8] A. Connes, *Noncommutative Geometry*, Academic Press, London and San Diego, 1994.

[9] A. Connes and G. Landi, *Noncommutative manifolds: the instanton algebra and isospectral deformations*, Commun. Math. Phys. **221** (2001), 141–159.

[10] L. Dabrowski and A. Sitarz, *Noncommutative circle bundles and new Dirac operators*, Commun. Math. Phys. **318** (2013), 111–130.

[11] L. Dabrowski, A. Sitarz, and A. Zucca, *Dirac operators on noncommutative principal circle bundles*, Int. J. Geom. Methods Mod. Phys. **11** (2014).

[12] J. M. Gracia-Bondía, J. C. Várilly, and H. Figueroa, *Elements of Noncommutative Geometry*, Birkhäuser, Boston, 2001.

[13] Nigel Higson and John Roe, *Analytic K-Homology*, Oxford University Press, Oxford, 2000.

[14] J. Kaad and M. Lesch, *Spectral flow and the unbounded Kasparov product*, Adv. Math. **248** (2013), 495–530.

[15] G. G. Kasparov, *The operator K-functor and extensions of C^*-algebras*, Math. USSR Izv. **16** (1981), 513–572.

[16] D. Kucerovsky, *The KK-product of unbounded modules*, J. K-Theory **11** (1997), 17–34.

[17] ———, *A lifting theorem giving an isomorphism of KK-products in bounded and unbounded KK-theory*, J. Operator Theory **44** (2000), 255–275.

[18] E. C. Lance, *Hilbert C^*-Modules*, CUP, Cambridge, 1995.

[19] H. B. Lawson and M.-L. Michelsohn, *Spin Geometry*, PUP, Princeton, 1989.

[20] B. Mesland, *Unbounded bivariant K-theory and correspondences in noncommutative geometry*, J. reine angew. Math. **691** (2014), 101–172.

[21] B. Mesland and A. Rennie, *Nonunital spectral triples and metric completeness in unbounded KK-theory*, J. Funct. Anal **271** (2016), no. 9, 2460–2538.

[22] M. Reed and B. Simon, *Methods of Modern Mathematical Physics. I. Functional Analysis. 2nd ed.*, Academic Press, New York, 1980.

[23] D. Pask and A. Rennie, *The noncommutative geometry of graph C^*-algebras I: The Index Theorem*, J. Funct. Anal. **233** (2006), 92–134.

[24] D. Pask, A. Rennie, and A. Sims, *The noncommutative geometry of k-graph C^*-algebras*, J. K-Theory **1** (2008), 259–304.

[25] N. C. Phillips, *Equivariant K-Theory and Freeness of Group Actions on C^*-Algebras*, Springer-Verlag, Berlin, 1987.

[26] I. Raeburn and D. P. Williams, *Morita Equivalence and Continuous-Trace C^*-algebras*, AMS, Providence, 1998.

[27] B. L. Reinhart, *Foliated manifolds with bundle-like metrics*, Ann. Math. **69** (1959), 119–132.

[28] M. A. Rieffel, *Deformation Quantization for Actions of $\mathbb{R}^d$*, Mem. Amer. Math. Soc. **106** (1993), no. 506.

[29] K. Schwieger and S. Wagner, *Free actions of compact groups on C^*-algebras, part I*, Adv. Math. **317** (2017), 224–266.

[30] S. Slebarski, *Dirac operators on a compact Lie group*, Bull. London Math. Soc. **17** (1985), 579–583.

[31] J. C. Várilly, *An Introduction to Noncommutative Geometry*, J. Eur. Math. Soc., Zürich, 2006.

[32] A. Zucca, *Dirac Operators on Quantum Principal G-Bundles*, SISSA, Digital Library, 2013.

Mathematical Sciences Institute
Australian National University,
Canberra, Australia
& School of Mathematics and Applied Statistics
University of Wollongong,
Wollongong, Australia
& Institut für Analysis
Leibniz Universität Hannover, Germany

School of Mathematics and Applied Statistics
University of Wollongong,
Wollongong, Australia