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High sensitive sensing by a laser diode with dual optical feedback operating at period-one oscillation

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Abstract

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High sensitive sensing by a laser diode with dual optical feedback operating at period-one oscillation

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Laser dynamics have great potentials for various applications, ranging from cryptography to microwave photonics and instrumentation. This letter presents a design for achieving high sensitive sensing and measurement using the dynamics of a laser diode (LD) with dual external cavity. In the design, one cavity is used to control the dynamics, making the LD operate at the period-one (P1) oscillation state, and the other one is associated with the quantities to be measured. The Lang-Kobayashi (L-K) equations are modified and solved to develop a bifurcation diagram for the design, from which we determine P1 state and investigate the sensing performance within this state. It is shown that, when operating in P1, the laser intensity exhibits an oscillation with its amplitude modulated by a traditional optical feedback interferometric (OFI) signal (generated with a single cavity and LD operating at steady state). It is also observed that the modulation depth is remarkably larger than the magnitude of a traditional OFI signal. This leads to significant increase in the sensitivity of sensing and measurement and hence provides an attractive solution for the detection of very small or weak physical quantities. An experimental system is designed and experimental results verify the high sensitive sensing performance of the proposed design.

The behavior of a laser diode (LD) can be significantly affected by its external optical feedback. This may give rise to negative effect on the LD performance, e.g., degrading the modulation response characteristics, enhancing laser intensity noise, etc. ¹ Meanwhile, it also enables many applications. For example, a class of laser interferometry, termed optical feedback interferometry (OFI) or self-mixing interferometry (SMI), utilizes such external optical feedback effect in an LD ². As a promising non-contact sensing technology, OFI has attracted much attention of researchers in recent decades due to its merits of minimum part-count scheme, low cost in implementation and ease in optical alignment. A sensing system with the OFI technique consists of an LD, a photodetector (PD) packaged at the rear of the LD, a lens and a target that is associated with the quantities to be measured. In general, the LD undergoes a weak level optical feedback from the target and operates at a steady state. Various OFI-based sensing applications have been reported, including measurement of displacement, velocity, vibration, laser related parameters, thickness, mechanical resonance ³⁻¹², etc. Recently, OFI based sensing has been extended for imaging, material parameter measurement, near-field microscopy, chaotic radar, acoustic detection, biomedical applications etc. ^{5,13,14}

An OFI system for displacement measurement provides the same resolution as the traditional two-beam interferometer ¹⁵ i.e., for a sensing signal generated by an OFI (called OFI signal hereafter), each fringe variation corresponds to a target displacement of a half of the laser wavelength. It is noted that the magnitude of the OFI signal mainly depends on the optical feedback level.

In some practical applications, the target surface to be measured has very low reflectivity and thus it is unable to generate adequate feedback light reflected back into the LD, leading to weak and blurred OFI signal. In this situation, the sensing sensitivity of the system is severely degraded and the OFI system may even lose its sensing ability. A general way to address this problem is to affix a

mirror or a piece of material with high reflectivity on the target, so that a high enough feedback level can be generated to get a clear OFI signal. However, in some applications, it is inconvenient or even impossible to affix high reflectivity materials on the target, e.g., the target to be measured is a living organism, a fluid field, or acoustic emission. Therefore, it is highly desirable to develop a method which enables the OFI system to achieve high sensitive sensing.

A few researchers proposed to use a dual-cavity configuration to improve OFI performance, e.g. the work in [16] shows that one of the cavities can be used to compensate for frequency fluctuation. In [17] and [18], 2-D vibration measurement and moving detection of multiple targets are achieved using dual-cavity OFI systems, respectively. The work in [19] proposes to use a second cavity (reference cavity) to realize nanoscale displacement sensing. However, these works did not explicitly address the improvement of sensing sensitivity. Only the work in [20] explored that the displacement sensitivity can be enhanced by 5 times compared to the signal cavity OFI with an accurate location control for one of the two cavities. In addition, for the above mentioned OFI systems, no matter with single or dual cavity configuration, the LD is always designed to operate at a steady state.

With the increase of the optical feedback level, an LD will leave the steady state and enter other operation states such as period-one (P1) oscillation, multi-periodic oscillation and chaos, and rich dynamics can then be observed. In recent years, LD dynamics have been investigated and found their various potential applications in optical communications, defense and security, and detection ^{1,21}. Our work in [22] showcased the sensing capability of an OFI system with a moving target and high level optical feedback operating above the steady boundary. In this case, the laser intensity signal contains very high frequency components and its amplitude is modulated and exhibited a clear envelop, which contains the displacement information of the external target. Our work in [22] opened a way to achieve more sensitive sensing using laser dynamics. However, the problem with the work is that the

laser intensity exhibits very complicated waveform due to the moving target with high feedback. It is not easy to ensure that the system has a stable performance during the measurement because the LD may switch among different states.

To solve the problem presented above, we propose a design, which enables the system to achieve stable operation in the P1 state and greatly enhances the sensing sensitivity. In this design, we use an LD with a dual-cavity configuration. By choosing the cavity parameters properly, we are able to use one cavity to control the laser dynamics and ensure that the LD operates at P1 during the measurement process. The other cavity is associated with the physical parameters to be measured. With such a design, we can achieve an ultra-high sensing sensitivity. In this work, we firstly modify the Lang-Kobayashi (L-K) equations to describe our proposed design and develop its bifurcation diagram through solving the L-K equations. The bifurcation diagram indicates the region of the parameters, making the LD of the system operate in the P1 state. The sensing performance is analyzed, and then an experimental system is built to verify the proposed design.

A dual-cavity OFI system is depicted in Fig. 1. One cavity with an initial cavity length L_{01} is formed by Target-1, which generates the displacement (or other quantities) to be measured. Target-1 is in a continuous movement and has a weak reflective surface. The other cavity with an initial cavity length L_{02} is formed by Target-2, which provides a pre-feedback to control the optical feedback level so that the LD is enabled to operate in the P1 state.

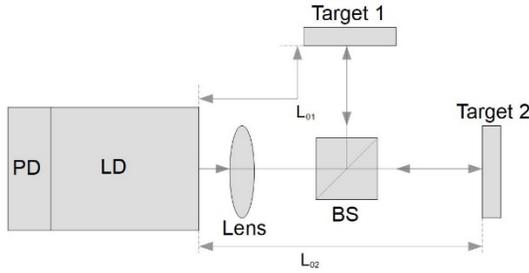


Fig. 1. Dual-cavity OFI

The behaviour of an LD with single cavity is described by the well-known L-K equations²³. A laser with dual-cavity can be modelled as below by modifying the L-K equations²⁴.

$$\begin{aligned} \frac{dE(t)}{dt} = & \frac{1}{2} \left\{ G[N(t), E(t)] - \frac{1}{\tau_p} \right\} E(t) \\ & + \frac{\kappa_1}{\tau_{in}} \cdot E(t - \tau_1) \cdot \cos[\omega_0 \tau_1 + \phi(t) - \phi(t - \tau_1)] \\ & + \frac{\kappa_2}{\tau_{in}} \cdot E(t - \tau_2) \cdot \cos[\omega_0 \tau_2 + \phi(t) - \phi(t - \tau_2)] \end{aligned} \quad (1)$$

$$\begin{aligned} \frac{d\phi(t)}{dt} = & \frac{1}{2} \alpha \left\{ G[N(t), E(t)] - \frac{1}{\tau_p} \right\} \\ & - \frac{\kappa_1}{\tau_{in}} \cdot \frac{E(t - \tau_1)}{E(t)} \cdot \sin[\omega_0 \tau_1 + \phi(t) - \phi(t - \tau_1)] \\ & - \frac{\kappa_2}{\tau_{in}} \cdot \frac{E(t - \tau_2)}{E(t)} \cdot \sin[\omega_0 \tau_2 + \phi(t) - \phi(t - \tau_2)] \end{aligned} \quad (2)$$

$$\frac{dN(t)}{dt} = J - \frac{N(t)}{\tau_s} - G[N(t), E(t)] E^2(t) \quad (3)$$

In Eq.(1)-Eq.(3), there are three variables, i.e., electric field amplitude $E(t)$, electric field phase $\phi(t)$ and carrier density $N(t)$. $\phi(t)$ is can be expressed as $\phi(t)=[\omega(t) - \omega_0] t$, where $\omega(t)$ is the instantaneous optical angular frequency for an LD with external feedback, and ω_0 is the unperturbed optical angular frequency for a solitary LD.

$G[N(t), E(t)] = G_N [N(t) - N_0] [1 - \varepsilon \Gamma E^2(t)]$ is the modal gain per unit time. The physical meanings of the symbols appearing in Eq.(1)-Eq.(3) and the typical values of the parameters are shown in Table 1²⁵.

Table 1: Physical meaning of symbols in Eq.(1)-Eq.(3)

Symbol	Physical Meaning	Value
G_N	Modal gain coefficient	$8.1 \times 10^{-13} m^3 s^{-1}$
N_0	Carrier density at transparency	$1.1 \times 10^{24} m^{-3}$
ε	Nonlinear gain compression coefficient	$2.5 \times 10^{-23} m^3$
Γ	Confinement factor	0.3
τ_p	Photon life time	$2.0 \times 10^{-12} s$
τ_{in}	Internal cavity round-trip time	$8.0 \times 10^{-12} s$
τ_s	Carrier life time	$2.0 \times 10^{-9} s$
κ	Feedback strength	
τ	External cavity round trip time $\tau = 2L / c$, where L is external cavity length, c is speed of light	
ω_0	Angular frequency of solitary laser $\omega_0 = 2\pi / \lambda_0$, where λ_0 is the wavelength of the LD	
α	Line-width enhancement factor	3.0
J	Injection current density	

The varying laser intensity is called an OFI signal when the LD operates at steady state, and it is called an OFI-P1 signal when the LD operates at P1 state. An important parameter called optical feedback factor (denoted by C) can be used to characterize the OFI

and OFI-P1 signals. C is expressed as $C = \kappa\tau\sqrt{1+\alpha^2} / \tau_{in}$. The parameter settings for the system depicted in Fig. 1 are as follows. We set $J = 1.3J_{th}$, where J_{th} the injection current density threshold, and the LD wavelength is $\lambda_0 = 780nm$. Target-1 related parameters are $L_{01} = 0.10m$ and $C_1 = 0.1$. The initial cavity length for Target-2 is $L_{02} = 0.24m$, and the values of C_2 are chosen within the P1 region. The values for other parameters in the Eq. (1)-Eq. (3) are listed in Table 1. By solving Eq.(1)-Eq.(3), a bifurcation diagram for the configuration in Fig. 1 can be obtained and it is shown in Fig. 2. Note that, in the following simulations, a relative laser intensity $(E^2(t) - \overline{E^2(t)}) / E_0^2$ is used to represent an OFI or OFI-P1 signal, where $\overline{E^2(t)}$ is the mean of $E^2(t)$ and E_0^2 is the light intensity of the LD without optical feedback.

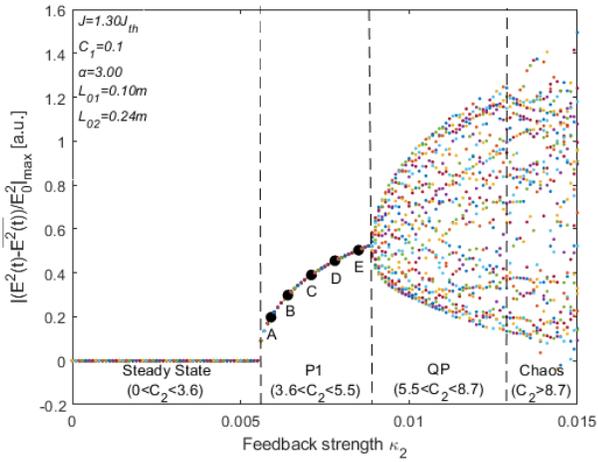


Fig. 2. Bifurcation diagram for dual-cavity OFI

We first compare the OFI signal (only Target-1 is used and the LD operates at steady state) and the OFI-P1 signal (both Target-1, Target-2 are used and the LD operates at P1 state). For the system with only Target-1, Target-1 is set in reciprocating movement with a triangular displacement waveform as shown in Fig. 3(a). By removing all Target-2 related terms in Eq. (1)-Eq.(3), we can get an OFI signal corresponding to Target-1's movement. Fig. 3(b). shows the corresponding OFI signal at steady state with $C_1 = 0.1$, and the peak-peak value is 6.92×10^{-4} (a.u.).

Then, we consider the system shown in Fig. 1, where both Target-1 and Target-2 are involved and the LD operates at P1 state. Target-1 related parameters remain unchanged, while Target-2 is set in stationary with $C_2 = 4.0$. The same displacement signal shown in Fig. 3(a) is applied on Target-1. The OFI-P1 signal corresponding to the displacement is presented in Fig. 4, where we find that OFI-P1 exhibits a high frequency oscillation at 2.28 GHz with its amplitude modulated by a slow-varying signal.

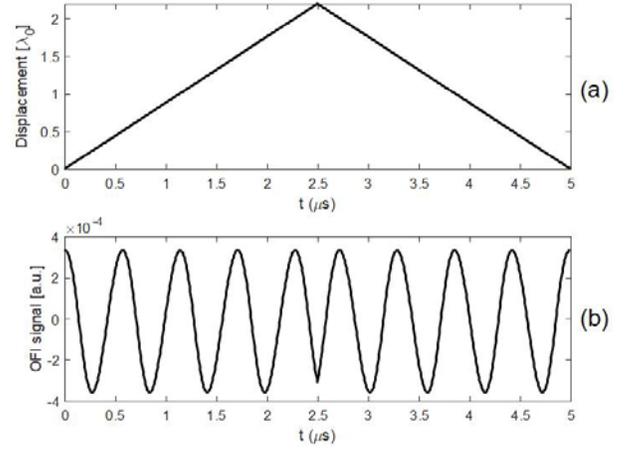


Fig. 3. (a) A triangular displacement waveform applied on Target-1. (b) OFI signal with $L_{01} = 0.10m$ and $C_1 = 0.1$.

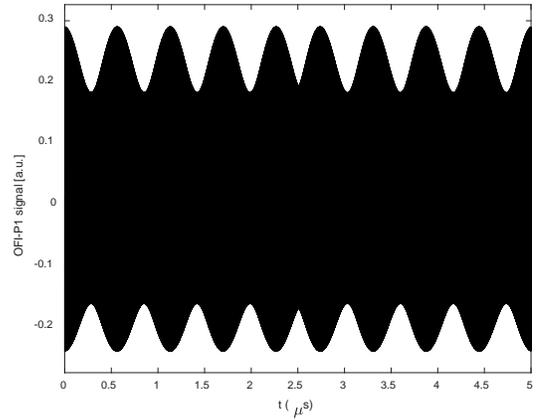


Fig. 4. OFI-P1 signal with $L_{02} = 0.24m$ and $C_2 = 4.0$.

We normalized both the OFI and the envelop of OFI-P1 signals, which are shown in Fig. 5(a). It is found that the waveforms are identical, i.e., the OFI-P1 with dual cavities can generate an amplitude-modulated (AM) sensing signal with P1 signal as the carrier modulated by an OFI signal. This indicates that OFI-P1 signal has the same sensing resolution (half laser wavelength) as the single cavity OFI signal. We also found that a significantly larger sensing signal can be obtained using the dual-cavity OFI system at P1. Fig. 5(b) shows the comparison of the two signals with single and dual cavities respectively. The peak-peak value of the envelop of OFI-P1 is 0.107 (a.u.), which is contrasted to 6.92×10^{-4} (a.u.) for OFI signal. The former is 155 times of the latter. It can be seen that the proposed design leads to significant increase in the sensitivity of sensing and measurement with respect to the single cavity OFI systems and it is able to provide an effective solution for the detection of very small or weak physical metrological quantities.

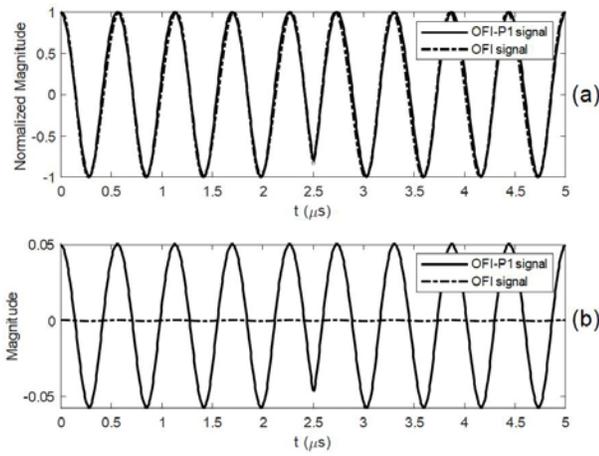


Fig. 5. (a) Normalized OFI and envelop of OFI-P1 have the same waveform. (b) Comparison of OFI-P1 envelop and OFI signal

To measure the sensitivity, we define R as the enhancement ratio of OFI-P1 envelop to OFI. The larger the modulation depth (envelop), the more sensitive an OFI-P1 for measurement. We further test the sensitivity within P1 for different C_2 values, which are marked as A-E in Fig. 2. The results for the enhancement ratio R are summarized in Table 2. It can be seen that, within P1, a smaller C_2 leads to a larger R . It is noted that C_2 can be easily adjusted in practice using a variable attenuator (VA) inserted in-between Target-2 and the LD.

Table 2: Sensitivity within P1 for different C_2 values

C_2	A: 3.73	B: 4.14	C: 4.55	D: 4.96	E: 5.38
R	268	129	93	76	64

Furthermore, we explored the influence of the system parameters (linewidth enhancement factor α , injection current density J and pre-feedback cavity length L_{02}) on the measurement sensitivity. If increasing α to 6 while keeping other parameters the same as those shown in Fig. 2 within P1 state, we have C_2 varying from 3.41 to 6.33 and their corresponding R changes from 200 to 33. If increasing J and keeping other parameters unchanged, we have C_2 varying from 7.65 to 8.85 and their corresponding R changes from 150 to 50. It can be seen that, if only allowing the LD to operate at P1 state, the proposed system can achieve remarkable amplification. However, an LD with large α or a high J can decrease the amplification ability for such system. We also tested the system with a long pre-feedback cavity length ($L_{02} = 0.75m$). We found that the long cavity leads to a very narrow P1 region. Although the amplification feature still exists, the narrow region will make it difficult to maintain the system at P1 state.

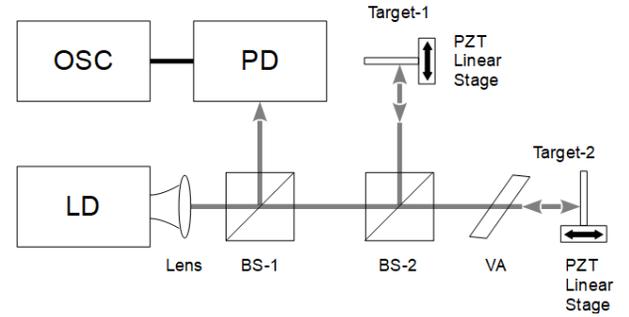


Fig. 6. Schematic layout of the experimental setup.

According to the structure of the dual-cavity OFI system in Fig. 1, an experimental system shown in Fig. 6 is built to further verify the proposed system. A single mode laser diode (Hatachi HL8325G, $\lambda=830\text{ nm}$, output power $P_0 = 40\text{ mW}$) is employed in this physical system. The LD is driven and temperature-stabilized by an LD controller (Thorlabs, ITC4001) with the injection current of 75mA and the temperature of $23 \pm 01^\circ\text{C}$. The light emitted by the LD is focused by a lens and split into two light beams by a beam splitter (BS-1) with splitting ratio of 50:50. One beam is directed to the Target-2 with a mirror surface for providing high enough optical feedback to the LD. A variable attenuator (VA) is inserted in-between the lens and Target-2 to adjust the feedback amount to ensure that the LD operates at P1 state. We also use it to adjust the modulation depth of OFI-P1 signals. The other beam from BS-1 is directed to the Target-1 which is a piezoelectric transducer (PZT) (PI P-841.20) with a very weak reflective surface. Since the OFI-P1 has a very high frequency, it needs a fast photodetector (PD) for its detection. Commonly, a commercial LD has an inbuilt PD which can be used for picking up OFI signals. However, the bandwidth of the inbuilt PD is less than 1GHz, which cannot meet the requirement of our design. Therefore, in this experiment, we use an external PD with a bandwidth of 9.5 GHz (Thorlabs, PDA8GS). Therefore, a second beam splitter (BS-2) is needed to direct the part of light into this external PD. To make a direct experimental comparison, both OFI and OFI-P1 signals are detected using the same PD, and they are then visualized by a fast oscilloscope (Tektronix DAS 70804).

One set of the recorded experimented waveforms are presented in Fig. 7. Fig. 7(a) is a voltage controlling signal applied on the PZT (Target-1) which drives Target-1 to move in a triangular waveform. The controlling voltage is provided by a signal generator with a frequency of 200Hz and amplitude of 0.85V. This makes Target-1 move with a displacement of 2500nm. Fig. 7(b) shows the corresponding OFI signal with only Target-1 in the experimental system. When recording this OFI signal, the light to Target-2 must be blocked. It can be seen that the OFI signal is nearly buried in noise due to the surface with very weak reflectivity. The peak-peak value of OFI signal is about 1.0 mv. Fig. 7(c) shows the OFI-P1 with both Target-1 and Target-2. It can be seen that this signal is large and has a clear envelop. The peak-peak of the envelop is about 50 mv. Therefore, the enhancement ratio R is 50 in this experiment. By carefully choosing the pre-feedback cavity related parameters, the higher enhancement ratio R can be achieved.

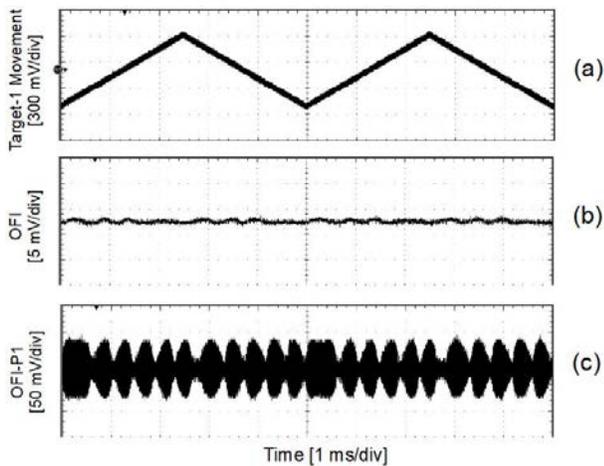


Fig. 7. Experimental results: (a) The triangular displacement waveform applied on Target-1. (b) The corresponding OFI signal. (c) The corresponding OFI-P1

This letter investigates the use of laser dynamics to improve the OFI sensing sensitivity and presents a dual-cavity OFI system to achieve ultra-high sensing sensitivity. A proof-of-concept prototyping system is designed by using a dual-cavity OFI operating at P1 state. The proposed design can boost the sensing sensitivity, which is demonstrated by both simulations and experiments. By introducing the 2nd cavity (pre-feedback cavity), it is easy to control the laser dynamics and ensure the LD to operate at P1 during the measurement. It does not require an accurate control of pre-feedback cavity length. By adjusting the feedback strength, the system can be set at a sensitive working point for sensing and measurement. Hence, this work lifts the restrictions in previous works^{20,22} and provides an effective solution for the detection of very small or weak physical quantities.

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