Time to ponding in one & two dimensional infiltration systems

Janine M. Stewart
University of Wollongong
TIME TO PONDING IN
ONE & TWO DIMENSIONAL
INFILTRATION SYSTEMS

*A thesis submitted in fulfilment of the
requirements for the award of the degree

MASTER OF SCIENCE (HONOURS)

from

UNIVERSITY OF WOLLONGONG

by

JANINE M. STEWART, BMath(Hons), GradDipEd

DEPARTMENT OF MATHEMATICS
1997
Dedicated to the memory of Uncle Maurice BE(Civil), NSW
Contents

A. Abstract 1

1. Introduction 3

2. Infiltration In One Dimension 11
   (2.1) Constant Supply Rate Boundary Conditions
   (2.2) Time Dependent Boundary Conditions
       a/ Linearly Increasing Time Dependent

3. Infiltration In Two Dimensions 28
   (3.1) Constant Supply Rate Boundary Conditions
       a/ Infiltration Boundary Conditions
       b/ Infiltration & Evaporation Boundary Conditions
       c/ Fractal Boundary Conditions
   (3.2) Time Dependent Boundary Conditions
       a/ Linearly Increasing Time Dependent
       b/ Periodic Boundary Conditions

4. Steady Infiltration In Sloping Porous Domains 60

5. Conclusions 70

6. Appendix 74

7. References 87
A. Abstract

Infiltration is the process whereby water enters soil through the surface. This can be a naturally occurring process, such as in rainfall, or can be artificially induced in engineering or agricultural applications.

In most cases, fluid is infiltrated into soil that is unsaturated. As water infiltrates drier unsaturated soil, the water molecules fill the smallest soil pores where they are bound tightly by capillary forces. In the transition to saturated soil, the capillary forces become less dominant and free water appears. Surface ponding is characterised by the appearance of this free water pooling on the surface of the soil and can occur even if the soil is dry at depth. Surface ponding is an important hydrological phenomenon with applications relevant to many fields from agriculture to civil engineering. With excessive irrigation techniques, once arable soils become water logged, the rising water table brings with it geological salts which kill vegetation rendering fertile soils effectively useless. However, ponding is a desirable phenomenon in areas of water catchment.

Before the emergence of highly versatile nonlinear analytic solution techniques for groundwater flow, reasonably accurate estimations for ponding times were available only with the use of numerical methods. Prior to this the linear and quasi linear models were applied to the problem of groundwater flow with mixed results.

An estimation for the time to surface ponding for a variety of one and two dimensional infiltration patterns is found using a number of analytic and numerical solution methods. It is found and is observable in the field that as the wetted proportion of the soil surface and the rate of surface infiltration increase the time to surface ponding decreases. It is found that this effect dominates over the spatial pattern of irrigation.

In this application horizontal and sloping fields are considered. In the case of a horizontal surface, it is found that surface ponding is unavoidable if the rate of surface
infiltration even locally exceeds the hydraulic conductivity at saturation. However, for an inclined surface, for a given basal inclination there exists a maximal surface infiltration rate for which basement saturation can be averted.
1. Introduction

The Darcy-Buckingham macroscopic theory of soil-water flow has endured the test of time as a successful scientific theory. In this theory, one neglects small-scale phenomena on the scale of single pores and grains. Just as for any fluid flow, groundwater flow through soil obeys the Equation of Continuity, expressed in this case as

\[
\frac{\partial \theta}{\partial t} + \nabla \cdot \mathbf{v} = 0
\]  

(1.1.1)

where \( \theta(x,z,t) \) represents the local volumetric concentration of the fluid in the soil with dimensions \([\text{length}]^3[\text{length}]^{-3}\) and \( \mathbf{v} \) is the Darcian volumetric flux density, measured in units of \([\text{length}[\text{time}]^{-1}\).

In order to derive an equation to model the flow of fluid through a uniform nonswelling soil, the continuity equation (1.1.1) is combined with Darcy's Law

\[
\mathbf{v} = -K(\theta)\nabla \Phi.
\]  

(1.1.2)

The hydraulic conductivity, \( K(\theta) \) is a measure of how well the soil transports fluid and has the dimensions \([\text{length}[\text{time}]^{-1}\). In the field, the hydraulic conductivity is a highly nonlinear function, varying over several orders of magnitude.

The potential energy per unit weight of water, \( \Phi \), is a function of the capillary potential and gravity. The capillary potential, \( \Psi \), is a measure of the energy state of water, and like the total potential or hydraulic head \( \Phi \), is measured in units of \([\text{length}]\).

In terms of gravity and the capillary potential, the total potential can be written,

\[
\Phi = \Psi(\theta) - z
\]  

(1.1.3)
where z is the vertical depth beneath the surface of the soil.

Combining equations (1.1.1) - (1.1.3), the resultant equation,

\[
\frac{\partial \theta}{\partial t} = \nabla \cdot (D(\theta) \nabla \psi(\theta)) - \frac{\partial K(\theta)}{\partial z}
\]

known as Richards' Equation, models groundwater flow through a uniform nonswelling soil. Assuming no hysteresis, there exists a one-to-one correspondence between the capillary potential \( \psi \), and the soil water concentration \( \theta \). This equation can be written in terms of a single dependent variable by noting that \( D(\theta) = K(\theta) \frac{d\psi}{d\theta} \), where \( D(\theta) \) is the diffusivity, with the dimensions \([\text{length}]^2[\text{time}]^{-1}\). Like the hydraulic conductivity, field occurring diffusivities are highly nonlinear. The diffusivity \( D(\theta) \) is related to capillary action rather than molecular diffusion. As fluid enters an initially dry soil, it is absorbed by the smallest pores first and is bound tightly by capillary forces. As a larger volume of fluid infiltrates into the soil, the new fluid is held by weaker capillary forces in larger pores as the soil becomes progressively saturated.

In terms of the water content \( \theta(x,z,t) \), Richards' Equation is written,

\[
\frac{\partial \theta}{\partial t} = \nabla \cdot (D(\theta) \nabla \theta) - \frac{\partial K(\theta)}{\partial z}
\]

Following the notation of Broadbridge and White (1988) Richards' Equation is rescaled in terms of dimensionless variables

\[
\frac{\partial \Theta}{\partial t^*} = \nabla_*(D_*(\Theta) \nabla_* \Theta) - \frac{\partial K_*(\Theta)}{\partial z^*}
\]

where
\[ (1.1.7a) \quad \Theta = \frac{\theta - \theta_n}{\theta_s - \theta_n} = \frac{\theta - \theta_n}{\Delta \theta} \]

\[ (1.1.7b) \quad K(\Theta) = \frac{K(\theta) - K_n}{K_s - K_n} = \frac{K(\theta) - K_n}{\Delta K} \]

\[ (1.1.7c) \quad D_*(\Theta) = D(\theta) \frac{t_s}{\lambda_s^2} \]

\[ (1.1.7d) \quad t_* = \frac{t}{t_s} \]

\[ (1.1.7e) \quad x_* = \frac{x}{\lambda_s} \quad z_* = \frac{z}{\lambda_s} \]

\[ (1.1.7f) \quad \lambda_s = \frac{D \Delta \theta}{\Delta K} \]

\[ (1.1.7g) \quad t_s = \overline{D} \left( \frac{\Delta \theta}{\Delta K} \right)^2 \]

\[ (1.1.7h) \quad \overline{D} = \frac{1}{\Delta \theta} \int_{\theta_s}^{\theta} D(\theta) d\theta. \]

Here, \( \overline{D} \) is the mean diffusivity, \( \theta_n \) is the initial concentration of fluid in the soil, \( \theta_s \) is the water content at saturation and \( K_n, K_s \) are the associated hydraulic conductivities given these soil water contents.

The capillary length scale \( \lambda_s \) is equal to a typical capillary rise and \( t_s \) is the associated gravity time scale, representing the time taken for a gravity-dominated travelling wave solution to propagate over a typical capillary length.
The physical system under consideration is that of fluid infiltrating into an initially dry soil, until surface ponding occurs. Surface ponding is characterised by the appearance of free water pooling at the surface of the soil which is an indication that the soil is locally saturated. In the case where the infiltration rate exceeds the value of the hydraulic conductivity at saturation, $K_s$, however, it is entirely possible for surface ponding to occur whilst the soil is dry at depth. The appearance of a free layer of fluid having depth $h$ ponding at the surface of the soil implies that the capillary potential $\Psi$ attains a non-negative value. Given that $\Psi$ is a monotonic decreasing function of the soil water content $\theta$ and $\Psi(\theta_s) = 0$, assuming no hysteresis, there exists a saturated zone at the surface of the soil even if the soil is dry at depth. Conversely, if the rate of surface infiltration is less than the hydraulic conductivity at saturation, unsaturated groundwater flow occurs in the absence of ponding. This is true because from equations (1.1.1)-(1.1.3), even in the absence of sorptive capillary action, water could be transported at the imposed rate $R < K_s$ by the flux $K(\theta) = R$ (for some $\theta < \theta_s$) which is due to gravity alone.

In agricultural applications and in civil construction work, the ponding phenomenon is generally best avoided as it may result in water run-off, a rise in the water table, increased soil salinity and soil erosion. During natural rainfall, some run-off is desirable for the purposes of water catchment. The quantity of water run-off must be estimated in the design of drainage systems. Because of the importance of the ponding phenomenon in agricultural and engineering applications, the prediction of ponding time is an important task in hydrological modelling and has come under the close scrutiny of many authors in a wide context of applications.

Rubin (1966) investigated the three kinds of infiltration due to rainfall: non-ponding, pre-ponding and post-ponding. A qualitative prediction of the changing characteristics of the soil moisture profile in terms of the depth, time and moisture content was established. Later authors however, investigated the ponding phenomenon exclusively.
Mein and Larson (1973) and later Swartzendruber (1974) modified the Green-Ampt equation to determine the time to ponding for steady rainfall periods. The Green-Ampt model is an over-simplified model that has a step function water concentration-depth profile at all times. As shown by Philip (1969), this profile arises from a delta function diffusivity that depends continuously on concentration. Chu (1978) extended the modified Green-Ampt equation to describe infiltration during periods of unsteady rain. In this case, two time parameters were utilised, ponding time and pseudotime which simply entails a shift in the time scale.

Knight (1983) also investigated pre-ponding and post-ponding infiltration. In this study, exact and approximate solutions of Richards' equation were utilised to express the time to ponding as a function of easily measurable soil water parameters. These include the soil water diffusivity, hydraulic conductivity at saturation and surface supply rate. Knight showed that the infiltration rate for post-ponding, unlike the Green-Ampt model used by Swartzendruber (1974), is not simply a translation of the curve for initial ponding. Knight further developed evidence that the cumulative infiltration can be used as an appropriate time-like variable in the case of variable surface flux.

Many problems involving variable surface flux can be difficult to solve analytically to obtain meaningful physical results. This problem has been considered however from an experimental perspective. Using an approximate analytical method, Parlange and Smith (1976) calculated the time to ponding under variable surface infiltration rates and expressed it in terms of soil water parameters easily measured in the field, namely the infiltration rate, sorptivity and saturated hydraulic conductivity. Kutilek (1980) also used heuristic techniques to calculate the time to ponding under conditions of constant infiltration. Here, ponding time was expressed also in terms of the infiltration rate and sorptivity. Unlike the Parlange and Smith model however, the time to ponding was also expressed in terms of the coefficient of t in Philip's (1969) power series expansion of the cumulative infiltration.
Chong (1983) applied the approximate solutions of both Parlange and Smith (1976) and Kutilek (1980) solution to estimate the sorptivity and then used this estimate to predict infiltration.

Previously, Hachum and Alfaro (1977) presented a physically based model to describe infiltration under any surface infiltration supply rate to implicitly predict infiltration after ponding. Experimental results were also used by Clothier et al. (1981b) to show that observable field infiltration phenomena agree with theoretical predictions resulting from the theory of constant flux. Clothier et al. further demonstrated that the time to surface saturation can be predicted but post-ponding fluid run-off is substantially more difficult to model as it is due to the influence of the particular soil matrix rather than external environmental factors. Perroux et al. (1981) further compared laboratory experiments with theoretical predictions for constant flux infiltration. These experiments were sufficiently accurate to become bench tests for later theoretical predictions of the moisture profile and the time to surface ponding.

Ponding formulae were also derived in the case of variable infiltration for high rainfall rates by Morel-Seytoux (1976, 1978, 1982). A power law relationship between the relative permeability of water as a function of the normalised water content was assumed. From these formulae, the depth of the cumulative infiltration and the ponding infiltration rate were calculated.

Using an exactly solvable model, Broadbridge and White (1987) introduced an exact expression for the time to ponding which encompasses soils with widely varying properties. After rescaling length and time variables, field and repacked soils can be expressed in terms of a single nonlinearity parameter which covers the spectrum from highly nonlinear soils, such as fine textured clays, to weakly nonlinear soils, like coarsely grained sands. A comparison for the time to ponding is made between the linear, Green-Ampt, Burgers and versatile nonlinear model. The time to ponding for each of these
models is parameterised in terms of readily measured field properties such as the
sorptivity, infiltration rate and saturated hydraulic conductivity.

Under investigation in this thesis, is the occurrence of ponding in time and space
subject to a variety of soil surface supply patterns and surface infiltration rates to
determine which soil water parameter has the most influence over the transition from
unsaturated to saturated flow. Throughout this thesis, it will be assumed that there exists
no hysteresis. That is, a one-to-one relationship between the capillary potential $\Psi$ and the
moisture content $\theta$ exists. Incipient ponding will then occur when $\Psi$ rises to the same
value as that for free water, taken to be when $\Psi = 0$. This corresponds to $\theta$ reaching its
saturated value $\theta_s$. We begin by considering one dimensional infiltration subject to a
constant supply rate imposed at the surface of the soil. Analytic expressions are presented
for the time to ponding by obtaining solutions to the linear, Burgers and nonlinear models
respectively. These models swathe the range of soil types which occur naturally in the
field and in the laboratory after repacking. A solution for one dimensional infiltration is
also presented for the time to ponding under conditions of surface infiltration which has a
significant time dependence. Previous work in this area has used only approximate
solutions. Our aim in this application is to find an exact analytic solution to this problem
and to utilise this solution to obtain an accurate prediction of surface ponding time
involving a time dependent surface supply rate.

In agricultural applications of infiltration, such as irrigation through a series of
parallel irrigation furrows, one dimensional models are inapplicable. In this instance a
two dimensional model which incorporates a series of parallel strips is considered. Batu
(1978) considered a similar physical system to model steady and variable infiltration. In
this application we present analytic solutions for constant and variable surface supply rates
for the linear model. A numerical solution is also presented for this physical system using
the versatile nonlinear model attributed to Broadbridge and White (1988). No previous
study of ponding involving fractionally wetted surfaces has been carried out.
Finally, we consider the problem of infiltration through a sloping porous domain where the effect of evapotranspiration is also evident. The aim is to determine whether the slope of the porous domain, the magnitude of the wetted fraction or the critical infiltration rate is the determining factor in the onset of saturation.

It is found that whilst the surface supply pattern has a significant influence on the time to ponding, it is the infiltration rate that exerts the most influence. We are led to surmise that in terms of modelling real-life hydrological events, there is little to be gained by considering complex surface geometries. Generally, in the case of realistic flux rates it is more computationally efficient to use the one dimensional model without significant losses in the accuracy of the results.
2. Infiltration In One Dimension

(2.1) Constant Rate Rainfall Boundary Conditions

Fluid flow through an initially dry soil, subject to a constant infiltration rate prescribed at the surface is modelled by Richards' Equation

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial z} \left( D(\theta) \frac{\partial \theta}{\partial z} \right) - \frac{\partial K(\theta)}{\partial z}
\]

where \( D(\theta) = K(\theta) \frac{d\psi}{d\theta} \) is the nonlinear soil water diffusivity and \( K(\theta) \) is the hydraulic conductivity.

In the field, both the diffusivity and the hydraulic conductivity may be highly nonlinear functions. Over the range of water contents occurring in the field, these functions often vary by several orders of magnitude.

Broadbridge and White (1988) deduced an exact analytical solution to the nonlinear one dimensional Richards' Equation when the soil water diffusivity and hydraulic conductivity functions had the form

(2.1.2) \[ D(\theta) = \frac{a}{(b - \theta)^2} \]

and

(2.1.3) \[ K(\theta) = \beta + \gamma (b - \theta) + \frac{\lambda}{2(b - \theta)^2} \]
with constants a, b, p, y, λ respectively. Similar models were devised by Sander et al (1988) with important differences explained by Broadbridge and White (1988). These models not only enable an exact analytic solution by transform methods, but also emulate data in the field to a high degree of accuracy.

To determine surface ponding time, the nonlinear flow equation (2.1.1) is solved subject to a uniform initial condition and a constant rate surface flux condition.

It is important to note that an exact nonlinear analytic solution is available for the constant rate surface flux condition only. Any other boundary condition normally precludes the successful application of the requisite transforms. Recently however, Broadbridge et al (1996) used an alternative transform method to solve equations (2.1.1) - (2.1.3) with variable flux boundary conditions. However the parameters of the variable flux boundary conditions can not be systematically varied in that case.

Broadbridge and White (1988) solved equation (2.1.1) subject to the following surface supply rate boundary condition and uniform initial condition

(2.1.4) \[ K(\theta) - D(\theta) \frac{\partial \theta}{\partial z} = R, \quad z = 0 \]

(2.1.5) \[ \theta = \theta_n, \quad \Psi(\theta_n) = \Psi_n, \quad t = 0. \]

The solution found by Broadbridge and White (1988) at the surface of the soil is written in terms of dimensionless variables,

(2.1.6) \[ \Theta(0, \tau) = c \left[ 1 - \left( 2p + 1 - u^{-1} \frac{\partial u}{\partial \zeta} \right)^{-1} \right] \]

with
\[ \rho = \frac{R_*}{4c(c-1)} \]  

\[ u = 2 \exp(p^2 \tau) \text{erf}(p \tau^{\frac{1}{2}}) + \exp(p(p + 1) \tau) \]  

\[ \frac{\partial u}{\partial \zeta} = \rho \left[ 2 \exp(p^2 \tau) - 2(1 + p) \frac{\tau}{2} \exp(p(p + 1) \tau) \text{erf}(p(p + 1) \tau^{\frac{1}{2}}) \right] + \rho \left[ 2 \exp(p^2 \tau) \text{erf}(p \tau^{\frac{1}{2}}) \right] \]

In the equations above,

\[ \tau = 4c(c-1) \tau_* \]

and

\[ \zeta = \frac{1}{2} \sqrt{4c(c-1)Z_*} \]

Here \( R_* \) is the dimensionless infiltration rate, \( \rho \) is the rescaled surface flux, \( \tau \) is the rescaled dimensionless time, \( \zeta \) is the rescaled depth variable and \( Z_* \) is the depth variable owing to the Storm transformation (Storm 1951).

Ponding occurs when free water first appears at the surface of the soil, and the time to ponding, \( \tau_p \), is found from \( \Theta(0, \tau_p) = 1 \). Under this condition, the analytic solution reduces to
(2.1.11) \[
\frac{2c}{R_*} = 1 - \exp\left(\frac{R_* \tau_p (c - 1)}{h}\right) \text{erfc}\left(-R_* \sqrt{\frac{\tau_p}{4h}}\right) 
+ \left(1 + \frac{4c(c-1)}{R_*}\right)^{\frac{1}{2}} \text{erfc}\left(\frac{(R_* + 4c(c-1))R_* \tau_p}{4h}\right)^{\frac{1}{2}}
\]

The parameter $c$ is an indication of the nonlinearity of the model. In the field this parameter generally ranges between 1.02 and 1.5. Highly nonlinear models are characterised by $c \to 1$, whilst for large values of $c$, the model is weakly nonlinear. In the field values where $c > 2$ are considered large.

The function $h(c)$ can be found from

(2.1.12) \[
\frac{1}{c} = \left(\frac{\pi}{4h(c)}\right)^{\frac{1}{2}} \exp\left[\left[4h(c)\right]^{-1}\right] \text{erfc}\left[\left[4h(c)\right]^{-\frac{1}{2}}\right]
\]

but is approximated to (within 1% accuracy) by Broadbridge and White (1988)

(2.1.13) \[
h(c) = c(c - 1) \left[\frac{\pi(c - 1) + 1.46147}{4(c - 1) + 2.92294}\right]
\]

For highly nonlinear models the solution (2.1.10) reduces to

(2.1.14) \[
R_* \tau_p = \frac{1}{2} \ln\left(\frac{R_*}{R_* - 1}\right)
\]

which is in agreement with Parlange and Smith (1976) while for weakly nonlinear models the solution (2.1.10) reduces to a Burgers' Equation solution.
Burgers' Equation is characterised by a constant diffusivity and quadratic
dependence of $K(\theta)$ on $\theta$ and has previously been shown by Clothier et al. (1981a) and
by White et al. (1979) to give a good prediction for ponding time.

If we consider $c$ large, then the soil water parameters are weakly nonlinear. For
large $c$ the fluid flow equation (2.1.1) reduces to a Burgers' model.

\begin{equation}
\frac{\partial \Theta}{\partial t_*} = \frac{D}{D_s} \frac{\partial^2 \Theta}{\partial z_*^2} - 2\Theta \frac{\partial \Theta}{\partial z_*}
\end{equation}

This is solved subject to constant rainfall rate surface flux condition

\begin{equation}
\Theta^2 - \frac{D}{D_s} \frac{\partial \Theta}{\partial z_*} = R_*, \quad z_* = 0
\end{equation}

and uniform initial condition

\begin{equation}
\Theta = 0, \quad t_* = 0
\end{equation}

Here the constant diffusivity is $D = \pi S_n^2 / 4(\Delta \theta)^2$, with scaling factor,

\begin{equation}
D_s = \lambda S_t s^{-1} = \frac{h(c)}{c(c-1)} \left( \frac{S_n}{\Delta \theta} \right)^2
\end{equation}

noting that as $c \to \infty$, for the Linear and Burgers' Equation models,
h(c) / c(c-1) \to \pi / 4$ and thus the dimensionless $D_s = 1$.

The sorptivity $S_n$ is directly measurable from the cumulative absorption, $i(t)$, of
water into soil at early times (Philip, 1957b).
\[ i(t) = \int_0^\infty (\theta - \theta_n)dz \]
\[ = S_n t^{\frac{1}{2}} + O(t) \]

The cumulative absorption \( i(t) \) represents the total amount of fluid which has infiltrated into the soil at time \( t \). In addition, equation (2.1.19) also represents a definition for the sorptivity of the soil, where the sorptivity is a representation of capillary influences which follow a change in the surface concentration of fluid, neglecting gravitational effects.

The Burgers' model (2.1.15) - (2.1.17) can be reduced to a linear diffusion model by applying the Hopf-Cole transformation (Hopf, 1950)

\[ \Theta = -\frac{\partial}{\partial z^*} (\ln u) \]

The resulting linear diffusion problem

\[ \frac{\partial u}{\partial t^*} = \frac{\partial^2 u}{\partial z^*^2} \]

(2.1.22) \quad u = \exp(R_t^*), \quad z^* = 0

(2.1.23) \quad u = 1, \quad t^* = 0

has solution

\[ u = \frac{1}{2} \exp(R_t^*) \left\{ F_+ + F_- \right\} - \text{erfc} \left( \frac{z^*}{2\sqrt{t^*}} \right) + 1 \]
with

\[(2.1.25) \quad F_\pm = \exp(\pm z* \sqrt{R*}) \text{erfc}\left(\frac{z*}{2\sqrt{t*}} \pm \sqrt{R*t*}\right)\]

The dimensionless water content \( \Theta \) is found by inverting equation (2.1.24) using the Hopf-Cole transformation (2.1.20). Thus,

\[(2.1.26) \quad \Theta = -u^{-1} \frac{\partial u}{\partial z*}\]

At the soil surface, the ponding time is found from

\[(2.1.27) \quad \Theta(0, \tau_p) = \sqrt{R*} \text{erf}\left(\sqrt{R*} \tau_p\right) = 1\]

which is solved for \( \tau_p \) and the relevant soil water parameters are rescaled so the time scale \( t_s \) is consistent with the nonlinear model, the result of which is in agreement with Broadbridge and White (1987: Equation 13),

\[(2.1.28) \quad R* \tau_p = \frac{\pi}{4} \left(\text{inverf}\left(\frac{1}{\sqrt{R*}}\right)\right)^2\]

The linear model for soil water flow uses the same constant soil water diffusivity as the Burgers model, but a linear dependence of \( K(\theta) \) on \( \theta \) is assumed (Braester, 1973). Recall however that the hydraulic conductivity is a highly nonlinear function. When the soil water content is low, as in the case of early infiltration times, so too is the hydraulic conductivity. The linear model, by assuming this linear dependence effectively over
estimates the hydraulic conductivity. Since in the absence of capillary action, the water flux due to gravity is identical with conductivity, this is equivalent to an over-statement of the importance of gravity. In this instance, the linear model predicts that a greater quantity of water is removed from the surface than is actually the case - hence the over estimation in the actual time to surface ponding. By determining a relationship between the linear and nonlinear solutions a quantitative assessment of this overestimation is deduced. This assessment can then be extended to physical systems where complex surface geometries preclude an exact analytic nonlinear solution. The level of compensation necessary to account for this modelling error in one dimension, will give us some indication of the required correction in two dimensions.

Figure 2.1 and Figure 2.2 display a qualitative comparison between the linear, Burgers' and nonlinear solution (for c=1.169) respectively.

The linear model does in fact overestimate the surface ponding times and the distortions caused by the enhancement of gravity in the physical system is evident. As the constant rate surface flux increases the linear and nonlinear models are in strong agreement, both agreeing asymptotically with \( \tau_p = \frac{1}{2}R^2 \), (White et al.; 1979). For large dimensionless flux rates \( R* \geq 3 \) the fluid infiltrates into the soil at a rate faster than the gravitational forces are able to remove it. Thus as the flux rate increases, the effect of vertical transport caused by gravitational effects is outweighed.

The weakly nonlinear Burgers' Equation model and the versatile nonlinear model of Broadbridge & White (1988) are in strong agreement for all surface flux rates and hence, the Burgers' model predicts surface ponding time with greater accuracy than the linear model. This is because in comparison to the linear model, the quadratic dependence of \( K(\theta) \) on \( \theta \) is physically a more realistic reflection of field hydraulic conductivities. This serves to correct the exaggerated gravitational effect in the linear model. Thus, the Burgers' model is an excellent predictor of surface ponding times as this model not only
retains the essential nonlinear features of the infiltration process but it is more elementary to solve than is the nonlinear model.

2.1 Time To Ponding Vs Infiltration Rate (Linear & Nonlinear Models)

2.2 Time To Ponding Vs Infiltration Rate (Linear & Burgers' Models)

Figure 2.3 and Figure 2.4 display the ratio between the surface ponding times for the linear and nonlinear and the linear and Burgers' models respectively. For low rates of
infiltration ($R_* = 1.5$) there is a large error in the ponding times between the linear and both the weakly nonlinear Burgers model and highly nonlinear models. This error rapidly decreases as the infiltration rate increases, and for large infiltration rates this constitutes an acceptable difference of approximately 5%.

2.3 Ratio Of Ponding Times Vs Infiltration Rate (Linear & Nonlinear Models)

2.4 Ratio Of Ponding Times Vs Infiltration Rate (Linear & Burgers' Models)
The cumulative infiltration is the total amount of water that is absorbed into the soil over a specified time period. This is an especially important soil water property in agricultural applications of infiltration, such as irrigation, for example.

2.5 Cumulative Infiltration Vs Infiltration Rate At Ponding

The linear model has the disadvantages not only of overestimating ponding times but for similar reasons, it also overestimates the cumulative infiltration, as shown in Figure 2.5 and in Figure 2.6. This overestimation can be calamitous in the field, environmentally in the case of ecologically sensitive crops and financially when the costs of water usage and poor crop yields are considered.

It is pleasing to note however, that the relative error between the linear and nonlinear models is of the same order of magnitude as the relative error when the time to ponding and cumulative infiltration is considered. This implies that the error is consistent across all the time-like soil water parameters and hence can be reasonably corrected when complex surface geometries preclude an exact analytic solution.
2.6 Cumulative Infiltration Vs Infiltration Rate At Ponding

(2.2) Time Dependent Boundary Conditions

(a) Burgers Equation Model

It is not entirely realistic that field occurring surface infiltration rates will be constant. In order to investigate the consequences of a variable water supply consideration is given to the case where the surface infiltration rate has a linear time dependence. An exact analytic solution is obtained for the Burgers' Equation model which incorporates this boundary condition.

The dimensionless boundary value problem to be solved is

\[
\frac{\partial \Theta}{\partial t^*} = \frac{\partial^2 \Theta}{\partial z^*^2} - 2\Theta \frac{\partial \Theta}{\partial z^*}
\]
\( (2.2.2) \quad \Theta^2 - \frac{\partial \Theta}{\partial z^*} = Q(t^*), \quad z^* = 0 \)

\( (2.2.3) \quad \Theta = 0, \quad t^* = 0 \)

where \( Q(t^*) = R^* t^*. \)

As in the previous constant rate surface flux condition case, the Hopf-Cole transformation (2.1.20) is applied to the boundary value problem, reducing the Burgers' equation model to a linear diffusion problem. Thus,

\( (2.2.4) \quad \frac{\partial u}{\partial t^*} = \frac{\partial^2 u}{\partial z^2} \)

\( (2.2.5) \quad u = \exp \left( \frac{R^* t^*}{2} \right), \quad z^* = 0 \)

\( (2.2.6) \quad u = 1, \quad t^* = 0 \)

The linear diffusion boundary value problem (2.2.4) - (2.2.6) is solved by taking Laplace Transforms with respect to the time variable \( t^* \), the result of which is

\( (2.2.7) \quad \tilde{u}(z^*,s) = \left[ \frac{1}{\sqrt{\frac{\pi}{2a}}} \exp \left( \frac{s^2}{4a} \right) \text{erfc} \left( \frac{s}{2\sqrt{a}} \right) - \frac{1}{s} \right] \exp \left( -z^* \sqrt{s} \right) + \frac{1}{s} \).

Noting the large \( x \) asymptotic expansion, from Abramowitz and Stegun (1964: Equation 7.1.23)
\[ (2.2.8) \quad \sqrt{\pi x^2} \exp\left(\frac{x^2}{2}\right) \text{erfc}(x) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{\left(2x^2\right)^m} \frac{1.3\ldots(2m-1)}{m} \]

equation (2.2.7) is inverted for \( u(z^*, t^*) \). Therefore,

\[ (2.2.9) \quad u(z^*, t^*) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{\left(2t^*\right)^m} \frac{1.3\ldots(2m-1)}{m} \text{erfc}\left(\frac{Z^*}{2\sqrt{t^*}}\right) \]

where

\[ (2.2.10) \quad i^n\text{erfc}(x) = \int_0^{\infty} i^{n-1}\text{erfc}(x)dx \]

is the repeated integral of the complementary error function.

At the soil surface equation (2.2.9) reduces to

\[ (2.2.11) \quad u(0, t^*) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{\left(2t^*\right)^m} \frac{1.3\ldots(2m-1)}{m} \text{erfc}(0) \]

and again we make use of Abramowitz and Stegun (1964: Equation 7.2.7), to wit

\[ (2.2.12) \quad i^n\text{erfc}(0) = \frac{1}{2^n\Gamma\left(\frac{n}{2} + 1\right)} \]

The dimensionless water content \( \Theta(0, t^*) \) is easily obtained by inverting equation (2.2.11) by making use of the Hopf-Cole transformation. Hence the ponding times are deduced from
(2.2.13) \[ \Theta(0, t^*) = \frac{\exp\left(\frac{at^2}{2}\right)}{\sqrt{\pi t^*}} \sum_{m=1}^{\infty} (-1)^m a^m 2^m \left[1.3\ldots(2m-1)\right] t^* \frac{2m}{(4m)!} \]

and setting \( \Theta(0, t^*) = 1 \), the results of which are shown in Figure 2.7 and Figure 2.8 below.

Figure 2.7 displays the difference in ponding times for the constant rate and the linearly time dependent surface flux conditions. The time to surface ponding is greatly increased when a linearly time dependent infiltration rate is imposed on the boundary. For early time this has the effect of decreasing the amount of fluid delivered at the soil surface. This early time effect is amplified with lower infiltration rates. As a result the ponding process is delayed because the gravitational and capillary forces acting within the soil system have ample time to remove the fluid from the surface of the soil.

2.7 Cumulative Infiltration Vs Infiltration Rate At Ponding
2.8 Cumulative Infiltration At Ponding Vs Infiltration Rate At Ponding

Figure 2.8 displays the cumulative infiltration at ponding for the linearly time dependent surface flux condition. Given a soil profile, if the sorptivity is held constant and the nonlinearity parameter \( c \) is varied the predicted time to ponding is bounded above by the Burgers equation model, given by equation (2.1.27) and below by the versatile nonlinear model, given by equation (2.1.14). Although the widely varying nonlinearity parameter encompass all soil types encountered in the field or repacked in the laboratory, the range of the ponding times which are bounded by the above formulations are exceedingly narrow. This result, for constant surface infiltration rates, was extended by Broadbridge and White (1987) to include the case where the infiltration rate has a significant time dependence. Therefore, the lower bound for the cumulative infiltration comes from the nonlinear model with constant rate surface flux condition in the case that \( c \to 1 \). Hence the lower bound for the time to surface ponding (Broadbridge and White 1987: Equation 19) is

\[
(2.2.14) \quad R\tau_m = \frac{1}{2} \ln \left[ R \left( \frac{\tau_p}{\tau_m} \right) / \left( R \tau_m - 1 \right) \right]
\]
which is in agreement with Parlange and Smith (1976). This lower bound provides a reasonable estimate of the cumulative infiltration for the time dependent case. However, as shown in Figure 2.7, it is reasonable to say that the errors are considerable which contradicts the "time-condensation" postulate (Eagleson 1978) which predicts the condensation of ponding curves when the cumulative infiltration is used as the time-like variable. As expected the cumulative infiltration for the linearly time dependent surface flux condition is greater than the corresponding constant rate boundary condition.
3 Infiltration In Two Dimensions

(3.1) Constant Supply Rate Surface Boundary Conditions

(a) Infiltration Boundary Conditions

Periodic strip sources are a common method of irrigating agricultural fields. The furrows are spaced equally apart and the distance between furrows is dependent on the crop type. Unlike flooding methods, furrows can be utilised in slopes with grades as steep as 10% and by directing flow diagonally down the hill, the effective grade is reduced. This factor can help minimise water run-off and as a result, prevent soil erosion and soil degradation.

Physically irrigation is accomplished by supplying water at a constant rate through each furrow which has a half width of length \( w \). The water content \( \theta(x,z,t) \) is an even periodic function in \( x \), with period \( 2L \), where \( x \) is taken to be the horizontal direction. There are axes of symmetry at \( x=0 \) and \( x=L \) and hence, flow through the central furrow is considered.

3.1 Two Dimensional Infiltration System
The system described above is modelled by the two dimensional Richards' Equation,

\[
\frac{\partial \theta}{\partial t} = \frac{\partial}{\partial x}\left(D(\theta)\frac{\partial \theta}{\partial x}\right) + \frac{\partial}{\partial z}\left(D(\theta)\frac{\partial \theta}{\partial z}\right) - \frac{\partial K(\theta)}{\partial z}
\]

subject to a constant surface supply rate boundary condition. Water is supplied at a constant rate within the furrow and an unirrigated drier region exists between furrows,

\[
K(\theta) - D(\theta)\frac{\partial \theta}{\partial z} = \begin{cases} R & 0 \leq x \leq w, \\
0 & w < x < L, \\
\end{cases} \
\]

A uniform initial water content is also assumed,

\[
\theta(x, z, 0) = \theta_n.
\]

To preserve the symmetry of the system, the boundary condition

\[
\frac{\partial \theta}{\partial x} = 0 \quad x=0, \ L
\]

is imposed. This condition ensures that there is nil water flux across the axes of symmetry.

It is also assumed that the steady state is maintained by gravity-driven transport at infinite depth \(z\). This is true since at large \(z\), diffusion will have negated lateral variations, and the semi-infinite one dimensional steady state is trivial. Thus,

\[
\lim_{z \to \infty} \frac{\partial \theta}{\partial z} = 0.
\]
These equations are rescaled and expressed in terms of dimensionless variables. Consequently,

\begin{equation}
\frac{\partial \Theta}{\partial t_*} = \frac{\partial}{\partial x_*} \left( D_*(\Theta) \frac{\partial \Theta}{\partial x_*} \right) + \frac{\partial}{\partial z_*} \left( D_*(\Theta) \frac{\partial \Theta}{\partial z_*} \right) - \frac{\partial K_*(\Theta)}{\partial z_*}
\end{equation}

is solved subject to the boundary conditions,

\begin{equation}
K_*(\Theta) - D_*(\Theta) \frac{\partial \Theta}{\partial z_*} = \begin{cases} R_* & 0 \leq x_* \leq \omega \\ 0 & \omega < x_* \leq \lambda \end{cases}
\end{equation}

\begin{equation}
\frac{\partial \Theta}{\partial x_*} = 0 \quad x_* = 0, \ \lambda
\end{equation}

\begin{equation}
\lim_{z_* \to \infty} \frac{\partial \Theta}{\partial z_*} = 0
\end{equation}

and initial condition,

\begin{equation}
\Theta(x_*, z_*, 0) = 0.
\end{equation}

This boundary value problem can be solved exactly by utilising the linear model. Braester (1973) linearised the nonlinear flow equation by assuming a constant diffusivity of the form

\begin{equation}
D_* = \frac{\pi}{4} \left( \frac{S_n}{\Delta \theta} \right)^2
\end{equation}
and a linear dependence of the hydraulic conductivity \( K_*(\Theta) \) on the water content \( \Theta \).

This constant diffusivity is selected as it ensures that the linearised and exact models of the flow equation predict identical cumulative infiltration at early times during a constant pressure experiment. This diffusivity will result in an exact solution in which the sorptivity is equivalent to the measured value of the sorptivity \( S_n \).

The linear flow equation,

\[
\frac{\partial \Theta}{\partial t_*} = \frac{\partial^2 \Theta}{\partial x_*^2} + \frac{\partial^2 \Theta}{\partial z_*^2} - \frac{\partial \Theta}{\partial z_*}
\]

is solved subject to (3.1.8)-(3.1.10) and revised surface flux condition

\[
\Theta - \frac{\partial \Theta}{\partial z_*} = \begin{cases} R_* & 0 \leq x_* \leq \omega \\ 0 & \omega < x_* \leq \lambda \end{cases} \quad z_* = 0
\]

by taking Laplace transforms and using separation of variables. Batu (1978) deduced a solution for a single and for periodic strip sources using similar techniques.

The series solution is of the form,

\[
\Theta(x_*, z_*, t_*) = \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi x_*}{\lambda}\right)
\]

with \( A_0 = A_0(z_*, t_*) \) and \( A_n = A_n(z_*, t_*) \). Batu (1978) used this solution to study the two dimensional flow pattern and Philip (1984) used a related flow pattern from a spatially periodic source to study leaching patterns. So far this solution has not been used to estimate ponding times.

Surface ponding first occurs at the centre of the wet strip \( x_* = 0 \). Thus to deduce the time to surface ponding, we consider the expression \( \Theta(0,0,t_*) \), where
(3.1.15) \[ \Theta(0,0,t^*) = \left( \frac{R\omega}{\lambda} \right) \left[ 1 + \left( \frac{t^*}{\pi} \right)^2 \exp \left( -\frac{t^*}{4} \right) - \left( 1 + \frac{t^*}{2} \right) \text{erfc} \left( \frac{t^*}{\sqrt{2}} \right) \right] \]

\[ + \sum_{n=1}^{\infty} \frac{2R^*}{n\pi} \sin \left( \frac{n\pi\omega}{\lambda} \right) \left[ \frac{1}{2} (F_- + F_+) + G \right] \]

with

(3.1.16) \[ F_{\pm} = \frac{\text{erfc} \left( \pm \sqrt{\frac{n^2\pi^2}{\lambda^2} + \frac{1}{4}} \right)}{1 \over 2\sqrt{D^*}} \frac{1}{n} \sqrt{\frac{n^2\pi^2}{\lambda^2} + \frac{1}{4}} \]

and

(3.1.17) \[ G = \frac{\lambda^2}{2n^2\pi^2} \exp \left( -\frac{n^2\pi^2t^*}{\lambda^2} \right) \text{erfc} \left( \sqrt{\frac{t^*}{4}} \right) \]

respectively.

The time to surface ponding \( \tau_p \), is subsequently evaluated by solving \( \Theta(0,0,\tau_p) = 1 \), the results of which are shown in Figure 3.2 and Figure 3.3.

It is evident that although the linear model is expected to overestimate \( \tau_p \) for all values of \( \omega \), the time to surface ponding through equally spaced irrigation furrows is substantially higher than the time required for ponding in the corresponding one dimensional infiltration system. In the two dimensional case, water is not only transported from the soil surface by gravity and vertical diffusion, but also laterally by horizontal diffusion. This additional degree of freedom results in longer ponding times.
3.2 Time To Ponding Vs Infiltration Rate \( \{ \lambda = 2.54 \text{ (Linear Model)} \} \)

3.3 Cumulative Infiltration Vs Infiltration Rate At Ponding \( \{ \lambda = 2.54 \} \)

Similarly the local cumulative infiltration at ponding is much higher through periodic furrows than in the comparable one dimensional case. Figure 3.3 shows the cumulative infiltration as function of the infiltration rate at ponding for the \( x_\ast \) averaged infiltration rate (that is, \( R_\ast \omega / \lambda \)). As the width of the furrow increases, so too does the wetted fraction. Therefore the cumulative infiltration and ponding times decrease. An increased half width
indicates that the water is supplied over a larger area of the soil surface. As a result the lateral transport effects are minimised. As the half width \( \omega \) approaches the furrow half-gap \( \lambda \), the two dimensional diffusion-convection problem reduces to the corresponding one dimensional flow system. Therefore the ponding time through equally spaced irrigation furrows will be the same as the ponding time in a vertical column of soil.

Figures 3.4 and 3.5 compare both the time to ponding and cumulative infiltration for the one dimensional and reduced two dimensional infiltration systems. The round-off error which is apparent for \( R_* < 3 \) is a numerical artefact which will give an estimate of the round-off error which appears in the nonlinear solution obtained using a numerical partial differential equation solver.

![Graph showing Time to Ponding Vs Infiltration Rate (1D & 2D Solutions)](image)

### 3.4 Time To Ponding Vs Infiltration Rate (1D & 2D Solutions)

It is important to recognise that as \( \omega \to \lambda \), \( \sin(n\pi \omega/\lambda) \to \sin(n\pi) = 0 \) and the two dimensional solution given by equation (3.1.15) reduces to

\[
\Theta(0,0,t^*) = R_* \left[ 1 + \left( \frac{t^*}{\pi} \right)^{\frac{1}{2}} \exp\left(-\frac{t^*}{4}\right) - \left(1 + \frac{t^*}{2}\right) \text{erfc}\left(\frac{\sqrt{t^*}}{\sqrt{4}}\right) \right]
\]
which after some minor rearranging is identical to the one dimensional linear solution.

![Graph showing cumulative infiltration vs infiltration rate at ponding (1D & 2D solutions)](image)

### 3.5 Cumulative Infiltration Vs Infiltration Rate At Ponding

*(1D & 2D Solutions)*

As with the one dimensional model, the two dimensional linear model overestimates ponding time. Again this is due to the over effect of gravity resulting from the linear dependence of the hydraulic conductivity on the water content. To obtain a superior measure of surface ponding time we look to the nonlinear flow model. It is unfortunate however that unlike in the one dimensional model, the two dimensional nonlinear flow equation is not solvable analytically. In this instance a numerical solution is applied.

The two dimensional boundary value problem is solved using PDETW0 (Software by Melgaard and Sincovec, 1981). This software reduces the nonlinear partial differential equation to a system of ordinary differential equations using a method of lines with continuous time variable and a discrete spatial discretisation. The resulting system of ordinary differential equations is solved numerically by the method of Runge-Kutta.
To test the accuracy of the numerical partial differential equation solver and to find the optimal spatial discretisation, the numerical package was tested against the exact analytic solution for the two dimensional linear model. A variety of mesh points and grid spacings were implemented in the numerical partial differential equation solver PDETWOTWO. Figure 3.6 is a comparison between the analytic and the numerical solution for the two dimensional linear problem. These two solutions are indistinguishable and as such, the solution to the corresponding nonlinear problem may be viewed with confidence. In the solution to the nonlinear problem, the same spatial discretisation and continuous time variable is used in order to maintain consistency.

The soil Yolo Light Clay is considered. This is a highly nonlinear soil as the nonlinearity parameter $c$ has the value 1.169. Recall, highly nonlinear soils have a nonlinearity parameter close to unity, whilst for weakly nonlinear soils, the nonlinearity parameter has values which approach infinity. In the field, soils for which $c$ is greater than two are considered weakly nonlinear.

3.6 Time To Ponding Vs Infiltration Rate (Comparison Analytic & Numerical PDE Solver for Linear Model)
3.7 Time To Ponding Vs Infiltration Rate ($\omega=1.09$)

As expected the linear model substantially over estimates the ponding time for low surface supply rates. However for larger surface supply rates ($R_s > 2.5$) the linear and nonlinear models agree with a high level of accuracy.

3.8 Cumulative Infiltration Vs Infiltration Rate At Ponding ($\omega=1.09$)
Figure 3.8 shows the cumulative infiltration plotted against the infiltration rate for the linear and nonlinear models. For constant surface supply rates, the cumulative infiltration is simply the product of the time to surface ponding and the surface infiltration rate. As the linear model overestimates ponding times for lower surface supply rates, this effect is carried over and the linear model will also overestimate the cumulative infiltration for these lower supply rates. Again for dimensionless infiltration rates $R_* > 2.5$, the cumulative infiltration for the linear and nonlinear models agree.

When the nonlinear model alone is considered, the same trends are evident as in the linear model. As the furrow width increases, the surface ponding time is decreased. Similarly as the surface supply rates increase, the ponding times for furrows of varying widths reach agreement, as shown in Figure 3.9.

![Graph of Time To Ponding Vs Infiltration Rate (Nonlinear Model)]

3.9 Time To Ponding Vs Infiltration Rate (Nonlinear Model)
By assuming infiltration at the surface of the soil only, a plethora of environmental influences are ignored, notably atmospheric evaporation and transpiration by plants. To redress this situation an evaporative effect is incorporated into the surface supply rate boundary condition.

The same array of periodic furrows is considered, again each furrow has half width $\omega$ and axes of symmetry at $x_*=0$ and $x_* = \lambda$.

As in the previous section, water is supplied at a constant rate $R_*$ through each furrow. In addition to this water is evaporated from the dry region at a rate $E_*$ which is set at a fraction of the infiltration rate $R_*$. Evaporation from a bare soil is considered, the effect of transpiration via plant roots is neglected. Within both regions of the surface supply pattern it is assumed that the initial water content $\Theta(x_*,z_*,0) = 0$.

3.10 Two Dimensional Infiltration & Evaporation System

The two dimensional linear flow equation (3.1.13) is solved subject to (3.1.8) - (3.1.10) with surface flux condition

\[
\Theta - \frac{\partial \Theta}{\partial z_*} = \begin{cases} 
R_*, & 0 \leq x_* \leq \omega \\
E_*, & \omega < x_* \leq \lambda \\
0, & z_* = 0
\end{cases}
\]
This surface flux condition is perhaps physically more realistic than the surface flux condition incorporating infiltration only as it encompasses vapour loss from the soil as a result of evaporative effects. Beneath the evaporation surface, the dimensionless soil water $\Theta$ attains a minimum value $\Theta_{\text{min}}$ that may be negative. However, our solution allows the initial volumetric water content $\theta_n$ to be any specified non-negative value. If $\theta_n$ is chosen to be large enough, then the negative value $\Theta_{\text{min}}$ still corresponds to a non-negative volumetric water content $\theta_{\text{min}} = \Theta(\theta_s - \theta_n) + \theta_n$.

Laplace transforms are taken and separation of variables achieves the solution with ease. In this case the dimensionless surface water content,

$$\Theta(0,0,t^*) = \left(\frac{R_s - E_s}{\lambda}\right) \left[ 1 + \left(\frac{t^*}{\pi}\right)^{\frac{1}{2}} \exp\left(-\frac{t^*}{4}\right) - \left(1 + \frac{t^*}{2}\right) \text{erfc}\left(\frac{t^*}{2\sqrt{4}}\right) \right]$$

$$+ \sum_{n=1}^{\infty} \frac{2(R_s - E_s)}{n\pi} \sin\left(\frac{n\pi\omega}{\lambda}\right) \left[ \frac{1}{2} (F_- + F_+) + G \right]$$

where $F_-$ and $G$ are as defined by equations (3.1.16) and (3.1.17) respectively. Again, the time to ponding is deduced from the solution to $\Theta(0,0,t^*) = 1$.

Figures 3.11 and 3.12 display the results obtained with a fixed exfiltration rate and varying furrow half widths. As in the previous infiltration only case, the time to surface ponding and cumulative infiltration is decreased as the half width of the furrow increases. Similarly, as the width of the half width approaches the length of the furrow $\lambda$, the two dimensional system reduces to a one dimensional system. In addition for an exfiltration rate greater than 5% of the infiltration rate, ponding is delayed compared to the infiltration only system. This effect is noticeably so as the half width of the furrow decreases. The longer ponding times for low surface supply rates are attributed to the large effect that evaporation from the dry region has on the system. As the surface supply rates increase
however, the atmospheric evaporation from the dry region has a negligible effect on ponding time because fluid is supplied at a rate which far exceeds the losses induced by evaporation. Also, ponding occurs before the influence of the distant evaporative surface can be felt at the irrigated surface. For surface supply rates exceeding $R_* = 2$ ponding time is determined solely by diffusive effects ($\tau_p = S_f^2/2R^2$, Perroux et al., 1981) before gravity or multidimensionality begin to have any importance. With high infiltration rates, large gradients in water content are maintained for a considerable time, resulting in capillary driven diffusive effects being the dominant transport mechanism. Fluid then pools near the soil surface before subterranean drainage transports a significant amount of water. The absence of convection implies that the infinite soil profile can be modelled as a finite soil column as a result of the "pooling" effect and lower soil levels are not affected. In cases such as this, ponding will occur at the surface of the soil before "basement" saturation is reached (Broadbridge et al. 1988).

3.11 Time To Ponding Vs Infiltration Rate (Fixed $E_*$)
3.12 Cumulative Infiltration Vs Infiltration Rate At Ponding

It is anticipated that higher exfiltration rates will lead to longer ponding times, especially in the case of low surface supply rates. The influence of exfiltration is shown in Figure 3.13 and in Figure 3.14. In this case the half width is kept at a fixed level and the exfiltration rate is varied.

3.13 Time To Ponding Vs Infiltration Rate (Fixed \( \omega \))
For surface supply rates greater than $R* = 2$, the rate of exfiltration has a minimal influence on the time taken for surface ponding. Again, this is because fluid is supplied at a rate which far exceeds the evaporation rate. For low surface supply rates this is not the case however. As expected, higher exfiltration rates lead to higher times for surface ponding and as a consequence, higher levels of cumulative infiltration.

As with the infiltration-only boundary value problem, the linear model which incorporates evaporative effects will over estimate the time to surface ponding. Therefore the numerical partial differential equation solver PDETW0 is used to solve the boundary value problem with nonlinear diffusivity and nonlinear hydraulic conductivity. The discrete spatial discretisation and continuous time variables are identical to the previous infiltration-only model. The only alteration is in the subroutine which evaluates the horizontal boundary condition. This subroutine is altered to take evaporative effects into account.

3.14 Cumulative Infiltration Vs Infiltration Rate At Ponding (Fixed $\omega$)
3.15 Time To Ponding Vs Infiltration Rate (Linear & Nonlinear Models)

The time to surface ponding is overestimated by the linear model, even more so with evaporative effects taken into account. For large surface supply rates however, the ponding time estimates provided by the linear and nonlinear models agree.

3.16 Time To Ponding Vs Infiltration Rate: Nonlinear Model (Fixed $\omega$)

When the nonlinear model is considered with varying evaporation levels in the dry region there is essentially very little difference in the time to surface ponding. Of course,
in this model, fluid is evaporated from the dry region at a rate which is 5% or 10% of the supply rate. The trend is still evident however; for low surface supply rates, higher evaporative levels lead to marginally longer surface ponding times.

In the model considered above the realistic irrigation rate of 5 cm/day ($R_\ast = 4.72$ for Yolo Light Clay) would entail a dimensionless evaporation rate $E_\ast$ ranging between 0.236 and 0.472 (that is, 0.25 cm/day and 0.5 cm/day). Referring to Thornwaite (1948) who quotes an evapotranspiration rate of 13.5 cm/month based on a temperature of 26.5°C, the above rates of evaporation are entirely feasible.

(c) Fractal Boundary Conditions

In the field, soil grains are not uniform in size. In addition, surface pore sizes differ on a macroscopic level and these differences may affect fluid infiltration. To take soil pore sizes at the surface into account a fractal model is considered.

Again two dimensional flow occurs through an array of equidistant periodic irrigation furrows. Infiltration occurs in the wet strip $0 \leq x_\ast \leq \omega$ whilst there exists a dry region for $\omega < x_\ast \leq \lambda$.

In the wet region, a Cantor Set boundary is adopted where each "middle third" is successively removed (Devaney 1989). The surface supply rate is adjusted however, so the same average flux $R_\ast$ occurs throughout the wet region.

The water content $\Theta(x_\ast, z_\ast, t_\ast)$ is again an even periodic function with axes of symmetry at $x_\ast = 0$ and $x_\ast = \lambda$. Hence flow is considered through the central furrow only.

This system is modelled by the two dimensional flow equation (3.1.13) subject to initial and boundary conditions (3.1.8) - (3.1.10). An adjustment is made in the constant surface supply rate boundary condition (3.1.13), the middle third is removed from the wet
strip and the infiltration rate is adjusted in order that the same average flux is absorbed into the soil. Thus,

\[
\begin{align*}
(3.1.21) \quad \Theta - \frac{\partial \Theta}{\partial z_*} &= \begin{cases} 
\frac{3}{2} R_* & 0 \leq x_* \leq \frac{1}{3} \omega \\
\frac{1}{3} R_* & \frac{1}{3} \omega < x_* \leq \frac{2}{3} \omega \\
\frac{3}{2} R_* & \frac{2}{3} \omega < x_* \leq \omega \\
0 & \omega < x_* \leq \lambda 
\end{cases} \\
& \quad z_* = 0
\end{align*}
\]

At the soil surface in the centre of the wet strip, the soil water content is given by

\[
(3.1.22) \quad \Theta(0,0,t_*) = \left( \frac{R_* \omega}{\lambda} \right) \left[ 1 + \left( \frac{t_*}{\pi} \right)^{\frac{1}{2}} \exp\left( -\frac{t_*}{4} \right) - \left( 1 + \frac{t_*}{2} \right) \text{erfc}\left( \sqrt{\frac{t_*}{4}} \right) \right] \\
+ \sum_{n=1}^{\infty} \frac{3R_*}{n\pi} \left( \sin\left( \frac{n\pi \omega}{\lambda} \right) - \sin\left( \frac{2n\pi \omega}{3\lambda} \right) \sin\left( \frac{n\pi \omega}{3\lambda} \right) \right) \left[ \frac{1}{2}(F_- + F_+) + G \right]
\]

with $F_\pm$ and $G$ defined by equations (3.1.16) and (3.1.17) respectively.

A second and third iteration of the Cantor Set surface boundary condition is taken. The middle thirds of the remaining wet strips are removed and the readjustment in the surface supply rate is made. These revised infiltration systems are solved, the results of which are shown in Figure 3.17.

A comparison is made between the time to surface ponding for the uniform wet region and the adjusted Cantor Set wet region. The same volumetric water content is delivered at the soil surface, the only difference is in the distribution of the fluid flow within the wet strip.
3.17 Time To Ponding Vs Infiltration Rate (Fractal Model)

With the first middle third removed, the time to surface ponding and the cumulative infiltration decrease substantially for low surface flux rates. This is due to a number of factors. Firstly, a greater quantity of fluid is delivered over a smaller surface area. As these areas are essentially flooded, more fluid is delivered than can be transported laterally, thus the effect of horizontal transport is decreased. The effect of gravity is also decreased because the rate at which fluid is delivered exceeds the level at which gravitational forces remove it.

With a further iteration, ponding time decreases again, but after two iterations the time to surface ponding does not change, regardless of how many iterations are taken. This is due to the self-similarity inherent within the Cantor Set. The time to ponding approaches a limit as the number of iterations increase. This makes physical sense because the average volume of fluid infiltrated into the soil at any given time does not change and after two iterations of the Cantor Set, the change in the boundary profile is inconsequential. In summary, ponding is sped up because the densest part of the Cantor supply surface is irrigated at a higher rate. However, the ponding time approaches a non-
zero limit as the fractal supply surface becomes more diffuse (having zero Lebesgue measure in the limiting Cantor Set) and these two effects eventually balance.

3.18 Cumulative Infiltration Vs Infiltration Rate At Ponding (Fractal Model)

(3.2) Time Dependent Boundary Conditions

(a) Linearly Increasing Time Dependent Boundary Conditions

To further investigate similarities and disparities between ponding times for the one and two dimensional models, a linearly increasing time dependence is imposed at the surface of the soil. In this case both the two dimensional linear model and the two dimensional Burgers equation model are considered.
(i) Linear Model

The linear two dimensional fluid flow equation (3.1.12) is solved subject to initial condition (3.1.10) and boundary conditions (3.1.8) - (3.1.9) with the surface supply condition,

\[
\Theta - \frac{\partial \Theta}{\partial z^*} = \begin{cases} 
Q(t^*) & 0 \leq x^* \leq \omega \\
0 & \omega < x^* \leq \lambda \\
& z^* = 0
\end{cases}
\]

once again by taking Laplace transforms and using separation of variables. Here, the supply rate \( Q(t^*) = R^*t^* \).

The solution at the soil surface \( z^* = 0 \) at the centre of the wet strip \( x^* = 0 \) is given by

\[
\Theta(0,0,t^*) = \frac{R^*\omega}{\lambda} \exp\left(-\frac{t^*}{4}\right) \left[ \frac{-4}{3} \left( \frac{1}{\pi t^*} \right)^{\frac{1}{2}} + \frac{8}{3} G\left(\frac{t^*}{4}\right) - 2\sqrt{2} G\left(\frac{t^*}{2}\right) - \exp\left(\frac{t^*}{2}\right) (2\sqrt{2} - 2) \right] \\
+ \sum_{n=1}^{\infty} \frac{2R^*}{n\pi} \sin\left(\frac{n\pi\omega}{\lambda}\right) \exp(-\alpha^2) \left\{ -16 \left( \frac{1}{\pi t^*} \right)^{\frac{1}{2}} - \frac{1}{2} G\left(\frac{t^*}{4}\right) - \frac{\exp(\alpha t^*)}{\alpha(4\alpha - 1)} \right\} \\
+ \sum_{n=1}^{\infty} \frac{2R^*}{n\pi} \sin\left(\frac{n\pi\omega}{\lambda}\right) \exp(-\alpha^2) \left\{ 2 \left( \frac{1}{\pi t^*} \right)^{\frac{1}{2}} - 2\sqrt{2} G\left(\sqrt{\alpha t^*}\right) + 2\sqrt{\alpha} \exp(\alpha t^*) \right\}
\]

where,

\[
G(x) = \exp(x) \text{erfc}(\sqrt{x})
\]
and after appropriate scaling,

\[(3.2.4) \quad \alpha^2 = \frac{n^2 \pi^2}{\lambda^2} + \frac{1}{4}\]

To deduce the time to surface ponding, equation (3.2.2) is set equal to unity and this expression is evaluated for \( \tau_p \), the values of which are used to determine the cumulative infiltration at ponding.

As the surface supply rate increases, the cumulative infiltration decreases. In addition, as with the previous constant surface supply rate case, as the half width of the furrow \( \omega \) increases the time to surface ponding and hence the cumulative infiltration decreases. However, the differences between the respective values for the cumulative infiltration are not of the same order of magnitude as in the constant surface supply rate case. This implies that if there is a significant time dependence imposed at the surface, such as in the linearly increasing case, the cumulative infiltration and subsequent ponding times are influenced more by the surface supply rate than by the surface irrigation pattern.

![Graph](image)

3.19 Cumulative Infiltration Vs Infiltration Acceleration Rate
(ii) Burgers Equation Model

The two dimensional Burgers equation,

\[ \frac{\partial \Theta}{\partial t_*} = \frac{\partial^2 \Theta}{\partial x_*^2} + \frac{\partial^2 \Theta}{\partial z_*^2} - 2\Theta \frac{\partial \Theta}{\partial z_*} \]  

is solved subject to initial and boundary conditions (3.1.8) - (3.1.10) and surface flux condition

\[ \Theta^2 - \frac{\partial \Theta}{\partial z_*} = \begin{cases} Q(t_*) & 0 \leq x_* \leq \omega \\ 0 & \omega < x_* \leq \lambda \end{cases} \quad z_* = 0 \]

numerically by the partial differential equation solver PDETWO. Again, \( Q(t_*) = R_*t_* \).

It is encouraging to note that the values for the cumulative infiltration decrease as the furrow width increases. The results obtained by the weakly nonlinear Burgers Equation model are similar to those obtained by the Linear model. The significant differences in the values for the cumulative infiltration are influenced more by the supply rate imposed at the surface than by the surface supply pattern.

It would appear that the surface structure does not have a major influence on ponding times when the infiltration rate is strongly increasing, as in the linearly increasing case. When the infiltration rate is held constant or is decreasing however, the structure of the infiltration system at the surface of the soil imparts a significant influence on the ponding times.
3.20 Cumulative Infiltration Vs Infiltration Acceleration Rate

When the linear and Burgers models are compared as in Figure 3.21, the linear model overestimates the cumulative infiltration, as is expected. This overestimation is greater in the linearly increasing time dependent model than in the previous constant surface supply rate case.

3.21 Time To Ponding Vs Infiltration Acceleration Rate
The interesting results are observable when the results from the one dimensional Burgers model are compared with the associated results in two dimensions. As expected, the one dimensional system ponds faster than the associated two dimensional system. However, unlike the constant rate infiltration case, when a significantly increasing irrigation rate is imposed at the boundary, the differences in the ponding times are not as pronounced. This lends credence to the hypothesis that if there is an accelerating irrigation rate, the surface supply rate has a much greater influence on ponding times than the surface supply pattern.

3.22 Time To Ponding Vs Infiltration Acceleration Rate

In many circumstances where there is a significant time dependence, the surface flux is the most important component of the infiltration system. Therefore, the use of a one dimensional model can be justified. The one dimensional models are often easier to solve analytically than the two dimensional models and for certain surface boundary conditions, there is no significant difference between the obtained results.
Field applications of infiltration rarely encompass a constant surface supply rate for all times \( t^* > 0 \). Likewise, it is impractical financially and environmentally to design an irrigation system with a surface supply rate that has a strongly increasing linear time dependence. Physically, field infiltration is highly varied, sprinkler systems are activated and deactivated as demand dictates and periods of naturally occurring rainfall can be highly mercurial. A number of periodic time dependent surface infiltration conditions are considered, to model two dimensional infiltration where the surface supply rate is highly variable.

Within the irrigated strip \( x^* \in [0, \omega] \), fluid is delivered at an oscillatory time dependent rate. This models the physical system in which an irrigation system is switched on or off at periodic time intervals. Two such cases are considered.

Firstly, we consider a surface supply rate which consists of both a constant component and a steady oscillation. In this situation the two dimensional linear fluid flow equation (3.1.12) is solved subject to initial and boundary conditions (3.1.8) - (3.1.10) with surface supply rate boundary condition

\[
\Theta - \frac{\partial \Theta}{\partial z^*} = \begin{cases} 
R^* - \alpha \cos(\mu t^*) & 0 \leq x^* \leq \omega \\
0 & \omega < x^* \leq \lambda \\
\end{cases} \quad z^* = 0
\]

by taking Laplace transforms and utilising the method of separation of variables. As there exists an extra time dependent term in the oscillatory component of the surface supply rate boundary condition, this leads to an extra time dependent term in the solution, found by employing the Laplacian Convolution Theorem.

The time to surface ponding \( \tau_p \) at the centre of the wet strip, \( x^* = 0 \) is found by solving \( \Theta(0,0,\tau_p) = 1 \), where
(3.2.8) \[ \Theta(0,0,t^*) = \frac{\omega R_*}{\lambda} \left( 1 + \left( \frac{t^*}{\pi} \right)^2 \exp \left( -\frac{t^*}{4} \right) - \left( 1 + \frac{t^*}{2} \right) \text{erfc} \left( \frac{t^*}{\sqrt{4}} \right) \right) \]

\[ - \frac{\omega R_*}{\lambda} \int_0^{t^*} \alpha \cos(\mu(t^* - \tau)) \left( \left( \frac{1}{\pi \tau} \right)^2 \exp \left( -\frac{\tau}{4} \right) - \frac{1}{2} \text{erfc} \left( \frac{\tau}{\sqrt{4}} \right) \right) d\tau \]

\[ + \sum_{n=1}^{\infty} \frac{2R_*}{n \pi} \sin \left( \frac{n \pi \omega}{\lambda} \right) \left[ \frac{1}{2} \left( F_- + F_+ \right) + G \right] - \frac{2R_*}{n \pi} \sin \left( \frac{n \pi \omega}{\lambda} \right) \]

\[ \times \int_0^{t^*} \alpha \cos(\mu(t^* - \tau)) \left( \left( \frac{1}{\pi \tau} \right)^2 \exp \left( -\left( \frac{n^2 \pi^2}{\lambda^2} + \frac{1}{4} \right) \tau \right) \right) - G_1 \right) d\tau \]

with \( F_\pm \) and \( G \) defined by equations (3.1.16) and (3.1.17) respectively and

(3.2.9) \[ G_1 = \frac{1}{2} \exp \left( -\frac{n^2 \pi^2 t^*}{\lambda^2} \right) \text{erfc} \left( \frac{t^*}{\sqrt{4}} \right). \]

The two quantities which may affect the time to surface ponding in two dimensions are the periodic strip half width and the magnitude of the surface infiltration rate.

The half widths considered are \( \omega \) approximately equal to the intrinsic length scale \( \lambda_s \) in the first case and this wetted fraction is increased so that \( \omega \) is greater than \( \lambda_s \) in the second case. Both of these cases have an identical surface infiltration rate imposed so that the effect of the alteration in the wetted soil surface structure can be scrutinised.

Figure 3.23 indicates that the time to surface ponding is generally independent of the surface wetting pattern when the infiltration rate has a significant time dependence imposed. The time to surface ponding \( \tau_p \), for each case is small when compared to the period of the oscillation \( 2\pi/\mu \) which would explain the lack of oscillatory effects. When the period of oscillation is decreased however, the time to surface ponding is not significantly altered.
The second factor which may affect the time to ponding is the rate of infiltration at the surface of the soil. To appraise the full effect of the surface supply rate and to investigate diffusive effects at depth, the half width $\omega$ is fixed and three distinct infiltration rates are considered.

Figure 3.24 shows the depth of the soil water profile $z_*$ against the soil water content $\Theta$ for fixed half width $\omega$ and varying surface supply rates.

As the initial infiltration rate increases, the relative importance of lateral transport is diminished and thus the time to surface ponding is decreased. The surface supply rate has a much greater influence on the time to surface ponding than does the pattern of the irrigation supply at the surface. This is also the case when the system has a constant surface supply rate.

3.23 Time To Ponding Vs Infiltration Rate
3.24 Depth Vs Soil Water Content At Incipient Ponding (Fixed $\omega$)

To increase evaporative effects in the wet strip, the surface supply rate boundary condition is again modified.

\[
\Theta - \frac{\partial \Theta}{\partial z_*} = \begin{cases} 
R_* \cos(\mu t_*) & 0 \leq x_* \leq \omega \\
0 & \omega < x_* \leq \lambda \\
\end{cases} \quad z_* = 0
\]

and the two dimensional linear fluid flow equation (3.1.12) is solved subject to initial and boundary conditions (3.1.8) - (3.1.10) and the surface supply condition (3.2.10) above.

Again, the solution is found by utilising the Laplacian Convolution Theorem, and in this case the time to surface ponding $\tau_p$ at the centre of the wet strip $x_* = 0$ is found by evaluating $\Theta(0,0,\tau_p) = 1$, where

\[
\Theta(0,0,t_*) = \frac{R_* \omega}{\mu} \int_0^{t_*} \left[ 1 + \left( \frac{t_*}{\pi} \right)^2 \exp\left( \frac{t_*}{4} \right) - \left( 1 + \frac{t_*}{2} \right) \text{erfc}\left( \frac{t_*}{\sqrt{2}} \right) \right] \cos(\mu(t_* - \tau)) \, d\tau \\
+ \sum_{n=1}^{\infty} \frac{2R_*}{n\pi} \sin\left( \frac{n\pi \omega}{\lambda} \right) \int_0^{t_*} \left[ \frac{1}{2} \left( F_- + F_+ \right) + G \right] \cos(\mu(t_* - \tau)) \, d\tau
\]
with \( F_\pm \) and \( G \) defined by equations (3.1.16) and (3.1.17) respectively.

As with the previous case, the two physical properties altered are the periodic strip half width and the surface supply rate. Once again a half width approximately equal to the intrinsic length scale and a half width greater than the intrinsic length scale are considered whilst imposing an identical surface supply rate.

Unlike the previous periodic case, the surface supply pattern imparts a significant influence on the time to ponding for lower initial supply rates. Lateral transport has an effect; for lower initial supply rates significant diffusion occurs both laterally and at depth. As a greater volume of fluid is transported from the soil’s surface, consequently a greater volume must be infiltrated in order that ponding takes place.

![Graph showing Time To Ponding Vs Infiltration Rate](image)

**3.25 Time To Ponding Vs Infiltration Rate**

Subsequently to investigate the effect of the surface supply rate, the periodic half width \( \omega \) is held constant and two different infiltration rates are imposed at the surface of the soil.
The time to surface ponding is greater for lower initial rates of surface infiltration. For each value $\tau_p$ the soil water profile is plotted as shown above in Figure 3.26. Here the non ponding part of the infiltration cycle has an exaggerated effect in the case of lower initial surface supply rates, $R^* < 2$. For higher initial rates of surface supply, ponding is virtually instantaneous which prevents the evaporative part of the infiltration cycle from having a significant effect on ponding times. That is to say, for higher initial rates of surface supply, the system ponds before the evaporative phase of the infiltration cycle is reached.

This has well known obvious field applications, in particular for crop management. As Figure 3.26 indicates, for the higher initial surface supply rate $R^* = 3.2$, ponding is virtually instantaneous. In some horticultural applications it is usually not feasible to irrigate fields by flooding as this method exacerbates water run-off, soil erosion and as a result does not adequately irrigate crops.
4. Steady Infiltration In Sloping Porous Domains

Under consideration is groundwater flow through a sloping porous domain. The cross section of the region is assumed to be a long thin parallelogram, the vertical sides and base of which are impermeable, as shown in Figure 4.1.

At the surface of the soil there exists a wetted fraction through which fluid is recharged and through the remaining proportion of the soil surface evaporation and transpiration (evapotranspiration) takes place. The rate of recharge is held equal to the rate of evapotranspiration so that the system is in equilibrium and the total soil water content remains constant.

Recently Read & Broadbridge (1996) solved the steady quasi linear unsaturated flow problem through porous domains with an arbitrary shape modelled by a stream function. Using the stream function method, the matrix flux potential and hydraulic head are available as series expansions. The matrix flux potential,

\[(4.1.1) \quad \mu = \int_0^\theta D(\theta) d\theta = \int_{-\infty}^{\psi} K(\psi) d\psi\]

sometimes referred to as the matric flux potential, is the horizontal flux potential and is due to the characteristics of the soil matrix rather than gravitational effects. It is significant that the matrix flux potential is not uniquely determined by the stream function as each stream function can correspond to many moisture distributions. Read & Broadbridge (1996) demonstrated however that in a finite porous domain there exists only one moisture distribution that has an emergent saturated zone. In common with the time to ponding studies of the previous section this phenomenon involves the prediction of a nascent
saturated zone. However, in this case of two dimensional flow, we are concerned with the location of this nascent saturated zone in space rather than in time.

The flow solution for flow through a finite porous domain is dependent on many soil water parameters including the size of the wetted fraction, the aspect ratio (that is, the ratio between the length and depth of the vadose zone), the rate of recharge, the dimensionless sorptive number and the basal inclination or slope. If the basal inclination is positive, we assume recharge at the summit and evapotranspiration at the foot of the porous region. A negative slope however, implies that evapotranspiration occurs at the summit and recharge at the foot of the porous region.

The analytic series solution has the flexibility of allowing arbitrary boundary conditions for the recharge representation. In addition it is computationally efficient and effectively models seepage geometries for which the aspect ratios are significantly larger than current numerical schemes allow.

For the system under consideration, the basement inclination is w. Given this slope, the impermeable base is at z=wx and the soil surface is z=wx+D, where D is the depth of the vadose zone. The system is recharged between x=S and x=L by a uniform rate R and evapotranspiration occurs between x=0 and x=S.

The dimensionless fluid flow equation modelling flow through the unsaturated zone

\[ \nabla . (K_\cdot \nabla \Psi_\cdot) + \frac{\partial K_\cdot}{\partial z_\cdot} = 0 \]

is simplified by making use of the Quasi linear approximation

\[ K_\cdot = \exp(\alpha_\cdot \Psi_\cdot) \]
Here, $K_* = K/K_s$ is the hydraulic conductivity, $\Psi_* = \Psi/D$, the moisture potential and the sorptive number $\alpha_* = \alpha/D$, all in terms of dimensionless variables.

The quasi-linear approximation assumes an exponential relationship between the hydraulic conductivity and capillary potential. Applying the quasi-linear approximation to equation (4.1.2) simplifies this complex nonlinear equation to an elementary form, yet still retains the essential nonlinear characteristics of the hydraulic conductivity and capillary potential.

4.1 Cross Section Of The Porous Domain (Tritscher 1996)
Therefore, applying the dimensionless Kirchhoff transformation, where the Kirchhoff variable $\mu$ is the matrix flux potential,

\begin{equation}
(4.1.4) \quad \mu = \int_{-\infty}^{\infty} K_n(\Psi) d\Psi = \int_{-\infty}^{\infty} \exp(\alpha, \Psi) d\Psi
\end{equation}

linearises the transport equation, given by equation (4.1.2) to the simple linear form,

\begin{equation}
(4.1.5) \quad \nabla^2 \mu + \alpha_s \frac{\partial \mu}{\partial z_s} = 0.
\end{equation}

The subsequent use of the Cauchy-Riemann Equations formulates the problem in terms of the stream function. The stream function similarly satisfies the linearised governing equation (4.1.5), (Raats, 1970). Thus in terms of the stream function $\Psi_s$, the linearised transport equation is

\begin{equation}
(4.1.6) \quad \frac{\partial \Psi_s}{\partial x_s} = -\left( \frac{\partial \mu}{\partial z_s} + \alpha_s \mu \right), \quad \frac{\partial \Psi_s}{\partial z_s} = \frac{\partial \mu}{\partial x_s}.
\end{equation}

We assume the soil surface is subject to a uniform vertical recharge at rate $R_s$ between $x_s = S_s$ and $x_s = L_s$, the remaining proportion of the soil surface from $x_s = 0$ to $x_s = S_s$ is subject to the process of evaporation and transpiration. The stream function varies linearly along the soil surface

\begin{equation}
(4.1.7) \quad \Psi_s(x_s, \alpha_x, +1) = R_s(x_s) = \begin{cases} 
R_s \frac{(L_s - S_s)}{S_s} x_s, & 0 \leq x_s \leq S_s \\
R_s \frac{L_s - x_s}{S_s} & S_s < x_s \leq L_s 
\end{cases}
\end{equation}
Here, \( \beta = (L_\ast - S_\ast)/L_\ast \), the wetted fraction, is the proportion of the soil surface under recharge.

As there is nil matrix flux through the base or across the impermeable vertical barriers,

\[
(4.1.8) \quad \Psi_\ast(0, z_\ast) = \Psi_\ast(x_\ast, \omega_x) = \Psi_\ast(L_\ast, z_\ast) = 0
\]

The linearised transport equation (4.1.6) is solved subject to boundary conditions (4.1.7) and (4.1.8) by separation of variables. Thus,

\[
(4.1.9) \quad \Psi_\ast(x_\ast, z_\ast) = \exp\left(-\frac{\alpha_\ast z_\ast}{2}\right) \sum_{n=1}^{\infty} \left[ A_n \sinh(\gamma_n z_\ast) + B_n \cosh(\gamma_n z_\ast) \right] \sin\left(\frac{n\pi x_\ast}{L_\ast}\right)
\]

with

\[
(4.1.10) \quad \gamma_n = \sqrt{\frac{\alpha_\ast^2}{4} + \frac{n^2 \pi^2}{L_\ast^2}}
\]

In terms of the matrix flux potential,

\[
(4.1.11) \quad \mu(x_\ast, z_\ast) = B_0 \exp(-\alpha_\ast z_\ast) - \exp\left(-\frac{\alpha_\ast z_\ast}{2}\right) \sum_{n=1}^{\infty} \frac{L_\ast}{n\pi} \left( -\frac{A_n}{2} + \gamma_n B_n \right) \sinh(\gamma_n z_\ast) \cos\left(\frac{n\pi x_\ast}{L_\ast}\right)
\]

\[
- \exp\left(-\frac{\alpha_\ast z_\ast}{2}\right) \sum_{n=1}^{\infty} \frac{L_\ast}{n\pi} \left( \gamma_n A_n - \frac{B_n}{2} \right) \cosh(\gamma_n z_\ast) \cos\left(\frac{n\pi x_\ast}{L_\ast}\right)
\]
Here, the constants $A_n$ and $B_n$ are determined by the least squares method as outlined by Read & Broadbridge (1996). The series coefficients $A_n$ and $B_n$ are therefore found from,

\[ (4.1.12) \sum_{i=1}^{\infty} \left[ (\cosh(\gamma_i) - \cosh(\gamma_n))k^u_{ni} - \sum_{j=1}^{\infty} \sinh(\gamma_j)k^u_{nj}k^u_{ji} + \delta_{ij} \sinh(\gamma_n) \right] A_i = k^R_n, \]

and

\[ (4.1.13) B_n = -\sum_{i=1}^{\infty} k^u_{ni} A_i \]

respectively, where $\delta_{ij}$ is the Kronecker delta.

The arbitrary parameter $B_0$ arises from setting $n=0$ in the eigenfunction expansion (4.1.13) and is determined by specifying one other physical quantity. In this application the value of $B_0$ is deduced from the field observation that the minimum water content in the region is zero.

As outlined previously, the emphasis is on determining the location of the saturated zone in space rather than in time. It is also of interest to determine the relationship, if any, between the critical infiltration rate, the inclination of the vadose zone and the dimensionless sorptive number.

The dimensionless sorptive number $\alpha_*$, is the ratio between the geometric length scale (or maximum depth of the vadose zone $D$) and the intrinsic length scale. The critical infiltration rate is defined as the infiltration rate at the onset of saturation.

Figure 4.2 (Tritscher, 1996) displays the critical infiltration rate as a function of the inclination of the sloping porous domain for a variety of dimensionless sorptive numbers. Highly nonlinear soils, such as fine textured clays are characterised by low dimensionless
sorptive numbers, whilst weakly nonlinear soils, such as coarsely grained sands generally have high dimensionless sorptive numbers. The six sorptive numbers plotted (in order from top to bottom) are $\alpha_\ast = 0.01$, $\sqrt[10]{10} \times 10^{-2}$, $0.1$, $\sqrt[10]{10} \times 10^{-1}$, $\sqrt[10]{10}$ and 10. The level curves displayed in Figure 4.2 represent soil samples which encompass the range of field occurring soils from highly nonlinear clays (at the top of the graphs) to weakly nonlinear sands (at the bottom of the graphs). The results displayed in Figure 4.2 ((a) - (i)) are for various wetted fractions and aspect ratios\(^1\). Recall that the aspect ratio of the porous domain is the ratio between the depth and length of the vadose zone.

![Graphs showing dimensionless recharge rate vs basal inclination](image)

**4.2 Dimensionless Recharge Rate Vs Basal Inclination**

\(^1\) (a)-(c) Coverage $\frac{1}{8}$, Aspect 25, 50, 100; (d)-(f) Coverage $\frac{1}{2}$, Aspect 25, 50, 100; (g)-(i) Coverage $\frac{7}{8}$, Aspect 25,50,100.
As shown in Figure 4.2, the critical infiltration rate is a decreasing function of the aspect ratio. As the aspect ratio increases, the critical infiltration rate decreases. Increasing the aspect ratio has the same effect of increasing the effective area under recharge, without necessarily increasing the size of the wetted fraction. A consequence of this increase in the effective area under recharge is an increase in the water pressure potential. As the effective area under recharge increases, it stands to reason that the flow rate through the region increases and from Darcys' Law (1.1.2), so too does the gradient of the total potential. In order to avert saturation, the critical infiltration rate must decrease.

Figure 4.3 (Tritscher, 1996) shows the horizontal coordinate of the saturation point for the same dimensionless sorptive numbers, aspect ratios and wetted fractions as in Figure 4.2.

As shown in previous sections, for a horizontal surface, ponding is inevitable if the rate of surface infiltration $R_*$ exceeds the hydraulic conductivity at saturation $K_s$. In a sloping domain with an impermeable base and impermeable vertical sides, saturation may occur at the surface of the soil or at depth. Recall, if there is a negative inclination, evapotranspiration occurs at the summit of the slope and recharge at the base. A positive basal inclination however implies evapotranspiration at the base and recharge at the summit.

If in the sloping porous domain, the basal inclination of the field is too steep in the negative direction, gravitational transport assists a pressure build up underneath the wetted fraction and basement ponding occurs. To avoid ponding the critical infiltration rate needs to be decreased. Similarly, if the inclination of the field is too steep in the positive direction, gravity will transport water to the lowest vertex, where saturation will first occur. This implies that given a certain recharge rate, there exists an optimal inclination for the avoidance of ponding; hence the local maximum in Figure 4.2.
For highly nonlinear soils such as fine textured clays, the basal inclination has negligible effect on the maximum infiltration rate before the onset of saturation. If the soil is weakly nonlinear however, such as in coarsely grained sands, the inclination of the domain has a significant influence on the critical infiltration rate at the onset of saturation. The dimensionless sorptive number, which characterises all soil types, is an indication of the relative dominance of capillary and gravitational forces. In a coarse soil, capillary action is relatively ineffective in transporting water laterally to the evapotranspiration zone and then upwards to the evaporation surface. Gravity forces dominate, accounting for the pressure build-up and basement saturation. A surprising result is that a saturated zone will always emerge in a sand sample regardless of the rate of infiltration or the inclination of the slope. The dominant gravitational forces transport fluid away from the surface until it

4.3 Horizontal Coordinate of the Saturation Point Vs Basal Inclination
reaches the impermeable basement barrier. As capillary action is ineffective in soils with high dimensionless sorptive numbers, there is very little lateral transport and extremely large pressure gradients are required to transport the fluid upwards through the evapotranspiration region to the evaporative surface.

For all soil types, as the coverage increases the critical infiltration rate decreases. This is due to the fact that as the surface area under recharge increases, so does the fluid flow within the region. This increased fluid flow increases the water pressure potential and the likelihood of saturation. To offset this, the critical infiltration rate before the onset of saturation must decrease.

A common relationship for all soil types appears to be the rate of surface infiltration and the proportion of soil through which fluid is delivered. As the rate of infiltration and the proportion of soil under recharge increases, overall the time taken for the onset of saturation decreases. For weakly nonlinear soils, the effect of increasing the surface coverage through which fluid is delivered, although reduces the critical infiltration rate, this is not as pronounced as altering the aspect ratio.
5. Conclusions

Many soil water parameters have an effect on infiltration and hence the time to surface ponding. In this application, the parameters concentrated on were the surface supply pattern and the surface infiltration rate for a variety of the standard groundwater flow models.

It was found that the surface supply pattern imparts a significant influence on ponding times especially in the case where the surface infiltration rate is held constant. As the wetted fraction increases, the time to surface ponding in a two dimensional array of periodic strips converges to reasonably agree with the associated time to ponding in one dimension. As the wetted fraction decreases however, the time to surface ponding increases due to the effect of horizontal diffusion. In addition, as the surface ponding time decreases, so too does the cumulative infiltration, that is, the amount of water absorbed into the soil during the transition from unsaturated to saturated flow.

In terms of variations to the constant rate rainfall surface boundary condition, it was determined that alterations in the infiltration rate consequently alter the time to surface ponding. With an evaporative effect acting between furrows in the two dimensional array of periodic strips, the ponding times increase marginally. As the rate of surface infiltration increases, the effect of evaporation from the drier unirrigated region is negated as the ponding phenomenon occurs before the influence of the evaporative surface is felt at the irrigated surface.

The other variation with regards to the two dimensional periodic array with a constant rate rainfall surface infiltration condition is the imposition of a fractal model at the surface. To compensate for the alteration at the surface, the rate of infiltration is increased. At all times however, the same average surface supply is delivered through the standard irrigation array and the fractal array. The surface ponding time decreases through
the Cantor Set fractal array, due to the increased local irrigation rate through the narrower wetted strips, but does in fact approach a non-zero limit even as the fractal supply surface becomes more diffuse.

It is pleasing to note that as the surface supply rate far exceeds the hydraulic conductivity at saturation, the ponding time for the linear model, well known for over estimating ponding times, converged to the ponding times for the weakly nonlinear Burgers equation model and the versatile nonlinear model. The relative error is approximately the same for both time and cumulative infiltration at ponding, and hence can be compensated for when an exact analytic solution is precluded by complex surface geometries.

Although the surface supply pattern has an influence over ponding times, if the surface supply rate is accelerating significantly, the influence of the surface structure is marginal at best.

Again, the linear model over-estimates ponding times for the case where the surface supply rate is linearly increasing over the irrigation cycle, but as this supply rate increases the linear model converges to agree reasonably well with the weakly nonlinear Burgers equation model. Unlike the constant rate rainfall infiltration boundary condition, the differences in ponding times for the one and two dimensional models are not as pronounced when the surface supply rate is linearly increasing.

For the linearly increasing time dependent flux boundary condition, the ponding time and hence the cumulative infiltration is higher than for the corresponding constant rate rainfall condition. In addition there is a considerable error between the Burgers equation time dependent infiltration and constant rate infiltration models when the cumulative infiltration as a function of the infiltration rate at ponding is considered. This contradicts the well-known postulate which asserts that if the cumulative infiltration is regarded as the time-like variable then all the ponding curves condense. Therefore, the so-called "time-condensation" phenomenon appears to be erroneous.
To fully test the influence of the surface supply rate, a further time dependent supply rate is considered. The periodic surface supply rates were chosen to model field occurring infiltration in agricultural applications, for example, the regular periodic flow as a result of sprinkler irrigation. It was found, that as with the previous time dependent infiltration case, the influence of the surface structure is marginal at best.

Hence, the prevailing factor in surface ponding for horizontal field applications is the magnitude of the surface supply rate. This helps to justify the widespread use of one dimensional models for this purpose.

In ponding applications involving a sloping field, under the influence of both infiltration and evaporation and transpiration, a number of soil water parameters impart an influence. One of the most significant of these parameters is the inclination of the sloping field. It is found that the surface supply rate is a decreasing function of the ratio between the length and the depth of the vadose zone, called the aspect ratio. For a given basal inclination there exists a maximum recharge rate, for which basement saturation is averted, as the recharge component and the evapotranspiration component of the infiltration system are in equilibrium. Furthermore, there is an optimal basal inclination, typically of the order of two degrees for avoiding saturation. At higher slopes, the soil saturates at the bottom of the slope whereas at lower slopes, the soil saturates directly beneath the recharge zone.
Acknowledgment

From time to time there emerges an awe inspiring individual who stands head and shoulders above the rest, who sets the standards others try to emulate; William Shakespeare, John Donne, Vincent Van Gogh and my supervisor Professor Philip Broadbridge.

I have had the rare privilege of working with a supervisor worth more than his weight in gold and I can not adequately express my gratitude. Thanks Phil for your fantastic support and encouragement, invaluable advice and inexhaustible patience.
6. Appendix

(6.1) Fortran77 Code for Fluid Flow Problems

(a) One Dimensional Infiltration

c **This Program Calculates The Time To Surface
Ponding and the Ponding Time Ratios for both the
Linear & Nonlinear 1-Dimensional Models
dimension r(45)
pi=4.*atan(1.)
c **Do loop sets the dimensionless surface
infiltration rates
do 10 i=1,46
r(i)=1.5+4.*(i-1.)/45.
rho=r(i)
c **Call statements refer the program to the relevant
solution (Linear or Nonlinear) for evaluation
call linear (rho,t1)
call nonlinear (rho,t2)
c **Call statement refers program to determine the
ratio for the ponding times for the Linear &
Nonlinear models
call ratio (t1,t2,rat)
c **Print statement prints the numerical output
print*,rho,t1,t2,rat
10 continue
stop
end

c subroutine linear(rho,t1)
data dk,dth,s/1.059,.2574,3.686/
pi=4.*atan(1.)
c **tlow & tu sets the lower & upper bounds for the surface ponding times
tlow=.5/rho**2
tu=log(rho/(rho-1.))/rho
c **t1 takes the average of the upper & lower time bounds->bisection method
t1=.5*(tlow+tu)
c **f1 & f2 incorporate the analytic solution for the linear model
f1=(rho-1.)/rho
f2=(1.+2.*t1/pi)*f(sqrt(t1/pi))-(2./pi)*sqrt(t1)*exp(-1.*t1/pi)
c **gl finds the absolute value of the difference between the two parts of the solution->the time interval is successively narrowed down by using the bisection method
gl=abs(f2-f1)
if(g1.le..0001)then
  go to 20
else if(f2.lt.f1)then
  tu=t1
else if(f2.gt.f1)then
  tlow=t1
end if
go to 25

20 continue
return
end

subroutine nonlinear(rho,t2)
**t2 is the explicit time to surface ponding for the
one dimensional nonlinear model

t2=.5*(log(rho/(rho-1.)))/rho
return
end

subroutine ratio(t1,t2,rat)
**The ratio between the linear & nonlinear ponding
time is determined

rat=t1/t2
return
end

function erfc(x)
**This subroutine determines an approximation to
the Complementary Error Function (Abramowitz &
Stegun 1964): Equation 7.1.27

real x

data a1,a2,a3,a4/.278393,.230389,.000972,.078108/
erfc=1./(1.+a1*x+a2*x*x+a3*x*x*x+a4*x*x*x*x)**4
return
(b) Two Dimensional Infiltration: Analytic Solution

**This Program Calculates The Time To Surface Ponding In Two Dimensions for the Linear Model (Analytic Solution)**

real l, ls, lstar
dimension r(50)

**Data sets the values of the various soil water parameters**
data dk, dth, s, w, l, ls /1.059, .2574, 3.686, 30., 70., 27.6/
pi = 4.*atan(1.)

**om = dimensionless furrow half width**
**lstar = dimensionless distance between furrows**
**ls = capillary length scale**

om = w/ls
lstar = l/ls

**Do loop sets the dimensionless surface infiltration rates**
do 50 i=1, 51
r(i) = 1.5 + 4. * (i-1)/50.
rho = r(i)

**t is the dimensionless time of infiltration**
t = 0.
10 t = t + .0001
sum = 0.
term = 0.
**Do loop evaluates the series component of $\theta_n$**

```plaintext
do 20 n=1,20
al=.25+(n*pi/lstar)**2
a2=.5
u1=2.*sqrt(al)-2.*sqrt(al)*f(sqrt(al*t))
ul=.5*u1/(n*pi/lstar)**2
u2=exp(t*(n*pi/lstar)**2)*f(sqrt(.25*t))
u22=u2/(2.*(n*pi/lstar)**2)
term=2.*rho*sin(n*pi*om/lstar)*(ul+u22)/(n*pi)
sum=sum+term
continue
```

```plaintext
gl=l.+sqrt(t/pi)*exp(-.25 *t)
g2=(l.+.5*t)*f(sqrt(.25*t))
**th0 evaluates $\theta_0$**

```plaintext
th0=rho*om*(gl-g2)/lstar
th=th0+sum
```

**Logical operators continue program loop until surface saturation, defined by $\theta = 1$ is reached**

```plaintext
if(th.lt.1)then
go to 10
else if(th.ge.1)then
go to 30
end if
```

```plaintext
**print statement prints the numerical output**

```plaintext
print*,rho,t
```

50 continue
stop
end
**This subroutine determines an approximation to the Complementary Error function (Abramowitz & Stegun 1964): Equation 7.1.27**

```fortran
function f(x)
real x

data a1,a2,a3,a4/.278393,.230389,.000972,.078108/
f=1./(1.+a1*x+a2*x*x+a3*x*x*x+a4*x*x*x*x)**4
return
end
```

(c) **Two Dimensional Infiltration: Numerical Solution**

**This Program Calculates The Time To Surface Ponding In Two Dimensions (Numerical Solution)**

real 1,ls,lstar

**Dimension statements set the size of the array for the relevant parameters:**

- `u=soil water content, r=surface infiltration rate`
- `x=horizontal mesh points, y=vertical mesh points`

```fortran
dimension u(1,21,101),x(21),y(101),r(100)
dimension work(157117),iwork(2121)
common om,lstar,rho,diff
```

**Data sets the values of the various soil water parameters**

```fortran
data dk,dth,s,w,l,ls/1.059,.2574,3.686,70.,70.,27.6/
```
pi=4.*atan(1.)
c  **om=dimensionless furrow half width
c  **lstar=dimensionless distance between furrows
c  **ls=capillary length scale
om=w/ls
lstar=1/ls
c  **Diff sets the dimensionless diffusivity
Linear & Burgers models->diff=1.
c  Nonlinear model->diff=c(c-1)/(c-u(1))**2
diff=1.
c  **Do loop sets the values of the dimensionless
surface infiltration rates
do 250  k=1,101
  r(k)=1.5+20.*((k-1)/100.
  rho=r(k)
c  **npde=number of coupled pdes to be solved by PDETWO
npde=1
morder=5
c  **nx=number horizontal discretisation points
c  **ny=number vertical discretisation points
nx=21
ny=101
node=nx*ny*npde
c  **y(ny)=dimensionless vertical depth
y(ny)=40.
dy=y(ny)/float(ny-1)
c  **do loops j=1,ny set the vertical spacial
discretisation
do 10 j=1,6
  y(j)=.1*(j-1)*dy
10 continue
do 11 j=7,12
  y(j)=.2*(j-1)*dy
11 continue
dy=(y(ny)-y(12))/(ny-12)
do 13 j=13,ny
  y(j)=y(j-1)+dy
13 continue
c  **x(nx)=dimensionless horizontal width
  x(nx)=lstar
  dx=x(nx)/float(nx-1)
c  **do loops i=1,nx set the horizontal spacial
c  discretisation
do 15 i=1,nx
  x(i)=(i-1)*dx
15 continue
c  set the initial condition
tin=0.
do 20 i=1,nx
do 20 j=1,ny
  u(1,i,j)=tin
20 continue
h=.001
eps=.01
mf=22
index=1
nout=1
iwork(1)=npde
iwork(2)=nx
iwork(3)=ny
iwork(4)=morder
iwork(5)=157117
iwork(6)=2121

**t0 & tout are the input & output times respectively

continue

t0=0
tout=.25/rho**2

25 continue

**drivep is the subroutine which calls the integrator in PDETWO

call drivep(node,t0,h,u,tout,eps,mf,
       . index,work,iwork,x,y)

**set the soil water content as the subject of the numerical solution ie $\theta = u(1,1,1)$

th=u(1,1,1)

**reset the time variable

t0=tout
tout=tout+.01

**Logical operators continue program loop until surface saturation is reached, ie $u(1,1,1)=1$

if(th.ge.1)then
go to 30
else if(th.lt.1)then
go to 25
end if
30 continue

c **Print statement prints the numerical output
print*,rho,tout
250 continue
stop
end

c

c **this subroutine sets the Horizontal
Boundary Conditions (surface & at depth)
subroutine bndryh(t,x,y,u,ah,bh,ch,npde)
c **set boundary condition coefficients

ah(i)u_i + bh(i) \frac{\partial u_i}{\partial y} = ch(i)

dimension u(npde),ah(npde),bh(npde),ch(npde)

common om,lstar,rho,diff
real lstar

c **y=0->surface boundary condition
if (y.eq.0) go to 35
ah(1)=1
bh(1)=0
ch(1)=0
go to 40
c **set surface infiltration pattern
35 if(x.gt.0..and.x.le.om)then
flux=rho
else if(x.gt.om.and.x.lt.lstar)then
flux=0.
end if
**cond=K(th): hydraulic conductivity

Linear K=1, Burgers K=u(1)**2,

Nonlinear K=c^2(c-1)/(c-u(1)) - 2c(c-1) + (c-1)(c-u(1))

ah(1)=0.

bh(1)=-diff

ch(1)=flux-cond

40 continue

return

diffusion

**this subroutine sets the Vertical Boundary Conditions

subroutine bndryv(t,x,y,u,av,bv,cv,npde)

dimension u(npde),av(npde),bv(npde),cv(npde)

common om,lstar,rho,diff

real lstar

**set boundary condition coefficients

\[ av(i)u_i + bv(i) \frac{\partial u_i}{\partial x} = cv(i) \]

av(1)=0.

bv(1)=1.

cv(1)=0.

return

end

**this subroutine sets the horizontal diffusion

subroutine diffh(t,x,y,u,dh,npde)

dimension u(npde),dh(npde,npde)
common om,lstar,rho,diff  
real lstar  
c **diff=diffusion  
c Linear,Burgers-> diff=1  
c Nonlinear-> diff=c(c-1)/(c-u(1))**2  
dh(1,1)=diff  
return  
end  
c  
c **this subroutine sets the vertical diffusion  
subroutine diffv(t,x,y,u,dv,npde)  
dimension u(npde),dv(npde,npde)  
common om,lstar,rho,diff  
real lstar  
c **diffusion is uniform horizontally & vertically,  
c therefore recall diffh & rename for diffv  
call diffh(t,x,y,u,dv,npde)  
return  
end  
c  
c **this subroutine defines the governing equation  
subroutine f(t,x,y,u,ux,uy,duxx,duyy,dudt,npde)  
dimension ux(npde),uy(npde),duxx(npde,npde)  
dimension duyy(npde,npde),dudt(npde),u(npde)  
common om,lstar,rho,diff  
real lstar  
c **set coefficients for the governing equation
c dcond=DK(th)

Linear-> dcond=1, Burgers-> dcond=2.*u(1)

Nonlinear-> dcond=c^2(c-1)/(c-u(1))^2-(c-1)

\[ dcond=2.*u(1) \]

\[ dudt(1)=duxx(1,1)+duyy(1,1)-dcond*uy(1) \]

return

end
7. References


