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Diffraction of nonlinear short-crested waves around a vertical circular cylinder

Pornchai Satravaha

University of Wollongong

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DIFFRACTION OF NONLINEAR SHORT-CRESTED WAVES
AROUND A VERTICAL CIRCULAR CYLINDER

A thesis submitted in partial fulfilment of the
requirements for the award of the degree

MASTER OF SCIENCE (HONOURS)

THE UNIVERSITY OF WOLLONGONG

by

Pornchai Satravaha B.Sc., M.Sc. (C.U.)

Department of Mathematics

1992
This Thesis is submitted to The University of Wollongong and has not been submitted for a higher degree to any other University or Institution.

Pornchai Satravaha

1992
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I trust that all people who I have not personally mentioned here are aware of my deep appreciation.
ABSTRACT

In this thesis, a complete solution is presented in closed-form for the velocity potential, up to the second-order of wave amplitude, resulting from the diffraction of nonlinear short-crested waves around a vertical circular cylinder. Second-order analytical solutions are developed and expressed in the form of series and integrals. When numerical results are required, the Hankel transform is then employed to simplify the integrals in combination with other extensive numerical techniques employed to significantly reduce the computational effort. Hydrodynamic forces at second-order are examined by comparing with those obtained by the adoption of linear diffraction theory. Second-harmonic terms resulting from self-interaction of incident waves are found to play an important role in the solutions. It will be shown that second-order forces exerted by short-crested waves on solid structures can be up to two times greater than those by plane waves. The present nonlinear analysis will also show that the excess of forces from those predicted by linear diffraction theory can be as much as 45%. This large increment is an important contribution toward the total wave-induced forces and therefore should be taken into consideration in offshore engineering design and operation processes.
# NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>cylinder radius</td>
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<tr>
<td>(A)</td>
<td>wave amplitude</td>
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<tr>
<td>(A_{n0}, A_{nj})</td>
<td>coefficients in scattered wave solutions</td>
</tr>
<tr>
<td>(A_{mn}, B_{mn})</td>
<td>coefficients in first-order scattered wave solutions</td>
</tr>
<tr>
<td>(c)</td>
<td>celerity of wave propagation for short-crested wave in the positive (x)-direction</td>
</tr>
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<td>(C^*s)</td>
<td>coefficients and functions for second-order velocity potential</td>
</tr>
<tr>
<td>(d)</td>
<td>water depth</td>
</tr>
<tr>
<td>(\frac{dF_z}{dz})</td>
<td>total horizontal force per unit length in the direction of wave propagation (THFPUL)</td>
</tr>
<tr>
<td>(D_l)</td>
<td>wave number spectrum</td>
</tr>
<tr>
<td>(f^{II}, f^{IS}, f^{SS})</td>
<td>quadratic forcing in the second-order free surface boundary condition</td>
</tr>
<tr>
<td>(F_l)</td>
<td>radial function for the wave number spectrum</td>
</tr>
<tr>
<td>(F_{mnpq}^{jIS}, F_{mnpq}^{jSS})</td>
<td>radial functions for (F_l) defined in Appendix A</td>
</tr>
<tr>
<td>(F_z)</td>
<td>total horizontal force (THF)</td>
</tr>
<tr>
<td>(g)</td>
<td>gravitational acceleration</td>
</tr>
<tr>
<td>(G_{mnpq}^j)</td>
<td>function of (k_x, k_y, k, a, z) and (d) for the second-order total horizontal force per unit length resulting from first-order solutions</td>
</tr>
<tr>
<td>(\overline{G}_{mnpq}^j)</td>
<td>function of (k_x, k_y, k, a, d) for the second-order total horizontal force resulting from first-order solutions</td>
</tr>
<tr>
<td>(H_n)</td>
<td>Hankel function of the first kind of order (n)</td>
</tr>
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<td>(I_n)</td>
<td>modified Bessel function of the first kind of order (n)</td>
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<tr>
<td>Symbol</td>
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<tr>
<td>$J_n$</td>
<td>Bessel function of the first kind of order $n$</td>
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<td>$k$</td>
<td>first-order wave number</td>
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<tr>
<td>$k_x$</td>
<td>first-order wave number in the $x$-direction</td>
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<td>$k_y$</td>
<td>first-order wave number in the $y$-direction</td>
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<td>$k_2$</td>
<td>second-order wave number</td>
</tr>
<tr>
<td>$k_{2x}$</td>
<td>second-order wave number in the $x$-direction</td>
</tr>
<tr>
<td>$k_{2y}$</td>
<td>second-order wave number in the $y$-direction</td>
</tr>
<tr>
<td>$K_n$</td>
<td>modified Bessel function of the second kind of order $n$</td>
</tr>
<tr>
<td>$N_{ij,mnpq}^n$</td>
<td>function defined in Appendix A</td>
</tr>
<tr>
<td>$p$</td>
<td>nonuniform pressure distribution applied to the free surface</td>
</tr>
<tr>
<td>$P$</td>
<td>pressure</td>
</tr>
<tr>
<td>$Q_{mn}$</td>
<td>radial and azimuthal function for the first-order velocity potential</td>
</tr>
<tr>
<td>$r, \theta, z$</td>
<td>cylindrical coordinates with origin at the centre of a cylinder</td>
</tr>
<tr>
<td>$R, R_0, R_n$</td>
<td>functions of $k_x, k_y, k$ and $\alpha$ for the first-order total force</td>
</tr>
<tr>
<td>$R, S, U, V, W$'s</td>
<td>coefficients in second-order scattered wave solutions</td>
</tr>
<tr>
<td>$R, \alpha, z$</td>
<td>cylindrical coordinates with origin at the source point</td>
</tr>
<tr>
<td>$Re$</td>
<td>real part of a complex quantity</td>
</tr>
<tr>
<td>$S^*, \overline{S}, S$</td>
<td>functions of $k_x, k_y, k, k_2, \alpha, z$ and $d$ for the second-order total horizontal force per unit length resulting from second-order solutions</td>
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<tr>
<td>$t$</td>
<td>time</td>
</tr>
<tr>
<td>$V^*, \overline{V}, V$</td>
<td>functions of $k_x, k_y, k, k_2, \alpha$ and $d$ for the second-order total horizontal force resulting from second-order solutions</td>
</tr>
<tr>
<td>$x, y, z$</td>
<td>rectangular coordinates with origin at the centre of a cylinder</td>
</tr>
<tr>
<td>$Y_n$</td>
<td>Bessel function of the second kind of order $n$</td>
</tr>
<tr>
<td>$\delta_{lj,mnpq}$</td>
<td>modified Kronecker delta</td>
</tr>
</tbody>
</table>
\( \epsilon_m \) coefficient defined in Equation (2.22)
\( \eta \) free surface elevation
\( \kappa \) general wave number
\( \kappa_j \) first-order wave number for evanescent modes
\( \kappa_{2\beta} \) second-order wave number for evanescent modes
\( \xi, \gamma, 0 \) \( x, y, z \) coordinates of a source point
\( \rho \) water density
\( \phi \) velocity potential without time-dependence
\( \Phi \) velocity potential with time-dependence
\( \omega \) wave frequency
\( f \) Cauchy Principal Value
TABLE OF CONTENTS

CHAPTER 1  Introduction  1

CHAPTER 2  Problem Formulation  7
   2.1 General Mathematical Formulation  7
   2.2 First-order Equations and Solution  11
   2.3 Formulation of the Second-order Problem  15

CHAPTER 3  Solution for the Second-order Problem  18
   3.1 Conceptual Solution  18
   3.2 Solution for the Incident Potential $\phi_{2P}^{II}$  20
   3.3 Solution for $\phi_{2P}^{IS}$ and $\phi_{2P}^{SS}$  21
   3.4 Solution for the Scattered Potential $\phi_{2}^{H}$  30
   3.5 Complete Solution for the Second-order Velocity Potential $\Phi_2$  32
   3.6 Further Simplification of $\Phi_2$  34

CHAPTER 4  Hydrodynamic Forces on a Vertical Circular Cylinder  39
   4.1 Perturbation Expansion of Pressure  39
   4.2 Total Horizontal Force per Unit Length (THFPUL)  39
      4.2.1 First-order Total Horizontal Force per Unit Length  40
      4.2.2 Second-order Total Horizontal Force per Unit Length  41
   4.3 Depth Integrated Total Horizontal Force (THF)  43

CHAPTER 5  Numerical Results and Discussion  46
   5.1 Computational Procedures  46
   5.2 Discussion of the Results  51
      5.2.1 Total Horizontal Force per Unit Length  52
      5.2.2 Depth Integrated Total Horizontal Force  60
CHAPTER 1

INTRODUCTION

To the author's knowledge, no one seems to have discussed the interaction of second-order \textit{short-crested} waves with a vertical circular cylinder. However, the study of wave forces exerted by \textit{plane} waves on a vertical surface-piercing cylinder has been a subject of interest for almost four decades due to its practical importance. In 1954, MacCamy and Fuchs found an exact solution for linear plane waves being diffracted by a large vertical circular cylinder. Their analytical solution has been employed by many investigators to calculate wave loads on a vertical cylinder induced by linear plane waves ever since. All these analyses, however, are restricted to linear wave theory, in which only the first term of a perturbation expansion is used. This restriction is severe and, in fact, it was found by Lighthill (1979) that the linear theory is inadequate to estimate wave loadings because the real character of ocean waves is generally nonlinear. Therefore, the neglected nonlinear terms must be taken into account in order to produce more realistic results.

Even though an ideal complete nonlinear theory seems to be unattainable, a number of second-order theories, in which the second-order of a perturbation expansion is included, have been proposed, especially in the last decade, to provide an improved evaluation of wave forces. Nevertheless, the solution for the second-order velocity potential has been filled with difficulties and controversies. There have been numerous authors, such as Lighthill (1979) and Molin (1979), who suggested methods to obtain forces without first finding the second-order velocity potential through the use of
Green's second identity whereby the forces are only expressed in terms of first-order potentials. Unfortunately, these proposed solutions in determining second-order wave loads involved a free surface integral for which it was extremely difficult to obtain a convergent solution. For more general application, however, many authors still made attempts to find the second-order velocity potential. In 1985, Sabuncu and Goren proposed a general solution which was used to study vertical and horizontal wave forces on axisymmetric bodies but it failed to satisfy even free surface boundary conditions or the radiation condition. A more advanced solution of this problem by Kim and Yue (1989) was based on numerical methods using Green's functions and seemed to satisfy the free surface boundary conditions at second-order. The results obtained by Kim and Yue were lately shown in good agreement with the analytical solution given by Chau and Eatock Taylor (1992), which was also based on Green's functions.

Recently, Kriebel (1990) presented a complete closed-form solution for the velocity potential resulting from the interaction of second-order plane waves with a large vertical circular cylinder. He obtained the second-order velocity potential \( \phi_2 \) by separating it into a linear sum of \( \phi_2^P \) and \( \phi_2^H \), where \( \phi_2^P \) denotes a velocity potential that satisfies the nonhomogeneous free surface boundary condition and \( \phi_2^H \) denotes a velocity potential that satisfies the homogeneous form of the free surface boundary condition. \( \phi_2^P \) is further separated as \( \phi_2^P = \phi_{2P}^{II} + \phi_{2P}^{IS} + \phi_{2P}^{SS} \) according to the sources of the quadratic forcing terms in the nonhomogeneous free surface boundary condition. Then the solution for \( \phi_{2P}^{II} \) can be obtained straightforwardly from the free surface boundary condition according to its associated quadratic forcing term. The rest of the quadratic forcing terms in the free surface boundary condition may then be regarded as a nonuniform pressure distribution so that the solutions for \( \phi_{2P}^{IS} \) and \( \phi_{2P}^{SS} \) can be obtained to represent the corresponding water waves generated by this nonuniform pressure distribution. To satisfy the no-flow condition on the surface of a
cylinder, the solution for $\phi_2^H$ is required and is constructed using the complete set of eigenfunctions of the second-order differential equation. The final complete solution for $\phi_2$ appears to satisfy all boundary conditions at second-order. With this solution, water particle kinematics, hydrodynamic pressures, water surface elevation or other important physical quantities, such as force and momentum, can be determined. Kriebel’s solution convincingly demonstrated the importance of nonlinear effects in the calculation of wave forces. However, this solution was still limited to plane waves.

Wind-generated ocean waves are not only nonlinear but also short-crested. Mar-<ref>chant and Roberts (1987) compared the maximum short-crested wave heights from their results to the maximum plane wave height given by Cokelet (1977) showing that short-crested waves can be much, up to two times, higher than plane waves. Similar comparisons have also been reported by Roberts (1983) and Le Mehaute (1986). All these works have shown that plane waves break at a smaller amplitude than short-crested waves. Therefore larger wave forces may be exerted on offshore structures by short-crested waves. A natural question arises as whether or not these nonlinear short-crested waves can exert larger forces on offshore structures than plane waves with the same total wave number. To correctly and more realistically calculate wave loads on offshore structures, the effects from short-crested waves must be taken into consideration.

Short-crested waves may be generated in many different ways. For example, they could be generated by

(i) wind blowing over the water surface of an open ocean,
(ii) the reflection of incoming waves off a sea wall or jetty,
(iii) the diffraction behind an obstacle, for example an island, or
(iv) the arrival of swell from two different storm centres.

In the present study, we assume their generation by some method and that they propagate
The theory of short-crested waves was first developed by Jeffreys (1924) and extended considerably by Fuchs (1952) to a second approximation. Fuchs described short-crested waves as waves being periodic in both the direction of propagation and the direction along the crest. There are two associated wave lengths, or wave numbers, which are of the same order of magnitude. Waves whose crest wave lengths are much longer than the wave lengths in the direction of propagation, are called long-crested waves. The long-crested limit is plane wave.

Since the work of Fuchs, various authors have employed the perturbation theory to study the properties of short-crested waves. Most of the studies on short-crested waves are related to the reflection of obliquely-incident waves onto a maritime structure, particularly a seawall or jetty. Short-crested waves from this type of problem are formed when two progressive wave trains of equal amplitude and frequency propagate at an angle to each other. Of many solutions using perturbation expansions related to the ratio of wave steepness, Fuchs (1952) derived a second-order solution while Chappelear (1961) and Hsu *et al.* (1979) obtained third-order solutions. However, Fenton (1985) found later that there were some mistakes in the expressions given by Hsu *et al.* Roberts (1983) investigated this problem in great detail by computing his solution via a perturbation expansion of 27th order in wave steepness. Marchant and Roberts (1987) extended the study further to the 35th order of a perturbation expansion. However, Roberts' solution and the above third-order solutions did not consider the loads exerted on a vertical wall by short-crested waves. Battjes (1982) presented expressions based on linear theory for the forces produced by short-crested waves angularly adjacent on long structures while Fenton (1985) obtained a solution at third-order. A numerical solution, involving truncated Fourier series, has also been reported by Roberts and
Schwartz (1983) for the investigation of kinematic and dynamic properties of short-crested waves.

These earlier works show that forces due to obliquely-incident waves can exceed considerably those due to normally-incident waves (see Kuznetzov). A similar phenomenon was also reported by Fenton (1985), and Marchant and Roberts (1987) who have shown that waves approaching a vertical wall at an 80° angle exert 21% greater force than the equivalent standing waves. Furthermore, for sufficiently deep water and high waves, the second-order second-harmonic terms can play an important role in the solution as shown by Rundgren (1958). Although these works are all closely related to wave forces induced by short-crested waves, none of the previous authors seems to have discussed the properties of short-crested waves being diffracted around a vertical cylinder.

Zhu (1992) was the first one who examined this problem and conjectured that wave loads on a vertical cylinder due to short-crested waves could be greater than those due to plane waves. However, with a linear analytical solution for the diffraction of short-crested waves around a vertical circular cylinder, he only showed that forces due to plane waves are always the largest. For the cylinder of other cross-sections, Zhu and Moule (1993) however found that larger linear forces on an elliptical cylinder can be induced by short-crested waves. Also, since the studies of wave forces on a vertical wall show that short-crested waves produce larger forces on the wall than plane waves, one may make a further conjecture that short-crested waves could exert larger second-order forces on a vertical circular cylinder than plane waves with the same total wave number do.

The aim of this study is to present a complete solution for the velocity potential
resulting from the interaction of nonlinear short-crested waves with a vertical circular cylinder which is resting on the ocean floor and piercing the free water surface. The nonlinear diffraction problem is first formulated. Subsequently, in conjunction with the second-order approximation of short-crested waves given by Fuchs (1952), the perturbation theory is utilised in order to find an approximate solution to this problem up to the second-order. Second-order analytical solutions are obtained in closed-form and expressed in terms of series and integrals. The solution given by Kriebel (1990) can now be regarded as a limiting case of the solution proposed here.

In order to efficiently sum up the quaduple series and evaluate the singular integrals involved in the solution, certain numerical techniques must be developed so that quantitative numerical results, such as hydrodynamic forces on a circular cylinder at second-order, can be obtained from the analytical solution. It will be shown that upon constructing a Hankel transform to simplify integrals, and adopting the \( \epsilon \)-algorithm to accelerate the convergence of the integrals and all series involved, considerable reduction of the computational effort and the required CPU time were achieved. After carrying out all the numerical calculations on a Sun 4/470 Sparc Sever, we were able to present the calculated hydrodynamic forces and compare the results from the second-order solution with those obtained by Zhu (1992) from the linear diffraction theory.

The present nonlinear analysis will show that the above conjecture is true and that wave-induced forces due to short-crested waves at second-order can be up to two times greater than those due to plane waves. Second-harmonic terms resulting from the self-interaction of incident waves will be shown to play an important role in the solution. It will also be shown later that linear theory underestimates wave forces where the discrepancy of the estimation between second-order theory and linear theory can be up to 45%.
2.1 GENERAL MATHEMATICAL FORMULATION.

Consider the problem where a fixed surface-piercing, vertical circular cylinder of radius $a$ is placed on the ocean bed and incident short-crested waves of frequency $\omega$ are propagating toward the cylinder in the positive $x$-direction in an ocean of uniform depth $d$, as shown in Figure 1. The nature of incident short-crested waves can be seen and compared with incident plane waves in Figure 2. A cylindrical polar coordinate system is adopted, with the origin placed at the centre of the cylinder on the mean surface level (MSL) and the $z$-axis pointing vertically upwards. Then, with the assumption that the fluid is irrotational and incompressible, the governing equation for the diffracted wave field is the Laplace equation

$$\nabla^2 \Phi = \Phi_{rr} + \frac{1}{r} \Phi_r + \frac{1}{r^2} \Phi_{\theta\theta} + \Phi_{zz} = 0,$$

within the region

$$a \leq r < \infty, \quad -d \leq z \leq \eta, \quad -\pi \leq \theta \leq \pi,$$

where $\eta$ is the free surface elevation and subscripts are used to denote partial derivatives.

The velocity potential $\Phi$ must also satisfy the following boundary conditions:
Figure 1. Definition sketch of coordinate system
Figure 2. Two types of incident waves

(a) plane waves,  (b) short-crested waves
Bottom boundary condition

For \( z = -d \),

\[
\Phi_z = 0. \tag{2.2}
\]

Kinematic free surface boundary condition

For \( z = \eta \) and \( r \geq a \),

\[
\eta_t + \Phi_r \eta_r + \frac{1}{r^2} \Phi_\theta \eta_\theta - \Phi_z = 0. \tag{2.3}
\]

Dynamic free surface boundary condition

For \( z = \eta \) and \( r \geq a \),

\[
\Phi_t + g \eta + \frac{1}{2} \left[ (\Phi_r)^2 + \left( \frac{1}{r} \Phi_\theta \right)^2 + (\Phi_z)^2 \right] = 0, \tag{2.4}
\]

where \( g \) is the acceleration due to gravity.

No-flow condition around the cylinder boundary

For \( r = a \) and \( -d \leq z \leq \eta \),

\[
\Phi_r = 0. \tag{2.5}
\]

Moreover, a far-field radiation condition must be specified to ensure that scattered waves behave as outgoing waves away from the cylinder. Sommerfeld (1949) gave the radiation condition as

\[
\lim_{r \to \infty} r^{\frac{1}{2}} (\Phi_r^S - i\kappa \Phi^S) = 0, \tag{2.6}
\]

where \( \kappa \) is the scattered wave number, which may differ from that of the incident wave.

The above differential system is highly nonlinear with because of the free surface boundary conditions. To find a solution, we adopt perturbation expansions to the velocity
potential and the free surface elevation as:
\[
\Phi = \Phi_1 + \Phi_2 + \Phi_3 + \cdots ,
\]
\[
\eta = \eta_1 + \eta_2 + \eta_3 + \cdots ,
\tag{2.7}
\]
where a perturbation parameter, \( \epsilon \), being chosen to be of the order of wave steepness, \( kA \), is implicitly included in the expansion such that first-order terms are of order \( kA \) while second-order terms are of order \( (kA)^2 \), in which \( k \) is the first-order wave number. Upon carrying out Taylor’s expansion, about the mean surface level \( z = 0 \), in the nonlinear boundary conditions (2.3) and (2.4), and substituting the expansions of (2.7) into (2.1) – (2.6), the governing equation together with boundary conditions for the first-order and the second-order problem can be formed separately by comparing the terms that implicitly contain the corresponding power of \( \epsilon \).

In the following sections, the first-order boundary value problem is first formed and then solved. Once the solution to this problem is obtained, it is used in the formulation of the second-order boundary value problem. Due to more complicated free surface boundary conditions and uncertainty about the far-field radiation condition, the second-order velocity potential cannot simply be split into the incident and scattered wave potentials as before. Therefore, an appropriate treatment of this potential must be made and this will be discussed in Chapter 3.

2.2 FIRST-ORDER EQUATIONS AND SOLUTION.

The first-order boundary value problem is formed as
\[
\nabla^2 \Phi_1 = \Phi_{1rr} + \frac{1}{r} \Phi_{1r} + \frac{1}{r^2} \Phi_{1\theta\theta} + \Phi_{1zz} = 0 ,
\tag{2.8}
\]
\[ \Phi_{1z} = 0 \quad \text{on} \quad z = -d, \quad \text{(2.9)} \]
\[ \Phi_{1tt} + g\Phi_{1z} = 0 \quad \text{on} \quad z = 0, \quad \text{(2.10)} \]
\[ \Phi_{1r} = 0 \quad \text{on} \quad r = a, \quad \text{(2.11)} \]
\[ \lim_{r \to \infty} r^{\frac{1}{2}}(\Phi_{1r}^S - i\kappa\Phi_{1}^S) = 0, \quad \text{(2.12)} \]

where \( \Phi_{1}^S \) is the first-order scattered wave potential.

Note that the first-order term of free surface elevation, \( \eta_1 \), appearing in the boundary conditions

\[ \eta_{1t} - \Phi_{1z} = 0 \quad \text{on} \quad z = 0, \quad \text{(2.13)} \]
\[ g\eta_1 + \Phi_{1t} = 0 \quad \text{on} \quad z = 0, \quad \text{(2.14)} \]

resulting from (2.3) and (2.4), was eliminated to yield the combined free surface boundary condition (2.10) in term of \( \Phi_{1} \).

An analytical solution of this problem has been given by Zhu (1992) and is reproduced here for easy reference and completeness of the diffraction problem. The first-order velocity potential can be obtained as the sum of incident and scattered wave potentials as

\[ \Phi_1 = \Phi_1^I + \Phi_1^S. \quad \text{(2.15)} \]

For the first-order incident short-crested waves propagating in the positive \( x \)-direction, the velocity potential \( \Phi_1^I \) can be found in Fuchs (1952), as the real part of

\[ \Phi_1^I = -\frac{igA}{\omega} \cosh k(z + d) e^{i(k_x x - \omega t)} \cos k_y y, \quad \text{(2.16)} \]
where \( A \) is the amplitude of an incident wave, \( i = \sqrt{-1} \), \( k_x \) and \( k_y \) are wave numbers in the \( x \)- and \( y \)-directions respectively, and \( k \) is the total first-order wave number related to \( k_x \) and \( k_y \) by the equation

\[
k^2 = k_x^2 + k_y^2.
\] (2.17)

The wave number, \( k \), and the frequency, \( \omega \), are linked via a linear dispersion relation from (2.10) as

\[
\omega^2 = g k \tanh kd.
\] (2.18)

The wavespeed of wave propagation for short-crested waves in the \( x \)-direction is given by

\[
c^2 = \frac{g k}{k_x^2} \tanh kd,
\] (2.19)

and one can see that

\[
c^2 \geq \frac{g}{k_x} \tanh k_x d,
\] (2.20)

which means long-crested waves propagate more slowly than short-crested ones. In particular, with the wave number fixed in the direction of wave propagation and the same water depth, plane waves travel slowest of all.

The velocity potential given in (2.16) can be rewritten in cylindrical coordinates (see Watson, 1962) as

\[
\Phi_I = -\frac{igA}{2\omega} \frac{\cosh k(z + d)}{\cosh kd} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \varepsilon_n i^m J_m(k_x r) J_2n(k_y r) \times \\
[ \cos(m + 2n)\theta + \cos(m - 2n)\theta ] e^{-i\omega t},
\] (2.21)

where

\[
\varepsilon_m = \begin{cases} 
1, & \text{if } m = 0; \\
2, & \text{if } m \geq 1,
\end{cases}
\] (2.22)
and $J_m(kr)$ is the Bessel function of the first kind of order $m$.

Now following Dean and Dalrymple (1984) or Mei (1989), the scattered wave potential, $\Phi^S_1$, may be obtained from the complete set of eigensolutions as

$$\Phi^S_1 = \sum_{n=0}^{\infty} A_{n0} H_n(kr) \cosh k(z + d) \cos n\theta e^{-i\omega t}$$

$$+ \sum_{n=0}^{\infty} \sum_{j=1}^{\infty} A_{nj} K_n(\kappa_j r) \cos \kappa_j(z + d) \cos n\theta e^{-i\omega t},$$

where $H_n(kr)$ is the Hankel function of the first kind of order $n$, defined by

$$H_n(kr) = J_n(kr) + iY_n(kr),$$

in which $J_n(kr)$ and $Y_n(kr)$ are the Bessel functions of the first and second kind, respectively, and $K_n(\kappa_j r)$ is the modified Bessel function of the second kind of order $n$. These $H_n$'s represent outgoing waves, or progressive wave modes, while $K_n$'s represent standing waves, or local evanescent modes, with wave numbers $\kappa_j$, given by

$$\omega^2 = -g\kappa_j \tan \kappa_j d.$$}

After adjusting (2.23) and demanding the total solution to satisfy the no-flow condition (2.11), the unknown coefficients in (2.23) are then determined with the utilisation of orthogonal properties of $\cosh k(z + d)$ and $\cos \kappa_j(z + d)$. Therefore, the solution for scattered waves is given by the real part of

$$\Phi^S_1 = \frac{-igA}{2\omega} \frac{\cosh k(z + d)}{\cosh kd} e^{-i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \varepsilon_n i^{m+n} \times$$

$$\left[ A_{mn} H_{m+2n}(kr) \cos(m + 2n)\theta + B_{mn} H_{m-2n}(kr) \cos(m - 2n)\theta \right],$$

where

$$A_{mn} = \frac{k_x J'_m(k_x a) J_{2n}(k_y a) + k_y J'_m(k_x a) J_{2n}(k_y a)}{k H'_{m+2n}(ka)},$$

$$B_{mn} = \frac{k_x J'_m(k_x a) J_{2n}(k_y a) + k_y J'_m(k_x a) J_{2n}(k_y a)}{k H'_{m-2n}(ka)},$$
in which the prime denotes the derivative of the Bessel function, $J_n(s)$, or Hankel function, $H_n(s)$, with respect to its argument $s$. We can see that the first-order scattered short-crested waves propagate outwardly without local evanescent modes.

The complete first-order wave potential can then be written as
\[
\Phi_1 = -\frac{igA \cosh k(z + d)}{2\omega \cosh kd} e^{-i\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \varepsilon_n i^m Q_{mn}(r, \theta),
\]
(2.29)
in which the function $Q_{mn}(r, \theta)$ is defined as
\[
Q_{mn}(r, \theta) = \left[ J_m(kr) J_{2n}(kr) - A_{mn} H_{m+2n}(kr) \right] \cos(m + 2n)\theta
+ \left[ J_m(kr) J_{2n}(kr) - B_{mn} H_{m-2n}(kr) \right] \cos(m - 2n)\theta.
\]
(2.30)

2.3 FORMULATION OF THE SECOND-ORDER PROBLEM.

In a similar manner to that used in the first-order problem, we now compare terms that implicitly contain the second power of $\varepsilon$, thus leading to the governing equation and boundary conditions of the second-order problem. This problem is then given by
\[
\nabla^2 \Phi_2 = \Phi_{2rr} + \frac{1}{r} \Phi_{2r} + \frac{1}{r^2} \Phi_{2\theta\theta} + \Phi_{2zz} = 0,
\]
(2.31)
\[
\Phi_{2z} = 0 \quad \text{on} \quad z = -d,
\]
(2.32)
\[
\Phi_{2tt} + g\Phi_{2z} = \frac{1}{g} \Phi_{1t} (\Phi_{1tt} + g\Phi_{1zz})
- 2 \left( \Phi_{1r} \Phi_{1rt} + \frac{1}{r^2} \Phi_{1\theta} \Phi_{1\theta t} + \Phi_{1z} \Phi_{1zt} \right) \quad \text{on} \quad z = 0,
\]
(2.33)
\[
\Phi_{2r} = 0 \quad \text{on} \quad r = a.
\]
(2.34)

The free surface boundary condition (2.33) is again the combination of the kinematic and dynamic free surface boundary conditions
\[
\eta_{2t} + \Phi_{1r} \eta_{1r} + \frac{1}{r^2} \Phi_{1\theta} \eta_{1\theta} - (\Phi_{2z} + \eta_1 \Phi_{1zz}) = 0 \quad \text{on} \quad z = 0,
\]
(2.35)
\[ g\eta_2 + \Phi_{2t} + \eta_1 \Phi_{1tx} + \frac{1}{2} \left[ (\Phi_{1r})^2 + \left( \frac{1}{r}\Phi_{1\theta} \right)^2 + (\Phi_{1r})^2 \right] = 0 \quad \text{on } z = 0, \quad (2.36) \]

in which the first- and second-order terms of the free surface elevation, \( \eta_1 \) and \( \eta_2 \) respectively, are eliminated.

The Sommerfeld radiation condition, however, cannot be simply applied at second-order since it does not apply to forced wave motions. An alternative far-field behaviour of \( \Phi_2 \) is demanded as an additional condition. This alternative radiation condition will be discussed in detail in the next chapter.

By comparing the first-order boundary value problem with the second-order boundary value problem, one can see the evident similarities in both problems: the governing Laplace equation and the no-flow conditions at the ocean bed and on the surface of a cylinder. On the other hand, the two major differences are:

(i) the combined free surface boundary condition given by Equation (2.33) for the second-order problem is nonhomogeneous while the combined free surface boundary condition for the first-order problem is homogeneous, and

(ii) the radiation condition for the second-order problem is somewhat more complicated than that for the first-order problem, where it is the Sommerfeld radiation condition given in Equation (2.6).

As suggested by Lighthill (1979), the quadratic forcing terms on the right hand side of the Equation (2.33) may be regarded as an oscillatory pressure applied to the free surface which generates forced wave motion. If we now substitute the complete first-order solution from Equations (2.29) and (2.30) into the right hand side of Equation (2.33), the combined free surface boundary condition may be expressed in the form

\[ \Phi_{2tt} + g\Phi_{2z} = f^{II} + f^{IS} + f^{SS} = f(r, \theta, t), \quad (2.37) \]
in which three forcing types, \( f^{II}, f^{IS} \) and \( f^{SS} \), can be defined as follows:

(i) \( f^{II} \) is the forcing component due to the self-interaction of the first-order incident short-crested waves involving \( J_n \) alone, or products of \( J_m \) and \( J_n \), which leads to the short-crested and plane second-harmonic waves travelling together in the \( x \)-direction;

(ii) \( f^{SS} \) is the forcing component due to the self-interaction of the first-order scattered waves involving products of \( H_m \) and \( H_n \);

(iii) \( f^{IS} \) is the forcing component due to the cross-interaction of the first-order incident and scattered waves involving products of \( J_m \) and \( H_n \).

These forcing components have all been determined once the first-order solution is completed; they are given in Appendix A because they are all quite lengthy. Since these quadratic forcing terms are all periodic in time and oscillate at twice the frequency of the incident waves, the time-dependence of the second-order velocity potential can be written as

\[
\Phi_2(r, \theta, z, t) = \text{Re} \left[ \phi_2(r, \theta, z)e^{-i2\omega t} \right]. \tag{2.38}
\]

After dropping the time-dependence, the combined free surface boundary condition can now be written in the form

\[
\phi_{2z} - \frac{4\omega^2}{g} \phi_2 = \frac{1}{g} \left[ f^{II}(r) + f^{IS}(r) + f^{SS}(r) \right], \tag{2.39}
\]

which gives rise to the main difficulty in finding the solution of the second-order problem.
3.1 CONCEPTUAL SOLUTION.

As demonstrated in Chapter 2, the traditional method to obtain the solution for the velocity potential in the first-order problem is to separate $\phi_1$ into a sum of an incident wave potential together with a scattered one. Previous second-order theories for plane waves, e.g. Chen and Hudspeth (1982), suggested splitting $\phi_2$ in the same manner as $\phi_1$, namely into an incident wave potential to satisfy the $f^{II}$ forcing term and a "scattered" wave potential. Unfortunately, this does not simplify the problem because, unlike the first-order problem in which we clearly know the origin of the scattered waves, we have additional forced wave motions on the free surface generated by the remaining nonhomogeneous terms, $f^{IS}$ and $f^{SS}$, in Equation (2.39). Hence, the second-order "scattered" wave potential has to be split into a form such that the forced wave motions can be taken into account easily.

Following Molin (1979), or Kriebel (1990), the solution for the second-order velocity potential can be found by decomposing $\phi_2$ into a particular solution, $\phi_2^P$, and a complementary or a homogeneous solution, $\phi_2^H$, as

$$\phi_2 = \phi_2^P + \phi_2^H. \tag{3.1}$$

The particular solution represents forced wave motions according to the nonlinear interactions of the first-order incident and scattered waves. This solution must satisfy the
Laplace equation, the bottom boundary condition, and the nonhomogeneous free surface boundary condition. In accordance with the form of the forcing terms in Equation (2.39), the particular solution can be further decomposed as

$$
\phi_2^P = \phi_{2P}^{II} + \phi_{2P}^{IS} + \phi_{2P}^{SS},
$$

where $\phi_{2P}^{II}$, $\phi_{2P}^{IS}$, and $\phi_{2P}^{SS}$ are velocity potentials corresponding to quadratic forcing terms $f^{II}$, $f^{IS}$ and $f^{SS}$, respectively.

The homogeneous solution or, in other words, the scattered wave solution represents free waves resulting from the interaction of the forced waves with the fixed cylinder. The solution must satisfy the Laplace equation, the bottom boundary condition, the Sommerfeld radiation condition, and the homogeneous form of the free surface boundary condition. From the last condition, the second-order dispersion relation can be derived as

$$
4\omega^2 = g k_2 \tanh k_2 d,
$$

where $k_2$ is the second-order wave number of free waves oscillating at double the fundamental frequency, $2\omega$, of linear waves and in the same direction of linear wave propagation. This second-order wave number $k_2$ has $k_{2x}$ and $k_{2y}$ as its associated wave numbers in the $x$- and $y$-directions respectively, and they are linked by

$$
k_2^2 = k_{2x}^2 + k_{2y}^2.
$$

From Equations (2.17), (2.18), (2.19), (3.3) and (3.4), one can see that, in deep water, $k_2 = 4k$ and $k_{2x} = 4k_x$ causing second-order free waves in the $x$-direction to travel at only half the speed of the linear waves. In shallow water, $k_2 = 2k$ and $k_{2x} = 2k_x$, and therefore, in the $x$-direction, second-order free waves propagate at the same velocity as linear waves.
The scattered wave solution mentioned above can be found using the same method as employed for the first-order solution.

3.2 SOLUTION FOR THE INCIDENT POTENTIAL $\phi_{2P}^{IIV}$.

Based on Equations (2.39) and (3.2), the solution for the forced wave motion generated by the $f^{IIV}$ forcing term, $\phi_{2P}^{IIV}$, can be found by demanding $\phi_{2P}^{IIV}$ satisfying the Laplace equation, the no-flow condition at the ocean bottom and the following free surface boundary condition:

$$\phi_{2P z}^{IIV} - \frac{4\omega^2}{g} \phi_{2P}^{IIV} = \frac{1}{g} f^{IIV}$$

$$= -i \frac{3gk^2 A^2}{4\omega} (\tanh^2 kd - 1) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \varepsilon_n i^m \times$$

$$J_m(2k_x r) J_{2n}(2k_y r) \left[ \cos(m + 2n) \theta + \cos(m - 2n) \theta \right]$$

$$- i \frac{g A^2}{2\omega} \left\{ 3k^2 (\tanh^2 kd - 1) + 4k_y^2 \right\} \times$$

$$\sum_{m=0}^{\infty} \varepsilon_m i^m J_m(2k_x r) \cos m \theta.$$ (3.5)

Such a solution can be easily found as

$$\phi_{2P}^{IIV} = -i \frac{3gA^2}{4\omega} \frac{k^2 (\tanh^2 kd - 1)}{2k \tanh 2kd - k_2 \tanh k_2 d} \cosh 2k(z + d) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \varepsilon_m \varepsilon_n i^m \times$$

$$J_m(2k_x r) J_{2n}(2k_y r) \left[ \cos(m + 2n) \theta + \cos(m - 2n) \theta \right]$$

$$- i \frac{g A^2}{2\omega} \left\{ 3k^2 (\tanh^2 kd - 1) + 4k_y^2 \right\} \cosh 2k_x (z + d) \times$$

$$\sum_{m=0}^{\infty} \varepsilon_m i^m J_m(2k_x r) \cos m \theta.$$ (3.6)

In (3.6), the double-summation terms represent short-crested second-harmonic waves and the single-summation terms represent plane second-harmonic waves. There-
fore, the above second-order wave potential represents short-crested second-harmonic waves which have twice the wave number of the linear waves travelling together with plane second-harmonic waves in the $x$-direction. One can clearly see that only plane second-harmonic waves will be represented when the first-order incident waves are plane waves since $k_y$ becomes zero. It is interesting to observe, from Equation (3.6) that the incident wave potential at second-order in deep water is of different forms or represents different wave types, depending on the first-order incident waves. For incident plane waves, the plane second-harmonic waves vanish as the water depth increases. On the other hand, for deep water incident short-crested waves, waves at second-order become plane second-harmonic waves with the numerator of the corresponding short-crested second-harmonic wave potential approaching zero. In shallow water, the denominator of the first term on the right hand side of Equation (3.6) approaches zero and resonance take place. Resonance occurs when $k_2 = 2k$ and will not be discussed since it is beyond the scope of the present study. The limiting condition in shallow water is the same for both the incident short-crested and plane waves.

3.3 SOLUTION FOR $\phi_{2P}^{IS}$ AND $\phi_{2P}^{SS}$.

The methods in the literature for the solutions corresponding to the $f^{IS}$ and $f^{SS}$ forcing terms can be categorised into two groups:

(i) those that assume a discrete wave number or a set of discrete wave numbers, and

(ii) those that assume a continuum of wave numbers.

In the first group, for instance, Chakrabarti (1983) and Sabuncu and Goren (1985) assumed a $2k$ wave number dependence whereas Chen and Hudspeth (1982) used a set of discrete wave numbers. Unfortunately, the results from their methods failed
to satisfy the second-order free surface boundary condition. Typical examples of the second group of solutions, which are based on the Fourier-Bessel integral, include those presented by Hunt and Baddour (1981), Rahman and Heaps (1983), Kim and Yue (1989) and Kriebel (1990). They adopted a continuum of wave numbers in order to satisfy the nonlinear free surface boundary condition over the entire domain.

In addition, both of the solutions for forced wave motions generated on the free surface must satisfy a radiation condition in some form as noted by Garrison (1979), Sabuncu and Goren (1985) and Kim and Yue (1989). Although the proper far-field condition for these forced wave motions is not clear, it should be somehow related to the behavior of radiating progressive free waves. The solutions obtained by Chen and Hudspeth (1982), Chakrabarti (1983) and Rahman and Heaps (1983) for plane waves do not satisfy a radiation condition and yield standing wave motions in the far-field. The most appropriate treatment of the far-field condition was completed by Kriebel (1990) which was based on the general solutions for water waves resulting from a nonuniform pressure distribution applied to the free surface. Kriebel successfully made the solution at far-field to represent local forced wave motions together with outwardly propagating free waves.

In this study, the solutions for $\phi_{2P}^{IS}$ and $\phi_{2P}^{SS}$ will be obtained by a similar approach to that of Kriebel (1990), in which nonhomogeneous terms, $f^{IS}$ and $f^{SS}$, are taken into account as a nonuniform pressure distribution applied to the free surface.

Let a point source be some point on the free surface and a field point be any other point on the free surface. Consider an impulsive disturbance of pressure, $p^*$, concentrated at the point source being oscillated at frequency $2\omega$ with the magnitude
and phase defined by a Dirac delta function as shown in Figure 3. The \( p^* \) defined by

\[
p^*(\xi, \gamma, z, t) = p(\xi, \gamma) \delta(x - \xi) \delta(y - \gamma) e^{-i2\omega t} \quad \text{on } z = 0,
\]

where \( p \) is a distributed pressure on the free surface.

In the global coordinate system with the origin being placed at the centre of a cylinder, the point source is located at \((\xi, \gamma, 0)\) or, on the other hand, \((r', \theta', 0)\) while the field point is located at \((x, y, 0)\) or \((r, \theta, 0)\). To find a solution corresponding to \( p^* \), let us temporarily define a local coordinate system with the origin located at the point source and the coordinate of the field point may be defined as \((R, \alpha, z)\) where

\[
R = \left[ (x - \xi)^2 + (y - \gamma)^2 \right]^{\frac{1}{2}} = \left[ r^2 + r'^2 - 2rr' \cos(\theta - \theta') \right]^{\frac{1}{2}}.
\]

After the solution with respect to this local coordinate system is found, it will then be converted back to yield the desired result in the global coordinate system.

The velocity potential according to this single oscillating point source must satisfy the Laplace equation

\[
\nabla^2 \Phi = \Phi_{RR} + \frac{1}{R} \Phi_R + \frac{1}{R^2} \Phi_{\alpha\alpha} + \Phi_{zz} = 0,
\]

(3.9)

together with the bottom boundary condition and dynamic and kinematic free surface boundary conditions as

\[
\Phi_z = 0 \quad \text{on } z = -d,
\]

(3.10)

\[
g\eta + \Phi_t = -\frac{p^*}{\rho} \quad \text{on } z = 0,
\]

(3.11)

\[
\eta_t - \Phi_z = 0 \quad \text{on } z = 0,
\]

(3.12)

where \( \rho \) is the water density and \( p^* \) is taken in cylindrical coordinate form in the local coordinate system.
Figure 3. Definition sketch of point source
Since the point source has axial symmetry, we shall define $\delta(R)$ by

$$\delta(x - \xi)\delta(y - \gamma) = \frac{\delta(R)}{2\pi R},$$  

(3.13)

in the sense that the area integrals of both sides are equal, that is,

$$\int_0^{2\pi} \int_0^\infty \frac{\delta(R)}{2\pi R} R dR d\alpha = 1 = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - \xi)\delta(y - \gamma) dx dy. \quad (3.14)$$

Substituting Equation (3.12) into the result after differentiating Equation (3.11) and using the expression in (3.13), yields the combined free surface boundary condition as

$$\Phi_{tt} + g\Phi_z = i \frac{2\omega}{\rho} p \frac{\delta(R)}{2\pi R} e^{-i2\omega t}. \quad (3.15)$$

It can now be seen that the solution for $\Phi$ should be independent of $\alpha$.

After dropping the time-dependence, the governing equation and boundary conditions become

$$\phi_{RR} + \frac{1}{R} \phi_R + \phi_{zz} = 0, \quad (3.16)$$

$$\phi_z = 0 \quad \text{on} \quad z = -d, \quad (3.17)$$

$$-4\omega^2 \phi + g\phi_z = i \frac{2\omega}{\rho} p \frac{\delta(R)}{2\pi R} \quad \text{on} \quad z = 0, \quad (3.18)$$

where $\phi$ represents the velocity potential with the removal of time-dependence.

Because of axial symmetry, the Hankel transform of order 0 with Bessel function $J_0(\kappa R)$ as the weighting function can be applied to Equations (3.16) – (3.18). The transformed governing equation and the boundary conditions are of the form

$$\overline{\phi}_{zz} - \kappa^2 \overline{\phi} = 0, \quad (3.19)$$

$$\overline{\phi}_z = 0 \quad \text{on} \quad z = -d, \quad (3.20)$$

$$g\overline{\phi}_z - 4\omega^2 \overline{\phi} = i \frac{\omega p}{2\rho\pi} \quad \text{on} \quad z = 0, \quad (3.21)$$
where the Hankel transform of order 0 (see Sneddon, 1951) is defined as

$$f(\kappa) = \int_0^\infty R J_0(\kappa R) f(R) dR,$$

and the inverse transform is defined accordingly as

$$f(R) = \int_0^\infty \kappa J_0(\kappa R) f(\kappa) d\kappa.$$

Now, $\overline{\phi}$ can be found by solving Equations (3.19) – (3.21) as

$$\overline{\phi} = i \frac{\omega p}{2\rho g \pi} \frac{\cosh \kappa(z + d)}{\kappa \tanh \kappa d - k_2 \tanh k_2 d}.$$

To obtain $\phi$, the following inverse Hankel transform must be performed.

$$\phi = \int_0^\infty \kappa J_0(\kappa R) \overline{\phi}(\kappa) d\kappa$$

$$= i \frac{\omega p}{2\rho g \pi} \int_0^\infty \frac{\kappa \cosh \kappa(z + d)}{\cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} J_0(\kappa R) d\kappa,$$

which is a singular integral with singularity at $k_2$.

The solution for $\phi$ must eventually satisfy the radiation condition at infinity or, in other words, represent outgoing progressive waves. The asymptotic form at infinity of $\phi$ can be obtained with the utilisation of the residue theorem as

$$\phi = i \frac{\omega p}{2\rho g \pi} \int_0^\infty \frac{\kappa \cosh \kappa(z + d) \cos(\kappa R - \frac{\pi}{4})}{\cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} \sqrt{\frac{2}{\pi \kappa R}} d\kappa$$

$$= -i \frac{\omega p}{2\rho g \pi} \frac{\pi k_2 \cosh k_2 d \cosh k_2(z + d) \sin(k_2 R - \frac{\pi}{4})}{2k_2 d + \sinh 2k_2 d} \sqrt{\frac{2}{\pi k_2 R}}$$

which represents cylindrical standing waves with wave number $k_2$. Therefore, this solution cannot satisfy the radiation condition. To overcome this problem, a second
standing wave potential is intuitively added to $\phi$ so that at infinity, $\phi$ takes the form

$$
\phi = i \frac{\omega p}{2 \rho g \pi} \left[ \frac{\pi 2k_2 \cosh k_2d \cosh k_2(z + d) \cos(k_2 R - \frac{\pi}{4})}{2k_2 d + \sinh 2k_2d} \sqrt{\frac{2}{\pi k_2 R}} - \frac{\pi 2k_2 \cosh k_2d \cosh k_2(z + d) \sin(k_2 R - \frac{\pi}{4})}{2k_2 d + \sinh 2k_2d} \right] \cosh k_2 \left( z + d \right) \sin \left( \frac{\pi}{4} - f \right) / 2 + \sinh k_2d \ \sum_{-\infty}^{\infty} e^{i(k_2 R - \frac{\pi}{4})}.
$$

(3.28)

Now, not only is $\phi$ a solution representing waves generated by applying an oscillating pressure $p(\xi, \gamma)$ at point $(\xi, \gamma)$ on the free surface, but also the generated waves are guaranteed to radiate outwardly as progressive waves toward infinity.

With the velocity potential due to a single oscillating point source of a pressure on the water surface being written as

$$
\phi(R, \alpha, z; \xi, \gamma) = i \frac{\omega p(\xi, \gamma)}{2 \rho g \pi} \left[ i \frac{\pi 2k_2 \cosh k_2d \cosh k_2(z + d) J_0(k_2 R)}{2k_2 d + \sinh 2k_2d} \cosh k_2(z + d) J_0(k_2 R) 
+ \int_{0}^{\infty} \frac{\kappa \cosh \kappa(z + d)}{\cosh \kappa d(\kappa \tanh \kappa d - k_2 \tanh k_2 d)} J_0(\kappa R) d\kappa \right],
$$

(3.29)

the velocity potential due to a distributed pressure applied on the water surface can be obtained by summing up the solution for $\phi(R, \alpha, z; \xi, \gamma)$ in Equation (3.29) over the spatial domain over which the pressure $p(\xi, \gamma)$ is prescribed, namely

$$
\phi(r, \theta, z) = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \phi(R, \alpha, z; \xi, \gamma) d\xi d\gamma.
$$

(3.30)

By using a change of variables of integration, namely $\xi = r' \cos \theta'$ and $\gamma = r' \sin \theta'$ where $a \leq r' < \infty$ and $-\pi \leq \theta' \leq \pi$, together with (3.8), the velocity
potential at any field point can be written as

\[
\phi(r, \theta, z) = \frac{1}{4\pi} \frac{i2\omega}{\rho g} \left[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \varepsilon_m \varepsilon_n \varepsilon_p \varepsilon_q i^{m+p} \times \left( F_{mnpq}^{jIS} + F_{mnpq}^{jSS} \right) \cos N_{mnpq}^j \right]
\]

where \(F_{mnpq}^{jIS}, F_{mnpq}^{jSS}\) and \(N_{mnpq}^j\) are all given in Appendix A. It is worth noting here that the last summation in (3.32) contains 8 terms only, which result from the combination of indices from the first four summations in the process of obtaining \(F_{mnpq}^{jIS}\) and \(F_{mnpq}^{jSS}\), and described by \(N_{mnpq}^j\)'s.

Fortunately, the solution in (3.31) can be further simplified by employing the Addition Theorem for Bessel Functions (see Watson, 1962),

\[
J_0 \left( \kappa \sqrt{r^2 + r'^2 - 2rr' \cos(\theta - \theta')} \right) = \sum_{l=0}^{\infty} \varepsilon_l J_l(\kappa r) J_l(\kappa r') \cos l(\theta - \theta'). \quad (3.33)
\]
After carrying out the integration over $\theta'$, by using the orthogonal property of cosine, we obtained the particular solutions $\phi_{2P}^{IS}$ and $\phi_{2P}^{SS}$ in the combined form $\phi_{2P}^{IS} + \phi_{2P}^{SS}$, which is of the form

$$
\phi_{2P}^{IS} + \phi_{2P}^{SS} = -i \frac{g k A^2}{16 \omega} \sum_{l=0}^{\infty} \cos l \theta \left[ i \frac{2 \pi k_2 \cosh k_2 d}{k (2 k_2 d + \sinh 2 k_2 d)} \cosh k_2 (z + d) J_l(k_2 r) D_l(k_2) + \int_0^\infty \frac{\kappa \cosh \kappa (z + d)}{k \cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} J_l(\kappa r) D_l(\kappa) d \kappa \right],
$$

(3.34)

where the function $D_l(\kappa)$ represents a nondimensional wave number spectrum and is defined by

$$
D_l(\kappa) = k^2 \int_a^\infty r' F_l(r') J_l(\kappa r') dr',
$$

(3.35)

in which

$$
F_l(r') = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=1}^{8} \varepsilon_m \varepsilon_n \varepsilon_p \varepsilon_q i^{m+p} \tilde{\delta}_{lj mnpq} \times 
\left[ F_{mnpq}^{IS}(r') + F_{mnpq}^{SS}(r') \right],
$$

(3.36)

and the modified Kronecker delta function is defined as

$$
\tilde{\delta}_{lj mnpq} = \begin{cases} 
1, & \text{if } |N_{mnpq}^j| = l; \\
0, & \text{if } |N_{mnpq}^j| \neq l.
\end{cases}
$$

(3.37)

The definition of $D_l(\kappa)$ can be considered as analogous to that of the usual amplitude spectrum obtained by the Fourier transform (see Sneddon, 1951) and in this case it is simply the Hankel transform of the applied quadratic forcing. Also, it should be noted that the solution in (3.34) for the forced waves due to the quadratic forcing terms $f^{IS}$ and $f^{SS}$ was written in a similar form to Equation (3.29) which represented forced standing waves with selective radiation of outwardly propagating second-order free waves.
3.4 SOLUTION FOR THE SCATTERED POTENTIAL $\phi_2^H$.

The second-order scattered wave solution can be constructed from the complete set of eigenfunctions in the same way as for the first-order scattered waves. Since all forced wave motions oscillate at frequency $2\omega$, the associated free wave motions must also oscillate at the same frequency. Without explicitly showing the time dependence, $\phi_2^H$ can be written as

$$
\phi_2^H = \sum_{l=0}^{\infty} W_{l0} \cosh k_2(z + d)H_l(k_2r) \cos l\theta
$$

$$
+ \sum_{l=0}^{\infty} \sum_{\beta=1}^\infty W_{l\beta} \cos \kappa_2\beta(z + d)K_l(\kappa_2\beta r) \cos l\theta
$$

$$
+ \sum_{l=0}^{\infty} \sum_{\alpha=0}^\infty \cosh k_2(z + d) \left[ R_{l\alpha} H_{l+2\alpha}(k_2r) \cos(l + 2\alpha)\theta + S_{l\alpha} H_{l-2\alpha}(k_2r) \cos(l - 2\alpha)\theta \right],
$$

$$
+ \sum_{l=0}^{\infty} \sum_{\alpha=0}^\infty \sum_{\beta=1}^\infty \cos \kappa_2\beta(z + d) \left[ U_{l\alpha\beta} K_{l+2\alpha}(\kappa_2\beta r) \cos(l + 2\alpha)\theta + V_{l\alpha\beta} K_{l-2\alpha}(\kappa_2\beta r) \cos(l - 2\alpha)\theta \right],
$$

where $W_{l0}$, $W_{l\beta}$, $R_{l\alpha}$, $S_{l\alpha}$, $U_{l\alpha\beta}$, and $V_{l\alpha\beta}$'s are some constants which will be defined from the no-flow condition on the surface of the cylinder.

However, the wave numbers obtained here differ from those obtained at first-order. For propagating wave modes, the second-order dispersion relation is given by Equation (3.3), as

$$
4\omega^2 = g k_2 \tanh k_2 d. \tag{3.39}
$$

For the evanescent wave modes, the wave numbers are defined by

$$
4\omega^2 = -g \kappa_2\beta \tan \kappa_2\beta d, \tag{3.40}
$$
which has infinitely many positive roots, \( \kappa_{2\beta} \), where

\[
(\beta - \frac{1}{2})\pi \leq \kappa_{2\beta}d \leq \beta\pi, \quad \beta = 1, 2, 3, \ldots
\]

Imposing the total solution to satisfy the no-flow condition on the cylinder boundary and utilising the orthogonal property of \( \cosh k_2(z + d) \) and \( \cos \kappa_{2\beta}(z + d) \), all coefficients in (3.38) can now be determined and the \( \phi_2^H \) finally takes the form

\[
\phi_2^H = i\frac{gkA^2}{16\omega} \sum_{l=0}^{\infty} \cos l\theta \left\{ \cosh k_2(z + d) \frac{H_{l}(k_2 r)}{H_{l}'(k_2 a)} \times \right. \\
\left. \left( \varepsilon_l i^l C_1^2 J_1'(2k_2 a) + C_2^1 J_1'(k_2 a)D_l(k_2) + C_2^2 \int_0^\infty \frac{\kappa^2 J_1'(\kappa a)D_l(\kappa)}{k(\kappa^2 - k_2^2)} d\kappa \right) \right\} \\
+ \sum_{\beta=1}^{\infty} \cos \kappa_{2\beta}(z + d) \frac{K_l(\kappa_{2\beta} r)}{K_l'(\kappa_{2\beta} a)} \times \\
\left( \varepsilon_l i^l C_1^3 J_1'(2k_2 a) + C_2^3 \int_0^\infty \frac{\kappa^2 J_1'(\kappa a)D_l(\kappa)}{k(\kappa^2 + \kappa_{2\beta}^2)} d\kappa \right) \right\} \\
+ \varepsilon_l i^l \sum_{\alpha=0}^{\infty} \cos(l + 2\alpha)\theta C_{l\alpha} \left( C_4^2 \cosh k_2(z + d) \frac{H_{l+2\alpha}(k_2 r)}{H_{l+2\alpha}'(k_2 a)} + \\
\sum_{\beta=1}^{\infty} C_4^3 \cos \kappa_{2\beta}(z + d) \frac{K_{l+2\alpha}(\kappa_{2\beta} r)}{K_{l+2\alpha}'(\kappa_{2\beta} a)} \right) \\
\left. + \cos(l - 2\alpha)\theta C_{l\alpha} \left( C_4^2 \cosh k_2(z + d) \frac{H_{l-2\alpha}(k_2 r)}{H_{l-2\alpha}'(k_2 a)} + \\
\sum_{\beta=1}^{\infty} C_4^3 \cos \kappa_{2\beta}(z + d) \frac{K_{l-2\alpha}(\kappa_{2\beta} r)}{K_{l-2\alpha}'(\kappa_{2\beta} a)} \right) \right\},
\]

where all of the constants, \( C \)'s, are given in the next section. The Hankel functions, \( H_l \)'s, in the above expression represent second-order plane progressive wave modes while \( H_{l+2\alpha} \)'s together with \( H_{l-2\alpha} \)'s represent second-order short-crested progressive wave modes. Similarly, the modified Bessel functions, \( K_l \)'s, represent second-order plane evanescent wave modes whereas \( K_{l+2\alpha} \)'s together with \( K_{l-2\alpha} \)'s represent second-order short-crested evanescent wave modes.
Unlike the first-order solution, the evanescent wave modes are necessary at second-order, since the depth-dependence of fluid motion in the second-order forced waves differs significantly from that of the freely propagating scattered waves.

### 3.5 COMPLETE SOLUTION FOR THE SECOND-ORDER VELOCITY POTENTIAL $\Phi_2$

The complete solution for the second-order velocity potential is finally obtained by adding together the particular solutions for the forced wave motions given by Equations (3.6) and (3.34) with the homogeneous solution in Equation (3.41). With the time-dependence being added for completeness, the final solution for the second-order velocity potential is given by the real part of

$$
\Phi_2 = -i \frac{g k A^2}{16 \omega} e^{-i \omega t} \sum_{l=0}^{\infty} \cos l \theta \left\{ \varepsilon_l^l \left[ C^1_l \cosh 2k_x (z + d) J_l(2k_x r)ight. \\
- C^2_l \cosh k_2 z J'_l(2k_x a) \frac{H_l(k_2 r)}{H'_l(k_2 a)} \\
- \sum_{\beta=1}^{\infty} C^3_{1 \beta} \cos \kappa_\beta (z + d) J'_l(2k_x a) \frac{K_l(\kappa_\beta r)}{K'_l(\kappa_\beta a)} \\
+ C^4_{1 \beta} \cosh k_2 (z + d) D_l(k_2) \left( J_l(k_2 r) - J'_l(k_2 a) \frac{H_l(k_2 r)}{H'_l(k_2 a)} \right) \\
+ \left( \int_0^\infty C^3_3(\kappa) \cosh \kappa (z + d) J_l(\kappa r) D_l(\kappa) \, d\kappa \right) \\
- C^2_2 \cosh k_2 (z + d) \frac{H_l(k_2 r)}{H'_l(k_2 a)} \int_0^\infty C^2_3(\kappa) J'_l(\kappa a) D_l(\kappa) \, d\kappa \\
- \sum_{\beta=1}^{\infty} C^3_{2 \beta} \cos \kappa_\beta (z + d) \frac{K_l(\kappa_\beta r)}{K'_l(\kappa_\beta a)} \int_0^\infty C^3_{3 \beta}(\kappa) J'_l(\kappa a) D_l(\kappa) \, d\kappa \right\} 
$$

(3.42)
\[ + \varepsilon_l l' \sum_{\alpha=0}^{\infty} \varepsilon_{\alpha} \left\{ \cos(l + 2\alpha) \theta \left( C^1_4 \cosh 2k(z + d) J_l(2kz) J_{2\alpha}(2ky) \right. \right. \\
& \left. \left. - C^2_4 \cosh 2k(z + d) \frac{H_{l+2\alpha}(k^2r) H'_{l+2\alpha}(k^2a)}{H_{l+2\alpha}(k^2a)} \right) \right. \\
& \left. \left. + \sum_{\beta=1}^{\infty} C^3_{4\beta} \cos \kappa_{2\beta}(z + d) \frac{K_{l+2\alpha}(\kappa_{2\beta}r)}{K'_{l+2\alpha}(\kappa_{2\beta}a)} \right) \right) \\
& \left. + \cos(l - 2\alpha) \theta \left( C^1_4 \cosh 2k(z + d) J_l(2kz) J_{2\alpha}(2ky) \right. \right. \\
& \left. \left. - C^2_4 \cosh 2k(z + d) \frac{H_{l-2\alpha}(k^2r) H'_{l-2\alpha}(k^2a)}{H_{l-2\alpha}(k^2a)} \right) \right. \\
& \left. \left. + \sum_{\beta=1}^{\infty} C^3_{4\beta} \cos \kappa_{2\beta}(z + d) \frac{K_{l-2\alpha}(\kappa_{2\beta}r)}{K'_{l-2\alpha}(\kappa_{2\beta}a)} \right) \right) \right), \]

where all coefficients are defined as

\[ C^1_4 = \frac{8}{k} \frac{3k^2(\tanh^2 kd - 1) + 4k^2_y}{\cosh 2k \left( 2k \tanh 2kz d - k_2 \tanh k_2 d \right)}, \]
\[ C^2_4 = \frac{64k_x}{k} \frac{3k^2(\tanh^2 kd - 1) + 4k^2_y}{(4k_x^2 - k_2^2)(2k_2 d + \sinh 2k_2 d)}, \]
\[ C^3_{1\beta} = \frac{64k_x}{k} \frac{3k^2(\tanh^2 kd - 1) + 4k^2_y}{(4k_x^2 + \kappa_{2\beta}^2)(2\kappa_{2\beta} d + \sin 2\kappa_{2\beta} d)}, \]
\[ C^1_2 = i2\pi \frac{k \cosh k_2 d}{k (2k_2 d + \sinh 2k_2 d)}, \]
\[ C^2_2 = \frac{4 \cosh k_2 d}{(2k_2 d + \sinh 2k_2 d)}, \]
\[ C^3_{2\beta} = \frac{4 \cos \kappa_{2\beta} d}{(2\kappa_{2\beta} d + \sin 2\kappa_{2\beta} d)}, \]
\[ C^3_{3\beta}(\kappa) = \frac{k^2}{k (\kappa^2 - k_2^2)}, \]
\[ C^3_{3\beta}(\kappa) = \frac{k^2}{k (\kappa^2 + \kappa_{2\beta}^2)}, \]
\[ C^4_1 = \frac{12k(\tanh^2 k d - 1)}{\cosh 2kd (2k \tanh 2kd - k_2 \tanh k_2 d)}, \]
\[ C^4_2 = \frac{96k(\tanh^2 k d - 1) \cosh k_2 d}{(4k^2 - k_2^2)(2k_2 d + \sinh 2k_2 d)}, \]
The second-order velocity potential is now derived in closed-integral form including four integrals with

\[ I_1 = D_1(k_2) \]
\[ = k^2 \int_a^\infty r' F_1(r') J_1(k_2 r') \, dr', \quad (3.43) \]

\[ I_2 = \int_0^\infty \frac{\kappa \cosh \kappa (z + d)}{k \cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} J_1(\kappa r) D_1(\kappa) \, d\kappa \]
\[ = k \int_0^\infty \frac{\kappa \cosh \kappa (z + d) J_1(\kappa r)}{\cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} \int_a^\infty r' F_1(r') J_1(\kappa r') \, dr' \, d\kappa, \quad (3.44) \]

\[ I_3 = \int_0^{\kappa^2} \frac{\kappa^2}{k(\kappa^2 - k_2^2)} J'_1(\kappa a) D_1(\kappa) \, d\kappa \]
\[ = k \int_0^{\kappa^2} \frac{\kappa^2}{(\kappa^2 - k_2^2)} J'_1(\kappa a) \int_a^\infty r' F_1(r') J_1(\kappa r') \, dr' \, d\kappa, \quad (3.45) \]

\[ I_4 = \int_0^{\kappa^2} \frac{\kappa^2}{k(\kappa^2 + \kappa_2^2)} J'_1(\kappa a) D_1(\kappa) \, d\kappa \]
\[ = k \int_0^{\kappa^2} \frac{\kappa^2}{(\kappa^2 + \kappa_2^2)} J'_1(\kappa a) \int_a^\infty r' F_1(r') J_1(\kappa r') \, dr' \, d\kappa. \quad (3.46) \]

Extensive numerical integration is required for the evaluation of these integrals. Several methods of computing the integrals are discussed by Garrison (1978), Rahman and Heaps (1983), and Kim and Yue (1989). The \( r' \)-integration in the wave number
spectra is difficult because of the oscillation of the integrand. A previous method used by Eatock Taylor and Hung (1987) cannot easily be used here since the integrand does not reduce into simple form as before and it contains four repeated series. Therefore, new techniques need to be developed to evaluate these integrals.

Note that all listed integrals which are mentioned above, except for the first one, are double integrals. These integrals can alternatively be evaluated by changing the order of the integration so that integrals in Equations (3.44) – (3.46) become

\[
I_2 = k \int_a^\infty \int_0^\infty r' F_1(r') \frac{\kappa \cosh \kappa (z + d) J_1(\kappa r)}{\cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} J_1(\kappa r') \, d\kappa \, dr',
\]

(3.47)

\[
I_3 = k \int_a^\infty \int_0^\infty r' F_1(r') \frac{\kappa^2}{(\kappa^2 - k_2^2)} J_1'(\kappa a) J_1(\kappa r') \, d\kappa \, dr',
\]

(3.48)

\[
I_4 = k \int_a^\infty \int_0^\infty r' F_1(r') \frac{\kappa^2}{(\kappa^2 + k_2^2)} J_1'(\kappa a) J_1(\kappa r') \, d\kappa \, dr'.
\]

(3.49)

Even after the changing the order of the integration, the convergence of the above integrals is still a problem. Their convergence seems to be slower than those in Equations (3.44) – (3.46) since all integrals currently contain a "double-oscillatory" integrand. However, detailed examination of each integrand leads to success in simplifying the problems as will be described hereafter.

For the double integral \( I_3 \), its inner singular-integral can be simplified by rewriting it into a sum of two integrals as

\[
\int_0^\infty \frac{\kappa^2}{(\kappa^2 - k_2^2)} J_1'(\kappa a) J_1(\kappa r') \, d\kappa = \int_0^\infty J_1'(\kappa a) J_1(\kappa r') \, d\kappa
\]

\[
+ \int_0^\infty \frac{k_2^2}{(\kappa^2 - k_2^2)} J_1'(\kappa a) J_1(\kappa r') \, d\kappa.
\]

(3.50)
Using an identity for the derivative of Bessel functions, the first integral on the right hand side of Equation (3.50) can be further decomposed into a difference of two integrals as

\[
\int_0^\infty J_1^l(\kappa a) J_1(\kappa r') \, d\kappa = \frac{1}{2} \int_0^\infty J_{l-1}(\kappa a) J_1(\kappa r') \, d\kappa - \frac{1}{2} \int_0^\infty J_1(\kappa r') J_{l+1}(\kappa a) \, d\kappa, \quad (3.51)
\]

which can now be integrated analytically by means of the Mellin transform or the Hankel transform. On the other hand, the second integral in Equation (3.50) still needs to be integrated numerically. However, the numerical convergence is now much faster than for the integral in the original form. Therefore, the original singular-integral becomes

\[
\int_0^\infty \frac{k^2}{(k^2 - k_2^2)} J_1^l(\kappa a) J_1(\kappa r') \, d\kappa = \begin{cases} 
0 & \text{if } l = 0 \\
\frac{1}{2r'} \left(\frac{a}{r'}\right)^{l-1} & \text{if } l > 0
\end{cases} + \int_0^\infty \frac{k_2^2}{(k^2 - k_2^2)} J_1^l(\kappa a) J_1(\kappa r') \, d\kappa. \quad (3.52)
\]

For the double integral \( I_4 \), we managed to integrate the inner integral analytically. By again applying an identity for the derivative of Bessel functions to this inner integral, the result can be shown as

\[
\int_0^\infty \frac{k^2}{(k^2 + k_2^2)} J_1^l(\kappa a) J_1(\kappa r') \, d\kappa = \int_0^\infty \frac{k^2}{(k^2 + k_2^2)} \left\{ \frac{l}{\kappa a} J_1(\kappa a) - J_{l+1}(\kappa a) \right\} J_1(\kappa r') \, d\kappa
\]

\[
= \frac{l}{a} \int_0^\infty \frac{k}{(k^2 + k_2^2)} J_1(\kappa a) J_1(\kappa r') \, d\kappa - \int_0^\infty J_{l+1}(\kappa a) J_1(\kappa r') \, d\kappa
\]

\[+ \int_0^\infty \frac{k_2^2}{(k^2 + k_2^2)} J_{l+1}(\kappa a) J_1(\kappa r') \, d\kappa. \quad (3.53)
\]
All the above integrals can be integrated analytically if one notices that these integrals are already in the form of a Hankel transform with \(J_i(\kappa r')\) being their kernel (see Oberhettinger, 1972), and the integration process is rather straightforward thereafter. Consequently, the result can be expressed in very simple form as

\[
\int_0^\infty \frac{\kappa^2}{(\kappa^2 + \kappa_{2\beta}^2)} J_i'(\kappa a) J_i(\kappa r') \, d\kappa = \frac{l}{a} I_i(\kappa_{2\beta} a) K_i(\kappa_{2\beta} r') + 0 + \kappa_{2\beta} I_{i+1}(\kappa_{2\beta} a) K_i(\kappa_{2\beta} r')
\]

\[
= \left\{ \frac{l}{a} I_i(\kappa_{2\beta} a) + \kappa_{2\beta} I_{i+1}(\kappa_{2\beta} a) \right\} K_i(\kappa_{2\beta} r'), \quad (3.54)
\]

where \(I_i\) and \(K_i\) are modified Bessel functions of the first and second kind of order \(i\), respectively. It should be noticed here that the results in Equations (3.52) and (3.54) are valid for \(r' > a\) only. However, in evaluating \(I_3\) and \(I_4\) numerically, the point \(r' = a\) does not occur in the integration rule used in this study anyway.

The Cauchy Principal Value integral in \(I_2\) can be computed by removing the singularity as suggested by Monacella (1966) in which two types of integrals need to be evaluated. The first type is a singular integral with a removable singularity which will be integrated from \(k_2 - \zeta\) to \(k_2 + \zeta\) for a chosen small positive real number \(\zeta\). The second type consists of two regular integrals which will be integrated from 0 to \(k_2 - \zeta\) and from \(k_2 + \zeta\) to \(\infty\). Despite regular behaviour across the singularity, the difficulty arises as the integrand is not well behaved far away from the singular point. The “double-oscillatory” behaviour of the inner integrand in \(I_2\) resulting from the product of \(J_i(\kappa r)\) and \(J_i(\kappa r')\) can be visualised in Figure 4 for an example where \(l = 2, d = 1.2, z = -0.1, k_2 = 7.86939896, r = 3.7, \) and \(r' = 1.9\). A similar problem was also observed in previous second-order diffraction theories, for instance in Eatock Taylor and Hung (1987). Numerical techniques to cope with all the difficulties which emerged in the evaluation of integrals will be discussed in Chapter 5.
Figure 4. Double-oscillatory behaviour of an inner integrand in $I_2$
CHAPTER 4

HYDRODYNAMIC FORCES ON A VERTICAL CIRCULAR CYLINDER

4.1 PERTURBATION EXPANSION FOR PRESSURE.

The pressure \( P(r, \theta, z, t) \) may be determined from Bernoulli's equation

\[
\frac{P}{\rho} + gz + \Phi_t + \frac{1}{2} \left[ \Phi_r^2 + \frac{1}{r^2} \Phi_{\theta}^2 + \Phi_z^2 \right] = 0. \tag{4.1}
\]

After the perturbation expansion, the pressure up to the second-order can be written as the real part of

\[
P_1 + P_2 = -\rho gz - \rho \Phi_{1t} - \rho \left[ \Phi_{2t} + \frac{1}{2} \left\{ \Phi_{1r}^2 + \frac{1}{r^2} \Phi_{1\theta}^2 + \Phi_{1z}^2 \right\} \right], \tag{4.2}
\]

in which \( P_1 \) and \( P_2 \) denote the first- and second-order terms of the pressure, respectively.

4.2 TOTAL HORIZONTAL FORCE PER UNIT LENGTH.

The total horizontal force per unit length (THFPUL), in the direction of wave propagation, is now determined by

\[
\left( \frac{dF_z}{dz} \right)_1 + \left( \frac{dF_z}{dz} \right)_2 = -a \int_0^{2\pi} [P_1 + P_2]_{r=a} \cos \theta \, d\theta, \tag{4.3}
\]

in which \( \left( \frac{dF_z}{dz} \right)_1 \) and \( \left( \frac{dF_z}{dz} \right)_2 \) are the first- and second-order terms of the THFPUL, respectively. It is worth noting that the first term in \( P_1 + P_2 \), namely the hydrostatic term \( \rho gz \), can be neglected since it contains no \( \cos \theta \) term, and the no-flow condition
4.2.1 First-order Total Horizontal Force per Unit Length.

Based on the complete solution of the first-order velocity potential in (2.29) and (2.30), the first-order total horizontal force per unit length (1-THFPUL) is given by

\[
\left( \frac{dF_z}{dz} \right)_1 = a \rho \int_0^{2\pi} \Phi_1 t + \left\{ \Phi_2 t + \frac{1}{2} \left( \frac{1}{a^2} \Phi_{1\theta}^2 + \Phi_{1z}^2 \right) \right\} r=a \cos \theta \, d\theta,
\]

where

\[
R_n(k, r, \xi) = \frac{j_2n}{j_2n - kH_1'(ka)} \left( J_2n(k_x a)J_2n(k_y a) - B_{2n+1,n}H_1(ka) \right) - \frac{j_{2n-1}}{j_{2n-1} - kH_1'(ka)} \left( J_{2n-1}(k_x a)J_{2n-1}(k_y a) - B_{2n-1,n}H_1(ka) \right).
\]

and

\[
R_0(k, r, \xi) = J_1(k_x a)J_0(k_y a)
\]

\[
- \frac{k_x J_1'(k_x a)J_0(k_y a) + k_y J_1(k_x a)J_0'(k_y a)}{kH_1'(ka)} H_1(ka),
\]

\[
R_n(k, r, \xi) = i^{2n} \left\{ J_{2n+1}(k_x a)J_{2n}(k_y a) - B_{2n+1,n}H_1(ka) \right\}
\]

\[
- \left\{ J_{2n-1}(k_x a)J_{2n}(k_y a) - B_{2n-1,n}H_1(ka) \right\}.
\]
4.2.2 Second-order Total Horizontal Force per Unit Length.

The second-order total horizontal force per unit length (2-THFPUL) contains two components. The first component, \( \frac{dF_x}{dz} \), resulting from the first-order velocity potential, is called the quadratic second-order total horizontal force per unit length (Q-2-THFPUL), which can be written as

\[
\left( \frac{dF_x}{dz} \right)_{21} = -\frac{a \rho}{2} \int_0^{2\pi} \left( \frac{1}{\alpha^2} \Phi_1^2 + \Phi_1^2 \right) \cos \theta \, d\theta.
\]  

(4.9)

The second component, \( \frac{dF_x}{dz} \), resulting from the second-order velocity potential, is called the complementary second-order total horizontal force per unit length (C-2-THFPUL), which is given by

\[
\left( \frac{dF_x}{dz} \right)_{22} = a \rho \int_0^{2\pi} \Phi_2(\theta) \cos \theta \, d\theta
\]

\[
= -i2\omega a \rho \int_0^{2\pi} \Phi_2(\theta, z, t) \cos \theta \, d\theta.
\]  

(4.10)

By direct substitution of the first-order solutions (2.29) and (2.30) into (4.9), the quadratic second-order total horizontal force per unit length can be obtained as

\[
\left( \frac{dF_x}{dz} \right)_{21} = -\frac{a \rho g A^2 \pi}{8k \sinh 2kd} e^{-i2\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=1}^{8} \epsilon_m \epsilon_n \epsilon_p \epsilon_q \times
\]

\[
i^{m+p} \bar{\delta}_{ijmnqp} G_{mnpq}^j(k_x, k_y, k, \alpha, z, d),
\]  

(4.11)

where functions \( G_{mnpq}^j \) are given in Appendix B. As already noted in Section 3.3 for the functions \( F_{mnpq}^{jS} \) and \( F_{mnpq}^{jS} \), the summation with respect to \( j \) in (4.11) contains only 8 terms corresponding to the 8 combinations of indices from the first four summations appearing during the process for obtaining \( G_{mnpq}^j \).
The complementary second-order total horizontal force per unit length can be found by substitution of (3.42), in association with (3.52) and (3.54), into (4.10) which yields

\[
\left( \frac{dF_z}{dz} \right)_{22} = \frac{a \rho g k A^2 \pi}{4} e^{-i2\omega t} \left\{ S^*(k_x, k_y, k, k_2, a, z, d) + 2S(k_x, k_y, k, a, z, d) - 2S(k_x, k_y, k, k_2, a, z, d) \right\},
\]

where

\[ S^*(k_x, k_y, k, k_2, a, z, d) \]

\[ = i \left\{ \sum_{\beta=1}^{\infty} C_1^2 \frac{\cosh 2k(z + d)J_1(2kza)}{J_1(2kza)} - C_1^1 \cosh k_2(z + d)J_1'(2kza) \frac{H_1(k_2a)}{H_1'(k_2a)} \right\} \]

\[ + C_2^1 \cosh k_2(z + d)D_1(k_2) \left( J_1(k_2a) - J_1'(k_2a) \frac{H_1(k_2a)}{H_1'(k_2a)} \right), \]

\[ + \left\{ k \int_a^{\infty} r'F_1(r') \int_0^{\infty} \frac{\cosh \kappa(z + d)J_1(\kappa r')}{\cosh \kappa d} \frac{K_1(\kappa \beta a)}{K_1'(\kappa \beta a)} dr' d\kappa \right\} \]

\[ - \sum_{\beta=1}^{\infty} C_2^2 \cosh k_2(z + d) \frac{H_1(k_2a)}{H_1'(k_2a)} \times \]

\[ k \left\{ \frac{1}{2} \int_a^{\infty} F_1(r') dr' + \int_a^{\infty} r'F_1(r') \int_0^{\infty} \frac{k_2^2}{\kappa^2 - k_2^2} J_1'(\kappa \beta a)J_1(\kappa r') d\kappa dr' \right\} \]

\[ - \sum_{\beta=1}^{\infty} C_2^3 \cosh k_2(z + d) \frac{K_1(\kappa \beta a)}{K_1'(\kappa \beta a)} \times \]

\[ k \left\{ \frac{1}{a} I_1(\kappa \beta a) + \kappa \beta I_2(\kappa \beta a) \right\} \int_a^{\infty} r'F_1(r')K_1(\kappa_2 \beta r') dr' \right\} \]

\[ \overline{S}(k_x, k_y, k, a, z, d) \]

\[ = i \left[ \overline{S}_0(k_x, k_y, a) + \sum_{\alpha=1}^{\infty} \overline{S}_\alpha(k_x, k_y, a) \right] C_4^1 \cosh 2k(z + d), \]
4.3 DEPTH INTEGRATED TOTAL HORIZONTAL FORCE.

The depth integrated total horizontal force (THF) is

\[
F_z = \int_{-d}^{d} \frac{dF_z}{dz} \, dz.
\]

(4.18)

Based on perturbation theory, the upper limit of the above \( z \) -integral may be taken at \( z = 0 \) rather than \( z = \eta_1 + \eta_2 + \ldots \) which would only create terms of order higher than \( \epsilon^2 \). Therefore, to the second-order, the depth integrated total horizontal force may be obtained as

\[
F_{z1} + F_{z2} = \int_{-d}^{0} \left[ \left( \frac{dF_z}{dz} \right)_1 + \left( \frac{dF_z}{dz} \right)_2 \right] \, dz,
\]

(4.19)

in which \( F_{z1} \) and \( F_{z2} \) are the first- and second-order terms of the THF, respectively.

The first-order depth integrated total horizontal force (1-THF) is then given by

\[
F_{z1} = \int_{-d}^{0} \left( \frac{dF_z}{dz} \right)_1 \, dz
= -2\pi \rho g A d \frac{\tanh kd}{kd} e^{-i\omega t} R(k_x, k_y, k, \alpha).
\]

(4.20)
The second-order depth integrated total horizontal force (2-THF) can be written with regard to the contributions of the first- and second-order solutions, as

\[ F_{z2} = F_{z21} + F_{z22}, \quad (4.21) \]

where

\[ F_{z21} = \int_{-d}^{0} \left( \frac{dF_z}{dz} \right)_{21} dz, \quad (4.22) \]

and

\[ F_{z22} = \int_{-d}^{0} \left( \frac{dF_z}{dz} \right)_{22} dz. \quad (4.23) \]

Substituting (4.11) into (4.22), we obtain the quadratic second-order depth integrated total horizontal force (Q-2-THF), as

\[ F_{z21} = -\frac{a \rho g A^2 \pi}{8k \sinh 2kd} e^{-i2\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{8} \varepsilon_m \varepsilon_n \varepsilon_p \varepsilon_q \times \]

\[ i^{m+p} \delta_{1j} \delta_{mn} \delta_{pq} \overline{G}_{mnpq}^j(k_x, k_y, k, a, d), \quad (4.24) \]

where functions \( \overline{G}_{mnpq}^j \) are given in Appendix B. Extra care must be taken when dealing with the last summation as mentioned in the previous section.

\( F_{z22}, \) namely the complementary second-order depth integrated total horizontal force (C-2-THF), can be found by substituting (4.12) – (4.17) into (4.23). With a little work, \( F_{z22} \) can be written as

\[ F_{z22} = \frac{a \rho g k A^2 \pi}{4} e^{-i2\omega t} \left\{ V^*(k_x, k_y, k, k_2, a, d) \right. \]

\[ + 2\overline{V}(k_x, k_y, k, a, d) - 2V(k_x, k_y, k, k_2, a, d) \right\}, \quad (4.25) \]

where

\[ V^*(k_x, k_y, k, k_2, a, d) \]
\[ V(k_x, k_y, k, a, d) = \frac{i}{4} \left[ C_{10} + \sum_{\alpha=1}^{\infty} i^{2\alpha} (C_{2\alpha+1,\alpha} - C_{2\alpha-1,\alpha}) \right] \times \]
\[ \left\{ C_4^2 \frac{\sinh k_2 d}{k_2} \frac{H_1(k_2 a)}{H'_1(k_2 a)} + \sum_{\beta=1}^{\infty} C_4^3 \frac{\sin \kappa_2 \beta d}{\kappa_2 \beta} \frac{K_1(\kappa_2 \beta a)}{K'_1(\kappa_2 \beta a)} \right\} \]

and
\[ V_0(k_x, k_y, a) = J_1(2k_x a) J_0(2k_y a), \]
\[ V_\alpha(k_x, k_y, a) = i^{2\alpha} \left[ J_{2\alpha+1}(2k_x a) - J_{2\alpha-1}(2k_x a) \right] J_{2\alpha}(2k_y a). \]
5.1 COMPUTATIONAL PROCEDURES.

The numerical evaluation of the integrals requires various special techniques and these are discussed in the current chapter. The notation is the same as that used in the previous chapters. There are three single integrals and two double integrals which need to be evaluated in order to obtain numerical results for a given cylinder and a given incident wave train. These single integrals are

\[ S_I = \frac{1}{k_2} D_1(k_2) = \int_a^\infty r' F_1(r') J_1(k_2 r') \, dr', \]  
\[ (5.1) \]

\[ S_{I_2} = \int_a^\infty F_1(r') \, dr', \]  
\[ (5.2) \]

and

\[ S_{I_3\beta} = \int_a^\infty r' F_1(r') K_1(\kappa_{2\beta} r') \, dr', \quad \beta = 1, 2, 3, \ldots \]  
\[ (5.3) \]

The double integrals are

\[ D_I = \int_a^\infty r' F_1(r') \int_0^\infty \frac{k_2^2}{(\kappa^2 - k_2^2)} J_1(\kappa a) J_1(\kappa r') \, d\kappa \, dr', \]  
\[ (5.4) \]

and

\[ D_{I_2} = \int_a^\infty r' F_1(r') \int_0^\infty \frac{\kappa \cosh \kappa(z + d) J_1(\kappa r)}{\cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} J_1(\kappa r') \, d\kappa \, dr'. \]  
\[ (5.5) \]

Although the inner singular-integrals in both double integrals are well behaved
across their singularities, considerable difficulties beyond these singular points are encountered. The difficulties originate from the highly double-oscillatory integrands which are badly behaved far away from the singular point, and it is necessary to overcome these problems before the complete double integral can be calculated. Furthermore, all of these single and double integrals are of the infinite range type.

Methods previously used by Hunt and Baddour (1981), Rahman and Heaps (1983) or Eatock Talor and Hung (1987) truncate these integrals at some finite upper limits (hereafter called the truncation points), supposedly large enough for the contributions from the rest of the range to be neglected. However, our numerical experience shows that using the above mentioned methods to evaluate the integrals is very time consuming when applied to the current problem since the truncation points cannot be simply determined. In fact, the positive and negative contributions to the integral certainly do not cancel each other out and many oscillations must be included before the difference between the contributions is negligible.

These difficulties, however, can be resolved by employing the $\varepsilon$-algorithm proposed by Wynn (1956). As a result, a suitable truncation point can be determined simply with a rapid rate of convergence of the summation involved in calculating the integral. The advantage of using the $\varepsilon$-algorithm to accelerate the convergence of the infinite integral (3.26), with an oscillatory singular integrand and analytical solution (3.27), namely

$$
\int_0^\infty \frac{\kappa \cosh \kappa (z + d) \cos (\kappa R - \frac{\pi}{4})}{\cosh \kappa d (\kappa \tanh \kappa d - k_2 \tanh k_2 d)} \sqrt{\frac{2}{\pi \kappa R}} \, d\kappa
= - \frac{\pi 2k_2 \cosh k_2 d \cosh k_2 (z + d) \sinh (k_2 R - \frac{\pi}{4})}{2k_2 d + \sinh 2k_2 d} \sqrt{\frac{2}{\pi k_2 R}},
$$

(5.6)

is illustrated in Table 1.
Table 1. Example of the $\varepsilon$-algorithm for a test case with $k_2 = 3$, $d = 1.2$ and $z = -0.1$

<table>
<thead>
<tr>
<th>R</th>
<th>Integral Range</th>
<th>Approx. Integral</th>
<th>Analytical Result</th>
</tr>
</thead>
<tbody>
<tr>
<td>9.2</td>
<td>0 - 8.793045</td>
<td>-1.0448521699292</td>
<td>-1.0448521816649</td>
</tr>
<tr>
<td>14.7</td>
<td>0 - 6.571699</td>
<td>0.5149791265392</td>
<td>0.5149791265402</td>
</tr>
<tr>
<td>23.8</td>
<td>0 - 5.114988</td>
<td>-0.6519910291695</td>
<td>-0.6519910291684</td>
</tr>
<tr>
<td>37.5</td>
<td>0 - 4.335398</td>
<td>0.5115540893840</td>
<td>0.5115540893846</td>
</tr>
<tr>
<td>41.9</td>
<td>0 - 4.180043</td>
<td>0.3354765727186</td>
<td>0.3354765727193</td>
</tr>
</tbody>
</table>

It is obvious from the table that, instead of an infinite range, a small finite range (column 2), leads to the approximate and analytical results (columns 3 and 4) being in excellent agreement. Note also that with increasing values of $R$, the integral range decreases even further.

The subroutine $dqelg$ from QUADPACK based on the $\varepsilon$-algorithm was used to obtain the results listed in Table 1 and was used throughout this study. Also, all regular integrals in this study were computed using a modification of the subroutine $dqag$ from QUADPACK based on Gauss-Kronrod integration rules while Cauchy Principal Values were evaluated by the subroutine $dqawc$ using the modified Clenshaw-Curtis and Gauss-Kronrod integration rules.

To evaluate double integrals, the inner singular-integral with respect to $\kappa$ was broken into the sum of three integrals, for a preset small positive real number $\zeta$, as

$$\int_0^\infty f(\kappa, r') \, d\kappa = \int_0^{k_2-\zeta} f(\kappa, r') \, d\kappa + \int_{k_2+\zeta}^{k_2+\zeta} f(\kappa, r') \, d\kappa + \int_{k_2+\zeta}^\infty f(\kappa, r') \, d\kappa, \quad (5.7)$$

in which

$$f(\kappa, r') = \frac{k_2^2}{(\kappa^2 - k_2^2)} J_1'(\kappa) J_1(\kappa r'), \quad (5.8)$$

or
where (5.8) and (5.9) correspond to the inner singular-integrands of (5.4) and (5.5), respectively.

The third integral in (5.7), namely $I_{3rd}(r')$, was further split into an infinite sum of integrals with finite ranges, that is

$$I_{3rd}(r') = \int_{k_2 + \zeta}^{\kappa_1'} f(\kappa, r') \, d\kappa + \int_{\kappa_1'}^{\kappa_2'} f(\kappa, r') \, d\kappa + \ldots + \int_{\kappa_{n-1}'}^{\kappa_n'} f(\kappa, r') \, d\kappa + \ldots, \quad (5.10)$$

where $\kappa_1'$ is the first root of $J_1(\kappa r')$ which is greater than $k_2 + \zeta$ and the $\kappa_n$'s (for $n \geq 2$) are the subsequent roots of $J_1(\kappa r')$. In practice, the first root of $J_1(\kappa r')$ which is greater than $k_2$ can be substituted for $k_2 + \zeta$. In conjunction with the $\varepsilon$-algorithm, $I_{3rd}(r')$ can be obtained by computing only a finite number of integrals in (5.10) as

$$I_{3rd}(r') = \int_{k_2 + \zeta}^{\kappa_1'} f(\kappa, r') \, d\kappa + \int_{\kappa_1'}^{\kappa_2'} f(\kappa, r') \, d\kappa + \ldots + \int_{\kappa_{n-1}'}^{\kappa_n'} f(\kappa, r') \, d\kappa, \quad (5.11)$$

where the truncation point $\kappa_n'$ is the upper limit of the last integral by which the convergence of the $\varepsilon$-algorithm is achieved.

With this method, not only is the computational time shortened but the integration range is also greatly reduced. In fact, as $r'$ increases, the truncation point becomes closer to (but of course remains greater than) the singular point $k_2$. Since the case for $r'$ is analogous to the case for $R$, this trend is clearly observed from the first and second columns of Table 1. Furthermore, an extra benefit of this method is to have avoided the evaluation of the Bessel function, $J_1$, at large argument since such evaluation consumes much more time than that at small argument. Nonetheless, total time spent in the computation of this Bessel function is still high. However, this time
can be reduced enormously by introducing an interpolation technique, but the higher the accuracy required for the interpolation, the bigger the computer storage needed. In other words, speed and accuracy are difficult to achieve simultaneously. However, to achieve reasonable results for both demands, a fourth-order Lagrange interpolation (e.g., see Davis and Rabinowitz, 1975) in which a Bessel function is discretised at 50 points per unit length was found to be most appropriate. Commercial IMSL subroutines were used for the computation of Bessel functions.

For the evaluation of the outer integrals of the double integrals and also the single integrals (5.1) – (5.3) with respect to \( r' \), one can see that every \( r' \)-integral is in the form

\[
I = \int_{a}^{\infty} r' F_1(r') h(r') \, dr',
\]

(5.12)

where a function \( h \) corresponds to a function in each integrand in (5.1) – (5.5). This observation allows the integrations to be performed simultaneously. The \( \epsilon \)-algorithm is again used to obtain the convergent solutions of these \( r' \)-integrals, and all of these integrals need to be broken into a sum of integrals as

\[
I = \int_{a}^{b} r' F_1(r') h(r') \, dr' \, d\kappa + \int_{b}^{r_1'} r' F_1(r') h(r') \, dr' + \int_{r_1'}^{r_2'} r' F_1(r') h(r') \, dr' + \ldots + \int_{r_{n-1}'}^{r_n'} r' F_1(r') h(r') \, dr' + \ldots,
\]

(5.13)

where \( b \) and \( r_n' \)'s (for \( n \geq 1 \)) are appropriate real numbers.

From numerical experiments, it was found that the 15-31 points Gauss-Kronrod rule was the most suitable in evaluating the integrals in (5.13) with \( b \) chosen as the first even number which is greater than \( a \), and then the \( r_n' \)'s (for \( n \geq 1 \)) as the subsequent
even numbers. Another possible combination could be the 7-15 points Gauss-Kronrod rule with \( b \) being the first root of \( J_1(k_2 r') \) in \( SI_1 \) which is beyond \( a \), and the \( r'_n \)'s (for \( n \geq 1 \)) being the subsequent roots of \( J_1(k_2 r') \). Other combinations, of course, may be used. The advantage of this method is that, when the integration rule is applied, the information about \( r'F_1(r') \) at any point \( r' \) is computed once and can then be used for every \( r' \)-integral. Moreover, the values of \( r'F_1(r') \) need not be calculated repeatedly, thus leading to a great saving in CPU time.

A further comment may be made regarding the convergence of these \( r' \)-integrals which can be improved by introducing a second \( \varepsilon \)-algorithm. The output from the first \( \varepsilon \)-algorithm, which is more stable than its input, is passed to the second algorithm. This concept for the acceleration in the convergence of integrals can be easily modified to obtain an acceleration in the convergence of all series in the present solution. One extra comment is that \( SI_{3\beta} \)'s occur inside of a summation, and thus a certain number of \( SI_{3\beta} \)'s have to be evaluated in advance before the summation can be treated because \( r' \)-integrals need to be calculated simultaneously.

The computation of the results in the next section was undertaken on a SUN 4/470 Sparc Server at The University of Wollongong where all formulae were programmed in Fortran 77 in double precision mode.

5.2 DISCUSSION OF THE RESULTS.

With the success in evaluating the integrals and series involved, hydrodynamic forces exerted on a fixed, surface-piercing, circular cylinder can now be determined.
To explore these forces, an incident wave train with the amplitude of 0.1 m, namely \( A = 0.1 \), has been chosen and used throughout the demonstrations with the variation of other physical parameters. All forces presented here are given as maxima from the moduli of complex quantities, in nondimensional form, and are plotted in two sets of Figures. In Figures 5-28 and 39-52, forces are plotted against the ratio of \( k_x/k \) with \( k \) being fixed in each case, whereas in Figures 29-38 they are plotted against the ratio of \( k_y/k_x \) with \( k_x \) being fixed in each case. For the clarity of presentation, full title of all forces will be used and the abbreviations will occasionally be used in the discussion and figures. The corresponding Kriebel's results for the forces due to plane waves are marked by a dot (o) in all the figures.

### 5.2.1 Total Horizontal Force per Unit Length.

The total horizontal force per unit length (THFPUL), in the direction of wave propagation, varies with water depth. All results of THFPUL were evaluated at a point 0.1 m beneath the mean surface level, namely \( z = -0.1 \), with varying water depth \( d \), cylinder radius \( a \) and wave number \( k \). With two of the wave numbers \( k, k_x \) and \( k_y \) given, the other can be determined from

\[
k^2 = k_x^2 + k_y^2,
\]

where \( k_x \) and \( k_y \) are wave numbers in the \( x \)- and \( y \)-directions, respectively.

For a given first-order wave number, \( k \), of an incident wave train, the second-order wave number, \( k_2 \), of the second-order progressive wave modes can be calculated from the equation

\[
k_2 \tanh k_2 d = 4k \tanh kd,
\]
which is a direct result from combining Equations (2.18) and (3.39). Then second-order wave numbers, \( \kappa_{2\beta} \)'s, of the second-order evanescent wave modes, can be obtained as the positive roots of

\[
\kappa_{2\beta} \tan \kappa_{2\beta} d + k_2 \tanh k_2 d = 0,
\]

which results from eliminating \( \omega^2 \) and \( g \) in Equations (3.39) and (3.40).

The second-order total horizontal force per unit length (2-THFPUL), will be compared with the first-order one (1-THFPUL), given by Zhu (1992) which can be calculated from Equation (4.5). There are two components of 2-THFPUL which need to be computed; one is the quadratic second-order total horizontal force per unit length (Q-2-THFPUL), resulting from the first-order potential as shown in (4.11), and the other one is the complementary second-order total horizontal force per unit length (C-2-THFPUL), originating from the second-order potential given in (4.12). The contributions of Q-2-THFPUL and C-2-THFPUL toward 2-THFPUL are shown in the even-numbered figures from Figures 5 to 18 while the contributions of 1-THFPUL and 1-THFPUL + 2-THFPUL are shown in odd-numbered figures. Throughout the demonstration, waves are “short-crested” when \( 0 < \gamma / 2k \). Otherwise they are “long-crested”, with plane waves being a special case when \( k_x = k \).

It has been pointed out by Zhu (1992) that the behaviour of the first-order total horizontal force per unit length is “linearly” increased from “short-crested” wave \( (k_x/k = 0) \) to plane wave \( (k_x/k = 1) \). In other words, with the above notation, the first-order total horizontal force per unit length induced by “long-crested” waves is always larger than that by “short-crested” waves and this first-order force reaches its maximum when incident waves are plane waves. As expected, the same behaviour occurs in the quadratic second-order total horizontal force per unit length. The only
difference is that this quadratic second-order force is no longer "linear" and is actually curved with nearly quadratic variation. However, this increasing behaviour ceases to be true in the complementary second-order total horizontal force per unit length or even in the second-order total horizontal force per unit length as can be observed in Figures 8, 14 and 16. It is shown in Figure 16 that the complementary second-order force induced by "short-crested" waves of some particular $k_x$ and $k_y$, for instance $k_x = 0.3$ and $k_y = \sqrt{0.91}$, can be up to four times greater than that by plane waves. On the other hand, the same "short-crested" waves can exert the second-order total horizontal force per unit length on a cylinder up to two times larger than plane waves do. Nevertheless, after adding the second-order force to the first-order one, Figure 15 shows that the total horizontal force per unit length of plane waves is still a maximum.

The relative importance of the quadratic second-order total horizontal force per unit length and the complementary one is determined by their contributions toward the second-order total horizontal force per unit length. In Figure 6, we can see that the quadratic second-order force dominates the complementary one as $k_x/k$ varies from 0 to about 0.6 where both of them have the same magnitude. The point at which the quadratic second-order force and the complementary one have the same magnitude will be referred to as a QCSM point hereafter. Beyond the QCSM point, the complementary second-order force rapidly soars and becomes dominant. Completely different behaviour occur in the cases shown in Figures 8, 14 and 16; the complementary second-order force starts to dominate the quadratic one and then the quadratic second-order force becomes dominant after the QCSM point is passed. It is of interest that the QCSM point moves to the right when $kd$ becomes larger. As can be seen in Figures 18 and 20, the complementary second-order forces entirely dominate the quadratic ones. Although there is no substantial verification, it has been shown, for most of cases illustrated here, especially in Figures 18 and 20, that the complementary second-order total horizontal
force per unit length plays a more important role in second-order total horizontal force per unit length than the quadratic one does. Despite the amplitude being increased, forces shown in Figures 9 and 10 still have the same behaviour as those in Figures 7 and 8, respectively. Of course the former are much bigger. This is due to the fact that terms in second-order solutions are of order \((kA)^2\) and consequently waves with larger amplitude exert larger forces on a cylinder. Figures 11 and 12 show that doubling wave amplitude will increase the total horizontal force per unit length by 120%.

It might be expected by the present nonlinear theory that in deep water, the second-harmonic terms in the incident potential given in (3.6) could play an important role in the solution. To investigate this hypothesis, let the potential in (3.6) be rewritten again as

\[
\phi_{2P}^H = -i \frac{3gA^2}{4\omega} \frac{k^2(tanh^2 kd - 1)}{2k tanh 2kd - k_2 tanh k_2 d} \cosh 2k(z + d) \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \epsilon_n \frac{m^m \times}{J_m(2kz r)J_n(2ky r)\left[ \cos(m + 2n)\theta + \cos(m - 2n)\theta \right]} \cosh 2kd \\
- i \frac{gA^2}{2\omega} \frac{3k^2(tanh^2 kd - 1) + 4k_y^2}{2k_z tanh 2k_x d - k_2 tanh k_2 d} \cosh 2k_z(z + d) \times \sum_{m=0}^{\infty} \epsilon_m \epsilon_n \frac{m^m}{J_m(2kz r) \cos m\theta}.
\]

As described in section 3.2, one notices that the double-summation terms represent short-crested second-harmonic waves while the single-summation terms represent plane second-harmonic waves. Generally speaking, the above wave potential represents short-crested and plane second-harmonic waves travelling together in the \(x\)-direction. As \(k_y\) decreases, or the incident-wave length in the crest direction increases, one can see that short-crested second-harmonic waves gradually become long-crested second-harmonic waves. Finally, for incident plane waves, the potential in (5.17) reduces to

\[
\phi_{2P}^H = -i \frac{3gA^2}{2\omega} \frac{k^2(tanh^2 kd - 1)}{2k tanh 2kd - k_2 tanh k_2 d} \cosh 2k(z + d) \times \sum_{m=0}^{\infty} \epsilon_m \epsilon_n \frac{m^m}{J_m(2kr) \cos m\theta,
\]
in which only the plane second-harmonic waves are represented.

The effect of the depth on the wave forces exerted on a cylinder is now clearly seen to depend on incident wave type. In (5.17), as \( kd \) increases, the term \( \tanh^2 kd - 1 \) approaches zero causing short-crested second-harmonic waves to become smaller and eventually vanish leaving plane second-harmonic waves alone. These plane second-harmonic waves, the amplitude of which depends on \( k_y \), are the ones that play a key role in the second-order total horizontal force per unit length. Figure 22 shows the comparison of the second-order total horizontal forces per unit length from Figures 8, 14 and 16 with \( ka = 1.0, kA = 0.1 \) and three different values of \( kd \). It is evident from these four figures that the second-order total horizontal force per unit length becomes larger as \( kd \) increases; especially, when \( kx/k \) is between 0 and 0.7 or incident waves are "short-crested". On the other hand, this means relative water depth affects second-order force induced by "short-crested" waves rather than that by "long-crested" waves. One can then conclude that, in deep water, there should be only a slight change in second-order force for incident plane waves because the plane second-harmonic waves in (5.18) disappear. Figure 21 conspicuously displays that total horizontal force per unit length is almost invariant when \( kx \) approaches \( k \). Although there is no theoretical verification, the analysis from the present calculations seems reasonable.

Points which should be mentioned here for the following discussions are:

(i) total horizontal forces per unit length in Figures 23, 25 and 27 are obtained from those in Figures 7 and 19, Figures 17 and 19, and Figures 7 and 17, respectively, and

(ii) second-order total horizontal forces per unit length in Figures 24, 26 and 28 are obtained from those in Figures 8 and 20, Figures 18 and 20, and Figures 8 and 18, respectively.
Figures 23 and 24 show the ratios of total horizontal force per unit length and second-order total horizontal force per unit length to cylinder radii, respectively. It is shown in Figure 23 that total wave load on a small cylinder is greater than that on a larger cylinder if all the rest of the physical parameters are kept the same. Unlike total force, only waves with some particular wave numbers will exert larger second-order forces on a small cylinder than those on a larger cylinder which is shown in Figure 24. It is interesting to notice that when wave amplitude, $A$, and water depth, $d$, are kept constant but wave number, $k$, and cylinder radius, $a$, vary such that $ka$ is fixed, then it is found that second-order total horizontal force per unit length exerted on a small cylinder is greater than that on a large cylinder, as can be seen in Figure 26. However, it is shown in Figure 25 that this second-order force does not affect total horizontal force per unit length as a large cylinder still receives bigger total horizontal force per unit length than a small cylinder does. This is because the first-order total horizontal force per unit length absolutely dominates the second-order one. When the radius of a cylinder is compared with wave length (or wave number instead), it is indicated in Figure 28 that as the cylinder radius, wave amplitude and water depth are fixed, then the second-order total horizontal force per unit length from waves with shorter wave length is greater than that from waves with long wave length. However, this second-order force only slightly affects total horizontal force per unit length as we can see in Figure 27.

For a fixed wave number, $k_z$, in the $x$-direction, the second-order theory predicts the total horizontal force per unit length exerted on the cylinder will become smaller as $k_y$ becomes bigger, or the wave length in the $y$-direction becomes shorter. On the other hand, despite the increase of the total wave number $k$ with the same $k_z$, the maximum wave load on the cylinder is achieved when incident waves are plane waves and can be observed in Figures 29, 31, 33 and 35. This prediction is the same as that
pointed out by Zhu (1992) in the linear theory. It is of interest to notice further that, with the wave crests being slightly shorter than that of plane waves, for example the ratio $k_y/k_x$ being 0.5, the wave loads could reduce by 12\%.

The decreasing behaviour also occurs in the quadratic second-order total horizontal force per unit length (but not in the complementary one) as expected. As shown in Figures 30, 32, 34 and 36, the complementary second-order force seems to increase as the ratio $k_y/k_x$ increases from 0 to 1. Even though it is not clear about the contributions from the other parts of the solution, the main contribution toward the complementary second-order total horizontal force per unit length may be due to the incident potential in (5.17). As we can see from Equation (5.17), the short-crested second-harmonic together with the plane second-harmonic raise the contribution toward the complementary second-order force when $k_y$ increases. However, the total wave number, $k$, must be taken into account as it increases along with $k_y$ causing $\tanh kd$ to approach 1. As a result, the main contribution towards the complementary second-order force will then be concentrated at the plane second-harmonic which contains $k_y$ as its factor.

Although the behaviour of the complementary second-order total horizontal force per unit length seems to be unpredictable when $k_y/k_x > 1$, the second-order total horizontal force per unit length is presumed to achieve its maximum when $k_x = k_y$ which may be seen in Figures 30, 32, 34, 36 and 38. Note that, with the ratio $k_y/k_x$ being 1, the total horizontal force per unit length could exceed the first-order force by approximately 30\%. It should also be noted that with the ratio of $k_y/k_x$ being 1.5, the short-crested wave load could drop from a maximum value by as much as 64\% at first order. However, after adding the second-order force to the first-order one, that difference becomes 47\%. These facts help to confirm the importance of including the
nonlinear effects in the calculation of wave forces. Furthermore, the nonlinear terms seem to affect forces induced by short-crested waves more than those by plane waves as we can see in Figures 30, 32, 34 and 36 where the second-order total horizontal force per unit length does not decrease as $k_y/k_x$ increases.

Figure 37 shows the comparison of total horizontal forces per unit length from Figures 31, 33 and 35 while Figure 38 shows the comparison of second-order total horizontal forces per unit length from Figures 32, 34 and 36. As can be seen in these two figures, when the water depth, $d$, is the main consideration, the second-order theory predicts no significant change in forces as $d$ varies with the same wave number $k_x$ and cylinder radius $a$.

From the second-order solution presented here, it appears that the increment of the total horizontal force per unit length from the first-order one predicted by the linear theory can be up to 161% for fixed wave number $k$ (e.g., when $k_x/k = 0.1$ in Figure 17). However, this case might scarcely happen since corresponding to this wave number, wave crest has become very long in the direction of wave propagation and very short in the direction normal to wave propagation; short-crested wave may have broken long before. However, due to limited experiments carried out in the past on short-crested waves, a realistic wave length ratio for short-crested waves is not clear presently. On the other hand, short-crested waves having $k_x$ and $k_y$ of the same magnitude, say $k_x/k = 0.7$, have been observed in oceans. For this case, our solution shows that forces can increase as much as 45%. For wave number $k_x$ being fixed, forces can increase from those at first-order up to 30% when $k_y/k_x = 1.0$. The importance of including the effects of nonlinear terms which are ignored in the linear diffraction theory is therefore quite clear. Including second-order forces may become crucial for ocean engineers in designing offshore structures. If the linear theory was used, it might
cause disaster to the ocean industry because of the underestimated wave forces. It is clearly seen that if the plane wave diffraction theory is utilised to calculate wave forces exerted on a cylinder in an ocean full of short-crested waves, it will always be safe since wave loads are always overestimated. However, if ocean engineers carefully embed the nonlinear short-crested wave diffraction theory into their design criteria for offshore structures, the design safety margin could be better predicted and thereby reduce the design and construction costs substantially.

### 5.2.2 Depth Integrated Total Horizontal Force.

The depth integrated total horizontal force (THF), may be calculated according to its components. The first-order depth integrated total horizontal force (1-THF), given by Zhu (1992) may be evaluated from Equation (4.20). There are two components of the second-order depth integrated total horizontal force (2-THF):

(i) the quadratic second-order depth integrated total horizontal force (Q-2-THF), resulting from the first-order potential, and

(ii) the complementary second-order depth integrated total horizontal force (C-2-THF), originating from the second-order potential,

which were evaluated from Equations (4.24) and (4.25), respectively. The contributions of Q-2-THF and C-2-THF toward 2-THF are shown in even-numbered figures from Figures 39 to 48 while the contributions of 1-THF and 1-THF + 2-THF are shown in odd-numbered figures.

Since the comparison between the second- and first-order depth integrated total horizontal forces is similar to that discussed in the previous section between the second-
and first-order total horizontal forces per unit length, a similar explanation of all terms can be applied here. However, there are two interesting points which are worthwhile mentioning. One is that the depth integrated total horizontal force and the second-order depth integrated total horizontal force will increase and finally approach their limit in deep water as can be seen in Figures 49 and 50. Note that the depth integrated total horizontal forces in Figure 49 are obtained from those in Figures 39, 41 and 43 while the second-order depth integrated forces in Figure 50 are obtained from those in Figures 40, 42 and 44. Another point of interest is that when cylinder radius, \( a \), and water depth, \( d \), are kept constant, the depth integrated total horizontal force exerted on a cylinder decreases as wave number, \( k \), increases, as shown in Figure 51. In other words, wave-induced forces exerted on a cylinder become smaller as incident waves become shorter. This is as expected. Contrarily, completely different behaviour occurred in the second-order depth integrated total horizontal force as can be observed in Figure 52. Note also that each depth integrated total horizontal force in Figure 51 can be seen in Figures 39, 45 and 47 while the second-order depth integrated forces in Figure 52 may be investigated individually in Figures 40, 46 and 48.

The present nonlinear analysis shows that the complementary second-order depth integrated total horizontal force induced by "short-crested" waves can sometimes be greater than that by "long-crested" waves. However, plane waves still produce the largest second-order depth integrated total horizontal force and also depth integrated total horizontal force. When the second-order theory is taken into consideration, it is shown that the depth integrated total horizontal force can increase up to 63% more than that predicted by the linear theory for waves with \( k_x/k = 0.1 \) and 12% for waves with \( k_x/k = 0.7 \).
Figure 5. THFPUL with $ka = 0.63$, $kA = 0.07$ and $kd = 0.84$

Figure 6. 2-THFPUL with $ka = 0.63$, $kA = 0.07$ and $kd = 0.84$
Figure 7. THFPUL with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 1.2$

Figure 8. 2-THFPUL with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 1.2$
Figure 9. THFPUL with $ka = 1.0$, $kA = 0.2$ and $kd = 1.2$

Figure 10. 2-THFPUL with $ka = 1.0$, $kA = 0.2$ and $kd = 1.2$
Figure 11. Comparison of THFPUL's with $k = 1.0(m^{-1})$, $a = 1.0(m)$ and $d = 1.2(m)$

Figure 12. Comparison of 2-THFPUL's with $k = 1.0(m^{-1})$, $a = 1.0(m)$ and $d = 1.2(m)$
Figure 13. THFPUL with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 3.0$

Figure 14. 2-THFPUL with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 3.0$
Figure 15. THFPUL with $k_A = 1.0$, $k_A = 0.1$ and $k_d = 5.0$

Figure 16. 2-THFPUL with $k_A = 1.0$, $k_A = 0.1$ and $k_d = 5.0$
Figure 17. THFPUL with $ka = 2.0$, $kA = 0.2$ and $kd = 2.4$

Figure 18. 2-THFPUL with $ka = 2.0$, $kA = 0.2$ and $kd = 2.4$
Figure 19. THFPUL with $ka = 2.0$, $kA = 0.1$ and $kd = 1.2$

Figure 20. 2-THFPUL with $ka = 2.0$, $kA = 0.1$ and $kd = 1.2$
Figure 21. Comparison of THFPUL's with $k_a = 1.0$ and $k_A = 0.1$.

Figure 22. Comparison of 2-THFPUL's with $k_a = 1.0$ and $k_A = 0.1$. 
Figure 23. Comparison of THFPUL's with $k_A = 0.1$ and $k_d = 1.2$

Figure 24. Comparison of 2-THFPUL's with $k_A = 0.1$ and $k_d = 1.2$
Figure 25. Comparison of THFPUL's with \( A = 0.1(m) \) and \( d = 1.2(m) \)

Figure 26. Comparison of 2-THFPUL's with \( A = 0.1(m) \) and \( d = 1.2(m) \)
Figure 27. Comparison of THFPUL's with $a = 1.0(m)$, $A = 0.1(m)$ and $d = 1.2(m)$

Figure 28. Comparison of 2-THFPUL's with $a = 1.0(m)$, $A = 0.1(m)$ and $d = 1.2(m)$
Figure 29. THFPUL with $k_x = 1.0(m^{-1})$, $a = 1.0(m)$, $A = 0.1(m)$ and $d = 1.2(m)$

Figure 30. 2-THFPUL with $k_x = 1.0(m^{-1})$, $a = 1.0(m)$, $A = 0.1(m)$ and $d = 1.2(m)$
Figure 31. THFPUL with $k_x = 1.2 \, (m^{-1})$, $a = 0.8 \, (m)$, $A = 0.1 \, (m)$
and $d = 1.2 \, (m)$

Figure 32. 2-THFPUL with $k_x = 1.2 \, (m^{-1})$, $a = 0.8 \, (m)$, $A = 0.1 \, (m)$
and $d = 1.2 \, (m)$
Figure 33. THFPUL with $k_x = 1.2(m^{-1})$, $a = 0.8(m)$, $A = 0.1(m)$ and $d = 3.0(m)$.

Figure 34. 2-THFPUL with $k_x = 1.2(m^{-1})$, $a = 0.8(m)$, $A = 0.1(m)$ and $d = 3.0(m)$. 
Figure 35. THFPUL with $k_x = 1.2(m^{-1})$, $a = 0.8(m)$, $A = 0.1(m)$

and $d = 5.0(m)$

Figure 36. 2-THFPUL with $k_x = 1.2(m^{-1})$, $a = 0.8(m)$, $A = 0.1(m)$

and $d = 5.0(m)$
Figure 37. Comparison of THFPUL's with $k_x = 1.2(m^{-1})$, $a = 0.8(m)$ and $A = 0.1(m)$

Figure 38. Comparison of 2-THFPUL's with $k_z = 1.2(m^{-1})$, $a = 0.8(m)$ and $A = 0.1(m)$
Figure 39. THF with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 1.2$

Figure 40. 2-THF with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 1.2$
Figure 41. THF with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 3.0$

Figure 42. 2-THF with $k_a = 1.0$, $k_A = 0.1$ and $k_d = 3.0$
Figure 43. THF with \( k_a = 1.0, k_A = 0.1 \) and \( k_d = 5.0 \)

Figure 44. 2-THF with \( k_a = 1.0, k_A = 0.1 \) and \( k_d = 5.0 \)
Figure 45. THF with $k_d = 1.5$, $k_A = 0.15$ and $kd = 1.8$

Figure 46. 2-THF with $k_d = 1.5$, $k_A = 0.15$ and $kd = 1.8$
Figure 47. THF with \( k_A = 2.0 \), \( k_A = 0.2 \) and \( k_d = 2.4 \)

Figure 48. 2-THF with \( k_A = 2.0 \), \( k_A = 0.2 \) and \( k_d = 2.4 \)
Figure 49. Comparison of THF's with $k_a = 1.0$ and $k_A = 0.1$

Figure 50. Comparison of 2-THF's with $k_a = 1.0$ and $k_A = 0.1$
Figure 51. Comparison of THF's with $a = 1.0(m)$, $A = 0.1(m)$

and $d = 1.2(m)$

Figure 52. Comparison of 2-THF's with $a = 1.0(m)$, $A = 0.1(m)$

and $d = 1.2(m)$
CONCLUSIONS

An endeavour has been made in this study to investigate the diffraction of short-crested waves impinging on a vertical, surface-piercing, circular cylinder. An analytical solution to this problem has been derived and presented in closed-form. At first-order, the solution is the linear diffraction theory presented by Zhu (1992). At second-order, the solution is found to include both short-crested second-harmonic and plane second-harmonic waves travelling together in the positive $x$-direction, as well as more complicated forced wave motions. The latter include local standing waves near the cylinder and outgoing progressive second-order free waves in the far field. Furthermore, to satisfy the no-flow condition on the surface of a cylinder, an additional set of scattered wave solutions is required, including freely progressive wave modes and local evanescent wave modes of second-order short-crested waves as well as those of second-order plane waves.

Even though closed-form integrals developed in this second-order solution were simplified via several special integration techniques, considerable difficulties in numerical evaluation of these integrals were still encountered. To enable the integrals to be calculated, the order of integration was changed in all double integrals. A Hankel transform was then used to further simplify the inner integrals. One of them can be integrated analytically while another inner integral can be reduced to an analytical solution plus an associated integral which is easier to compute than the original one. However, awkward convergence of these integrals was encountered. To allow the numerical integrations to proceed, the $\varepsilon$-algorithm was then employed to accelerate the convergence of the integrals. The appropriate application of these techniques resulted in remarkable
reductions of computational effort as well as CPU time required in carrying out the numerical calculations. The same concept was used in accelerating the convergence of other series which occurred in the computations.

Following the success in dealing with all integrals and series involved in numerical calculations, the complete solution for the velocity potential was then used to determine hydrodynamic forces of short-crested waves exerted on a vertical circular cylinder. The second-order theory has shown that the increasing-property of the first-order total horizontal force per unit length (1-THFPUL), is still obtained at second-order in the quadratic second-order total horizontal force per unit length (Q-2-THFPUL), resulting from the first-order potential. However, this property ceases to be true in the complementary second-order total horizontal force per unit length (C-2-THFPUL), which originates from the second-order potential. It was found that the second-order total horizontal force per unit length (2-THFPUL), (or C-2-THFPUL) of short-crested waves can sometimes be two (or four) times greater than that of long-crested waves. It was also shown that C-2-THFPUL plays a more important role in 2-THFPUL than Q-2-THFPUL does. The result of C-2-THFPUL was emphasised as being most affected from the potential given in (3.6). More specifically, the short-crested second-harmonic and plane second-harmonic in this potential resulting from self-interaction of incident waves, are those which make key contributions toward C-2-THFPUL.

The present nonlinear analysis has shown the importance of including the effects of the nonlinear terms in the total forces exerted on a cylinder. Through our calculations, the total horizontal force per unit length can increase from that predicted by the linear theory up to 45%. For depth integrated total horizontal force, the increment could reach up to 12% for short-crested wave of equal wave lengths in both direction. Such a large increment in nonlinear forces clearly shows that the inclusion of the second-order

- 87 -
forces may become important when ocean engineers design offshore structures. If the linear theory was used, it might cause disaster to those offshore structures because of the underestimated wave forces. This nonlinear theory also predicts that if the plane wave diffraction theory is utilised to calculate wave forces exerted on a cylinder in an ocean full of short-crested waves, it will always be safe since wave loads are always overestimated. However, if ocean engineers carefully embed the nonlinear short-crested wave diffraction theory into their design criteria for offshore structures, the design safety margin could be better predicted and thereby reduce the design and construction costs substantially.

Finally, when the water particle kinematic, water surface elevation and other quantities are of interest, this second-order solution can be directly used to determine them, although the numerical integrations involved in the calculations may be quite tedious and time consuming.
APPENDIX A

SERIES FORM OF QUADRATIC FORCING TERMS

The quadratic forcing terms in the second-order combined free surface boundary condition from Equation (2.33) are

\[ f(r, \theta, t) = \frac{1}{g} \Phi_{1t}(\Phi_{1zz} + g\Phi_{1zz}) - 2 \left( \Phi_{1r} \Phi_{1rt} + \frac{1}{r^2} \Phi_{1\theta} \Phi_{1\theta} + \Phi_{1z} \Phi_{1zt} \right), \]  

(A1)

and may be decomposed into three individual forcing terms as

\[ f(r, \theta, t) = f^{II} + f^{IS} + f^{SS}. \]

Based on the idea of decomposing the first-order velocity potential into the sum of incident and scattered velocity potentials, each quadratic forcing term can be obtained by substituting \( \Phi_{1}^{I} \) expressed in Equation (2.21) and \( \Phi_{1}^{S} \) in Equation (2.26) into Equation (A1).

The function \( f^{II} \), resulting from \( \Phi_{1}^{I} \), represents the forcing term according to the self-interaction of the first-order incident waves involving \( J_{m}(s) \) only, or the product terms like \( J_{m}(s)J_{n}(s) \), and may be written as

\[ f^{II}(r, \theta, t) = -i \frac{3g^2 A^2}{4\omega} \frac{k^2(\tanh^2 kd - 1) \cosh 2k(z + d)}{2k \tanh 2kd - k_2 \tanh k_2 d} \cosh 2kd \times \]

\[ \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \epsilon_m \epsilon_n \frac{i^m J_m(2kr) J_2 n(2kr)}{\cosh 2kz} [\cos (m + 2n) \theta + \cos (m - 2n) \theta] \]  

(A2)

\[ - i \frac{g^2 A^2}{2\omega} \frac{3k^2(\tanh^2 kd - 1) + 4k_2^2}{2k \tanh 2kd - k_2 \tanh k_2 d} \cosh 2kd \times \]

\[ \sum_{m=0}^{\infty} \epsilon_m i^m J_m(2kz) \cos m \theta. \]
The function $f^{IS}$, resulting from the product of $\Phi_1^I$ and $\Phi_1^S$ in Equation (A1), represents the forcing term according to the cross-interaction of the first-order incident and scattered waves involving the product terms like $J_m(s)H_n(s)$. The function $f^{SS}$, resulting from $\Phi_1^S$ in Equation (A1), represents the forcing term according to the self-interaction of the first-order scattered waves involving the product terms like $H_m(s)H_n(s)$. These two functions can be found in the combined form as

$$f^{IS}(r, \theta, t) + f^{SS}(r, \theta, t)$$

$$= -i \frac{g^2 k^2 A^2}{8\omega} e^{-i2\omega t} \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{j=1}^{8} \varepsilon_m \varepsilon_n \varepsilon_p \varepsilon_q \epsilon^{m+p} \times$$

$$\delta_{l,mnpq} (F_{mnpq}^{IS}(r) + F_{mnpq}^{SS}(r)) \cos N_{mnpq}^j \theta,$$

where the modified Kronecker delta function $\delta_{l,mnpq}$ has been defined in Equation (3.37) and

$$N_{mnpq}^1 = m + 2n - p - 2q,$$
$$N_{mnpq}^2 = m + 2n - p + 2q,$$
$$N_{mnpq}^3 = m - 2n - p - 2q,$$
$$N_{mnpq}^4 = m - 2n - p + 2q,$$
$$N_{mnpq}^5 = m + 2n + p - 2q,$$
$$N_{mnpq}^6 = m + 2n + p + 2q,$$
$$N_{mnpq}^7 = m - 2n + p + 2q,$$
$$N_{mnpq}^8 = m - 2n + p - 2q,$$

$$F_{mnpq}^{IS}(r) = F_{mnpq}^{IS1}(r) + F_{mnpq}^{IS5}(r),$$
$$F_{mnpq}^{SS}(r) = F_{mnpq}^{SS1}(r) + F_{mnpq}^{SS5}(r),$$
$$F_{mnpq}^{2IS}(r) = F_{mnpq}^{2IS2}(r) + F_{mnpq}^{2IS6}(r),$$
$$F_{mnpq}^{2SS}(r) = F_{mnpq}^{2SS2}(r) + F_{mnpq}^{2SS6}(r),$$
$$F_{mnpq}^{3IS}(r) = F_{mnpq}^{3IS3}(r) + F_{mnpq}^{3IS7}(r),$$
$$F_{mnpq}^{3SS}(r) = F_{mnpq}^{3SS3}(r) + F_{mnpq}^{3SS7}(r),$$
$$F_{mnpq}^{4IS}(r) = F_{mnpq}^{4IS4}(r) + F_{mnpq}^{4IS8}(r),$$
$$F_{mnpq}^{4SS}(r) = F_{mnpq}^{4SS4}(r) + F_{mnpq}^{4SS8}(r),$$
$$F_{mnpq}^{5IS}(r) = F_{mnpq}^{5IS1}(r) - F_{mnpq}^{5IS5}(r),$$
$$F_{mnpq}^{5SS}(r) = F_{mnpq}^{5SS1}(r) - F_{mnpq}^{5SS5}(r),$$
$$F_{mnpq}^{6IS}(r) = F_{mnpq}^{6IS2}(r) - F_{mnpq}^{6IS6}(r),$$
$$F_{mnpq}^{6SS}(r) = F_{mnpq}^{6SS2}(r) - F_{mnpq}^{6SS6}(r),$$
$$F_{mnpq}^{7IS}(r) = F_{mnpq}^{7IS3}(r) - F_{mnpq}^{7IS7}(r),$$
$$F_{mnpq}^{7SS}(r) = F_{mnpq}^{7SS3}(r) - F_{mnpq}^{7SS7}(r),$$
$$F_{mnpq}^{8IS}(r) = F_{mnpq}^{8IS4}(r) - F_{mnpq}^{8IS8}(r),$$
$$F_{mnpq}^{8SS}(r) = F_{mnpq}^{8SS4}(r) - F_{mnpq}^{8SS8}(r).$$
in which

\[
F_{mnpq}^{IS1}(r) = \frac{2}{(kr)^2} \left \{ (IS_{mnpq}^{IS1}(r) + IS_{pqmn}^{IS1}(r)) (m + 2n)(p + 2q) \right \},
\]

\[
F_{mnpq}^{IS2}(r) = \frac{2}{(kr)^2} \left \{ (IS_{mnpq}^{IS2}(r) + IS_{pqmn}^{IS2}(r)) (m + 2n)(p + 2q) \right \},
\]

\[
F_{mnpq}^{IS3}(r) = \frac{2}{(kr)^2} \left \{ (IS_{mnpq}^{IS3}(r) + IS_{pqmn}^{IS3}(r)) (m + 2n)(p + 2q) \right \},
\]

\[
F_{mnpq}^{IS4}(r) = \frac{2}{(kr)^2} \left \{ (IS_{mnpq}^{IS4}(r) + IS_{pqmn}^{IS4}(r)) (m + 2n)(p + 2q) \right \},
\]

\[
F_{mnpq}^{SS1}(r) = \frac{2}{(kr)^2} f_{mnpq}^{SS1}(r)(m + 2n)(p + 2q),
\]

\[
F_{mnpq}^{SS2}(r) = \frac{2}{(kr)^2} f_{mnpq}^{SS2}(r)(m + 2n)(p - 2q),
\]

\[
F_{mnpq}^{SS3}(r) = \frac{2}{(kr)^2} f_{mnpq}^{SS3}(r)(m + 2n)(p + 2q),
\]

\[
F_{mnpq}^{SS4}(r) = \frac{2}{(kr)^2} f_{mnpq}^{SS4}(r)(m - 2n)(p + 2q),
\]
and

\[ f_{mnpq}^{I_{S11}}(r) = J_m(k_x r) J_{2n}(k_y r) A_{pq} H_{p+2q}(kr), \]

\[ f_{mnpq}^{I_{S12}}(r) = J_m(k_x r) J_{2n}(k_y r) B_{pq} H_{|p-2q|}(kr), \]

\[ f_{mnpq}^{I_{S21}}(r) = \{ k_x J_m'(k_x r) J_{2n}(k_y r) + k_y J_m(k_x r) J_{2n}'(k_y r) \} A_{pq} H_{p+2q}'(kr), \]

\[ f_{mnpq}^{I_{S22}}(r) = \{ k_x J_m'(k_x r) J_{2n}(k_y r) + k_y J_m(k_x r) J_{2n}'(k_y r) \} B_{pq} H_{|p-2q|}'(kr), \]

\[ f_{mnpq}^{S_{S11}}(r) = A_{mn} A_{pq} H_{m+2n}(kr) H_{p+2q}(kr), \]

\[ f_{mnpq}^{S_{S12}}(r) = A_{mn} B_{pq} H_{m+2n}(kr) H_{|p-2q|}(kr), \]

\[ f_{mnpq}^{S_{S3}}(r) = B_{mn} B_{pq} H_{|m-2n|}(kr) H_{|p-2q|}(kr), \]

\[ f_{mnpq}^{S_{S1d}}(r) = A_{mn} A_{pq} H_{m+2n}(kr) H_{p+2q}'(kr), \]

\[ f_{mnpq}^{S_{S2d}}(r) = A_{mn} B_{pq} H_{m+2n}(kr) H_{|p-2q|}'(kr), \]

\[ f_{mnpq}^{S_{S3d}}(r) = B_{mn} B_{pq} H_{|m-2n|}(kr) H_{|p-2q|}'(kr). \]
APPENDIX B

FUNCTIONS $G^j_{mnpq}$ AND $G^j_{mnpq}$ IN THE SECOND-ORDER DEPTH INTEGRATED TOTAL HORIZONTAL FORCE

Functions $G^j_{mnpq}$ in Equation (4.11) are given by

\[
\begin{align*}
G^1_{mnpq} &= G^1_{mnpq} + G^1_{mnpq}, \\
G^2_{mnpq} &= G^2_{mnpq} + G^2_{mnpq}, \\
G^3_{mnpq} &= G^3_{mnpq} + G^3_{mnpq}, \\
G^4_{mnpq} &= G^4_{mnpq} + G^4_{mnpq}, \\
G^5_{mnpq} &= G^1_{mnpq} - G^1_{mnpq}, \\
G^6_{mnpq} &= G^2_{mnpq} - G^2_{mnpq}, \\
G^7_{mnpq} &= G^3_{mnpq} - G^3_{mnpq}, \\
G^8_{mnpq} &= G^4_{mnpq} - G^4_{mnpq},
\end{align*}
\]

where

\[
\begin{align*}
G^1_{mnpq} &= k^2 \sinh^2 k(z + d) g^1_{mnpq}, \\
G^2_{mnpq} &= k^2 \sinh^2 k(z + d) g^2_{mnpq}, \\
G^3_{mnpq} &= k^2 \sinh^2 k(z + d) g^3_{mnpq}, \\
G^4_{mnpq} &= k^2 \sinh^2 k(z + d) g^4_{mnpq}, \\
G^{1\theta}_{mnpq} &= \frac{1}{a^2} \cosh^2 k(z + d)(m + 2n)(p + 2q) g^{1\theta}_{mnpq}, \\
G^{2\theta}_{mnpq} &= \frac{1}{a^2} \cosh^2 k(z + d)(m + 2n)(p - 2q) g^{2\theta}_{mnpq}, \\
G^{3\theta}_{mnpq} &= \frac{1}{a^2} \cosh^2 k(z + d)(m - 2n)(p + 2q) g^{3\theta}_{mnpq}, \\
G^{4\theta}_{mnpq} &= \frac{1}{a^2} \cosh^2 k(z + d)(m - 2n)(p - 2q) g^{4\theta}_{mnpq},
\end{align*}
\]

and

\[
\begin{align*}
g^1_{mnpq} &= J_{mnpq} + 2J_{mn}(AH)_{pq} + (AH)_{mn}(AH)_{pq}, \\
g^2_{mnpq} &= J_{mnpq} + 2J_{mn}(BH)_{pq} + (AH)_{mn}(BH)_{pq}, \\
g^3_{mnpq} &= J_{mnpq} + 2J_{mn}(AH)_{pq} + (BH)_{mn}(AH)_{pq}, \\
g^4_{mnpq} &= J_{mnpq} + 2J_{mn}(BH)_{pq} + (BH)_{mn}(BH)_{pq},
\end{align*}
\]
in which
\[ J_{mn} = J_m(k_x r) J_{2n}(k_y r), \]
\[ J_{mnpq} = J_{mn} J_{pq}, \]
\[ (AH)_{mn} = A_{mn} H_{m+2n}(kr), \]
\[ (BH)_{mn} = B_{mn} H_{|m-2n|}(kr). \]

Constants \( A_{mn} \) and \( B_{mn} \) are given in Equations (2.27) and (2.28).

Functions \( G^j_{mnpq} \) in Equation (4.24) are defined similar to \( G^j_{mnpq} \) except for the following changes:

\[
G^{1z}_{mnpq} = \frac{k^2}{2} \left\{ \frac{\sinh 2kd}{2k} - d \right\} g^1_{mnpq}, \\
G^{2z}_{mnpq} = \frac{k^2}{2} \left\{ \frac{\sinh 2kd}{2k} - d \right\} g^2_{mnpq}, \\
G^{3z}_{mnpq} = \frac{k^2}{2} \left\{ \frac{\sinh 2kd}{2k} - d \right\} g^3_{mnpq}, \\
G^{4z}_{mnpq} = \frac{k^2}{2} \left\{ \frac{\sinh 2kd}{2k} - d \right\} g^4_{mnpq},
\]

and
\[
G^{1\theta}_{mnpq} = \frac{1}{2a^2} \left\{ \frac{\sinh 2kd}{2k} + d \right\} (m + 2n)(p + 2q) g^1_{mnpq}, \\
G^{2\theta}_{mnpq} = \frac{1}{2a^2} \left\{ \frac{\sinh 2kd}{2k} + d \right\} (m + 2n)(p - 2q) g^2_{mnpq}, \\
G^{3\theta}_{mnpq} = \frac{1}{2a^2} \left\{ \frac{\sinh 2kd}{2k} + d \right\} (m - 2n)(p + 2q) g^3_{mnpq}, \\
G^{4\theta}_{mnpq} = \frac{1}{2a^2} \left\{ \frac{\sinh 2kd}{2k} + d \right\} (m - 2n)(p - 2q) g^4_{mnpq}. 
\]
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