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Abstract

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Keywords. Finance; Continuous-time portfolio choice; Return predictability; Linear price impact; Quadratic transaction cost.

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1 Introduction

Portfolio selection problem is a recurrent topic in financial economics. In absence of trading frictions, the trade-off paradigm between risk and return has been well understood since the seminal works of Markowitz (1952), Samuelson (1969) and Merton (1969, 1971). One main implication of their works is that when asset returns are independent and identically distributed (i.i.d.), the optimal trading strategies for the agents with mean-variance, constant-relative-risk-aversion (CRRA), or constant-absolute-risk-aversion (CARA) preferences, are always *myopic* in nature, i.e. the current optimal portfolio is independent of the distribution of asset returns in future periods. For a comprehensive overview on the classical portfolio selection problems, readers are referred to Brandt (2009), Detemple (2014) and Markowitz (2014).

On the other hand, the empirical literature, such as Campbell (1987); Campbell and Shiller (1988); Fama and French (1988, 1989), has documented extensively that asset returns are non-stationary, i.e. the i.i.d assumption is violated. To reconcile with the non-stationarities of asset returns, Kim and Omberg (1996) proposed a continuous-time optimal portfolio selection model where risk premium follows an Ornstein-Uhlenbeck (O-U) process; Campbell and Viceira (1999) considered an optimal investment and consumption problem of an infinitely-lived agent with return predictability and Epstein-Zin-Weil utility; and Wachter (2002) derived a closed-form solution for the continuous-time optimal investment and consumption problem faced by a CRRA agent with return predictability. These aforementioned works have demonstrated that return predictability induces *nonmyopic* behavior as there is a demand for the agent to hedge against risk-premium uncertainty. Liu (2007) developed a unified framework to solve continuous-time portfolio selection problems under stochastic environments. Recently, Liu and Muhle-Karbe (2013) reviewed portfolio choice in settings where investment opportunities are stochastic due to stochastic volatility or return predictability. Battauz et al. (2017) is another recent example of a non-myopic behaviour in case of non-stationary asset return.

Optimal portfolio selection problem with return predictability and transaction costs was first studied by Balduzzi and Lynch (1999)[1]. By incorporating proportional and fixed transaction

[1]For the case of single risky asset, portfolio selection with transaction costs was first studied in Constantinides (1986) and was rigorously analyzed in Davis and Norman (1990) under the assumption of constant investment opportunity sets. Liu (2004) obtained a closed-form solution to the optimal portfolio problem with multiple,

costs, Balduzzi and Lynch (1999) showed that substantial loss in utility would occur when the agent ignores return predictability and transaction costs. Lynch and Balduzzi (2000) examined the impact of return predictability and transaction costs on the agent's rebalancing behavior. Lynch and Tan (2010) solved the optimal portfolio selection problem for the case of two correlated risky assets with return predictability and transaction costs. However, extensions beyond two risky assets with return predictability and transaction costs remain very challenging.

When transaction costs are of quadratic form, Gârleanu and Pedersen (2013) obtained a closed-form solution to the portfolio selection problem of an infinitely-lived mean-variance agent with multiple assets and multiple return-predicting signals in a discrete-time framework. Due to transaction costs, the optimal strategy of the mean-variance agent is to rebalance his portfolio partially towards an aim portfolio, which is a weighted sum of the expected Markowitz portfolios on all future dates with weights to account for return predictability and transaction costs.

The highly tractable framework in Gârleanu and Pedersen (2013) has attracted another tide of extensions recently. Representative works include Glasserman and Xu (2013) and DeMiguel et al. (2015) for their extensive examinations of robustness in model uncertainty; Collin-Dufresne et al. (2015) for the inclusion of multi-variate stochastic volatilities; Moallemi and Sağlam (2017) for the proposal of the linear rebalancing rules to obtain a near-optimal portfolio with trading constraints; Zhang et al. (2017) for the impact of the ambiguity aversion on returns and return-predicting signals; Mei and Nogales (2018) for the portfolio selection with multiple risky assets; and Collin-Dufresne et al. (2019) for the considerations of regime-switching dynamics in expected returns, volatilities, and trading costs.

Gârleanu and Pedersen (2016) developed the continuous-time version from the limit of the discrete-time model in Gârleanu and Pedersen (2013). Chan and Sircar (2015) revisited Gârleanu and Pedersen (2016) with stochastic volatility and derived an approximate optimal trading strategy using multi-scale asymptotic expansion. Bouchard et al. (2018) studied the impact of transaction costs on the equilibrium return in continuous time. To the best of our knowledge, analyses beyond the case of the mean-variance agent have not been well addressed.

In this paper, we study the continuous-time portfolio selection problem of a finitely-lived

mutually independent assets for the case of a CARA agent.

CARA agent with return predictability and quadratic transaction costs. We adopt the model dynamics of return predictability and quadratic transaction costs from Gârleanu and Pedersen (2016). Different from Gârleanu and Pedersen (2016), we consider the case of the CARA agent who maximizes the expected utility of his terminal wealth. Indeed, embracing the high tractability of CARA preferences, Liu (2004) solved the multi-asset portfolio selection problem with transaction costs analytically when asset returns are stationary. In this respect, this paper shows that the desired tractability is maintained when asset returns are predictable. Specifically, when return-predicting signals follow general stochastic processes, we characterize the solution to the optimal portfolio selection problem in terms of the solution to a coupled system of partial differential equations (PDEs). When return-predicting signals follow a multi-variate O-U process, we analytically solve the optimal portfolio selection problem up to the solution of a coupled system of Riccati differential equations. More importantly, we establish the existence and uniqueness of the solution to the associated Riccati system and solve it in closed form. Similar to the case of the mean-variance agent in Gârleanu and Pedersen (2016), our optimal trading strategy also has an analogous feature of trading gradually toward an aim portfolio.

Our optimal trading strategy departs from that in Gârleanu and Pedersen (2016) in two aspects. First, the aim portfolio of the CARA agent is a weighted sum of the expected future Merton portfolios with weights to account for return predictability and transaction costs. In addition, our finite-time portfolio optimization problem introduces a time-dependent effect to the corresponding trading strategy, which is absent in Gârleanu and Pedersen (2016). Moreover, our aim portfolio gradually converges to the Merton portfolio as time approaches the terminal date. The other significant difference with Gârleanu and Pedersen (2016) is that not only the mean-reversion rates but also the volatility of return-predicting signals would affect the trading strategy for the CARA agent, whereas the volatility effect is absent in Gârleanu and Pedersen (2016). In addition, our results indicate the CARA agent invests more into the risky assets with more persistent signals and with less transaction costs. High persistence in return-predicting signals is interpreted from either low mean-reversion rates or low volatilities.

The rest of this paper is organized as follows. In Section 2, a dynamic model with return predictability and transaction costs is presented, and a corresponding utility maximization problem for a CARA agent is also provided. In Section 3, the main results of this paper are provided. In Section 4, we numerically illustrate our main theoretical results. Section 5 concludes the paper. Appendix A contains proofs of all results in Section 3.

2 Model

2.1 Asset dynamics and return predictability

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete filtered probability space, where $\mathcal{F} = (\mathcal{F}_s)_{0 \le t \le T}$ denotes the filtration generated by a standard n-dimensional Brownian motion $\{\mathbf{W}_p(t)\}_{t \ge 0}$ and a standard k-dimensional Brownian motion $\{\mathbf{W}_f(t)\}_{t \ge 0}$.

Consider a financial market with n risky assets $\mathbf{p}(t) = (p_1(t), ..., p_n(t))^\top$ and one risk-free asset $p_0(t)$ that are traded continuously on a finite horizon $[0, T]$, where $\top$ denotes the transpose of a matrix or a vector. The dynamics of the risk-free asset p_0 is given as

$$dp_0(t) = rp_0(t)dt, \tag{2.1}$$

where $r > 0$ denotes the constant risk-free interest rate. The price of the risky assets $\mathbf{p}(t) = [p_1(t), p_1(t), \cdots, p_n(t)]^\top$ has the following dynamics [2]

$$d\mathbf{p}(t) = [r\mathbf{p}(t) + \mathbf{B}\mathbf{f}(t)]dt + \boldsymbol{\sigma}_p d\mathbf{W}_p(t), \tag{2.2}$$

where $\boldsymbol{\sigma}_p$ is an $n \times n$ constant non-degenerate matrix with $\boldsymbol{\Sigma}_p \triangleq \boldsymbol{\sigma}_p^\top \boldsymbol{\sigma}_p$, $\mathbf{B}$ is an $n \times k$ matrix of factor loadings and $\mathbf{f}(t)$ is a $k \times 1$ factor vector. From the dynamic (2.2), the excess return of the risky assets $\mathbf{p}(t)$ is $\mathbf{B}\mathbf{f}(t)$, and hence factors $\mathbf{f}(t) \triangleq (f_1(t), ..., f_k(t))^\top$ can be interpreted as the return-predicting factor, as done in Campbell and Viceira (1999); Balduzzi and Lynch (1999);

[2] In this paper, we model the return of the risky assets as the *absolute changes* in price levels, in contrast to the *proportional changes* in price levels, which can be seen from the dynamics of $\mathbf{p}(t)$ in (2.2). As in the case of the mean-variance portfolio selection problem in Gârleanu and Pedersen (2016), modelling the return dynamics of the risky assets as (2.2) allows us to solve a portfolio selection problem for a CARA agent analytically (see Theorem 1). Indeed, both Forsyth et al. (2012) and Collin-Dufresne et al. (2015, 2019) have shown that the optimal solution of the optimal portfolio/execution problem under the arithmetic Brownian motion framework turns out to be an excellent approximation to the same problem under the geometric Brownian motion framework.

Lynch and Balduzzi (2000)[3]. The return-predicting factor $\mathbf{f}(t)$ has its own dynamics as follows

$$d\mathbf{f}(t) = \boldsymbol{\mu}_f\left(t, \mathbf{f}(t)\right) dt + \boldsymbol{\sigma}_f\left(t, \mathbf{f}(t)\right) d\mathbf{W}_f(t), \tag{2.3}$$

where $\boldsymbol{\mu}_f : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}^k$ and $\boldsymbol{\sigma}_f : \mathbb{R}_+ \times \mathbb{R}^k \to \mathbb{R}^{k\times k}$ denote the drift and the volatility of return-predicting factors with $\boldsymbol{\Sigma}_f(t, \mathbf{f}(t)) \triangleq \boldsymbol{\sigma}_f^\top(t, \mathbf{f}(t))\boldsymbol{\sigma}_f(t, \mathbf{f}(t))$. For the sake of analytical tractability, we assume that $\{\mathbf{W}_p(t)\}_{t\geq 0}$ and $\{\mathbf{W}_f(t)\}_{t\geq 0}$ are mutually independent for $t \in [0, T]$[4] . In practice, the matrix of factor loading $\mathbf{B}$ is calibrated from empirical data using a panel regression. Therefore, the factors are required to be non-multicollinear, i.e. $\mathbf{B}$ has full rank.

2.2 Transaction cost and wealth dynamics

Denote $\mathbf{x}(t) = (x_1(t), x_2(t), \cdots, x_n(t))^\top \in \mathbb{R}^n$ to be an agent's portfolio, which contains the number of shares of the risky assets $\mathbf{p}(t)$ at time $t \in [0, T]$. The agent chooses his trading intensity $\boldsymbol{\tau}(t) \triangleq (\tau_1(t), ..., \tau_n(t))^\top \in \mathbb{R}^n$, which determines the changes of his portfolio $\mathbf{x}(t)$, i.e.

$$d\mathbf{x}(t) = \boldsymbol{\tau}(t)dt, \tag{2.4}$$

where $\boldsymbol{\tau}$ is hereafter referred to as a *trading strategy*.

Equation (2.4) implies that we only consider smooth or absolutely continuous portfolio $\mathbf{x}(t)$. In fact, Proposition 4 of Gârleanu and Pedersen (2016) (hereafter, the GP model) shows that non-smooth changes in portfolios, such as discrete jump or quadratic variation (e.g. Brownian motion) in position, would result in infinite transaction costs under the continuous-time framework.

Next, we specify the concept of quadratic transaction costs in this paper, which is originated from the optimal execution problem first studied by Almgren and Chriss (2001). Following Gârleanu and Pedersen (2016), we assume that when the agent trades $\boldsymbol{\tau} dt$ units of the asset $\mathbf{p}(t)$,

[3] Recent literature also shows that co-integrating and macroeconomic factors can also be used as predicting factors for asset returns. See, for example, Çanakoğlu and Özekici (2010, 2012); Chiu and Wong (2011, 2012) for the optimal portfolio selection in the aforementioned directions without transaction costs.

[4] For the case where the asset and return-predicting factors are correlated, a cross-derivative term would appear in the HJB equation (3.1). To the best of our knowledge, closed-form solution is not available in this case and we need to resort to numerical scheme (Ma et al., 2018).

his trade has a *transient linear price impact* on the asset price $\mathbf{p}$ as shown below:

$$\mathbf{p}^{\mathrm{E}}(t) \triangleq \mathbf{p}(t) + \frac{1}{2}\mathbf{\Lambda}\boldsymbol{\tau}(t), \tag{2.5}$$

where the matrix $\mathbf{\Lambda}$ is an $n \times n$ symmetric positive-definite matrix that measures the level of trading cost, and is often referred to as the multi-dimensional version of Kyle's lambda. In (2.5), $\mathbf{p}^{\mathrm{E}} = (p_1^{\mathrm{E}}, ..., p_n^{\mathrm{E}})^\top$ denotes the vector of *execution prices* and p_i^{E} represents the amount the agent pays (receives) for each unit of his buy (sell) order $\tau_i dt$, for $i = 1, ..., n$. The transient nature of the linear price impact in (2.5) can be interpreted from the fact that the agent's trade size $\boldsymbol{\tau} dt$ affects only the execution price and his trade has no permanent impact on the asset price dynamics in (2.2). In terms of the execution price, the total cost per time unit of trading with intensity $\boldsymbol{\tau}(t)$ is expressed as follows

$$\mathrm{Cost}(\boldsymbol{\tau}(t)) \triangleq \boldsymbol{\tau}(t)^\top \mathbf{p}^{\mathrm{E}}(t) = \boldsymbol{\tau}(t)^\top \mathbf{p}(t) + \underbrace{\frac{1}{2}\boldsymbol{\tau}(t)^\top \mathbf{\Lambda}\boldsymbol{\tau}(t)}_{\mathrm{TC}(\boldsymbol{\tau}(t))}. \tag{2.6}$$

In (2.6), the first term denotes the cost of trading $\boldsymbol{\tau}(t)$ units of the assets at the market price $\mathbf{p}(t)$, while the second term captures the *transaction costs* of trading $\boldsymbol{\tau}(t)$ units. As a result, the quadratic transaction cost is indeed a consequence of the linear price impact assumption (2.5), which has been supported by many empirical studies such as Breen et al. (2002); Greenwood (2005); Cartea et al. (2015); Gârleanu and Pedersen (2016).

With the specifications of the asset price dynamics and transaction costs in place, we now derive the wealth dynamics of the agent. Let $y(t)$ be the agent's wealth process that represents the dollar amount of his allocation at time t, i.e.

$$y(t) = x_0(t)p_0(t) + \mathbf{x}(t)^\top \mathbf{p}(t), \tag{2.7}$$

where $x_0(t)$ is the number of shares of risk-free asset at time t. By applying Itô's lemma and self-financing condition, we derive the dynamics of the agent's wealth as

$$dy(t) = \left[ry(t) + \mathbf{x}(t)^\top \mathbf{B}\mathbf{f}(t) - \frac{1}{2}\boldsymbol{\tau}(t)^\top \mathbf{\Lambda}\boldsymbol{\tau}(t) \right] dt + \mathbf{x}(t)^\top \boldsymbol{\sigma}_p d\mathbf{W}_p(t). \tag{2.8}$$

Remark 1. The wealth dynamics in (2.8) can also be used to recover the agent's mean-variance preference on the excess return of his wealth over an infinitesimal period, which was extensively studied in Gârleanu and Pedersen (2016). Consider a time interval $[t, t + dt]$. Using wealth dynamics in (2.8), the mean and variance of the change of agent's wealth over $[t, t + dt]$ become

$$\mathbf{E}_t[dy(t) - ry(t)dt] = \left[\mathbf{x}(t)^\top \mathbf{B}\mathbf{f}(t) - \frac{1}{2}\boldsymbol{\tau}(t)^\top \boldsymbol{\Lambda}\boldsymbol{\tau}(t)\right] dt, \tag{2.9}$$

$$\mathbf{Var}_t[dy(t) - ry(t)dt] = \mathbf{x}(t)^\top \boldsymbol{\Sigma}_p \mathbf{x}(t)dt. \tag{2.10}$$

In terms of (2.9) and (2.10), the mean-variance preference on the wealth change on $[t, t + dt]$ is

$$\mathbf{E}_t[dy(t) - ry(t)dt] - \frac{\gamma}{2}\mathbf{Var}_t[dy(t) - ry(t)dt] = \left[\mathbf{x}(t)^\top \mathbf{B}\mathbf{f}(t) - \frac{1}{2}\boldsymbol{\tau}(t)^\top \boldsymbol{\Lambda}\boldsymbol{\tau}(t)\right] dt - \frac{\gamma}{2}\mathbf{x}(t)^\top \boldsymbol{\Sigma}_p \mathbf{x}(t)dt, \tag{2.11}$$

where γ denotes the risk-aversion parameter of the agent in the GP model. Aggregating the present value of the agent's mean-variance preferences in (2.11) from t to ∞, we have

$$\begin{aligned} & \int_t^\infty e^{-\rho(s-t)} \Big(\mathbf{E}_s[dy(s) - ry(s)ds] - \frac{\gamma}{2}\mathbf{V}ar_s[dy(s) - ry(s)ds]\Big) \\ = & \int_t^\infty e^{-\rho(s-t)} \left[\mathbf{x}(s)^\top \mathbf{B}\mathbf{f}(s) - \frac{1}{2}\boldsymbol{\tau}(s)^\top \boldsymbol{\Lambda}\boldsymbol{\tau}(s) - \frac{\gamma}{2}\mathbf{x}(s)^\top \boldsymbol{\Sigma}_p \mathbf{x}(s)\right] ds, \end{aligned} \tag{2.12}$$

where ρ is the discount rate and γ is the risk aversion coefficient. This coincides with the objective function considered in Gârleanu and Pedersen (2016).

2.3 Utility maximization problem

In this paper, we consider a utility maximization problem of a CARA agent[5]. More specifically, the agent chooses his trading strategy $\boldsymbol{\tau}(t)$ to maximize the expected exponential utility of his terminal wealth.

[5] Portfolio selection problem for a CARA agent with transaction costs was studied in Liu (2004) when asset returns are i.i.d. Recently, Bodnar et al. (2015) studied a multi-period portfolio choice problem of a CARA agent with return predictability in a frictionless market. Collin-Dufresne et al. (2019) also mentioned the possible extension to the CARA case when they explored the liquidity regimes in their Appendix B.3, but without considering return predictability. In this respect, this paper serves as a first work that unifies these aforementioned works in a tractable framework.

Problem 1. The agent seeks to solve the following problem

$$\max_{\boldsymbol{\tau}\in\mathcal{A}} \mathbf{E}\left[-\frac{1}{\eta}e^{-\eta y(T)}|\mathbf{x}(0)=\mathbf{x}, y(0)=y, \mathbf{f}(0)=\mathbf{f}\right], \tag{2.13}$$

with portfolio $\mathbf{x}(t)$ in (2.4), wealth dynamics $y(t)$ in (2.8), the dynamics of the return-predicting factor $\mathbf{f}(t)$ in (2.3), and $\boldsymbol{\tau}\in\mathcal{A}$, a collection of admissible trading strategies satisfying the integrable condition and the well-defined condition as follow:

$$\mathbf{E}\int_0^T \|\boldsymbol{\tau}(s)\|_1 ds < \infty; \quad \mathbf{E}\left[e^{-\eta y^{\boldsymbol{\tau}}(T)}\right] < \infty. \tag{2.14}$$

The study of Problem 1 results in two significant departures from the mean-variance analysis in Gârleanu and Pedersen (2016). First, we consider a finite-time utility maximization problem as opposed to infinite-time utility maximization problem in Gârleanu and Pedersen (2016). Finite-time utility maximization problem allows us to study the time-dependent effect of the agent's portfolio selection in presence of transaction costs. In addition, the time-dependent effect also plays a pivotal role in assessing the impact of return-predicting factors on the agent's optimal portfolio with respect to time. In contrast, Gârleanu and Pedersen (2016) studied the infinite-horizon utility maximization problem, and hence time plays no role in their optimal portfolio. Consequently, the impact of the return-predicting factors on their optimal trading strategy is constant. However, investment horizon is finite in practice. The study of the finite-time utility maximization problem is therefore necessary, yet is more challenging.

Secondly, as we shall see in Section 3, the optimal trading strategy to Problem 1 would depend on both the drift $\boldsymbol{\mu}_f$ and the volatility $\boldsymbol{\Sigma}_f$ of the return-predicting signal $\mathbf{f}$. On the other hand, the optimal trading strategy of the mean-variance agent in the GP model is influenced only by the drift of the return-predicting signal. Mathematically, it implies that the solution methodology in the GP model cannot be directly applied to solve Problem 1. We shall return to these issues in the next section, where we highlight the economic consequences of the aforementioned departures.

3 Solution

In this section, we derive an explicit solution to Problem 1 first and then compare our trading strategy with a finite-horizon version of Gârleanu and Pedersen (2016).

3.1 Analytical solution to our problem

Consider the following Hamilton-Jacobi-Bellman (HJB) equation

$$\max_{\boldsymbol{\tau}\in\mathbf{R}^n}\left\{\frac{\partial V}{\partial t}+\mathcal{D}^{\boldsymbol{\tau}}V(t,\mathbf{x},y,\mathbf{f})\right\}=0, \tag{3.1}$$

where

$$\mathcal{D}^{\boldsymbol{\tau}}V(t,\mathbf{x},y,\mathbf{f})=(\frac{\partial V}{\partial \mathbf{x}})^{\top}\boldsymbol{\tau}+\frac{\partial V}{\partial y}(ry+\mathbf{x}^{\top}\mathbf{B}\mathbf{f}-\frac{1}{2}\boldsymbol{\tau}^{\top}\boldsymbol{\Lambda}\boldsymbol{\tau})+(\frac{\partial V}{\partial \mathbf{f}})^{\top}\boldsymbol{\mu}_f+\frac{1}{2}\frac{\partial^2 V}{\partial y^2}\mathbf{x}^{\top}\boldsymbol{\Sigma}_p\mathbf{x}+\frac{1}{2}\mathrm{tr}(\frac{\partial^2 V}{\partial \mathbf{f}^2}\boldsymbol{\Sigma}_f), \tag{3.2}$$

and $V(T,\mathbf{x},y,\mathbf{f})=-\frac{1}{\eta}e^{-\eta y}$ with $\mathrm{tr}(\cdot)$ representing the trace of a matrix. Theorems 1 and 2 summarize the main results of this paper, and show that the solution to Problem 1 admits a tractable representation[6]. Theorem 1 shows that the solution to the HJB equation (3.1) is expressed in terms of the solutions to the coupled system of PDEs. Theorem 2 verifies that the solution of the HJB equation (3.1) is indeed the solution to Problem 1.

To understand Theorem 1 below, we use the following notation as a reference [7] to the classical results of Merton (1969):

$$\mathbf{Merton}(t)\triangleq\frac{e^{-r(T-t)}}{\eta}\boldsymbol{\Sigma}_p^{-1}\mathbf{B}\mathbf{f}(t). \tag{3.3}$$

We shall hereafter refer to (3.3) as *the Merton portfolio.* The Merton portfolio is directly proportional to the expected excess return $\mathbf{B}\mathbf{f}(t)$ per unit of the variance of risky assets $\boldsymbol{\Sigma}_p$, multiplied by the reciprocal of the coefficient of the absolute risk aversion η.

[6] Our work can be extended to HARA utility functions and the HJB equation (3.1) remains the same only with a different terminal condition. However, to the best of our knowledge, no analytical solution is available in the case of HARA utility and a numerical scheme needs to be resorted to. We leave it as future work since the focus of this paper is on analytical solution.

[7] The notation in (3.3) represents the CARA agent's optimal portfolio in the absence of transaction costs. Indeed, the strategy in (3.3) is *myopic* in the sense that $\mathbf{Merton}(t)$ only depends on the current value of $\mathbf{f}(t)$. See Gârleanu and Pedersen (2013); Glasserman and Xu (2013); Gârleanu and Pedersen (2016) for the analogous notation for the case of mean-variance agent.

Now we come to our main results of this paper:

Theorem 1. (Main results)

(1) The solution to the HJB equation (3.1) admits an analytical representation:

$$V(t,\mathbf{x},y,\mathbf{f}) = -\frac{1}{\eta} e^{-h(t)y+\frac{1}{2}\mathbf{x}^\top \mathbf{A}_{xx}(t)\mathbf{x}-\mathbf{x}^\top \mathbf{A}_x(t,\mathbf{f})+\mathbf{A}(t,\mathbf{f})}, \tag{3.4}$$

where $h(t)=\eta e^{r(T-t)}$ and $\mathbf{A}_{xx}(t)$, $\mathbf{A}_x(t,\mathbf{f})$ and $\mathbf{A}(t,\mathbf{f})$ are the solutions of the coupled system of PDEs in (A.3–A.5). The optimal trading strategy of the CARA agent is of a feedback form as

$$\boldsymbol{\tau}^*(t) = \frac{1}{h(t)}\boldsymbol{\Lambda}^{-1}\left[\mathbf{A}_x(t,\mathbf{f}(t)) - \mathbf{A}_{xx}(t)\mathbf{x}(t)\right] = \bar{\mathbf{M}}^{\mathrm{rate}}(t)[\bar{\mathbf{M}}^{\mathrm{aim}}(t,\mathbf{f}(t)) - \mathbf{x}(t)], \tag{3.5}$$

where the tracking rate $\bar{\mathbf{M}}^{\mathrm{rate}}(t)$ and the aim portfolio $\bar{\mathbf{M}}^{\mathrm{aim}}(t,\mathbf{f}(t))$ are given as

$$\bar{\mathbf{M}}^{\mathrm{rate}}(t) \triangleq \frac{1}{h(t)}\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xx}(t), \quad \bar{\mathbf{M}}^{\mathrm{aim}}(t,\mathbf{f}(t)) \triangleq \mathbf{A}_{xx}^{-1}(t)\mathbf{A}_x(t,\mathbf{f}(t)). \tag{3.6}$$

(2) The aim portfolio $\bar{\mathbf{M}}^{\mathrm{aim}}(t,\mathbf{f})$ admits an intuitive representation

$$\bar{\mathbf{M}}^{\mathrm{aim}}(t,\mathbf{f}) = \mathbf{A}_{xx}^{-1}(t)\int_t^T e^{-\int_t^s q(u)du} l(s)\mathbf{E}[\mathbf{Merton}(s)|\mathbf{f}(t)=\mathbf{f}]ds, \tag{3.7}$$

where $q(t) \triangleq \frac{1}{h(t)}\mathbf{A}_{xx}(t)\boldsymbol{\Lambda}^{-1}$ and $l(t) \triangleq h^2(t)\boldsymbol{\Sigma}_p$.

(3) Under the following assumption

$$\boldsymbol{\mu}_f(t,\mathbf{f}(t)) = -\boldsymbol{\Phi}\mathbf{f}(t), \quad \boldsymbol{\sigma}_f(t,\mathbf{f}(t)) = \boldsymbol{\sigma}_f, \tag{3.8}$$

where $\boldsymbol{\Phi}$ is a $k\times k$ diagonal matrix with non-negative elements and $\boldsymbol{\sigma}_f$ is a $k\times k$ non-degenerate matrix, $\mathbf{f}(t)$ is a multi-variate Ornstein-Uhlenbeck (O-U) process. Then, the solution to the HJB equation (3.1) can be further simplified as follows:

$$V(t,\mathbf{x},y,\mathbf{f}) = -\frac{1}{\eta} e^{-h(t)y+\frac{1}{2}\mathbf{x}^\top \mathbf{A}_{xx}(t)\mathbf{x}-\mathbf{x}^\top \mathbf{A}_{xf}(t)\mathbf{f}+\frac{1}{2}\mathbf{f}^\top \mathbf{A}_{ff}(t)\mathbf{f}+A_0(t)}, \tag{3.9}$$

where $\mathbf{A}_{xx}(t)$, $\mathbf{A}_{xf}(t)$, $\mathbf{A}_{ff}(t)$, and $A_0(t)$ are the solutions of the coupled system of Riccati equations in (A.7). Moreover, the corresponding optimal trading strategy $\boldsymbol{\tau}^*$ is given as (3.5) with $\mathbf{A}_x(t, \mathbf{f}(t)) = \mathbf{A}_{xf}(t)\mathbf{f}(t)$.

Theorem 1 shows that the optimal trading strategy $\boldsymbol{\tau}^*(t)$ of the CARA agent in (3.5) is to trade *gradually* toward his aim portfolio $\bar{\mathbf{M}}^{\mathrm{aim}}(t, \mathbf{f}(t))$ in (3.6). To see this, note that the optimal trading strategy $\boldsymbol{\tau}^*(t)$ in (3.5) has the same sign, element-wise, as $\bar{\mathbf{M}}^{\mathrm{aim}}(t, \mathbf{f}(t)) - \mathbf{x}(t)$. Economically, the agent should purchase more of Asset d, for $1 \le d \le n$, at time t, when his position $x_d(t)$ is below his corresponding aim portfolio $\bar{\mathrm{M}}_d^{\mathrm{aim}}(t, \mathbf{f}(t))$, where $\bar{\mathrm{M}}_d^{\mathrm{aim}}(t, \mathbf{f}(t))$ denotes the d-th element in $\bar{\mathbf{M}}^{\mathrm{aim}}(t, \mathbf{f}(t))$. On the other hand, when the agent's current position $x_d(t)$ is above $\bar{\mathrm{M}}_d^{\mathrm{aim}}(t, \mathbf{f}(t))$, it indicates that the agent has over-invested in Asset d. In this case, he should gradually reduce his position of Asset d. To conclude, the aim portfolio guides the agent to rebalance his portfolio over time.

Secondly, the representation in (3.7) shows that the aim portfolio is a weighted sum of the expected future Merton portfolios, denoted in (3.3), with weights to account for return-predicting signals' persistence and transaction costs. This representation indicates that the aim portfolio tracks the evolution of the Merton portfolio, yet it cannot reach the Merton portfolio directly due to transaction costs. Indeed, rebalancing motive occurs when the return-predicting factor $\mathbf{f}(t)$ changes, but the level of transaction costs $\boldsymbol{\Lambda}$ inhibits trading in large positions. In this respect, the aim portfolio can be interpreted as a conservative form of the Merton portfolio that incorporates both return predictability and transaction costs. Together with the optimal trading strategy $\boldsymbol{\tau}^*$ in (3.5), the agent optimally trades toward his dynamic aim portfolio, which in turn follows the Merton portfolio conservatively.

It is essential to highlight that, under the assumption in (3.8), the optimal trading strategy of the CARA agent is influenced by both the mean-reversion rates and the volatilities of return-predicting factors. On the other hand, Gârleanu and Pedersen (2016) showed that the mean-reversion rates of return-predicting factors provide a sound financial interpretation on the optimal trading strategy of the mean-variance agent. Yet, the volatilities of return-predicting factors do not appear in their optimal trading rules in the GP model. In other words, even when the variances of return-predicting factors are high, the optimal trading strategy remains unchanged in the case of the mean-variance agent. However, for the CARA agent case, Theorem 1 indicates that the CARA agent's optimal portfolio are sensitive to both the mean-reversion rates and the volatilities of return-predicting factors.

According to Equation (2.4), the optimal trading strategy $\boldsymbol{\tau}^*$ (3.6) in Theorem 1 immediately yields an explicit representation of the optimal portfolio $\mathbf{x}^*(t)$.

Corollary 1. The optimal portfolio is given as

$$\mathbf{x}^*(t) = e^{\int_0^t q(s)ds}\left[\mathbf{x}_0 + \frac{1}{\eta}\int_0^t e^{-\int_0^s q(u)du} e^{-r(T-s)}\boldsymbol{\Lambda}^{-1}\mathbf{A}_x(s,\mathbf{f}(s))ds\right]. \tag{3.10}$$

Moreover, under the assumption in (3.8), optimal portfolio $\mathbf{x}^*$ can be expressed in terms of weighted sum of the past Merton portfolios, i.e.

$$\mathbf{x}^*(t) = e^{\int_0^t q(s)ds}\left[\mathbf{x}_0 + \int_0^t e^{-\int_0^s q(u)du} a(s)\mathbf{Merton}(s)ds\right], \tag{3.11}$$

where $a(t) = \boldsymbol{\Lambda}^{-1}\mathbf{A}_{xf}(t)\mathbf{B}^{+}\boldsymbol{\Sigma}_p$ with $\mathbf{B}^{+}$ being the Moore-Penrose inverse.

Next, Proposition 1 establishes the existence and uniqueness of the solutions to the coupled system of Riccati Equations (A.7) in Theorem 1. In addition, Proposition 1 also shows that, under the assumption in (3.8), the optimal trading strategy $\boldsymbol{\tau}^*$ (3.5) and the value function V (3.9) in Theorem 1 are well-defined.

Proposition 1. Let $\mathbf{A}:\mathbb{R}_+ \to \mathbb{R}^{(n+k)\times(n+k)}$ be the matrix-valued function satisfying the following differential matrix Riccati equation:

$$\dot{\mathbf{A}}(t) = \mathbf{D}(t) + \mathbf{A}(t)\mathbf{K} + \mathbf{K}^\top\mathbf{A}(t) + \mathbf{A}(t)\mathbf{N}(t)\mathbf{A}(t), \tag{3.12}$$

where

$$\mathbf{A}(t) = \begin{pmatrix} \mathbf{A}_{xx}(t) & \mathbf{A}_{xf}(t) \\ \mathbf{A}_{xf}^\top(t) & \mathbf{A}_{ff}(t) \end{pmatrix}, \mathbf{D}(t) = \begin{pmatrix} -h(t)^2\boldsymbol{\Sigma}_p & -h(t)\mathbf{B} \\ -h(t)\mathbf{B}^\top & \mathbf{0}_{k\times k} \end{pmatrix}, \tag{3.13}$$

$$\mathbf{K} = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times k} \\ \mathbf{0}_{k\times n} & \boldsymbol{\Phi} \end{pmatrix}, \mathbf{N}(t) = \begin{pmatrix} \frac{1}{h(t)}\boldsymbol{\Lambda}^{-1} & \mathbf{0}_{n\times k} \\ \mathbf{0}_{k\times n} & -\boldsymbol{\Sigma}_f \end{pmatrix}, \tag{3.14}$$

with terminal condition $\mathbf{A}(T) = \mathbf{0}_{(n+k)\times(n+k)}$ and $\mathbf{A}_{xx}(t)$, $\mathbf{A}_{xf}(t)$ and $\mathbf{A}_{ff}(t)$ are the solutions of the coupled system of Riccati equations in (A.7). Then,

(1) there exists a unique bounded solution $\mathbf{A}(t)$ on $[0,T]$. Moreover, $\mathbf{A}_{xx}(t)$ is positive definite and $\mathbf{A}_{ff}(t)$ is negative definite on $[0,T)$.

(2) $\mathbf{A}(t)$ has an explicit form:

$$\mathbf{A}(t) = \mathbf{R}_2(t)\mathbf{R}_1^{-1}(t), \tag{3.15}$$

where $\begin{pmatrix} \mathbf{R}_1(t) \\ \mathbf{R}_2(t) \end{pmatrix} = \mathbf{e}^{-\int_t^T \mathbf{L}(s)ds} \begin{pmatrix} \mathbf{I}_{n+k} \\ \mathbf{0}_{n+k} \end{pmatrix}$ with $\mathbf{L}(t) = \begin{pmatrix} -\mathbf{K}^\top & -\mathbf{N}(t) \\ \mathbf{D}(t) & \mathbf{K} \end{pmatrix}$.

Note that the differential matrix Riccati equation in (3.12) differs from the algebraic Riccati equation in Gârleanu and Pedersen (2016), which results from the fact that Problem 1 is a utility maximization problem in finite time. As we shall see in Section 4, the differential matrix Riccati equation introduces the time-dependent effect on the agent's optimal trading strategy.

Finally, Theorem 2 verifies that the value function is indeed the solution to Problem 1.

Theorem 2. (Verification theorem)

(1) Let $\mathbf{x}^*(t)$ and $y^*(t)$ to be the optimal portfolio and wealth processes in (2.4) and (2.8), respectively, corresponding to the optimal trading strategy $\boldsymbol{\tau}^*$ (3.5). If the family

$$\mathfrak{V} \triangleq \{V(\mathfrak{s}, \mathbf{x}^*(\mathfrak{s}), y^*(\mathfrak{s}), \mathbf{f}(\mathfrak{s})) : \mathfrak{s} \text{ is stopping time with values in } [0,T]\} \tag{3.16}$$

is uniformly integrable, and value function V satisfies the HJB equation (3.1), then

$$V(0,\mathbf{x},y,\mathbf{f}) = \max_{\boldsymbol{\tau}\in\mathcal{A}} \mathbf{E}\left[-\frac{1}{\eta}e^{-\eta y(T)}\Big|\mathbf{x}(0)=\mathbf{x}, y(0)=y, \mathbf{f}(0)=\mathbf{f}\right], \tag{3.17}$$

and the corresponding optimal trading strategy $\boldsymbol{\tau}^*$ in (3.5) is the solution to Problem 1.

(2) Under the assumption in (3.8), if C is a positive constant satisfying

$$4CT \max_{0\le s\le T} [\mathcal{R}(\boldsymbol{\Sigma}_H^{\frac{1}{2}}(s))]^2 < 1, \tag{3.18}$$

where $\mathcal{R}(\mathbf{M})$ denotes the spectral radius of positive-definite matrix $\mathbf{M}$, $\mathbf{M}^{\frac{1}{2}}$ denotes the square root of $\mathbf{M}$, and $\boldsymbol{\Sigma}_H(t) \triangleq \int_0^t e^{\boldsymbol{\Phi}(s-t)}\boldsymbol{\Sigma}_f e^{\boldsymbol{\Phi}^\top(s-t)}ds$, and that the value function V satisfies the HJB equation (3.1), then (3.17) holds, and the corresponding optimal trading strategy $\boldsymbol{\tau}^*$ in (3.5) is the solution to Problem 1.

3.2 Comparisons with Gârleanu and Pedersen (2016)

To effectively highlight the distinctions of our work from Gârleanu and Pedersen (2016), we first provide the finite-time version of the Gârleanu and Pedersen (2016)'s model in Problem 2 below.

Problem 2. An agent seeks to maximize an objective functional, over a finite horizon, as

$$\max_{\boldsymbol{\tau}\in\mathcal{A}} \mathbf{E}\int_t^T e^{-\rho(s-t)}\left[\mathbf{x}(s)^\top\mathbf{B}\mathbf{f}(s)-\frac{1}{2}\boldsymbol{\tau}(s)^\top\boldsymbol{\Lambda}\boldsymbol{\tau}(s)-\frac{\gamma}{2}\mathbf{x}(s)^\top\boldsymbol{\Sigma}_p\mathbf{x}(s)\right]ds. \tag{3.19}$$

Similar to the derivation in last section, the value function $V^G(t,\mathbf{x},\mathbf{f})$ of Problem 2 satisfies the following HJB equation

$$\max_{\boldsymbol{\tau}\in\mathbf{R}^n}\left\{\frac{\partial V^G}{\partial t}+\mathbf{x}^\top\mathbf{B}\mathbf{f}-\frac{1}{2}\boldsymbol{\tau}^\top\boldsymbol{\Lambda}\boldsymbol{\tau}-\frac{\gamma}{2}\mathbf{x}^\top\boldsymbol{\Sigma}_p\mathbf{x}+\mathcal{D}_f^{\boldsymbol{\tau}}V^G\right\}=0, \tag{3.20}$$

where

$$\mathcal{D}_G^{\boldsymbol{\tau}}V^G(t,\mathbf{x},\mathbf{f})=(\frac{\partial V^G}{\partial \mathbf{x}})^\top\boldsymbol{\tau}+(\frac{\partial V^G}{\partial \mathbf{f}})^\top\boldsymbol{\mu}_f+\frac{1}{2}\mathrm{tr}(\frac{\partial^2 V^G}{\partial \mathbf{f}^2}\boldsymbol{\Sigma}_f)-\rho V^G, \tag{3.21}$$

and $V^G(t,\mathbf{x},\mathbf{f})=0$. The solution of Problem 2 is derived in the following theorem:

Theorem 3. Under the assumption (3.8), the solution to the HJB equation (3.20) is found to be

$$V^G(t,\mathbf{x},\mathbf{f})=-\frac{1}{2}\mathbf{x}^\top\mathbf{A}_{xx}^G(t)\mathbf{x}+\mathbf{x}^\top\mathbf{A}_{xf}^G(t)\mathbf{f}+\frac{1}{2}\mathbf{f}^\top\mathbf{A}_{ff}^G(t)\mathbf{f}+A_0^G(t), \tag{3.22}$$

where $\mathbf{A}_{xx}^G(t)$, $\mathbf{A}_{xf}^G(t)$, $\mathbf{A}_{ff}^G(t)$, and $A_0^G(t)$ are the solutions of the coupled system of Riccati equations in (A.28). The optimal trading strategy for Problem 2 is also of a feedback form as

$$\boldsymbol{\tau}_G^*(t)=\boldsymbol{\Lambda}^{-1}\left[\mathbf{A}_{xf}^G(t)\mathbf{f}(t)-\mathbf{A}_{xx}^G(t)\mathbf{x}(t)\right]. \tag{3.23}$$

Obviously, when the investment horizon T approaches infinity, Problem 2 degenerates to the original problem in Gârleanu and Pedersen (2016). As T tends to infinity, the disappearance of time derivative in (A.28) results in an algebraic Riccati system, which explains the stationarity of the optimal investment strategies in Gârleanu and Pedersen (2016). A comparison between the original problem in Gârleanu and Pedersen (2016) and Problem 2 shows that the time-dependent effect for the trading strategy results from the finite investment horizon.

A more fundamental distinction between our work and the GP model can be readily seen by comparing Problems 1 and 2, which illustrates how the choice of utility function affects the trading strategy. Although the optimal trading strategies in (3.5) and (3.23) for Problems 1 and 2, respectively, are of a similar feedback form, an essential difference lies in coefficient functions. By comparing the systems (A.7) and (A.28), the pair (A_{xx}, A_{xf}) depends on the value of $\mathbf{\Sigma}_f$, the volatility of the return-predicting factor, whereas the pair (A^G_{xx}, A^G_{xf}) is independent of $\mathbf{\Sigma}_f$. Financially, it means that the optimal trading strategy of the CARA agent in our setting is affected by the value of $\mathbf{\Sigma}_f$, while the optimal trading strategy of the agent in Gârleanu and Pedersen (2016) is independent of $\mathbf{\Sigma}_f$. In other words, the CARA agent exhibits risk aversion to the uncertainty of the return-predicting factor, while the mean-variance agent is risk-neutral to such an uncertainty (Collin-Dufresne et al., 2019). Such a dependence effect of optimal strategy on the volatility of return-predicting factor is a key contribution to the existing literature.

Mathematically, solving the pair (A_{xx}, A_{xf}, A_{ff}) in (A.7) is more difficult than solving the pair $(A^G_{xx}, A^G_{xf}, A^G_{ff})$ in (A.28). Indeed, a direct observation of (A.28) reveals that A^G_{xx} is decoupled from (A^G_{xf}, A^G_{ff}) and can therefore be solved separately. On the other hand, (A_{xx}, A_{xf}, A_{ff}) in (A.7) is a coupled system and must be solved simultaneously. In this respect, another contribution of this paper is to prove rigorously that (A_{xx}, A_{xf}, A_{ff}) in (A.7) also admits a closed-form solution.

4 Numerical analysis

In this section, we provide two representative examples to illustrate our main results in Section 3. Unless stated otherwise, the assumption in (3.8) is in force for the remaining section. Moreover, we shall use the terms *factor* and *signal* interchangeably. Example 1 studies a case with one risky asset and one return-predicting factor. This simple example allows us to delineate effectively the time-dependent impact of transaction costs and return-predicting factor's persistence on the optimal trading strategy. As portfolios typically consist of more than one risky asset, it is imperative to study the impact of signals' persistence and transaction costs on agent's optimal portfolio with multiple risky assets. Without loss of generality, it suffices to study the corresponding portfolio selection problem with two risky assets.

4.1 Example 1: A single risky asset

Consider the market with one risk-free asset and one risky asset. Asset return is driven by one return-predicting factor. Figure 1 displays the Merton portfolio in (3.3), the aim portfolio $\bar{\mathrm{M}}^{\mathrm{aim}}(t, f(t))$ in (3.6), and the optimal portfolio $x^*(t)$ in (3.11) over one simulated path of return-predicting factor f on $[0, T]$. Unless specified otherwise, Example 1 is based on the following set of parameters[8]:

$$\Phi = 1, \Sigma_p = 0.04, \Sigma_f = 0.16, f_0 = 0.1, r = 0.05, \eta = 1, B = 1, T = 1, x_0 = 0, \Lambda = 0.001. \qquad (4.1)$$

Figure 1 shows that both the Merton portfolio and the aim portfolio follow the same trend with non-smooth paths. The reason is that both of them are driven by the same return-predicting factor, albeit with different coefficients, as shown in (3.3) and (3.6). Since the return-predicting factor follows an O-U process, both the Merton portfolio and the aim portfolio follow the Brownian motion W_f in (2.3). Moreover, Figure 1 also indicates that the aim portfolio is more conservative than the Merton portfolio. As discussed in Section 3, the Merton portfolio depends only on the current position of the return-predicting factor and includes no transaction cost. On the other hand, the aim portfolio incorporates both transaction costs and the agent's anticipation that the return-predicting factor will mean-revert in the future. These effects are consistent with the case for the mean-variance agent in Gârleanu and Pedersen (2016).

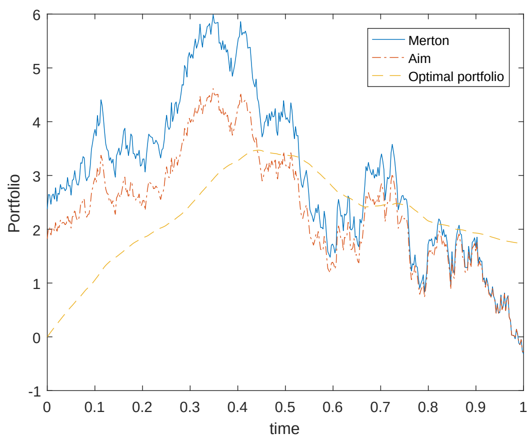


Figure 1: A simulated path for the Merton portfolio, the aim portfolio and the optimal portfolio.

[8]To avoid any notational confusion, we reserve the boldfaced symbols for vectors and matrices. Hence, non-boldfaced symbols represent scalars.

As the remaining time $T-t$ approaches zero, Figure 1 also shows that the difference between the Merton portfolio and the aim portfolio dissipates gradually. Although the aim portfolio incorporates the agent's anticipation of return-predicting factor, such anticipation becomes less influential as time approaches the terminal date. It should be pointed out that this time-dependent effect is absent in the GP model as the coefficients of their aim portfolio are constant.

Different from the Merton portfolio and the aim portfolio, the optimal portfolio in Figure 1 is smooth, as shown in Corollary 1. More importantly, Figure 1 shows that the optimal portfolio is increasing when it is below the aim portfolio, while it is decreasing when it is above the aim portfolio. This is consistent with our discussions following Theorem 1. Indeed, Figure 1 shows that the agent would optimally *decrease* his asset holding when his current optimal portfolio is above the aim portfolio, even when the current excess return of the risky asset, Bf, is positive. Although the current excess return of the risky asset is positive, the agent may optimally reduce his portfolio holding if his aim portfolio indicates that the return-predicting factor would mean-revert in the future. Due to transaction costs, the agent should gradually reduce his current portfolio.

To study the impact of transaction costs on the optimal portfolio, we now turn to Figure 2. Recall that the parameter Λ represents the level of transaction costs. Figure 2 shows the optimal portfolio with respect to different values of Λ, while other parameters in (4.1) remain unchanged. When Λ is small, the level of transaction costs for trading is low. Hence, the agent would rebalance his portfolio more aggressively so that his optimal portfolio would follow more closely to the Merton portfolio. As the value of Λ increases, high level transaction cost prevents agent from trading in large positions. For example, when $\Lambda = 0.1$, we see that agent's optimal portfolio becomes rather sluggish with respect to the Merton portfolio, with almost no change in portfolio across time.

To effectively compare the Merton portfolio and the aim portfolio, we apply Theorem 1 to the ratio $\frac{\bar{\mathrm{M}}^{\mathrm{aim}}(t,f(t))}{\mathrm{Merton}(t,f(t))}$ and obtain

$$\frac{\bar{\mathrm{M}}^{\mathrm{aim}}(t,f(t))}{\mathrm{Merton}(t,f(t))} = \begin{cases} \frac{\Sigma_p h(t)}{B}\frac{A_{xf}(t)}{A_{xx}(t)}, & \text{for} \quad t\in[0,T), \\ 1, \quad \text{for} \quad t=T. \end{cases} \tag{4.2}$$

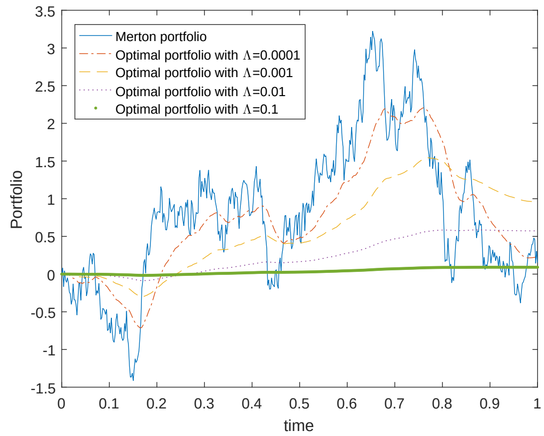


Figure 2: A simulated path for the Merton portfolio and optimal portfolio with different levels of transaction costs.

As shown in (4.2), such a ratio only depends on time t. For this reason, we hereafter denote it as Ratio(t). In addition, we find that the agent's aim portfolio converges to the Merton portfolio as time approaches the terminal date, which also applies to the multi-asset case. The asymptotic analysis between the aim portfolio and the Merton portfolio is left in Appendix A.5.

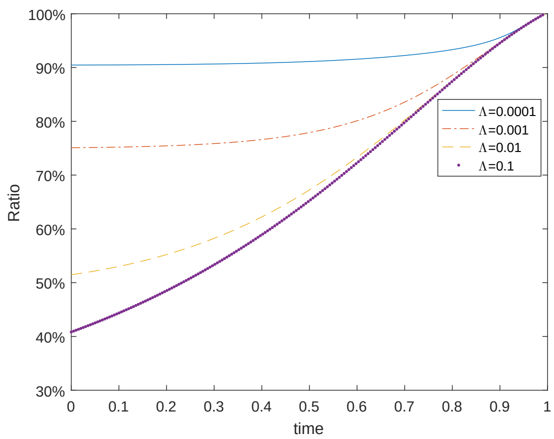


Figure 3: The evolution of Ratio(t) with different levels of transaction costs.

We first study the impact of transaction costs on the aim portfolio in Figure 3. From Figure 3, Ratio(t) is less than one for all $t \in [0, T]$ and converges to one as time approaches the terminal date, which is consistent with Figure 1 that the aim portfolio is more conservative than the Merton portfolio. Analogous to Figure 2, the higher the level of transaction costs, the more conservative the aim portfolio becomes, and the further the aim portfolio deviates from the Merton portfolio.

To analyze the impact of the signal's persistence on the aim portfolio, we now turn to Figures 4 and 5. Figure 4 studies the impact of the mean-reversion rate of the return-predicting factor on the aim portfolio. In Figure 4, the level of Ratio(t) decreases as the mean-reversion rate

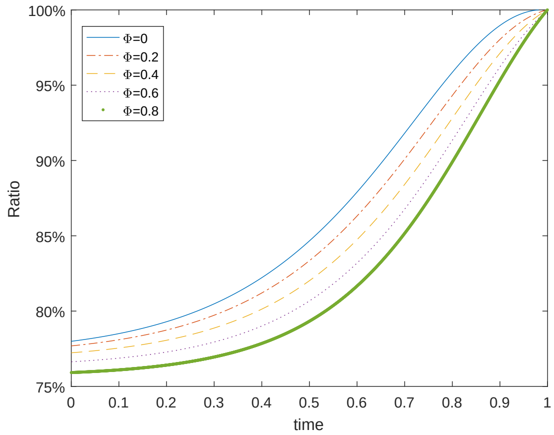


Figure 4: The evolution of Ratio(t) with different values of the mean-reversion rate Φ.

increases, *ceteris paribus*. A large mean-reversion rate indicates that the factor would return to zero quickly, implying that the factor is not persistent. In other words, even though the current level of return-predicting signal is high, such a level is not sustainable. Consequently, in the presence of transaction costs, the aim portfolio informs the agent to gradually decrease his holding of the risky asset.

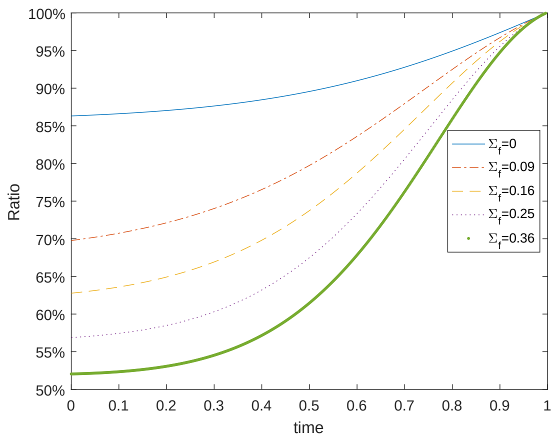


Figure 5: The evolution of Ratio(t) with different values of the volatility of f.

Figure 5 studies the impact of the volatility of return-predicting signal to the aim portfolio. Similar to the case of mean-reversion rate in Figure 4, the level of Ratio(t) decreases as the value of Σ_f increases. A large value of Σ_f indicates high variance of the return-predicting factor, implying that the signal is not persistent. In this case, the aim portfolio would become more conservative in comparison with the Merton portfolio even when the current level of the signal is high. When Σ_f degenerates to zero, the return-predicting factor becomes a deterministic process. In this case, future demand of rebalancing is deterministic, and hence there is only a small difference between the Merton portfolio and the aim portfolio when the agent rebalances his portfolio deterministically.

It is essential to highlight that the ratio between the aim portfolio of a mean-variance agent and the corresponding Markowitz portfolio in Gârleanu and Pedersen (2016) is *not* influenced by the volatility of the return-predicting factor. Mathematically, this results from the fact that Σ_f does not enter into equations A_{xx}, A_{xf}, and A_{ff} in the GP model. Consequently, the optimal portfolio of the mean-variance agent incorporates the mean-reversion rate, but not the volatility, of the return-predicting signal.

4.2 Example 2: Two risky assets

In this section, we consider the case of two risky assets, each of which has its own return-predicting factor f_i for $i = 1, 2$. Unless stated otherwise, the parameters in this section are set to be

$$\mathbf{\Sigma}_p = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.1 \end{pmatrix}, \mathbf{f}_0 = (1,1)^\top, r = 0.05, \eta = 1, \mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, T = 1. \tag{4.3}$$

Remaining parameters, namely the level of transaction costs $\mathbf{\Lambda}$, the mean-reversion rate $\mathbf{\Phi}$ and volatility $\mathbf{\Sigma}_f$ of return-predicting factor $\mathbf{f}$, and initial portfolio position $\mathbf{x}_0$, are specified in Figures 6, 7, and 8 below.

Instead of displaying the evolutions of the Merton portfolio, the aim portfolio, the optimal trading portfolio, and the optimal strategy with respect to one simulated path of factor, we shall display their corresponding *expected* values, defined as follows:

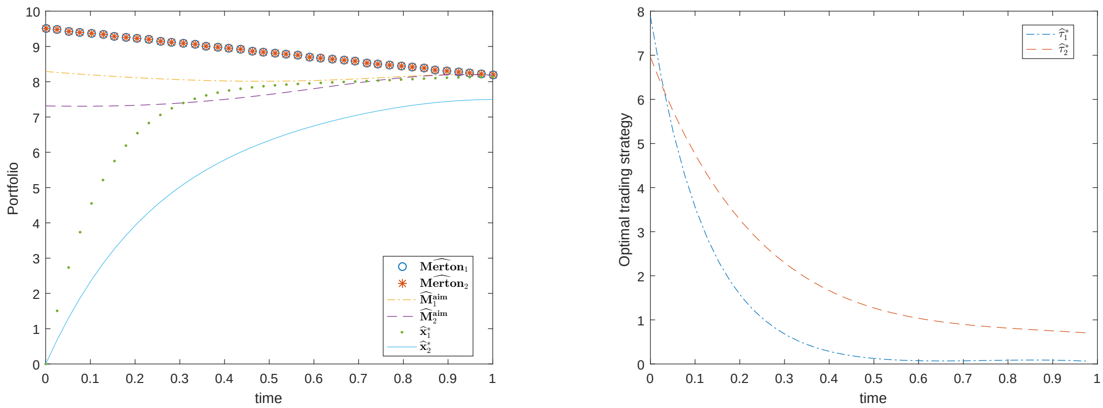
$$\widehat{\mathbf{Merton}}(t) \triangleq \mathbf{E}\left[\mathbf{Merton}(t)|\mathbf{f}(0) = \mathbf{f}_0\right], \qquad \widehat{\boldsymbol{\tau}}^*(t) \triangleq \mathbf{E}\left[\boldsymbol{\tau}^*(t)|\mathbf{f}(0) = \mathbf{f}_0\right], \tag{4.4}$$

$$\widehat{\mathbf{M}}^{\text{aim}}(t) \triangleq \mathbf{E}\left[\bar{\mathbf{M}}^{\text{aim}}(t, \mathbf{f}(t))|\mathbf{f}(0) = \mathbf{f}_0\right], \qquad \widehat{\mathbf{x}}^*(t) \triangleq \mathbf{E}\left[\mathbf{x}^*(t)|\mathbf{f}(0) = \mathbf{f}_0\right]. \tag{4.5}$$

Figure 6 explores the impact of transaction costs on the expected optimal trading strategy $\widehat{\boldsymbol{\tau}}^* = (\widehat{\tau}_1^*, \widehat{\tau}_2^*)$ and the optimal portfolio position $\widehat{\mathbf{x}}^* = (\widehat{x}_1^*, \widehat{x}_2^*)$ with respect to the parameters in (4.3) and

$$\Phi = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \boldsymbol{\Sigma}_f = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} 0.02 & 0 \\ 0 & 0.08 \end{pmatrix}, \mathbf{x}_0 = (0,0)^\top. \tag{4.6}$$

From the parameters (4.6), each element of $\widehat{\mathbf{Merton}}$ is equal, i.e. $\widehat{\mathrm{Merton}}_1 = \widehat{\mathrm{Merton}}_2$. The only difference between Assets 1 and 2 lies in their respective levels of transaction cost $\boldsymbol{\Lambda}$. In other words, trading Asset 2 is four times as costly as trading Asset 1. Figure 6(a) displays the expected Merton portfolio, the expected aim portfolio, and the expected optimal portfolio for each asset, whereas Figure 6(b) displays the corresponding expected optimal trading strategy $\widehat{\tau}_i^*$ with respect to time. In Figure 6(a), the expected aim portfolio on Asset 2 lies below that of Asset 1 due to higher transaction cost in Asset 2. Consequently, the expected aim portfolio in Asset 2 is more conservative than that in Asset 1.



(a) (b)

Figure 6: The expected Merton portfolio, the expected aim portfolio, and the expected optimal portfolio (left); the expected optimal trading strategy (right).

Figure 6(a) shows that the expected optimal portfolio in Asset 1 reaches its expected aim portfolio faster than that in Asset 2. In Figure 6(b), the trading strategy for Asset 1 decreases faster than that for Asset 2. Since the agent reaches his expected aim portfolio in Asset 1 earlier, he would then optimally slow down his investment in Asset 1 across time. It is worthy to note that the expected optimal trading strategy in Asset 2 does not decay to zero as time approaches terminal date, which implies that the expected optimal portfolio has not reached its expected aim portfolio in ultimatum due to large transaction costs.

In Figures 7 and 8, we examine the persistence effects of return-predicting factor on optimal allocation between Assets 1 and 2. The horizontal and vertical axes in Figures 7 and 8 correspond to the allocations in Assets 1 and 2, respectively. The expected Merton portfolio, the expected aim portfolio, and the expected optimal portfolio evolve in the directions of arrows shown in the Figures 7 and 8.

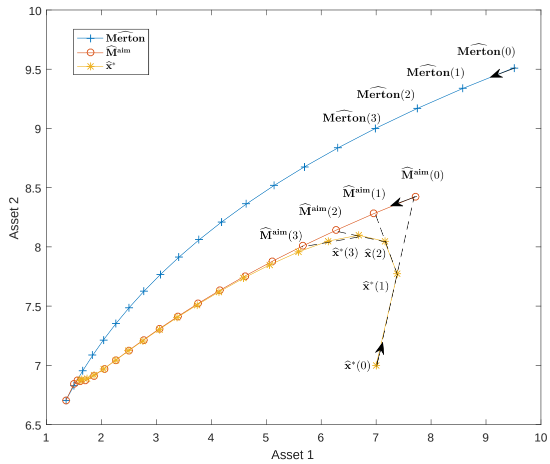


Figure 7: The expected portfolios for two assets with different mean-reversion speed.

Figure 7 compares the expected optimal portfolio against the expected Merton and the expected aim portfolio with respect to the parameters in (4.3) and

$$\mathbf{\Phi} = \begin{pmatrix} 2 & 0 \\ 0 & 0.4 \end{pmatrix}, \mathbf{\Sigma}_f = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix}, \mathbf{\Lambda} = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}, \mathbf{x}_0 = (7,7)^\top. \tag{4.7}$$

From the parameter specifications in (4.7), the only difference between return-predicting factors in Figure 7 lies in their mean-reversion rates.

Figure 7 shows that the expected Merton, the expected aim portfolio, and the expected optimal portfolio generally decay in time. Indeed, the expected Merton portfolio can be further simplified as

$$\widehat{\mathbf{Merton}}(t) \quad = \quad \frac{e^{-r(T-t)}}{\eta}\mathbf{\Sigma}_p^{-1}\mathbf{B}\mathbf{E}\left[\mathbf{f}(t)|\mathbf{f}(0)=\mathbf{f}_0\right] = \frac{e^{-r(T-t)}}{\eta}\mathbf{\Sigma}_p^{-1}\mathbf{B}e^{-\mathbf{\Phi}t}\mathbf{f}_0, \tag{4.8}$$

which shows that the expected Merton portfolio decreases in time. Simultaneously, the expected aim portfolio follows the expected Merton portfolio and therefore decreases gradually over time. Finally, since the expected optimal trading strategy follows the expected aim portfolio, the expected optimal portfolio also decays in time.

In addition, Figure 7 indicates that the expected Merton and the expected aim portfolio are *concave* with respect to the allocation in Asset 1. Based on the parameter setting in (4.7), return-predicting signal for Asset 1 is less persistent than that for Asset 2. Consequently, the expected aim portfolio informs the agent to sell more Asset 1 than Asset 2 per unit time. Same reasoning applies to explain the nature of the concavity in the expected Merton portfolio.

The expected optimal portfolio moves in the direction of the expected aim portfolio, as shown in Figure 7. Heuristically, we can deduce from the optimal trading strategy $\boldsymbol{\tau}^*$ in (3.5) that, for $[t, t+dt]$,

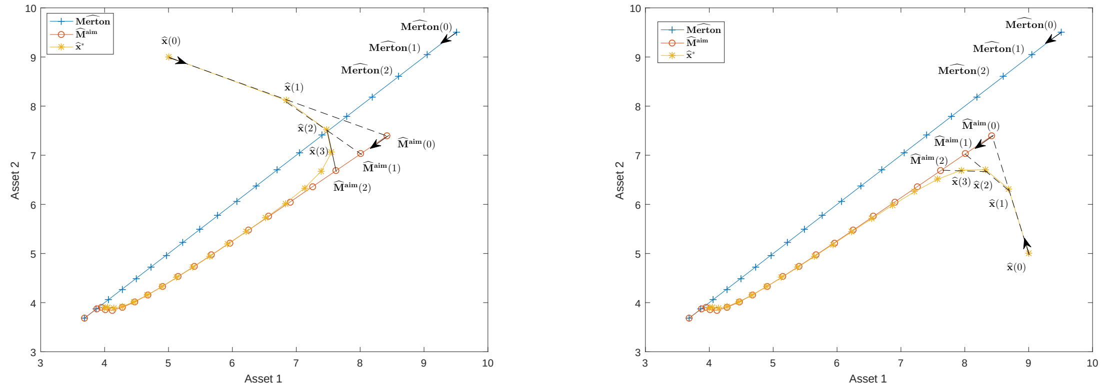
$$\begin{aligned}\mathbf{x}^*(t+dt) = \mathbf{x}^*(t) + \boldsymbol{\tau}^*(t)dt &= \mathbf{x}^*(t) + \bar{\mathbf{M}}^{\text{rate}}(t)\left(\bar{\mathbf{M}}^{\text{aim}}(t, \mathbf{f}(t)) - \mathbf{x}^*(t)\right) dt \\ &= \left(\mathbf{I} - \bar{\mathbf{M}}^{\text{rate}}(t)dt\right)\mathbf{x}^*(t) + \bar{\mathbf{M}}^{\text{rate}}(t)\bar{\mathbf{M}}^{\text{aim}}(t, \mathbf{f}(t))dt. \qquad (4.9)\end{aligned}$$

That is, for an infinitesimal time dt, the dynamics of the optimal portfolio $\mathbf{x}^*(t+dt)$ can be expressed as the linear combination of the current optimal position $\mathbf{x}^*(t)$ and the current aim portfolio $\bar{\mathbf{M}}^{\text{aim}}$. Since $\mathbf{x}(0) = (7,7)^\top$, i.e. the initial positions in Assets 1 and 2 are the same, the evolutions of the dash lines in Figure 7 would result when taking expectation for (4.9) with respect to current time $t = 0$. For example, the expected optimal portfolio on unit time Δt, denoted by $\hat{\mathbf{x}}^*(1)$, is a linear combination of current portfolio $\widehat{\mathbf{x}}^*(0)$ and current aim portfolio $\widehat{\mathbf{M}}^{\text{aim}}(0)$, as indicated by the dash line in Figure 7.

Figure 8 studies the impact of the factor's volatility, $\boldsymbol{\Sigma}_f$, on the expected Merton portfolio, the expected aim portfolio, and the expected optimal portfolio with respect to the parameters in (4.3) and

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \boldsymbol{\Sigma}_f = \begin{pmatrix} 0.1 & 0 \\ 0 & 0.8 \end{pmatrix}, \boldsymbol{\Lambda} = \begin{pmatrix} 0.001 & 0 \\ 0 & 0.001 \end{pmatrix}. \qquad (4.10)$$

In this case, the difference for Assets 1 and 2 lies in the volatilities of their respective return-predicting factors, $\boldsymbol{\Sigma}_f$.



(a) $\mathbf{x}_0 = (5, 9)^\top$ (b) $\mathbf{x}_0 = (9, 5)^\top$

Figure 8: The expected Merton portfolio, the expected aim portfolio, and the expected optimal portfolio with different values of $\mathbf{x}_0$.

From Figure 8, the expected Merton and the expected aim portfolio are independent of initial position $\mathbf{x}_0$. Moreover, the expected Merton portfolio lies on the 45° line, which can be explained in (4.8). On the other hand, the expected aim portfolio is below the expected Merton portfolio, indicating that the expected aim portfolio is convex with respect to the allocation in Asset 1. The expected aim portfolio is influenced by return-predicting factor's mean-reversion rate and volatility. For a given mean-reversion rate, signal's persistence decreases as its volatility increases. From parameter setting in (4.10), we know that the return-predicting signal for Asset 1 is more persistent than that for Asset 2. Therefore, the expected aim portfolio would decrease its holdings in Asset 2 faster than it would in Asset 1 per unit time, explaining the convexity in Figure 8.

Finally, we illustrate the relationship between the initial position $\mathbf{x}_0$ and the expected path of agent's optimal portfolio in Figures 8(a) and 8(b). In Figure 8(a), the current position $\widehat{\mathbf{x}}^*(0)$ indicates that the agent has under-invested in Asset 1 and over-invested in Asset 2, compared with the current aim position $\widehat{\mathbf{M}}^{\text{aim}}(0)$. Hence, the agent needs to rebalance his portfolio to $\widehat{\mathbf{x}}^*(1)$, with its exact value to be a linear combination between the current position $\widehat{\mathbf{x}}^*(0)$ and the current aim position $\widehat{\mathbf{M}}^{\text{aim}}(0)$ shown in (4.9). Following the same line of reasoning, we can deduce the directional movement of the dash lines in Figure 8(a). On the other hand, in Figure 8(b), the current position $\widehat{\mathbf{x}}^*(0)$ implies that the agent has over-invested in Asset 1 and under-invested in Asset 2. Therefore, he needs to rebalance his portfolio to $\widehat{\mathbf{x}}^*(1)$ based on Equation (4.9).

5 Conclusions

This paper studies the continuous-time optimal portfolio selection problem of a CARA agent with return predictability and transaction costs in finite time. We assume that asset returns are driven by signals that follow a general stochastic process, and that transaction costs are of quadratic form. We derive a closed-form solution to the finite-time utility maximization problem for the CARA agent, whereas Gârleanu and Pedersen (2016) solved the corresponding problem for an infinitely-lived mean-variance agent.

Our closed-form optimal trading strategy indicates that the agent should trade gradually toward a dynamic aim portfolio, which is expressed as a weighted sum of the expected future Merton portfolios with weights to account for return-predicting signals' persistence and transaction costs. The financial interpretation of our results is that return predictability induces the agent to rebalance his portfolio aggressively. However, the presence of transaction costs prohibits the agent from making large trades. The agent should gradually increase (decrease) his allocation when his current position is below (above) the aim portfolio. Another key consequence of our optimal trading strategy depends on the volatility of the return-predicting factor volatility, while such an effect is absent in the finite horizon problem of Gârleanu and Pedersen (2016).

Our numerical analysis also offers new insights to the existing literature. First, our solution exhibits a time-dependent effect and the agent's aim portfolio converges to the Merton portfolio as the terminal date approaches. Secondly, the agent rebalances his portfolio toward the assets with less transaction costs. Third, the agent invests more into the assets with return-predicting signals exhibiting higher persistence, i.e. slower mean-reversion rates and/or lower volatilities.

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Appendix A Proofs

A.1 Proof of Theorem 1

(1) The first-order condition on *Ansatz* in (3.9) with respect to $\boldsymbol{\tau}$ yield (3.5). The second-order condition can be checked easily to prove the optimality of $\boldsymbol{\tau}^*$. Computing the HJB equation (3.1) in terms of the *Ansatz*, we have

$$\begin{aligned}
0 = & \frac{\partial h(t)}{\partial t} y + \frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial t} - \mathbf{x}^\top \frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial t} + \frac{1}{2}\mathbf{x}^\top \frac{\partial \mathbf{A}_{xx}}{\partial t}\mathbf{x} - h(t)(ry + \mathbf{x}^\top \mathbf{B}\mathbf{f}) \\
& + \frac{1}{2}h^2(t)\mathbf{x}^\top \boldsymbol{\Sigma}_p \mathbf{x} - \frac{1}{2h(t)}[\mathbf{A}_{xx}(t)\mathbf{x} - \mathbf{A}_x(t,\mathbf{f})]^\top \boldsymbol{\Lambda}^{-1}[\mathbf{A}_{xx}(t)\mathbf{x} - \mathbf{A}_x(t,\mathbf{f})] \\
& + \left[\frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}} - \frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}}\mathbf{x}\right]^\top \mu_f + \frac{1}{2}\mathrm{tr}\Bigg(\left[\frac{\partial^2 \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}^2} - \frac{\partial^2 \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}^2}\mathbf{x}\right]\boldsymbol{\Sigma}_f \\
& + \left[\frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}} - \frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}}\mathbf{x}\right]\left[\frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}} - \frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}}\mathbf{x}\right]^\top \boldsymbol{\Sigma}_f \Bigg).
\end{aligned} \tag{A.1}$$

Matching the coefficients for $(\cdot)y$, $\mathbf{x}^\top(\cdot)\mathbf{x}$, $\mathbf{x}^\top(\cdot)$ and the constant yields an ordinary differential equation (ODE)

$$\frac{\partial h(t)}{\partial t} + rh(t) = 0, \tag{A.2}$$

with $h(T) = \eta$ and a coupled system of PDEs:

$$\frac{\partial \mathbf{A}_{xx}(t)}{\partial t} = -h^2(t)\boldsymbol{\Sigma}_p + \frac{1}{h(t)}\mathbf{A}_{xx}(t)\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xx}(t) - (\frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}})^\top \boldsymbol{\Sigma}_f(t,\mathbf{f})\frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}}, \tag{A.3}$$

$$\begin{aligned}
\frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial t} \quad = \quad & -h(t)\mathbf{B}\mathbf{f} + \frac{1}{h(t)}\mathbf{A}_{xx}(t)\boldsymbol{\Lambda}^{-1}\mathbf{A}_x(t,\mathbf{f}) - (\frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}})^\top \mu_f - \frac{1}{2}\frac{\partial^2 \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}^2} \cdot \boldsymbol{\Sigma}_f(t,\mathbf{f}) \\
- \quad & (\frac{\partial \mathbf{A}_x(t,\mathbf{f})}{\partial \mathbf{f}})^\top \boldsymbol{\Sigma}_f(t,\mathbf{f})\frac{\mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}},
\end{aligned} \tag{A.4}$$

where $\mathbf{Q} \cdot \mathbf{M}$ denotes the product of an $n \times k \times k$ tensor $\mathbf{Q}$ and a $k \times k$ matrix $\mathbf{M}$ with definition $(\mathbf{Q} \cdot \mathbf{M})^i \triangleq \sum_{j_1,j_2} \mathbf{Q}^i_{j_1,j_2}\mathbf{M}_{j_1,j_2}$, $i = 1, 2, \cdots, n$, and

$$\frac{\partial \mathbf{A}(t,f)}{\partial t} = \frac{1}{2h(t)}\mathbf{A}_x^\top(t,\mathbf{f})\boldsymbol{\Lambda}^{-1}\mathbf{A}_x(t,\mathbf{f}) - (\frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}})^\top \mu_f - \frac{1}{2}\mathrm{tr}\left(\left[\frac{\partial^2 \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}^2} + \frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}}\left(\frac{\partial \mathbf{A}(t,\mathbf{f})}{\partial \mathbf{f}}\right)^\top\right]\boldsymbol{\Sigma}_f\right), \tag{A.5}$$

with terminal conditions $\mathbf{A}_{xx}(T) = \mathbf{0}_{n\times n}$, $\mathbf{A}_x(T,\mathbf{f}) = \mathbf{0}_{n\times 1}$ and $\mathbf{A}(T,\mathbf{f}) = 0$.

(2) Applying Feynman-Kac formula, we interpret $\mathbf{A}_x(t,\mathbf{f})$ as the following representation

$$\mathbf{A}_x(t,\mathbf{f}) = \mathbf{E}\left[\int_t^T e^{-\int_t^s \frac{\mathbf{A}_{xx}(s)\mathbf{\Lambda}^{-1}}{h(t)}ds} h(t)\mathbf{B}\mathbf{f}(s)ds|\mathbf{f}(t) = \mathbf{f}\right]. \tag{A.6}$$

(3) Under the assumption in (3.8), $\mu_f(t,\mathbf{f}) = -\mathbf{\Phi}\mathbf{f}$, and hence $\mathbf{A}_x(t,\mathbf{f}) = \mathbf{A}_{xf}(t)\mathbf{f}(t)$. Similar to the proof of Theorem 1 (1), we yield a coupled differential Riccati system:

$$\begin{cases} \dot{\mathbf{A}}_{xx} = -h^2(t)\mathbf{\Sigma}_p + \frac{1}{h(t)}\mathbf{A}_{xx}\mathbf{\Lambda}^{-1}\mathbf{A}_{xx} - \mathbf{A}_{xf}\mathbf{\Sigma}_f\mathbf{A}_{xf}^\top, \\ \dot{\mathbf{A}}_{xf} = -h(t)\mathbf{B} + \frac{1}{h(t)}\mathbf{A}_{xx}\mathbf{\Lambda}^{-1}\mathbf{A}_{xf} + \mathbf{A}_{xf}\mathbf{\Phi} - \mathbf{A}_{xf}\mathbf{\Sigma}_f\mathbf{A}_{ff}, \\ \dot{\mathbf{A}}_{ff} = \frac{1}{h(t)}\mathbf{A}_{xf}^\top\mathbf{\Lambda}^{-1}\mathbf{A}_{xf} + 2\mathbf{A}_{ff}\mathbf{\Phi} - \mathbf{A}_{ff}\mathbf{\Sigma}_f\mathbf{A}_{ff}, \\ \dot{A}_0 = -\frac{1}{2}\mathrm{tr}\left(\mathbf{A}_{ff}\mathbf{\Sigma}_f\right), \end{cases} \tag{A.7}$$

where $\dot{\mathbf{A}} \triangleq \frac{d\mathbf{A}}{dt}$,with $\mathbf{A}_{xx}(T) = \mathbf{0}_{n\times n}$, $\mathbf{A}_{xf}(T) = \mathbf{0}_{n\times k}$, $\mathbf{A}_{ff}(T) = \mathbf{0}_{k\times k}$ and $A_0(T) = 0$.

A.2 Proof of Proposition 1

To demonstrate the existence of the coupled Riccati Equations, we first provide the standard comparison principle, which is proved in Abou-Kandil et al. (2012).

Lemma 1. (Comparison principle) For $i = 1, 2$, let $\mathbf{A}_i$ be the solution to the following matrix Riccati equation:

$$\dot{\mathbf{A}}_i(t) = \mathbf{D}_i(t) + \mathbf{A}_i(t)\mathbf{K}_i + \mathbf{K}_i^\top\mathbf{A}_i(t) + \mathbf{A}_i(t)\mathbf{N}_i(t)\mathbf{A}_i(t), \quad \text{for all} \quad t \in [0,T], \tag{A.8}$$

where $\mathbf{D}_i(t)$ and $\mathbf{N}_i(t)$ are symmetric on $[0,T]$. If

$$\begin{pmatrix} \mathbf{D}_2(t) & \mathbf{K}_2^\top \\ \mathbf{K}_2 & \mathbf{N}_2(t) \end{pmatrix} \leq \begin{pmatrix} \mathbf{D}_1(t) & \mathbf{K}_1^\top \\ \mathbf{K}_1 & \mathbf{N}_1(t) \end{pmatrix}, \quad \text{for all} \quad t \in [0,T], \tag{A.9}$$

and $\mathbf{A}_1(T) \leq \mathbf{A}_2(T)$, then $\mathbf{A}_1(t) \leq \mathbf{A}_2(t)$ on $t \in [0,T]$.

Proof of Proposition 1

(1) Let $\mathbf{A}_i$, for $i = 1, 2$, be the solutions to the following matrix Riccati equations:

$$\begin{cases} \dot{\mathbf{A}}_i(t) = \mathbf{D}(t) + \mathbf{A}_i(t)\mathbf{K} + \mathbf{K}^\top \mathbf{A}_i(t) + \mathbf{A}_i(t)\mathbf{N}_i(t)\mathbf{A}_i(t), \quad t \in [0, T), \\ \mathbf{A}_i(T) = \mathbf{A}(T), \end{cases} \tag{A.10}$$

with

$$\mathbf{N}_1(t) = \begin{pmatrix} \frac{1}{h(t)}\mathbf{\Lambda}^{-1} & \mathbf{0}_{n\times k} \\ \mathbf{0}_{k\times n} & \mathbf{0}_{k\times k} \end{pmatrix}, \quad \mathbf{N}_2 = \begin{pmatrix} \mathbf{0}_{n\times n} & \mathbf{0}_{n\times k} \\ \mathbf{0}_{k\times n} & -\mathbf{\Sigma}_f \end{pmatrix}; \tag{A.11}$$

By the definitions of $\mathbf{A}_1$ and $\mathbf{A}_2$, it follows that

$$\begin{pmatrix} \mathbf{D}(t) & \mathbf{K}^\top \\ \mathbf{K} & \mathbf{N}_2(t) \end{pmatrix} \le \begin{pmatrix} \mathbf{D}(t) & \mathbf{K}^\top \\ \mathbf{K} & \mathbf{N}(t) \end{pmatrix} \le \begin{pmatrix} \mathbf{D}(t) & \mathbf{K}^\top \\ \mathbf{K} & \mathbf{N}_1(t) \end{pmatrix}, \quad t \in [0, T], \tag{A.12}$$

from which comparison principle in Lemma 1 yields

$$\mathbf{A}_1(t) \le \mathbf{A}(t) \le \mathbf{A}_2(t), \quad \text{for all} \quad t \in [0, T]. \tag{A.13}$$

In light of (A.13), it suffices to show that $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ are bounded for $t \in [0, T]$. We only show $\mathbf{A}_1(t)$ is bounded on $[0, T]$, for the boundedness of $\mathbf{A}_2(t)$ follows similarly. Write

$$\mathbf{A}_1(t) = \begin{pmatrix} \mathbf{A}_{1,xx}(t) & \mathbf{A}_{1,xf}(t) \\ \mathbf{A}_{1,xf}^\top(t) & \mathbf{A}_{1,ff}(t) \end{pmatrix}, \quad \text{for all} \quad t \in [0, T]. \tag{A.14}$$

Together with (A.11), it is clear that $\mathbf{A}_{1,xx}$, $\mathbf{A}_{1,xf}$, and $\mathbf{A}_{1,ff}$ in (A.14) are the solutions to the following coupled differential Riccati equations:

$$\dot{\mathbf{A}}_{1,xx} = -h^2(t)\mathbf{\Sigma}_p + \frac{1}{h(t)}\mathbf{A}_{1,xx}\mathbf{\Lambda}^{-1}\mathbf{A}_{1xx}, \tag{A.15}$$

$$\dot{\mathbf{A}}_{1,xf} = -h(t)\mathbf{B} + \frac{1}{h(t)}\mathbf{A}_{1,xx}\mathbf{\Lambda}^{-1}\mathbf{A}_{1,xf} + \mathbf{A}_{1,xf}\mathbf{\Phi}, \tag{A.16}$$

$$\dot{\mathbf{A}}_{1,ff} = \frac{1}{h(t)}\mathbf{A}_{1,xf}^\top\mathbf{\Lambda}^{-1}\mathbf{A}_{1,xf} + 2\mathbf{A}_{1,ff}\mathbf{\Phi}. \tag{A.17}$$

Applying Lemma 1 to function $\mathbf{A}_{1,xx}(t)$, we have

$$\mathbf{A}_{1,xx}^{\text{lower}}(t) \leq \mathbf{A}_{1,xx}(t) \leq \mathbf{A}_{1,xx}^{\text{upper}}(t), \tag{A.18}$$

where $\mathbf{A}_{1,xx}^{\text{lower}}(t)$ and $\mathbf{A}_{1,xx}^{\text{upper}}(t)$ satisfy

$$\dot{\mathbf{A}}_{1,xx}^{\text{lower}} = \frac{1}{h(t)}\mathbf{A}_{1,xx}^{\text{lower}}\mathbf{\Lambda}^{-1}\mathbf{A}_{1,xx}^{\text{lower}}, \tag{A.19}$$

$$\dot{\mathbf{A}}_{1,xx}^{\text{upper}} = -h^2(t)\mathbf{\Sigma}_p, \tag{A.20}$$

with $\mathbf{A}_{1,xx}^{\text{lower}}(T) = \mathbf{A}_{1,xx}^{\text{upper}}(T) = \mathbf{A}_{1,xx}(T) = \mathbf{0}_{n\times n}$. It is easy to see that $\mathbf{A}_{1,xx}^{\text{lower}}(t) \equiv \mathbf{0}_{n\times n}$ and $\mathbf{A}_{1,xx}^{\text{upper}}(t) = \int_t^T h^2(s)ds\mathbf{\Sigma}_p$. Consequently, we demonstrate that $\mathbf{A}_{1,xx}(t)$ is bounded on $[0,T]$. Since Equations (A.16) and (A.17) are both linear, the solutions $\mathbf{A}_{1,xf}(t)$ and $\mathbf{A}_{1,ff}(t)$ are bounded on $[0,T]$. Therefore, $\mathbf{A}_1(t)$ is bounded on $[0,T]$. Applying Lemma 1 again, the uniqueness of solution $\mathbf{A}(t)$ can be deduced easily. The same arguments can be applied to show that $\mathbf{A}_{xx}$ is positive definite and $\mathbf{A}_{ff}$ is negative definite on $[0,T)$.

(2) Next, we show that $\mathbf{A}(t)$ in (3.12) admits the explicit form in (3.15). Indeed, taking derivative of the right-hand side of (3.15) with respect to t yields

$$\frac{d}{dt}\left(\mathbf{R}_2(t)\mathbf{R}_1^{-1}(t)\right) = \mathbf{D}(t) + \mathbf{K}^\top\mathbf{A}(t) + \mathbf{A}(t)\mathbf{K} + \mathbf{A}(t)\mathbf{N}(t)\mathbf{A}(t) = \dot{\mathbf{A}}(t), \tag{A.21}$$

which shows that $\mathbf{A}(t) = \mathbf{R}_2(t)\mathbf{R}_1^{-1}(t)$ for all $t \in [0,T]$. This concludes the proof.

A.3 Proof of Theorem 2

(1) If the family $\mathfrak{V}$ in (3.16) is uniformly integrable, we can now apply the standard verification arguments as shown in Fleming and Soner (2006) to yield the desired result.

In order to prove the (2) of Theorem 2, we need the following lemma:

Lemma 2. (Uniformly integrability) Under the assumption in (3.8), if process $\mathbf{f}(t)$ satisfies the following integrability condition

$$\mathbf{E}[e^{C\int_0^T \|\mathbf{f}(t)\|_k^2}] < \infty, \tag{A.22}$$

where C is a positive constant given later and $\|\cdot\|_k$ denotes the Euclidean norm in $\mathbb{R}^k$, then the optimal trading strategy $\boldsymbol{\tau}^*$ (3.5) is admissible. Furthermore, the family

$$\mathfrak{U} \triangleq \{V(\mathfrak{s}, \mathbf{x}^*(\mathfrak{s}), y^*(\mathfrak{s}), : \mathfrak{s} \text{ is stopping time with values in } [0,T]\}, \tag{A.23}$$

is uniformly integrable.

Proof. We first show that the dynamics (2.4) and (2.8) admit strong solutions corresponding to the optimal trading strategy $\boldsymbol{\tau}^*$ (3.5). The return-predicting factors $\mathbf{f}(t)$ can be written as follows

$$\mathbf{f}(t) = e^{-\boldsymbol{\Phi}t}\mathbf{f}(0) + \int_0^t e^{\boldsymbol{\Phi}(s-t)}\boldsymbol{\sigma}_f d\mathbf{W}_f(s). \tag{A.24}$$

Consequently, any $p-$th order finite moment of process $\mathbf{f}(t)$ always exists, i.e. $\mathbf{f}(t) \in L^p_{\mathcal{F}}(0,T;\mathbb{R}^k)$ which denotes the set of all $\mathcal{F}_t$ adopted $\mathbb{R}^k$ processes $\mathbf{f}(t)$ such that $\|\mathbf{f}\|^p_{\mathcal{F}_T} \triangleq \mathbf{E}\int_0^T \|\mathbf{f}(t)\|^p_k dt < \infty$. Expressing τ^* into (2.4) and integrating from 0 to T yields,

$$\mathbf{x}^*(t) = e^{-\int_0^t \frac{\boldsymbol{\Lambda}^{-1}}{h(s)}\mathbf{A}_{xx}(s)ds}\left[\mathbf{x}(0) + \int_0^t e^{\int_0^s \frac{\boldsymbol{\Lambda}^{-1}}{h(u)}\mathbf{A}_{xx}(u)du}\frac{\boldsymbol{\Lambda}^{-1}}{h(s)}\mathbf{A}_{xf}(s)\mathbf{f}(s)ds\right]. \tag{A.25}$$

According to Proposition 1, $\mathbf{A}_{xx}(t)$, $\mathbf{A}_{xf}(t)$ and $\mathbf{A}_{ff}(t)$ are bounded on $[0,T]$. Therefore, we have

$$\|\mathbf{x}^*\|^p_{\mathcal{F}_T} \leq C_1(1 + \|\mathbf{f}\|^p_{\mathcal{F}_T}), \tag{A.26}$$

where C_1 is a positive constant, which also shows $\mathbf{x}^*(t) \in L^p_{\mathcal{F}}(0,T;\mathbb{R}^n)$, for any $p \geq 1$. Consequently, it also leads to the fact that $\boldsymbol{\tau}^*(t) \in L^p_{\mathcal{F}}(0,T;\mathbb{R}^n)$. Here we consider a specific case with $p = 4$. All $\mathbf{x}^*(t), \mathbf{f}(t), \boldsymbol{\tau}^*(t) \in L^4_{\mathcal{F}}(0,T;\mathbb{R}^n)$, which results in $\mathbf{x}^{*\top}\mathbf{B}\mathbf{f} - \frac{1}{2}\boldsymbol{\tau}^{*\top}\boldsymbol{\Lambda}\boldsymbol{\tau}^* \in L^2_{\mathcal{F}}(0,T;\mathbb{R})$. Consequently, the dynamic (2.8) corresponding to $\boldsymbol{\tau}^*$ admits a unique strong solution

$$y^*(t) = e^{rt}\left\{y(0) + \int_0^t e^{-rs}\left[\mathbf{x}^{*\top}(s)\mathbf{B}\mathbf{f}(s) - \frac{1}{2}\boldsymbol{\tau}^{*\top}(s)\boldsymbol{\Lambda}\boldsymbol{\tau}^*(s)\right]ds + \int_0^t e^{-rs}\mathbf{x}^{*\top}(s)\boldsymbol{\sigma}_p d\mathbf{W}_p(s)\right\}. \tag{A.27}$$

Consequently, the feedback optimal control $\boldsymbol{\tau}^*$ (3.5) is admissible and all the corresponding dynamics $(\mathbf{x}^*(t), y^*(t))$ exist. On the other hand, we have

$$\mathbf{E}|V(\mathfrak{s}, \mathbf{x}^*(\mathfrak{s}), y^*(\mathfrak{s}), \mathbf{f}(\mathfrak{s}))| \leq C_2\mathbf{E}e^{-h(\mathfrak{s})y^*(\mathfrak{s}) + C_3\|f(\mathfrak{s})\|^2_k} \leq C_2\left[\mathbf{E}e^{-2h(\mathfrak{s})y^*(\mathfrak{s})}\right]^{\frac{1}{2}}\left[\mathbf{E}e^{2C_3\|f(\mathfrak{s})\|^2_k}\right]^{\frac{1}{2}}.$$

If condition (A.22) holds and $C \geq 2C_3$, the uniformly integrability of family $\mathfrak{U}$ is sufficient to demonstrate $\mathbf{E}e^{-2h(\mathfrak{s})y^*(\mathfrak{s})} < \infty$, which has been proved in Lemma 4 of Ma and Zhu (2019). $\square$

We also provide another useful lemma proved by Ma and Zhu (2019).

Lemma 3. (A sufficient condition) Condition (3.18) is sufficient to ensure the integrability condition (A.22).

With two useful lemmas in hand, let us go back to the proof of (2). Under the assumption in (3.8), Lemma 3 shows that the integrability condition (A.22) holds under condition (3.18). Hence, by Lemma 2, it follows that $\mathfrak{U}$ in (A.23) is a uniformly-integrable family. With the uniform-integrability property established, applying the standard verification arguments as shown in Fleming and Soner (2006) yields the desired result.

A.4 Proof of Theorem 3

Analogous to the proof of Theorem 1, we also derive the coefficient functions in (3.22) satisfy the following ordinary differential equation (ODE) system:

$$\begin{cases} \dot{\mathbf{A}}_{xx}^G = -\gamma\boldsymbol{\Sigma}_p + \mathbf{A}_{xx}^G\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xx}^G + \rho\mathbf{A}_{xx}^G, \\ \dot{\mathbf{A}}_{xf}^G = -\mathbf{B} + \mathbf{A}_{xx}^G\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xf}^G + \mathbf{A}_{xf}^G\boldsymbol{\Phi} + \rho\mathbf{A}_{xf}^G, \\ \dot{\mathbf{A}}_{ff}^G = -(\mathbf{A}_{xf}^G)^\top\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xf}^G + 2\mathbf{A}_{ff}^G\boldsymbol{\Phi} + \rho\mathbf{A}_{ff}^G, \\ \dot{A}_0^G = -\frac{1}{2}\mathrm{tr}\left(\mathbf{A}_{ff}^G\boldsymbol{\Sigma}_f\right), \end{cases} \tag{A.28}$$

with terminal conditions $\mathbf{A}_{xx}^G(T) = \mathbf{0}_{n\times n}$, $\mathbf{A}_{xf}^G(T) = \mathbf{0}_{n\times k}$, $\mathbf{A}_{ff}^G(T) = \mathbf{0}_{k\times k}$ and $A_0^G(T) = 0$.

A.5 Asymptotic analysis between aim and Merton portfolios

In this section, we show that the aim portfolio converges to the Merton portfolio as time t approaches the investment horizon T. Applying Taylor's expansion to $\mathbf{A}_{xx}(t)$ and $\mathbf{A}_{xf}(t)$ of the

ODE system in (A.7), we have, for all $t \le T$,

$$\begin{aligned}
\mathbf{A}_{xx}(t) &= \mathbf{A}_{xx}(T) + (t-T)\dot{\mathbf{A}}_{xx}(T) + o(t-T) \\
&= (t-T)\left[-h^2(T)\boldsymbol{\Sigma}_p + \frac{1}{h(T)}\mathbf{A}_{xx}(T)\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xx}(T) - \mathbf{A}_{xf}(T)\boldsymbol{\Sigma}_f\mathbf{A}_{xf}^\top(T)\right] + o(t-T) \\
&= -\eta^2\boldsymbol{\Sigma}_p(t-T) + o(t-T), \qquad \text{(A.29)} \\
\mathbf{A}_{xf}(t) &= \mathbf{A}_{xf}(T) + (t-T)\dot{\mathbf{A}}_{xf}(T) + o(t-T) \\
&= (t-T)\left[-h(T)\mathbf{B} + \frac{1}{h(T)}\mathbf{A}_{xx}(T)\boldsymbol{\Lambda}^{-1}\mathbf{A}_{xf}(T) + \mathbf{A}_{xf}(T)\boldsymbol{\Phi} - \mathbf{A}_{xf}(T)\boldsymbol{\Sigma}_f\mathbf{A}_{ff}(T)\right] + o(t-T) \\
&= -\eta\mathbf{B}(t-T) + o(t-T). \qquad \text{(A.30)}
\end{aligned}$$

Combining (A.29) and (A.30), we achieve the desired results as follows:

$$\lim_{t\to T}\bar{\mathbf{M}}^{\mathrm{aim}}(t,\mathbf{f}) = \lim_{t\to T}\mathbf{A}_{xx}^{-1}(t)\mathbf{A}_x(t)\mathbf{f} = \frac{1}{\eta}\boldsymbol{\Sigma}_p^{-1}\mathbf{B}\mathbf{f} \triangleq \mathbf{Merton}(T). \qquad \text{(A.31)}$$

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