Vanishing Fourier transforms and generalized differences in $L^2(\mathbb{R})$

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Vanishing Fourier transforms and generalized differences in $L^2(\mathbb{R})$

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Let $\alpha, \beta \in \mathbb{R}$ and $s \in \mathbb{N}$ be given. Let $\delta_x$ denote the Dirac measure at $x \in \mathbb{R}$, and let $\ast$ denote convolution. If $\mu$ is a measure, $\mu \ast$ is the measure that assigns to each Borel set $A$ the value $\mu(A)$. If $u \in \mathbb{R}$, we put 

$$
\mu_{\alpha, \beta, u} = e^{iu(\alpha - \beta)/2} \delta_0 - e^{iu(\alpha + \beta)/2} \delta_u.
$$

Then, we call a function $g \in L^2(\mathbb{R})$ a generalised $(\alpha, \beta)$-difference of order $2s$ if for some $u \in \mathbb{R}$ and $h \in L^2(\mathbb{R})$ we have 

$$
g = [\mu_{\alpha, \beta, u} + \mu\ast_{\alpha, \beta, u}] \ast h.
$$

We denote by $D_{\alpha, \beta, s}(\mathbb{R})$ the vector space of all functions $f$ in $L^2(\mathbb{R})$ such that $f$ is a finite sum of generalised $(\alpha, \beta)$-differences of order $2s$. It is shown that every function in $D_{\alpha, \beta, s}(\mathbb{R})$ is a sum of $4s + 1$ generalised $(\alpha, \beta)$-differences of order $2s$. Letting $\hat{f}$ denote the Fourier transform of a function $f \in L^2(\mathbb{R})$, it is shown that $f \in D_{\alpha, \beta, s}(\mathbb{R})$ if and only if $\hat{f}$ "vanishes" near $\alpha$ and $\beta$ at a rate comparable with $(x - \alpha)^{2s}(x - \beta)^{2s}$. In fact, $D_{\alpha, \beta, s}(\mathbb{R})$ is a Hilbert space where the inner product of functions $f$ and $g$ is 

$$
\int_{-\infty}^{\infty} (1 + (x - \alpha)^{2s}(x - \beta)^{2s}) \hat{f}(x)\overline{\hat{g}(x)} dx.
$$

Letting $D$ denote differentiation, and letting $I$ denote the identity operator, the operator $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ is bounded with multiplier $(-1)^s(x - \alpha)^s(x - \beta)^s$, and the Sobolev subspace of $L^2(\mathbb{R})$ of order $2s$ may be given a norm equivalent to the usual one so that $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ becomes an isometry onto the Hilbert space $D_{\alpha, \beta, s}(\mathbb{R})$. So, a space $D_{\alpha, \beta, s}(\mathbb{R})$ may be regarded as a type of Sobolev space having a negative index.
1. Introduction

Let \( \mathbb{R} \) denote the set of real numbers, let \( \mathbb{T} \) denote the set of complex numbers of modulus 1, and let \( G \) denote either \( \mathbb{R} \) or \( \mathbb{T} \). Note that in some contexts \( \mathbb{T} \) may be identified with the interval \([0, 2\pi)\) under the mapping \( t \mapsto e^{it} \) (some comments on this are in [9, p. 1034]). Then \( G \) is a group and its identity element we denote by \( e \), so that \( e = 0 \) when \( G = \mathbb{R} \) and \( e = 1 \) when \( G = \mathbb{T} \). Let \( \mathbb{N} \) denote the set of natural numbers, \( \mathbb{Z} \) the set of integers, and let \( s \in \mathbb{N} \). The Fourier transform of \( f \in L^2(G) \) is denoted by \( \hat{f} \), and is given by \( \hat{f}(n) = (2\pi)^{-1} \int_0^{2\pi} f(e^{it})e^{-int}dt \) for \( n \in \mathbb{Z} \) (in the case of \( \mathbb{T} \)), and by the extension to all of \( L^2(\mathbb{R}) \) of the transform given by \( \hat{f}(x) = \int_{-\infty}^{\infty} e^{-ixu}f(u)\,du \) for \( x \in \mathbb{R} \) (in the case of \( \mathbb{R} \)). Let \( M(G) \) denote the family of bounded Borel measures on \( G \). If \( x \in G \) let \( \delta_x \) denote the Dirac measure at \( x \), and let \( * \) denote convolution in \( M(G) \).

We call a function \( f \in L^2(G) \) a difference of order \( s \) if there is a function \( g \in L^2(G) \) and \( u \in G \) such that \( f = (\delta_u - \delta_u)^s \ast g \). The functions in \( L^2(G) \) that are a sum of a finite number of differences of order \( s \) we denote by \( \mathcal{D}_s(G) \). Note that \( \mathcal{D}_s(G) \) is a vector subspace of \( L^2(G) \). Now in the case of \( \mathbb{T} \) it was shown by Meisters and Schmidt [5] that

\[
\mathcal{D}_1(\mathbb{T}) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(0) = 0 \right\},
\]

and that every function in \( \mathcal{D}_1(\mathbb{T}) \) is a sum of 3 differences of order 1. It was shown in [6] that, for all \( s \in \mathbb{N} \),

\[
\mathcal{D}_s(\mathbb{T}) = \mathcal{D}_1(\mathbb{T}) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(0) = 0 \right\},
\]

and that every function in \( \mathcal{D}_s(\mathbb{T}) \) is a sum of \( 2s + 1 \) differences of order \( s \). It was also shown in [6] that

\[
\mathcal{D}_s(\mathbb{R}) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{|x|^{2s}} \,dx < \infty \right\},
\]

and again, that every function in \( \mathcal{D}_s(\mathbb{R}) \) is a sum of \( 2s + 1 \) differences of order \( s \). Further results related to the work of Meisters and Schmidt in [5] may be found in [1, 2, 3, 4, 7].

The Sobolev space of order \( s \) in \( L^2(G) \) is the space of all functions \( f \in L^2(G) \) such that \( D^s(f) \in L^2(G) \), where \( D \) denotes differentiation in the sense of Schwartz distributions. Then, \( D^s \) is a multiplier operator on \( W^s(\mathbb{T}) \) with multiplier \((ix)^s\), in the sense that \( D^s(f)(n) = (in)^s\hat{f}(n) \), for all \( f \in W^s(\mathbb{T}) \) and \( n \in \mathbb{Z} \). Also, \( D^s \) is a multiplier operator on \( W^s(\mathbb{R}) \) with multiplier \((ix)^s\), in the sense that \( D^s(f)(x) = (ix)^s\hat{f}(x) \), for almost all \( x \in \mathbb{R} \) for \( f \in W^s(\mathbb{R}) \). Note that \( W^s(\mathbb{T}) \) is a Hilbert space where the inner product of \( f, g \in W^s(\mathbb{T}) \) is \( \sum_{n=-\infty}^{\infty} (1 + |n|^{2s}) \hat{f}(n)\overline{\hat{g}(n)} \). Note also that \( W^s(\mathbb{R}) \) is a Hilbert space for which the usual inner product is given by

\[
\langle f, g \rangle_{W^s} = \int_{-\infty}^{\infty} (1 + |x|^{2s}) \hat{f}(x)\overline{\hat{g}(x)} \,dx, \text{ for } f, g \in W^s(\mathbb{R}).
\]
Using these observations, together with Plancherel’s Theorem, it is easy to verify that

(1.4) \[ D^s(W^s(\mathbb{T})) = \left\{ f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(0) = 0 \right\}, \]

and that

(1.5) \[ D^s(W^s(\mathbb{R})) = \left\{ f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{|x|^{2s}} \, dx < \infty \right\}. \]

In view of (1.4) and (1.5), (1.1) together with (1.2) can be regarded as describing the ranges of \( D^s \) upon \( W^s(\mathbb{T}) \) and \( W^s(\mathbb{R}) \) as spaces consisting of finite sums of differences of order \( s \). Corresponding results have been obtained in [8] for operators \((D^2 - i(\alpha + \beta)D - \alpha\beta I)^s\) acting on \( W^{2s}(\mathbb{T})\), where \( \alpha, \beta \in \mathbb{Z} \) and \( I \) denotes the identity operator. In this paper, the main aim is to derive corresponding results for the operator \((D^2 - i(\alpha + \beta)D - \alpha\beta I)^s\), where \( \alpha, \beta \in \mathbb{R} \), for the non-compact case of \( \mathbb{R} \) in place of the compact group \( \mathbb{T} \). Note that, in general, the range of a multiplier operator depends upon the behaviour of Fourier transforms at or around the zeros of the multiplier of the operator, as in (1.4) and (1.5). Note also that on \( \mathbb{R}, \) \((D^2 - i(\alpha + \beta)D - \alpha\beta I)^s\) is a multiplier operator whose multiplier is \((-1)^s(x-\alpha)^s(x-\beta)^s\), which has zeros at \( \alpha \) and \( \beta \).

Given \( \alpha, \beta \in \mathbb{R} \) and \( s \in \mathbb{N} \), a \textit{generalised \((\alpha, \beta)\)-difference of order} \( 2s \) is a function \( f \in L^2(\mathbb{R}) \) such that for some \( g \in L^2(\mathbb{R}) \) and \( u \in \mathbb{R} \) we have

(1.6) \[ f = \left[ \left( e^{iu(\frac{\alpha-\beta}{2})} + e^{-iu(\frac{\alpha-\beta}{2})} \right) \delta_0 - \left( e^{iu(\frac{\alpha+\beta}{2})} \delta_u + e^{-iu(\frac{\alpha+\beta}{2})} \delta_{-u} \right) \right]^s * g. \]

It may be called also an \((\alpha, \beta)\)-\textit{difference of order} \( 2s \), or simply a \textit{generalised difference}. The vector space of functions in \( L^2(\mathbb{R}) \) that can be expressed as some finite sum of \((\alpha, \beta)\)-differences of order \( 2s \) is denoted by \( D_{\alpha,\beta,s}(\mathbb{R}) \). Thus, \( f \in D_{\alpha,\beta,s}(\mathbb{R}) \) if and only if there are \( m \in \mathbb{N} \), \( u_1, u_2, \ldots, u_m \in \mathbb{R} \) and \( f_1, f_2, \ldots, f_m \in L^2(\mathbb{R}) \) such that

\[ f = \sum_{j=1}^{m} \left[ \left( e^{iu_j(\frac{\alpha-\beta}{2})} + e^{-iu_j(\frac{\alpha-\beta}{2})} \right) \delta_0 - \left( e^{iu_j(\frac{\alpha+\beta}{2})} \delta_{u_j} + e^{-iu_j(\frac{\alpha+\beta}{2})} \delta_{-u_j} \right) \right]^s * f_j. \]

We prove that if \( f \in L^2(\mathbb{R}) \), \( f \in D_{\alpha,\beta,s}(\mathbb{R}) \) if and only if \( \hat{f} \) is “vanishing” near \( \alpha \) and \( \beta \) in the sense that

\[ \int_{-\infty}^{\infty} (x-\alpha)^{-2s}(x-\beta)^{-2s} |\hat{f}(x)|^2 \, dx < \infty, \]

in which case \( f \) is a sum of \( 4s + 1 \) \((\alpha, \beta)\)-differences of order \( 2s \). It follows that \( D_{\alpha,\beta,s}(\mathbb{R}) \) is a Hilbert space where the inner product of \( f, g \in D_{\alpha,\beta,s}(\mathbb{R}) \) is

\[ \int_{-\infty}^{\infty} \left( 1 + (x-\alpha)^{-2s}(x-\beta)^{-2s} \right) \hat{f}(x) \overline{g(x)} \, dx. \]

In fact, it follows straightforwardly from the above that the usual norm on \( W^{2s}(\mathbb{R}) \), as derived from (1.3), may be replaced by a natural equivalent norm in which the operator \((D^2 - i(\alpha + \beta)D - \alpha\beta I)^s(\mathbb{R})\) is an isometry from \( W^{2s}(\mathbb{R}) \) onto \( D_{\alpha,\beta,s}(\mathbb{R}) \). Consequently, the space \( D_{\alpha,\beta,s}(\mathbb{R}) \) may be thought of a “Sobolev-type” space with a negative index, consisting of sums of generalised differences associated with the operator.
2. Preliminaries and Proof of the Main Result

We need the following result, which characterises those functions which are a sum of convolutions of other functions by given measures.

**Theorem 2.1.** Let \( f \in L^2(\mathbb{R}) \) and let \( \mu_1, \mu_2, \ldots, \mu_r \in M(\mathbb{R}) \). Then the following conditions (i) and (ii) are equivalent.

(i) There are \( f_1, f_2, \ldots, f_r \in L^2(\mathbb{R}) \) such that \( f = \sum_{j=1}^{r} \mu_j * f_j \).

(ii) 
\[
\int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}} dx < \infty.
\]

**Proof.** This is essentially proved in [5, pp. 411-412], but see also [6, pp. 77-88] and [7, p. 23]. \( \square \)

**Lemma 2.2.** Let \( J, K \) be two closed intervals of positive length such that \( J \cap K \) also has positive length. Let \( \xi \in J \) and \( \eta \in K \) be given. If \( \xi \in J \cap K \) put \( \xi = \xi \), and if \( \xi \notin J \cap K \), let \( \xi \) be the end point of \( J \cap K \) that is closest to \( \xi \). If \( \eta \in J \cap K \) put \( \eta = \eta \), and if \( \eta \notin J \cap K \) let \( \eta \) be the endpoint of \( J \cap K \) that is closest to \( \eta \). Then,
\[
|x - \xi| \cdot |x - \eta| \geq |x - \xi| \cdot |x - \eta|, \quad \text{for all } x \in J \cap K.
\]

**Proof.** The result is immediate from the observation that for all \( x \in J \cap K \), \( |x - \xi| \geq |x - \xi| \) and \( |x - \eta| \geq |x - \eta| \). \( \square \)

The main aim in this paper is to prove the following. In the proof we will use the notation that \( A^c \) denotes the complement of the set \( A \).

**Theorem 2.3.** Let \( s \in \mathbb{N} \) and let \( \alpha, \beta \in \mathbb{R} \). Let \( \mathcal{D}_{\alpha,\beta,s}(\mathbb{R}) \) be the vector space of functions in \( L^2(\mathbb{R}) \) that can be expressed as some finite sum of generalised \((\alpha, \beta)\)-differences of order \( 2s \). Then the following conditions (i) - (iii) are equivalent for a function \( f \in L^2(\mathbb{R}) \).

(i) 
\[
\int_{-\infty}^{\infty} \frac{|\hat{f}(x)|^2}{(x-\alpha)^{2s}(x-\beta)^{2s}} dx < \infty.
\]

(ii) \( f \in \mathcal{D}_{\alpha,\beta,s}(\mathbb{R}) \).

(iii) There are \( u_1, u_2, \ldots, u_{4s+1} \in \mathbb{R} \) and \( f_1, f_2, \ldots, f_{4s+1} \in L^2(\mathbb{R}) \) such that
\[
f = \sum_{j=1}^{4s+1} \left( e^{iu_j(\alpha^{2s})} + e^{-iu_j(\alpha^{2s})} \right) \delta_{0} - \left( e^{iu_j(\beta^{2s})} \delta_{u_j} + e^{-iu_j(\beta^{2s})} \delta_{-u_j} \right)^{s} * f_j.
\]

Furthermore the following statements (iv), (v) and (vi) hold.

(iv) When the conditions (i)-(iii) hold for a given function \( f \in L^2(\mathbb{R}) \), for almost all \( (u_1, u_2, \ldots, u_{4s+1}) \in \mathbb{R}^{4s+1} \), there are \( f_1, f_2, \ldots, f_{4s+1} \in L^2(\mathbb{R}) \) such that (2.2) holds.

(v) The vector space \( \mathcal{D}_{\alpha,\beta,s}(\mathbb{R}) \) is a Hilbert space with the inner product \( \langle \cdot, \cdot \rangle_{\alpha,\beta,s} \) given by
\[
\langle f, g \rangle_{\alpha,\beta,s} = \int_{-\infty}^{\infty} \left( 1 + \frac{1}{(x-\alpha)^{2s}(x-\beta)^{2s}} \right) \hat{f}(x) \overline{\hat{g}(x)} dx, \quad \text{for } f, g \in \mathcal{D}_{\alpha,\beta,s}(\mathbb{R}).
\]
(vi) For \( f, g \in W^{2s}(\mathbb{R}) \), put \( \langle f, g \rangle_{W^{2s,\alpha,\beta}} = \int_{-\infty}^{\infty} (1+(x-\alpha)^{2s}(x-\beta)^{2s}) \hat{f}(x) \overline{\hat{g}(x)} \, dx \). Then \( \langle \cdot, \cdot \rangle_{W^{2s,\alpha,\beta}} \) is an inner product on \( W^{2s}(\mathbb{R}) \) that is equivalent to the usual one on \( W^{2s}(\mathbb{R}) \) as given in (1.3). The operator \( (D^2 - i(\alpha + \beta)D - \alpha \beta I)^s \) has the multiplier \((-1)^s(x-\alpha)^s(x-\beta)^s\) and it is an isometry that maps \( W^{2s}(\mathbb{R}) \) with the inner product \( \langle \cdot, \cdot \rangle_{W^{2s,\alpha,\beta}} \) onto \( D_{\alpha,\beta,s}(\mathbb{R}) \).

Proof. If (iii) holds then (ii) holds, by definition.

Let (ii) hold. If \( u \in \mathbb{R} \), define \( \lambda_u \in M(\mathbb{R}) \) by

\[
\lambda_u = \frac{1}{2} \left[ e^{iu(\frac{a+b}{2})} + e^{-iu(\frac{a+b}{2})} \right] \delta_0 - \frac{1}{2} \left[ e^{iu(\frac{a-b}{2})} \delta_u + e^{-iu(\frac{a-b}{2})} \delta_{-u} \right].
\]

The Fourier transform \( \hat{\lambda}_u \) of \( \lambda_u \) is given for \( x \in \mathbb{R} \) by

\[
\hat{\lambda}_u(x) = 2 \sin \left( \frac{u(x - \alpha)}{2} \right) \sin \left( \frac{u(x - \beta)}{2} \right).
\]

So, if \( u \in \mathbb{R} \) and \( f, g \in L^2(\mathbb{R}) \) are such that \( f = \lambda_u^* g \), we have

\[
\int_{-\infty}^{\infty} \frac{\hat{f}(x)^2}{(x-\alpha)^{2s}(x-\beta)^{2s}} \, dx = 2^s \int_{-\infty}^{\infty} \frac{\sin^2 \left( \frac{u(x - \alpha)/2}{2} \right) \sin^2 \left( \frac{u(x - \beta)/2}{2} \right)}{(x-\alpha)^{2s}(x-\beta)^{2s}} \, |\hat{g}(x)|^2 \, dx < \infty.
\]

Using (2.3), we deduce that (ii) implies (i).

Now, we assume that (i) holds, and we will prove that (iii) holds. Let \( x \in \mathbb{R} \) be given but with \( x \notin \{\alpha, \beta\} \). Note that it may happen that \( \alpha = \beta \). Put, for each \( k \in \mathbb{Z} \),

\[
\text{(2.5)} \quad a_k = \frac{k\pi}{|x - \alpha|}, \quad b_k = \frac{k\pi}{|x - \beta|}, \quad a'_k = \frac{(k-1/2)\pi}{|x - \alpha|} \quad \text{and} \quad b'_k = \frac{(k-1/2)\pi}{|x - \beta|}.
\]

Then put, again for each \( k \in \mathbb{Z} \),

\[
\text{(2.6)} \quad A_k = [a'_k, a'_{k+1}] \quad \text{and} \quad B_k = [b'_k, b'_{k+1}].
\]

Note that \( a_k \) is the mid-point of \( A_k \) and \( b_k \) is the mid-point of \( B_k \). The points \( a_k \) are the zeros of \( u \mapsto \sin(u(x - \alpha)) \), while the \( b_k \) are the zeros of \( u \mapsto \sin(u(x - \beta)) \).

Using (2.5) and (2.6), we see that for each \( k \in \mathbb{Z} \),

\[
\text{(2.7)} \quad \lambda(A_k) = \frac{\pi}{|x - \alpha|} \quad \text{and} \quad \lambda(B_k) = \frac{\pi}{|x - \beta|}.
\]

We will use the notation that \( d_\mathbb{R}(w) \) denotes the distance from \( w \in \mathbb{R} \) to the nearest integer. Note that \( d_\mathbb{R}(w) = |w| \) if and only if \(-1/2 \leq w \leq 1/2\). Note also that \( |\sin(\pi w)| \geq 2d_\mathbb{R}(w) \) for all \( w \in \mathbb{R} \) (for example see [7, p. 89] or [10, p. 233]).

Now

\[
u \in A_j \implies \frac{(j - 1/2)\pi}{|x - \alpha|} \leq u \leq \frac{(j + 1/2)\pi}{|x - \alpha|} \implies -1/2 \leq |x - \alpha| \left| \frac{u}{\pi} - \frac{j}{|x - \alpha|} \right| \leq 1/2.
\]
So, for \( u \in A_j \),
\[
| \sin(u(x - \alpha)) | = \left| \sin \left( \frac{\pi}{\alpha} - \frac{j}{u \alpha} \right) \right| \\
\geq 2 \left| \frac{\pi}{\alpha} - \frac{j}{u \alpha} \right| \\
= \frac{2}{\pi} | x - \alpha | - \frac{j}{| x - \alpha |} \\
(2.8)
\]
Similarly, for \( u \in B_k \),
\[
(2.9) \\
| \sin(u(x - \beta)) | \geq \frac{2}{\pi} | x - \beta | \left| \frac{u}{| x - \beta |} \right| - \frac{k\pi}{| x - \beta |} \\
We see from (2.8) and (2.9) that for all \( u \in A_j \cap B_k \) we have
\[
| \sin(u(x - \alpha)) | \sin(u(x - \beta)) | \geq \frac{4}{\pi^2} | (x - \alpha)(x - \beta) | \left| \frac{u}{| x - \alpha |} \right| \left| \frac{u}{| x - \beta |} \right| \\
(2.10)
\]
where \( a_j \) and \( b_k \) are the points as given in (2.5).
Now, recall that \( x \notin \{ \alpha, \beta \} \) has been given. Let also \( c > 0 \) be given, and let the intervals \( A_j \) such that \( \lambda(A_j \cap [-c, c]) > 0 \) be \( A_{m_1}, \ldots, A_{m_{r-1}} \), and let the intervals \( B_k \) such that \( \lambda(B_k \cap [-c, c]) > 0 \) be \( B_{m_2}, \ldots, B_{m_{s-1}} \).
Then, put
\[
(2.11) \quad \mathcal{P}_1 = \{ A_{m_1}, A_{m_1+1}, \ldots, A_{m_{r-1}} \} \quad \text{and} \quad \mathcal{P}_2 = \{ B_{m_2}, B_{m_2+1}, \ldots, B_{m_{s-1}} \}.
\]
Note that in (2.11), \( \mathcal{P}_1 \) is a partition of some closed interval into closed subintervals in the sense described in [8, p.1430]. The same comment applies to \( \mathcal{P}_2 \). We put
\[
(2.12) \quad \mathcal{A} = \{(j,k) : 0 < j < r - 1, 0 \leq k \leq s - 1, \quad \text{and} \quad \lambda(A_{m_{1+j}} \cap B_{m_{2+k}}) > 0 \},
\]
\[
(2.13) \quad \mathcal{P} = \{ A_{m_{1+j}} \cap B_{m_{2+k}} : (j,k) \in \mathcal{A} \},
\]
and we observe that
\[
[-c, c] \subseteq \bigcup_{(j,k) \in \mathcal{A}} A_{m_{1+j}} \cap B_{m_{2+k}}.
(2.14)
\]
The family \( \mathcal{P} \) of closed intervals in (2.13) is a partition of some closed interval into closed subintervals, and by (2.12) and Lemma 3.2 in [8], we have:
\[
(2.15) \quad (\text{the number of intervals in } \mathcal{P}) = (\text{the number of elements of } \mathcal{A}) \leq r + s - 1.
\]
Now, from (2.7) we see that all lengths of the \( r \) intervals in the closed-interval partition \( \mathcal{P}_1 \) equal \( \pi/|x - \alpha| \), so that \( (r-2)\pi/|x - \alpha| < 2c \). Hence,
\[
(2.16) \quad 1 < r < \frac{2c|x - \alpha|}{\pi} + 2 = \frac{2c}{\pi} \left( 1 + \frac{\pi}{c|x - \alpha|} \right) |x - \alpha|.
\]
Let $0 < \delta < 1/2$. Then, if $|x - \alpha| > \pi \delta/c$, we have from (2.16) that

$$1 \leq r < \frac{2c}{\pi} \left(1 + \frac{1}{\delta}\right) |x - \alpha|.$$  

(2.17)

On the other hand, if $|x - \alpha| \leq \pi \delta/c$, as $0 < \delta < 1/2$ we have $2c < \pi/|x - \alpha|$, and it follows from (2.7) that $[-c, c] \subseteq A_0$, so that $m_1 = 0$ and

$$r = 1.$$  

(2.18)

Again let $0 < \delta < 1/2$. Then, as in the preceding argument, but with $\beta$ replacing $\alpha$, if $|x - \beta| > \pi \delta/c$ we have

$$1 \leq s < \frac{2c}{\pi} \left(1 + \frac{1}{\delta}\right) |x - \beta|,$$

while if $|x - \beta| \leq \pi \delta/c$, we have

$$s = 1.$$  

(2.20)

Now, again let $0 < \delta < 1/2$. We see now from (2.17), (2.18), (2.19) and (2.20) that if either $|x - \alpha| > \pi \delta/c$ or $|x - \beta| > \pi \delta/c$ (with both perhaps holding) then we have

$$r + s - 1 < 2 \max\{r, s\}$$

$$\leq 2 \max\left\{\frac{2c}{\pi} \left(1 + \frac{1}{\delta}\right) |x - \alpha|, \frac{2c}{\pi} \left(1 + \frac{1}{\delta}\right) |x - \beta|\right\}$$

$$= \frac{4c}{\pi} \left(1 + \frac{1}{\delta}\right) \max\{|x - \alpha|, |x - \beta|\}.$$  

(2.21)

Also, observe that if $0 < \delta < 1/2$, $|x - \alpha| \leq \pi \delta/c$ and $|x - \beta| \leq \pi \delta/c$, we have from (2.18) and (2.20) that

$$r = s = 1.$$  

(2.22)

Note that in the above, $a_k, b_k, A_k, B_k$ and so on, depend upon $x$ and $c$. Also, $r$ and $s$ depend upon $x$ and $c$.

We now take $m \in \mathbb{N}$ with $m \geq 4s + 1$, and we estimate the integral

$$\int_{[-c, c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^m \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)},$$

allowing for the different values $x$ may be, but recall that $x \notin \{\alpha, \beta\}$. We let $\mathcal{P}_1$, $\mathcal{P}_2$ be the partitions as given in (2.11) and let $\mathcal{P}$ be the partition as in (2.13). We
have, using the definitions and (2.5), (2.10), (2.13) and (2.14),

\[
\int_{[-c,c]^m} du_1 du_2 \ldots du_m \\
\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)
\]

\[
\leq \sum_{(j_1,k_1),\ldots,(j_m,k_m) \in A} \int \prod_{t=1}^{m} A_{m+1+j_t} \cap B_{m+2+k_t} \\
\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)
\]

\[
\leq \left( \frac{\pi^{4s}}{2^{4s}(x-\alpha)^{2s}(x-\beta)^{2s}} \right)^{\frac{1}{2}}.
\]

(2.23)

\[
\sum_{(j_1,k_1),\ldots,(j_m,k_m) \in A} \int \prod_{t=1}^{m} A_{m+1+j_t} \cap B_{m+2+k_t} \\
\sum_{j=1}^{m} (u_j - a_{m+1+j_t}^2)(u_j - b_{m+2+k_t}^2)
\]

Now in (2.23), we have \(a_{m+1+j_t} \in A_{m+1+j_t}\) and \(b_{m+2+k_t} \in B_{m+2+k_t}\), but neither \(a_{m+1+j_t}\) nor \(b_{m+2+k_t}\) necessarily belongs to \(A_{m+1+j_t} \cap B_{m+2+k_t}\). If \(a_{m+1+j_t} \in A_{m+1+j_t} \cap B_{m+2+k_t}\) put \(\tilde{a}_{m+1+j_t} = a_{m+1+j_t}\); otherwise let \(\tilde{a}_{m+1+j_t}\) be the endpoint of \(A_{m+1+j_t} \cap B_{m+2+k_t}\) closest to \(a_{m+1+j_t}\). If \(b_{m+2+k_t} \in A_{m+1+j_t} \cap B_{m+2+k_t}\) put \(\tilde{b}_{m+2+k_t} = b_{m+2+k_t}\); otherwise let \(\tilde{b}_{m+2+k_t}\) be the endpoint of \(A_{m+1+j_t} \cap B_{m+2+k_t}\) closest to \(b_{m+2+k_t}\). Then, from Lemma 2.2, for all \(t \in \{1, 2, \ldots, m\}\), we have that in (2.23)

(2.24)

\[
|(u-a_{m+1+j_t})(u-b_{m+2+k_t})| \geq |(u-\tilde{a}_{m+1+j_t})(u-\tilde{b}_{m+2+k_t})|, \text{ for all } u \in A_{m+1+j_t} \cap B_{m+2+k_t}.
\]

Now let \(0 < \delta < 1/2\) and assume that we have either \(|x-\alpha| > \pi\delta/c\) or \(|x-\beta| > \pi\delta/c\). Then from (2.15), the right hand side of (2.21) gives an upper bound for the number of elements in \(P\). Using (2.23) and (2.24), and then using (2.21), the assumption
So far, we have in this case that

\[
\int_{[-c,c]^m} \sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta) \leq \frac{\pi^{4s}}{24^s (x - \alpha)^{2s} (x - \beta)^{2s}}.
\]

Then, define disjoint intervals \(J, K\) such that

\[
\max \left\{ \max \left\{ |x - \alpha|^m, |x - \beta|^m \right\}, \min \left\{ \frac{\pi^{m-4s}}{|x - \alpha|^{m-4s}}, \frac{\pi^{m-4s}}{|x - \beta|^{m-4s}} \right\} \right\},
\]

where we have used (2.7),

\[
(2.25)
\]

so that \(m \geq 4s + 1\), and Lemma 4.1 in [8], we have in this case that

\[
\int_{[-c,c]^m} \sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta) \leq \frac{\pi^{4s}}{24^s (x - \alpha)^{2s} (x - \beta)^{2s}}.
\]

As well, \((2.26)\) holds for all \(x \in \mathbb{R}\) such that either \(|x - \alpha| > \pi \delta / c\) or \(|x - \beta| > \pi \delta / c\). We now consider the cases where \(\alpha \neq \beta\) and \(\alpha = \beta\).

CASE I: \(\alpha \neq \beta\).

In this case, choose \(\delta\) so that

\[
0 < \delta < \min \left\{ \frac{1}{2}, \frac{c |\alpha - \beta|}{2 \pi} \right\}.
\]

Then, define disjoint intervals \(J, K\) by putting

\[
J = \left[ \alpha - \frac{\pi \delta}{c}, \alpha + \frac{\pi \delta}{c} \right] \quad \text{and} \quad K = \left[ \beta - \frac{\pi \delta}{c}, \beta + \frac{\pi \delta}{c} \right].
\]

Clearly, there is \(C_1 > 0\) such that

\[
(2.26) \quad \max \left\{ \frac{(x - \alpha)^{2s}}{(x - \beta)^{2s}}, \frac{(x - \beta)^{2s}}{(x - \alpha)^{2s}} \right\} \leq C_1, \quad \text{for all } x \in (J \cup K)^c.
\]

As well, \((x - \beta)^{2s}\) is bounded on \(J\), so we see that there is \(C_2 > 0\) such that

\[
(2.27) \quad \max \left\{ \frac{(x - \alpha)^{2s}}{(x - \beta)^{2s}}, \frac{(x - \beta)^{2s}}{(x - \alpha)^{2s}} \right\} \leq C_2, \quad \text{for all } x \in J \cap \{\alpha\}^c.
\]
And, as \((x - \alpha)^{-2s}\) is bounded on \(K\), there is \(C_3 > 0\) such that

\[
\max \left\{ \frac{(x - \alpha)^{2s}}{(x - \beta)^{2s}}, \frac{(x - \beta)^{2s}}{(x - \alpha)^{2s}} \right\} (x - \beta)^{2s} \leq C_3, \text{ for all } x \in K \cap \{\beta\}^c.
\]

We now have from (2.25), (2.26), (2.27) and (2.28), that

\[
\int_{-\infty}^{\infty} \left( \int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)} \right) |\hat{f}(x)|^2 dx
\]

\[
\leq C_1 \int_{(J \cup K)^c} |\hat{f}(x)|^2 dx + C_2 \int_{J} |\hat{f}(x)|^2 dx + C_3 \int_{K} |\hat{f}(x)|^2 dx < \infty,
\]

(2.29)

as we are assuming that \(\int_{-\infty}^{\infty} |\hat{f}(x)|^2 (x - \alpha)^{-2s} (x - \beta)^{-2s} dx < \infty\).

CASE II. \(\alpha = \beta\).

Let’s assume that \(\alpha \in (-c, c)\) and that

\[
\delta < \min \left\{ \frac{1}{2}, \frac{c(c - |\alpha|)}{\pi} \right\}.
\]

Put \(L = (\alpha - \pi \delta/c, \alpha + \pi \delta/c)\), and observe that because of (2.30), \(L \subseteq (-c, c)\). Let \(x \in L\) be given. Then, \(|x - \alpha| < \pi \delta/c\) and as \(\delta < 1/2\), it follows that \(c < \pi/2|x - \alpha|\). Consequently, using the definitions of \(A_0\) and \(B_0\) as given by (2.5) and (2.6), we see that \((-c, c) \subseteq A_0 = B_0\). Note that although \(A_0\) and \(B_0\) each depends upon \(x\), \((-c, c) \subseteq A_0 = B_0\) occurs regardless of \(x \in L\). Putting \(j = k = 0\) in (2.10), we now deduce that for all \(u \in (-c, c)\) and all \(x \in L\),

\[
|\sin(u(x - \alpha))| \geq \frac{2}{\pi} |u| \cdot |x - \alpha|.
\]

Let \(C > 0\) be such that

\[
\sum_{j=1}^{m} u_j^{4s} \geq C \left( \sum_{j=1}^{m} u_j^{2s} \right)^{2s}, \text{ for all } (u_1, u_2, \ldots, u_m) \in \mathbb{R}^m.
\]
We now have from (2.31) and (2.32) that if $m \geq 4s + 1$ and $x \in L$, 
\[
\int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{4s} u_j (x - \alpha)} 
\leq \frac{\pi^{4s}}{2^{4s}(x - \alpha)^{4s}} \int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} u_j^{4s}} 
\leq \frac{1}{C} \frac{\pi^{4s}}{2^{4s}(x - \alpha)^{4s}} \int_{[-c,c]^m} \left( \sum_{j=1}^{m} u_j^{2s} \right)^{2s} 
\leq \frac{D}{C} \frac{\pi^{4s}}{2^{4s}(x - \alpha)^{4s}} \int_{0}^{c} \sqrt{m} u_j^{m-4s-1} dr,
\]
for some $D > 0$, by [10, pp. 394-395],

(2.33)

\[
\leq \frac{G}{(x - \alpha)^{4s}},
\]

for some $G > 0$ that is independent of $x \in L \cap \{\alpha\}^c$.

On the other hand, if $x \notin L$ we have $|x - \alpha| \geq \pi \delta / c$, so that if we apply (2.25) with $\alpha = \beta$ we have

(2.34)

\[
\int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{4s} u_j (x - \alpha)} \leq Q < \infty.
\]

Assuming that $|\alpha| < c$, we now have, using (2.33) and (2.34), that

\[
\int_{-\infty}^{\infty} \left( \int_{[-c,c]^m} \frac{du_1 du_2 \ldots du_m}{\sum_{j=1}^{m} \sin^{4s} u_j (x - \alpha)} \right) |\hat{f}(x)|^2 dx 
\leq G \int_{L} \frac{|\hat{f}(x)|^2}{(x - \alpha)^{4s}} dx + Q \int_{L^c} |\hat{f}(x)|^2 dx < \infty,
\]

(2.35)

as $\alpha = \beta$ and we are assuming that $\int_{-\infty}^{\infty} |\hat{f}(x)|^2 (x - \alpha)^{-2s}(x - \beta)^{-2s} dx < \infty$.

We have considered both the cases $\alpha \neq \beta$ and $\alpha = \beta$. The dénouement results from using Fubini’s Theorem, (2.29) and (2.35). We see that provided $|\alpha| < c$ and $m \geq 4s + 1$, in both cases we have

\[
\int_{[-c,c]^m} \left( \int_{-\infty}^{\infty} \frac{\sum_{j=1}^{m} |\hat{f}(x)|^2 dx}{\sum_{j=1}^{m} \sin^{2s} u_j (x - \alpha) \sin^{2s} u_j (x - \beta)} \right) du_1 du_2 \ldots du_m < \infty.
\]
We conclude from this that for almost all \((u_1, u_2, \ldots, u_m) \in [-c, c]^m\),
\begin{equation}
(2.36) \quad \int_{-\infty}^{\infty} \frac{\left|\hat{f}(x)\right|^2 dx}{\sum_{j=1}^{m} \sin^{2s}(u_j(x - \alpha)) \sin^{2s}(u_j(x - \beta))} < \infty.
\end{equation}

By letting \(c\) tend to \(\infty\) through a sequence of values, we deduce that, in fact, the inequality in (2.36) holds for almost all \((u_1, u_2, \ldots, u_m) \in \mathbb{R}^m\). But then, using (2.3), (2.4) and Theorem 2.1, we see that provided \(m \geq 4s + 1\), for almost all \((u_1, u_2, \ldots, u_m) \in \mathbb{R}^m\) there are \(f_1, f_2, \ldots, f_m \in L^2(\mathbb{R})\) such that

\[
f = \sum_{j=1}^{m} \left( e^{iu_j(\alpha/2) + e^{-iu_j(\alpha/2)})} \delta_0 - e^{iu_j(\alpha/2)} \delta_{u_j} + e^{-iu_j(\alpha/2)} \delta_{-u_j} \right) * f_j.
\]

We deduce that (i) implies (ii) in Theorem 2.3 and, by taking \(m = 4s + 1\), we see that (i) implies (iii).

We have now proved that (i), (ii) and (iii) are equivalent. Also, we have proved statement (iv) that (iii) is possible for almost all \((u_1, u_2, \ldots, u_{4s+1}) \in \mathbb{R}^{4s+1}\).

The final statements (v) and (vi) now follow in a routine way, using as needed the equivalence of the statements (i), (ii) and (iii). This completes the proof of Theorem 2.3. 

\[\square\]

Note that in Theorem 2.3, if we take the special case \(\alpha = \beta = 0\) we obtain the identity (1.2) for the case \(s = 2\), proved originally in [6] and [7].

In the case when \(\alpha, \beta \in \mathbb{Z}\), and if we identify \(\mathbb{T}\) with \([0, 2\pi)\) in the usual way, we can define a generalised \((\alpha, \beta)\)-difference of order \(s\) in \(L^2(\mathbb{T})\) to be a function as given in (1.6), but with \(g \in L^2([0, 2\pi))\) and \(u \in [0, 2\pi)\). Then, by analogy with \(D_{\alpha,\beta,s}(\mathbb{R})\), make the definition that \(D_{\alpha,\beta,s}(\mathbb{T})\) is the vector subspace of \(L^2(\mathbb{T})\) consisting of finite sums of generalised \((\alpha, \beta)\)-differences of order \(s\) in \(L^2(\mathbb{T})\). It was proved in [8, Theorem 2.3] that
\begin{equation}
(2.37) \quad D_{\alpha,\beta,s}(\mathbb{T}) = \{f : f \in L^2(\mathbb{T}) \text{ and } \hat{f}(\alpha) = \hat{f}(\beta) = 0\}.
\end{equation}

There is an obvious similarity between this fact and the result derived from Theorem 2.3 which is that
\begin{equation}
(2.38) \quad D_{\alpha,\beta,s}(\mathbb{R}) = \left\{f : f \in L^2(\mathbb{R}) \text{ and } \int_{-\infty}^{\infty} \frac{\left|\hat{f}(x)\right|^2}{(x - \alpha)^{2s}(x - \beta)^{2s}} dx < \infty \right\}.
\end{equation}

However, note in (2.37) that the right hand side is independent of \(s\) whereas in (2.38) the right hand side depends upon \(s\). At first sight this may seem surprising, but each equality expresses a condition that \(\hat{f}\) “vanishes” at or near \(\alpha\) and \(\beta\). Since the dual \(\mathbb{Z}\) of \(\mathbb{T}\) is discrete, the only way this can occur in the case of \(\mathbb{T}\) is if \(\hat{f}\) actually vanishes at \(\alpha\) and \(\beta\), and this forces the independence from \(s\) in the right hand side of (2.37). In the case of \(\mathbb{R}\), however, because the dual of \(\mathbb{R}\) is itself and so is a continuum, there is an infinity of possible behaviours of \(\hat{f}\) near \(\alpha\) and \(\beta\) expressing the idea that \(\hat{f}\) “vanishes” near \(\alpha\) and \(\beta\), and we observe a dependence upon \(s\) in the right hand side of (2.38).

Another difference between \(D_{\alpha,\beta,s}(\mathbb{T})\) and \(D_{\alpha,\beta,s}(\mathbb{R})\) is that the former has finite algebraic codimension in \(L^2(\mathbb{T})\) while the latter has infinite algebraic codimension in \(L^2(\mathbb{R})\). Note further that when \(\alpha, \beta \in \mathbb{Z}\), it has been shown [8, Theorem 2.3] that \((D^2 - i(\alpha + \beta)D - \alpha \beta I)^s\) maps \(W^{2s}(\mathbb{T})\) onto \(D_{\alpha,\beta,s}(\mathbb{T})\) (which is independent
of $s$), while here we have seen that $(D^2 - i(\alpha + \beta)D - \alpha\beta I)^s$ maps $W^{2s}(\mathbb{R})$ onto $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$.

In [5] Meisters and Schmidt showed that every translation invariant linear form on $L^2(\mathbb{T})$ is continuous, but in [3] Meisters showed that there are discontinuous translation invariant linear forms on $L^2(\mathbb{R})$, and this latter result may also be deduced from the identity (1.2) in the case $s = 1$. The following introduces, in the present context, a notion corresponding to translation invariant linear forms.

**Definition.** Let $\alpha, \beta \in \mathbb{R}$ and let $s \in \mathbb{N}$. Then a linear form $T$ on $L^2(\mathbb{R})$ is called $(\alpha, \beta, s)$-invariant if, for all $f \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$,

$$T \left( \left[ e^{iu(\frac{\alpha - \delta}{2} + \beta)} + e^{-iu(\frac{\alpha - \delta}{2})} \right] \delta_u - \left( e^{iu(\frac{\alpha + \delta}{2})} \delta_u + e^{-iu(\frac{\alpha + \delta}{2})} \delta_{-u} \right) \right)^s \ast f \right) = 0.$$  

Equivalently, the linear form $T$ on $L^2(\mathbb{R})$ is $(\alpha, \beta, s)$-invariant when $T(\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})) = \{0\}$.

A linear form $T$ on $L^2(\mathbb{R})$ is $(\alpha, -\alpha, 1)$-invariant when, for all $f \in L^2(\mathbb{R})$ and $u \in \mathbb{R}$,

$$T \left( 2^{-1}(\delta_u + \delta_{-u}) \ast f \right) = \cos \alpha \, T(f),$$

from which we see that if $T$ is a translation invariant linear form on $L^2(\mathbb{R})$ it is $(0, 0, 1)$-invariant.

When $\alpha, \beta \in \mathbb{Z}$, we may also introduce the corresponding notion of $(\alpha, \beta, s)$-invariant linear forms on $L^2(\mathbb{T})$. It was shown in [8, Theorem 7.1] that an $(\alpha, \beta, 1)$-invariant linear form on $L^2(\mathbb{T})$ is continuous and, in fact, any $(\alpha, \beta, s)$-invariant linear form on $L^2(\mathbb{T})$ is continuous (proved by the technique used for the case $s = 1$ in [8]). However, the following corollary to Theorem 2.3 shows that the situation pertaining to translation invariant linear forms on $L^2(\mathbb{R})$ is mirrored by that for $(\alpha, \beta, s)$-invariant linear forms on $L^2(\mathbb{R})$.

**Corollary 2.4.** Let $\alpha, \beta \in \mathbb{R}$ and let $s \in \mathbb{N}$. Then, there are discontinuous $(\alpha, \beta, s)$-invariant linear forms on $L^2(\mathbb{R})$.

**Proof.** It is a consequence of Theorem 2.3 that $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$ has infinite algebraic codimension in $L^2(\mathbb{R})$. Consequently there are discontinuous linear forms on $L^2(\mathbb{R})$ that vanish on $\mathcal{D}_{\alpha,\beta,s}(\mathbb{R})$, and such forms are $(\alpha, \beta, s)$-invariant. \qed

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**References**


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