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Nirmalendu Chaudhuri  
*University of Wollongong, chaudhur@uow.edu.au*
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Bounds for the Best Constant in an Improved Hardy-Sobolev Inequality

N. Chaudhuri

Abstract. In this note we show that the best constant $C$ in the improved Hardy-Sobolev inequality of Adimurthi, Chaudhuri and Ramaswamy [1] for $2 \leq p < n$ is bounded by $\frac{p-1}{p} \left( \frac{n-p}{p} \right)^{p-2} \leq C \leq \frac{p-1}{2} \left( \frac{n-p}{p} \right)^{p-2}.$

Keywords: Hardy-Sobolev inequality, best constant in inequality

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1. Introduction

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$ ($n \geq 2$) with $0 \in \Omega$. Adimurthi, Chaudhuri and Ramaswamy in [1] have obtained the following improved Hardy-Sobolev inequality. Let $1 < p < n$ and let $R \geq e^2 \sup_{\Omega} |x|$. Then there exists a constant $C > 0$ such that

$$\int_{\Omega} |\nabla u|^p dx \geq \left( \frac{n-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx + C \int_{\Omega} \frac{|u|^p}{|x|^p} \left( \log \frac{R}{|x|} \right)^{-2} dx \quad (1.1)$$

holds for all $u \in W_0^{1,p}(\Omega)$. In his book on Sobolev Spaces [14: Section 2.1.6] Maz'ya discovered that the classical multi-dimensional Hardy-type inequalities with sharp constant can be improved by adding different additional positive integrals. However, the above inequality have applications in proving existence, non-existence and regularity of solutions for differential equations involving the potential $\frac{1}{|x|^p}$ (see [1, 3, 10 - 12, 15]). Adimurthi and Esteban [2] extended the above inequality for $W^{1,p}$ functions and found interesting applications to the Schrödinger operator. However, finding the best constant in inequality (1.1) remains open. In this article we find interesting bounds for the best

N. Chaudhuri: Australian Nat. Univ., Centre for Math. and its Appl., Canberra, ACT 0200, Australia; chaudhuri@maths.anu.edu.au

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constant $C(n, p, R, \Omega)$, defined in (1.4) below. In [1: Theorem 1.2] it has been shown that for $0 < \mu < \left( \frac{n-2}{p} \right)^p$ the eigenvalue problem

$$
- \left( \text{div}(|\nabla u|^{p-2} \nabla u) + \frac{\mu}{|x|^p} |u|^{p-2} u \right) = \lambda \frac{|u|^{p-2}}{|x|^p \left( \log \frac{R}{|x|} \right)^2} u \quad \text{in } \Omega \\
\quad \left. u = 0 \quad \text{on } \partial \Omega \right)
$$

(1.2)

admits a positive weak solution $u \in W_0^{1,p}(\Omega)$ corresponding to the eigenvalue $\lambda = \lambda_1^{\mu} > 0$. Moreover, $\lambda_1^{\mu} \to C(n, p, R, \Omega)$ as $\mu \to \left( \frac{n-2}{p} \right)^p$. Thus the bounds on the best constant in inequality (1.1) gives bounds on the limiting behaviour of the first eigenvalue for the eigenvalue problem (1.2).

In [1], the following $n$-dimensional version of the Hardy-Sobolev inequality also has been established. For any bounded domain $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) with $0 \in \Omega$,

$$
\int_{\Omega} |\nabla u|^n dx \geq \left( \frac{n-1}{n} \right)^n \int_{\Omega} \frac{|u|^n}{|x|^n} \left( \log \frac{R}{|x|} \right)^{-n} dx
$$

(1.3)

holds for every $u \in W_0^{1,n}(\Omega)$. Adimurthi and Sandeep [3] proved that the best constant herein is indeed $\left( \frac{n-1}{n} \right)^n$. For some interesting improvements of the classical Hardy-Sobolev inequality and their applications see [5 - 9, 13].

Before stating our theorem we define the best constant $C(n, p, R, \Omega)$ in inequality (1.1) by

$$
C(n, p, R, \Omega) = \inf_{0 \neq u \in W_0^{1,p}(\Omega)} Q_{\Omega, R}(u)
$$

(1.4)

where

$$
Q_{\Omega, R}(u) = \frac{\int_{\Omega} |\nabla u|^p dx - \left( \frac{n-p}{p} \right)^p \int_{\Omega} \frac{|u|^p}{|x|^p} dx}{\int_{\Omega} \frac{|u(x)|^p}{|x|^p} \left( \log \frac{R}{|x|} \right)^{-2} dx}.
$$

It is also known (see [1]) that the best constant in $C(n, p, R, \Omega)$ is not achieved.

2. Result

In this article we prove the following

**Theorem.** The constant $C(n, p, R, \Omega)$ defined by (1.4) is independent of the domain $\Omega$ and the choice of the constant $R \geq e^{\frac{1}{2}} \sup_{\Omega} |x|$. For $2 \leq p < n$,

$$
\frac{p-1}{p^2} \left( \frac{n-p}{p} \right)^{p-2} \leq C(n, p) \leq \frac{p-1}{2} \left( \frac{n-p}{p} \right)^{p-2}.
$$

It appears to the author that, for the case $2 \leq p < n$, the constant $C(n, p)$ herein is indeed $\frac{p-1}{p^2} \left( \frac{n-p}{p} \right)^{p-2}$.
Proof of the theorem. We prove the independence and the bounds for the best constant through the following steps.

Step 1. We first prove that if $B_i$ $(i = 1,2)$ are concentric balls centered at origin of radii $T_i$, then $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$, where $R_i = \alpha T_i$ with $\alpha \geq e^{1/2}$. For this take $u \in W_0^{1,p}(B_2)$ and define $v(x) = u(T_2^i x)$ for $|x| < T_1$. Then

$$Q_{B_1,R_1}(v) = \frac{\int_{B_1} |\nabla v|^p dx - (\frac{n-p}{p})^p \int_{B_1} \frac{|v|^p}{|x|^p} dx}{\int_{B_1} \frac{|v|^p}{|x|^p} (\log \frac{\alpha T_1}{|x|})^{-2} dx}$$

$$= \frac{\int_{B_2} |\nabla u|^p dx - (\frac{n-p}{p})^p \int_{B_2} \frac{|u|^p}{|x|^p} dx}{\int_{B_2} \frac{|u|^p}{|x|^p} (\log \frac{\alpha T_1}{|x|})^{-2} dx}$$

$$= Q_{B_2,R_2}(u)$$

and hence $C(n,p,R_1,B_1) = C(n,p,R_2,B_2)$.

Step 2. Now we prove that $C(n,p,R,\Omega) = C(n,p,R,\Omega^*)$, where $\Omega^* = B(0,T)$ is the ball of radius $T = (\frac{|\Omega|}{|B(0,1)|})^{1/n}$, $| \cdot |_n$ denoting the $n$-dimensional Lebesgue measure. Indeed, for any $u \in W_0^{1,p}(\Omega)$, $|u| \in W_0^{1,p}(\Omega^*)$, where $|u|$ is the symmetric decreasing rearrangement of $|u|$. By standard symmetrization arguments (see [4]) we conclude that, for any $u \in W_0^{1,p}(\Omega)$, $Q_{n,R}(u) \geq Q_{n,R}(u^*)$ and hence

$$C(n,p,R,\Omega) \geq C(n,p,R,\Omega^*) .$$

To prove the other inequality, take $s > 0$ such that $B_s = B(0,s) \subseteq \Omega$. Then, clearly, $C(n,p,R,\Omega) \leq C(n,p,R,B_s)$ and hence, by Step 1, $C(n,p,R,\Omega) = C(n,p,R,\Omega^*)$.

Now if $\Omega_i$ $(i = 1,2)$ are two bounded domains with $R_i \geq e^{1/2} \sup_{\Omega_i} |x|$, by Steps 1 and 2, $C(n,p,R_1,\Omega_1) = C(n,p,R_2,\Omega_2)$ and hence the constant is independent of the domain and the choice of $R$. We shall denote this constant simply by $C(n,p)$.

Step 3. Lower Bound: The lower bound for the best constant $C(n,p)$ essentially follows from [1: Proof of Theorem 1.1], but for sake of completeness we include a proof. Since $C(n,p)$ is independent of the domain, without loss of generality we assume $\Omega$ to be the unit ball $B = B(0,1)$. Let $R \geq e^{1/2}$. For $0 < u \in C_0^\infty(B)$ radially non-increasing we define

$$v(r) = u(r) r^{\frac{n-2}{p}} \quad (r = |x|). \quad (2.1)$$
Here without loss of generality we as well assume \( u'(r) < 0 \) (replacing \( u \) by \( u + \varepsilon(1 - r) \) for \( \varepsilon > 0 \) sufficiently small). Now we observe that

\[
\int_B |\nabla u|^p dx - \left( \frac{n - p}{p} \right)^p \int_B \frac{|u(x)|^p}{|x|^p} dx
\]

\[
= \omega_n \int_0^1 \left| \frac{n - p}{p} r^{-\frac{n-2}{p}} v(r) - r^{1-\frac{p}{n}} v'(r) \right|^p r^{n-1} dr
\]

\[
- \left( \frac{n - p}{p} \right)^p \omega_n \int_0^1 \frac{v^p(r)}{r} dr
\]

\[
= \omega_n \left( \frac{n - p}{p} \right)^p \int_0^1 v^p(r) \left\{ 1 - \frac{pv'(r)r}{(n-p)v(r)} \right\} ^p r^{n-1} dr
\]

where \( \omega_n \) is the volume of the \((n-1)\)-dimensional sphere. Since \( u \) is a decreasing function, from (2.1) we have \( v'(r) - \frac{(n-p)v(r)}{pr} < 0 \). We set \( x(r) = -\frac{pv'(r)r}{(n-p)v(r)} \) so that \( x(r) > -1 \). By using the inequality \((1 + x)^p \geq 1 + px + (p - 1)x^2\) for all \( x \geq -1 \) and for all \( p \geq 2 \) we obtain

\[
\int_B |\nabla u|^p - \left( \frac{n - p}{p} \right)^p \int_B \frac{|u(x)|^p}{|x|^p} dx
\]

\[
\geq \omega_n (p - 1) \left( \frac{n - p}{p} \right)^{p-2} \int_0^1 v^{p-2}(r)|v'(r)|^2 r dr
\]

\[
- \omega_n p \left( \frac{n - p}{p} \right)^{p-1} \int_0^1 v^{p-1}(r)v'(r) dr
\]

\[
= \frac{4\omega_n (p - 1)}{p^2} \left( \frac{n - p}{p} \right)^{p-2} \int_0^1 \left| (v^{p/2}(r))' \right|^2 r dr
\]

since \( v \in C^1_0(0, T) \). By applying the \( n \)-dimensional Hardy inequality (1.3) with \( n = 2 \) for the function \( v^{\frac{p}{2}} \), we obtain

\[
\int_0^1 \left| (v^{p/2}(r))' \right|^2 r dr \geq \frac{1}{4} \int_0^1 \left( \frac{v^{p/2}(r)}{r \log R} \right)^2 r dr
\]

\[
= \frac{1}{4} \int_0^1 \frac{u^p(r)}{r^p} \left( \log \frac{R}{r} \right)^{-2} r^{n-1} dr
\]

\[
= \frac{1}{4\omega_n} \int_B \frac{|u(x)|^p}{|x|^p} \left( \log \frac{R}{|x|} \right)^{-2} dx.
\]

Hence for all radially non-increasing functions \( 0 < u \in C^2_0(B) \) we have

\[
\int_B |\nabla u|^p - \left( \frac{n - p}{p} \right)^p \int_B \frac{|u(x)|^p}{|x|^p} dx
\]

\[
\geq \frac{p - 1}{p^2} \left( \frac{n - p}{p} \right)^{p-2} \int_B \frac{|u(x)|^p}{|x|^p} \left( \log \frac{R}{|x|} \right)^{-2} dx.
\]
Now by standard approximation and symmetrization this inequality holds for all \( u \in W_0^{1,p}(B) \) and hence \( C(n, p) \geq \frac{p - 1}{p^2} \left( \frac{n - p}{p} \right)^{p - 2} \).

Step 3. Upper Bound: Here our idea is to construct a family of functions \( \{u_{\epsilon,k}\}_{0<\epsilon<1} \) in \( W_0^{1,p}(B) \), where \( B = B(0,1) \) is the unit ball, and then to estimate \( Q_{B,R} \) for this family. Similar to the family found in [1], for any \( 0 < \epsilon < 1 \) and for \( 2 \leq k \in \mathbb{N} \) we define

\[
 u_{\epsilon,k}(r) = \begin{cases} 
 0 & \text{for } r \leq \epsilon^k \\
 \frac{\log \frac{r}{\epsilon^k}}{(k-1)r^{\frac{n-p}{p}} \log \frac{1}{\epsilon}} & \text{for } \epsilon^k \leq r \leq \epsilon \\
 \frac{\log \frac{1}{r}}{r^{\frac{n-p}{p}} \log \frac{1}{\epsilon}} & \text{for } \epsilon \leq r \leq 1.
\end{cases}
\]

Clearly, \( u_{\epsilon,k} \in W_0^{1,p}(B) \) is continuous and differentiable a.e., and its derivative is given by

\[
 u_{\epsilon,k}'(r) = \begin{cases} 
 0 & \text{for } 0 \leq r \leq \epsilon^k \\
 -\frac{1}{(k-1)r^{\frac{n-p}{p}} \log \frac{1}{\epsilon}} \left[ 1 - \frac{n-p}{p} \frac{\log \frac{r}{\epsilon^k}}{\epsilon^k} \right] & \text{for } \epsilon^k \leq r \leq \epsilon \\
 -\frac{1}{r^{\frac{n-p}{p}} \log \frac{1}{\epsilon}} \left[ 1 + \frac{n-p}{p} \frac{\log \frac{1}{r}}{r} \right] & \text{for } \epsilon \leq r \leq 1.
\end{cases}
\]

Since \( \epsilon > 0 \) is sufficiently small, after a change of variables and the use of Neumann series we have the estimates

\[
 \int_B |\nabla u_{\epsilon,k}|^p dx = \frac{\omega_n}{(\log \frac{1}{\epsilon})^p} \left[ \frac{1}{(k-1)^p} \int_{\epsilon^k}^{\epsilon} \left| \frac{n-p}{p} \frac{\log \frac{r}{\epsilon^k}}{r} - 1 \right|^p \frac{dr}{r} \\
 + \int_{\epsilon}^{1} \left| 1 + \frac{n-p}{p} \frac{\log \frac{1}{r}}{r} \right|^p \frac{dr}{r} \right] \\
 = \frac{\lambda_{n,p} \omega_n}{p+1} \log \frac{1}{\epsilon} \left[ (k-1) \left( 1 - \frac{p}{(k-1)(n-p) \log \frac{1}{\epsilon}} \right)^{p+1} \\
 + \left( 1 + \frac{p}{(n-p) \log \frac{1}{\epsilon}} \right)^{p+1} \right] \\
 = \frac{\lambda_{n,p} \omega_n}{p+1} \log \frac{1}{\epsilon} \left[ (k-1) - \frac{p(p+1)}{(n-p) \log \frac{1}{\epsilon}} \\
 + \frac{p(p+1)}{2(k-1)(n-p) \log \frac{1}{\epsilon}} \left( \frac{p}{(n-p) \log \frac{1}{\epsilon}} \right)^2 \right]
\]
\begin{equation}
+ O\left( \frac{1}{(k-1)^2 \left( \log \frac{1}{\varepsilon} \right)^3} \right) + 1 + \frac{p(p+1)}{(n-p) \log \frac{1}{\varepsilon}}
+ \frac{p(p+1)}{2} \left( \frac{p}{(n-p) \log \frac{1}{\varepsilon}} \right)^2 + O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^3 \right]
= \frac{k\lambda_n p \omega_n}{p+1} \log \frac{1}{\varepsilon} + \frac{k\lambda_n p}{2(k-1)} \left( \frac{n-p}{p} \right)^{p-2} \left( \log \frac{1}{\varepsilon} \right)^{-1}
+ O\left( \frac{1}{(k-1) \log \frac{1}{\varepsilon}} \right)^2 + O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^2.
\end{equation}

Then we have
\begin{equation}
\int_B \frac{|u_{\varepsilon,k}|^p}{|x|^p} \, dx = \frac{\omega_n}{(\log \frac{1}{\varepsilon})^p} \left[ \frac{1}{(k-1)} \int_{\varepsilon^k}^{e} \left( \log \frac{r}{\varepsilon^k} \right)^p \frac{dr}{r} + \int_{\varepsilon}^{1} \left( \log \frac{1}{r} \right)^p \frac{dr}{r} \right]
= \frac{\omega_n}{(p+1)(\log \frac{1}{\varepsilon})^p} \left[ \frac{1}{(k-1)} \int_{e}^{e^k} \frac{d}{dr} \left( \log \frac{r}{\varepsilon^k} \right)^{p+1} \frac{dr}{r} \right]
- \int_{\varepsilon}^{1} \frac{d}{dr} \left( \log \frac{1}{r} \right)^{p+1} \frac{dr}{r}
= \frac{k\lambda_n p \omega_n}{2(k-1)} \left( \frac{n-p}{p} \right)^{p-2} \left( \log \frac{1}{\varepsilon} \right)^{-1} + O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^2.
\end{equation}

Thus (2.2) - (2.3) yield
\begin{equation}
\int_B |\nabla u_{\varepsilon,k}|^p - \left( \frac{n-p}{p} \right)^p \int_B \frac{|u_{\varepsilon,k}|^p}{|x|^p} \frac{dr}{r}
= \frac{k\lambda_n p \omega_n}{2(k-1)} \left( \frac{n-p}{p} \right)^{p-2} \left( \log \frac{1}{\varepsilon} \right)^{-1} + O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^2.
\end{equation}

Finally, let us try to find a “good” estimate of the integral
\begin{equation}
I_p = \int_B \frac{|u_{\varepsilon,k}|^p}{|x|^p} \left( \log \frac{R}{|x|} \right)^{-2} \, dx
= \frac{\omega_n}{(\log \frac{1}{\varepsilon})^p} \left[ \frac{1}{(k-1)} \int_{e^k}^{e} \left( \log \frac{r}{\varepsilon^k} \right)^p \frac{dr}{r} + \int_{\varepsilon}^{1} \left( \log \frac{1}{r} \right)^p \frac{dr}{r} \right].
\end{equation}

By change of variable \( r \mapsto \log \frac{R}{r} \) and denoting \( a_\varepsilon = \log \frac{R}{\varepsilon} \), \( b_\varepsilon = \log \frac{R}{e^k} \) and \( c = \log R \) we get
\begin{equation}
I_p = \frac{\omega_n}{((k-1) \log \frac{1}{\varepsilon})^p} \int_{a_\varepsilon}^{b_\varepsilon} \frac{\left( \log \frac{R}{\varepsilon^k} \right)^p}{r^2} \frac{dr}{r^2}
+ \frac{\omega_n}{(\log \frac{1}{\varepsilon})^p} \int_{c}^{a_\varepsilon} \frac{\left( \log \frac{\varepsilon'}{R} \right)^p}{r^2} \frac{dr}{r^2}
=: I_1^p + I_2^p.
\end{equation}
For the integrals $I_p^1$ and $I_p^2$ we get the estimations

\[
I_p^1 = \int_{a_\varepsilon}^{b_\varepsilon} \left( \log \frac{R}{\varepsilon} - r \right)^p \frac{dr}{r^2} \\
= b_\varepsilon^p \int_{a_\varepsilon}^{b_\varepsilon} \left( 1 - \frac{r}{b_\varepsilon} \right)^p \frac{dr}{r^2} \\
\geq b_\varepsilon^p \int_{a_\varepsilon}^{b_\varepsilon} \left( 1 - \frac{pr}{b_\varepsilon} + \frac{(p - 1)r^2}{b_\varepsilon^2} \right) \frac{dr}{r^2} \\
= \frac{b_\varepsilon^p}{a_\varepsilon} \left[ \left( 1 - \frac{a_\varepsilon}{b_\varepsilon} \right) \left( 1 + (p - 1)\frac{a_\varepsilon}{b_\varepsilon} \right) - \frac{pa_\varepsilon}{b_\varepsilon} \log \frac{b_\varepsilon}{a_\varepsilon} \right]
\]

and

\[
I_p^2 = \int_c^{a_\varepsilon} (r - \log R)^p \frac{dr}{r^2} \\
= \int_c^{a_\varepsilon} r^{p-2} \left( 1 - \frac{c}{r} \right)^p \frac{dr}{r} \\
\geq \int_c^{a_\varepsilon} r^{p-2} \left( 1 - \frac{pc}{r} + \frac{(p - 1)c^2}{r^2} \right) \frac{dr}{r} \\
= \begin{cases} 
\frac{a_\varepsilon}{2} \left[ 1 - \left( \frac{c}{a_\varepsilon} \right)^2 \right] + 2 \left( \frac{c}{a_\varepsilon} \right)^2 \log \frac{a_\varepsilon}{c} + o(1) & \text{for } p = 2 \\
\frac{a_\varepsilon^{p-1}}{p-1} \left[ 1 - \left( \frac{c}{a_\varepsilon} \right)^{p-1} \right] + o(1) & \text{for } p \neq 2, p \neq 3 
\end{cases}
\]

where $o(1) \to 0$ as $\varepsilon \to 0$. From these estimations for $I_p^1$ and $I_p^2$ we obtain

\[
I_p \geq J_{k,\varepsilon} := \frac{\omega_n}{((k - 1)\log \frac{1}{\varepsilon})^p} \frac{b_\varepsilon^p}{a_\varepsilon} \left[ \left( 1 - \frac{a_\varepsilon}{b_\varepsilon} \right) \left( 1 + (p - 1)\frac{a_\varepsilon}{b_\varepsilon} \right) - \frac{pa_\varepsilon}{b_\varepsilon} \log \frac{b_\varepsilon}{a_\varepsilon} \right] \\
+ \frac{\omega_n}{(\log \frac{1}{\varepsilon})^p} \frac{a_\varepsilon^{p-1}}{p-1} \left[ \frac{1}{p-1} + o(1) \right].
\]

Hence from (2.4) we obtain

\[
Q_{B,\varepsilon}(w_{\varepsilon,k}) \leq \frac{pk}{2(k - 1)} \left( \frac{n - p}{p} \right)^{p-2} \left( \log \frac{1}{\varepsilon} \right)^{p-1} \\
\times \left[ \frac{b_\varepsilon^p}{(k - 1)^p a_\varepsilon} \left\{ \left( 1 - \frac{a_\varepsilon}{b_\varepsilon} \right) \left( 1 + (p - 1)\frac{a_\varepsilon}{b_\varepsilon} \right) \right\} \right]
\]
\[
\quad + a_{\varepsilon}^{p-1} \left\{ \frac{1}{p-1} + o(1) \right\} \right]^{-1} + J_{k,\varepsilon}^{-1} \left[ O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^2 \right]
\]
\[
= \frac{pk}{2(k-1)} \left( \frac{n-p}{p} \right)^{p-2}
\times \left[ \frac{(k-1)-p b_{\varepsilon}^p}{a_{\varepsilon} \left( \log \frac{1}{\varepsilon} \right)^{p-1}} \left\{ \left( 1 - \frac{a_{\varepsilon}}{b_{\varepsilon}} \right) \left( 1 + (1-p) \frac{a_{\varepsilon}}{b_{\varepsilon}} \right) \right\}
\right.
\]
\[
\left. + \left( \frac{a_{\varepsilon}}{\log \frac{1}{\varepsilon}} \right)^{p-1} \left\{ \frac{1}{p-1} + o(1) \right\} \right]^{-1} + J_{k,\varepsilon}^{-1} \left[ O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^2 \right].
\]

Here we note that \( \frac{b_{\varepsilon}^p}{a_{\varepsilon}} \left( \log \frac{1}{\varepsilon} \right)^{p-1} \rightarrow k^p \) as \( \varepsilon \rightarrow 0 \) and hence \( J_{k,\varepsilon}^{-1} \left[ O\left( \frac{1}{\log \frac{1}{\varepsilon}} \right)^2 \right] \rightarrow 0 \) as either \( \varepsilon \rightarrow 0 \) or \( k \rightarrow \infty \). Thus
\[
Q_{B,R}(u_{\varepsilon,k}) \rightarrow \frac{pk}{2(k-1)} \left( \frac{n-p}{p} \right)^{p-2}
\times \left[ \left( \frac{k}{k-1} \right)^p \left\{ \left( 1 - \frac{1}{k} \right) \left( 1 + \frac{p-1}{k} \right) \right\}
\right.
\]
\[
\left. + \frac{p}{k \log \frac{1}{k}} + \frac{1}{p-1} \right]^{-1} \quad (\varepsilon \rightarrow 0)
\]
\[
\rightarrow \frac{p}{2} \left( \frac{n-p}{p} \right)^{p-2} \left[ 1 + \frac{1}{p-1} \right]^{-1} \quad (k \rightarrow \infty)
\]
\[
= \frac{p-1}{2} \left( \frac{n-p}{p} \right)^{p-2}.
\]

Since \( C(n,p) \leq Q_{B,R}(u_{\varepsilon,k}) \) for all \( k \geq 2 \) and for any sufficiently small \( \varepsilon > 0 \), by passing through the limits as \( \varepsilon \rightarrow 0 \) and \( k \rightarrow \infty \) we get \( C(n,p) \leq \frac{p-1}{2} \left( \frac{n-p}{p} \right)^{p-2} \)
and hence the theorem is proved \( \blacksquare \)

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