The energy levels and spacings of resonances

Arthur Joseph Gilks

Wollongong University College
THE ENERGY LEVELS AND SPACINGS OF RESONANCES

by

Arthur Joseph Gilks

Submitted for the Degree of Master of Science in the School of Mathematics at Wollongong University College, The University of New South Wales

November, 1968
STATEMENT

This work has not been submitted for a degree or similar award to any other University or Institution.

A.J. Gilks
18th November, 1968.
SUMMARY

It is known that the quantum states of the nucleus determine the energy levels at which resonances occur. This thesis describes the collision matrix and Wigner's R-matrix and derives the Breit-Wigner single level resonance formula in order to give the quantum mechanics background to resonance phenomena. A review of the early statistical descriptions of resonances is also given.

In this review it is shown that the random matrix hypothesis allows the development of the Porter-Thomas probability density function, \( f(y) = (2\pi y)^{-\frac{1}{2}} \exp(-y/2) \), for the reduced resonance widths. It is also shown that with a further assumption, the same hypothesis leads to Wigner's spacing distribution function, \( f(x) = \frac{1}{\sqrt{\pi}} x \exp(-\pi x^2/4) \).

Although Wigner's function is known to be a very good approximation to the exact distribution, the possibility of replacing it by one of the chi-squared functions,

\[
f_\nu(x) = (\frac{1}{2})^{\nu/2} \frac{1}{\sqrt{\pi}} \nu x^{\nu-1} \exp(-\nu x) / \Gamma(\nu),
\]

is considered and the resonance density is derived for the case where \( \nu \) is even.

Renewal theory is used to extend the work on resonance spacing by deriving the spacing density function for the superposition of \( N \) independent resonance sequences. In particular,
the case of two sequences, each with Wigner's spacing function or with particular chi-squared functions is considered.

From an analogy with the use of synthetic kernels in slowing down theory, a synthetic kernel for the spacing probability density function is used to evaluate infinitely dilute resonance integrals and the results are compared with those of Reichel and Wilkins (1964).

Schmidt's (1966) set of $^{238}\text{U} \ell = 0$ resonance parameters is shown to be statistically acceptable when it is compared with the Wigner and Porter-Thomas theories. The data are then matched, with chi-squared tests, against the chi-squared functions and those with $v = 6, 7, 8$ are shown to fit the spacing data well. These functions are recommended as reasonably accurate alternatives to Wigner's function.
PREFACE

There is quite a large gap between some of the recent statistical developments of nuclear resonance theory and the quantum mechanics theory upon which they are fundamentally based. This thesis is an attempt to bridge the gap, indicating where the great mathematicians and physicists such as Wigner, Porter and Mehta have contributed to our present understanding of the subject. It has proved to be a fascinating study, drawing on so many fields of mathematics.

The candidate wishes to express deep gratitude to his supervisor, Professor C.A. Wilkins (Wollongong University College) for his continual encouragement and help, Mr. J.L. Cook (Australian Atomic Energy Commission) for frequent discussions and helpful suggestions and to Mr. J.P. Pollard (Australian Atomic Energy Commission) for his assistance with computing.

The encouragement given by the Mathematics staff of the Royal Australian Naval College has been appreciated and the Department of the Navy is thanked for enabling the candidate to pursue this thesis.
# CONTENTS

## 1. NUCLEAR REACTION THEORY
- 1.1 Introduction
- 1.2 Channels
- 1.3 The Collision Matrix
- 1.4 Wigner's R-matrix
- 1.5 The Collision Matrix in terms of the R-matrix
- 1.6 Energy Level Wave Functions
- 1.7 The R-matrix in terms of Reduced Width Amplitudes
- 1.8 The Breit-Wigner Single-Level Resonance Formula

## 2. RESONANCE WIDTHS AND SPACINGS
- 2.1 General Remarks
- 2.2 Historical Development
  - 2.2.1 Level Spacings
  - 2.2.2 Reduced Widths
- 2.3 The Random Matrix Hypothesis
  - 2.3.1 The Hamiltonian Matrix
  - 2.3.2 The Statistical Properties of \( \mathcal{H} \)
- 2.4 The Distribution of Resonance Widths
- 2.5 The Distribution of Resonance Spacings
  - 2.5.1 Wigner's Surmise
  - 2.5.2 Mehta and Gaudin's Exact Work

## 3. RESONANCE SPACINGS - SIMPLIFYING APPROACHES
- 3.1 The Conditional Resonance Density Function in the Unresolved Region
  - 3.1.1 The Use of Chi-Squared Functions to Evaluate \( \Omega(x) \)
  - 3.1.2 The Use of Wigner's Function to Evaluate \( \Omega(x) \)
- 3.2 The Superposition of Resonance Sequences
  - 3.2.1 Distribution of the Spacings of the Superposed System in terms of \( D \)
  - 3.2.2 Distribution of the Spacings of the Superposed System in terms of \( x \)
1. NUCLEAR REACTION THEORY

1.1 Introduction

Nuclear reactions caused by bombardment are considered. The compound nucleus theory is studied from the quantum mechanics viewpoint with matrices used to relate the amplitude of the incoming wave function with properties of the outgoing wave functions which represent the various modes of break-up of the compound nucleus. This chapter concludes with the derivation of the Breit-Wigner single level resonance formula.

In Chapter 2 a further examination of the wave functions leads to the random matrix hypothesis which may be regarded as the starting point of the statistical theory of nuclear energy levels.

1.2 Channels

Nuclear reaction theory is built upon the concept of "channels" which, according to Blatt and Weisskopf (1952) was first used by Wigner and Eisenbud in unpublished work and by Breit in 1940.

\( X_\alpha \) and \( Y_\alpha \) are used to denote the incident particle and the target nucleus which collide to form a compound nucleus where the subscript \( \alpha \) identifies the particles and their energy levels. \( X_{\alpha'} \) and \( Y_{\alpha'} \) are used to denote a partition pair into which the compound nucleus decays.

The relative motion of \( Y_\alpha \) and \( X_\alpha \) accounts for the angular momentum vector \( L \) which is determined by the quantum numbers \( l, m \) (as far as the uncertainty principle will
allow by
\[
\begin{align*}
|L_z| &= \sqrt{L(L+1)\hbar^2} \\
L_z &= m\hbar
\end{align*}
\] (1.1)

In addition to this, there may be angular momentum due to the intrinsic spin of each particle and their vector sum is denoted $s$ which is specified by quantum numbers $s, \mu$ where
\[
\begin{align*}
|s| &= s\hbar \\
s_z &= \mu\hbar
\end{align*}
\] (1.2)

The fundamental principle of conservation of angular momentum deems that the total "spin" $J$ for the reaction remains constant where
\[
J = L + s
\] (1.3)

The "entrance channel", $c$, is then specified by the numbers $l, m, s, \mu$ and by $\alpha$. Any possible disintegration of the compound nucleus is specified by $\alpha'$ and by the corresponding quantum numbers $l', m', s', \mu'$ of $\nu'_{\alpha'}, \nu_{\alpha'}$. Such combinations, consistent with the principles of conservation of energy and total angular momentum are the "reaction channels", $c'$.

1.3 The Collision Matrix

The wave function, $\psi_c$, for a collision with only one entrance channel $c$ is a linear combination of the incoming wave function of the entrance channel $\mathcal{I}_c$ and the outgoing wave functions $\mathcal{O}_{c'}$ of all possible exit channels. That is
\[
\psi_c = \mathcal{I}_c - \sum_{c'} \psi_{c'} \mathcal{O}_{c'}.
\] (1.4)
The collision matrix $U$ is defined as the matrix whose elements are the coefficients $U_{c'c}$ of equation 1.4. However, to allow for more than one entrance channel, equation 1.4 is generalized to

$$
\psi = \sum_c C_c (\mathcal{J}_c - \sum_{c'} U_{c'c} \Theta_{c'})
$$

(1.5)

Preston (1962) expresses the incoming and outgoing wave functions in terms of the solutions of the free particle radial wave equation and the wave functions of the partition pairs involving the energy of the relative motion and the separation vector of the particles and the quantum numbers of the channels. It was originally shown by Breit (1940) that invariance of nuclear reaction theory under time reversal and the principle of conservation of probability imply that $U$ is both symmetric and unitary.

Wigner (1957a) states that the formal theory of resonance absorption has been largely the work of Eisenbud whose objective was the calculation of the collision matrix. He then explains that it is easier to calculate another matrix which he called the derivative matrix and which is now referred to as Wigner's $R$-matrix.

1.4 Wigner's $R$-matrix

The wave functions $J_c, \Theta_c$ may be separated into parts which are functions of the radial distance $r_c$ between $y_\alpha, y_\sigma$ and parts which are functions of the polar angles of their
separation vector. We let $I_c v_c^{-1/2}$ and $O_c v_c^{-1/2}$ be the radial parts of $I_c$ and $O_c$ where $v_c$ is the relative velocity at infinite separation of $X$ and $Y$. The factors $v_c^{-1/2}$ have been introduced to make $I_c$ and $O_c$ have unit flux. $I_c$ and $O_c$ are called the incoming and outgoing radial wave functions and Preston (ibid) shows that they are complex conjugate.

Taking the radial parts of equation 1.5 for channel $c$ we obtain

$$\psi_c(r_c) = (C_c I_c - \sum_{c'} C_{cc'} O_{cc'} v_c^{-1/2})$$

where $\psi_c(r_c)$ is the radial wave function in channel $c$. The total wave function $\psi$ of equation 1.5 may be expressed as the sum of the individual channel radial wave functions by

$$\psi = \sum_c \phi_c \psi_c$$

where $\phi_c$ is a set of orthonormal functions called the channel functions. Hence

$$\int \phi_c^* \psi \, ds = \int \sum_{c'} \phi_{c'}^* \phi_c \psi_{c'} \, ds$$

$$= \psi_c$$

where, in the integral, $r_c$ is kept constant and all other variables run over their complete ranges.

The standard approach, developed by Wiener and Eisenbud (1947), divides the configuration space into two parts. In the internal region (or reaction zone) any type of interaction between the colliding or separating particles...
may take place. However, in the external region, the interactions must leave the structures of the two particles unchanged so that the wave function is unchanged. The channel radius $a_c$ is defined as the value of $r_c$ which separates the two regions.

The constants $\Lambda_c$ are now introduced by

$$\Lambda_c = \frac{d}{dr_c} \psi_c(r_c) \bigg|_{r_c=a_c}$$

$$= v_c^{-\frac{1}{2}} a_c (c' \frac{d}{dr} - \sum_{c'} C_{c',c} U_{cc',0'c}) \bigg|_{r_c=a_c}$$

where $c', 0'$ are the radial derivatives of $I_c$, $n_c$. The $n$-matrix is then defined by its elements $R_{cc'}$, given by

$$(M_{c,c'})^{-\frac{1}{2}} \psi_c (a_c) = \sum_{c'} R_{cc'} (M_{c,c'})^{-\frac{1}{2}} \psi_{c'} \quad (1.10)$$

where $M_c$ is the reduced mass of $\alpha$, $\alpha$ given by

$$\frac{M_{\alpha} M_{\alpha}}{M_{\alpha} + M_{\alpha}}$$

$M_{\alpha}, M_{\alpha}$ are the masses of $\alpha, \alpha$.

In effect, $R$ relates the value of the entrance channel wave function at the channel radius with the derivatives of the wave functions in all the channels at their channel radii. On the other hand, the collision matrix relates the amplitudes of the incoming and outgoing waves.

1.5 The Collision matrix in terms of the $R$-matrix.

By substituting for $\psi_c (a_c)$ and $\Lambda_c$, in equation 1.10 the connection between $R$ and $U$ is found to be
\[
(M_a \cdot v_c)^{-\frac{1}{2}} (C_c I_c - \sum_{c'} C_{cc'}^0) \bigg| r_c = a_c
\]

\[
= \sum_{c''} R_{cc''} (M_{cc''} a_{cc''} v_{cc''})^{-\frac{1}{2}} a_{cc''} (C_c I_c' - \sum_{c'} C_{cc'}^0 a_{cc''}) \bigg| r_{cc''} = a_{cc''}
\]

that is

\[
k_c^{-\frac{1}{2}} a_c^{-\frac{1}{2}} (C_c I_c - \sum_{c'} C_{cc'}^0) \bigg| r_c = a_c
\]

\[
= \sum_{c''} R_{cc''} k_{cc''}^{-\frac{1}{2}} a_{cc''} (C_c I_c' - \sum_{c'} C_{cc'}^0 a_{cc''}) \bigg| r_{cc''} = a_{cc''}
\]

(1.12)

where \( k_c \) is the wave number of channel \( c \) given by

\[
k_c = \left( \frac{2 M_c E_c}{\hbar^2} \right)^{\frac{1}{2}} = \frac{M_c v_c}{\hbar}.
\]

(1.13)

In the last equation, \( E_c \) is the energy of the relative motion of the partition pair of channel \( c \) and \( \hbar \) is Planck's constant divided by \( 2\pi \).

Let \( C \) be a column matrix with elements \( C_c \). Also let \( k_{cc'}^{\frac{1}{2}} \) be a diagonal matrix with elements \( k_{cc'}^{\frac{1}{2}} = k_c^{\delta_{cc'}} \), and let similar equations define the diagonal matrices \( a_{cc'}^{\frac{1}{2}} \), \( I \), \( 0 \), \( I' \) and \( 0' \). The inverse matrices of \( k_{cc'}^{\frac{1}{2}} \) and \( a_{cc'}^{\frac{1}{2}} \), written \( k_{cc'}^{-\frac{1}{2}} \) and \( a_{cc'}^{-\frac{1}{2}} \) are also diagonal and have elements given by \( k_{cc'}^{-\frac{1}{2}} = k_c^{-\delta_{cc'}} \), and \( a_{cc'}^{-\frac{1}{2}} = a_c^{-\delta_{cc'}} \).

Equation 1.12 may now be written in terms of these matrices:-
\[ \text{whence} \]

\[ U = (k^2 a_{00}^{-1})^{-1} (k^2 a_{00}^{-1} \mathbf{I})^{-1} \]

\[ = (k^2 a_{00}^{-1})^{-1} (k^2 a_{00}^{-1} \mathbf{I})^{-1} = (k a_{00}^{-1})^{-1} (k a_{00}^{-1})^{-1} = k a_{00}^{-1} \]

Introduce the diagonal matrix \( D \) by \( D_{cc'} = a_{cc'}^{0} \delta_{cc'} \)

\[ \text{and its Hermitian adjoint} \]

\[ D^*_{cc'} = a_{cc'}^{0} \delta_{cc'} ^* \text{ so that} \]

\[ D = a_{00}^{-1} \text{ and } D^* = a_{11} \]

Using the commutative law of multiplication for diagonal matrices, we follow Preston (ibid) and write:

\[ \]

\[ U = (k a_{00}^{-1})^{-1} (1 - R a_{00}^{-1})^{-1} (1 - R a_{11}^{-1})^{-1} = (k a_{00}^{-1})^{-1} (1 - R L^{-1} (1 - R L^{-1})^{-1} \]

\[ = (k a_{00}^{-1})^{-1} (1 - R L)^{-1} (1 - R L^*)^{-1} (k a_{00}^{-1})^{-1} \]

where \( L \) is the identity matrix, \( (k a_{00}^{-1})^{-1} = \frac{1}{k} a_{00}^{-1} \) and

\[ (k a_{00}^{-1})^{-1} = \frac{1}{k} a_{00}^{-1} \frac{1}{k} = \frac{1}{k^2 a_{00}^{-1}}. \]

### 1.6 Energy Level Wave Functions

Let \( \psi_{\lambda} \) be the wave function of the compound system for the energy level \( \Gamma_{\lambda} \) where the subscript \( \lambda \) indicates all quantum numbers of the system. Such a function will satisfy a boundary condition and Schrödinger's equation which is

\[ H \psi_{\lambda} = E_{\lambda} \psi_{\lambda}. \]

The operator \( H \) is given by

\[ H = \sum_{i} \frac{\mathbf{p}^2}{m_i} + V \]

\[ (1.12) \]
where $V$ is the short range potential function, $m_j$ is the mass of each particle of the interactions of the channels and the Laplacian operator $\nabla^2$ is applied in the co-ordinate spaces of all the particles.

The part of the radial function in channel $c$ which $\chi_\lambda$ contributes to is denoted by $\psi_{\chi\lambda}(r_c)$. The wave functions $\psi_c$ are used to express the continuity condition between the internal and external regions by

$$\chi_\lambda = \sum_c \psi_c \psi_{\chi\lambda}(a_c) \tag{1.20}$$

where $\chi_\lambda$ is evaluated at the boundary so that

$$\psi_{\chi\lambda}(a_c) = \int \psi_c \chi_\lambda \, ds \tag{1.21}$$

with the integration as in equation 1.8.

The constant $\Delta_{\chi\lambda}$ which measures the radial derivative of $\psi_{\chi\lambda}(r_c)$ at the channel radius is introduced by analogy with equation 1.9, viz...

$$\Delta_{\chi\lambda} = a_c \frac{d\psi_{\chi\lambda}}{dr_c} \bigg|_{r_c=a_c} \tag{1.22}$$

Time reversal invariance may be used to show that $\psi_{\chi\lambda}(a_c)$ and hence $\Delta_{\chi\lambda}$ are real.

Boundary conditions are imposed on the channel radial wave functions in state $\lambda$ by

$$\xi_c = \frac{\Delta_{\chi\lambda}}{\psi_{\chi\lambda}(a_c)} \tag{1.23}$$
where $B_c$ is a constant so that

$$\psi_{\lambda}^\prime c (a_c) \Delta_c \psi_{\lambda}^\prime c - \psi_{\lambda}^\prime c (a_c) \lambda^\prime c = 0. \quad (1.24)$$

Now, because $E_\lambda$ is real, the complex conjugate of $\psi_{\lambda}$ is also a solution of equation 1.12 so we write

$$-\sum_{i=1}^{\infty} \frac{1}{2\pi i} \psi^2 \chi^\prime \psi_{\lambda} + \psi_{\lambda} \chi^\prime = \psi_{\lambda} \chi^\prime$$

Consider also the eigenfunction $X_{\lambda}$, which is associated with the eigenvalue $E_{\lambda}$, so that

$$-\sum_{i=1}^{\infty} \frac{1}{2\pi i} \chi^2 X_{\lambda} + \psi_{\lambda} \chi^\prime = \psi_{\lambda} \chi^\prime.$$

Multiplying the first of these by $X_{\lambda}$, the second by $X_{\lambda}$, subtracting and integrating over the entire interior region $\tau$, we get

$$\int_{\tau} d\tau \left\{ \psi^2 \chi^2 + \psi_{\lambda} \psi_{\lambda} \chi^\prime^2 \right\} = (E_{\lambda} - E_{\lambda}) \left\{ \psi_{\lambda} \chi^\prime^2 \right\} \tau. \quad (1.25)$$

Green's Theorem can be applied to the left side of equation 1.25 so that

$$(E_{\lambda} - E_{\lambda}) \int_{\tau} X_{\lambda} X_{\lambda} \chi^\prime = \int_{\tau} \frac{X_{\lambda}^2 (X_{\lambda} \chi^\prime - X_{\lambda} \chi^\prime) \tau}{c} \text{d}^2 \xi_{\lambda} \quad (1.25)$$

where $S_c$ is the surface of a sphere of radius $a_c$. Solving equation 1.20 to the right side of equation 1.26 leads to
\[ (E_\lambda - E'_\lambda, \int_T X^*_\lambda X_\lambda \, d\tau) \]

\[ = \frac{\hbar^2}{2M} \int_S \left( \sum_{c'} \psi^*_{c'} \phi_{c'} c' \psi_{c'} - \sum_{c''} \psi^*_{c''} \phi_{c'} c'' \psi_{c''} \phi_{c'} - \sum_{c'} \psi^*_{c'} \phi_{c'} c'' \psi_{c''} \phi_{c'} - \sum_{c''} \psi^*_{c''} \phi_{c'} c' \psi_{c'} \phi_{c''} \right) dS \]

\[ = \frac{\hbar^2}{2M} \left( \psi_{c'} \phi_{c'} c' \psi_{c'} \phi_{c'} - \psi_{c''} \phi_{c'} c'' \psi_{c''} \phi_{c'} - \psi_{c'} \phi_{c'} c'' \psi_{c'} \phi_{c''} - \psi_{c''} \phi_{c'} c' \psi_{c'} \phi_{c''} \right) \mid_{r_c = a_c} \]

\[ = 0 \quad (1.27) \]

where the orthonormality of the wave functions \( \phi_{c'} \) and boundary condition given by equation 1.24 have been used. Equation 1.27 implies the orthonormality of the level functions and they may be normalized so that

\[ \int_T X^*_\lambda X_\lambda \, d\tau = \delta_{\lambda \lambda'}, \quad (1.28) \]

1.7 The R-matrix in Terms of Reduced Width Amplitudes

For the interior region, any solution \( \Psi \) of equation 1.18 can be expanded in terms of the level functions by

\[ \Psi = \sum_{\lambda} A_{\lambda} \phi_{\lambda} \quad (1.29) \]

where the coefficients \( A_{\lambda} \) are given by

\[ A_{\lambda} = \int_T \phi_{\lambda}^* \Psi \, d\tau. \quad (1.30) \]

Because \( \Psi \) is a solution of Schrödinger's equation, the procedure
which led to equation 1.26 can be followed to obtain

$$\left( E_\lambda - E \right) \int \frac{X_\lambda^* \psi \, dt}{r} = \sum c \int \frac{\hbar^2}{2 M_c} (X_\lambda^* \frac{\partial}{\partial r_c} \psi - \psi \frac{\partial}{\partial r_c} X_\lambda^*) \, ds$$

(1.31)

where $E$ is the internal energy of the system.

Substitution for $\psi$ in the right side gives

$$\left( E_\lambda - E \right) \int \frac{X_\lambda^* \psi \, dt}{r}$$

$$= \sum c \int \frac{\hbar^2}{2 M_c} (\sum \psi_{a, c} \delta_{a, c} \sum \psi_{c, c'} \frac{d}{dr} \phi_{c, c'} - \sum \psi_{c, c'} \phi_{c, c'} \sum \psi_{c, c'} \frac{d}{dr} \phi_{c, c'} \, ds$$

$$= \sum c \int \frac{\hbar^2}{2 M_c} \psi_{a, c} \Delta_{a, c} - \psi_{c, c'} \Delta_{c, c'}$$

(1.32)

where $\Delta_{c, c'} = \frac{\delta_{a, c} - \psi_{c, c'} \delta_{a, c'}}{c}$.

Putting $A_\lambda$ for $\int \frac{X_\lambda^* \psi \, dt}{r}$ in the left side and rearranging gives

$$A_\lambda = \frac{1}{E_\lambda - E} \sum c \int \frac{\hbar^2}{2 M_c} \psi_{a, c} \Delta_{a, c}$$

(1.33)

which, when substituted into equation 1.25, yields

$$\psi = \int \frac{\hbar^2}{2 M_c} \psi_{a, c} \Delta_{a, c} \sum c \int \frac{\psi_{a, c} X_\lambda^*}{E_\lambda - E}$$

(1.34)

Multiplying both sides of equation 1.34 by $\phi_{c}^*$ and integrating over the surface $S_c$ gives

$$\int S_c \psi_{a, c}^* \, ds = \sum c \int \frac{\hbar^2}{2 M_c} \psi_{c, c'} \frac{d}{dr} \phi_{c, c'} \, ds$$

(1.35)

so that

$$\psi_c(r_c) = \sum c \int \frac{\hbar^2}{2 M_c} \psi_{c, c'} \frac{d}{dr} \phi_{c, c'} \, ds$$

(1.36)

Now equation 1.10, which is the defining equation of the R-matrix,
may be rearranged to the form:

$$\psi_c(a_c) = (M_c a_c)^{1/2} \sum_{c'} R_{cc'} (M_{c'} a_{c'})^{-1/2} \Delta^0_{c'c}.$$  \hspace{1cm} (1.37)

Setting $B_c = 0$ so that $\Delta^0_{c'c} = \Delta_{c'c}$, a comparison between equations 1.36 and 1.37 yields

$$R_{cc'} = \frac{\hbar^2}{2(M_c a_c)^{1/2}(M_{c'} a_{c'})^{1/2}} \sum_{\lambda} \frac{\psi_{\lambda c'}(a_{c'}) \psi_{\lambda c}(a_c)}{E_{\lambda} - E}$$

$$= \sum_{\lambda} \frac{\gamma_{\lambda c} \gamma_{\lambda c'}}{E_{\lambda} - E} \quad (1.38)$$

where $\gamma_{\lambda c} = \left( \frac{\hbar^2}{2M_c a_c} \right)^{1/2} \psi_{\lambda c}(a_c)$. \hspace{1cm} (1.39)

The same result may be obtained without this simplification by modifying the definition of the $R$-matrix. This in turn leads to a change in the value of $L$ as used in equation 1.14. (See Preston (ibid)).

Porter (1965) calls $\gamma_{\lambda c}$ the reduced width amplitude, but Blatt and Weisskopf (ibid) reserve that name for $\gamma^2_{\lambda c}$. Eisenbud and Wigner (1958) refer to $a_{\lambda c}^2 \gamma_{\lambda c}$ as the transition strength for the characteristic function $X_{\lambda}$ in channel $c$, indicating that $\frac{2M_c a_c}{\hbar^2} \gamma^2_{\lambda c}$ can be thought of as the probability that a stationary wave state $X_{\lambda}$ consists, on the surface, of the partition pair of channel $c$. 
Lane and Thomas (1958), Breit (1959) and Preston (ibid) give more detailed derivations of equation 1.38. The review article by Lane and Thomas (ibid) also gives Wigner's original derivation as well as a derivation from a variation principle.

1.8 The Breit-Wigner Single-Level Resonance Formula

In the derivation of this formula it is assumed that the constants $\gamma_{\lambda c}$ are real. Porter (ibid) shows that the reality of $\gamma_{\lambda c}$ depends upon time reversal and rotational symmetry properties of the nuclear reaction. According to Dyson (1962) this applies to the neutron bombardment of heavy nuclei. It should be noted however, that the more recent work on spectra, initiated by Dyson (ibid) allows for situations where these symmetries are poor and the $\gamma_{\lambda c}$'s are either complex or real quaternion.

The $R$-matrix may be expressed as the sum of two matrices, only one of which is dependent upon a particular $\lambda$. That is

$$R_{cc'} = \sum_{\lambda} y_{\lambda c} y_{\lambda c'} (E_{\lambda c} - E)^{-1} + y_{\lambda c} y_{\lambda c'} (E_{\lambda c} - E)^{-1}$$

which is expressed in matrix form by

$$R = R^0 + R^\lambda.$$  \hspace{1cm} (1.40)

The diagonal elements of $R^0$ contain the positive factors $y_{\lambda c}^2$ whereas the other elements have factors of
the form $\gamma_{\lambda}^{\prime}c^{\prime}\gamma_{\lambda}^{\prime}c^{\prime}$ which will be positive for some $\lambda'$ and negative for others. It is therefore reasonable to expect the diagonal elements to have larger magnitude than the off-diagonal elements and the approximation

$$R_{cc}^{0} = \sum_{\lambda} \gamma_{\lambda}^{\prime}c^{\prime}\gamma_{\lambda}^{\prime}c^{\prime}(E_{\lambda} - E)^{-1}\delta_{cc}$$

which is used later is justified.

The matrices $S$ and $P$ are introduced as the real and imaginary parts of $L$ so that

$$L - L^{*} = 2iP$$

Now, $O_{c}$ and $I_{c}$, being solutions of the same equation will have their Wronskian equal to a constant. They can be normalized so that

$$O_{c}^{'I_{c}} - I_{c}^{'O_{c}} = 2ik_{c}.$$  

Using this and the fact that $O_{c}$ and $I_{c}$ are complex conjugate it can be shown that the c-c element of the diagonal matrix $kaI^{-1}O^{-1}$ is the c-c element of $P$.

That is

$$kaI^{-1}O^{-1} = P.$$  

It is also convenient to define $\Omega$ by

$$\Omega = I^{\frac{1}{2}}O^{-\frac{1}{2}}.$$
noting that the modulus of $\Omega_{cc}^*$ is unity because $\Omega_{cc}$ is the identity matrix. Equation 1.17 is now rewritten in terms of the new matrices:

$$U = (ka)^{-\frac{1}{2}} [1 + 2i(1-RL)^{-1}RP]I(ka)^{-\frac{1}{2}}$$

$$= \Omega^2 + 2i\Omega P^\frac{1}{2}(1-RL)^{-1}RP^\frac{1}{2}\Omega$$

$$= \Omega^2 + 2i\Omega P^\frac{1}{2}(1-R^O L)^{-1}R^O P^\frac{1}{2}\Omega + 2i\Omega P^\frac{1}{2}XP^\frac{1}{2}\Omega$$

where $X = (1-RL)^{-1}R - (1-R^O L)^{-1}R^O$.  

A manipulation of the last equation gives

$$(1-RL)X = R - (1-R^O L - R^L)(1-R^O L)^{-1}R^O$$

$$= R^\lambda + R^\lambda L(1-R^O L)^{-1}R^O$$

$$= R^\lambda (1-R^O L)^{-1}$$

where the diagonal approximation of $R^O$ has been assumed in order to make the commutative multiplication of the matrices valid. By introducing the background collision matrix $U^O$, equation 1.46 is rewritten

$$U = U^O + 2i\Omega P^\frac{1}{2}XP^\frac{1}{2}\Omega.$$  

It is clear that $U^O$ is diagonal and independent of the particular $\lambda$ which was used to separate $R$ into its two parts.

Using the substitution $\gamma_{\lambda c}^\gamma_{\lambda c'}(E_{\lambda} - E)^{-1}$ for $R^\lambda_{cc'}$, the $c-c'$ element of the left and right sides of equation 1.48 are
equated and a rearrangement gives

\[
(1-R_{c'}^{Oc})X_{cc'} = \sum_{c''} \gamma_{\lambda c''} \gamma_{\lambda c'} (E_{\lambda} - E)^{-1} L_{c''c'} X_{c''c'} + \frac{\gamma_{\lambda c'c'} (E_{\lambda} - E)}{(1-R_{c'}^{Oc}) L_{c'} c'} \tag{1.50}
\]

(where the single subscript \(c\) is being used as an abbreviation for \(cc\) for the diagonal matrices), whence

\[
\frac{(E_{\lambda} - E)(1-R_{c'}^{Oc})X_{cc'}}{\gamma_{\lambda c}} = \sum_{c''} \gamma_{\lambda c''} L_{c''c'} X_{c''c'} + \frac{\gamma_{\lambda c'c'}}{1-R_{c'}^{Oc} L_{c'} c'} \tag{1.51}
\]

The right side of this equation is independent of \(c\). Hence

\[
X_{cc'} = \alpha_{\lambda c'} c', \tag{1.52}
\]

where

\[
\alpha_{\lambda c} = \frac{\gamma_{\lambda c}}{1-R_{c}^{Oc c}} \tag{1.53}
\]

and \(\beta_{c'}\) is found by substituting \(X_{cc'}\) for \(X_{cc'}\) in equation 1.51 to be

\[
\alpha_{\lambda c'} [(E_{\lambda} - E) - \sum_{c''} \gamma_{\lambda c''} L_{c''c'} \alpha_{\lambda c''}]^{-1}.
\]

Therefore,

\[
X_{cc'} = \frac{\alpha_{\lambda c'} \gamma_{\lambda c'}}{(E_{\lambda} - E) - \sum_{c''} \gamma_{\lambda c''} L_{c''c'} \alpha_{\lambda c''}}. \tag{1.54}
\]

Because \(\gamma_{\lambda c}\) is real, \(\gamma_{\lambda c} = \alpha_{\lambda c}^* (1-R_{c}^{Oc c})\) so that
\[ X_{cc'} = \frac{\alpha_{\lambda c}\alpha_{\lambda c'}}{(E_\lambda - E) - \sum_{c''} |\alpha_{\lambda c''}|^2 (1-R^{0*}_{c''}L_{c''}^*)L_{c''}}. \] (1.55)

Now, using equations 1.55 and 1.49

\[ U_{cc'} = U^0_{cc'} + 2i \frac{\Omega_{c'}\alpha_{\lambda c}^*\alpha_{\lambda c'}^*P_{c'}^2\Omega'}{(E_\lambda - E) - \sum_{c''} |\alpha_{\lambda c''}|^2 (1-R^{0*}_{c''}L_{c''}^*)L_{c''}} \]
\[ = U^0_{cc'} + \frac{i\Omega_{c'}\Omega^*_{c'}(\Gamma_{\lambda c}\Gamma_{\lambda c'})^{1/2}e^{-i(\chi_c + \chi_{c'})}}{E_\lambda - E + \Delta_\lambda - \frac{1}{2}i\Gamma_\lambda} \] (1.56)

where

\[
\begin{align*}
\Gamma_{\lambda c} &= 2P_c |\alpha_{\lambda c}|^2 \\
\Gamma_\lambda &= \sum_{c} 2P_c |\alpha_{\lambda c}|^2 = \sum_{c} \Gamma_{\lambda c} \\
\Delta_\lambda &= \sum_{c} |\alpha_{\lambda c}|^2 [S_c + R_c (S_c^2 + P_c^2)]
\end{align*}
\] (1.57)

and
\[ \chi_c = \arg (1-R^{0*}_{c}L_{c}) \]

Putting \( e^{-i\phi_c} \) for \( \Omega_{c} \), the elements of the background matrix are

\[ U^0_{cc'} = e^{-2i\phi_c} \begin{pmatrix} 1 + \frac{2iP_c R^{0*}_{c}}{1-R^{0*}_{c}L_{c}^*} \delta_{cc'} \\
\end{pmatrix} \]
\[ = e^{-2i\phi_c} \frac{1-R^{0*}_{c}L_{c}^*}{1-R^{0*}_{c}L_{c}^*} \delta_{cc'} \]
\[ = e^{-2i(\phi_c + \chi_c)} \delta_{cc'}. \] (1.58)
Hence

\[ U_{cc'} = e^{-2i\theta_c \delta_{cc'}} + \frac{e^{-i(\theta_c + \theta_{c'})} (\Gamma_{\lambda_c \lambda_{c'}})^{1/2}}{E - E + \Lambda - \frac{1}{2} i \Gamma_{\lambda_c}} \]  

(1.59)

where \( \theta_c = \phi_c + \chi_c \).

Now, the experimentalist may be unable to
determine the quantum numbers of a channel, save the modulus
of the total angular momentum \( J \) (measured in units of \( \hbar \)). So,
for the cross-section \( \sigma_{\alpha\alpha'} \), which measures the probability of
the reaction \( \alpha \rightarrow \alpha' \) for a particular \( J \), the well established
formula (Preston equation 16-42) is used:-

\[ \sigma_{\alpha\alpha'} = \frac{\pi}{k^2} g^J_{\alpha} \sum_{l,l',s,s'} |U_{\alpha's'_{\alpha},sl}^J|^2 \quad (\alpha \neq \alpha') \]  

(1.60)

where \( g^J_{\alpha} \) is the "statistical factor" given by

\[ g^J_{\alpha} = \frac{2J+1}{(2J_X^\alpha+1)(2J_Y^\alpha+1)} \]  

(1.61)

(\( J_X^\alpha \), \( J_Y^\alpha \) being the moduli of the total angular momenta
measured in units of \( \hbar \) of \( X^\alpha \), \( Y^\alpha \)) and where

\[ U_{\alpha's'_{\alpha},sl}^J = \Sigma_{c'c} U_{c'c} \]  

(1.62)

with summation over all \( m, \mu, m', \mu' \) such that the total
angular momentum \( J \) is conserved.
Another well established formula (Preston equation 16-44) gives the elastic cross-section:

$$\sigma_{\alpha\alpha} = \frac{\pi}{k^2} g_\alpha^J \sum \left\{ 1 - 2\text{Re}U_{\alpha S}^J, \bar{a}_\lambda \sum_{\lambda'} \left| U_{\alpha S}^J, \bar{a}_\lambda' \bar{a}_\lambda \right|^2 \right\} \tag{1.63}$$

Substitution of $U_{cc}'$, given by equation 1.59 into equation 1.61 gives

$$\sigma_{\alpha\alpha'} = \frac{\pi}{k^2} g_\alpha^J \frac{\Sigma \Gamma_\alpha \Sigma \Gamma_\alpha'}{E_\lambda - E + \Delta_\lambda} \left( \frac{1}{4} \Gamma_\lambda^2 \right) \tag{1.64}$$

where the primes indicate that summation is over all variables consistent with $L + S = L' + S' = J$. Equation 1.64 is the celebrated Breit-Wigner single level resonance formula. It is assumed that all terms of the right side are either constant or only slightly energy dependent in the vicinity of $E_\lambda$ so that $\sigma_{\alpha\alpha'}$ has maximum value of

$$\frac{\pi g_\alpha^J \Sigma \Gamma_\alpha \Sigma \Gamma_\alpha'}{\lambda c \lambda c'} \left( \frac{1}{4} \Gamma_\lambda^2 \right)$$

when $E = E_\lambda + \Delta_\lambda$. It is only a useful formula when the shift $\Delta_\lambda$ is small, when $E$ is close to $E_\lambda$ and when the width $\Gamma_\lambda$ is small compared with the average energy spacing of the $E_\lambda$'s. We note too that $\sigma_{\alpha\alpha'}$ has half its maximum value when $E$ is $E_\lambda + \Delta_\lambda \pm \frac{1}{2} \Gamma_\lambda$ thus
explaining why $\Gamma_{\lambda}$ is called the width of the resonance.

Substitution for $U_{cc}$, in equation 1.63 is more tedious and gives

$$
\sigma_{\alpha\alpha} = \frac{\pi}{k^2} g_{\alpha} \left\{ \sum_{\lambda,s} 4\sin^2 \theta_c - \sum_{\lambda} \frac{2(E_{\lambda} - E + \Delta_{\lambda})\sin 2\theta_c - \Gamma_{\lambda}(1 - \cos 2\theta_c)}{(E_{\lambda} - E + \Delta_{\lambda})^2 + \frac{1}{4}\Gamma_{\lambda}^2} \right\}
$$

(1.65)

The three terms of this expression are referred to as the potential scattering, interference and resonance scattering terms respectively.
2. RESONANCE WIDTHS AND SPACINGS

2.1 General Remarks

Because the structure of the nucleus determines the energy levels \((E_A + \Delta_A)\) at which resonances occur it is of interest to measure the actual distribution law of the level spacings for comparison with theory. For the same reason, the variation in resonance widths from level to level for various modes if decay of the compound nucleus may be used to test theories. There is also the practical use in the design of reactors where cross sections are predicted for high energy incident neutrons once the width and spacing frequency functions have been established at lower energies.

It should be emphasized, however, that the resonance energy levels are well above the ground state of the compound nucleus because of the contribution of the binding energy of the neutron. The experimentalist observes resonances of the \(^{239}\text{U}\) compound nucleus over a 4000 ev range which follow an unobservable sequence which extends over \(488 \times 10^4\) ev (Wigner 1957(b)). This feature dismisses the feasibility of calculating the energy levels at which resonance occurs and draws attention to the necessity of a statistical approach.

It could be mentioned here that the unobserved sequence of resonances with energy less than the binding energy
of the neutron are referred to as the negative energy levels. Egelstaff (1959) has studied the properties of negative levels of 20 isotopes by considering their effects on the lowest positive levels. He concludes that the resonance widths for neutron emission and the resonance spacing distribution functions appear to be the same as for positive energy levels.

2.2 Historical Development

2.2.1 Level Spacings

The early assumption, for want of a better one, was that energy levels were distributed at random so that any interval $dE$ of the energy scale has probability $\frac{dE}{D}$ of containing an energy level, where $D$ stands for the mean value of the spacing $D$ between adjacent levels. Now, the probability of a resonance spacing of between $D$ and $D + dD$ is the same as the probability of an absence of energy levels between $E_r$ and $E_r + D$ with a level between $E_r + D$ and $E_r + D + dD$, given an energy level at $E_r$. If $f(D)$ is the frequency function for the spacings then

$$f(D + dD) = f(D)(1 - \frac{dD}{D}) \quad (2.1)$$

so that

$$f(D) = \frac{1}{D} \exp (-D/D) \quad (2.2a)$$

This function, along with the uniform spacing function

$$f(D) = \delta(D - D) \quad (2.2b)$$

and

$$f(D) = \frac{4D}{D^2} \exp (-2D/D) \quad (2.2c)$$
were used by Goertzel (1955) for Doppler effect calculations, 
but it was realized that of these, only equation 2.2c had 
experimental backing.

The cumulative frequency function $F(D)$ is defined by

$$ F(D) = \int_0^D f(D')dD' $$ (2.3)

and so the functions

$$ F(D) = 1 - \exp\left(-\frac{D}{\bar{D}}\right) $$ (2.4a)

$$ F(D) = H(\bar{D}) $$ (2.4b)

$$ F(D) = 1 - (1+2D/\bar{D}) \exp\left(-\frac{2D}{\bar{D}}\right) $$ (2.4c)

correspond to the density functions given by equation 2.2.

Gurevich and Pevsner (1957), after studying 
the spacing of levels of 11 nuclides dismissed the negative 
exponential function because of the "repulsion of levels" which 
were observed. Their experiments made it clear that the true 
frequency function would have the features $f(0) = 0$.

Resolution of cross sections was also carried out 
by Harvey (1955) and he also dismissed the random distribution of 
the energy levels, suggesting the use of equation 2.4. Harvey and 
Hughes (1958) elaborate on Harvey's work of 1955 and draw attention 
to the fact that in 1929, von Neumann and Wigner stated that many-
body problems in quantum mechanics would always give rise to a 
repulsion of the energy levels.
The experimental determination of energy level spacings is a difficult task. If the spacings are small and the widths large, many levels may be unobserved so that the observed spacings form a biased sample. Unless the target nucleus has zero spin, the observed levels correspond to different spin states (i.e. different J) with independent distributions. The superposed system of the virtually indistinguishable sequences disguises the properties of the individual distributions. Lane and Lynn (1957) suggest corrections to account for these difficulties. The properties of the spacing frequency function for the resonances formed by two independent sequences is considered in detail in a later chapter.

The first theoretically based distribution for the spacings was given by Wigner (1957 (a), (b) ) as

\[ f_{\text{W}}(D) = \frac{\pi D}{2D^2} \exp\left(-\frac{\pi D^2}{4D^2}\right) \]  

(2.5)

although he conceded that equation 2.2c fitted the existing experimental data better. He realized that the true spacing distribution would be found as the distribution of the spacings between the energy eigenvalues of the Hamiltonian matrix and it was Mehta (1960) who first made progress with this problem with Gaudin (1961) obtaining the solution numerically, noting that Wigner's function (equation 2.5) was remarkably close to being correct.

2.2.2 Reduced Widths

The neutron width \( \Gamma_n \) is defined as \( \sum_{\lambda c} \Gamma_{\lambda c} \).
of equation 1.64 where c is a channel representing neutron emission.

The reduced width \( \Gamma_n^0 \) is defined by

\[
\Gamma_n^0 = \Gamma_n / E_o^{1/2}
\]

(2.6)

where \( E_o \) is the resonance energy of an \( \ell = 0 \) angular momentum channel thus making \( \Gamma_n^0 \) energy independent.

Hughes and Harvey (1955) attempted to find the distribution of \( \Gamma_n^0 / \bar{\Gamma}_n^0 \), where \( \bar{\Gamma}_n^0 \) is the mean value of \( \Gamma_n^0 \), by normalizing the data for different nuclides in order to get a larger sample size. Attention is drawn to Bethe's early suggestion of

\[
f(y) = \frac{1}{2\sqrt{y}} \exp(-\sqrt{y})
\]

(2.7)

where \( y = \Gamma_n^0 / \bar{\Gamma}_n^0 \) and of Porter and Thomas' suggestion of

\[
f(y) = (2\pi y)^{-1/2} \exp(-y/2)
\]

(2.8)

(both private communication with Hughes and Harvey). The examination of the data gave support to the latter distribution with the simple exponential distribution of \( e^{-y} \) being put forward as another possibility. It was with these two distributions in mind that Porter and Thomas (1956) set out to solve the problem theoretically.
2.3 The Random Matrix Hypothesis

2.3.1 The Hamiltonian Matrix

Quantum mechanics theory (see for example Schiff (1955)) shows that all physically measurably dynamical variables can be represented by Hermitian matrices whose eigenvalues are the possible values of the variable. The Hamiltonian matrix \( [H_{\alpha \beta}] \) determines the possible values \( E_\lambda \) of the energy variable \( E \) as does the Hamiltonian operator \( H \) of equation 1.19. Using the notation of Porter and Rosenzweig (1960) the matrix is defined in terms of \( H \) by

\[
H_{\alpha \beta} = \int \psi_\alpha H \psi_\beta \, d\tau \quad (2.9)
\]

where the integration is over the many-body configuration space of the nucleus and \( \psi_\alpha, \psi_\beta \) belong to a complete set of orthonormal vectors. Time reversal invariance may be shown to imply the reality of \( H_{\alpha \beta} \) which, with the Hermitian property, implies that the matrix is symmetric. The more recent Dyson (ibid) papers develop a mathematical model which applied when time reversal is not invariant and the matrix elements are not real.

The matrix \( H \) is of infinite rank but is approximated by a square matrix of order \( N \) where \( N \) is large. The problem of finding the energy levels of a compound nucleus is now seen to be equivalent to finding the eigenvalues of a corresponding matrix \( H \).
which is assumed to be drawn from an ensemble of Hamiltonian matrices, each with the same statistical properties.

2.3.2 The Statistical Properties of $H$

By dividing the configuration space $\tau$ into cells $\Delta \tau$ of identical volume we approximate

$$H_{\alpha\beta} \approx \Delta \tau \sum_i (\psi_\alpha H \psi_\beta)_i$$

(2.10)

and, because of the complex nature of the wave functions of the interior region, the contribution $(\psi_\alpha H \psi_\beta)_i$ is expected to vary randomly, with random sign, independent of the contribution $(\psi_\gamma H \psi_\delta)_i$ of $H_{\gamma\delta}$. Now the central limit theorem of statistics (see Cramér (1945)) can be applied in this situation so that each matrix element of $H$ is expected to have a normal distribution. The random matrix hypothesis requires each of the elements $H_{\alpha\beta}(\alpha \leq \beta \leq N)$ to be an independent Gaussian (i.e. normal) random variable where the diagonal and superdiagonal elements are taken as the $\frac{1}{2}N(N+1)$ independent elements. It also requires the joint distribution function $P_N(H_{11}, H_{12}, \ldots, H_{\alpha\beta}, \ldots, H_{NN})$, which is introduced to describe the probability that $H_{\alpha\beta}$ is in the interval $dH_{\alpha\beta}$ at $H_{\alpha\beta}$ for all $\alpha \leq \beta \leq N$, to be invariant under all transformations of the type $H \rightarrow S^{-1}HS$ where $S$ is a real orthogonal matrix.

The assumption of independence allows us to write
Using the assumption of invariance Porter and Rosenzweig (ibid) show that

\[
f_{\alpha\alpha}(H_{\alpha\alpha}) = (4\pi a^2)^{-\frac{1}{2}} \exp\left\{-(H_{\alpha\alpha} - E_0)^2/4a^2\right\}
\]

\[
f_{\alpha\beta}(H_{\alpha\beta}) = (2\pi a^2)^{-\frac{1}{2}} \exp\left\{-H_{\alpha\beta}^2/2a^2\right\} \text{ for } \alpha \neq \beta
\]

(2.11)

where \(a^2\) is the common variance of the off-diagonal elements of \(H\) and \(E_0\) is a constant, referred to by Porter (1963) as the energy origin. These results lead to

\[
P_N(H_{11}, ..., H_{\alpha\beta}, ..., H_{NN}) = A \exp\left\{-\sum_{\alpha} (H_{\alpha\alpha} - E_0)^2/4a^2 - \sum_{\alpha<\beta} H_{\alpha\beta}^2/2a^2\right\}
\]

\[
= A \exp\left\{-\text{Tr}(H-E_0I)^2/4a^2\right\}
\]

(2.12)

where the normalizing factor \(A\) is \((4\pi a^2)^{-\frac{1}{2}}N(N+1)^{\frac{1}{2}}\).

The joint distribution function of the eigenvalues \(\theta_1, \theta_2, ..., \theta_N\) of a real symmetric matrix, in which every element has a Gaussian distribution, is the Wishart distribution (see for example Wilks (1943)); that is

\[
P(\theta_1, \theta_2, ..., \theta_N) = B \exp\left\{-\sum_\alpha \theta_\alpha^2\right\} \prod_{\alpha<\beta} |\theta_\alpha - \theta_\beta|
\]

(2.13)

(2.14)

with the constant \(B\) being first given by Hsu(1939) as
The derivation of equation (2.8) for the distribution of reduced neutron widths and the extent to which it fits the experimental data may be taken as verification (in part) of the random matrix hypothesis. Porter and Thomas (ibid) proceed along the following lines:

From equations (1.21) and (1.39) it is apparent that:

\[ \gamma_{\lambda c} = \left( \frac{\hbar^2}{2Mc} \right)^{\frac{1}{2}} \int \bar{\phi}_c \phi_{\lambda} \, dS. \quad (2.15) \]

The complex nature of the wave functions \( \phi_{\lambda} \) suggests the division of the configuration space into cells with random contributions from each cell so that \( \gamma_{\lambda c} \) (compare with \( H_{\alpha\beta} \)) is expected to have a normal distribution. From equations (1.53) and (1.57) it is seen that \( \Gamma_{\lambda} \) is the sum of the squares of variables with normal distributions, and it is known (see Cramér(ibid)) that the chi-squared distribution

\[ f_{\nu}(y) = \frac{y^{\frac{\nu}{2}-1} \exp(-\frac{y}{2})}{2^{\frac{\nu}{2}} \Gamma \left( \frac{\nu}{2} \right)} \]
should apply where \( y \) is the ratio of \( \Gamma_\lambda \) to its mean value and \( \nu \) is the number of terms of the summation which determines \( \Gamma_\lambda \).

The definition of \( y^2 \) as used in equation 2.16 implies that its mean value \( \bar{y} \) is unity. The variance \( \sigma^2 \) may be expressed by

\[
\sigma^2 = \bar{y}^2 - y^2. \tag{2.17}
\]

where \( \bar{y}^2 \) is the mean value of \( y^2 \). For the chi-squared distribution, \( \bar{y}^2 \) is evaluated by

\[
\int_0^\infty y^2 \frac{1}{\nu} \frac{1}{\Gamma \left( \frac{1}{2}\nu \right)} \exp \left( -\frac{y}{\nu} \right) dy
= \frac{\nu + 2}{\nu \Gamma \left( \frac{1}{2}\nu \right)} \int_0^\infty y^{\frac{1}{2}\nu - 1} \exp (-y) dy
= 1 + 2/\nu
\]

so that

\[
\sigma^2 = 2/\nu. \tag{2.18}
\]

The significance of this result is that the variation in observations of \( y \) will be slight if \( \nu \) is large and great if \( \nu \) is small. The most probably value of \( y \) is \( 1 - 2/\nu \) for \( \nu \gtrsim 2 \), with the function becoming infinite at \( y = 0 \) for \( \nu < 2 \) (see Figure 1).

Now, it is expected that at energies up to a few keV, the only channel open to neutron emission is the
incident channel. So, to find the distribution of $\Gamma_n^o / \Gamma_n^o$, we put $\nu = 1$ in equation 2.16 and get equation 2.8, which is frequently referred to as the Porter-Thomas distribution. This fits the data well at low energies and accounts for the big variation in neutron widths most satisfactorily.

Further weight is lent to this argument when the distribution of radiation widths is considered. As many channels are expected to give rise to radiation in heavy nuclei-neutron reactions the relevant values of $\Gamma^\lambda$ are the sums of the same number of squares, implying a chi-squared distribution with $\nu$ large. It is not therefore surprising to find that the total radiation width for a given spin state which is formed in neutron capture is essentially constant. Porter and Thomas also note that equation 2.16, with $\nu = 2.5$, is a good fit for fission widths of $^{235}U$, thus indicating that there are only a few fission channels open at the energies considered.

2.5 The Distribution of Resonance Spacings

2.5.1 Wigner's Surmise

Wigner's spacing frequency function (equation 2.5) is based on the random matrix hypothesis. It is readily seen that the difference between the eigenvalues of the real symmetric matrix
is \( \sqrt{(a_{11} - a_{22})^2 + 4a_{12}^2} \), which is the distance of the point \((a_{11} - a_{22}, 2a_{12})\) from the origin in plane geometry. Wigner (1957(a),(b)) used the above result to gain insight to the problem of the eigenvalue spacings of a real symmetric matrix with random elements. Now, the probability that \((a_{11} - a_{21}, 2a_{12})\) should be distant \(D\) from the origin is proportional to \(D\) for \(D\) small because the area of the annulus with radii \(D\) and \(D+dD\) is proportional to \(D\). Wigner then surmised that the probability of a spacing would be proportional to the size of spacing for large spacings also, and quoted equation 2.5 which has this property.

A derivation of this equation is as follows:-

Consider an energy level at \(E_r\) and suppose that the next energy level is at \(E_r + \xi\). Denote the probability that \(\xi > D\) by \(\mathcal{I}(D)\) so that

\[
\mathcal{I}(D) = \int_D^{\infty} f(D') \, dD' \tag{2.19}
\]

where \(f(D)\) is the spacing frequency function which is assumed to be proportional to \(D\). Now

\[
\mathcal{I}(D+dD) = \mathcal{I}(D)(1-KDdD) \tag{2.20}
\]
where $K$ is a constant and $1-KDdD$ is the probability that $\xi$ does not have a value between $D$ and $D + dD$. Expanding $\mathcal{F}(D+dD)$ and omitting second and higher powers of $dD$ leads to

$$
\mathcal{F}'(D) = -KDf(D)
$$

(2.21)

$$
\therefore \quad f(D) = Ae^{-KD^2/2}.
$$

(2.22)

Consideration of $\mathcal{F}(0)$ shows that the constant $A$ is unity.

Hence

$$
f(D) = KDe^{-KD^2/2}.
$$

(2.23)

$K$ can be expressed in terms of the mean spacing $\overline{D}$ by using

$$
\overline{D} = \int_0^\infty Df(D)dD
$$

(2.24)

so that

$$
f(D) = \frac{\pi D}{2\overline{D}^2} \exp\left(\frac{-\pi D^2}{4\overline{D}}\right)
$$

(2.25)

which is the same as equation 2.5 (see Figure 1).

Using the techniques of statistical mechanics, Wigner (1957(b)) derived the density function for a particular eignevalue $\theta$ of a Hamiltonian matrix. It is known as the semi-circle law and has the form:

$$
P(\theta) = \begin{cases} 
\frac{2}{N\pi} \sqrt{N - \theta^2}, & |\theta| < \sqrt{N} \\
0, & |\theta| > \sqrt{N}
\end{cases}
$$

(2.26)

where $N$ is large and where a transformation has been
applied to make $E_0$ and $a$ (of equation 2.12) 0 and $\frac{1}{2}$ respectively.

Monte-Carlo studies by Porter and Rosenzweig (ibid), which generated random matrices up to $N = 20$, gave backing to both the semi-circle law and Wigner's spacing distribution.

### 2.5.2 Mehta and Gaudin's Exact Work

In this section a brief outline is given of papers by Mehta (ibid), Mehta and Gaudin (1960) and Gaudin (ibid). The first of these papers expresses the probability density function of the eigenvalue $\theta_1$ of a Hamiltonian matrix (assumed real and symmetric) in terms of the Wishart distribution (equation 2.14). That is

$$P(\theta_1) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\theta_1, \theta_2 \ldots \theta_N) d\theta_2 \ldots d\theta_N. \tag{2.27}$$

The immediate difficulty is that the integrand is not analytic due to the presence of the absolute values. However, Mehta overcomes the problem by integrating over alternate variables and using the properties of determinants.

The density function at the origin, $P(0)$, is found to have the limit $\frac{2}{\pi \sqrt{N}}$ as $N \to \infty$ and, because there are $N$ eigenvalues, the level density function is found by multiplying by $N$. Here we see agreement with the semi-circle law. When the moments of $P(\theta_1)$ are studied, it is found that the odd moments vanish and the even moments approach, in the
In order to calculate the spacing distribution, Mehta considers the function:

\[ P(\theta_1, \theta_2) = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} P(\theta_1, \theta_2, \ldots, \theta_N) d\theta_3 \cdots d\theta_N \]  
\hspace{1cm} (2.28)

where the primes on the integral signs indicate that when integrating over \( \theta_3, \theta_4, \ldots, \theta_N \) the interval between \( \theta_1 \) and \( \theta_2 \) is excluded. Then, by putting \( \theta_1 = \theta_2 - D \) and integrating over \( \theta_2 \) the cumulative distribution function for the spacings, \( F(D) \), may, in general, be found. In particular the cumulative frequency function at the energy origin is found by

\[ F(D) = N(N-1) \int_0^D P(\frac{D'}{2} - \frac{D'}{2}) dD' \]  
\hspace{1cm} (2.29)

where it is assumed that the distribution is symmetric about the origin. The mean spacing \( \overline{D} \), is evaluated at the centre of the distribution and the factor \( N(N-1) \) is introduced because any two of the \( N \) eigenvalues could bound the spacing of \( D \).

Mehta proves that

\[ 1 - \exp \left( -\frac{\pi D}{4D} \right) \leq F(D) \leq 1 - \exp \left( -\frac{\pi D}{4D} \right) \cdot \left\{ 1 - \frac{1}{3} \left( \frac{\pi D}{4D} \right)^2 \right\} \]  
\hspace{1cm} (2.30)

thus establishing fairly restrictive upper and lower bounds for \( F(D) \), and he shows that the cumulative function derived from Wigner's density function
\[ F_w(D) = \int_0^D f_w(D') dD' = 1 - \exp \left[ \frac{\pi D^2}{4D^2} \right] \] (2.31)

lies between these bounds also. However he proves that the Wigner surmise is not the true distribution by showing that the 2\textsuperscript{nd} and 4\textsuperscript{th} derivatives of $F(D)$ evaluated at $D = 0$ are not equal to the corresponding derivatives of $F_w(D)$.

Mehta and Gaudin study the eigenvalue density function $P(\theta)$ in more detail, expressing in it terms of one-dimensional harmonic oscillator wave functions. When the order of the matrix becomes infinite they show that the semi-circle density function is obtained. This work is extended by Gaudin who evaluates the spacing distribution function $F(D)$ numerically, confirming that although Wigner's distribution function is not exact it may be regarded as a very good approximation for $F(D)$. 
3. RESONANCE SPACINGS - SIMPLIFYING APPROACHES

Prior to the publication of the papers by Mehta and Gaudin (ibid) which confirmed the worth of Wigner's spacing distribution function (equation 2.5), Nicholson (1960) assumed that a chi-squared function (equation 2.16) could be found which was as good a fit for resonance spacing data as was the former function. He concluded that the function with 10 degrees of freedom (i.e. $v = 10$) was the best function to use. Hwang (1963, 1965) considered the effect of resonance interactions on Doppler-effect calculations on the assumption that the chi-squared function with 8 degrees of freedom was good enough. Garg et al. (1964) state "there is no good reason to believe that the level spacings follow a formula of this type". But a comparison between the results developed in section 3.1.1, Hwang's results and that of Reichel and Wilkins (1964) show that a chi-squared function may lead to calculations which are more manageable than those resulting from the use of Wigner's function while at the same time, the end results are sometimes sufficiently accurate for practical purposes.

In this chapter the level densities of single and superposed sequences of resonances are investigated in the light of this observation. The use of synthetic kernels is also discussed.
3.1 The Conditional Resonance Density Function in the Unresolved Region

It is assumed that a resonance is known to exist at the energy \( E_r \) and that the positions of the resonances for which \( E > E_r \) are unknown. The density of a single sequence of resonances (e.g. the \( l = 0 \) sequence) in this unresolved region is developed on the assumption that the mean spacing \( \overline{D} \) is constant. It is convenient to measure the energy in units of \( \overline{D} \) and denote \((E - E_r)/\overline{D}\) by \( x \). Each of the spacing density functions \( f(D) \) of Chapter 2 will be replaced by \( f(x) \) where \( f(x) = \overline{D}f(D) \) in order to be consistent.

The conditional resonance density function \( \Omega(x) \) is now defined as the density of the probability of finding a resonance at \( x \) given a resonance at \( E_r \). This is the sum of the probability of having a spacing of \( x \) and the joint probability of having a resonance at \( x - t \) followed of a spacing of \( t \) for all \( t \) such that \( 0 < t < x \). Hence

\[
\Omega(x) = f(x) + \int_0^x \Omega(x-t)f(t)\,dt. \quad (3.1)
\]

3.1.1 The Use of Chi-Squared Functions

To Evaluate \( \Omega(x) \)

By using the chi-squared function

\[
f_\nu(x) = \frac{x^{\nu/2-1} \exp(-x/2)}{\sqrt{\nu} \Gamma(1/2\nu)} \quad (3.2)
\]
for \( f(x) \) in equation 3.1, Hwang (1963,1965) found \( \Omega(x) \) for \( \nu = 2, 4, 8 \) and gave the results

\[
\begin{align*}
\Omega(x) &= \begin{cases} 
1, & \nu = 2 \\
1 - e^{-4x}, & \nu = 4 \\
1 - e^{-8x} - 2e^{-4x} \sin 4x, & \nu = 8.
\end{cases}
\end{align*}
\]

(3.3)

A general result is given below.

The Convolution Theorem is used to take one-sided Laplace transforms of equation 3.1 to give

\[
\overline{\Omega}(p) = \overline{f}(p) + \overline{\omega}(p)\overline{f}(p)
\]

(3.4)

where \( \overline{\Omega}(p), \overline{f}(p) \) are the transform of \( \Omega(x), f(x) \). Now the transform of \( \overline{f}(x) \), \( \overline{f}(p) \) is given by

\[
\overline{f}(p) = \int_0^\infty \frac{\nu^{1+\nu} x^{1+\nu-1} \exp\left[-(p+i\nu x)\right]}{2^{2\nu} \Gamma(\nu)} \, dx
\]

\[
= \left( \frac{\nu}{2p+\nu} \right)^{1/\nu}
\]

(3.5)

and substitution of \( \overline{f}(p) \) for \( \overline{f}(p) \) in equation 3.4 leads to

\[
\overline{\Omega}(p) = \frac{(1+\nu)^{1+\nu}}{(p+1+\nu)^{1/\nu} - (1+\nu)^{1/\nu}}
\]

(3.6)

so that

\[
\Omega(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{px} \rho^{\rho}}{(p+\rho)^{\rho} - \rho^{\rho}} \, dp
\]

(3.7)
where $\rho = \frac{1}{\omega}$ and $c > 0$. Only those cases for which $\rho$ is integral are considered as they give rise to expressions for $\Omega(x)$ in closed form which are therefore more manageable and it is a manageable approximation which we seek.

From complex variable theory, it is apparent that for $x > 0$, the integral in equation 3.7 is $2\pi i$ multiplied by the sum of the residues at the poles of the integrand at points with real part less that $c$. The denominator $(p+\rho)^\rho - \rho^\rho$ is factorized to $\prod_{k=1}^{\rho} [p - \rho(\omega^k - 1)]$ where $\omega = \text{cis}(2\pi/\rho)$ indicating that the sum of all residues is required.

At the pole where $p = \rho(\omega^k - 1)$ the calculation of the residue involves

$$\prod_{k \neq k'} \left( p(\omega^{k'} - \omega^k) \right) = \left( \frac{\rho^{-1} / \omega^{k'}}{\prod_{j=1}^{\rho-1} (1-\omega^j)} \right)$$

which is

$$\left( \frac{2\rho^{\rho-1} / \omega^{k'}}{\prod_{j=1}^{\rho-1} 2(1 - \text{Re} \omega^j)} \right), \rho \text{ even}$$

and

$$\left( \rho^{\rho-1} / \omega^{k'} \right)^{\frac{1}{2}(\rho-2)}$$

These may be expressed in terms of $\sin(j\pi/\rho)$ and use of the results
\[
\begin{align*}
\sum_{j=1}^{n-1} \sin \frac{j\pi}{2n} &= \frac{\sqrt{n}}{2^{n-1}} \\
\sum_{j=1}^{n} \sin \frac{j\pi}{2n+1} &= \frac{\sqrt{2n+1}}{2^n}
\end{align*}
\] (3.9)

(see, for example, Archbold (1958)), shows that the expression in equation 3.8 is \(\rho^p / \omega^k\) for \(\rho\) both even and odd. The residue at \(p = \rho(\omega^k - 1)\) is therefore \(\omega^k\exp[\rho x (\omega^k - 1)]\). Hence

\[
\Omega(x) = \sum_{k=1}^{\rho} \omega^k \exp[\rho x (\omega^k - 1)].
\] (3.10)

The summation is best carried out by separating the term for which \(k = \rho\) and, if \(\rho\) is even, the term for which \(k = \frac{1}{2}\rho\) and taking the remaining terms in conjugate pairs so that

\[
\Omega(x) = 1 - \delta_{\rho, 2r} e^{-2\rho x} + 2e^{-\rho x} \sum_{k=1}^{r-1} \left[ \omega^k \exp(\rho x \omega^k) + \omega^{-k} \exp(\rho x \omega^{-k}) \right] (3.11)
\]

where \(r = \begin{cases} \frac{1}{2}\rho & \text{if } \rho \text{ is even} \\ \frac{1}{2}(\rho+1) & \text{if } \rho \text{ is odd.} \end{cases}\)

The expansion of \(\omega^k\) and its complex conjugate \(\omega^{\rho-k}\) in terms of the sine and cosine of \(2k\pi/\rho\) enables a further simplification to be made so that

\[
\Omega(x) = 1 - \delta_{\rho, 2r} e^{-2\rho x} + 2e^{-\rho x} \sum_{k=1}^{r-1} \cos\left[\frac{2k\pi}{\rho} + \rho x \sin\left(\frac{2k\pi}{\rho}\right)\right] \exp[\rho x \cos\left(\frac{2k\pi}{\rho}\right)].
\] (3.12)
Substitution of \( p = 1, 2, 4 \) gives
Hwang's results (equation 3.3). Three more results are easily obtained and they are

\[
\Omega(x) = \begin{cases} 
1 - 2e^{-9x/2} \sin(3\sqrt{3}x/2 + \pi/6) \quad & v=6 \\
1 + 2e^{-5x} \{ e^{5xcos(2\pi/5)} \cos[5x \sin(2\pi/5) + 2\pi/5] \} \\
\quad + e^{5xcos(4\pi/5)} \cos[5x \sin(4\pi/5) + 4\pi/5] \}, v=10 \\
1 - e^{-12x} + 2e^{-6x} \{ e^{3x \cos(3\sqrt{3} + \pi/3)} - e^{-3x} \cos(3\sqrt{3}x - \pi/3) \}, v=12 \end{cases} \quad (3.13)
\]

The graphs of \( \Omega(x) \) for \( v = 4, 6, 8, 10, 12 \) are shown in Figure 2. The details of the fluctuations and the ways in which the asymptotic value of 1 is reached are more obvious from the values of \( \Omega(x) \) given in Table 1.

3.1.2. The Use of Wigner's Function to Evaluate \( \Omega(x) \).

Wigner's spacing density function (equation 2.5) has been used to generate \( \Omega(x) \) by Reichel and Wilkins (1964). The Laplace transform of \( f_w(x) \) is \( 1 - pe^{x^2/\pi} \text{erfc}(p/\sqrt{\pi}) \) which is substituted for \( \tilde{f}(p) \) in equation 3.4 so that \( \tilde{\Omega}(p) \) can be found. Term by term inversion of the series expansion of \( \tilde{\Omega}(p) \) for large \( p \) gives the asymptotic form of \( \Omega(x) \) for small \( x \) (Tauberian Theorem) which Reichel and Wilkins (ibid) give as

\[
\Omega(x) = \frac{\pi x}{2} - \frac{2\pi^2 x^3}{2^2 \cdot 3!} + \frac{10\pi^3 x^5}{2^3 \cdot 5!} - \frac{74 \cdot 4 \cdot 7 x^7}{2^4 \cdot 7!} + ... \\
+ \frac{706 \pi^5 x^9}{2^5 \cdot 9!} - \frac{8162 \pi^6 x^{11}}{2^6 \cdot 11!} + ... \quad (3.14)
\]

from which they tabulate \( \Omega(x) \) and draw its graph. The table of
values and graph are reproduced as parts of Table 1 and Figure 2.

The unwieldy form of this result makes further calculation very difficult thus explaining Hwang's choice (ibid) of a simpler but less accurate closed form. The question then arises as to which chi-squared function (with $\nu$ even) should be taken as spacing distribution to minimize the errors, and this is discussed in Chapter 4.

3.2 The Superposition of Resonance Sequences

We now extend the work of the last section and allow for more than one spin state. The distribution of spacings resulting from the random superposition of $N$ sequences of resonances is considered, following the techniques of Rosenzweig and Porter (1960). Each sequence is taken to have the spacing density function $f(x)$ where the variable $x$ is the ratio of the spacing between resonances of the $i^{th}$ sequence $D_i$ to the mean spacing of the $i^{th}$ sequence $\overline{D}_i$. It is assumed that the density of levels for each sequence has reached its asymptotic form so that the (unconditional) level density of the superposed system $1/\overline{D}$ is given by

$$1/\overline{D} = \sum_{i=1}^{N} 1/\overline{D}_i$$

(3.15)
3.2.1. Distribution of the Spacings of the Superposed System in terms of $D$.

Suppose that on the energy scale, the lower and upper points which determine a spacing of the superposed system are marked by $i^{th}$ type and $j^{th}$ type resonances respectively.

Let $D' = 0$ at the lower point and write $\phi_j(D)dD$ for the probability that there is a $j^{th}$ type resonance in the interval $D < D' < D + dD$ and no $j^{th}$ type in the interval $0 < D' < D$. Hence

$$\phi_i(D)dD = \frac{1}{D_i} f(D/D_i)dD$$  \hspace{1cm} (3.16)

$$\phi_j(D)dD = \frac{1}{D_j} \int_0^\infty \frac{1}{D_j} \frac{f(a+D/D_j)}{D_j} da dD$$  \hspace{1cm} \text{(j\neq i)}

$$= \frac{1}{D_j} \mathcal{J}(D/D_j)dD$$  \hspace{1cm} (3.17)

where the function $\mathcal{J}(D/D_j)$ is defined as $\int_D^\infty \frac{1}{D_j} f(D'/D_j)dD'$.

For a spacing of the kind described above, we consider the joint probability of a $j^{th}$ type resonance in $D < D' < D + dD$ (given an $i^{th}$ type resonance at $D' = 0$ for the case $j = i$) and no other levels in $0 < D' < D$.

The probability that a $j^{th}$ type $(j\neq i)$ does not occur in the interval $0 < D' < D$ is

$$\int_D^\infty \frac{1}{D_j} \mathcal{J}(D'/D_j)dD' = \int_0^\infty \mathcal{J}(b+D/D_j) db$$

$$= \int_0^\infty \int_0^\infty f(a+b+D/D_j) da db. \hspace{1cm} (3.18)$$
Using the transformation \( s = a - b \), \( t = a + b \) the integral may be written

\[
\int_0^\infty \int_{-t}^t f(t + D/D_j) \cdot s^2 ds \, dt = \int_0^\infty tf(t + D/D_j) \, dt
\]

\[= D(D/D_j) \quad \text{say.} \quad (3.19)\]

Now, introducing \( p_{ij}(D)dD \) to denote the conditional probability that, given an \( i \) th type resonance at the lower end, a spacing with magnitude between \( D \) and \( D+dD \) is found with \( j \) th type at the upper end we get

\[
P_{ii}(D)dD = \left(1/D_i\right) f(D/D_i) \sum_{k=1}^{N} \frac{1}{D(D/D_k)} dD \quad (3.20)
\]

and for \( j \neq i \)

\[
P_{ij}(D)dD = \left(1/D_j\right) g(D/D_j) \sum_{k=1}^{N} \frac{1}{D(D/D_k)} f(D/D_k) \int_0^\infty \frac{1}{D(D/D_i)} f(D'/D_i) dD' \, dD
\]

\[= \left(1/D_j\right) g(D/D_j) \sum_{k=1}^{N} \frac{1}{D(D/D_k)} \frac{1}{D(D/D_i)} dD \quad (3.21)\]

The total probability of finding a spacing with magnitude between \( D \) and \( D+dD \), irrespective of the type of resonance which bounds the spacing, \( P*(D)dD \), is given by a weighted summation

\[
P*(D)dD = \sum_i \sum_{j \neq i} p_{ij}(D)dD \quad (3.22)
\]
where the weight factor $q_i$ measures the probability that a resonance chosen at random is of the $i$th type. Obviously

$$q_i = (1/D_i)/(1/D) = D/D_i$$

and $\sum_i q_i = 1$.

It then follows from equations 3.20, 3.21 and 3.22 that

$$P^*(D) = \sum_{i=1}^{n} \frac{q_i}{D(D/D_i)} \left\{ (1/D_i) f(D/D_i) \right\}$$

$$+ \sum_{j \neq i} (1/D_j) \frac{\mathcal{Y}(D/D_j) \mathcal{Y}(D/D_i)}{D(D/D_j)}$$

(3.23)

A special case of this last result occurs in renewal theory when the probability density function for the spacings of $N$ superposed systems each with the same mean spacing is required. Putting $D_1 = D_2 = \ldots D_N = ND$ in equation 3.23 gives

$$P^*(D) = \frac{1}{ND} f\left(\frac{D}{ND}\right) \cdot \left( D \left( \frac{D}{ND} \right) \right)^{N-1}$$

$$+ \frac{N-1}{ND} \left( \mathcal{Y} \left( \frac{D}{ND} \right) \right)^2 \cdot \left( D \left( \frac{D}{ND} \right) \right)^{N-2}$$

$$= -\frac{d}{dD} \left[ \mathcal{Y} \left( \frac{D}{ND} \right) \left\{ \int_0^\infty \frac{1}{ND} \left( \frac{D'}{ND} \right) dD' \right\} \right]^{N-1}$$

(3.24)

which is a result given by Cox (1962).
3.2.2 Distribution of the Spacings of the
Superposed System in Terms of x

In units of $\overline{D}$, letting $D/\overline{D} = x$ and $P^*(D)dD = P(x)dx$, 

$$P(x) = \frac{N}{\sum_{k=1}^{N} D(q_k x)} \left\{ \frac{q_i^2 f(q_i x)}{D(q_i x)} + \sum_{j \neq i} \frac{q_i q_j f(q_i x)}{D(q_i x)} \right\}$$

$$= \frac{N}{\sum_{k=1}^{N} D(q_k x)} \left\{ \frac{q_i^2 f(q_i x)}{D(q_i x)} + \left( \sum_{i} \frac{q_i f(q_i x)}{D(q_i x)} \right)^2 - \sum_{i} \left( \frac{q_i f(q_i x)}{D(q_i x)} \right)^2 \right\} \quad (3.25)$$

The results $f(o) = D(o) = 1$ and $\sum_{i} q_i = 1$ are used to show that 

$$P(o) = 1 - \sum_{i} q_i^2 \quad (3.26)$$

where the original spacing density function $f(x)$ is such that $f(o) = o$ (i.e. such that there is a repulsion of levels). For this type of density function $f'(o) = o$ and by putting 

$$D(y) = \int_{y}^{\infty} xf(x)dx - y f(y) \quad (3.27)$$

it is easily shown that $D'(o) = -1$. By differentiating equation 3.25 and applying these results we see that 

$$P'(o) = -1 + 3\sum_{i} q_i^2 + (f'(o) - 2)\sum_{i} q_i^3 \quad (3.28)$$

Rosenzweig and Porter (ibid) use equations 3.26 and 3.28 to indicate the way in which an exponential form for the distribution of
spacings is approached when a large number of resonance sequences are superposed. Cox (ibid) also shows how the superposition of independent and different sequences may give rise to an asymptotically exponential spacing density function.

3.2.3 The Superposition of Two Sequences

Using reasoning similar to that of the previous sections, a formula for the superposed spacing density function of two sequences with different spacing density functions is derived. The result is given by Lane and Lynn (1957) as

\[
P_2^s(D) = \frac{D}{D_1 D_2} \left\{ \int_{\infty}^{D} \int_{\infty}^{D'} f_1(D'') dD' f_2(0) + 2 \int_{\infty}^{D} f_1(D') dD' \int_{\infty}^{D} f_2(D') dD' \right. \\
+ \left. f_1(D) \int_{\infty}^{D} dD' \int_{\infty}^{D'} f_2(D'') dD'' \right\} 
\]

(3.29)

where \( \overline{D}, \overline{D_1}, \overline{D_2} \) are as before, \( P_2^s(D) \) is \( P^s(D) \) with \( N = 2 \) and \( f_1(D), f_2(D) \) are the spacing density functions for the first and second sequences respectively.

The conditional probability for a spacing with size between \( D \) and \( D + dD \), given type 1 resonance at the lower end is considered as before so that

\[
P_{11}(D) dD = f_1(D) \int_{0}^{\infty} dD' \int_{\infty}^{\infty} (1/2) f_2(a+D') da dD' \\
= (1/2) f_1(D) \int_{0}^{\infty} dD' \int_{\infty}^{\infty} f_2(D'') dD'' dD 
\]

(3.30)
\[ p_{12}(D) dD = \int_0^\infty \frac{1}{D^2} f_2(a+D) da \int_D^\infty f_1(D') dD' \]
\[ = \frac{1}{D^2} \int_D^\infty f_1(D') dD' \int_D^\infty f_2(D') dD' \quad (3.31) \]

with similar expressions for \( p_{21} \) and \( p_{22} \). Taking the weighted sum,
\[ p^*(D) = \left( \frac{D}{D_1} \right) \left( p_{11}(D) + p_{12}(D) \right) + \left( \frac{D}{D_2} \right) \left( p_{21}(D) + p_{22}(D) \right) \]
\[ (3.32) \]

which yields equation 3.29.

The case is now taken where the sequences have the same normalized spacing probability density function.

That is
\[ \bar{D}_i f_i(D) = f(D/D_i) \quad i = 1, 2. \quad (3.33) \]

Therefore
\[ p^*(D) = \left( \frac{D}{D_1 D_2} \right) \left\{ \int_D^\infty \int_D^\infty f(D''/D_1) dD'' f(D/D_1) \right\} \]
\[ + 2 \int_D^\infty \int_D^\infty f(D'/D_1) dD' \int_D^\infty f(D'/D_2) dD' + \int_D^\infty \int_D^\infty f(D''/D_2) dD'' f(D/D_2) \]
\[ = \left( \frac{D}{D_1 D_2} \right) \left\{ \left( \frac{D_1}{D_2} \right) D(D/D_1) f(D/D_1) + 2 \mathcal{F}(D/D_1) \mathcal{F}(D/D_2) \right\} \]
\[ + \left( \frac{D_2}{D_1} \right) D(D/D_2) f(D/D_1) \} \quad (3.34) \]

which is readily seen to be the \( N = 2 \) case of equation 3.23.
3.2.4 The Superposition of Two Sequences

Using Wigner's Spacing Density Function.

In order to apply the result of equation 3.34 to two sequences of resonances which have spacings distributed according to Wigner's function we first put

\[ f(x) = \frac{1}{2\pi x} \exp\left(-\frac{1}{2}\pi x^2\right) \]

\[ \mathcal{F}(x) = \exp\left(-\frac{1}{2}\pi x^2\right) \]

\[ D(x) = \text{erfc}\left(\frac{1}{2}\sqrt{\pi x}\right) \]

so that

\[ P_2^*(D) = \frac{1}{2\pi D_1}D_1^{-3} \text{erfc}\left(\sqrt{\pi D/2D_1}\exp\left(-\pi D^2/4D_1^2\right)\right) \]

\[ + \frac{1}{2D_2}D_2^{-3} \text{erfc}\left(\sqrt{\pi D/2D_2}\exp\left(-\pi D^2/4D_2^2\right)\right) \]

\[ + 2\exp\left(-\frac{1}{2}\pi D^2(D_1^{-2} + D_2^{-2})\right) \]  

(3.35)

In order to assess the likelihood of missing resonances when making observations, the probability of a spacing of the pooled sequence \( D \) being less than a resonance width \( \Gamma \) is of interest and it is found by integration:

\[ P(D \leq \Gamma) = \int_0^\Gamma P_2^*(D) \, dD \]

\[ = \frac{(\overline{D}/\overline{D}_1)}{\overline{D}_1} \{1 - \text{erfc}(\sqrt{\pi \Gamma/2D_2}\exp(-\pi \Gamma^2/4D_1^2))\} \]

\[ + \frac{(\overline{D}/\overline{D}_2)}{\overline{D}_2} \{1 - \text{erfc}(\sqrt{\pi \Gamma/2D_1}\exp(-\pi \Gamma^2/4D_2^2))\} \]

(3.37)
which is the form used by Musgrove (unpublished). Lane and Lynn (ibid) consider the special case in which $\bar{D}_1 = \bar{D}_2 (=2\bar{D})$. This results in considerable simplifications, in particular, equation 3.36 becomes

$$P^*(D) = \left(\frac{\pi D}{8D^2}\right) \exp\left(-\frac{\pi D^2}{16D^2}\right) \text{erfc}\left(\sqrt{\pi D/4D}\right) + \left(\frac{1}{2D}\right) \exp\left(-\frac{\pi D^2}{8D^2}\right)$$

(3.38)

and equation 3.37 becomes

$$P(D \leq \Gamma | \bar{D}_1=\bar{D}_2) = 1-\text{erfc}\left(\sqrt{\pi D/4D}\right) \exp\left(-\frac{\pi \Gamma^2}{16D^2}\right). \quad (3.39)$$

It is interesting to note that Lane and Lynn (ibid) also considered the equivalent of equation 3.38 derived from the assumption that the resonance spacing density function for each sequence was a chi-squared function with either 2 or 4 degrees of freedom. For the case where $\nu = 2$ (i.e. Poisson distribution)

$$f(x) = \mathcal{N}(x) = D(x) = e^{-x} \quad (3.40)$$

and equation 3.25 reduces to $P(x) = e^{-x}$ whence

$$P^*(D) = \left(\frac{1}{D}\right) e^{-D/\bar{D}} \quad (3.41)$$

irrespective of the number of resonance sequences which are superposed.

When $\nu = 4$, considering only two sequences with equal mean spacings,

$$f(x) = 4x \exp(-2x)$$

$$\mathcal{N}(x) = (2x+1)\exp(-2x)$$

$$D(x) = (x+1)\exp(-2x) \quad (3.42)$$
so that using either equation 3.24 or equation 3.34,

\[
P^*(D) = \frac{1}{2D} \left( \frac{2D^2}{D^2} + \frac{4D}{D} + 1 \right) \exp \left( \frac{-2D}{D} \right) \tag{3.43}
\]

For low energy s-wave neutron bombardment of nuclei with spin \( I \geq 5/2 \), two sequences of resonances with approximately equal mean spacing are superposed. Lane and Lynn (ibid), comparing the above results with the 1955 data of Hughes and Harvey found the likelihood of equation 3.43 being correct was 3.6 times that of equation 3.41.

### 3.3 Kernels and Synthetic Kernels

#### 3.3.1 Kernels in General Reactor Theory

In Nuclear Reactor Theory, the transport kernel \( H(x, x', E, E', \Omega, \Omega', t-t') \) gives the flux of neutrons at time \( t \) in a unit volume at \( x \) and in a unit energy interval at \( E \) with velocity vectors in a unit solid angle at \( \Omega \) which is produced by fission, elastic collision or some other process due to the presence of a single neutron at \( (x', E', \Omega', t) \) (see for example Weinberg (1961)). If we only concern ourselves with the number of neutrons in unit energy interval at \( E \) resulting from a single neutron at \( E' \) we get the slowing down kernel

\[
K(E, E') = \int \int \int \int H(x, x', E, E', \Omega, \Omega', t-t') dx \, dx' \, d\Omega \, d\Omega' \, dt' \tag{3.44}
\]
The lethargy \( u \) of a neutron is defined in terms of its energy \( E \) by

\[
  u = \ln\left(\frac{E_o}{E}\right)
\]

where \( E_o \) is some reference energy taken above the energy of the neutron source. The slowing down equation, in terms of \( u \), is

\[
  F(u) = S(u) + \int_0^u k(u,u')F(u')du'
\]

where \( F(u) \) is the neutron density per unit lethargy per cc. per sec, \( S(u) \) is the source function which gives the rate at which neutrons are injected into the system and \( k(u,u') \) is the kernel for the density of neutrons with lethargy \( u \) resulting from a neutron with lethargy \( u' \).

The form of \( k(u,u') \) is dependent upon the geometry of the reactor, the properties of the moderator and many other factors. However, by assuming an infinite non-absorbing homogeneous moderating medium with nuclei of mass \( A \) and assuming that the only moderating process is spherically symmetric, elastic scattering, the kernel which gives the density of neutrons at \( u \) resulting from a single collision of a neutron with lethargy \( u' \) may be shown to depend upon the lethargy change
and \( A \) only, and it takes the form

\[
f(u-u') = \begin{cases} 
  e^{-(u-u')(1-\alpha)^{-1}}, & u' < u < u' + \ln(1/\alpha) \\
  0, & u > u' + \ln(1/\alpha) 
\end{cases} \quad (3.47)
\]

where \( \alpha = \left(\frac{\Lambda - 1}{\Lambda + 1}\right)^2 \) (Goertzel and Greuling (1960)). It is therefore possible to represent slowing down under these conditions by

\[
F(u) = S(u) + \int_0^u F(u-x)f(x)dx \quad (3.48)
\]

### 3.3.2 Synthetic Slowing Down Kernels

There are three classical synthetic kernels which may replace the kernel \( f(x) \) of equation 3.48, each making further calculations simpler without introducing intolerable errors. They are the Fermi kernel, \( \delta(x) - \xi \delta'(x) \), the Wigner kernel, \( \frac{1}{\overline{\xi}} \exp(-x/\overline{\xi}) \) and the Goertzel-Greuling kernel, \( (1 - \gamma_0^{-1})\delta(x) + (\gamma_0^{-2} \overline{\xi})^{-1} \exp(-x/\gamma_0 \overline{\xi}) \) where \( \overline{\xi} \) is the average lethargy gain per collision and \( \gamma_0 \) is \( \overline{\xi}^2 / 2 \overline{\xi}^2 \) where \( \overline{\xi}^2 \) is the average squared lethargy gained per collision. Wilkins and Keane (1966) give a derivation of these synthetic kernels by taking the Laplace transform of equation 3.48, truncating the series expansion of the transform of the collision density function and inverting. Truncation after one term leads to the Fermi kernel. After two terms it gives the Goertzel-Greuling kernel with the Wigner kernel being the \( \gamma_0 = 1 \) case of the latter. Wilkins and Keane (ibid) also indicate which kernel should be used when the source function is decaying rapidly,
slowly, or not decaying.

3.3.3 Synthetic Kernels for the Conditional Resonance Density Function

As was seen in the previous section, a synthetic kernel is a function which, as a mathematical convenience, replaces a theoretically justified probability density function (that is, a kernel) without causing undue loss of accuracy. The synthetic kernel itself may not be a probability density function. Another example of a synthetic kernel is the function $\Omega(x)$ as derived in Section 3.1.1 where the theoretically unjustified chi-squared functions were used as spacing distribution functions. The use of such synthetic kernels is now discussed with particular reference to the calculation of infinitely dilute resonance integrals.

3.3.4 The Calculation of Infinitely Dilute Resonance Integrals

For the conditions detailed in Dresner (1959), Reichel (1962) and Reichel and Wilkins (ibid) the infinitely dilute resonance integral $I(E_r)$ for a sequence of $\ell = 0$ resonances in the unresolved region $E > E_r$ ($E_r$ being the energy of the last resolved $\ell = 0$ resonance) is

$$I(E_r) = \frac{1}{D} \int_{E_r}^{\infty} \Omega \left( \frac{E-E_r}{D} \right) g(E) dE \quad (3.49)$$
where \( \Omega(x) \) is the conditional resonance density function as defined in Section 3.1 and \( \bar{D} \) is the mean spacing. The function \( g(E) \) is found by averaging over the Porter-Thomas distribution the infinitely dilute resonance integrals each of which is given by

\[
I_\infty = \frac{gh^2}{4mE^2} \cdot \frac{\Gamma \Gamma^0}{\bar{D} + \Gamma} \tag{3.50}
\]

where \( g \) is a statistical weighting factor, \( h \) is Planck's constant, \( m \) is the neutron mass, \( \Gamma \) is the gamma width (all assumed constant), \( \Gamma^0 \) is the reduced neutron width as defined in Section 2.2.2 and \( E \) is the energy at which the resonance occurs. Hence

\[
g(E) = \frac{gh^2 \Gamma}{4mE^2} \int_0^\infty \frac{y f(y) \, dy}{y + (\Gamma / \Gamma^0/E)} \tag{3.51}
\]

where \( f(y) \) is the Porter-Thomas probability density function as given by equation 2.8, \( y \) being the ratio of \( \Gamma^0 \) to its mean value \( \bar{\Gamma}^0 \).

The substitutions

\[
\begin{align*}
A &= \frac{\Gamma^0}{\Gamma/E} \\
t &= \sqrt{\frac{y}{A}}
\end{align*}
\tag{3.52}
\]

as suggested by Dresner (ibid) enable the integral in equation 3.51 to be written

\[
\left(\frac{2A}{\pi}\right)^{\frac{1}{2}} \int_0^\infty \frac{t^2 \exp\left(-\frac{1}{2}At^2\right)}{1 + t^2} \, dt
\]

which separates to
\[ 1 - \sqrt{\frac{2A}{\pi}} \int_0^\infty \frac{\exp\left(-\frac{1}{2}At^2\right)}{1 + t^2} \, dt. \]

Taking the Laplace transform with respect to \(A\) of the last integrand, integrating with respect to \(t\) from 0 to infinity and then inverting yields \(\frac{1}{2}\pi \exp\left(\frac{1}{2}A\right) \text{erfc}\left(\frac{1}{\sqrt{2}A}\right)\) so that ultimately

\[ g(E) = \frac{gh^2\Gamma}{4mE^2} \left\{ 1 - \sqrt{\frac{1}{2}\pi \exp\left(\frac{1}{2}A\right) \text{erfc}\left(\frac{1}{\sqrt{2}A}\right)} \right\}. \quad (3.53) \]

Reichel (ibid) shows that

\[ \int_{E_r}^\infty g(E) dE = \frac{gh^2\Gamma}{2mE_r} \left\{ \frac{1}{2} + \sum_{n=1}^{\infty} \frac{B^n}{(2n-1)!!(n+2)} \right\} \quad (3.54) \]

where \(B = \frac{\Gamma_0}{\Gamma n\sqrt{E_r}}\) and \((2n-1)!! = (2n)!/2^n n!\), which gives the value of \(\Omega(E)\) when \(E > E_r\). The value of \(I\) for all \(E > E_r\).

An improved method for calculating \(I\) uses

\[ I(E_r) = \frac{1}{D} \int_{E_r}^{E_r+kD} \Omega\left(\frac{E-E_r}{D}\right) g(E) dE + \frac{1}{D} \int_{E_r+kD}^\infty g(E) dE \quad (3.55) \]

where \(k\) is the value of \(x\) for which \(\Omega(x)\) is assumed to have taken its asymptotic value. Because the Wigner spacing function has been shown to be very accurate, the form of \(\Omega(x)\) as derived by Reichel and Wilkins (ibid) indicates an appropriate value of \(k\). These authors, in fact, suggest \(k = 1.5\). With \(k\) equal to this number they evaluate \(I(1,000) - I(30,000)\) for the \(^{238}U\) \(l = 0\) resonance sequence taking \(D = 18.5\) ev and using the series expansion of equation 3.54 to evaluate the second integral of equation 3.55 and their tabulated form of \(\Omega(x)\) to evaluate the
first integral. Their result of 2.821 barns compares with Reichel's result of 2.838 barns for \( \frac{1}{D} \int_{1,000}^{30,000} g(E) \, dE \).

Now it is proved by Smith (1954) that

\[
\int_0^\infty (\Omega(x) - 1) \, dx = x^2(2x^2) - 1
\]

\[
= \gamma - 1 \text{ (say)} \quad (3.56)
\]

where \( x \) and \( x^2 \) are the first and second moments of the spacing function used to generate \( \Omega(x) \), and it is the difference between \( \Omega(x) \) and 1 for small \( x \) which accounts for the difference in the above numerical results. For \( \Omega(x) \) as derived from Wigner's function, \( \gamma = 2/\pi \) because the first and second moments of \( \frac{1}{\pi}x \exp(-\frac{1}{\pi}x^2) \) are 1 and \( 4/\pi \). Wilkins et al (1968) put

\[
\Omega^*(x) = (\gamma - 1) \delta(x) + 1 \quad (3.57)
\]

(with \( \gamma \) as given above) which amounts to drawing all the deviations of \( \Omega(x) \) from 1 back to \( x = 0 \) and they find that

\[
(\gamma - 1)g(1,000) = -0.017 \text{ barns.}
\]

When added to Reichel's result of 2.838 barns this gives the Reichel and Wilkins value of \( I(1,000) - I(30,000) \), thus showing the equivalence (in this particular calculation) of using \( \Omega(x) \) as given by equation 3.14 or using the synthetic kernel \( \Omega^*(x) \).
More generally, if \( \phi(E) \) is a function which varies slowly for 
\[ E_r < E < E_r + 1.5D, \] equation 3.57 is of use in calculations 
of the form
\[
\int_{E'_r}^{E_r'} \Omega \left( \frac{E-E_r}{D} \right) \phi(E) dE
\]
where \( E' > E_r + 1.5D \).

It should be noted that \( \Omega(x) \) does not 
have all the properties of a density function. It is obvious 
that a positive number \( \varepsilon \) can be found such that
\[
\int_{0}^{\varepsilon} \Omega(x) dx < 0
\]
which is not the case for a true kernel.

### 3.3.5 Synthetic Spacing Kernel

It is of interest to find the pseudo spacing 
density function \( f^*(x) \) which gives rise to the resonance density 
function \( \Omega^*(x) \). Substitution of \((\gamma-1)d(x) + 1\) for \( \Omega(x) \) and 
\( f^*(x) \) for \( f(x) \) in equation 3.1 (the defining equation for \( \Omega(x) \)) 
yields
\[
(\gamma-1)d(x) + 1 = \gamma f^*(x) + \int_{0}^{x} f^*(t) dt.
\] (3.58)
Taking Laplace transforms leads to
\[
\bar{f}^*(p) = (p\gamma-p+1)(p\gamma+1)^{-1}
\]
\[
= 1-\gamma^{-1}+\gamma^{2}(p+\gamma^{-1})^{-1}
\] (3.59)
whence

\[ f^*(x) = (1 - \gamma^{-1}) \delta(x) + \gamma^{-2} \exp(-x/\gamma). \quad (3.60) \]

This synthetic kernel has the same form as the Goertzel-Greuling kernel (Section 3.3.2) and has been derived in much the same way as the latter. In fact the similarity may be traced back to the similar roles played by the spacing distribution function and the slowing down kernel in equations 3.1 and 3.48 respectively. Like the Goertzel-Greuling kernel, \( f^*(x) \) is not a probability function because of the "negative probability" which results in calculating \( \int_0^\epsilon f^*(x) \, dx \) when \( \epsilon \) is small and positive.
4.1 Scope of the Analysis

It is assumed that the Porter-Thomas distribution correctly describes the variation in the reduced neutron widths and that Wigner's function is a satisfactory fit for the distribution of the spacings. Chi-squared goodness of fit tests compare the data with these functions to see the extent to which the data can be relied upon. The resonance spacing data are then compared with chi-squared distribution functions to investigate which (if any) may be used in place of Wigner's function without undue loss of accuracy. Only the \( \ell = 0 \) sequence of resonances is studied.

4.2 Experimental Measurements

The \( ^{238} \text{U} \) isotope has an even-even nucleus with zero spin (i.e. \( I = 0 \)) and fairly abundant resonances in the lower part of the energy range. However, at higher energies, where the \( \ell = 0 \) resonance peaks are smaller, there is an increased chance of counting two or more levels as one level and omitting other levels. Further to this, the p-wave (\( \ell = 1 \)) resonances become more significant at the higher energies and may be mistaken for s-wave (\( \ell = 0 \)) resonances.

The reduced width \( r_n^0 \) and energy \( E \) of each resonance are taken from the table compiled by Schmidt (1966).
but he is not sure whether the 10.25 ev, 153 ev and 173 ev resonances are s or p-wave interactions. An attempt is made to assess the effect of omitting or including these resonances by applying the chi-squared goodness of fit test without and then with them in the $\lambda = 0$ sequence.

A list of the parameters used is given in Table 2.

4.3 Energy Ranges

To examine the energy ranges over which the data compares well with the theory, the spectrum has been treated in cumulative sections, each with lower limit of 6.68 ev, the energy of the first resonance. The upper limits are taken as the greatest resonance energies less than 200$n$ ev, $n = 4, 5, \ldots$, smaller ranges being omitted because intervals much less than 800 ev provide insufficient resonances to make the statistical procedures sound. The last section extends to the last resonance of the data.

4.4 The Chi-Squared Test

With the variates expressed in units of their mean, ten class intervals are determined by $x_i$ where

$$\int_0^{x_i} f(x)dx = i/10 \quad i = 1, 2, \ldots 9.$$  \hspace{1cm} (4.1)
The probability for a variate falling in the intervals \((0,x_1),(x_1,x_2),\ldots,(x_9,\infty)\) is 0.1 and consequently the expected number of variates in each interval is \(n/10\) where \(n\) is the total number of variates. The statistic \(\chi^2\) compares the observed frequencies with the expected frequencies and, in this case, has the form

\[
\chi^2 = \left(\frac{n}{10}\right)^{-1} \sum_{k=1}^{10} (r_k - n/10)^2
\]

where \(r_k\) is the observed frequency of the \(k^{th}\) class interval.

4.4.1. \(^{238}\)U \(\ell = 0\) Resonance Spacings

The equation \(x_i = D_i/\bar{D}\) has been used to express the observed spacings in units of their mean with \(\bar{D}\) obtained for each section of the energy spectrum by the maximum likelihood method, which for Wigner's distribution gives

\[
\bar{D} = \sqrt{\frac{n}{4n}} \sum_{i=1}^{n} D_i^2
\]

and for the chi-squared functions gives the arithmetic mean,

\[
\bar{D} = \frac{1}{n} \sum_{i=1}^{n} D_i
\]

With nine independent range intervals and one estimated parameter \((\bar{D})\), there are eight degrees of freedom with "partial recovery" of one degree as the parameter was estimated before grouping (see Kendal and Stuart (1947) p.440). Because of the "partial recovery", the values of \(\chi^2\) should be
compared against the critical values $\chi^2_{v,p}$ for both eight and nine degrees of freedom (i.e. $v = 8, 9$). The relevant values are given in Section 4.5 and they are defined by

$$\int_{\chi^2_{v,p}}^{\infty} f_{v}(x)dx = p$$

(4.5)

where $f_{v}(x)$ is the chi-squared distribution function as given by equation 2.16 and $p$ is the probability used to determine the significance of the results.

The computed values of $\chi^2$ obtained by comparing the data with Wigner's function and each of the chi-squared functions with $v = 2, 3 \ldots 14$ for each energy section are given in Table 3(a) when the "doubtful" resonances are excluded and in Table 3(b) when they are included. The values of $D$ as obtained from equations 4.3 and 4.4 are also given in these tables.

4.4.2. $^{238}$U Reduced Neutron Widths

Because the Porter-Thomas distribution is a chi-squared function, the maximum likelihood method yields the arithmetic mean of the reduced neutron widths for $\Gamma_n^0$.

When the values of $\Gamma_n^0$ are used without taking their possible errors into account, it is found that very few variates fall into the smallest class interval and for energy ranges extending beyond 1400 ev, the chi-squared test
rejects the data at the 5% confidence level. However, when we note that the possible error is often up to 50% of the value used when $\Gamma_n^0$ is very small, we are not surprised to find the discrepancy. To overcome the problem, the two smallest classes are treated together and equation 4.2 is modified to

$$\chi^2 = (n/5)^{-1}(r_1^2 + r_2^2 - n/5)^2 + (n/10)^{-1} \sum_{k=3}^{10} (r_k - n/10)^2. \quad (4.6)$$

Hence, with only eight independent range intervals, it is necessary to compare $\chi^2$ with the critical values of both seven and eight degrees of freedom. For the same energy sections as before the values of $\chi^2$ obtained by way of equation 4.6 using Schmidt's data both without and with the doubtful resonances are given in Table 3 with values of $\Gamma_n^0$.

### 4.5. Critical Values

The table below, extracted from Fisher (1941), gives the critical values of $\chi^2_{\nu,p}$ for seven, eight and nine degrees of freedom corresponding to probabilities of 20%, 10%, 5%, 2% and 1%.

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>$\nu$</th>
<th>0.2</th>
<th>0.1</th>
<th>0.05</th>
<th>0.02</th>
<th>0.01</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>9.80</td>
<td>12.02</td>
<td>14.07</td>
<td>16.62</td>
<td>18.48</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>11.03</td>
<td>13.36</td>
<td>15.51</td>
<td>18.17</td>
<td>20.09</td>
<td></td>
</tr>
</tbody>
</table>
These critical values are to be read in conjunction with the values of $\chi^2$ given in Table 3.

4.6 Conclusions

The chi-squared test provides no grounds for rejecting the data provided that the possible errors in the small reduced widths are taken into account. In fact, the modified test (Section 4.4.2) shows that the data fit the Porter-Thomas distribution function very well, all results being within the 10% confidence limit.

Whilst realizing that the Wigner function is not the exact probability density function for the resonance spacings it is regarded as good enough to test the data. Again, the data cannot be rejected, this time all results being within the 5% confidence limit.

Unfortunately the test is not sufficiently sensitive to determine whether the doubtful resonances should be counted as $l=0$ or $l=1$ resonances. It therefore seems prudent to assume that either set of data could be taken to test the fit of the other spacing functions, despite the fact that one set must be erroneous and, on the basis of the chi-squared test, there is no justification for preferring one set to the other.

The small variation in mean spacing levels for the cumulative energy ranges indicate that few (if any)
resonances are being missed at higher energies. We therefore conclude that the quality of both sets of data is quite adequate for use in further chi-squared tests for all of the energy ranges.

The values of $\chi^2$ obtained by comparing the data with $f_\nu(x)$ are generally large for $\nu = 2, 3, 12, 13, 14$ so that these distributions are rejected at the 1% level for most ranges. When large energy ranges are considered $f_4(x)$ fails whilst $f_{11}(x)$ fails for several ranges of the order of 2500 ev.

Values of $\chi^2$ obtained for the other values of $\nu$ for both sets of data compare quite favourable with those obtained when testing the data against Wigner's function indicating that $f_6(x)$, $f_7(x)$ and $f_8(x)$ fit the data particularly well. It is therefore recommended that in calculations where Wigner's function is unmanageable, the chi-squared functions with $\nu = 6, 7$ and 8 should be considered as a reasonably accurate alternative. Of these $f_6(x)$ and $f_8(x)$ are easiest to handle.
<table>
<thead>
<tr>
<th>$x$</th>
<th>$v=4$</th>
<th>$v=6$</th>
<th>$v=8$</th>
<th>$v=10$</th>
<th>$v=12$</th>
<th>Abridged from Reichel &amp; Wilkins</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
<td>0.0000</td>
</tr>
<tr>
<td>0.1</td>
<td>0.3297</td>
<td>0.1000</td>
<td>0.0286</td>
<td>0.0079</td>
<td>0.0021</td>
<td>0.1562</td>
</tr>
<tr>
<td>0.2</td>
<td>0.5507</td>
<td>0.2974</td>
<td>0.1534</td>
<td>0.0766</td>
<td>0.0374</td>
<td>0.3076</td>
</tr>
<tr>
<td>0.3</td>
<td>0.6988</td>
<td>0.5000</td>
<td>0.3478</td>
<td>0.2354</td>
<td>0.1562</td>
<td>0.4498</td>
</tr>
<tr>
<td>0.4</td>
<td>0.7981</td>
<td>0.6694</td>
<td>0.5556</td>
<td>0.4521</td>
<td>0.3614</td>
<td>0.5788</td>
</tr>
<tr>
<td>0.5</td>
<td>0.8647</td>
<td>0.7958</td>
<td>0.7356</td>
<td>0.6723</td>
<td>0.6063</td>
<td>0.6919</td>
</tr>
<tr>
<td>0.6</td>
<td>0.9093</td>
<td>0.8828</td>
<td>0.8692</td>
<td>0.8537</td>
<td>0.8315</td>
<td>0.7875</td>
</tr>
<tr>
<td>0.7</td>
<td>0.9392</td>
<td>0.9386</td>
<td>0.9556</td>
<td>0.9769</td>
<td>0.9961</td>
<td>0.8650</td>
</tr>
<tr>
<td>0.8</td>
<td>0.9592</td>
<td>0.9719</td>
<td>1.0031</td>
<td>1.0433</td>
<td>1.0871</td>
<td>0.9248</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9727</td>
<td>0.9904</td>
<td>1.0234</td>
<td>1.0658</td>
<td>1.1444</td>
<td>0.9683</td>
</tr>
<tr>
<td>1.0</td>
<td>0.9817</td>
<td>0.9996</td>
<td>1.0274</td>
<td>1.0610</td>
<td>1.0996</td>
<td>0.9977</td>
</tr>
<tr>
<td>1.1</td>
<td>0.9877</td>
<td>1.0034</td>
<td>1.0232</td>
<td>1.0439</td>
<td>1.0654</td>
<td>1.0154</td>
</tr>
<tr>
<td>1.2</td>
<td>0.9918</td>
<td>1.0043</td>
<td>1.0163</td>
<td>1.0247</td>
<td>1.0295</td>
<td>1.0240</td>
</tr>
<tr>
<td>1.3</td>
<td>0.9945</td>
<td>1.0040</td>
<td>1.0097</td>
<td>1.0091</td>
<td>1.0021</td>
<td>1.0261</td>
</tr>
<tr>
<td>1.4</td>
<td>0.9963</td>
<td>1.0031</td>
<td>1.0047</td>
<td>0.9990</td>
<td>0.9865</td>
<td>1.0239</td>
</tr>
<tr>
<td>1.5</td>
<td>0.9975</td>
<td>1.0022</td>
<td>1.0014</td>
<td>0.9943</td>
<td>0.9814</td>
<td>1.0194</td>
</tr>
<tr>
<td>1.6</td>
<td>0.9983</td>
<td>1.0015</td>
<td>0.9996</td>
<td>0.9933</td>
<td>0.9836</td>
<td>1.0140</td>
</tr>
<tr>
<td>1.7</td>
<td>0.9989</td>
<td>1.0009</td>
<td>0.9989</td>
<td>0.9944</td>
<td>0.9890</td>
<td>1.0087</td>
</tr>
<tr>
<td>1.8</td>
<td>0.9992</td>
<td>1.0005</td>
<td>0.9988</td>
<td>0.9963</td>
<td>0.9949</td>
<td>1.0043</td>
</tr>
<tr>
<td>1.9</td>
<td>0.9995</td>
<td>1.0003</td>
<td>0.9990</td>
<td>0.9982</td>
<td>0.9959</td>
<td>1.0009</td>
</tr>
<tr>
<td>2.0</td>
<td>0.9997</td>
<td>1.0001</td>
<td>0.9993</td>
<td>0.9996</td>
<td>1.0021</td>
<td>0.9987</td>
</tr>
<tr>
<td>2.1</td>
<td>0.9998</td>
<td>1.0000</td>
<td>0.9996</td>
<td>1.0003</td>
<td>1.0030</td>
<td>0.9974</td>
</tr>
<tr>
<td>2.2</td>
<td>0.9998</td>
<td>1.0000</td>
<td>0.9998</td>
<td>1.0007</td>
<td>1.0027</td>
<td>0.9970</td>
</tr>
<tr>
<td>2.3</td>
<td>0.9999</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0007</td>
<td>1.0018</td>
<td>0.9971</td>
</tr>
<tr>
<td>2.4</td>
<td>0.9999</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0005</td>
<td>1.0009</td>
<td>0.9975</td>
</tr>
<tr>
<td>2.5</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0003</td>
<td>1.0001</td>
<td>0.9981</td>
</tr>
<tr>
<td>2.6</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0001</td>
<td>0.997</td>
<td>0.9987</td>
</tr>
<tr>
<td>2.7</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9995</td>
<td>0.9993</td>
</tr>
<tr>
<td>2.8</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9996</td>
<td>0.9997</td>
</tr>
<tr>
<td>2.9</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.9997</td>
<td>0.9999</td>
</tr>
<tr>
<td>3.0</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.9998</td>
<td>1.0000</td>
</tr>
</tbody>
</table>

* Reichel and Wilkins give $\Omega(x)$ for $0 \leq x \leq 3$ at intervals of 0.02.
<table>
<thead>
<tr>
<th>Resonance Energy (eV)</th>
<th>Reduced Width (meV)</th>
<th>Resonance Energy (eV)</th>
<th>Reduced Width (meV)</th>
<th>Resonance Energy (eV)</th>
<th>Reduced Width (meV)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6.8</td>
<td>0.590</td>
<td>661.2</td>
<td>4.7</td>
<td>1245.1</td>
<td>6.55</td>
</tr>
<tr>
<td>*10.25</td>
<td>0.00047</td>
<td>693.2</td>
<td>1.31</td>
<td>1267.0</td>
<td>0.75</td>
</tr>
<tr>
<td>21.0</td>
<td>1.9</td>
<td>708.5</td>
<td>0.76</td>
<td>1273.2</td>
<td>0.80</td>
</tr>
<tr>
<td>36.8</td>
<td>5.1</td>
<td>721.8</td>
<td>0.041</td>
<td>1298.4</td>
<td>0.09</td>
</tr>
<tr>
<td>66.1</td>
<td>3.00</td>
<td>732.3</td>
<td>0.050</td>
<td>1317.2</td>
<td>0.11</td>
</tr>
<tr>
<td>80.0</td>
<td>0.22</td>
<td>765.1</td>
<td>0.19</td>
<td>1335.7</td>
<td>0.03</td>
</tr>
<tr>
<td>89.5</td>
<td>0.0090</td>
<td>779.1</td>
<td>0.05</td>
<td>1393.0</td>
<td>4.1</td>
</tr>
<tr>
<td>102.6</td>
<td>6.7</td>
<td>790.9</td>
<td>0.16</td>
<td>1405.1</td>
<td>2.05</td>
</tr>
<tr>
<td>116.8</td>
<td>2.38</td>
<td>821.6</td>
<td>2.05</td>
<td>1419.6</td>
<td>0.28</td>
</tr>
<tr>
<td>145.8</td>
<td>0.059</td>
<td>851.0</td>
<td>1.92</td>
<td>1427.7</td>
<td>0.80</td>
</tr>
<tr>
<td>*153</td>
<td>0.003</td>
<td>856.2</td>
<td>2.77</td>
<td>1444.1</td>
<td>0.57</td>
</tr>
<tr>
<td>165.4</td>
<td>0.23</td>
<td>866.5</td>
<td>0.10</td>
<td>1473.8</td>
<td>2.05</td>
</tr>
<tr>
<td>*173</td>
<td>0.0025</td>
<td>905.1</td>
<td>1.50</td>
<td>1523.1</td>
<td>5.3</td>
</tr>
<tr>
<td>190.0</td>
<td>10.8</td>
<td>909.9</td>
<td>0.03</td>
<td>1532.0</td>
<td>0.05</td>
</tr>
<tr>
<td>209.6</td>
<td>3.9</td>
<td>925.2</td>
<td>0.29</td>
<td>1550.0</td>
<td>0.03</td>
</tr>
<tr>
<td>237.4</td>
<td>1.91</td>
<td>936.9</td>
<td>5.0</td>
<td>1565.0</td>
<td>0.05</td>
</tr>
<tr>
<td>273.7</td>
<td>1.54</td>
<td>958.4</td>
<td>4.9</td>
<td>1598.2</td>
<td>8.0</td>
</tr>
<tr>
<td>291.1</td>
<td>0.97</td>
<td>991.8</td>
<td>11.1</td>
<td>1622.9</td>
<td>2.33</td>
</tr>
<tr>
<td>311.1</td>
<td>0.056</td>
<td>1000.3</td>
<td>0.04</td>
<td>1638.2</td>
<td>1.00</td>
</tr>
<tr>
<td>347.9</td>
<td>3.03</td>
<td>1011.3</td>
<td>0.032</td>
<td>1662.1</td>
<td>3.9</td>
</tr>
<tr>
<td>376.9</td>
<td>0.059</td>
<td>1023.0</td>
<td>0.24</td>
<td>1688.3</td>
<td>1.73</td>
</tr>
<tr>
<td>397.6</td>
<td>0.30</td>
<td>1029.1</td>
<td>0.10</td>
<td>1709.4</td>
<td>1.23</td>
</tr>
<tr>
<td>410.2</td>
<td>0.84</td>
<td>1053.9</td>
<td>1.94</td>
<td>1723.0</td>
<td>0.34</td>
</tr>
<tr>
<td>434.2</td>
<td>0.36</td>
<td>1068.1</td>
<td>0.02</td>
<td>1744.0</td>
<td>0.04</td>
</tr>
<tr>
<td>463.3</td>
<td>0.23</td>
<td>1098.4</td>
<td>0.35</td>
<td>1755.8</td>
<td>1.72</td>
</tr>
<tr>
<td>478.7</td>
<td>0.13</td>
<td>1108.9</td>
<td>0.90</td>
<td>1782.3</td>
<td>11.6</td>
</tr>
<tr>
<td>518.6</td>
<td>1.87</td>
<td>1131.5</td>
<td>0.06</td>
<td>1797.7</td>
<td>0.05</td>
</tr>
<tr>
<td>535.5</td>
<td>1.7</td>
<td>1140.4</td>
<td>6.5</td>
<td>1808.3</td>
<td>0.40</td>
</tr>
<tr>
<td>580.2</td>
<td>1.13</td>
<td>1167.5</td>
<td>2.25</td>
<td>1845.6</td>
<td>0.31</td>
</tr>
<tr>
<td>595.2</td>
<td>3.27</td>
<td>1177.6</td>
<td>1.72</td>
<td>1902.3</td>
<td>0.48</td>
</tr>
<tr>
<td>619.9</td>
<td>1.15</td>
<td>1195.0</td>
<td>2.40</td>
<td>1917.1</td>
<td>0.50</td>
</tr>
<tr>
<td>628.7</td>
<td>0.14</td>
<td>1210.9</td>
<td>0.26</td>
<td>1968.7</td>
<td>13.00</td>
</tr>
</tbody>
</table>

* \( l \) may be 0 or 1 for these resonances.
## TABLE 2 (Continued)

<table>
<thead>
<tr>
<th>Resonance Energy</th>
<th>Reduced Width (mev)</th>
<th>Resonance Energy (ev)</th>
<th>Reduced Width (mev)</th>
<th>Resonance Energy (ev)</th>
<th>Reduced Width (mev)</th>
<th>Resonance Energy (ev)</th>
<th>Reduced Width (mev)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1974.7</td>
<td>10.50</td>
<td>2620.6</td>
<td>0.80</td>
<td>3280.0</td>
<td>1.80</td>
<td>3295.0</td>
<td>0.15</td>
</tr>
<tr>
<td>2023.6</td>
<td>4.50</td>
<td>2672.8</td>
<td>3.40</td>
<td>3310.9</td>
<td>1.65</td>
<td>3321.3</td>
<td>1.42</td>
</tr>
<tr>
<td>2031.1</td>
<td>1.10</td>
<td>2695.6</td>
<td>0.45</td>
<td>3334.0</td>
<td>1.00</td>
<td>3355.7</td>
<td>1.30</td>
</tr>
<tr>
<td>2088.6</td>
<td>0.30</td>
<td>2716.8</td>
<td>1.36</td>
<td>3371.0</td>
<td>0.05</td>
<td>3387.8</td>
<td>0.14</td>
</tr>
<tr>
<td>2096.5</td>
<td>0.22</td>
<td>2750.1</td>
<td>0.75</td>
<td>3409.0</td>
<td>1.80</td>
<td>3436.9</td>
<td>3.25</td>
</tr>
<tr>
<td>2124.4</td>
<td>0.10</td>
<td>2761.9</td>
<td>0.30</td>
<td>3459.1</td>
<td>6.50</td>
<td>3484.3</td>
<td>2.00</td>
</tr>
<tr>
<td>2146.0</td>
<td>0.75</td>
<td>2787.9</td>
<td>0.20</td>
<td>3526.0</td>
<td>0.18</td>
<td>3561.5</td>
<td>2.40</td>
</tr>
<tr>
<td>2152.8</td>
<td>3.80</td>
<td>2806.2</td>
<td>0.13</td>
<td>3574.0</td>
<td>4.00</td>
<td>3593.0</td>
<td>0.26</td>
</tr>
<tr>
<td>2172.0</td>
<td>0.05</td>
<td>2828.6</td>
<td>0.17</td>
<td>3611.0</td>
<td>0.05</td>
<td>3625.0</td>
<td>0.05</td>
</tr>
<tr>
<td>2186.0</td>
<td>7.80</td>
<td>2866.1</td>
<td>1.48</td>
<td>3630.0</td>
<td>3.60</td>
<td>3693.0</td>
<td>4.00</td>
</tr>
<tr>
<td>2201.4</td>
<td>2.40</td>
<td>2882.9</td>
<td>9.80</td>
<td>3717.7</td>
<td>1.00</td>
<td>3733.3</td>
<td>2.50</td>
</tr>
<tr>
<td>2230.0</td>
<td>0.10</td>
<td>2897.8</td>
<td>0.50</td>
<td>3764.7</td>
<td>0.56</td>
<td>3783.7</td>
<td>4.50</td>
</tr>
<tr>
<td>2235.7</td>
<td>0.10</td>
<td>2923.6</td>
<td>0.08</td>
<td>3832.0</td>
<td>0.10</td>
<td>3858.1</td>
<td>5.50</td>
</tr>
<tr>
<td>2259.1</td>
<td>1.38</td>
<td>2932.3</td>
<td>0.46</td>
<td>3871.3</td>
<td>4.00</td>
<td>3895.0</td>
<td>0.08</td>
</tr>
<tr>
<td>2266.4</td>
<td>3.05</td>
<td>2956.3</td>
<td>0.28</td>
<td>3904.4</td>
<td>3.60</td>
<td>3904.4</td>
<td>3.60</td>
</tr>
<tr>
<td>2281.3</td>
<td>2.30</td>
<td>2967.4</td>
<td>0.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2288.7</td>
<td>0.05</td>
<td>2987.4</td>
<td>0.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2315.9</td>
<td>0.30</td>
<td>3003.1</td>
<td>1.70</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2337.4</td>
<td>0.10</td>
<td>3015.0</td>
<td>0.13</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2352.0</td>
<td>1.30</td>
<td>3029.0</td>
<td>2.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2356.0</td>
<td>1.30</td>
<td>3041.0</td>
<td>0.05</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2392.5</td>
<td>0.23</td>
<td>3060.2</td>
<td>0.50</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2410.2</td>
<td>0.09</td>
<td>3081.1</td>
<td>0.08</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2426.5</td>
<td>1.65</td>
<td>3109.4</td>
<td>1.80</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2446.2</td>
<td>2.25</td>
<td>3133.2</td>
<td>0.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2454.0</td>
<td>0.05</td>
<td>3149.0</td>
<td>1.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2489.8</td>
<td>1.10</td>
<td>3169.0</td>
<td>0.18</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2520.7</td>
<td>0.20</td>
<td>3179.4</td>
<td>1.10</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2548.7</td>
<td>6.80</td>
<td>3189.0</td>
<td>0.77</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2559.3</td>
<td>4.30</td>
<td>3206.0</td>
<td>1.00</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2580.7</td>
<td>4.80</td>
<td>3226.0</td>
<td>0.40</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2598.7</td>
<td>11.0</td>
<td>3249.2</td>
<td>0.20</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
TABLE 3(a) \( ^{238}\text{U} \) \( \ell=0 \) resonance data excluding doubtful resonances

<table>
<thead>
<tr>
<th>Upper limit of energy range (ev)</th>
<th>Number of Spacings</th>
<th>Mean Spacing (ev)</th>
<th>( \chi^2 ) for Spacings using Wigner distribution</th>
<th>Mean Reduced Width (mev)</th>
<th>( \chi^2 ) for Reduced Widths using Porter-Thomas distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using equation 4.3</td>
<td>Using equation 4.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>790.0</td>
<td>36</td>
<td>21.01</td>
<td>21.78</td>
<td>11.22</td>
<td>1.646</td>
</tr>
<tr>
<td>991.8</td>
<td>46</td>
<td>20.53</td>
<td>21.42</td>
<td>8.35</td>
<td>1.883</td>
</tr>
<tr>
<td>1195.0</td>
<td>59</td>
<td>19.50</td>
<td>20.14</td>
<td>10.66</td>
<td>1.751</td>
</tr>
<tr>
<td>1393.0</td>
<td>67</td>
<td>20.21</td>
<td>20.69</td>
<td>11.35</td>
<td>1.726</td>
</tr>
<tr>
<td>1598.2</td>
<td>77</td>
<td>20.31</td>
<td>20.67</td>
<td>10.92</td>
<td>1.724</td>
</tr>
<tr>
<td>1797.7</td>
<td>87</td>
<td>19.95</td>
<td>20.59</td>
<td>13.11</td>
<td>1.773</td>
</tr>
<tr>
<td>1974.7</td>
<td>93</td>
<td>20.88</td>
<td>21.16</td>
<td>9.47</td>
<td>1.924</td>
</tr>
<tr>
<td>2186.0</td>
<td>102</td>
<td>20.93</td>
<td>21.37</td>
<td>7.61</td>
<td>1.893</td>
</tr>
<tr>
<td>2392.5</td>
<td>114</td>
<td>20.57</td>
<td>20.93</td>
<td>11.44</td>
<td>1.784</td>
</tr>
<tr>
<td>2598.7</td>
<td>124</td>
<td>20.45</td>
<td>20.90</td>
<td>11.16</td>
<td>1.898</td>
</tr>
<tr>
<td>2787.9</td>
<td>131</td>
<td>20.73</td>
<td>21.23</td>
<td>10.15</td>
<td>1.847</td>
</tr>
<tr>
<td>2987.4</td>
<td>141</td>
<td>20.57</td>
<td>21.14</td>
<td>8.72</td>
<td>1.808</td>
</tr>
<tr>
<td>3189.0</td>
<td>153</td>
<td>20.19</td>
<td>20.80</td>
<td>9.16</td>
<td>1.721</td>
</tr>
<tr>
<td>3387.8</td>
<td>164</td>
<td>19.94</td>
<td>20.62</td>
<td>13.07</td>
<td>1.656</td>
</tr>
<tr>
<td>3593.0</td>
<td>174</td>
<td>19.86</td>
<td>20.61</td>
<td>13.36</td>
<td>1.669</td>
</tr>
<tr>
<td>3783.7</td>
<td>182</td>
<td>20.12</td>
<td>20.75</td>
<td>15.47</td>
<td>1.684</td>
</tr>
<tr>
<td>3904.4</td>
<td>187</td>
<td>20.04</td>
<td>20.88</td>
<td>13.89</td>
<td>1.700</td>
</tr>
</tbody>
</table>
### TABLE 3(a) Continued

<table>
<thead>
<tr>
<th>Upper limit of energy range (ev)</th>
<th>$\chi^2$ for Spacings using chi-squared distributions</th>
<th>v=2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
<th>13</th>
<th>14</th>
</tr>
</thead>
<tbody>
<tr>
<td>790.9</td>
<td></td>
<td>31.8</td>
<td>26.2</td>
<td>15.1</td>
<td>20.7</td>
<td>11.8</td>
<td>7.4</td>
<td>9.6</td>
<td>11.9</td>
<td>8.4</td>
<td>13.4</td>
<td>11.8</td>
<td>10.1</td>
<td>11.8</td>
</tr>
<tr>
<td>991.8</td>
<td></td>
<td>33.1</td>
<td>22.7</td>
<td>14.4</td>
<td>14.4</td>
<td>10.1</td>
<td>7.1</td>
<td>4.0</td>
<td>6.1</td>
<td>16.6</td>
<td>17.9</td>
<td>16.2</td>
<td>13.6</td>
<td>17.0</td>
</tr>
<tr>
<td>1195.0</td>
<td></td>
<td>40.2</td>
<td>26.6</td>
<td>20.5</td>
<td>11.0</td>
<td>11.3</td>
<td>11.4</td>
<td>14.4</td>
<td>10.8</td>
<td>10.0</td>
<td>10.3</td>
<td>14.4</td>
<td>16.7</td>
<td>20.5</td>
</tr>
<tr>
<td>1393.0</td>
<td></td>
<td>35.2</td>
<td>21.8</td>
<td>12.6</td>
<td>5.7</td>
<td>4.5</td>
<td>11.7</td>
<td>7.5</td>
<td>6.5</td>
<td>6.9</td>
<td>10.2</td>
<td>14.6</td>
<td>17.0</td>
<td>21.2</td>
</tr>
<tr>
<td>1598.2</td>
<td></td>
<td>41.8</td>
<td>22.6</td>
<td>13.8</td>
<td>6.0</td>
<td>6.5</td>
<td>14.1</td>
<td>11.4</td>
<td>9.5</td>
<td>10.9</td>
<td>13.8</td>
<td>18.5</td>
<td>21.5</td>
<td>25.7</td>
</tr>
<tr>
<td>1797.7</td>
<td></td>
<td>50.8</td>
<td>30.1</td>
<td>22.1</td>
<td>7.8</td>
<td>9.2</td>
<td>12.1</td>
<td>9.2</td>
<td>7.3</td>
<td>7.4</td>
<td>7.9</td>
<td>16.1</td>
<td>17.6</td>
<td>22.3</td>
</tr>
<tr>
<td>1974.7</td>
<td></td>
<td>47.3</td>
<td>29.3</td>
<td>15.9</td>
<td>12.9</td>
<td>10.5</td>
<td>7.3</td>
<td>5.4</td>
<td>9.0</td>
<td>12.3</td>
<td>16.8</td>
<td>22.2</td>
<td>23.0</td>
<td>26.9</td>
</tr>
<tr>
<td>2186.0</td>
<td></td>
<td>48.0</td>
<td>26.0</td>
<td>11.7</td>
<td>10.7</td>
<td>7.8</td>
<td>4.2</td>
<td>4.3</td>
<td>7.6</td>
<td>16.6</td>
<td>21.9</td>
<td>25.3</td>
<td>23.9</td>
<td>35.6</td>
</tr>
<tr>
<td>2392.5</td>
<td></td>
<td>48.2</td>
<td>30.6</td>
<td>15.3</td>
<td>5.0</td>
<td>11.8</td>
<td>12.7</td>
<td>10.6</td>
<td>9.8</td>
<td>14.9</td>
<td>33.2</td>
<td>31.8</td>
<td>40.3</td>
<td>36.9</td>
</tr>
<tr>
<td>2598.7</td>
<td></td>
<td>57.0</td>
<td>32.0</td>
<td>16.5</td>
<td>6.4</td>
<td>10.2</td>
<td>9.2</td>
<td>10.0</td>
<td>8.3</td>
<td>12.5</td>
<td>32.8</td>
<td>32.6</td>
<td>38.5</td>
<td>35.5</td>
</tr>
<tr>
<td>2787.9</td>
<td></td>
<td>57.5</td>
<td>33.0</td>
<td>16.6</td>
<td>11.0</td>
<td>8.9</td>
<td>6.9</td>
<td>6.8</td>
<td>6.5</td>
<td>16.7</td>
<td>22.5</td>
<td>29.5</td>
<td>31.0</td>
<td>45.7</td>
</tr>
<tr>
<td>2987.4</td>
<td></td>
<td>66.0</td>
<td>34.4</td>
<td>19.6</td>
<td>10.5</td>
<td>7.9</td>
<td>6.1</td>
<td>7.2</td>
<td>6.4</td>
<td>10.0</td>
<td>21.4</td>
<td>31.3</td>
<td>29.8</td>
<td>35.4</td>
</tr>
<tr>
<td>3189.0</td>
<td></td>
<td>78.7</td>
<td>41.4</td>
<td>24.2</td>
<td>8.7</td>
<td>10.1</td>
<td>9.0</td>
<td>6.4</td>
<td>7.3</td>
<td>8.9</td>
<td>21.9</td>
<td>23.7</td>
<td>32.3</td>
<td>31.6</td>
</tr>
<tr>
<td>3387.8</td>
<td></td>
<td>89.0</td>
<td>45.5</td>
<td>33.3</td>
<td>11.5</td>
<td>8.1</td>
<td>9.0</td>
<td>7.2</td>
<td>9.6</td>
<td>10.5</td>
<td>13.1</td>
<td>24.0</td>
<td>28.7</td>
<td>34.8</td>
</tr>
<tr>
<td>3593.0</td>
<td></td>
<td>95.9</td>
<td>49.3</td>
<td>34.3</td>
<td>13.6</td>
<td>7.8</td>
<td>8.9</td>
<td>6.5</td>
<td>4.8</td>
<td>9.3</td>
<td>12.2</td>
<td>23.1</td>
<td>27.6</td>
<td>33.5</td>
</tr>
<tr>
<td>3783.7</td>
<td></td>
<td>100.9</td>
<td>54.6</td>
<td>33.5</td>
<td>13.8</td>
<td>13.7</td>
<td>10.9</td>
<td>4.4</td>
<td>5.8</td>
<td>8.0</td>
<td>16.7</td>
<td>22.1</td>
<td>30.1</td>
<td>29.3</td>
</tr>
<tr>
<td>3904.4</td>
<td></td>
<td>104.0</td>
<td>57.2</td>
<td>31.5</td>
<td>14.8</td>
<td>14.8</td>
<td>21.1</td>
<td>4.3</td>
<td>5.3</td>
<td>5.4</td>
<td>20.2</td>
<td>21.7</td>
<td>33.4</td>
<td>27.5</td>
</tr>
</tbody>
</table>
### TABLE 3(b) \(^2\) resonance data including doubtful resonances

<table>
<thead>
<tr>
<th>Upper limit of energy range (ev)</th>
<th>Number of Spacings</th>
<th>Mean Spacing (ev)</th>
<th>(\chi^2) for Spacings</th>
<th>Mean Reduced Width (mev)</th>
<th>(\chi^2) for Reduced Widths using Porter-Thomas distribution</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Using equation 4.3</td>
<td>Using equation 4.4</td>
<td></td>
<td></td>
</tr>
<tr>
<td>790.0</td>
<td>39</td>
<td>19.93</td>
<td>20.11</td>
<td>10.49</td>
<td>1.522</td>
</tr>
<tr>
<td>991.8</td>
<td>49</td>
<td>19.68</td>
<td>20.10</td>
<td>14.88</td>
<td>1.770</td>
</tr>
<tr>
<td>1195.0</td>
<td>62</td>
<td>18.84</td>
<td>19.17</td>
<td>14.13</td>
<td>1.668</td>
</tr>
<tr>
<td>1393.0</td>
<td>70</td>
<td>19.63</td>
<td>19.80</td>
<td>8.57</td>
<td>1.653</td>
</tr>
<tr>
<td>1598.2</td>
<td>80</td>
<td>19.80</td>
<td>19.89</td>
<td>12.50</td>
<td>1.660</td>
</tr>
<tr>
<td>1797.7</td>
<td>90</td>
<td>19.50</td>
<td>19.90</td>
<td>8.89</td>
<td>1.715</td>
</tr>
<tr>
<td>1974.7</td>
<td>96</td>
<td>20.45</td>
<td>20.50</td>
<td>9.42</td>
<td>1.864</td>
</tr>
<tr>
<td>2186.0</td>
<td>105</td>
<td>20.54</td>
<td>20.76</td>
<td>8.24</td>
<td>1.839</td>
</tr>
<tr>
<td>2392.5</td>
<td>117</td>
<td>20.22</td>
<td>20.39</td>
<td>10.26</td>
<td>1.739</td>
</tr>
<tr>
<td>2598.7</td>
<td>127</td>
<td>20.13</td>
<td>20.41</td>
<td>9.14</td>
<td>1.854</td>
</tr>
<tr>
<td>2787.9</td>
<td>134</td>
<td>20.42</td>
<td>20.76</td>
<td>8.84</td>
<td>1.806</td>
</tr>
<tr>
<td>2987.4</td>
<td>144</td>
<td>20.28</td>
<td>20.70</td>
<td>6.56</td>
<td>1.771</td>
</tr>
<tr>
<td>3189.0</td>
<td>156</td>
<td>19.93</td>
<td>20.40</td>
<td>8.62</td>
<td>1.689</td>
</tr>
<tr>
<td>3387.8</td>
<td>167</td>
<td>19.70</td>
<td>20.25</td>
<td>8.87</td>
<td>1.626</td>
</tr>
<tr>
<td>3593.0</td>
<td>177</td>
<td>19.63</td>
<td>20.26</td>
<td>9.95</td>
<td>1.641</td>
</tr>
<tr>
<td>3783.7</td>
<td>185</td>
<td>19.90</td>
<td>20.42</td>
<td>11.38</td>
<td>1.657</td>
</tr>
<tr>
<td>3904.4</td>
<td>190</td>
<td>19.79</td>
<td>20.51</td>
<td>10.31</td>
<td>1.683</td>
</tr>
<tr>
<td>Upper limit of energy range (ev)</td>
<td>$\chi^2$ for Spacings using chi-squared distributions $v=2$</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>----------------------------------</td>
<td>-------------------------------------------------</td>
<td>---</td>
<td>---</td>
<td>---</td>
<td>---</td>
</tr>
<tr>
<td>790.9</td>
<td>29.5 20.2 9.5 12.0 10.0 7.4 12.5</td>
<td>11.4</td>
<td>8.4</td>
<td>9.0</td>
<td>12.5</td>
</tr>
<tr>
<td>991.8</td>
<td>31.2 20.2 10.0 13.2 10.8 16.5</td>
<td>13.1</td>
<td>11.6</td>
<td>12.5</td>
<td>16.9</td>
</tr>
<tr>
<td>1195.0</td>
<td>38.0 18.3 7.0 11.5 10.9 11.9</td>
<td>11.4</td>
<td>15.7</td>
<td>15.5</td>
<td>15.7</td>
</tr>
<tr>
<td>1393.0</td>
<td>35.7 18.0 9.4 8.8 8.6 7.4</td>
<td>11.7</td>
<td>11.1</td>
<td>12.0</td>
<td>15.1</td>
</tr>
<tr>
<td>1598.2</td>
<td>43.0 20.5 13.5 7.5 10.5 10.3 11.8</td>
<td>14.4</td>
<td>15.5</td>
<td>16.5</td>
<td>18.8</td>
</tr>
<tr>
<td>1797.7</td>
<td>51.3 27.3 16.4 7.4 8.7 9.4 9.3</td>
<td>10.2</td>
<td>10.9</td>
<td>12.0</td>
<td>13.6</td>
</tr>
<tr>
<td>1947.7</td>
<td>48.0 23.4 14.8 6.3 8.0 10.3 12.5</td>
<td>10.1</td>
<td>14.2</td>
<td>15.5</td>
<td>28.2</td>
</tr>
<tr>
<td>2186.0</td>
<td>44.8 21.6 13.8 5.0 6.3 6.8 10.9</td>
<td>12.3</td>
<td>15.3</td>
<td>25.4</td>
<td>32.0</td>
</tr>
<tr>
<td>2392.5</td>
<td>48.7 23.9 14.7 5.6 9.8 13.0 14.2</td>
<td>18.7</td>
<td>19.5</td>
<td>24.7</td>
<td>35.8</td>
</tr>
<tr>
<td>2598.7</td>
<td>56.4 24.7 14.8 5.9 7.1 9.9 15.9</td>
<td>18.5</td>
<td>18.6</td>
<td>22.4</td>
<td>35.1</td>
</tr>
<tr>
<td>2787.9</td>
<td>57.3 26.3 17.5 5.3 5.9 5.6 11.5</td>
<td>11.6</td>
<td>16.1</td>
<td>26.4</td>
<td>36.0</td>
</tr>
<tr>
<td>2987.4</td>
<td>63.6 29.1 17.1 5.9 3.2 5.2 11.1</td>
<td>9.6 13.9</td>
<td>19.5</td>
<td>34.2</td>
<td>40.4</td>
</tr>
<tr>
<td>3189.0</td>
<td>72.8 34.3 21.1 8.7 4.1 6.3 7.2</td>
<td>14.3</td>
<td>15.5</td>
<td>16.8</td>
<td>26.8</td>
</tr>
<tr>
<td>3387.8</td>
<td>86.2 44.9 29.7 15.0 7.2 6.5 11.0</td>
<td>12.9</td>
<td>10.8</td>
<td>15.0</td>
<td>24.9</td>
</tr>
<tr>
<td>3593.0</td>
<td>92.2 49.6 35.1 13.7 7.5 5.4 9.9</td>
<td>10.7</td>
<td>11.8</td>
<td>12.7</td>
<td>22.9</td>
</tr>
<tr>
<td>3783.7</td>
<td>95.4 47.4 27.8 13.0 5.3 7.8 7.7</td>
<td>9.9 12.1</td>
<td>13.1</td>
<td>22.6</td>
<td>30.0</td>
</tr>
<tr>
<td>3904.4</td>
<td>93.2 47.9 27.3 12.1 6.6 6.9 7.2</td>
<td>6.2 12.7</td>
<td>15.7</td>
<td>27.8</td>
<td>33.2</td>
</tr>
</tbody>
</table>
FIGURE 1

Graphs of $\bar{D} f_w(D)$ where $y = D/\bar{D}$ (equation 2.5) and $f_y(y)$ (equation 2.16).
Graphs a to e show $\Omega(x)$ derived with $\nu = 4, 6, 8, 10, 12$ respectively.

Graph f shows $\Omega(x)$ as derived by Reichel and Wilkins.
REFERENCES

Archbold J.W. (1961)-
Algebra
Pitman and Sons, Ltd. London.

Blatt J.M. and Weisskopf V.F. (1952)-
Theoretical Nuclear Physics
Wiley and Sons, New York.

Breit G. (1940)-
Phys. Rev. 58, 1068.

Cox D.R. (1962)-
Renewal Theory
Menthuen and Co.Ltd., London.

Cramér H. (1945)-
Mathematical Methods of Statistics

Dresner L. (1959)-
O.R.N.L. 2659.

Dyson F.J. (1962)-
J.Math.Phys. 3 140-175.

Egelstaff (1959)-

Eisenbud L. and Wigner E.P. (1958)-
Nuclear Structure

Fisher R.A. (1941)-
Statistical Methods for Research Workers
Oliver and Boyd, Edinburgh.


Gaudin M. (1961)-
Nucl.Phys. 25, 447.

Goertzel G. (1956)-
Goertzel G. and Greuling E. (1960)-

Gurevich I.I. and Pevsner M.I. (1957)-
Nucl. Phys. 2, 575.

Harvey J.A. (1955)-
Phys. Rev. 98, 1162.

Harvey J.A. and Hughes D.J. (1958)-
Phys. Rev. 109, 471.

Hsu, P.L. (1939)-

Hughes D.J. and Harvey J.A. (1955)-

Hwang, R.N. (1963)-
Proc. Conf. on Breeding, Economics and Safety in Large Fast Power Reactors
ANL – 6792 p727.

Hwang, R.N. (1965)-
Nucl. Sci. Eng. 21, 523.

Kendal M.G. and Stuart A. (1947)-
The Advanced Theory of Statistics
Griffin and Co. Ltd., London.

Lane A.M. and Lynn J.E. (1957)-

Lane A.M. and Thomas R.G. (1958)-
Rev. Mod. Phys. 30, 257.

Mehta, M.L. (1960)-
Nucl. Phys. 18, 395.

Mehta M.L. and Gaudin M. (1960)-
Nucl. Phys. 18, 420.

Musgrove A.R. (1968)-
(Unpublished).

Nicholson R.B. (1960)-

Porter C.E. (1963)-
Porter C.E. (1965)—
Statistical Theories of Spectra: Fluctuations

Porter C.E. and Rosenzweig N. (1960)—
Tiedeakat Toimituksia AVI, No. 44.

Porter C.E. and Thomas R.G. (1956)—

Preston M.A. (1962)—
Physics of the Nucleus
Addison-Wesley, Reading, Massachusetts.

Reichel A. (1962)—
A.E.E.W.-R 118, H.M.S.O.

Reichel A. and Wilkins C.A. (1964)—
J.Appl.Prob. 1, 335.

Rosenzweig N. and Porter C.E. (1960)—
Phys.Rev. 120, 1698.

Schiff L.I. (1955)—
Quantum Mechanics

Schmidt (1966)—
K.F.K. 120.

Smith (1954)—

Weinberg A.M. (1961)—
Amer. Maths. Soc.

Wigner E.P. (1957a)—
Gatlinberg Conference on Neutron Physics by Time of Flight.
O.R.N.L. 2039, 57.

Wigner E.P. (1957b)—

Wigner E.P. and Eisenbud L. (1947)—
Phys.Rev. 72, 29.
Wilkins C.A., Chiarella C., Gilks A.J. and Reichel A. (1968)

Wilkins C.A. and Keane A. (1966)-
A.A.E.C./E 155.

Wilks S.S. (1943)-
Mathematical Statistics
Wilkins C.A., Chiarella C., Gilks A.J. and Reichel A.  
"Nuclear Resonance Spacings and Synthetic Kernels," 