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The K-theory of twisted multipullback quantum odd spheres and complex projective spaces

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Abstract

We find multipullback quantum odd-dimensional spheres equipped with natural U.1/-actions that yield the multipullback quantum complex projective spaces constructed from Toeplitz cubes as noncommutative quotients. We prove that the noncommutative line bundles associated to multipullback quantum odd spheres are pairwise stably non-isomorphic, and that the K-groups of multipullback quantum complex projective spaces and odd spheres coincide with their classical counterparts.We show that theseK-groups remain the same for more general twisted versions of our quantum odd spheres and complex projective spaces.

THE K-THEORY OF TWISTED MULTIPULLBACK QUANTUM ODD SPHERES AND COMPLEX PROJECTIVE SPACES

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Abstract. We find multipullback quantum odd-dimensional spheres equipped with natural $U(1)$-actions that yield the multipullback quantum complex projective spaces constructed from Toeplitz cubes as noncommutative quotients. We prove that the noncommutative line bundles associated to multipullback quantum odd spheres are pairwise stably *non*-isomorphic, and that the K-groups of multipullback quantum complex projective spaces and odd spheres coincide with their classical counterparts. We show that these K-groups remain the same for more general twisted versions of our quantum odd spheres and complex projective spaces.

Contents

Introduction

Complex projective space is a fundamental object in topology and algebraic geometry. It also makes its mark in lattice theory as its affine covering provides a natural model of a free distributive lattice [3]. In [15], a noncommutative deformation of complex projective spaces preserving this lattice-theoretic property was introduced and studied. The new quantum complex projective space C^*-algebras $C(\mathbb{P}^N(\mathcal{T}))$ were defined as multipullback C^*-algebras [25] rather than as fixed-point subalgebras [31, 24].

In this paper, we solve the problem of constructing multipullback quantum-odd-sphere C^*-algebras $C(S_H^{2N+1})$ from which the C^*-algebras $C(\mathbb{P}^N(\mathcal{T}))$ emerge as fixed-point subalgebras for a natural circle action. Then we develop and utilise a presentation of $C(S_H^{2N+1})$ as the universal C^*-algebra generated by $N+1$ commuting isometries satisfying a sphere equation (see Theorem 2.3). We exploit this presentation to show that the K-goups of $C(S_H^{2N+1})$ and of $C(\mathbb{P}^N(\mathcal{T}))$ coincide with their classical counterparts. [1]

[1]Keywords: free action on C*-algebras, associated noncommutative line bundle, multipullback and higher-rank graph C*-algebras, noncommutative deformation. AMS codes: 46L80, 46L85.

The constructions and results described above admit the following generalisation. For each antisymmetric matrix $\theta \in M_{N+1}(\mathbb{R})$, we construct θ-twisted versions $C(S^{2N+1}_{H,\theta})$ and $C(\mathbb{P}^N_\theta(\mathcal{T}))$ of our quantum odd sphere C^*-algebra and our quantum complex projective space C^*-algebra. The twisted sphere algebra is universal for $N+1$ isometries commuting up to phases specified by the matrix θ and satisfying a sphere equation. The twisted projective space C^*-algebra is the fixed-point subalgebra of $C(S^{2N+1}_{H,\theta})$ for a natural diagonal $U(1)$-action. We prove that K-theory of these algebras is independent of θ.

To state our main result, we recall some background. Given a C^*-algebra A, we write $C(U(1), A)$ for the C^*-algebra of norm-continuous functions from $U(1)$ to A. Each action α of $U(1)$ on A determines a homomorphism

$$\delta : A \longrightarrow C(U(1), A) \quad \text{by} \quad \delta(a)(\lambda) := \alpha_\lambda(a), \qquad a \in A,\ \lambda \in U(1). \tag{0.1}$$

We say that α is *free* if and only if

$$\overline{\operatorname{span}}\{a\,\delta(b) \mid a, b \in A\} = C(U(1), A),$$

where $\overline{\operatorname{span}}$ stands for the closed linear span. The general definition of freeness of a quantum-group action on a C^*-algebra is due to Ellwood [11], and the special case of any compact Hausdorff topological group acting on a unital C^*-algebra looks exactly as above.

Given $\alpha : U(1) \curvearrowright A$ as above, for each character $m \in \widehat{U(1)} \cong \mathbb{Z}$, the *spectral subspace* A_m is

$$A_m := \{a \in A \mid \alpha_\lambda(a) = \lambda^m a \text{ for all } \lambda \in U(1)\}.$$

The subspace A_0 is the fixed-point subalgebra A^α (also denoted $A^{U(1)}$) of A, and since $A_m A_n \subseteq A_{m+n}$ for all m, n, the spectral subspaces are always A^α-bimodules. When α is free, they are finitely generated projective left A^α-modules [10, Theorem 1.2] encoding associated noncommutative line bundles.

By constructing a strong connection [13], we prove that the action of $U(1)$ on $C(S^{2N+1}_H)$ is free, so that its spectral subspaces $C(S^{2N+1}_H)_n$ are finitely generated projective left $C(\mathbb{P}^N(\mathcal{T}))$-modules. To prove that the characters of $U(1)$ defining these noncommutative line bundles are K_0-invariants, we derive a general method of pulling back noncommutative associated line bundles over equivariant maps (Theorem 5.1).

The key results of this paper can be summarized as follows:

Theorem 0.1. *Fix an integer $N \geq 1$ and a matrix $\theta \in M_{N+1}(\mathbb{R})$ that is antisymmetric in the sense that $\theta_{ij} = -\theta_{ji}$ for all i, j. Then:*

(1) $K_0(C(S^{2N+1}_{H,\theta})) \cong \mathbb{Z} \cong K_1(C((S^{2N+1}_{H,\theta}))$.
(2) $K_0(C(\mathbb{P}^N_\theta(\mathcal{T}))) = \mathbb{Z}^{N+1}$ *and* $K_1(C(\mathbb{P}^N_\theta(\mathcal{T}))) = 0$.
(3) *The spectral subspaces $C(S^{2N+1}_H)_m$, regarded as left $C(\mathbb{P}^N(\mathcal{T}))$-modules, are pairwise stably nonisomorphic. In particular, the module $C(S^{2N+1}_H)_{-1}$ of sections of the tautological line bundle is not stably free.*

Our multipullback approach to quantum odd spheres is based on the Heegaard-type splitting of a $(2N+1)$-dimensional sphere into N-dimensional solid tori. Each odd-dimensional sphere decomposes into a union of solid tori, along the lines of the Heegaard splitting of the 3-sphere [21]. Under this decomposition, the embedding of each component torus in the sphere is equivariant for the diagonal $U(1)$-action. Taking quotients by the $U(1)$-actions yields a covering of the complex projective space by quotients of solid tori, which is a closed restriction of the usual affine covering.

To obtain the untwisted ($\theta = 0$) sphere algebras $C(S_H^{2N+1})$, we study a noncommutative deformation of this decomposition, using the point of view from [22] that the Toeplitz algebra $\mathcal{T}$ can be regarded as the C^*-algebra of a noncommutative disc. In [5], the authors constructed a decomposition of a 3-dimensional quantum sphere along these lines by taking a pullback of two copies of the tensor product of the circle algebra and the Toeplitz algebra. The index pairing of noncommutative line bundles over the resulting pullback quantum complex projective line (mirror quantum sphere) was computed in [18]. Subsequently, in his Ph.D. thesis, Jan Rudnik extended the construction in [5] to five dimensions using multipullback C^*-algebras. One of his main results was establishing the stable nontriviality of the dual tautological line bundle over the multipullback complex quantum projective plane [19, Theorem 2.4]. In this paper, we carry this idea further to all odd integers bigger than one. Very recently, Albert Jeu-Liang Sheu showed in [27] that, for all dimensions, the multipullback quantum-complex-projective-space C^*-algebras can be realized as groupoid C^*-algebras.

The paper is organized as follows. In Section 1, we recall definitions and claims crucial for the formulation and proofs of new results. In Section 2, we construct our multipullback quantum-odd-sphere C^*-algebras and their twisted analogues. With the help of the theory of twisted higher-rank graph C^*-algebras [28], we establish that the twisted multipullback quantum-odd-sphere C^*-algebras can be presented in terms of a universal property (see Theorem 2.3). In Section 3, we construct quantum complex projective space C^*-algebras and their twisted analogues as fixed-point algebras for $U(1)$-actions on the corresponding sphere algebras. We identify the untwisted quantum-projective-space algebras obtained in this way with the ones constructed in [15] as multipullbacks. In Section 4, we prove parts (1) and (2) of Theorem 0.1. In Section 5, we use the Chern-Galois theory of [4] to prove Theorem 5.1, which then we use to show Theorem 0.1(3).

1. Background

1.1. **Multipushouts, multipullbacks and the cocycle condition.** In what follows, we will construct algebras of functions on quantum spaces as multipullbacks of C^*-algebras. To make sure that this construction corresponds via duality to the presentation of a quantum space as a union of closed subspaces (see [20]), we assume the cocycle condition (see Definition 1.1). First we need some auxiliary definitions.

Let $(\pi^i_j : A_i \to A_{ij})_{i,j\in J, i\neq j}$ be a finite family of surjective C^*-algebra homomorphisms, with $A_{ij} = A_{ji}$ for $i \neq j$. For all distinct $i,j,k \in J$, we define $A^i_{jk} := A_i/(\ker \pi^i_j + \ker \pi^i_k)$ and denote by $[\cdot]^i_{jk} : A_i \to A^i_{jk}$ the canonical surjections. For distinct $i,j,k \in J$, define

$$\pi^{ij}_k : A^i_{jk} \longrightarrow A_{ij}/\pi^i_j(\ker \pi^i_k), \quad \text{by} \quad [b_i]^i_{jk} \longmapsto \pi^i_j(b_i) + \pi^i_j(\ker \pi^i_k).$$

These π^{ij}_k are isomorphisms when the π^i_j are all surjective, as assumed herein.

Definition 1.1 (Proposition 9 in [5])**.** We say that a finite family $(\pi^i_j : A_i \to A_{ij})_{i,j\in J, i\neq j}$ of surjective C^*-homomorphisms satisfies the *cocycle condition* if and only if, for all distinct $i,j,k \in J$,

(1) $\pi^i_j(\ker \pi^i_k) = \pi^j_i(\ker \pi^j_k)$, and
(2) the isomorphisms $\phi^{ij}_k := (\pi^{ij}_k)^{-1} \circ \pi^{ji}_k : A^j_{ik} \to A^i_{jk}$ satisfy $\phi^{ik}_j = \phi^{ij}_k \circ \phi^{jk}_i$.

Theorem 1 of [20] implies that a finite family $(\pi^i_j : A_i \to A_{ij})_{i,j\in J, i\neq j}$ of C^*-algebra surjections satisfies the cocycle condition if and only if, for all $K \subsetneq J$, all $k \in J \setminus K$, and all $(b_i)_{i\in K} \in \bigoplus_{i\in K} A_i$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$ for all distinct $i,j \in K$, there exists $b_k \in A_k$ such that also $\pi^i_k(b_i) = \pi^k_i(b_k)$ for all $i \in K$. This corresponds in the classical setting to the idea that all partial pushouts of a collection of topological spaces embed in the total pushout.

1.2. **Heegaard-type splittings of odd spheres.** We recall the Heegaard-type splittings of odd-dimensional spheres into solid tori. We write

$$\mathbb{T} := \{c \in \mathbb{C} \mid |c| = 1\}$$

for the unit circle, $D := \{c \in \mathbb{C} \mid |c| \leq 1\}$ for the unit disc, and

$$S^{2N+1} := \{(z_i)_i \in \mathbb{C}^{N+1} \mid \sum_{i=0}^{N} |z_i|^2 := 1\}$$

for the unit $(2N+1)$-dimensional sphere. For $0 \leq i \leq N$, let

$$V_i := \big\{(z_0, \ldots, z_N) \in S^{2N+1} \mid |z_i| = \max\{|z_0|, \ldots, |z_N|\}\big\}.$$

Also, let $z := (z_0, \ldots, z_N)$ and $d := (d_0, \ldots, d_N)$. Then $\phi_i(z) := |z_i|^{-1} z$ determines a homeomorphism $\phi_i : V_i \to D^i \times \mathbb{T} \times D^{N-i} \subseteq \mathbb{C}^{N+1}$, with inverse given by $\phi_i^{-1}(d) = (1 + \sum_{j\neq i} |d_j|^2)^{-\frac{1}{2}}\, d$.

These homeomorphisms allow us to present S^{2N+1} as a multipushout of closed solid tori. Indeed, for each i, let $X_i := D^i \times \mathbb{T} \times D^{N-i}$, and for $i < j$, let

$$X_{i,j} := D^i \times \mathbb{T} \times D^{j-i-1} \times \mathbb{T} \times D^{N-j} = X_i \cap X_j.$$

Then S^{2N+1} is the multipushout of the solid tori $X_0, \dots X_N$ given by the diagrams (1.1).

$$
\begin{tikzcd}
 & & S^{2N+1} & & \\
X_i \arrow[urr, dashed] & V_i \arrow[l, "\phi_i"'] \arrow[ur, hook] & & V_j \arrow[r, "\phi_j"'] \arrow[ul, hook'] & X_j \arrow[ull, dashed] \\
X_{i,j} \arrow[u, hook] & & V_i \cap V_j \arrow[ll, "\phi_{ij}"'] \arrow[rr, "\phi_{ji}"] \arrow[ul, hook'] \arrow[ur, hook] & & X_{i,j} \arrow[u, hook']
\end{tikzcd}
\tag{1.1}
$$

So if $\sim$ is the equivalence relation on the disjoint union $\coprod_i X_i$ generated by $\phi_i(d) \sim \phi_j(d)$ for all $d \in V_i \cap V_j$ and all $i < j$, then $S^{2N+1} \cong \big(\coprod_i X_i\big)/{\sim}$. (Note that $\phi_{ji} \circ \phi_{ij}^{-1} = \mathrm{id}_{X_{i,j}}$.)

To motivate our definition of Heegaard quantum spheres later on, we dualize this multipushout picture of S^{2N+1} to obtain a multipullback presentation of $C(S^{2N+1})$. Let $\mathrm{res} : C(D) \to C(\mathbb{T})$ be the restriction map. For $i < j$, we write

$$
\begin{aligned}
\pi_j^i : C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-i} \\
\longrightarrow C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes j-i-1} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-j}
\end{aligned}
$$

for the surjection $\mathrm{id}^{\otimes j} \otimes \mathrm{res} \otimes \mathrm{id}^{\otimes N-j}$. Then $C(S^{2N+1})$ is naturally isomorphic to

$$
\big\{(f_0, \dots, f_N) \in \textstyle\bigoplus_{i=0}^N C(D)^{\otimes i} \otimes C(\mathbb{T}) \otimes C(D)^{\otimes N-i} \,|\, \pi_j^i(f_j) = \pi_i^j(f_i) \ \text{ for all } i < j\big\}.
$$

1.3. **Gauging diagonal actions and coactions.** Throughout this paper, we denote a right action of a group G on a space X by juxtaposition, that is $(x, g) \mapsto xg$. The general idea for converting between diagonal and rightmost actions of a group G on a topological space X is as follows. We regard $X \times G$ as a right G-space in two different ways, which we distinguish notationally as follows.

- We write $(X \times G)^R$ for the product $X \times G$ with G-action $(x, g) \cdot h := (x, gh)$.
- We write $X \times G$ for the same space with diagonal G-action $(x, g)h = (xh, gh)$.

There is a G-equivariant homeomorphism $\kappa : (X \times G)^R \to X \times G$ determined by $\kappa(x, g) := (xg, g)$, with inverse given by $\kappa^{-1}(x, g) = (xg^{-1}, g)$. In general, given any cartesian product of G-spaces, we will regard it as a G-space with the diagonal action, except for those of the form $(X \times G)^R$ just described.

In what follows, the tensor product means completed tensor product, and we use the Heynemann-Sweedler notation (with the summation sign suppressed) for this completed product. Since all C^*-algebras that we tensor are nuclear, this completion is unique. We often identify the unit circle $\mathbb{T}$ with the unitary group $U(1)$, and use the quantum group structure on $C(U(1))$. Even though we only use the classical compact Hausdorff group $U(1)$, we are forced to use the quantum-group language of coactions, etc., to write explicit formulas, and carry out computations.

Let G be a locally compact group, and let $H := C(G)$. Then $S\colon H \to H$, given by $S(h)(g) := h(g^{-1})$, is the antipode map, $\varepsilon(h) := h(e)$ defines the counit (e is the neutral element of G), and

$$\Delta\colon H \longrightarrow H \otimes H \cong C(G \times G),$$
$$\Delta(h)(g_1, g_2) := h(g_1 g_2) =: (h_{(1)} \otimes h_{(2)})(g_1, g_2) = h_{(1)}(g_1) h_{(2)}(g_2),$$

is a coproduct. If $\alpha : G \to \mathrm{Aut}(A)$ is a G-action on a unital C^*-algebra A, then there is a coaction $\delta\colon A \to A \otimes H \cong C(G, A)$ given by

$$\delta(a)(g) := \alpha_g(a) =: (a_{(0)} \otimes a_{(1)})(g) = a_{(0)} a_{(1)}(g).$$

Consider $A \otimes H$ as a C^*-algebra with the diagonal coaction

$$p \otimes h \longmapsto p_{(0)} \otimes h_{(1)} \otimes p_{(1)} h_{(2)},$$

and denote by $(A \otimes H)^R$ the same C^*-algebra with the coaction on the rightmost factor: $p \otimes h \mapsto p \otimes h_{(1)} \otimes h_{(2)}$. Then the following map is a G-equivariant (i.e., intertwining the coactions) isomorphism of C^*-algebras:

$$\widehat{\kappa} : (A \otimes H) \longrightarrow (A \otimes H)^R, \quad a \otimes h \longmapsto a_{(0)} \otimes a_{(1)} h. \tag{1.2}$$

Its inverse is explicitly given by

$$\widehat{\kappa}^{-1} : (A \otimes H)^R \longrightarrow (A \otimes H), \quad a \otimes h \longmapsto a_{(0)} \otimes S(a_{(1)}) h. \tag{1.3}$$

1.4. **Affine closed coverings of complex projective spaces.** The odd sphere S^{2N+1} is a $U(1)$-principal bundle. The diagonal action of $U(1)$ on S^{2N+1} is given by

$$(z_0, \ldots, z_N)\lambda := (z_0 \lambda, \ldots, z_N \lambda).$$

Since $\mathbb{T} \subseteq D$ is rotation-invariant, this action restricts to a $U(1)$-action on each $D^1 \times \mathbb{T} \times D^{N-1}$, so the multipushout given by (1.1) is $U(1)$-equivariant.

To obtain a multipushout presentation of $\mathbb{P}^N(\mathbb{C}) = S^{2N+1}/U(1)$, we need to gauge the diagonal actions to actions on the rightmost components. This will yield an alternative multipushout presentation of S^{2N+1}. Using the notation of Section 1.3, we write $\kappa : (D^N \times U(1))^R \to D^N \times U(1)$ for the gauging homeomorphism. Identify $U(1)$ with $\mathbb{T}$, and write $F_{i,N} : D^N \times U(1) \to D^i \times \mathbb{T} \times D^{N-i}$ for the map

$$F_{i,N}(d_0, \ldots, d_{i-1}, d_i, d_{i+1} \ldots, d_{N-1}, d_N) := (d_0, \ldots, d_{i-1}, d_N, d_{i+1} \ldots, d_{N-1}, d_i).$$

We obtain a $U(1)$-equivariant homeomorphism

$$h_i := F_{i,N} \circ \kappa : (D^N \times U(1))^R \longrightarrow D^i \times \mathbb{T} \times D^{N-i}.$$

Let $X_i^R := (D^N \times U(1))^R$ for all i. For $i < j < N$, let

$$X_{i,j}^R := \left(D^i \times \mathbb{T} \times D^{N-i-1} \times U(1)\right)^R, \quad X_{j,i}^R := \left(D^{j-1} \times \mathbb{T} \times D^{N-j} \times U(1)\right)^R,$$
$$\text{and } X_{i,j} := D^i \times \mathbb{T} \times D^{j-i-1} \times \mathbb{T} \times D^{N-j} =: X_{j,i}.$$

For $i \neq j$, we define $h_{ij} := h_i\big|_{X_{j,i}^R} : X_{j,i}^R \to X_{i,j} = X_{j,i}$.

We use the h_i and h_{ij} to transform the multipushout structure of S^{2N+1} described by (1.1). Explicitly, for $0 \le i < j \le N$, we obtain the commuting diagram (1.4).

$$
\begin{tikzcd}
X_i^R \arrow[r, "h_i"] & X_i & & X_j & X_j^R \arrow[l, "h_j"'] \\
X_{j,i}^R \arrow[u, hook] \arrow[drr, "h_{ij}"] & & & & X_{i,j}^R \arrow[u, hook] \arrow[dll, "h_{ji}"'] \\
& & X_{i,j} \arrow[uul, hook] \arrow[uur, hook] & &
\end{tikzcd}
\tag{1.4}
$$

For $i < j$, we define $\chi_{ij} := h_{ji}^{-1} \circ h_{ij} : X_{j,i}^R \to X_{i,j}^R$. (Note that, unlike in the previous multipushout presentation of S^{2N+1}, these maps are not identities.) With this notation, S^{2N+1} is homeomorphic to the quotient of the disjoint union

$$\coprod_{0 \le i \le N} (D^N \times U(1))^R = \coprod_{0 \le i \le N} X_i^R$$

by the smallest equivalence relation such that $d \sim \chi_{ij}(d)$ for all $d \in X_{j,i}^R$. The equivalence relation $\sim$ respects the $U(1)$-actions, so that we obtain a multipushout presentation of $S^{2N+1}/U(1) \cong \mathbb{P}^N(\mathbb{C})$ by everywhere restricting $U(1)$ to a point. This multipushout presentation of the complex projective space agrees with the multipushout presentation used in [15, Section 1.2] to obtain the multipullback noncommutative deformation of $\mathbb{P}^N(\mathbb{C})$.

2. Twisted multipullback quantum odd spheres

2.1. **Twisted quantum even balls.** Recall that we regard the Toeplitz algebra $\mathcal{T}$ as the quantum-disc C^*-algebra [22]. Let s be the generating isometry in $\mathcal{T}$ [6, 7] and u the generating unitary in $C(\mathbb{T})$. Let $\sigma : \mathcal{T} \to C(\mathbb{T})$, $s \mapsto u$, denote the symbol map. We use the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(\mathbb{T}) \longrightarrow 0$$

to regard the circle $\mathbb{T}$ as the boundary of the quantum disc, or two-dimensional quantum ball. Thus the one-dimensional quantum sphere then corresponds to the quotient $\mathcal{T}/\mathcal{K}$. From this perspective, $\mathcal{T}^{\otimes N}$ can be regarded as the algebra of a Cartesian product of N two-dimensional balls, and therefore as a copy of a $2N$-dimensional (non-round) quantum ball. The quotient $\mathcal{T}^{\otimes N+1}/\mathcal{K}^{\otimes N+1}$ is then viewed as the algebra of the boundary of the quantum ball, that is, a quantum sphere of dimension $2N+1$. In the same spirit, $\mathcal{T}^{\otimes N} \otimes C(\mathbb{T})$ is regarded as the algebra of the Cartesian product of a $2N$-ball and a circle, which is to say a $(2N+1)$-dimensional noncommutative solid torus.

By analogy with the Heegaard splitting of S^{2N+1} in the preceding section, we define the algebra $C(S_H^{2N+1})$ of continuous functions on the Heegaard quantum sphere as a multipullback of the C^*-algebras $\mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i}$ with respect to the maps

$$\pi^i_j \colon \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i} \longrightarrow \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes j-i-1} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-j}, \quad i < j,$$

given by $\pi^i_j := \mathrm{id}_{\mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{j-i-1}} \otimes \sigma \otimes \mathrm{id}_{\mathcal{T}^{\otimes N-j}}$.

In Section 4.2, we will realize $C(S^{2N+1}_H)$ as the special case where $\theta = 0$ of a multipullback of twisted tensor products of the same sort. We begin by defining the twisted Toeplitz algebras $\mathcal{T}^{N+1}_\theta$, which we view as twisted-quantum-ball C^*-algebras.

Definition 2.1. Fix $N > 0$, and suppose that $\theta = (\theta_{ij})^N_{i,j=0} \in M_{N+1}(\mathbb{R})$ is antisymmetric in the sense that $\theta_{ij} = -\theta_{ji}$. We define the twisted Toeplitz algebra $\mathcal{T}^{N+1}_\theta$ to be the universal C^*-algebra generated by isometries $\{w^\theta_0, \dots, w^\theta_N\}$ such that

$$w^\theta_j w^\theta_k = e^{2\pi i \theta_{jk}} w^\theta_k w^\theta_j \quad \text{and} \quad w^{\theta *}_j w^\theta_k = e^{-2\pi i \theta_{jk}} w^\theta_k w^{\theta *}_j \quad \text{for all } j \neq k.$$

With this in hand, we are ready to present our definition of the twisted Heegaard quantum sphere $S^{2N+1}_{H,\theta}$, which we view as the boundary of a twisted quantum ball. Thus we generalize the 3-dimensional case $S^3_{H,\theta}$ introduced and analyzed in [2].

Definition 2.2. For $0 \le i \le N$, let I^θ_i denote the ideal of $\mathcal{T}^{N+1}_\theta$ generated by $1 - w^\theta_i w^{\theta *}_i$, and for $i \neq j$, let $I^\theta_{ij} := I^\theta_i + I^\theta_j$. Let $B^\theta_i := \mathcal{T}^{N+1}_\theta / I^\theta_i$ and $B^\theta_{ij} := \mathcal{T}^{N+1}_\theta / I^\theta_{ij}$. Also, let

$$\sigma_i : \mathcal{T}^{N+1}_\theta \longrightarrow B^\theta_i \quad \text{and} \quad \pi^i_j : B^\theta_i \longrightarrow B^\theta_{ij} \tag{2.1}$$

be the natural quotient maps. We define the *twisted Heegaard quantum sphere* C^**-algebra* as the multipullback of the algebras B^θ_i over the homomorphisms π^i_j, that is

$$C(S^{2N+1}_{H,\theta}) := \Big\{ (b_0, \dots, b_N) \in \textstyle\bigoplus_{i=0}^N B^\theta_i \ \Big|\ \pi^i_j(b_i) = \pi^j_i(b_j) \text{ for all } 0 \le i < j \le N \Big\}.$$

To ease notation we define $w^{\theta;i}_k := \sigma_i(w^\theta_k)$ and $w^{\theta;ij}_k := w^\theta_k + I^\theta_i + I^\theta_j$ for all k and distinct i, j. We define $\mathbf{s}_i \in C(S^{2N+1}_{H,\theta})$ by

$$\mathbf{s}_i := \big(w^{\theta;0}_i, \dots, w^{\theta;N}_i\big).$$

For $i, j \in \{0, \dots, N\}$, we have

$$\mathbf{s}_i \mathbf{s}_j = e^{2\pi i \theta_{ij}} \mathbf{s}_j \mathbf{s}_i, \quad \mathbf{s}_i \mathbf{s}^*_j = e^{-2\pi i \theta_{ij}} \mathbf{s}^*_j \mathbf{s}_i, \quad \text{when } i \neq j,$$
$$\mathbf{s}^*_i \mathbf{s}_i = 1, \quad \text{and}$$
$$\prod_{k=0}^N (1 - \mathbf{s}_k \mathbf{s}^*_k) = 0. \tag{2.2}$$

The universal property of $\mathcal{T}^{N+1}_\theta$ yields a $U(1)^{N+1}$-action satisfying $(\lambda_0, \dots, \lambda_N) \cdot w^\theta_j = \lambda_j w^\theta_j$. We call this the *gauge action* on $\mathcal{T}^{N+1}_\theta$. This action descends to each

B_i and each B_{ij}, and hence induces a $U(1)^{N+1}$-action on $C(S^{2N+1}_{H,\theta})$, also called the gauge action. Restricting to the diagonal in $U(1)^{N+1}$ gives a $U(1)$-action α on $C(S^{2N+1}_{H,\theta})$ such that

$$\alpha_\lambda(b_0, \dots, b_N) = (\lambda \cdot b_0, \dots, \lambda \cdot b_N). \tag{2.3}$$

2.2. **A universal presentation.** We prove, using Whitehead's twisted relative Cuntz–Krieger algebras of higher-rank graphs [33] (see also [28]), that the twisted Heegaard quantum sphere C^*-algebra of Definition 2.2 enjoys a universal property.

Theorem 2.3. *Consider an integer $N \geq 1$ and a antisymmetric matrix $\theta \in M_{N+1}(\mathbb{R})$. Let $A_\theta(N+1)$ be the universal C^*-algebra generated by isometries $s_0, \dots, s_N$ satisfying*

$$s_i s_j = e^{2\pi i \theta_{ij}} s_j s_i \quad \text{and} \quad s_i s_j^* = e^{-2\pi i \theta_{ij}} s_j^* s_i, \tag{2.4}$$

and the sphere equation

$$\prod_{i=0}^{N} (1 - s_i s_i^*) = 0. \tag{2.5}$$

Then there is a $U(1)$-action on $A_\theta(N+1)$ such that $\lambda \cdot s_i = \lambda s_i$ for all i, and there is a $U(1)$-equivariant isomorphism $\phi_\theta : A_\theta(N+1) \to C(S^{2N+1}_{H,\theta})$ such that

$$\phi_\theta(s_i) = \mathbf{s}_i = \big(w_i^{\theta;0}, \dots, w_i^{\theta;N}\big) \quad \text{for all } i.$$

Furthermore, the maps $\pi^i_j : B_i \to B_{ij}$ satisfy the cocycle condition of Definition 1.1.

The existence of the $U(1)$-action on $A_\theta(N+1)$ and of the homomorphism ϕ_θ follows from the universal property of $A_\theta(N+1)$. We use the technology of twisted relative higher-rank graph C^*-algebras [28] to see that ϕ_θ is injective. For surjectivity, and to see that the cocycle condition is satisfied, we will need the following technical lemma.

Lemma 2.4. *Let A be a C^*-algebra and suppose that $I_0, \dots, I_n$ are ideals of A. Suppose that $a_0, \dots, a_n \in A$ satisfy $a_i + (I_i + I_j) = a_j + (I_i + I_j)$ for all i, j. Then there exists $a \in A$ such that $a + I_i = a_i + I_i$ for all i.*

Proof. We proceed by induction on n. The base case $n = 0$ is trivial. Suppose as an inductive hypothesis that there exists $a' \in A$ such that $a' + I_i = a_i + I_i$ for all $i < n$. Then $a' + (I_i + I_n) = a_n + (I_i + I_n)$ for all $i < n$, whence

$$a' - a_n \in \textstyle\bigcap_{i<n}(I_i + I_n). \tag{2.6}$$

Since the ideals of the C^*-algebra A form a distributive lattice with meet given by intersection and join given by sum, we have

$$\bigcap_{i<n}(I_i + I_n) = \Big(\bigcap_{i<n} I_i\Big) + \sum_{\emptyset \neq F \subseteq \{0,\dots,n-1\}} \Big(I_n \cap \bigcap_{i \notin F} I_i\Big) \subseteq \Big(\bigcap_{i<n} I_i\Big) + I_n.$$

Combining this with (2.6), we obtain $a' - a_n = b' - b_n$, where $b' \in \bigcap_{i=0}^{n-1} I_i$ and $b_n \in I_n$. Put $a := a' - b'$. Since $b' \in I_i$ for all $i \leq n-1$, we have $a + I_i = a' + I_i = a_i + I_i$ for $i \leq n-1$. Furthermore, $a = a' - b' = a_n - b_n$ and $b_n \in I_n$, so $a + I_n = a_n + I_n$ too. $\square$

To prove Theorem 2.3, we use twisted higher-rank graph C^*-algebras. The general theory of these objects requires significant background, but fortunately the only higher-rank graphs we need to consider are the following elementary examples.

Let Λ denote a copy of the monoid $\mathbb{N}^{N+1}$ under addition. This becomes an $(N+1)$-graph in the sense of [23, Definition 1.1] under the degree map $d : \Lambda \to \mathbb{N}^{N+1}$ given by the identity map on $\mathbb{N}^{N+1}$. We write $e_0, \ldots, e_N$ for the canonical generators of $\mathbb{N}^{N+1}$. Since we are viewing Λ as a category, we write $\mu\nu$ for the composition of elements μ, ν. This is really just $\mu + \nu$ when the two are regarded as elements of $\mathbb{N}^{N+1}$. The unique vertex of Λ is $0 \in \mathbb{N}^{N+1}$. For $\mu = (\mu_0, \ldots, \mu_N) \in \Lambda$, we write $|\mu| := \sum_{i=0}^{N} \mu_i$. A *cocycle* on Λ is a map $c : \Lambda \times \Lambda \to \mathbb{T}$ satisfying the cocycle identity $c(\mu, \nu)c(\lambda, \mu\nu) = c(\lambda, \mu)c(\lambda\mu, \nu)$ for all $\lambda, \mu, \nu \in \Lambda$. Since $\mathbb{N}^{N+1}$ is directed, every finite $F \subseteq \Lambda \setminus \{0\}$ is exhaustive as in [28, Section 2]. So given any collection $\mathcal{E}$ of finite subsets of $\Lambda \setminus \{0\}$, we can form the twisted relative Cuntz–Krieger algebra $C^*(\Lambda, c; \mathcal{E})$, which is generated by isometries $\{s^c_{\mathcal{E}}(\lambda) : \lambda \in \Lambda\}$ satisfying relations (TCK1)–(TCK4) and (CK) of [28, Section 3].

Lemma 2.5. *Let Λ denote $\mathbb{N}^{N+1}$ regarded as an $(N+1)$-graph as above. Fix a antisymmetric matrix $\theta \in M_{N+1}(\mathbb{R})$. There is a cocycle c on Λ given by*

$$c(\mu, \nu) := e^{\pi i (d(\mu)^T \theta d(\nu))}. \tag{2.7}$$

Let $\mathcal{E} := \{\{e_0, \ldots, e_N\}\}$. Then there is an isomorphism $A_\theta(N+1) \to C^*(\Lambda, c; \mathcal{E})$ that carries $w_i \in A_\theta(N+1)$ to $s^c_{\mathcal{E}}(e_i) \in C^*(\Lambda, c; \mathcal{E})$ for $0 \leq i \leq N$.

Proof. One checks that $A_\theta(N+1)$ and $C^*(\Lambda, c; \mathcal{E})$ have the same universal property. $\square$

Proof of Theorem 2.3. The relations (2.4) and (2.5) are invariant under multiplication of the s_i by any fixed $\lambda \in U(1)$. Thus the universal property of $A_\theta(N+1)$ yields the desired $U(1)$-action.

The universal property of $\mathcal{T}^{N+1}_\theta$ yields a homomorphism

$$\psi_\theta : \mathcal{T}^{N+1}_\theta \longrightarrow C(S^{2N+1}_{H,\theta}) \quad \text{given by} \quad \psi_\theta(a) = (\sigma_0(a), \sigma_1(a), \ldots, \sigma_N(a)).$$

Applying Lemma 2.4 to $A = \mathcal{T}^{N+1}_\theta$ and the ideals $I_i = \ker(\sigma_i)$ shows that

$$C(S^{2N+1}_{H,\theta}) = \{(\sigma_0(a), \sigma_1(a), \ldots, \sigma_N(a)) \mid a \in \mathcal{T}^{N+1}_\theta\},$$

so that ψ_θ is surjective. Since $\prod_{j=0}^{N}(1 - w_j w_j^*) \in \ker \sigma_i$ for each i, it belongs to $\ker \psi_\theta$, so ψ_θ descends to a surjective homomorphism $\phi_\theta : A_\theta(N+1) \to C(S^{2N+1}_{H,\theta})$ such that $\phi_\theta(s_i) = \mathbf{s}_i$ for all i.

By Lemma 2.5 it suffices to show that the homomorphism

$$\rho : C^*(\Lambda, c; \mathcal{E}) \longrightarrow C(S^{N+1}_{H,\theta})$$

satisfying $\rho(s^c_{\mathcal{E}}(e_i)) = \phi_\theta(s_i)$ is injective. For this, we aim to apply the gauge-invariant uniqueness theorem [28, Theorem 3.15] for $C^*(\Lambda, c; \mathcal{E})$.

The homomorphism ρ is equivariant for the gauge actions on $C(S^{2N+1}_{H,\theta})$ and $C^*(\Lambda, c; \mathcal{E})$. Since $s^c_{\mathcal{E}}(0)$ is the identity element of $C^*(\Lambda, c; \mathcal{E})$, we have

$$\rho(s^c_{\mathcal{E}}(0)) = (1, 1, \ldots, 1) \neq 0.$$

Hence, by [28, Theorem 3.15], it suffices to show that for each finite F in the complement of the satiation $\overline{\mathcal{E}}$ of $\mathcal{E}$ (see [28, page 837]),

$$\rho\Big(\prod_{\mu\in F}(s^c_{\mathcal{E}}(0) - s^c_{\mathcal{E}}(\mu)s^c_{\mathcal{E}}(\mu)^*)\Big) \neq 0.$$

The set

$$\mathcal{E}' := \{F \subset \Lambda \setminus \{0\} \mid \text{there exists } i > 0 \text{ such that } |p| > i \text{ implies } p \geq q \text{ for some } q \in F\}$$

satisfies (S1)–(S4) on page 87 of [28] and contains $\mathcal{E}$. An induction shows that any set containing $\mathcal{E}$ and satisfying (S1)–(S4) contains $\mathcal{E}'$. So $\mathcal{E}' = \overline{\mathcal{E}}$. So for a finite set $F \notin \overline{\mathcal{E}}$, there is a sequence (p^i) in Λ with $|p^i| \to \infty$ such that $p^i \ngeq q$ for all $q \in F$ and all $i \in \mathbb{N}$. By passing to a subsequence, we may assume that $p^i_j \to \infty$ for some $j \leq N$. Since $p^i \ngeq q$ for all $q \in F$ and all i, it follows that $q \in F$ implies $q_l > 0$ for some $l \neq j$. Hence there exists $l \neq j$ such that $q \geq e_l$, which forces

$$s^c_{\mathcal{E}}(q)s^c_{\mathcal{E}}(q)^* = s^c_{\mathcal{E}}(e_l)s^c_{\mathcal{E}}(q-e_l)s^c_{\mathcal{E}}(q-e_l)^*s^c_{E}e(e_l)^* \leq s^c_{\mathcal{E}}(e_l)s^c_{\mathcal{E}}(e_l)^*.$$

Thus

$$\rho\big(1 - s^c_{\mathcal{E}}(q)s^c_{\mathcal{E}}(q)^*\big) \geq \rho\big(1 - s^c_{\mathcal{E}}(e_l)s^c_{\mathcal{E}}(e_l)^*\big) = 1 - \mathbf{s}_l\mathbf{s}_l^*.$$

Applying this reasoning to each $q \in F$, we obtain

$$\rho\Big(\prod_{q\in F}(1 - s^c_{\mathcal{E}}(q)s^c_{\mathcal{E}}(q)^*)\Big) \geq \prod_{l\neq j}(1 - \mathbf{s}_l\mathbf{s}_l^*).$$

Since each $\mathbf{s}_l \in C(S^{2N+1}_{H,\theta}) \subseteq \bigoplus_{i=0}^N B_i$ (where $B_i = \mathcal{T}^{N+1}_\theta / I_i$), the jth coordinate of $\prod_{l\neq j}(1 - \mathbf{s}_l\mathbf{s}_l^*)$ is

$$\Big(\prod_{l\neq j}(1 - \mathbf{s}_l\mathbf{s}_l^*)\Big)_j = \sigma_j\Big(\prod_{l\neq j}(1 - w_l w_l^*)\Big). \tag{2.8}$$

So it suffices to show that the right-hand side of (2.8) is nonzero. Since $\sigma_j(\mathcal{T}^{N+1}_\theta)$ is universal for the same relations as the twisted relative Cuntz–Krieger algebra $C^*(\Lambda, c; \{e_j\})$, there is an isomorphism $\sigma_j(\mathcal{T}^{N+1}_\theta) \to C^*(\Lambda, c; \{e_j\})$ that carries

$\sigma_j(w_l)$ to $s^c_{\{e_j\}}(e_l)$ for each l. The satiation $\overline{\{e_j\}}$ of $\{e_j\}$ does not contain the set $\{e_l \mid l \neq j\}$, so [28, Proposition 3.9] implies that

$$\prod_{l\neq j}(1 - s^c_{\{e_j\}}(e_l)s^c_{\{e_j\}}(e_l)^*) \neq 0,$$

giving $\sigma_j\Big(\prod_{l\neq j}(1 - w_l w_l^*)\Big) \neq 0$ as required. This completes the proof that ϕ_θ is an isomorphism.

Since each $B_i = \mathcal{T}_\theta^{N+1}/I_i$ and $B_{ij} = \mathcal{T}_\theta^{N+1}/(I_i + I_j)$ by definition, the homomorphisms π^i_j are distributive in the sense of [20, Definition 2]. Lemma 2.4 shows in particular that given distinct i, j, k and elements $b_i \in B_i$ and $b_j \in B_j$ such that $\pi^i_j(b_i) = \pi^j_i(b_j)$, there exists $b_k \in B_k$ such that $\pi^k_i(b_k) = \pi^i_k(b_i)$ and $\pi^k_j(b_k) = \pi^j_k(b_j)$. Hence Theorem 1 of [20] implies that the π^i_j satisfy the cocycle condition of Definition 1.1. □

2.3. **Strong connections.** Since we focus on free $U(1)$-actions on unital C^*-algebras, we avoid the general coalgebraic formalism of strong connections of [4], and formulate the concept of a strong connection from [13] solely for $U(1)$-actions on unital C^*-algebras.

Let A be a unital C^*-algebra carrying a $U(1)$-action. For $m \in \mathbb{Z}$, recall that A_m denotes the spectral subspace $\{a \in A \mid \lambda \cdot a = \lambda^m a \text{ for all } \lambda \in U(1)\}$. We write $\mathbb{C}[u, u^*]$ for the *-algebra of Laurent polynomials. Let ℓ be a unital linear map

$$\ell : \mathbb{C}[u, u^*] \longrightarrow \big(\textstyle\bigoplus_{m\in\mathbb{Z}} A_m\big) \underset{\text{alg}}{\otimes} \big(\textstyle\bigoplus_{m\in\mathbb{Z}} A_m\big) \subseteq A \underset{\text{alg}}{\otimes} A,$$

where $\bigoplus_{m\in\mathbb{Z}} A_m$ denotes the algebraic direct sum of the spectral subspaces. We say that ℓ is a *strong connection* for the $U(1)$-action on A if, writing

$$m_A : A \otimes_{\text{alg}} A \longrightarrow A$$

for the multiplication map, we have

$$(m_A \circ \ell)(h) = h(1)1_A \quad \text{for all } h \in \mathbb{C}[u, u^*], \tag{2.9}$$

and

$$\ell(u^n) \in A_{-n} \otimes A_n \quad \text{for all } n \in \mathbb{Z}. \tag{2.10}$$

By [30] the existence of a strong connection is equivalent to strong grading:

$$A_m A_n = A_{m+n} \quad \text{for all } m, n \in \mathbb{Z}.$$

Moreover, by the main theorem of [1] combined with [4, Theorem 2.5(1)], the existence of a strong connection is equivalent to freeness.

2.3.1. *A strong connection on* $S^{2N+1}_{H,\theta}$. In what follows, we will also need the following family of $U(1)$-fixed elements of $C(S^{2N+1}_{H,\theta})$:

$$H_N = 1, \quad H_i = \prod_{j=i+1}^{N} (1 - \mathbf{s}_j \mathbf{s}_j^*), \quad i \in \{0, \dots, N-1\}.$$

Consider the linear map

$$\ell : \mathbb{C}[u, u^*] \longrightarrow \left(\bigoplus_{m\in\mathbb{Z}} C(S^{2N+1}_{H,\theta})_m\right) \underset{\text{alg}}{\otimes} \left(\bigoplus_{m\in\mathbb{Z}} C(S^{2N+1}_{H,\theta})_m\right)$$

defined inductively as follows:

$$\ell(1) := 1 \otimes 1, \qquad \ell(u^n) = \mathbf{s}_0^{*n} \otimes \mathbf{s}_0^n \text{ for } n > 0, \quad \text{and}$$

$$\ell(u^{n-1}) := \sum_{0\le k\le N} \Big((\mathbf{s}_k \otimes 1)\ell(u^n)(1 \otimes \mathbf{s}_k^* H_k) \Big) \text{ for } n \le 0. \tag{2.11}$$

Then ℓ is a strong connection for the $U(1)$-action on $C(S^{2N+1}_{H,\theta})$: Equation (2.10) for $n \ge 0$ is trivial, and for $n < 0$ follows from an elementary induction argument. Equation (2.9) for $n \ge 0$ is trivial because $\mathbf{s}_0$ is an isometry. To check it for $n < 0$, we first use the sphere equation (2.2) to see that $\sum_{k=0}^{N} \mathbf{s}_k \mathbf{s}_k^* H_k = 1$, and then employ a straightforward induction argument (see the proof of [17, Lemma 4.2]) using the recursive formula (2.11).

3. Twisted multipullback quantum complex projective spaces

Our twisted multipullback quantum odd sphere C^*-algebras (see Definition 2.2) yield a natural construction of a family of θ-twisted complex projective space C^*-algebras as fixed-point algebras. Using the $U(1)$-action α on $C(S^{2N+1}_{H,\theta})$ from equation (2.3), we define

$$C(\mathbb{P}^N_\theta(\mathcal{T})) := C(S^{2N+1}_{H,\theta})^\alpha.$$

To study $C(S^{2N+1}_{H,\theta})^\alpha$, we gauge the diagonal action α on $C(S^{2N+1}_{H,\theta})$ to an action on a single twisted component, where it is easy to determine the $U(1)$-invariant subalgebra. As in Section 2, restricting to the diagonal subgroup of $U(1)^{N+1}$ yields a diagonal action on $\mathcal{T}^{N+1}_\theta$ given by $\lambda \cdot w^\theta_j := (\lambda, \dots, \lambda) \cdot w^\theta_j = \lambda w^\theta_j$. We can also compose with the coordinate inclusions $U(1) \hookrightarrow U(1)^{N+1}$ to obtain actions $\cdot_i$ of $U(1)$ given by

$$\lambda \cdot_i w_j = \begin{cases} w_j & \text{if } i \neq j \\ \lambda w_i & \text{if } i = j. \end{cases}$$

Since that gauge action descends to the quotients by the I^θ_k and I^θ_{kj}, so do these $U(1)$-actions. We will consider B^θ_i and B^θ_{ij} to be endowed with the diagonal $U(1)$-action and we denote by $B^{\theta;R_k}_i$ and $B^{\theta;R_k}_{ij}$ the same C^*-algebras endowed with the $U(1)$-action on the k-th twisted component. Accordingly, we will write the generators of $B^{\theta;R_k}_i$ and $B^{\theta;R_k}_{ij}$ as $w^{\theta;i;R_k}_l$ and $w^{\theta;ij;R_k}_l$, respectively.

Lemma 3.1. *For any $(N+1)\times(N+1)$ antisymmetric real matrix θ and $0 \le i \le N$, define antisymmetric real matrices $\kappa_i(\theta)$ and $\kappa_i^{-1}(\theta)$ of the same size by*

$$\begin{aligned}\kappa_i(\theta)_{jk} &:= \theta_{ij} + \theta_{jk} + \theta_{ki}, \text{ if } j,k \neq i, \quad \kappa_i(\theta)_{ij} := \theta_{ij},\\ \kappa_i^{-1}(\theta)_{jk} &:= -\theta_{ij} + \theta_{jk} - \theta_{ki} \text{ if } jk \neq i, \quad \kappa_i^{-1}(\theta)_{ik} := \theta_{ik}.\end{aligned} \tag{3.1}$$

Then $\kappa_i^{-1}(\kappa_i(\theta)) = \theta = \kappa_i(\kappa_i^{-1}(\theta))$, and there exists a $U(1)$-equivariant C^*-isomorphism $\kappa_i : B_i^\theta \to B_i^{\kappa_i(\theta);R_i}$ such that

$$\begin{aligned}\kappa_i(w_k^{\theta;i}) &:= w_k^{\kappa_i(\theta);i;R_i} w_i^{\kappa_i(\theta);i;R_i} \text{ if } i \neq k, \quad \kappa_i(w_i^{\theta;i}) := w_i^{\kappa_i(\theta);i;R_i},\\ \kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i}) &:= w_k^{\theta;i}(w_i^{\theta;i})^* \text{ if } i \neq k, \quad \kappa_i^{-1}(w_i^{\kappa_i(\theta);i;R_i}) := w_i^{\theta;i}.\end{aligned} \tag{3.2}$$

Proof. The equalities $\kappa_i^{-1}(\kappa_i(\theta)) = \theta = \kappa_i(\kappa_i^{-1}(\theta))$ follow from elementary calculations using (3.1). To see that (3.2) defines *-homomorphisms, note that, by the universal property of $\mathcal{T}^\theta$ and the definition of I_i^θ, it suffices to check that the elements $\kappa_i(w_k^{\theta;i})$ and $\kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i})$ satisfy respectively the relations that determine B_i^θ and $B_i^{\kappa_i(\theta);R_i}$. Let i,j,k be all distinct (the cases where $k = i$ or $j = i$ are trivial).

(1) Since $(w_k^{\theta;i})^* w_k^{\theta;i} = 1$, we must have $\kappa_i((w_k^{\theta;i})^* w_k^{\theta;i}) = 1$. Furthermore,

$$\begin{aligned}\kappa_i((w_k^{\theta;i})^* w_k^{\theta;i}) &= \kappa_i(w_k^{\theta;i})^* \kappa_i(w_k^{\theta;i})\\ &= (w_i^{\kappa_i(\theta);i;R_i})^* (w_k^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i} w_i^{\kappa_i(\theta);i;R_i} = 1.\end{aligned}$$

(2) Since $(w_k^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i} = 1$, we must have

$$\kappa_i^{-1}\big((w_k^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i}\big) = 1.$$

Furthermore,

$$\begin{aligned}\kappa_i^{-1}\big((w_k^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i}\big) &= \kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i})^* \kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i})\\ &= w_i^{\theta;i}(w_k^{\theta;i})^* w_k^{\theta;i}(w_i^{\theta;i})^* = 1\end{aligned}$$

(3) Since $w_j^{\theta;i} w_k^{\theta;i} = e^{2\pi i\theta_{jk}} w_k^{\theta;i} w_j^{\theta;i}$, we must have

$$\kappa_i(w_j^{\theta;i} w_k^{\theta;i}) = e^{2\pi i\theta_{jk}} \kappa_i(w_k^{\theta;i} w_j^{\theta;i}).$$

Furthermore,

$$\begin{aligned}\kappa_i(w_j^{\theta;i}w_k^{\theta;i}) &= \kappa_i(w_j^{\theta;i})\kappa_i(w_k^{\theta;i})\\ &= w_j^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}w_k^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}\\ &= e^{2\pi i\kappa_i(\theta)_{ik}}w_j^{\kappa_i(\theta);i;R_i}w_k^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}\\ &= e^{2\pi i\left(\kappa_i(\theta)_{ik}+\kappa_i(\theta)_{jk}\right)}w_k^{\kappa_i(\theta);i;R_i}w_j^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}\\ &= e^{2\pi i\left(\kappa_i(\theta)_{ik}+\kappa_i(\theta)_{jk}+\kappa_i(\theta)_{ji}\right)}w_k^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}w_j^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}\\ &= e^{2\pi i\left(\kappa_i(\theta)_{ik}+\kappa_i(\theta)_{jk}+\kappa_i(\theta)_{ji}\right)}\kappa_i(w_k^{\theta;i})\kappa_i(w_j^{\theta;i})\\ &= e^{2\pi i\left(\kappa_i(\theta)_{ik}+\kappa_i(\theta)_{jk}+\kappa_i(\theta)_{ji}\right)}\kappa_i(w_k^{\theta;i}w_j^{\theta;i}).\end{aligned}$$

It remains to show that $\theta_{jk}=\kappa_i(\theta)_{ik}+\kappa_i(\theta)_{jk}+\kappa_i(\theta)_{ji}$. Since $\kappa_i(\theta)_{ik}=\theta_{ik}$ and $\kappa_i(\theta)_{ij}=\theta_{ij}$, we have $\theta_{jk}=\theta_{ik}+\kappa_i(\theta)_{jk}+\theta_{ji}$, so $\kappa_i(\theta)_{jk}=\theta_{ij}+\theta_{jk}+\theta_{ki}$ by antisymmetry of θ.

(4) Since $(w_j^{\theta;i})^*w_k^{\theta;i}=e^{-2\pi i\theta_{jk}}w_k^{\theta;i}(w_j^{\theta;i})^*$, we must have

$$\kappa_i\big((w_j^{\theta;i})^*w_k^{\theta;i}\big)=e^{-2\pi i\theta_{jk}}\kappa_i\big(w_k^{\theta;i}(w_j^{\theta;i})^*\big).$$

Furthermore,

$$\begin{aligned}&\kappa_i\big((w_j^{\theta;i})^*w_k^{\theta;i}\big)\\ &= \kappa_i(w_j^{\theta;i})^*\kappa_i(w_k^{\theta;i})\\ &= (w_i^{\kappa_i(\theta);i;R_i})^*(w_j^{\kappa_i(\theta);i;R_i})^*w_k^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}\\ &= e^{2\pi i\left(-\kappa_i(\theta)jk-\kappa_i(\theta)_{ji}-\kappa_i(\theta)_{ik}\right)}w_k^{\kappa_i(\theta);i;R_i}(w_i^{\kappa_i(\theta);i;R_i})^*w_i^{\kappa_i(\theta);i;R_i}(w_j^{\kappa_i(\theta);i;R_i})^*\\ &= e^{2\pi i\left(-\kappa_i(\theta)jk-\kappa_i(\theta)_{ji}-\kappa_i(\theta)_{ik}\right)}w_k^{\kappa_i(\theta);i;R_i}w_i^{\kappa_i(\theta);i;R_i}(w_i^{\kappa_i(\theta);i;R_i})^*(w_j^{\kappa_i(\theta);i;R_i})^*\\ &= e^{2\pi i\left(-\kappa_i(\theta)jk-\kappa_i(\theta)_{ji}-\kappa_i(\theta)_{ik}\right)}\kappa_i(w_k^{\theta;i})\kappa_i(w_j^{\theta;i})^*\\ &= e^{2\pi i\left(-\kappa_i(\theta)jk-\kappa_i(\theta)_{ji}-\kappa_i(\theta)_{ik}\right)}\kappa_i\big(w_k^{\theta;i}(w_j^{\theta;i})^*\big)\\ &= e^{-2\pi i\theta_{jk}}\kappa_i\big(w_k^{\theta;i}(w_j^{\theta;i})^*\big).\end{aligned}$$

(5) Since $w_j^{\kappa_i(\theta);i;R_i}w_k^{\kappa_i(\theta);i;R_i}=e^{2\pi i\kappa_i(\theta)_{jk}}w_k^{\kappa_i(\theta);i;R_i}w_j^{\kappa_i(\theta);i;R_i}$, we must have

$$\kappa_i^{-1}(w_j^{\kappa_i(\theta);i;R_i}w_k^{\kappa_i(\theta);i;R_i})=e^{2\pi i\kappa_i(\theta)_{jk}}\kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i}w_j^{\kappa_i(\theta);i;R_i}).$$

Furthermore,

$$\begin{aligned}\kappa_i^{-1}(w_j^{\kappa_i(\theta);i;R_i}w_k^{\kappa_i(\theta);i;R_i}) &= \kappa_i^{-1}(w_j^{\kappa_i(\theta);i;R_i})\kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i})\\ &= w_j^{\theta;i}(w_i^{\theta;i})^*w_k^{\theta;i}(w_i^{\theta;i})^*\\ &= e^{2\pi i\left(-\theta_{ik}+\theta_{jk}+\theta_{ij}\right)}w_k^{\theta;i}(w_i^{\theta;i})^*w_j^{\theta;i}(w_i^{\theta;i})^*\\ &= e^{2\pi i\left(\theta_{ij}+\theta_{jk}+\theta_{ki}\right)}\kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i})\kappa_i^{-1}(w_j^{\kappa_i(\theta);i;R_i})\\ &= e^{2\pi i\kappa_i(\theta)_{jk}}\kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i}w_j^{\kappa_i(\theta);i;R_i}).\end{aligned}$$

(6) Since $(w_j^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i} = e^{2\pi i - \kappa_i(\theta)_{jk}} w_k^{\kappa_i(\theta);i;R_i}(w_j^{\kappa_i(\theta);i;R_i})^*$, we see that

$$\kappa_i^{-1}\big((w_j^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i}\big) = e^{-2\pi i \kappa_i(\theta)_{jk}} \kappa_i^{-1}\big(w_k^{\kappa_i(\theta);i;R_i}(w_j^{\kappa_i(\theta);i;R_i})^*\big).$$

Furthermore,

$$\begin{aligned}\kappa_i^{-1}\big((w_j^{\kappa_i(\theta);i;R_i})^* w_k^{\kappa_i(\theta);i;R_i}\big) &= \kappa_i^{-1}(w_j^{\kappa_i(\theta);i;R_i})^* \kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i}) \\ &= w_i^{\theta;i}(w_j^{\theta;i})^* w_k^{\theta;i}(w_i^{\theta;i})^* \\ &= e^{2\pi i\left(-\theta_{jk}+\theta_{ik}+\theta_{ji}\right)} w_k^{\theta;i}(w_i^{\theta;i})^* w_i^{\theta;i}(w_j^{\theta;i})^* \\ &= e^{2\pi i\left(-\theta_{ij}-\theta_{jk}-\theta_{ki}\right)} \kappa_i^{-1}(w_k^{\kappa_i(\theta);i;R_i})^* \kappa_i^{-1}(w_j^{\kappa_i(\theta);i;R_i}) \\ &= e^{-2\pi i \kappa_i(\theta)_{jk}} \kappa_i^{-1}\big(w_k^{\kappa_i(\theta);i;R_i}(w_j^{\kappa_i(\theta);i;R_i})^*\big).\end{aligned}$$

Thus we have shown that κ_i and κ_i^{-1} are well defined *-homomorphisms. They are evidently $U(1)$-equivariant. Since $w_i^{\theta;i}$ and $w_i^{\kappa_i(\theta);i;R_i}$ are unitaries, κ_i and κ_i^{-1} are mutually inverse. $\square$

The maps κ_i, κ_i^{-1} descend to the B_{ij}^θ because they fix the generator

$$\sigma_i\big(1 - w_j^\theta (w_j^\theta)^*\big)$$

of $\sigma_i(I_j^\theta)$. It follows that κ_i induces an invertible $U(1)$-equivariant C^*-isomorphism $\kappa_{i;j} : B_{ij}^\theta \to B_{ij}^{\kappa_i(\theta);R_i}$ such that

$$\begin{aligned}\kappa_{i;j}(w_k^{\theta;ij}) = w_k^{\kappa_i(\theta);ij;R_i} w_i^{\kappa_i(\theta);ij;R_i} \text{ if } i \neq k, \quad \kappa_{i;j}(w_i^{\theta;ij}) = w_i^{\kappa_i(\theta);ij;R_i},\\ \kappa_{i;j}^{-1}(w_k^{\kappa_i(\theta);ij;R_i}) = w_k^{\theta;ij}(w_i^{\theta;ij})^* \text{ if } i \neq k, \quad \kappa_{i;j}^{-1}(w_i^{\kappa_i(\theta);ij;R_i}) = w_i^{\theta;ij}.\end{aligned}$$

So we obtain maps $\hat{\sigma}_j^i : B_i^{\kappa_i(\theta);R_i} \to B_{ij}^{\kappa_i(\theta);R_i}$ from the commuting diagram

$$\begin{tikzcd} B_i^{\kappa_i(\theta);R_i} \arrow[r, "\kappa_i^{-1}"] \arrow[rrr, bend right, "\hat{\sigma}_j^i"'] & B_i^\theta \arrow[r, "\pi_j^i"] & B_{ij}^\theta \arrow[r, "\kappa_{i;j}"] & B_{ij}^{\kappa_i(\theta);R_i}. \end{tikzcd}$$

For any $0 \leq k \leq N$, we have $\hat{\sigma}_j^i(w_k^{\kappa_i(\theta);i;R_i}) = w_k^{\kappa_i(\theta);ij;R_i}$.

3.1. **The multipullback structure of $C(S_{H,\theta}^{2N+1})^R$.** We define the twisted Heegaard sphere C^*-algebra $C(S_{H,\theta}^{2N+1})^R$ to be the image of $C(S_{H,\theta}^{2N+1})$ under $\prod_{i=0}^N \kappa_i$. We compute morphisms $\hat{\pi}_j^i$ that assemble the $B_i^{\kappa_i(\theta);R_i}$ into the multipullback C^*-algebra $C(S_{H,\theta}^{2N+1})^R$. Fix any $i < j$. We determine $\hat{\pi}_j^i$ and $\hat{\pi}_i^j$ through the commutative diagram

$$
\begin{tikzcd}
B_i^{\kappa_i(\theta);R_i} \arrow[d, "\kappa_i^{-1}"'] \arrow[drr, dashed, "\hat{\pi}_j^i"] & & & & B_j^{\kappa_j(\theta);R_j} \arrow[d, "\kappa_j^{-1}"] \arrow[dll, dashed, "\hat{\pi}_i^j"'] . \\
B_i^\theta \arrow[r, "\pi_j^i"'] & B_{ij}^\theta \arrow[r, "\kappa_{i;j}"'] & B_{ij}^{\kappa_i(\theta);R_i} & B_{ij}^\theta \arrow[l, "\kappa_{i;j}"] & B_i^\theta \arrow[l, "\pi_i^j"]
\end{tikzcd}
$$

Then $C(S_{H,\theta}^{2N+1})^R$ is equivariantly isomorphic to the multipullback C^*-algebra over the $\hat{\pi}_j^i$. Note that the above diagram can be rewritten as follows:

$$
\begin{tikzcd}
B_i^{\kappa_i(\theta);R_i} \arrow[d, dashed, bend right, "\hat{\pi}_j^i"'] \arrow[d, "\hat{\sigma}_j^i"] & & B_j^{\kappa_j(\theta);R_j} \arrow[dll, dashed, "\hat{\pi}_i^j"'] \arrow[d, "\hat{\sigma}_i^j"] . \\
B_{ij}^{\kappa_i(\theta);R_i} & B_{ij}^\theta \arrow[l, "\kappa_{i;j}"'] & B_{ij}^{\kappa_j(\theta);R_j} \arrow[l, "\kappa_{j;i}^{-1}"'] \arrow[ll, dashed, bend left, "\hat{\psi}_{ij}"]
\end{tikzcd}
\tag{3.3}
$$

Thus, for $i < j$, we have $\hat{\pi}_j^i := \hat{\sigma}_j^i$ and $\hat{\pi}_i^j := \hat{\psi}_{ij} \circ \hat{\sigma}_j^i$, where $\hat{\psi}_{ij} := \kappa_{i;j} \circ \kappa_{j;i}^{-1}$. We compute the images of the generators of $B_{ij}^{\kappa_j(\theta);R_j}$ under the $\hat{\psi}_{ij}$: for $i < j$ and $k \neq i, j$,

$$
\begin{aligned}
\hat{\psi}_{ij}(w_k^{\kappa_j(\theta);ij;R_j}) &:= \kappa_{i;j}\big(\kappa_{j;i}^{-1}(w_k^{\kappa_j(\theta);ij;R_j})\big) \\
&= \kappa_{i;j}\big(w_k^{\theta;ij}(w_j^{\theta;ij})^*\big) \\
&= \kappa_{i;j}(w_k^{\theta;ij})\kappa_{i;j}(w_j^{\theta;ij})^* \\
&= w_k^{\kappa_i(\theta);ij;R_i} w_i^{\kappa_i(\theta);ij;R_i} (w_i^{\kappa_i(\theta);ij;R_i})^* (w_j^{\kappa_i(\theta);ij;R_i})^* \\
&= w_k^{\kappa_i(\theta);ij;R_i} (w_j^{\kappa_i(\theta);ij;R_i})^*, \\
\hat{\psi}_{ij}(w_i^{\kappa_j(\theta);ij;R_j}) &:= \kappa_{i;j}\big(\kappa_{j;i}^{-1}(w_i^{\kappa_j(\theta);ij;R_j})\big) \\
&= \kappa_{i;j}\big(w_i^{\theta;ij}(w_j^{\theta;ij})^*\big) \\
&= w_i^{\kappa_i(\theta);ij;R_i} (w_i^{\kappa_i(\theta);ij;R_i})^* (w_j^{\kappa_i(\theta);ij;R_i})^* \\
&= (w_j^{\kappa_i(\theta);ij;R_i})^*, \quad \text{and} \\
\hat{\psi}_{ij}(w_j^{\kappa_j(\theta);ij;R_j}) &:= \kappa_{i;j}\big(\kappa_{j;i}^{-1}(w_j^{\kappa_j(\theta);ij;R_j})\big) \\
&= \kappa_{i;j}(w_j^{\theta;ij}) \\
&= w_j^{\kappa_i(\theta);ij;R_i} w_i^{\kappa_i(\theta);ij;R_i}.
\end{aligned}
$$

3.2. **The $U(1)$-fixed-point subalgebra of $C(S_{H,\theta}^{2N+1})^R$ as a multipullback.** For any antisymmetric $(N+1) \times (N+1)$ real matrix θ, let us denote by $\breve{\kappa}_i(\theta)$ the matrix obtained from $\kappa_i(\theta)$ by removing the i-th row and column. Re-index the remaining elements so that both row and column indices run from 1 to N.

For any $0 \leq i \leq N$, let $A_i := \mathcal{T}^N_{\check{\kappa}_i(\theta)}$. The isometries $v^i_1, \ldots, v^i_N$ generating A_i satisfy

$$v^i_j v^i_k = e^{2\pi i \check{\kappa}_i(\theta)_{jk}} v^i_k v^i_j, \quad (v^i_j)^* v^i_k = e^{-2\pi i \check{\kappa}_i(\theta)_{jk}} v^i_k (v^i_j)^*,$$

for all $1 \leq j, k \leq N$, $j \neq k$.

We claim that A_i is isomorphic as a C^*-algebra with the $U(1)$-invariant subalgebra of $B_i^{\kappa_i(\theta);R_i}$. To see this, observe that the universal property of A_i yields a C^*-homomorphism $\phi_i : A_i \to B_i^{\kappa_i(\theta);R_i}$ such that

$$\phi_i(v^i_k) = \begin{cases} w_{k-1}^{\kappa_i(\theta);i;R_i} & \text{if } k \leq i \\ w_k^{\kappa_i(\theta);i;R_i} & \text{if } k > i. \end{cases}$$

An argument using the gauge-invariant uniqueness theorem as in the proof of Theorem 2.3 shows that ϕ_i is injective. To see that it is surjective, first observe that $B_i^{\kappa_i(\theta);R_i}$ is densely spanned by elements of the form

$$(w_1^{\kappa_i(\theta);i;R_i})^{n_1} \cdots (w_N^{\kappa_i(\theta);i;R_i})^{n_N} (w_N^{\kappa_i(\theta);i;R_i})^{*m_N} \cdots (w_1^{\kappa_i(\theta);i;R_i})^{*m_1}.$$

Since $w_i^{\kappa_i(\theta);i;R_i}$ is unitary in $B^{\kappa_i(\theta);R_i}$, the expectation onto the $U(1)$-invariant subalgebra of $B_i^{\kappa_i(\theta);R_i}$, obtained by averaging over the $U(1)$-action, takes such a spanning element to

$$\delta_{n_i,m_i} \prod_{\substack{j=0 \\ j\neq i}}^{N} (w_j^{\kappa_i(\theta);i;R_i})^{n_j} \prod_{\substack{k=N \\ k\neq i}}^{0} (w_k^{\kappa_i(\theta);i;R_i})^{*n_k}.$$

Therefore, the $U(1)$-invariant subalgebra of $B_i^{\kappa_i(\theta);R_i}$ is spanned by elements of this form, and such elements are in the range of ϕ_i. Hence ϕ_i is surjective. For any $i \neq j$ we will denote the generators of $A_{i;j}$ (which are the images under the canonical quotient maps of the generators of A_i) by $v_1^{i;j}, \ldots v_N^{i;j}$. For $i < j$, the elements $v_j^{i;j} \in A_{i;j}$ and $v_{i+1}^{j;i} \in A_{j;i}$ are unitary. The inverse of ϕ_i satisfies

$$\phi_i^{-1}(w_k^{\kappa_i(\theta);i;R_i}) = \begin{cases} v^i_{k+1} & \text{if } k < i \\ v^i_k & \text{if } i < k. \end{cases}$$

Let J_j be the ideal of A_i generated by $(1 - v^i_j (v^i_j)^*)$. For $0 \leq i < j \leq N$, let

$$A_{i;j} := A_i / J_j, \quad \text{and} \quad A_{j;i} := A_j / J_{i+1}.$$

The isomorphisms ϕ_i^{-1} descend to isomorphisms

$$\phi_{ij}^{-1} : (B_{ij}^{\kappa_i(\theta);R_i})^{U(1)} \to A_{i;j}, \quad w_k^{\kappa_i(\theta);ij;R_i} \mapsto \begin{cases} v^{i;j}_{k+1} & \text{if } k < i \\ v^{i;j}_k & \text{if } i < k. \end{cases}$$

Using the isomorphisms ϕ_i and ϕ_{ij} we can transport the multipullback structure of the $U(1)$-fixed-point subalgebra of $C(S^{2N+1}_{H,\theta})^R$ as follows ($0 \leq i < j \leq N$):

$$
\begin{tikzcd}
A_i \arrow[r, "\phi_i"] \arrow[ddr, dashed, "\rho^i_j"'] & \big(B_i^{\kappa_i(\theta);R_i}\big)^{U(1)} \arrow[d, "\hat{\sigma}^i_j"'] & & \big(B_j^{\kappa_j(\theta);R_j}\big)^{U(1)} \arrow[d, "\hat{\sigma}^j_i"] & A_j \arrow[l, "\phi_j"'] \arrow[ddl, dashed, "\rho^j_i"] \\
& \big(B_{ij}^{\kappa_i(\theta);R_i}\big)^{U(1)} \arrow[d, "\phi_{ij}^{-1}"'] & & \big(B_{ij}^{\kappa_j(\theta);R_j}\big)^{U(1)} \arrow[ll, "\hat{\psi}_{ij}"'] \arrow[d, "\phi_{ji}^{-1}"] & \\
& A_{i;j} & & A_{j;i} \arrow[ll, dashed, "\psi_{ij}"] &
\end{tikzcd}
\tag{3.5}
$$

In the diagram (3.5), we have used the same symbols to denote the (co)-restrictions of the maps $\hat{\sigma}^i_j$, $\hat{\sigma}^j_i$ and $\hat{\psi}_{ij}$ to the respective $U(1)$-invariant subalgebras. Since all these maps are $U(1)$-equivariant, the restrictions corestrict as expected.

We will now explicitly write the values of maps ρ^i_j, ρ^j_i, ψ_{ij}, $0 \leq i < j \leq N$, defined by the commutative diagram above, on generators of respective domains. It is straightforward to verify that ρ^i_j and ρ^j_i are the canonical quotient maps given by

$$\rho^i_j(v^i_k) = v^{i;j}_k, \quad \text{and} \quad \rho^j_i(v^j_k) = v^{j;i}_k, \quad 1 \leq k \leq N.$$

In case of the isomorphisms $\psi_{ij} := \phi_{ij}^{-1} \circ \hat{\psi}_{ij} \circ \phi_{ji}$, $0 \leq i < j \leq N$, we will perform a careful case-by-case analysis. The first splitting into cases follows from the definition of $\hat{\psi}_{ij}$ (see (3.4)): either $k = i+1$ or $k \neq i+1$.

(1) For $k = i+1$:

$$
\begin{aligned}
\psi_{ij}(v^{j;i}_{i+1}) &= \phi_{ij}^{-1}\Big(\hat{\psi}_{ij}\big(\phi_{ji}(v^{j;i}_{i+1})\big)\Big) \\
&= \phi_{ij}^{-1}\Big(\hat{\psi}_{ij}\big(w_i^{\kappa_j(\theta);ij;R_j}\big)\Big) \\
&= \phi_{ij}^{-1}\Big(\big(w_j^{\kappa_i(\theta);ij;R_i}\big)^*\Big) \\
&= (v^{i;j}_j)^*.
\end{aligned}
$$

(2) For $k \neq i+1$: $\psi_{ij}(v^{j;i}_k) = \phi_{ij}^{-1}\Big(\hat{\psi}_{ij}\big(\phi_{ji}(v^{j;i}_k)\big)\Big) =: (*)$. Here the definition of ϕ_{ji} forces a split into cases $k > j$ or $k \leq j$.
   (a) For $k > j$:

$$
\begin{aligned}
(*) &= \phi_{ij}^{-1}\Big(\hat{\psi}_{ij}\big(w_k^{\kappa_j(\theta);ij;R_j}\big)\Big) \\
&= \phi_{ij}^{-1}\Big(w_k^{\kappa_i(\theta);ij;R_i}\big(w_j^{\kappa_i(\theta);ij;R_i}\big)^*\Big) \\
&= v^{i;j}_k (v^{i;j}_j)^*.
\end{aligned}
$$

(b) For $k \leq j$:

$$\begin{aligned}(*) &= \phi_{ij}^{-1}\Big(\hat{\psi}_{ij}\big(w_{k-1}^{\kappa_j(\theta);ij;R_j}\big)\Big)\\ &= \phi_{ij}^{-1}\Big(w_{k-1}^{\kappa_i(\theta);ij;R_i}\big(w_j^{\kappa_i(\theta);ij;R_i}\big)^*\Big)\\ &=: (**).\end{aligned}$$

Now we arrive at another split into cases: $k-1 > i$ or $k-1 < i$. (The case $k-1 = i$ was taken care of previously.)

(i) If $k-1 > i$, then $(**) = v_{k-1}^{i;j}(v_j^{i;j})^*$.

(ii) If $k-1 < i$, then $(**) = v_k^{i;j}(v_j^{i;j})^*$.

Summarizing, when $0 \leq i < j \leq N$ and $1 \leq k \leq N$, we obtain

$$\psi_{ij}(v_k^{j;i}) = \begin{cases} v_j^{i;j} & \text{if } k = i+1\\ v_k^{i;j}(v_j^{i;j})^* & \text{if } k > j \text{ or } k < i+1\\ v_{k-1}^{i;j}(v_j^{i;j})^* & \text{if } i+1 < k \leq j\end{cases}.$$

Consequently, the $U(1)$-fixed-point subalgebra of $C(S_{H,\theta}^{2N+1})^R$ is isomorphic to the multipullback of the algebras A_i with respect to the natural maps $A_i \to A_{i;j}$, $A_j \to A_{i;j}$, $i < j$, determined by the diagrams

$$\begin{tikzcd} A_i \arrow[d, "\rho_j^i"'] & A_j \arrow[d, "\rho_i^j"] \\ A_{i;j} & A_{j;i}. \arrow[l, "\psi_{ij}"] \end{tikzcd}$$

4. The K-groups of twisted multipullback quantum odd spheres and complex projective spaces

We begin by deriving a short exact sequence of commutative C^*-algebras whose noncommutative counterpart provides a basis for computing the K-groups of the twisted multipullback quantum complex projective spaces.

The $2N+1$-dimensional sphere S^{2N+1} is the closed subset of $\mathbb{C}^{N+1}$ defined by

$$S^{2N+1} = \Big\{(z_0, \ldots, z_N) \in \mathbb{C}^{N+1} \;\Big|\; \textstyle\sum_{i=0}^N |z_i|^2 = 1\Big\}.$$

Denote by $D := \{c \in \mathbb{C} \mid |c| \leq 1\}$ the unit disk, and by $D_0 := \{c \in \mathbb{C} \mid |c| < 1\}$ the interior of the unit disk. Next, we define a "non-round" odd sphere as follows:

$$S_D^{2N+1} := \Big\{(c_0, \ldots, c_N) \in D^{N+1} \;\Big|\; \textstyle\prod_{i=0}^N (1 - |c_i|^2) = 0\Big\}.$$

Since $\prod_{i=0}^N (1 - |c_i|^2) = 0$ if and only if $|c_i| = 1$ for some $i \in \{0, \ldots, N\}$, it follows that $\sum_{i=0}^N |c_i|^2 \geq 1$ for any $(c_0, \ldots, c_N) \in S_D^{2N+1}$. Also, $\sum_{i=0}^N |z_i|^2 = 1$ gives

$$\max\{|z_0|, \ldots, |z_N|\} \geq \frac{1}{\sqrt{N+1}}.$$

Hence there are well-defined maps

$$S_D^{2N+1} \ni (c_j)_{j=0}^N \longmapsto \left(\frac{c_j}{\sqrt{\sum_{i=0}^N |c_i|^2}} \right)_{j=0}^N \in S^{2N+1},$$
$$S^{2N+1} \ni (z_j)_{j=0}^N \longmapsto \left(\frac{z_j}{\max\{|z_0|, \dots, |z_N|\}} \right)_{j=0}^N \in S_D^{2N+1}.$$

These maps are mutually inverse and continuous, so that $S^{2N+1} \cong S_D^{2N+1}$.

Now consider the following splitting of S_D^{2N+1} into a pair of disjoint sets which are closed and open respectively:

$$S_D^{2N+1} = \{(c_i)_i \in S_D^{2N+1} \mid |c_N| = 1\} \coprod \{(c_i)_i \in S_D^{2N+1} \mid |c_N| < 1\}.$$

The condition in the first of these sets forces $\prod_{i=0}^N (1 - |c_i|^2) = 0$ regardless of the values of $(c_0, \dots, c_{N-1}) \in D^N$. Hence

$$\{(c_i)_i \in S_D^{2N+1} \mid |c_N| = 1\} = D^N \times S^1.$$

Furthermore, when $(c_i)_{i=0}^N$ is an element of the second set, then $\prod_{i=0}^{N-1}(1-|c_i|^2) = 0$ because $1 - |c_N|^2 > 0$. Consequently,

$$\{(c_i)_i \in S_D^{2N+1} \mid |c_N| < 1\} = S_D^{2N-1} \times D_0.$$

Summarizing, we obtain the decomposition

$$S_D^{2N+1} = \big(D^N \times S^1\big) \coprod \big(S_D^{2N-1} \times D_0\big).$$

For the diagonal actions of $U(1)$, this decomposition of S_D^{2N+1} induces the $U(1)$-equivariant short exact sequence

$$0 \longrightarrow C_0(S_D^{2N-1} \times D_0) \longrightarrow C(S_D^{2N+1}) \longrightarrow C(D^N \times S^1) \longrightarrow 0$$

of C^*-algebras. Finally, remembering that S_D^{2N-1} and S^{2N-1} are equivariantly homeomorphic for the diagonal $U(1)$-actions, and using standard identifications, we obtain the following $U(1)$-equivariant short exact sequence of C^*-algebras:

(4.1)
$$0 \longrightarrow C(S^{2N-1}) \otimes C_0(D_0) \longrightarrow C(S^{2N+1}) \longrightarrow C(D)^{\otimes N} \otimes C(S^1) \longrightarrow 0\ .$$

4.1. **Quantum odd spheres.** Recall that s denotes the isometry generating the Toeplitz algebra $\mathcal{T}$. The universal properties of the maximal tensor product and of the untwisted algebra $\mathcal{T}_0^{N+1}$ show that the map

$$\mathcal{T}_0^{N+1} \ni w_j \longmapsto 1^{\otimes j} \otimes s \otimes 1^{\otimes N-j} \in \mathcal{T}^{\otimes N+1} \tag{4.2}$$

is an isomorphism.

To see where Definition 2.2 comes from, and how it relates to noncommutative solid tori, recall first that σ denotes the symbol map from $\mathcal{T}$ to $C(\mathbb{T})$. When

$\theta = 0$, we denote $C(S^{2N+1}_{H,\theta})$ by $C(S^{2N+1}_{H})$. We have $\mathcal{T}^{N+1}_0 = \mathcal{T}^{\otimes N+1}$, and each I_i of Definition 2.2 is precisely the kernel of

$$\mathrm{id}^{\otimes i} \otimes \sigma \otimes \mathrm{id}^{\otimes N-i} : \mathcal{T}^{\otimes N+1} \longrightarrow B_i := \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i}, \tag{4.3}$$

and so each B_i is the noncommutative solid torus algebra $\mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-i}$. The algebras B_i and B_{ij} and the maps π^i_j of Definition 2.2 are then given by

$$B_{ij} := \mathcal{T}^{\otimes i} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes j-i-1} \otimes C(\mathbb{T}) \otimes \mathcal{T}^{\otimes N-j}, \quad i < j, \quad i, j \in \{0, 1, \ldots, N\},$$
$$B_{ij} := B_{ji}, \quad j < i, \quad i, j \in \{0, 1, \ldots, N\}, \quad \text{and}$$
$$\pi^i_j := \mathrm{id}^j \otimes \sigma \otimes \mathrm{id}^{N-j} : B_i \to B_{ij}, \quad i \neq j, \quad i, j \in \{0, 1, \ldots, N\}. \tag{4.4}$$

Thus our definition of $C(S^{2N+1}_{H,\theta})$ as the multipullback along the π^i_j is a natural noncommutative dual to the Heegaard-type splitting of S^{2N+1} described in Section 1.2.

To compute $K_*(C(S^{2N+1}_{H,\theta}))$, we first compute the K-theory of the untwisted quantum sphere $C(S^{2N+1}_H)$ by applying the Künneth theorem and then the six-term ideal-quotient exact sequence. We then apply results of [28] to see that the K-theory of $C(S^{2N+1}_{H,\theta})$ is identical to that of $C(S^{2N+1}_H)$. Since the cocycle c on Λ in Lemma 2.5 is induced by a group cocycle on $\mathbb{Z}^k$, the corresponding twisted multiplication on $C^*(\Lambda; \mathcal{E})$ can be realised using Rieffel's framework of twisted multiplicative structures on C^*-algebras arising from actions of $\mathbb{R}^k$ applied to the gauge action of $\mathbb{T}^k$ on $C^*(\Lambda; \mathcal{E})$ and the dense *-subalgebra $\operatorname{span}\{s_\mu s_\nu^* : \mu, \nu \in \Lambda\}$. So we could alternatively apply [26, Main Theorem (page 200)] to prove that the K-theory of $C(S^{2N+1}_{H,\theta})$ is identical to that of $C(S^{2N+1}_H)$.

Recall that $\mathcal{T}^{N+1}_0$ is canonically isomorphic to $\mathcal{T}^{\otimes N+1}$ via the map that carries the generator w_i of $\mathcal{T}^{N+1}_0$ to the elementary tensor $1 \otimes \cdots \otimes 1 \otimes s \otimes 1 \otimes \cdots \otimes 1$, where the s appears in the ith (counting from zero) tensor factor. Recall also that we have $K_0(\mathcal{T}) = \mathbb{Z}$ and $K_1(\mathcal{T}) = 0$ with the generator in K_0 being the class of the identity element. It then follows from the Künneth theorem (see, e.g., [32, Remarks 9.3.3]) that $K_0(\mathcal{T}^{N+1}_0) = \mathbb{Z}[1]$ and $K_1(\mathcal{T}^{N+1}_0) = 0$. Given $m = (m_0, m_1, \ldots, m_N) \in \mathbb{Z}^{N+1}$, we write W_m for the element $\prod_{i=0}^N w_i^{m_i}$ of $\mathcal{T}^{N+1}_0$. (By convention, $w_i^{-k} = (w_i^*)^k$ for $k \geq 0$.)

Lemma 4.1. *For $N \geq 0$, there is an isomorphism of $\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))$ onto the ideal I of $\mathcal{T}^{N+1}_0$ generated by $\prod_{j=0}^N (1 - w_j w_j^*)$ that carries the matrix unit E_{pq} to*

$$W_p\Big(\prod_{j=0}^N (1 - w_j w_j^*)\Big) W_q^*.$$

Proof. Let $R := \prod_{j=0}^N (1 - w_j w_j^*)$. As the w_i are commuting isometries, we see that $w_i^* R = 0 = R w_i$ for all i, and then we deduce that $W_p^* R = 0 = R W_p$ for all $p \in \mathbb{N}^{N+1} \setminus \{0\}$. Similarly, observe that

$$(W_p R W_q^*)(W_a R W_b^*) = (W_p R) W_q^* W_a R W_b^* = \delta_{q,a} w_p R w_b^*.$$

Since $(W_p R W_q^*)^* = W_q R W_p^*$, we see that the $W_p R W_q^*$ form a family of matrix units indexed by $\mathbb{N}^{N+1}$, and so there is a homomorphism of $\mathcal{K}(\ell^2(\mathbb{N}^{N+1})) \to I$ carrying each E_{pq} to $W_p R W_q^*$. Since R is nonzero, and since $\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))$ is simple, this homomorphism is injective. Surjectivity follows from

$$\textstyle\prod_{j=0}^{N}(1 - w_j w_j^*) = (1 - w_0 w_0^*)R = R - w_0 R w_0^*. \qquad \square$$

The following result generalizes [2, Theorem 4.1] and [19, Theorem 3.2]. It also contains statement (1) of Theorem 0.1, so the proof of this theorem also proves Theorem 0.1(1).

Theorem 4.2. *Consider an integer $N \geq 1$ and an antisymmetric matrix $\theta \in M_{N+1}(\mathbb{R})$. Then $K_1(C(S_{H,\theta}^{2N+1})) \cong \mathbb{Z}$ and there is an isomorphism $K_0(C(S_{H,\theta}^{2N+1})) \cong \mathbb{Z}$ that carries $[1_{C(S_{H,\theta}^{2N+1})}]$ to 1.*

Proof. We first consider the case where $\theta_{ij} = 0$ for all i, j. Theorem 2.3 combined with Lemma 4.1 and the isomorphism $\mathcal{T}_0^{N+1} \cong \mathcal{T}^{\otimes N+1}$ given in (4.2) implies that

$$C(S_H^{2N+1}) \;\cong\; \mathcal{T}_0^{N+1}/I \;\cong\; \mathcal{T}^{\otimes N+1}/\mathcal{K}(\ell^2(\mathbb{N}^{N+1})). \tag{4.5}$$

We claim that the inclusion $\iota : \mathcal{K}(\ell^2(\mathbb{N}^{N+1})) \to \mathcal{T}_0^{N+1}$ of Lemma 4.1 induces the zero map on K-theory. As $K_0(\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))) \cong \mathbb{Z}$ is generated by $[R]$, we just have to show that $[R] = 0$ in $K_0(\mathcal{T}_0^{N+1}) \cong \mathbb{Z}$. The isomorphism $\mathcal{T}_0^{N+1} \cong \mathcal{T}^{\otimes N+1}$ given by (4.2) carries R to $(1 - ss^*) \otimes (1 - ss^*) \otimes \cdots \otimes (1 - ss^*)$. Since s is an isometry, we have $[1 - ss^*] = [s^*s - ss^*] = 0$ in $K_0(\mathcal{T})$. As $K_1(\mathcal{T}) = 0$, the Künneth isomorphism implies that $[(1 - ss^*) \otimes (1 - ss^*) \otimes \cdots \otimes (1 - ss^*)] = 0$ in $K_0(\mathcal{T}^{\otimes N+1})$. Therefore $[R]$ is zero in $K_0(\mathcal{T}_0^{N+1})$ as claimed.

Since $K_1(\mathcal{K}(\ell^2(\mathbb{N}^{N+1}))) = 0 = K_1(\mathcal{T}_0^{N+1})$, Theorem 9.3.2 of [32] gives an exact sequence

$$\begin{tikzcd}
\mathbb{Z} \arrow[r, "0"] & \mathbb{Z} \arrow[r] & K_0(C(S_H^{2N+1})) \arrow[d] \\
K_0(C(S_H^{2N+1})) \arrow[u] & 0 \arrow[l] & 0. \arrow[l]
\end{tikzcd}$$

Hence $K_0(C(S_H^{2N+1})) = \mathbb{Z}[1]$ and $K_1(C(S_H^{2N+1})) \cong \mathbb{Z}$.

For general θ, we have $C(S_{H,\theta}^{2N+1}) \cong C^*(\Lambda, c; \mathcal{E})$ by Lemma 2.5. By (2.7), the cocycle c on Λ arises from exponentiation of an $\mathbb{R}$-valued cocycle. Hence [28, Theorem 6.1] gives

$$K_*(C(S_{H,\theta}^{2N+1})) \;\cong\; K_*(C^*(\Lambda, c; \mathcal{E})) \;\cong\; K_*(C^*(\Lambda, 1; \mathcal{E})) \;\cong\; K_*(C(S_H^{2N+1}))$$

via isomorphisms that preserve the K_0-class of the identity. $\square$

Remark 4.3. An alternative proof can be obtained using the exact sequence (4.6).

4.2. **Multipullback quantum complex projective spaces.** In our computation of the K-theory of $C(\mathbb{P}^N(\mathcal{T}))$, we will use two auxiliary results. The first result is a quantum version of the short exact sequence (4.1):

Lemma 4.4. *With respect to the diagonal $U(1)$-action, for any positive integer k, there exists a $U(1)$-equivariant short exact sequence of C^*-algebras*

$$0 \longrightarrow C(S_H^{2k-1}) \otimes \mathcal{K} \longrightarrow C(S_H^{2k+1}) \longrightarrow \mathcal{T}^{\otimes k} \otimes C(S^1) \longrightarrow 0. \tag{4.6}$$

Proof. The starting point is the Toeplitz extension, i.e., the exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow \mathcal{T} \xrightarrow{\sigma} C(S^1) \longrightarrow 0,$$

where σ is the symbol map. Since the Toeplitz algebra is nuclear, so is $\mathcal{T}^{\otimes k}$, whence the sequence of C^*-algebras

$$0 \longrightarrow \mathcal{T}^{\otimes k} \otimes \mathcal{K} \longrightarrow \mathcal{T}^{\otimes k} \otimes \mathcal{T} \xrightarrow{\mathrm{id}\otimes\sigma} \mathcal{T}^{\otimes k} \otimes C(S^1) \longrightarrow 0 \tag{4.7}$$

is also exact. Equation (4.5) gives $\big(\mathcal{T}^{\otimes k} \otimes \mathcal{K}\big)/\mathcal{K}^{\otimes k+1} \cong C(S_H^{2k-1}) \otimes \mathcal{K}$ by the nuclearity of $\mathcal{K}$. So taking quotients by $\mathcal{K}^{\otimes k+1}$ throughout (4.7) yields the exact sequence (4.6). The $U(1)$-equivariance follows from the fact that all the identifications used are $U(1)$-equivariant. □

The second result is a standard fact about compact-group actions, so we omit its proof.

Lemma 4.5. *Let G be a compact Hausdorff topological group and let A be a C^*-algebra with a pointwise norm continuous G-action $\alpha : G \to \mathrm{Aut}(A)$. Let $I \subseteq A$ be a closed two-sided G-invariant ideal of A. Then A/I admits the induced G-action, and the sequence of fixed-point algebras*

$$0 \longrightarrow I^G \longrightarrow A^G \longrightarrow (A/I)^G \longrightarrow 0$$

is exact.

To compute the K-groups of the invariant subalgebra $C(\mathbb{P}^N(\mathcal{T})) := C(S_H^{2N+1})^\alpha$, we first construct a family of short exact sequences. Fix $N \in \mathbb{N}$, $N \geq 1$. For all $k \in \{1, \ldots, N\}$ apply the exact functor $\_ \otimes \mathcal{K}^{\otimes N-k}$ to the sequence (4.6) to obtain the short exact sequence

$$\begin{aligned}0 \longrightarrow C(S_H^{2k-1}) \otimes \mathcal{K}^{\otimes N-k+1} \longrightarrow\ & C(S_H^{2k+1}) \otimes \mathcal{K}^{\otimes N-k} \\ & \longrightarrow \mathcal{T}^{\otimes k} \otimes C(S^1) \otimes \mathcal{K}^{\otimes N-k} \longrightarrow 0.\end{aligned}$$

By Lemma 4.5, the restriction of the above sequence to $U(1)$-invariant subalgebras is again exact:

$$0 \longrightarrow \Big(C(S_H^{2k-1}) \otimes \mathcal{K}^{\otimes N-k+1}\Big)^{U(1)} \longrightarrow \Big(C(S_H^{2k+1}) \otimes \mathcal{K}^{\otimes N-k}\Big)^{U(1)} \longrightarrow \Big(\mathcal{T}^{\otimes k} \otimes C(S^1) \otimes \mathcal{K}^{\otimes N-k}\Big)^{U(1)} \longrightarrow 0\,. \tag{4.8}$$

Our gauge trick (1.2)–(1.3) shows that $\mathcal{T}^{\otimes k} \otimes C(S^1) \otimes \mathcal{K}^{\otimes N-k}$ with diagonal $U(1)$-action is $U(1)$-equivariantly isomorphic with $\mathcal{T}^{\otimes k} \otimes C(S^1) \otimes \mathcal{K}^{\otimes N-k}$ where $U(1)$ acts only on the $C(S^1)$-component. Hence

$$(\mathcal{T}^{\otimes k} \otimes C(S^1) \otimes \mathcal{K}^{\otimes N-k})^{U(1)} \cong \mathcal{T}^{\otimes k} \otimes \mathcal{K}^{\otimes N-k}. \tag{4.9}$$

Next, let

$$S_k := \Big(C(S_H^{2k+1}) \otimes \mathcal{K}^{\otimes N-k}\Big)^{U(1)}, \quad k \in \{0, \dots, N\}.$$

Using this notation and (4.9), we can write the family of short exact sequences (4.8) as

$$0 \longrightarrow S_{k-1} \longrightarrow S_k \longrightarrow \mathcal{T}^{\otimes k} \otimes \mathcal{K}^{\otimes N-k} \longrightarrow 0, \tag{4.10}$$

where $k \in \{1, \dots, N\}$.

Theorem 4.6. *Let N be a positive integer. Then*

$$K_0(C(\mathbb{P}^N(\mathcal{T}))) = \mathbb{Z}^{N+1} \qquad \textit{and} \qquad K_1(C(\mathbb{P}^N(\mathcal{T}))) = 0.$$

Proof. Since $S_N = C(\mathbb{P}^N(\mathcal{T}))$, it suffices to prove that $K_0(S_k) = \mathbb{Z}^{k+1}$ and $K_1(S_k) = \{0\}$ for all $k \in \{1, \dots, N\}$. We do this by induction on k. For $k = 0$, the gauge trick gives

$$S_0 = \big(C(S^1) \otimes \mathcal{K}^{\otimes N}\big)^{U(1)} \cong \mathcal{K}^{\otimes N}.$$

Consequently,

$$K_0(S_0) \cong K_0(\mathcal{K}) = \mathbb{Z}, \quad K_1(S_0) \cong K_1(\mathcal{K}) = 0.$$

Now assume that $K_0(S_{k-1}) = \mathbb{Z}^k$ and $K_1(S_{k-1}) = 0$. The short exact sequence (4.10) of C^*-algebras induces the six-term exact sequence of Abelian groups:

$$\begin{tikzcd} K_0(S_{k-1}) \arrow[r] & K_0(S_k) \arrow[r] & K_0(\mathcal{T}^{\otimes k}) \arrow[d] \\ K_1(\mathcal{T}^{\otimes k}) \arrow[u] & K_1(S_k) \arrow[l] & K_1(S_{k-1}). \arrow[l] \end{tikzcd} \tag{4.11}$$

The Künneth theorem gives $K_0(\mathcal{T}^{\otimes k}) = \mathbb{Z}$ and $K_1(\mathcal{T}^{\otimes k}) = 0$. Combining this with the inductive hypothesis, the sequence (4.11) becomes

$$\begin{tikzcd} \mathbb{Z}^k \arrow[r] & K_0(S_k) \arrow[r] & \mathbb{Z} \arrow[d] \\ 0 \arrow[u] & K_1(S_k) \arrow[l] & 0. \arrow[l] \end{tikzcd}$$

Exactness gives $K_1(S_k) = 0$, and exactness combined with the projectivity of free abelian groups gives $K_0(S_k) = \mathbb{Z} \oplus \mathbb{Z}^k = \mathbb{Z}^{k+1}$. □

4.3. **Twisted multipullback quantum complex projective spaces.** We begin by establishing notation. Fix a positive integer N, and let $\theta \in M_{N+1}(\mathbb{R})$ be an antisymmetric real matrix. For $k, l \leq N$, define $\Theta_{kl} := e^{2\pi i \theta_{kl}}$. For $0 \leq k \leq l \leq N$, let $\mathcal{T}_{k,l}$ be the universal C^*-algebra generated by the isometries $s_k, \ldots, s_l$ satisfying the usual identities:

$$s_i s_j = \Theta_{ij} s_j s_i, \;\; s_i^* s_j = \overline{\Theta}_{ij} s_j s_i^*.$$

We will identify $\mathcal{T}_{k,l}$ with the corresponding subalgebra of $\mathcal{T}_{0,N}$. Let $\mathcal{K}_{(k,l)}$ be the ideal of $\mathcal{T}_{k,l}$ generated by the product $\prod_{i=k}^{l}(1 - s_i s_i^*)$. For each $k \leq N$, the universal property of $\mathcal{T}_{0,N}$ shows that the formula

$$\alpha_k(s_i) := \Theta_{ik} s_i$$

defines actions α_k of both $\mathbb{N}$ and $\mathbb{Z}$ on $\mathcal{T}_{0,N}$, and hence on each $\mathcal{T}_{l_1,l_2}$.

The idea of the computation is the same as in the untwisted case, with small changes due to the fact that the isometries generating the noncommutative sphere do not commute. We regard the twisted noncommutative sphere as the quotient of the twisted semigroup algebra of $\mathbb{N}^{N+1}$ by the ideal of compact operators: $C^*(\mathbb{N}^{N+1}, \Theta)/\mathcal{K}$. A convenient presentation of $C^*(\mathbb{N}, \Theta)$ that will be used below comes from the fact that

$$C^*(\mathbb{N}^{N+1}, \Theta) \cong (\ldots((\mathcal{T} \rtimes_{\alpha_1} \mathbb{N}) \rtimes \mathbb{N}) \ldots) \rtimes_{\alpha_N} \mathbb{N},$$

where the actions α_k are determined by the cocycle Θ. While there exists a considerable theory of semigroup C^*-algebras, we do not need to use it below. Instead, we will reduce the computation to the one done in the untwisted case.

Let $\mu = (\mu_k, \ldots, \mu_l) \in \mathbb{N}^{l+1-k}$ be a multi-index, and let $\{e_\mu\}_\mu$ be the standard orthonormal basis of $l^2(\mathbb{N}^{l+1-k})$. For $k \leq i \leq l$, let $\delta_i := (0, \ldots, 1, \ldots 0) \in \mathbb{N}^{l+1-k}$ with 1 in the slot labeled by i. Define

$$\pi_{(k,l)}(s_i) e_\mu := \prod_{k \leq i < j \leq l} \Theta_{ij}^{\mu_j} e_{\mu + \delta_i}.$$

Lemma 4.7. *Let $k \in \{1, \ldots, N\}$. In the decomposition*

$$l^2(\mathbb{N}^{N+1}) = l^2(\mathbb{N}^k) \otimes l^2(\mathbb{N}^{N+1-k}),$$

where the second factor corresponds to the last $N+1-k$ components in $\mathbb{N}^{N+1}$, the following equalities hold:

$$\pi_{(0,N)}(\mathcal{T}_{0,k-1}\mathcal{K}_{(k,N)}) = \pi_{(0,k-1)}(\mathcal{T}_{0,k-1}) \otimes \mathcal{K}_{(k,N)},$$
$$\pi_{0,N}(\mathcal{K}_{(0,N)}) = \pi_{(0,k-1)}(\mathcal{K}_{(0,k-1)}) \otimes \mathcal{K}_{(k,N)}.$$

Proof. By construction, for $i < k$,

$$\pi_{(0,N)}(s_i) \in \pi_{(0,k-1)}(\mathcal{T}_{0,k-1}) \otimes_{\min} B(l^2(\mathbb{N}^{N+1-k})),$$
$$\pi_{(0,N)}(\mathcal{K}_{(k,N)}) \subset \pi_{(0,k-1)}(\mathcal{T}_{0,k-1}) \otimes_{\min} \mathcal{K}(l^2(\mathbb{N}^{N+1-k})).$$

Now the claim of the lemma follows. $\square$

Corollary 4.8. *Let $k \in \{1, \ldots, N-1\}$. Put $C(S^{2k-1}_{H,\Theta_{0j}}) := \mathcal{T}_{0,j}/\mathcal{K}_{(0,j)}$. There exists a U(1)-equivariant short exact sequence of C^*-algebras:*

$$0 \longrightarrow C(S^{2k-3}_{H,\Theta_{0\,(k-1)}}) \otimes \mathcal{K}_{(k,N)} \longrightarrow C(S^{2k-1}_{H,\Theta_{0k}}) \otimes \mathcal{K}_{(k+1,N)}$$
$$\longrightarrow (\mathcal{T}_{0,k-1} \rtimes_{\alpha_k} \mathbb{Z}) \otimes \mathcal{K}_{(k+1,N)} \longrightarrow 0.$$

The action of U(1) is the one induced naturally from its diagonal action on $\mathcal{T}_{0,N}$.

Proof. Lemma 4.7 reduces the claim to the identity

$$\begin{aligned}\pi_{(0,k)}(\mathcal{T}_{0,k})/\pi_{(0,k)}(\mathcal{T}_{0,k-1}\mathcal{K}_{0,k})\\ = \pi_{(0,k)}(\mathcal{T}_{0,k-1}\mathcal{T}_{k,k})/\pi_{(0,k)}(\mathcal{T}_{0,k-1}\mathcal{K}_{(k,k)}) \cong \mathcal{T}_{0,k-1} \rtimes_{\alpha_k} \mathbb{Z},\end{aligned}$$

which immediately follows from the construction of $\pi_{(0,k)}$. $\square$

Proof of Theorem 0.1(2). For $0 \le k \le N$, let $T_k := (C(S^{2k+1}_{\Theta_{0k}}) \otimes \mathcal{K}_{(k+1,N)})^{U(1)}$. Since the crossed product $\mathcal{T}_{0,k-1} \rtimes_{\alpha_k} \mathbb{Z}$ contains the regular representation of $\mathbb{Z}$, and hence a copy of the regular representation of U(1) on $C^*(\mathbb{Z}) = C(S^1)$, we get, as in the untwisted case,

$$\big((\mathcal{T}_{0,k-1} \rtimes_{\alpha_k} \mathbb{Z}) \otimes \mathcal{K}_{(k+1,N)}\big)^{U(1)} \cong \mathcal{T}_{0,k-1} \otimes \mathcal{K}_{(k+1,N)}.$$

Finally, as $T_N \cong C(\mathbb{P}^N_\theta(\mathcal{T}))$ by Theorem 2.3, the rest of the argument is the same as in the untwisted case, with T_k in place of S_k. $\square$

5. Noncommutative line bundles over multipullback quantum complex projective spaces

5.1. **Equivariant homomorphisms and spectral subspaces.** Take a $U(1)$-equivariant *-homomorphism $f\colon A \to A'$ of unital $U(1)$-C^*-algebras, and suppose that the $U(1)$-action on A is free. Then there exists a strong connection ℓ on A. It is straightforward to check that $\ell' := (f \otimes f) \circ \ell$ is a strong connection on A', so that the $U(1)$-action on A' is also free. The $U(1)$-equivariance of f guarantees that its restriction to the fixed-point subalgebra $B := A^{U(1)}$ corestricts to the fixed-point subalgebra $B' := (A')^{U(1)}$. This f turns B' into a B'–B bimodule given by the usual multiplication on the left and the formula $b' \cdot b := b' f(b)$ on the right.

Since $f : A \to f(A)$ is a linear surjection over a field, it splits. So there exists a linear map $g\colon f(A) \to A$ such that $f \circ g = \mathrm{id}_{f(A)}$. We have $A = g(f(A)) \oplus \ker f$. Let $\{a'_j\}_j$ be an extension of a basis $\{e'_i\}_i$ of $f(A)$ to a basis of A'. Also, let $\{e_k\}_k$ be a basis of $\ker f$. Then $\{a_l\}_l := \{g(e'_i)\}_i \cup \{e_k\}_k$ is a basis of A, and $f(a_l) = a'_l$ or $f(a_l) = 0$. For any $n \in \mathbb{Z}$, we can write $\ell(u^n) = \sum_{l\in L} a_l \otimes r_l(u^n)$ and $\ell'(u^n) = \sum_{l\in L'} a'_l \otimes f(r_l(u^n))$. Here L' and L are respectively m' and m element sets, with $m' \leq m$, and $f(a_l) = a'_l$ for $l \leq m'$ and $f(a_l) = 0$ for $l > m'$.

It follows from the Chern-Galois theory of [4] that the existence of a strong connection guarantees that spectral subspaces are finitely generated projective as left modules over fixed-point C^*-algebras. Given a strong connection ℓ and a spectral subspace A_n, we have an explicit formula given in [4, Theorem 3.1] for an idempotent E^n representing the spectral subspace: $E^n_{kl} := r_k(u^n)a_l$. Hence $f(E^n_{kl}) = f(r_k(u^n))a'_l$ for $l \leq m'$ and $f(E^n_{kl}) = 0$ for $l > m'$ are the matrix coefficients of an idempotent representing $B' \otimes_B A_n$. Using the strong connection ℓ' and the linear basis $\{a'_l\}_l$, we conclude that the matrix coefficients of an idempotent representing A'_n are also $f(r_k(u^n))a'_l$, but with indices $k, l \in L'$.

To continue this reasoning and to take care of the range of indices, it is convenient to adopt the block-matrix notation. Let $\beta_n := (r_1(u^n), \ldots, r_m(u^n))$ and $\gamma := (a_1, \ldots, a_m)$. Much in the same way, let $\beta'_n := (f(r_1(u^n)), \ldots, f(r_{m'}(u^n)))$ and $\gamma' := (a'_1, \ldots, a'_{m'})$. Then $E^n = {\beta_n}^T\gamma \in M_m(B)$ is an idempotent matrix representing A_n, and $(E')^n = {\beta'_n}^T\gamma' \in M_{m'}(B')$ is an idempotent matrix representing A'_n. Finally, put

$$\beta''_n := (f(r_1(u^n)), \ldots, f(r_m(u^n)) =: (\beta'_n, \rho'_n) \qquad \text{and}$$
$$\gamma'' := (\gamma', 0, \ldots, 0) \qquad \text{(with } m - m' \text{ zeros at the end).}$$

Then $(E'')^n = {\beta''_n}^T\gamma'' \in M_m(B')$ is an idempotent matrix representing $B' \otimes_B A_n$.

The crux of our argument is that $(E')^n$ and $(E'')^n$ represent *isomorphic* left B'-modules. After extending $(E')^n$ by zeros to size m, we obtain a matrix conjugate[2] to $(E'')^n$:

$$\begin{pmatrix} 1 & 0 \\ -{\rho'_n}^T\gamma' & 1 \end{pmatrix} \begin{pmatrix} {\beta'_n}^T \\ {\rho'_n}^T \end{pmatrix} \begin{pmatrix} \gamma' & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ {\rho'_n}^T\gamma' & 1 \end{pmatrix} = \begin{pmatrix} {\beta'_n}^T \\ 0 \end{pmatrix} \begin{pmatrix} \gamma' & 0 \end{pmatrix}.$$

Here we used the fact that $\gamma'{\beta'_n}^T = 1$, which is condition (2.9) for the strong connection ℓ'. Following the reasoning of the previous paragraph we have arrived at:

Theorem 5.1. *Let $f\colon A \to A'$ be a $U(1)$-equivariant *-homomorphism of unital $U(1)$-C^*-algebras, and let B and B' be the respective fixed-point C^*-subalgebras.*

[2] We are grateful to Tomasz Maszczyk for pointing this out to us.

Assume that the $U(1)$-action on A is free. For each $n \in \mathbb{Z}$, let A_n and A'_n denote the n^{th} spectral subspaces of A and A' respectively. Then, for any $n \in \mathbb{Z}$, there is an isomorphism of finitely generated left B'-modules:

$$B' \underset{B}{\otimes} A_n \cong A'_n.$$

In particular, the induced map $(f|_B)_* : K_0(B) \to K_0(B')$ satisfies

$$(f|_B)_*\big([A_n]\big) = [A'_n] \quad \textit{for every } n \in \mathbb{Z}.$$

5.2. **Pairwise non-isomorphism.** The goal of this section is to prove Theorem 0.1(3), i.e. to show that the line bundles over the multipullback quantum complex projective space $\mathbb{P}^N(\mathcal{T})$ associated to the Heegaard odd quantum sphere S_H^{2N+1} are classified by their defining winding number. We will do it reducing the problem to the special case $N = 1$, which was already solved elsewhere. Here the main problem is that we do not have any $U(1)$-equivariant maps from $C(S_H^{2N+1})$ to $C(S_H^3)$. We overcome this difficulty by finding a wrong-way equivariant map that restricted to fixed-point subalgebras induces an isomorphism on the K-groups.

To begin with, we need to unravel the pullback structure of $C(S_H^{2N+1})$:

Lemma 5.2. *For any $N \in \mathbb{N}$, $N > 0$, the $U(1)$-C^*-algebra $C(S_H^{2N+1})$ can be presented as the following equivariant pullback:*

$$\begin{tikzcd}
& C(S_H^{2N+1}) \arrow[dl, "\mathrm{pr}_1^N"'] \arrow[dr, "\mathrm{pr}_2^N"] & \\
C(S_H^{2N-1}) \otimes \mathcal{T} \arrow[dr, "\mathrm{id}\otimes\sigma"'] & & \mathcal{T}^{\otimes N} \otimes C(S^1) \arrow[dl, "\boldsymbol{\sigma}\otimes\mathrm{id}"] \\
& C(S_H^{2N-1}) \otimes C(S^1)\,. &
\end{tikzcd}$$

*Here $\boldsymbol{\sigma}\colon \mathcal{T}^{\otimes N} \ni w \mapsto (\sigma_0(w), \ldots, \sigma_{N-1}(w)) \in C(S_H^{2N-1})$, and σ_i is defined by (4.3), which is the $\theta = 0$ case of (2.1). The defining *-homomorphisms are equivariant with respect to the diagonal action.*

Proof. We adopt the definitions from (4.3) and (4.4), but now we have to play with different N at the same time, whence the need for additional labeling:

$$\begin{aligned} B_i^N &:= \mathcal{T}^{\otimes i} \otimes C(S^1) \otimes \mathcal{T}^{\otimes (N-i)} \\ \pi_j^{i,N} &:= \mathrm{id}^j \otimes \sigma \otimes \mathrm{id}^{N-j}. \end{aligned}$$

Then, the definition of $C(S_H^{2N+1})$ becomes:

$$C(S_H^{2N+1}) := \Big\{(b_0, \ldots, b_N) \in \textstyle\bigoplus_{i=0}^N B_i^N \;\big|\; \forall\, 0 \le i < j \le N :\ \pi_j^{i,N}(b_i) = \pi_i^{j,N}(b_j)\Big\}.$$

Denoting $\forall\ 0 \le i < j \le M \le N:\ \pi_j^{i,N}(b_i) = \pi_i^{j,N}(b_j)$ by $P_M^N((b_i)_i)$, we can rewrite this formula as

$$C(S_H^{2N+1}) = \Big\{((b_i)_i, b_N) \in (\textstyle\bigoplus_{i=0}^{N-1} B_i^N) \oplus (\mathcal{T}^{\otimes N} \otimes C(S^1))\ \big| \\ P_{N-1}^N((b_i)_i) \wedge \big(\forall\, 0 \le i \le N-1:\ \pi_N^{i,N}(b_i) = (\sigma_i \otimes \mathrm{id})(b_N)\big)\Big\}. \tag{5.1}$$

Next, using the exactness of the tensor product $\_ \otimes \mathcal{T}$ (which follows from nuclearity of $\mathcal{T}$), we can write

$$\begin{aligned} C(S_H^{2N-1}) \otimes \mathcal{T} &= \Big\{(\tilde{b}_i)_i \in \textstyle\bigoplus_{i=0}^{N-1} B_i^{N-1} \mid P_{N-1}^{N-1}((\tilde{b}_i)_i)\Big\} \otimes \mathcal{T} \\ &= \Big\{(b_i)_i \in \textstyle\bigoplus_{i=0}^{N-1} B_i^{N} \mid P_{N-1}^{N}((b_i)_i)\Big\}. \end{aligned} \tag{5.2}$$

Combing (5.1) with (5.2), we arrive at:

$$C(S_H^{2N+1}) = \big\{((b_i)_i, b_N) \in \big(C(S_H^{2N-1}) \otimes \mathcal{T}\big) \oplus \big(\mathcal{T}^{\otimes N} \otimes C(S^1)\big) \mid \\ \forall\, 0 \le i \le N-1:\ \pi_N^{i,N}(b_i) = (\sigma_i \otimes \mathrm{id})(b_N)\big\}.$$

Finally, we obtain

$$C(S_H^{2N+1}) \\ = \big\{(x,y) \in \big(C(S_H^{2N-1}) \otimes \mathcal{T}\big) \oplus \big(\mathcal{T}^{\otimes N} \otimes C(S^1)\big) \mid (\mathrm{id} \otimes \sigma)(x) = (\boldsymbol{\sigma} \otimes \mathrm{id})(y)\big\},$$

which proves the lemma. □

The next step is to establish a wrong-way map with the right-way inverse in K-theory:

Lemma 5.3. *Consider $C(S_H^3) \otimes \mathcal{T}^{\otimes(N-1)}$ with the diagonal $U(1)$-action. Then*

$$\eta\colon C(S_H^3) \ni x \longmapsto x \otimes 1 \in C(S_H^3) \otimes \mathcal{T}^{\otimes(N-1)}$$

*is a $U(1)$-equivariant *-homomorphism whose restriction-corestriction $\bar{\eta}$ to the $U(1)$-invariant subalgebras induces an isomorphism of K-groups:*

$$\bar{\eta}_* : K_*\big(C(\mathbb{P}^1(\mathcal{T}))\big) \longrightarrow K_*\Big(\big(C(S_H^3) \otimes \mathcal{T}^{\otimes(N-1)}\big)^{U(1)}\Big).$$

Proof. The pullback presentation of $C(S_H^3)$ together with the exactness of tensoring with $\mathcal{T}^{\otimes(N-1)}$ yields two $U(1)$-equivariant pullback diagrams. We combine them in the following commutative diagram of $U(1)$-equivariant *-homomorphisms

(all considered with the diagonal $U(1)$-action):
(5.3)

$$
\begin{tikzcd}
 & C(S^3_H) \arrow[dl] \arrow[dr] \arrow[rrrrr, dashed, bend left, "\eta"] & & & & & C(S^3_H)\otimes\mathcal{T}^{\otimes(N-1)} \arrow[dl] \arrow[dr] & \\
\mathcal{T}\otimes C(S^1) \arrow[dr, "\sigma\otimes\mathrm{id}"'] \arrow[rrrrr, bend left, "\mathrm{id}\otimes 1"] & & C(S^1)\otimes\mathcal{T} \arrow[dl, "\mathrm{id}\otimes\sigma"] \arrow[rrrrr, bend right, "\mathrm{id}\otimes 1"'] & & & \mathcal{T}\otimes C(S^1)\otimes\mathcal{T}^{\otimes(N-1)} \arrow[dr, "\sigma\otimes\mathrm{id}"'] & & C(S^1)\otimes\mathcal{T}^{\otimes N} \arrow[dl, "\mathrm{id}\otimes\sigma\otimes\mathrm{id}"] \\
 & C(S^1)\otimes C(S^1) \arrow[rrrrr, bend right, "\mathrm{id}\otimes 1"'] & & & & & C(S^1)\otimes C(S^1)\otimes\mathcal{T}^{\otimes(N-1)} &
\end{tikzcd}
\quad .
$$

Using the gauge isomorphisms (1.2) together with some permutations of tensor factors, we transform the diagonal action (on the pullback components) to the action on the rightmost factor thus obtaining the following diagram:
(5.4)

$$
\begin{tikzcd}
 & C(S^3_H)^R \arrow[dl] \arrow[dr] \arrow[rrrrr, dashed, bend left, "\eta^R"] & & & & & (C(S^3_H)\otimes\mathcal{T}^{\otimes(N-1)})^R \arrow[dl] \arrow[dr] & \\
\mathcal{T}\otimes C(S^1) \arrow[dr, "\sigma\otimes\mathrm{id}"'] \arrow[rrrrr, bend left, "\mathrm{id}\otimes 1\otimes\mathrm{id}"] & & \mathcal{T}\otimes C(S^1) \arrow[dl, "\phi"] \arrow[rrrrr, bend right, "\mathrm{id}\otimes 1\otimes\mathrm{id}"'] & & & \mathcal{T}^{\otimes N}\otimes C(S^1) \arrow[dr, "\sigma\otimes\mathrm{id}"'] & & \mathcal{T}^{\otimes N}\otimes C(S^1) \arrow[dl, "\psi"] \\
 & C(S^1)\otimes C(S^1) \arrow[rrrrr, bend right, "\mathrm{id}\otimes 1\otimes\mathrm{id}"'] & & & & & C(S^1)\otimes\mathcal{T}^{\otimes(N-1)}\otimes C(S^1) &
\end{tikzcd}
\quad .
$$

Here the top line is $U(1)$-equivariantly isomorphic to the top line of the previous diagram, and ϕ and ψ are given by

$$
\begin{aligned}
\phi : \mathcal{T}\otimes C(S^1) &\longrightarrow C(S^1)\otimes C(S^1),\\
\phi : t\otimes u &\longmapsto u_{(1)}S(\sigma(t))\otimes u_{(2)}\,,\\
\psi : \mathcal{T}\otimes\mathcal{T}^{\otimes(N-1)}\otimes C(S^1) &\longrightarrow C(S^1)\otimes\mathcal{T}^{\otimes(N-1)}\otimes C(S^1),\\
\psi : t\otimes\bar{r}\otimes u &\longmapsto S(\sigma(t)\bar{r}_{(1)})u_{(1)}\otimes\bar{r}_{(0)}\otimes u_{(2)}\,.
\end{aligned}
$$

Finally, to pass to the restriction-corestriction of Diagram 5.3 to the $U(1)$-invariant subalgebras, it suffices to note that it is isomorphic the restriction-corestriction of Diagram 5.4, and that the latter is obtained by removing the rightmost factors from the pullback components:

$$
\begin{tikzcd}
 & C(\mathbb{P}^1(\mathcal{T})) \arrow[dl] \arrow[dr] \arrow[rrr, dashed, "\bar{\eta}"] & & & \Big(\big(C(S^3_H)\otimes\mathcal{T}^{\otimes(N-1)}\big)^R\Big)^{U(1)} \arrow[dl] \arrow[dr] & \\
\mathcal{T} \arrow[dr, "\sigma"'] \arrow[rrr, bend left, "\mathrm{id}\otimes 1"] & & \mathcal{T} \arrow[dl, "S\circ\sigma"] \arrow[rrr, bend right, "\mathrm{id}\otimes 1"'] & \mathcal{T}^{\otimes N} \arrow[dr, "\sigma\otimes\mathrm{id}"'] & & \mathcal{T}^{\otimes N} \arrow[dl, "\tilde{\psi}"] \\
 & C(S^1) \arrow[rrr, bend right, "\mathrm{id}\otimes 1"'] & & & C(S^1)\otimes\mathcal{T}^{\otimes(N-1)} & .
\end{tikzcd}
$$

Here

$$\tilde{\psi} : \mathcal{T}^{\otimes N} \ni t \otimes \bar{r} \longmapsto S(\sigma(t)\bar{r}_{(1)}) \otimes \bar{r}_{(0)} \in C(S^1) \otimes \mathcal{T}^{\otimes(N-1)}.$$

Due to the naturality of the Künneth formula, all three maps $\mathrm{id}\otimes 1_{\mathcal{T}^{\otimes(N-1)}}$ between the pullback components induce isomorphisms on K-groups. Hence, it follows from [12, Theorem 3.1] that also $\bar{\eta}$ induces an isomorphism on K-groups. $\square$

Proof of Theorem 0.1(3). Lemma 5.2 implies that

$$f := (\mathrm{pr}^2_1 \otimes \mathrm{id}_{\mathcal{T}^{\otimes(N-1)}}) \circ (\mathrm{pr}^3_1 \otimes \mathrm{id}_{\mathcal{T}^{\otimes(N-2)}}) \circ \cdots \circ \mathrm{pr}^N_1$$

is a surjective $U(1)$-equivariant *-homomorphism

$$f : C(S^{2N+1}_H) \longrightarrow C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)}.$$

Furthermore, by Lemma 5.3 we have a $U(1)$-equivariant *-homomorphism

$$\eta : C(S^3_H) \longrightarrow C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)},$$

which induces an isomorphism on K-groups.

Next, the freeness of the diagonal $U(1)$-action on $C(S^{2N+1}_H)$, which follows from Section 2.3.1 for $\theta = 0$, allows us to apply the final statement of Theorem 5.1 to infer that the equality of K_0-classes $[C(S^{2N+1}_H)_m] = [C(S^{2N+1}_H)_n]$ implies the equality of K_0-classes

$$
\begin{aligned}
(5.5)\quad \big[(C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)})_m\big] &= \bar{f}_*([(S^{2N+1}_H)_m]) \\
&= \bar{f}_*([(S^{2N+1}_H)_n]) = \big[(C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)})_n\big].
\end{aligned}
$$

Here by $\bar{f}$ we denoted the restriction-corestriction of f to $U(1)$-invariant subalgebras. Much in the same way, identifying the isomorphic C^*-algebras

$$\Big(\big(C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)}\big)^R\Big)^{U(1)} \cong \big(C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)}\big)^{U(1)},$$

we conclude that

$$\big[(C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)})_m\big] = \bar{\eta}_*\big([C(S^3_H)_m]\big), \tag{5.6}$$

$$\big[(C(S^3_H) \otimes \mathcal{T}^{\otimes(N-1)})_n\big] = \bar{\eta}_*\big([C(S^3_H)_n]\big). \tag{5.7}$$

Now, it follows from (5.5)–(5.7) and Lemma 5.3 that $[C(S^3_H)_m] = [C(S^3_H)_n]$. Finally, by an index-pairing calculation[16, Theorem 3.3], we obtain $m = n$. $\square$

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References

[1] Baum, P.F.; De Commer, K.; Hajac, P.M.: *Free actions of compact quantum groups on unital C^*-algebras*, Documenta Math. 22 (2017), 825–849.

[2] Baum, P.F.; Hajac, P.M. Matthes, R.; Szymański, W.: *The K-theory of Heegaard-type quantum 3-spheres*, K-Theory 35 (2005), 159–186.

[3] Birkhoff, G.: *Lattice Theory*, vol. XXV. American Mathematical Society, Providence, R.I., third edition, 1967.

[4] Brzeziński, T.; Hajac, P.M.: *The Chern-Galois character*, C.R. Acad. Sci. Paris **338** (2004), 113–116.

[5] Calow, D.; Matthes, R.: *Covering and gluing of algebras and differential algebras*, J. Geom. Phys. **32** (2000), 364–396.

[6] Coburn, L.A.: *The C^*-algebra of an isometry*, Bull. Amer. Math. Soc. **73** (1967), 722–726.

[7] Coburn, L.A.: *The C^*-algebra of an isometry II*, Trans. Amer. Math. Soc. **137** (1969), 211–217.

[8] Connes, A.: *Non-commutative differential geometry*, Inst. Hautes Études Sci. Publ. Math. **62** (1985), 257–360.

[9] D'Andrea, F.; Giovanni L.: *Bounded and unbounded Fredholm modules for quantum projective spaces*, J. K-Theory **6** (2010), 231–240.

[10] De Commer, K.; Yamashita, M.: *A construction of finite index C^*-algebra inclusions from free actions of compact quantum groups*. Publ. Res. Inst. Math. Sci. **49** (2013), 709–735.

[11] Ellwood, D.A.: *A new characterisation of principal actions*, J. Funct. Anal. **173** (2000), 49–60.

[12] Farsi, C.; Hajac, P.M.; Maszczyk, T.; Zieliński B.; *Rank-two Milnor idempotents for the multipullback quantum complex projective plane*, arXiv:1708.04426.

[13] Hajac, P.M.: *Strong connections on quantum principal bundles*, Comm. Math. Phys, **182** (1996), 579–617.

[14] Hajac, P.M.: *Bundles over quantum sphere and noncommutative index theorem*, K-theory **21** (2000): 141–150.

[15] Hajac, P.M.; Kaygun, A.; Zieliński, B.: *Quantum complex projective spaces from Toeplitz cubes*, J. Noncomm. Geom. **6** (2012), 603–621.

[16] Hajac, P.M.; Matthes, R.; Szymański, W.: *Chern numbers for two families of noncommutative Hopf fibrations*, C. R. Math. Acad. Sci. Paris **336** (2003), 925–930.

[17] Hajac, P.M.; Matthes, R.; Szymański, W.: *A locally trivial quantum Hopf fibration*, Algebr. Represent. Theory **9** (2006), 121–146.

[18] Hajac, P.M.; Matthes, R.; Szymański, W.: *Noncommutative index theory for mirror quantum spheres*, C. R. Math. Acad. Sci. Paris **343** (2006), 731–736.

[19] Hajac, P.M.; Rudnik, J.: *Noncommutative bundles over the multi-pullback quantum complex projective plane*, arXiv:1512.08772.

[20] Hajac, P.M.; Zieliński, B.: *The cocycle condition for multipullbacks of algebras*, Banach Center Publ. **98** (2012), 239–243.

[21] Heegaard, P.: *Forstudier til en topologisk Teori for de algebraiske Fladers Sammenhang*, Thesis (in Danish) 1898.

[22] Klimek, S.; Leśniewski, A.: *A two-parameter quantum deformation of the unit disc*, J. Funct. Anal. **115** (1993), 1–23.

[23] Kumjian, A.; Pask D.: *Higher rank graph C^*-algebras*, New York J. Math. **6** (2000), 1–20.

[24] Meyer, U.: *Projective quantum spaces*, Lett. Math. Phys., **35** 91–97, 1995.

[25] Pedersen, G. K.: *Pullback and pushout constructions in C^*-algebra theory*, J. Funct. Anal. **267** (1999), 243–344.

[26] Rieffel, M.A.: *K-groups of C^*-algebras deformed by actions of R^d*, J. Funct. Anal. **116** (1993), 199–214.

[27] Sheu, A. J.-L.: *Projective modules over quantum projective line*, Internat. J. Math. **28** (2017), 1750022, 14 pp.

[28] Sims, A.; Whitehead, A.; Whittaker, M.F. *Twisted C^*-algebras associated to finitely aligned higher-rank graphs*, Doc. Math. **19** (2014), 831–866.

[29] Szymański, W.: *Quantum lens spaces and principal actions on graph C^*-algebras*, Banach Center Publ. **61** (2003), 299–304.

[30] Ulbrich, K.H.: *Vollgraduierte Algebren*, Abh. Math. Sem. Univ. Hamburg **51** (1981), 136–148.

[31] Vaksman, L.L.; Soibel'man, Ya.S.: *Algebra of functions on the quantum group SU(n+1), and odd-dimensional quantum spheres,* (Russian) Algebra i Analiz **2** (1990), 101–120; translation in Leningrad Math. J. **2** (1991), 1023–1042.

[32] Wegge-Olsen, N.E.: *K-theory and C^*-algebras, A friendly approach*, The Clarendon Press Oxford University Press, New York, 1993.

[33] Whitehead, B.: *Twisted relative Cuntz-Krieger algebras associated to finitely aligned higher-rank graphs*, honours thesis, University of Wollongong, 2012. (arXiv:1310.7045).

[34] Woronowicz, S.L.: *Twisted SU(2) group. An example of a noncommutative differential calculus*, Publ. Res. Inst. Math. Sci. **23** (1987), 117–181.

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