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# THE EQUIVARIANT BRAUER GROUPS OF COMMUTING FREE AND PROPER ACTIONS ARE ISOMORPHIC

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Abstract. If $X$ is a locally compact space which admits commuting free and proper actions of locally compact groups $G$ and $H$, then the Brauer groups $\mathrm{Br}_H(G\backslash X)$ and $\mathrm{Br}_G(X/H)$ are naturally isomorphic.

Rieffel's formulation of Mackey's Imprimitivity Theorem asserts that if $H$ is a closed subgroup of a locally compact group $G$, then the group $C^*$-algebra $C^*(H)$ is Morita equivalent to the crossed product $C_0(G/H) \rtimes G$. Subsequently, Rieffel found a symmetric version, involving two subgroups of $G$, and Green proved the following *Symmetric Imprimitivity Theorem*: *If two locally compact groups act freely and properly on a locally compact space $X$, $G$ on the left and $H$ on the right, then the crossed products $C_0(G\backslash X) \rtimes H$ and $C_0(X/H) \rtimes G$ are Morita equivalent.* (For a discussion and proofs of these results, see [15].) Here we shall show that in this situation there is an isomorphism $\mathrm{Br}_H(G\backslash X) \cong \mathrm{Br}_G(X/H)$ of the equivariant Brauer groups introduced in [2].

Suppose $(G, X)$ is a second countable locally compact transformation group. The objects in the underlying set $\mathfrak{Br}_G(X)$ of the equivariant Brauer group $\mathrm{Br}_G(X)$ are dynamical systems $(A, G, \alpha)$, in which $A$ is a separable continuous-trace $C^*$-algebra with spectrum $X$, and $\alpha\colon G \to \mathrm{Aut}(A)$ is a strongly continuous action of $G$ on $A$ inducing the given action of $G$ on $X$. The equivalence relation on such systems is the equivariant Morita equivalence studied in [1], [3]. The group operation is given by $[A, \alpha] \cdot [B, \beta] = [A \otimes_{C(X)} B, \alpha \otimes \beta]$, the inverse of $[A, \alpha]$ is the conjugate system $[\overline{A}, \overline{\alpha}]$, and the identity is represented by $(C_0(X), \tau)$, where $\tau_s(f)(x) = f(s^{-1} \cdot x)$.

*Notation.* Suppose that $H$ is a locally compact group, that $X$ is a free and proper right $H$-space, and that $(B, H, \beta)$ is a dynamical system. Then $\mathrm{Ind}_H^X(B, \beta)$ will be the $C^*$-algebra (denoted by $GC(X, B)^\alpha$ in [13] and by $\mathrm{Ind}(B; X, H, \beta)$ in [11]) of bounded continuous functions $f\colon X \to B$ such that $\beta_h(f(x \cdot h)) = f(x)$, and $x \cdot H \mapsto \|f(x)\|$ belongs to $C_0(X/H)$.

We now state our main theorem.

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**Theorem 1.** *Let $X$ be a second countable locally compact Hausdorff space, and let $G$ and $H$ be second countable locally compact groups. Suppose that $X$ admits a free and proper left $G$-action, and a free and proper right $H$-action such that $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $x \in X$, $g \in G$, and $h \in H$. Then there is an isomorphism $\Theta$ of $\mathrm{Br}_H(G\backslash X)$ onto $\mathrm{Br}_G(X/H)$ satisfying*:

(1) *if $(A, \alpha)$ represents $\Theta[B, \beta]$, then $A \rtimes_\alpha G$ is Morita equivalent to $B \rtimes_\beta H$;*

(2) *$\Theta[B, \beta]$ is realised by the pair $(\mathrm{Ind}_H^X(B, \beta)/J, \tau \otimes \mathrm{id})$ in $\mathfrak{Br}_G(X/H)$, where $\tau \otimes \mathrm{id}$ denotes left translation and, if $\pi_{G\cdot x}$ is the element of $\widehat{B} = G\backslash X$ corresponding to $G \cdot x$,*

$$J = \{f \in \mathrm{Ind}_H^X(B, \beta)\colon \pi_{G\cdot x}(f(x)) = 0 \textit{ for all } x \in X\}.$$

Item (1) is itself a generalization of Green's symmetric imprimitivity theorem, and our proof of Theorem 1 follows the approach to Green's theorem taken in [3]: prove that both $C_0(G\backslash X) \rtimes H$ and $C_0(X/H) \rtimes G$ are Morita equivalent to $C_0(X) \rtimes_\alpha (G \times H)$, where $\alpha_{s,h}(f)(x) = f(s^{-1} \cdot x \cdot h)$, by noting that the Morita equivalences of $C_0(X) \rtimes G$ with $C_0(G\backslash X)$ and $C_0(X) \rtimes H$ with $C_0(X/H)$ ([7], [15, Situation 10]) are equivariant, and hence induce Morita equivalences

$$\begin{aligned} C_0(G\backslash X) \rtimes H &\sim (C_0(X) \rtimes G) \rtimes H \cong C_0(X) \rtimes (G \times H) \\ &\cong (C_0(X) \rtimes H) \rtimes G \sim C_0(X/H) \rtimes G. \end{aligned}$$

The same symmetry considerations show that it will be enough to prove that $\mathrm{Br}_H(G\backslash X) \cong \mathrm{Br}_{G\times H}(X)$. Since we already know that $\mathrm{Br}(G\backslash X) \cong \mathrm{Br}_G(X)$ [2, §6.2], we just have to check that this isomorphism is compatible with the actions of $H$.

Suppose $G$ acts freely and properly on $X$, and $p\colon X \to G\backslash X$ is the orbit map. If $B$ is a $C^*$-algebra with a nondegenerate action of $C_0(G\backslash X)$, then the pull-back $p^*B$ is the quotient of $C_0(X) \otimes B$ by the balancing ideal

$$I_{G\backslash X} = \overline{\mathrm{span}}\{f \cdot \phi \otimes b - \phi \otimes f \cdot b\colon \phi \in C_0(X), f \in C_0(G\backslash X), b \in B\};$$

in other words, $p^*B = C_0(X) \otimes_{C(G\backslash X)} B$. The nondegenerate action of $C_0(G\backslash X)$ on $B$ induces a continuous map $q$ of $\widehat{B}$ onto $G\backslash X$, characterized by $\pi(f \cdot b) = f(q(\pi))\pi(b)$. Then under the natural identification of $C_0(X) \otimes B$ with $C_0(X, B)$,

$$I_{G\backslash X} \cong \{f \in C_0(X, B)\colon \pi(f(x)) = 0 \text{ for all } x \in q(\pi)\},$$

so that $p^*B$ has spectrum

$$\widehat{p^*B} = \{(x, \pi) \in X \times \widehat{B}\colon G \cdot x = q(\pi)\}.$$

If $B$ is a continuous-trace algebra with spectrum $G\backslash X$, then $p^*B$ is a continuous-trace algebra with spectrum $X$.

The isomorphism $\Theta\colon \mathrm{Br}(G\backslash X) \cong \mathrm{Br}_G(X)$ is given by $\Theta[A] = [p^*A, \tau \otimes \mathrm{id}]$. To prove $\Theta$ is surjective in [2], we used [12, Theorem 1.1], which implies that if $(B, \beta) \in \mathfrak{Br}_G(X)$, then $B \rtimes_\beta G$ is a continuous-trace algebra with spectrum $G\backslash X$ such that $(B, \beta)$ is Morita equivalent to $(p^*(B \rtimes_\beta G), \tau \otimes \mathrm{id})$, and hence that $[B, \beta] = \Theta[B \rtimes_\beta G, \mathrm{id}]$. In obtaining the required equivariant version of [12, Theorem 1.1], we have both simplified the proof and mildly strengthened the conclusion (see Corollary 4 below). However, with all these different group actions around, the notation could get messy, and we pause to establish some conventions.

*Notation.* We shall be dealing with several spaces carrying a left action of $G$ and/or a right action of $H$. We denote by $\tau$ the action of $G$ by left translation on $C_0(G)$, $C_0(X)$ or $C_0(G\backslash X)$, and by $\sigma$ any action of $H$ by right translation; we shall also use $\sigma^G$ to denote the action of $G$ by right translation on $C_0(G)$. Restricting an action $\beta$ of $G \times H$ on an algebra $A$ gives actions $\alpha\colon G \to \mathrm{Aut}(A), \gamma\colon H \to \mathrm{Aut}(A)$ such that

$$\alpha_s(\gamma_h(a)) = \gamma_h(\alpha_s(a)) \quad \text{for all } h \in H, s \in G, a \in A. \tag{1}$$

Conversely, two actions $\alpha, \gamma$ satisfying (1) define an action of $G \times H$ on $A$, which we denote by $\alpha\gamma$; we write $\gamma$ for $\mathrm{id}\,\gamma$ since it will be clear from context whether an action of $H$ or $G \times H$ is called for. If $\Phi\colon (A, G, \alpha) \to (B, G, \beta)$ is an equivariant isomorphism (i.e. $\Phi(\alpha_s(a)) = \beta_s(\Phi(a))$), then we denote by $\Phi \rtimes \mathrm{id}$ the induced isomorphism of $A \rtimes_\alpha G$ onto $B \rtimes_\beta G$. Similarly, if $\alpha$ and $\gamma$ satisfy (1), we write $\alpha \rtimes \mathrm{id}$ for the induced action of $G$ on $A \rtimes_\gamma H$.

**Lemma 2.** *Suppose a locally compact group $G$ acts freely and properly on a locally compact space $X$, and that $A$ is a $C^*$-algebra carrying a non-degenerate action of $C_0(X)$. If $\alpha\colon G \to \mathrm{Aut}(A)$ is an action of $G$ on $A$ satisfying $\alpha_s(\phi\cdot a) = \tau_s(\phi)\cdot\alpha_s(a)$, then the map sending $f \otimes a$ in $C_0(X) \otimes A$ to the function $s \mapsto f \cdot \alpha_s^{-1}(a)$ induces an equivariant isomorphism $\Phi$ of $(C_0(X) \otimes_{C(G\backslash X)} A, G, \mathrm{id} \otimes \alpha)$ onto $(C_0(G, A), G, \tau \otimes \mathrm{id})$.*

*Remark* 3. For motivation, consider the case where $A = C_0(X)$. Then the map $\Psi\colon C_b(X \times X) \to C_b(G \times X)$ defined by $\Psi(f)(s, x) = f(x, s \cdot x)$ maps $C_0$ to $C_0$ precisely when the action is proper, has range which separates the points of $G \times Y$ precisely when the action is free, and has kernel consisting of the functions which vanish on the closed subset $\Delta = \{(x, y)\colon G \cdot x = G \cdot y\}$. Thus the free and proper actions are precisely those for which $\Psi$ induces an isomorphism of $C_0(X) \otimes_{C(G\backslash X)} C_0(X)$ onto $C_0(G) \otimes C_0(X)$.

*Proof of Lemma* 2. If $\phi \in C_0(G\backslash X)$, then $f \cdot \phi \otimes a$ and $f \otimes \phi \cdot a$ have the same image in $C_0(G, A)$, and the map factors through the balanced tensor product as claimed. Further, $\Phi$ is related to the map $\Psi$ in Remark 3 by

$$\Phi(f \otimes g \cdot a) = (\Psi(f \otimes g)(s, \cdot)) \cdot \alpha_s^{-1}(a). \tag{2}$$

Thus it follows from the remark that (2) defines an element of $C_0(G, A)$ and that the closure of the range of $\Phi$ contains all functions of the form $s \mapsto \xi(s) f \cdot \alpha_s^{-1}(a)$ for $\xi \in C_c(G)$, $f \in C_c(X)$, and $a \in A$. These elements span a dense subset of $C_0(G, A)$, and hence $\Phi$ is surjective. The nondegenerate action of $C_0(X)$ on $A$ induces a continuous equivariant map $q$ of $\widehat{A}$ onto $X$ such that $\pi(f \cdot a) = f(q(\pi))\pi(a)$, and the balanced tensor product $C_0(X) \otimes_{C(G\backslash X)} A$ has spectrum $\Delta = \{(x, \pi)\colon G \cdot x = G \cdot q(\pi)\}$. Since each representation $(q(\pi), s \cdot \pi) = (q(\pi), \pi \circ \alpha_s^{-1})$ in $\Delta$ factors through $\Phi$ and the representation $b \mapsto \pi(b(s))$ of $C_0(G, A)$, the homomorphism $\Phi$ is also injective. Finally, to see the equivariance, we compute:

$$\begin{aligned}\Phi(\mathrm{id} \otimes \alpha_s(h \otimes a))(t) &= h \cdot \alpha_t^{-1}(\alpha_s(a)) = \Phi(h \otimes a)(s^{-1}t) \\ &= \tau_s \otimes \mathrm{id}(\Phi(h \otimes a))(t). \quad \square\end{aligned}$$

**Corollary 4** (cf. [12, Theorem 1.1])**.** *Let $(G, X)$ and $\alpha\colon G \to \mathrm{Aut}(A)$ be as in Lemma 2. Then there is an equivariant isomorphism of $(p^*(A \rtimes_\alpha G), G, p^*\,\mathrm{id})$ onto $(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \mathrm{Ad}\,\rho)$.*

*Proof.* A routine calculation shows that the equivariant isomorphism $\Phi$ of Lemma 2 gives an equivariant isomorphism

$$\text{(3)}\qquad \Phi \rtimes \mathrm{id}\colon ((C_0(X) \otimes_{C(G\backslash X)} A) \rtimes_{\mathrm{id}\otimes\alpha} G, (\tau \otimes \mathrm{id}) \rtimes \mathrm{id}) \\ \to (C_0(G, A) \rtimes_{\tau\otimes\mathrm{id}} G, (\sigma^G \otimes \alpha) \rtimes \mathrm{id}).$$

We also have equivariant isomorphisms

$$\text{(4)}\qquad \begin{aligned}(C_0(G, A) \rtimes_{\tau\otimes\mathrm{id}} G, (\sigma^G \otimes \alpha) \rtimes \mathrm{id}) &\cong (A \otimes (C_0(G) \rtimes_\tau G), \alpha \otimes (\sigma^G \rtimes \mathrm{id})),\\ &\cong (A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \operatorname{Ad}\rho)\end{aligned}$$

and

(5)

$$(C_0(X) \otimes_{C(G\backslash X)} (A \rtimes_\alpha G), \tau \otimes \mathrm{id}) \cong ((C_0(X) \otimes_{C(G\backslash X)} A) \rtimes_{\mathrm{id}\otimes\alpha} G, (\tau \otimes \mathrm{id}) \rtimes \mathrm{id});$$

combining (3), (4), and (5) gives the result. □

**Lemma 5.** *In addition to the hypotheses of Lemma 2, suppose that $H$ is a locally compact group acting on the right of $X$, and that $(A, H, \gamma)$ is a dynamical system such that $\alpha$ and $\gamma$ commute and $\gamma_h(f \cdot a) = \sigma_h(f) \cdot \gamma_h(a)$ for $h \in H$, $f \in C_0(X)$, $a \in A$. Then the action $\tau\sigma \otimes \gamma$ of $G \times H$ on $C_0(X) \otimes A$ preserves the balancing ideal $I_{G\backslash X}$, and hence induces an action of $G \times H$ on $C_0(X) \otimes_{C(G\backslash X)} A$, also denoted $\tau\sigma \otimes \gamma$. The equivariant isomorphism of Lemma 2 induces an equivariant isomorphism*

$$((C_0(X) \otimes_{C(G\backslash X)} A) \rtimes_{\mathrm{id}\otimes\alpha} G, (\tau\sigma \otimes \gamma) \rtimes \mathrm{id}) \\ \cong (C_0(G, A) \rtimes_{\tau\otimes\mathrm{id}} G, (\sigma^G \otimes \alpha\gamma) \rtimes \mathrm{id}).$$

*Proof.* The first assertion is straightforward. For the second, we can consider the actions of $H$ and $G$ separately. We have already observed in (3) that $\Phi \rtimes \mathrm{id}$ intertwines the $G$-actions. On the other hand, if $h \in H$ and $t \in G$, then

$$\begin{aligned}\Phi(\sigma_h \otimes \gamma_h(f \otimes a))(t) &= \sigma_h(f) \cdot \alpha_t^{-1}(\gamma_h(a)) = \sigma_h(f) \cdot \gamma_h(\alpha_t^{-1}(a))\\ &= \gamma_h(\Phi(f \otimes a)(t)). \qquad \square\end{aligned}$$

**Corollary 6.** *Let ${}_GX_H$ and $\alpha\colon G \to \operatorname{Aut}(A)$, $\gamma\colon H \to \operatorname{Aut}(A)$ be as in the lemma. Denote by $p$ the orbit map of $X$ onto $G\backslash X$. Then there is an equivariant isomorphism*

$$(p^*(A \rtimes_\alpha G), G \times H, \tau\sigma \otimes (\gamma \rtimes \mathrm{id})) \cong (A \otimes \mathcal{K}(L^2(G)), G \times H, \alpha\gamma \otimes \operatorname{Ad}\rho).$$

*Proof.* Compose the isomorphism of Lemma 5 with (4) and (5). □

We are now ready to define our map of $\mathrm{Br}_H(G\backslash X)$ into $\mathrm{Br}_{G\times H}(X)$. Suppose $(B, \beta) \in \mathfrak{Br}_H(X)$. Then the action $\tau\sigma \otimes \beta$ of $G \times H$ preserves the balancing ideal $I_{G\backslash X}$: if $\phi \in C_0(G\backslash X)$, then

$$\begin{aligned}(\tau\sigma \otimes \beta)_{s,h}(f \cdot \phi \otimes b - f \otimes \phi \cdot b) &= \sigma_h(\tau_s(f \cdot \phi)) \otimes \beta_h(b) - \sigma_h(\tau_s(f)) \otimes \beta_h(\phi \cdot b)\\ &= \sigma_h(\tau_s(f)) \cdot \sigma_h(\phi) \otimes \beta_h(b) - \sigma_h(\tau_s(f)) \otimes \sigma_h(\phi) \cdot \beta_h(b).\end{aligned}$$

Since $p^*(B)$ is a continuous-trace $C^*$-algebra with spectrum $X$ [12, Lemma 1.2], and $\tau\sigma \otimes \beta$ covers the canonical $G \times H$-action on $X$, we can define $\theta\colon \mathfrak{Br}_H(G\backslash X) \to \mathfrak{Br}_{G\times H}(X)$ by $\theta(B, \beta) = (p^*(B), \tau\sigma \otimes \beta)$.

Similarly if $(A, \alpha\gamma) \in \mathfrak{Br}_{G\times H}(X)$, then $A \rtimes_\alpha G$ is a continuous-trace $C^*$-algebra with spectrum $G\backslash X$ by [12, Theorem 1.1]. Since $\gamma$ is compatible with $\sigma$, we have $\gamma_h(\phi \cdot z(s)) = \sigma_h(\phi) \cdot \gamma_h(z(s))$ for $z \in C_c(G, A)$, and hence $\gamma \rtimes \mathrm{id}$ covers the given action of $H$ on $X$. Thus we can define $\lambda\colon \mathfrak{Br}_{G\times H}(X) \to \mathfrak{Br}_H(G\backslash X)$ by $\lambda(A,\alpha\gamma) = (A \rtimes_\alpha G, \gamma \rtimes \mathrm{id})$.

**Proposition 7.** *Let $X$ be a second countable locally compact Hausdorff space, and let $G$ and $H$ be second countable locally compact groups. Suppose that $X$ admits a free and proper left $G$-action, and an $H$-action such that $(g\cdot x)\cdot h = g\cdot(x\cdot h)$ for all $x \in X$, $g \in G$, and $h \in H$. Then $\theta$ and $\lambda$ above preserve Morita equivalence classes, and define homomorphisms $\Theta\colon \mathrm{Br}_H(G\backslash X) \to \mathrm{Br}_{G\times H}(X)$ and $\Lambda\colon \mathrm{Br}_{G\times H}(X) \to \mathrm{Br}_H(G\backslash X)$. In fact, $\Theta$ is an isomorphism with inverse $\Lambda$, and if $\Theta[B,\beta] = [A,\alpha]$, then $B \rtimes_\beta H$ is Morita equivalent to $A \rtimes_\alpha (G \times H)$.*

*Proof.* If $(\mathcal{Y}, v)$ implements an equivalence between $(B,\beta)$ and $(B',\beta')$ in $\mathfrak{Br}_H(G\backslash X)$, then the external tensor product $\mathcal{Z} = C_0(X)\widehat{\otimes}\mathcal{Y}$, as defined in [9, §1.2] or [2, §2], is a $C_0(X)\otimes B - C_0(X)\otimes B'$-imprimitivity bimodule. A routine argument, similar to that in [2, Lemma 2.1], shows that the Rieffel correspondence [14, Theorem 3.1] between the lattices of ideals in $C_0(X)\otimes B$ and in $C_0(X)\otimes B'$ maps the balancing ideal $I = I_{C(G\backslash X)}$ in $C_0(X)\otimes B$ to the balancing ideal $J = J_{C(G\backslash X)}$ in $C_0(X)\otimes B'$. Thus [14, Corollary 3.2] implies that $\mathcal{X} = \mathcal{Z}/\mathcal{Z}\cdot J$ is a $p^*(B) - p^*(B')$-imprimitivity bimodule. Since $f\cdot x = x\cdot f$ for all $x \in \mathcal{X}$ and $f \in C_0(X)$, it follows from [10, Proposition 1.11] that $\mathcal{X}$ implements a Morita equivalence over $X$. More tedious but routine calculations show that the map defined on elementary tensors in $\mathcal{Z}_0 = C_0(X)\odot\mathcal{Y}$ by $u^0_{(s,h)}(f\otimes y) = \sigma_h(\tau_s(f))\otimes v_h(y)$ extends to the completion $\mathcal{Z}$, and defines a strongly continuous map $u\colon G\times H \to \mathrm{Iso}(\mathcal{X})$ such that $(\mathcal{X}, u)$ implements an equivalence between $(p^*(B), \tau\sigma\otimes\beta)$ and $(p^*(B'), \tau\sigma\otimes\beta')$. Thus $\Theta$ is well defined.

Observe that

$$\begin{aligned}\Theta([B,\beta][B',\beta']) &= \Theta([B\otimes_{C(G\backslash X)} B', \beta\otimes\beta']) \\ &= [p^*(B\otimes_{C(G\backslash X)} B'), \tau\sigma\otimes(\beta\otimes\beta')].\end{aligned} \tag{6}$$

But (6) is the class of

$$\begin{aligned}&(C_0(X)\otimes_{C(G\backslash X)} B\otimes_{C(G\backslash X)} B', \tau\sigma\otimes\beta\otimes\beta') \\ &\quad\sim (C_0(X)\otimes_{C(X)} C_0(X)\otimes_{C(G\backslash X)} B\otimes_{C(G\backslash X)} B', \tau\sigma\otimes\tau\sigma\otimes\beta\otimes\beta') \\ &\quad\sim (C_0(X)\otimes_{C(G\backslash X)} B\otimes_{C(X)} C_0(X)\otimes_{C(G\backslash X)} B', \tau\sigma\otimes\beta\otimes\tau\sigma\otimes\beta'),\end{aligned}$$

which represents the product of $\Theta[B,\beta]$ and $\Theta[B',\beta']$. Thus $\Theta$ is a homomorphism.

Now suppose that $(A,\alpha\gamma) \sim (A',\alpha'\gamma')$ in $\mathfrak{Br}_{G\times H}(X)$ via $(\mathcal{Z}, w)$. Then $u_s = w_{(s,e)}$ and $v_h = w_{(e,h)}$ define actions of $G$ and $H$, respectively, on $\mathcal{Z}$. In particular, $(\mathcal{Z}, u)$ implements an equivalence between $(A,\alpha)$ and $(A',\alpha')$ in $\mathfrak{Br}_G(X)$. It follows from [1, §6] that $\mathcal{Y}_0 = C_c(G,\mathcal{Z})$ can be completed to a $A\rtimes_\alpha G - A'\rtimes_{\alpha'} G$-imprimitivity bimodule $\mathcal{Y}$. One can verify that the induced $C_0(G\backslash X)$-actions on $\mathcal{Y}_0$ are given by $(\phi\cdot x)(t) = \phi\cdot(x(t))$ and $(x\cdot\phi)(t) = (x(t))\cdot\phi$, and [10, Proposition 1.11] implies that $\mathcal{Y}$ is an imprimitivity bimodule over $G\backslash X$. Now define $\tilde{v}^0_h$ on $\mathcal{Y}_0$ by $\tilde{v}^0_h(x)(t) = v_h(x(t))$. Using the inner products defined in [1, §6],

$$\begin{aligned}{}_{A\rtimes_\alpha G}\langle \tilde{v}^0_h(x), \tilde{v}^0_h(y)\rangle(t) &= \int_{G^A} \langle \tilde{v}^0_h(x)(s), \Delta(t^{-1}s)u_t(\tilde{v}^0_h(y)(t^{-1}s))\rangle\, ds \\ &= \int_{G^A} \langle v_h(x(s)), \Delta(t^{-1}s)u_t(v_h(y(t^{-1}s)))\rangle\, ds \\ &= \gamma_h({}_{A\rtimes_\alpha G}\langle x, y\rangle(t)),\end{aligned}$$

where, in the last equality, we use $u_s \circ v_h = v_h \circ u_s$. A similar computation shows that $\langle \tilde{v}^0_h(x), \tilde{v}^0_h(y)\rangle_{A'\rtimes_{\alpha'}G}(t) = \gamma'_h(\langle x, y\rangle_{A'\rtimes_{\alpha'}G}(t))$. Thus $\tilde{v}^0_h$ extends to all of $\mathcal{Y}$ and defines a map $\tilde{v}\colon H \to \mathrm{Iso}(\mathcal{Y})$, and it is not hard to verify that $\tilde{v}$ is strongly continuous. Therefore $(A \rtimes_\alpha G, \gamma \rtimes \mathrm{id}) \sim (A' \rtimes_{\alpha'} G, \gamma' \rtimes \mathrm{id})$ in $\mathfrak{Br}_{G\times H}(X)$, and $\Lambda$ is well defined.

Now it will suffice to show that, for $\mathfrak{a} \in \mathfrak{Br}_H(G\backslash X)$ and $\mathfrak{b} \in \mathfrak{Br}_{G\backslash H}(X)$, $\theta(\lambda(\mathfrak{b})) \sim \mathfrak{b}$ and $\lambda(\theta(\mathfrak{a})) \sim \mathfrak{a}$. For the first of these, suppose that $(A, \alpha\gamma) \in \mathfrak{Br}_{G\times H}(X)$. Then $\theta(\lambda(A, \alpha\gamma)) = (p^*(A \rtimes_\alpha G), (\tau\sigma \otimes \gamma) \rtimes \mathrm{id})$, which by Corollary 6 is equivalent to $(A \otimes \mathcal{K}(L^2(G)), \alpha\gamma \otimes \mathrm{Ad}\,\rho)$, and hence to $(A, \alpha\gamma)$. For the other direction, suppose that $(B, \beta) \in \mathfrak{Br}_H(G\backslash X)$. Then $\lambda(\theta(B, \beta)) = (p^*B \rtimes_{\tau\otimes\mathrm{id}} G$, $(\sigma \otimes \beta) \rtimes \mathrm{id})$. Now

$$p^*B \rtimes_{\tau\otimes\mathrm{id}} G \cong (C_0(X) \otimes_{C(G\backslash X)} B) \rtimes_{\tau\otimes\mathrm{id}} G \cong (C_0(X) \rtimes_\tau G) \otimes_{C(G\backslash X)} B,$$

which is Morita equivalent to $C_0(G\backslash X) \otimes_{C(G\backslash X)} B \cong B$. Because the Morita equivalence of $C_0(X) \rtimes G$ with $C_0(G\backslash X)$ is $H$-equivariant [3], it follows that

$$\lambda(\theta(B, \beta)) = (p^*B \rtimes_{\tau\otimes\mathrm{id}} G, (\sigma \otimes \beta) \rtimes \mathrm{id}) \sim (C_0(G\backslash X) \otimes_{C(G\backslash X)} B, \sigma \otimes \beta) \cong (B, \beta).$$

This shows that $\Lambda \circ \Theta$ is the identity, and also implies that

$$p^*B \rtimes_{\tau\sigma\otimes\beta} (G \times H) \cong (p^*B \rtimes_{\tau\otimes\mathrm{id}} G) \rtimes_{\sigma\otimes\beta} H \sim B \rtimes_\beta H,$$

which proves the last assertion. □

*Remark* 8. We showed that $\Lambda$ is a well-defined map of $\mathrm{Br}_{G\times H}(X)$ into $\mathrm{Br}_H(G\backslash X)$, and that it is a set-theoretic inverse for $\Theta$; since $\Theta$ is a group homomorphism, it follows that $\Lambda$ is also a homomorphism. This seems to be non-trivial: it implies that if $(A, \alpha)$, $(B, \beta)$ are in $\mathfrak{Br}_G(X)$, then $(A \otimes_{C(X)} B) \rtimes_{\alpha\otimes\beta} G$ is Morita equivalent to $(A \rtimes_\alpha G) \otimes_{C(G\backslash X)} (B \rtimes_\beta G)$. We do not know what general mechanism is at work here. Certainly, it is a Morita equivalence rather than an isomorphism: if $G$ is finite and the algebra commutative, one algebra is $|G|$-homogeneous and the other $|G|^2$-homogeneous. The only direct way we have found uses [8, Theorem 17], which seems an excessively heavy sledgehammer.

*Proof of Theorem* 1. It follows from Proposition 7 that there are isomorphisms $\Theta_H\colon \mathrm{Br}_H(G\backslash X) \to \mathrm{Br}_{G\times H}(X)$ and $\Lambda_G\colon \mathrm{Br}_{G\times H}(X) \to \mathrm{Br}_G(X/H)$. Therefore $\Lambda_G \circ \Theta_H$ is an isomorphism of $\mathrm{Br}_H(G\backslash X)$ onto $\mathrm{Br}_G(X/H)$. Assertion (1) also follows from Proposition 7. The isomorphism $\Lambda_G \circ \Theta_H$ maps the class of $(B, \beta)$ in $\mathfrak{Br}_H(G\backslash X)$ to the class of $(p^*(B) \rtimes_{\sigma\otimes\beta} H, (\tau \otimes \mathrm{id}) \rtimes \mathrm{id})$, so it remains to show that the latter is equivalent to $(A/J, \tau)$.

For convenience, write $I$ for the balancing ideal $I_{C(G\backslash X)}$ in $C_0(X) \otimes B$. Then

$$p^*(B) \rtimes_{\sigma\otimes\beta} H = ((C_0(X) \otimes B)/I) \rtimes_{\sigma\otimes\beta} H = (C_0(X, B) \rtimes_{\sigma\otimes\beta} H)/(I \rtimes_{\sigma\otimes\beta} H)$$

by, for example, [8, Proposition 12]. By [13, Theorem 2.2], $\mathcal{X}_0 = C_c(X, B)$ can be completed to a $C_0(X, B) \rtimes_{\sigma\otimes\beta} H - A$-imprimitivity bimodule $\mathcal{X}$. The irreducible representations of $A$ are given by $M_{(x,\pi_{G\cdot y})}(f)(x) = \pi_{G\cdot y}(f(x))$ [13, Lemma 2.6]. In the proof of [13, Theorem 2.5], it was shown that the representation $\mathcal{X}^{M_{(x,\pi_{G\cdot y})}}$ of $C_0(X, B) \rtimes_{\sigma\otimes\beta} H$ induced from $M_{(x,\pi_{G\cdot y})}$ via $\mathcal{X}$ is equivalent to $\mathrm{Ind}_{\{e\}}^G N_{(x,G\cdot y)}$, where $N_{(x,G\cdot y)}$ is the analogous irreducible representation of $C_0(X, B)$. Since the orbit space for a proper action is Hausdorff, [5] implies that

$(C_0(X,B),H,\sigma\otimes\beta)$ is regular. Since $R=\bigoplus_{x\in X}N_{(x,G\cdot x)}$ is a faithful representation of $p^*(B)$, it follows from [8, Theorem 24] that $\operatorname{Ind}_{\{e\}}^G(R)$ is a faithful representation of $p^*(B)\rtimes_{\sigma\otimes\beta}H$, and so has kernel $I\rtimes_{\sigma\otimes\beta}H$. On the other hand, $\operatorname{Ind}_{\{e\}}^G(R)$ is equivalent to $\bigoplus_{x\in X}\mathcal{X}^{M_{(x,G\cdot x)}}$. It follows from [14, §3] that ${}^I\mathcal{X}=\mathcal{X}/I\cdot\mathcal{X}$ is an $p^*(B)\rtimes_{\sigma\otimes\beta}H_{-X/H}A/J$-imprimitivity bimodule. Then the map $u_s^0\colon\mathcal{X}_0\to\mathcal{X}_0$ defined by $u_s^0(\xi)(x)=\xi(s^{-1}\cdot x)$ induces a map $u\colon G\to\operatorname{Iso}({}^I\mathcal{X})$ such that $({}^I\mathcal{X},u)$ implements the desired equivalence. □

We close with two interesting special cases where the isomorphism takes a particularly elegant form. Recall that if $B$ is a continuous-trace $C^*$-algebra with spectrum $X$, then we may view $B$ as the sections $\Gamma_0(\xi)$ of a $C^*$-bundle $\xi$ vanishing at infinity.

**Corollary 9.** *Suppose that $H$ is a closed subgroup of a second countable locally compact group $G$, and that $X$ is a second countable locally compact right $H$-space. Then $G\times X$ is a free and proper $H$-space via the diagonal action $(s,x)\cdot h=(sh,x\cdot h)$. Thus $(G\times X)/H$ is a locally compact $G$-space via $s\cdot[r,x]=[sr,x]$, and the map $(B,\beta)\mapsto(\operatorname{Ind}_H^G(B,\beta),\tau)$ induces an isomorphism of $\operatorname{Br}_H(X)$ onto $\operatorname{Br}_G((X\times G)/H)$.*

*Proof.* We apply Theorem 1 to ${}_G(G\times X)_H$, where $G$ acts on the left of the first factor, obtaining an isomorphism of $\operatorname{Br}_H(X)\cong\operatorname{Br}_H(G\backslash(G\times X))$ onto $\operatorname{Br}_G((G\times X)/H)$ sending the class of $(B,\beta)$ to the class of $\operatorname{Ind}_H^{G\times X}(B,\beta)/J$ where $J=\{f\colon f(s,x)(x)=0\}$.

Given $f\in\operatorname{Ind}_H^{G\times X}(B,\beta)$ and $s\in G$, let $\Phi(f)(s)$ be the function from $X$ to $\xi$ defined by $\Phi(f)(s)(x)=f(s,x)(x)$. We claim $\Phi(f)(s)\in\Gamma_0(\xi)$. If $x_0\in X$, then $x\mapsto f(s,x_0)(x)$ is in $\Gamma_0(\xi)$, and $\|\Phi(s)(x)-f(s,x_0)(x)\|$ tends to zero as $x\to x_0$. It follows from [6, Proposition 1.6 (Corollary 1)] that $\Phi(f)(s)$ is continuous. To see that $\Phi(f)(s)$ vanishes at infinity, suppose that $\{x_n\}\subset X$ satisfies

$$\|\Phi(f)(s)(x_n)\|\ge\varepsilon>0$$

for all $n$. Then $\|f(s,x_n)\|\ge\varepsilon$ for all $n$, and passing to a subsequence and relabeling if necessary, there must be $h_n\in H$ such that $(s\cdot h_n,x_n\cdot h_n)\to(r,x)$. Then $h_n\to s^{-1}r\in H$, and $x_n\to x\cdot(r^{-1}s)$. In sum, $\Phi(f)(s)\in\Gamma_0(\xi)=B$. Now the continuity of $f$ easily implies that $s\mapsto\Phi(f)(s)$ is continuous from $G$ to $B$. Furthermore, since $\beta$ covers $\sigma$ (i.e., $\beta_h(\phi\cdot b)(x)=\phi(x\cdot h)\beta_h(b)(x)$),

$$f(rh,x)(x)=\beta_h^{-1}(\Phi(f)(r))(x),$$

and $\Phi$ is a $*$-homomorphism of $\operatorname{Ind}_H^{G\times X}(B,\beta)$ into $\operatorname{Ind}_H^G(B,\beta)$, which clearly has kernel $J$.

Finally, it is not difficult (cf., e.g., [13, Lemma 2.6]) to see that $\Phi(\operatorname{Ind}_H^{G\times X}(B,\beta))$ is a rich subalgebra of $\operatorname{Ind}_H^G(B,\beta)$ as defined in [4, Definition 11.1.1]. Thus $\Phi$ is surjective by [4, Lemma 11.1.4]. □

**Corollary 10.** *Suppose that $X$ is a locally compact left $G$-space, and that $H$ is a closed normal subgroup of $G$ which acts freely and properly on $X$. Then there is an isomorphism of $\operatorname{Br}_{G/H}(H\backslash X)$ onto $\operatorname{Br}_G(X)$ taking $[B,\beta]$ to $[p^*(B),p^*(\beta)]=[p^*(B),\tau\otimes\beta]$.*

*Proof.* View $Y=X\times G/H$ as a left $G$-space via the diagonal action, and a right $G/H$-space via right translation on the second factor. Both actions are free, and the second action is proper. To see that the first action is proper, suppose that

$(x_n, t_nH) \to (x, tH)$ while $(s_n \cdot x_n, s_nt_nH) \to (y, rH)$. Then $s_nH \to sH$ for some $s \in G$. Passing to a subsequence and relabeling, we can assume that there are $h_n \in H$ such that $h_ns_n \to s$ in $G$. But then $s_n \cdot x_n \to y$ while $h_n \cdot (s_n \cdot x_n) \to s \cdot x$. Since the $H$-action is proper, we can assume that $h_n \to h$ in $H$. Thus $s_n \to h^{-1}s$, and this proves the claim.

The map $G \cdot (x, tH) \mapsto Ht^{-1} \cdot x$ is a bijection $\phi$ of $G\backslash Y$ onto $H\backslash X$. Further, $G\backslash Y$ is a right $G/H$-space and $H\backslash X$ is a left $G/H$-space with

$$\phi(v \cdot (s^{-1}H)) = sH \cdot \phi(v).$$

(That is, $\phi$ is equivariant when the $G/H$-action on $G\backslash Y$ is viewed as a left-action.) Therefore,

$$\mathrm{Br}_{G/H}(G\backslash Y) \cong \mathrm{Br}_{G/H}(H\backslash X). \tag{7}$$

Similarly, $Y/(G/H)$ and $X$ are isomorphic as left $G$-spaces so that

$$\mathrm{Br}_G(Y/(G/H)) \cong \mathrm{Br}_G(X). \tag{8}$$

Finally, Theorem 1 implies that

$$\mathrm{Br}_G(Y/(G/H)) \cong \mathrm{Br}_{G/H}(G\backslash Y). \tag{9}$$

Thus, Equations (7)–(9) imply that there is an isomorphism of $\mathrm{Br}_{G/H}(H\backslash X)$ onto $\mathrm{Br}_G(X)$ sending $(B, \beta)$ to $(\mathrm{Ind}_{G/H}^{X\times G/H}(B, \beta)/J, \tau \otimes \mathrm{id})$ with

$$J = \{f \in \mathrm{Ind}_{G/H}^{X\times G/H}(B, \beta) \colon f(x, rH)(Hr^{-1} \cdot x) = 0 \text{ for all } x \in X\}.$$

Define $\Phi\colon \mathrm{Ind}_{G/H}^{X\times G/H}(B, \beta) \to C_0(X, B)$ by $\Phi(f)(x) = f(x, H)$. Then $\Phi$ is onto (see, for example, the first sentence of the proof of [13, Lemma 2.6]). Since

$$\begin{aligned}\Phi(\tau_s \otimes \mathrm{id}(f))(x) &= \tau_s \otimes \mathrm{id}(f)(x, H) = f(s^{-1} \cdot x, s^{-1}H)\\ &= \beta_{sH}(f(s^{-1} \cdot x, H)) = \tau_s \otimes \beta_{sH}(\Phi(f))(x),\end{aligned}$$

$\Phi$ is equivariant, and it only remains to show that $\Phi$ induces a bijection of the quotient by $J$ with the quotient of $C_0(X, B)$ by the balancing ideal $I$.

However, if $\Phi(f) \in I$, then $f(x, H)(H \cdot x) = 0$ for all $x \in X$. But then $f(x, rH)(Hr^{-1} \cdot x) = \beta_{rH}^{-1}(f(x, H))(Hr^{-1} \cdot x)$, which is zero since $\beta$ covers the $G/H$-action on $X$, and $f \in J$. The argument reverses, so $\Phi(J) = I$, and the result follows. □

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