THE EQUIVARIANT BRAUER GROUPS OF COMMUTING FREE AND PROPER ACTIONS ARE ISOMORPHIC

ALEXANDER KUMJIAN, IAIN RAEBURN, AND DANA P. WILLIAMS

(Communicated by Palle E. T. Jorgensen)

Abstract. If $X$ is a locally compact space which admits commuting free and proper actions of locally compact groups $G$ and $H$, then the Brauer groups $\text{Br}_H(G \backslash X)$ and $\text{Br}_G(X/H)$ are naturally isomorphic.

Rieffel’s formulation of Mackey’s Imprimitivity Theorem asserts that if $H$ is a closed subgroup of a locally compact group $G$, then the group $C^*$-algebra $C^*(H)$ is Morita equivalent to the crossed product $C_0(G/H) \rtimes H$. Subsequently, Rieffel found a symmetric version, involving two subgroups of $G$, and Green proved the following Symmetric Imprimitivity Theorem: If two locally compact groups act freely and properly on a locally compact space $X$, $G$ on the left and $H$ on the right, then the crossed products $C_0(G \backslash X) \rtimes H$ and $C_0(X/H) \rtimes G$ are Morita equivalent. (For a discussion and proofs of these results, see [15].) Here we shall show that in this situation there is an isomorphism $\text{Br}_H(G \backslash X) \cong \text{Br}_G(X/H)$ of the equivariant Brauer groups introduced in [2].

Suppose $(G, X)$ is a second countable locally compact transformation group. The objects in the underlying set $\text{Br}_G(X)$ of the equivariant Brauer group $\text{Br}_G(X)$ are dynamical systems $(A, G, \alpha)$, in which $A$ is a separable continuous-trace $C^*$-algebra with spectrum $X$, and $\alpha: G \to \text{Aut}(A)$ is a strongly continuous action of $G$ on $A$ inducing the given action of $G$ on $X$. The equivalence relation on such systems is the equivariant Morita equivalence studied in [1], [3]. The group operation is given by $[(A, \alpha) \cdot (B, \beta)] = [A \otimes C(X) B, \alpha \otimes \beta]$, the inverse of $[A, \alpha]$ is the conjugate system $[A, \alpha]$, and the identity is represented by $(C_0(X), \tau)$, where $\tau_s(f)(x) = f(s^{-1} \cdot x)$.

Notation. Suppose that $H$ is a locally compact group, that $X$ is a free and proper $H$-space, and that $(B, H, \beta)$ is a dynamical system. Then $\text{Ind}_H^G(B, \beta)$ will be the $C^*$-algebra (denoted by $GC(X, B)^\alpha$ in [13] and by $\text{Ind}(B; X, H, \beta)$ in [11]) of bounded continuous functions $f: X \to B$ such that $\beta_h(f(x \cdot h)) = f(x)$, and $x \cdot H \mapsto \|f(x)\|$ belongs to $C_0(X/H)$.

We now state our main theorem.
Theorem 1. Let $X$ be a second countable locally compact Hausdorff space, and let $G$ and $H$ be second countable locally compact groups. Suppose that $X$ admits a free and proper left $G$-action, and a free and proper right $H$-action such that $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $x \in X$, $g \in G$, and $h \in H$. Then there is an isomorphism $\Theta$ of $\text{Br}_H(G \backslash X)$ onto $\text{Br}_G(X/H)$ satisfying:

1. if $(A, \alpha)$ represents $\Theta[B, \beta]$, then $A \rtimes_\alpha G$ is Morita equivalent to $B \rtimes_\beta H$;

2. $\Theta[B, \beta]$ is realised by the pair $(\text{Ind}_H^G(B, \beta)/J, \pi \otimes \text{id})$ in $\text{Br}_G(X/H)$, where $\pi \otimes \text{id}$ denotes left translation and, if $\pi_G \cdot x$ is the element of $\tilde{B} = G \backslash X$ corresponding to $G \cdot x$,

$$J = \{ f \in \text{Ind}_H^G(B, \beta) : \pi_G \cdot x(f(x)) = 0 \text{ for all } x \in X \}.$$

Item (1) is itself a generalization of Green’s symmetric imprimitivity theorem, and our proof of Theorem 1 follows the approach to Green’s theorem taken in [3]: prove that both $C_0(G \backslash X) \rtimes H$ and $C_0(X/H) \rtimes G$ are Morita equivalent to $C_0(X) \rtimes H$, where $\alpha_{s,h}(f)(x) = f(s^{-1} \cdot x \cdot h)$, by noting that the Morita equivalences of $C_0(X) \rtimes G$ with $C_0(G \backslash X)$ and $C_0(X) \rtimes H$ with $C_0(X/H)$ ([7], [15, Situation 10]) are equivariant, and hence induce Morita equivalences

$$C_0(G \backslash X) \rtimes H \sim (C_0(X) \rtimes G) \rtimes H \cong C_0(X) \rtimes (G \rtimes H) \cong (C_0(X) \rtimes H) \rtimes G.$$

The same symmetry considerations show that it will be enough to prove that $\text{Br}_H(G \backslash X) \cong \text{Br}_{G \times H}(X)$. Since we already know that $\text{Br}(G \backslash X) \cong \text{Br}_G(X)$ [2, §6.2], we just have to check that this isomorphism is compatible with the actions of $H$.

Suppose $G$ acts freely and properly on $X$, and $p : X \to G \backslash X$ is the orbit map. If $B$ is a $C^*$-algebra with a nondegenerate action of $C_0(G \backslash X)$, then the pull-back $p^* B$ is the quotient of $C_0(X) \rtimes B$ by the balancing ideal

$$I_{G \backslash X} = \{ f \cdot \phi \otimes b - \phi \otimes f \cdot b : \phi \in C_0(X), f \in C_0(G \backslash X), b \in B \};$$

in other words, $p^* B \cong C_0(X) \otimes C(G \backslash X) B$. The nondegenerate action of $C_0(G \backslash X)$ on $B$ induces a continuous map $q$ of $\tilde{B}$ onto $G \backslash X$, characterized by $\pi(f \cdot b) = f(q(\pi)) \pi(b)$. Then under the natural identification of $C_0(X) \otimes B$ with $C_0(X, B)$,

$$I_{G \backslash X} \cong \{ f \in C_0(X, B) : \pi(f(x)) = 0 \text{ for all } x \in q(\pi) \},$$

so that $p^* B$ has spectrum

$$p^* \tilde{B} = \{ (x, \pi) \in X \times \tilde{B} : G \cdot x = q(\pi) \}.$$

If $B$ is a continuous-trace algebra with spectrum $G \backslash X$, then $p^* B$ is a continuous-trace algebra with spectrum $X$.

The isomorphism $\Theta : \text{Br}(G \backslash X) \cong \text{Br}_G(X)$ is given by $\Theta[A] = [p^* A, \pi \otimes \text{id}]$. To prove $\Theta$ is surjective in [2], we used [12, Theorem 1.1], which implies that if $(B, \beta) \in \text{Br}_G(X)$, then $B \rtimes_\beta G$ is a continuous-trace algebra with spectrum $G \backslash X$ such that $(B, \beta)$ is Morita equivalent to $(p^* (B \rtimes_\beta G), \pi \otimes \text{id})$, and hence that $[B, \beta] = \Theta[B \rtimes_\beta G, \text{id}]$. In obtaining the required equivariant version of [12, Theorem 1.1], we have both simplified the proof and mildly strengthened the conclusion (see Corollary 4 below). However, with all these different group actions around, the notation could get messy, and we pause to establish some conventions.
the balanced tensor product
continuous equivariant map
Thus it follows from the remark that (2) defines an element of
Proof of Lemma
Remark
Suppose a locally compact group $G$ acts freely and properly on a locally compact space $X$, and that $A$ is a $C^\ast$-algebra carrying a non-degenerate action of $C_0(X)$. If $\alpha: G \to \text{Aut}(A)$ is an action of $G$ on $A$ satisfying $\alpha_s(\phi \cdot a) = \tau_s(\phi) \cdot \alpha_s(a)$, then the map sending $f \otimes a$ in $C_0(X) \otimes A$ to the function $s \mapsto f(\alpha_s(a))$ induces an equivariant isomorphism $\Phi$ of $(C_0(X) \otimes_{C(G \times X)} A, G, \text{id} \otimes \alpha)$ onto $(C_0(G), A, G, \tau \otimes \text{id})$.

Remark 3. For motivation, consider the case where $A = C_0(X)$. Then the map $\Psi: C_0(X) \times X \to C_0(G \times X)$ defined by $\Psi(f)(s, x) = f(s, x)$ maps $C_0$ to $C_0$ precisely when the action is proper, has range which separates the points of $G \times Y$ precisely when the action is free, and has kernel consisting of the functions which vanish on the closed subset $\Delta = \{(x, y): G \cdot x = G \cdot y\}$. Thus the free and proper actions are precisely those for which $\Psi$ induces an isomorphism of $C_0(X) \otimes_{C(G \times X)} C_0(X)$ onto $C_0(G) \otimes C_0(X)$.

Proof of Lemma 2. If $\psi \in C_0(G \times X)$, then $f \cdot \psi \cdot a$ and $f \otimes \psi \cdot a$ have the same image in $C_0(G, A)$, and the map factors through the balanced tensor product as claimed. Further, $\Phi$ is related to the map $\Psi$ in Remark 3 by
\[
\Phi(f \otimes g \cdot a) = (\Psi(f \otimes g)(s, \cdot)) \cdot \alpha_s^{-1}(a).
\]
Thus it follows from the remark that (2) defines an element of $C_0(G, A)$ and that the closure of the range of $\Phi$ contains all functions of the form $s \mapsto \xi(s)f^{-1}(a)$ for $\xi \in C_c(G)$, $f \in C_c(X)$, and $a \in A$. These elements span a dense subset of $C_0(G, A)$, and hence $\Phi$ is surjective. The nondegenerate action of $C_0(X)$ on $A$ induces a continuous equivariant map $q$ of $A$ onto $X$ such that $\pi(f \cdot a) = f(q(\pi))\pi(a)$, and the balanced tensor product $C_0(X) \otimes_{C(G \times X)} A$ has spectrum $\Delta = \{(x, \pi): G \cdot x = G \cdot q(\pi)\}$. Since each representation $(q(\pi), s \cdot \pi) = (q(\pi), \pi \cdot \alpha_s^{-1})$ in $\Delta$ factors through $\Phi$ and the representation $b \mapsto \pi(b(s))$ of $C_0(G, A)$, the homomorphism $\Phi$ is also injective. Finally, to see the equivariance, we compute:
\[
\Phi(\text{id} \otimes \alpha_s(h \otimes a))(t) = h \cdot \alpha_t^{-1}(\alpha_s(a)) = \Phi(h \otimes a)(\iota^{-1}t) = \tau_s \otimes \text{id}(\Phi(h \otimes a))(t).
\]

Corollary 4 (cf. [12, Theorem 1.1]). Let $(G, X)$ and $\alpha: G \to \text{Aut}(A)$ be as in Lemma 2. Then there is an equivariant isomorphism of $(p^\ast(A \ltimes_\alpha G), G, p^\ast \text{id})$ onto $(A \otimes \mathcal{K}(L^2(G)), G, \alpha \otimes \text{Ad} \rho)$. 

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Proof. A routine calculation shows that the equivariant isomorphism \(\Phi\) of Lemma 2 gives an equivariant isomorphism
\[
(3) \quad \Phi \times \text{id}: ((C_0(X) \otimes_{C(G \setminus X)} A) \times_{\text{id} \otimes \alpha} G, (\tau \otimes \text{id}) \times \text{id}) \\
\rightarrow (C_0(G, A) \times_{\tau \otimes \text{id}} G, (\sigma^G \otimes \alpha) \times \text{id}).
\]
We also have equivariant isomorphisms
\[
(4) \quad (C_0(G, A) \times_{\tau \otimes \text{id}} G, (\sigma^G \otimes \alpha) \times \text{id}) \cong (A \otimes (C_0(G) \times_{\tau} G), \alpha \otimes (\sigma^G \times \text{id})),
\]
and
\[
(5) \quad (C_0(X) \otimes_{C(G \setminus X)} (A \rtimes \alpha G), \tau \otimes \text{id}) \cong ((C_0(X) \otimes_{C(G \setminus X)} A) \times_{\text{id} \otimes \alpha} G, (\tau \otimes \text{id}) \times \text{id});
\]
combining (3), (4), and (5) gives the result.

\[\square\]

Lemma 5. In addition to the hypotheses of Lemma 2, suppose that \(H\) is a locally compact group acting on the right of \(X\), and that \((A, H, \gamma)\) is a dynamical system such that \(\alpha\) and \(\gamma\) commute and \(\gamma_h(f \cdot a) = \alpha_h(f) \cdot \gamma_h(a)\) for \(h \in H, f \in C_0(X), a \in A\). Then the action \(\tau \sigma \otimes \gamma\) of \(G \times H\) on \(C_0(X) \otimes A\) preserves the balancing ideal \(I_{G \setminus X}\), and hence induces an action of \(G \times H\) on \(C_0(X) \otimes C(G \setminus X)A\), also denoted \(\tau \sigma \otimes \gamma\). The equivariant isomorphism of Lemma 2 induces an equivariant isomorphism
\[
((C_0(X) \otimes_{C(G \setminus X)} A) \times_{\text{id} \otimes \alpha} G, (\tau \sigma \otimes \gamma) \times \text{id})
\]
\[
\cong (C_0(G, A) \times_{\tau \otimes \text{id}} G, (\sigma^G \otimes \alpha \gamma) \times \text{id}).
\]

Proof. The first assertion is straightforward. For the second, we can consider the actions of \(H\) and \(G\) separately. We have already observed in (3) that \(\Phi \times \text{id}\) intertwines the \(G\)-actions. On the other hand, if \(h \in H\) and \(t \in G\), then
\[
\Phi(\sigma_h \otimes \gamma_h(f \cdot a))(t) = \sigma_h(f) \cdot \alpha^{-1}_t(\gamma_h(a)) = \sigma_h(f) \cdot \gamma_h(\alpha^{-1}_t(a))
\]
\[
= \gamma_h(\Phi(f \cdot a)(t)). \quad \square
\]

Corollary 6. Let \(G \times_{\text{id}} A\) be as in the lemma. Denote by \(p\) the orbit map of \(X\) onto \(G \setminus X\). Then there is an equivariant isomorphism
\[
(p^*(A \rtimes \alpha G), G \times H, \tau \sigma \otimes (\gamma \times \text{id})) \cong (A \otimes K(L^2(G)), G \times H, \alpha \gamma \otimes \text{Ad} \rho).
\]

Proof. Compose the isomorphism of Lemma 5 with (4) and (5). \(\square\)

We are now ready to define our map of \(\text{Br}_H(G \setminus X)\) into \(\text{Br}_{G \times H}(X)\). Suppose \((B, \beta) \in \mathfrak{B}_{\tau \beta}(X)\). Then the action \(\tau \sigma \otimes \beta\) of \(G \times H\) preserves the balancing ideal \(I_{G \setminus X}\): if \(\phi \in C_0(G \setminus X)\), then
\[
(\tau \sigma \otimes \beta)_\phi(f \cdot \phi \cdot b - f \cdot \phi \cdot b) = \sigma_h(\tau_s(f \cdot \phi)) \cdot \beta_h(b) - \sigma_h(\tau_s(f)) \cdot \beta_h(b).
\]
Since \(p^*(B)\) is a continuous-trace \(C^*\)-algebra with spectrum \(X\) [12, Lemma 1.2], and \(\tau \sigma \otimes \beta\) covers the canonical \(G \times H\)-action on \(X\), we can define \(\theta: \mathfrak{B}_{\tau \beta}(G \setminus X) \rightarrow \mathfrak{B}_{\tau \sigma \otimes \beta}(X)\) by \(\theta(B, \beta) = (p^*(B), \tau \sigma \otimes \beta)\).

Similarly if \((A, \alpha \gamma) \in \mathfrak{B}_{\tau \sigma \otimes \beta}(X)\), then \((A \rtimes \alpha G)\) is a continuous-trace \(C^*\)-algebra with spectrum \(G\setminus X\) by [12, Theorem 1.1]. Since \(\gamma\) is compatible with \(\sigma\), we have
\[
\gamma_h(\phi \cdot z(s)) = \sigma_h(\phi) \cdot \gamma_h(z(s))\]
for \(z \in C_0(G, A)\), and hence \(\gamma \times \text{id}\) covers the
given action of $H$ on $X$. Thus we can define $\lambda: \mathfrak{B}_G \times \mathcal{H}(X) \to \mathfrak{B}_H(G \setminus X)$ by $\lambda(A, \alpha\gamma) = (A \times_{\alpha} G, \gamma \times \text{id})$.

**Proposition 7.** Let $X$ be a second countable locally compact Hausdorff space, and let $G$ and $H$ be second countable locally compact groups. Suppose that $X$ admits a free and proper left $G$-action, and an $H$-action such that $(g \cdot x) \cdot h = g \cdot (x \cdot h)$ for all $x \in X$, $g \in G$, and $h \in H$. Then $\Theta$ and $\lambda$ above preserve Morita equivalence classes, and define homomorphisms $\Theta: \text{Br}(G, X) \to \text{Br}(G \times H, X)$ and $\Lambda: \text{Br}(G \times H, X) \to \text{Br}(G(X))$. In fact, $\Theta$ is an isomorphism with inverse $\Lambda$, and if $\Theta[B, \beta] = [A, \alpha]$, then $B \times_{\beta} H$ is Morita equivalent to $A \times_{\alpha} (G \times H)$.

**Proof.** If $(\mathcal{Y}, \nu)$ implements an equivalence between $(B, \beta)$ and $(B', \beta')$ in $\mathfrak{B}_G(G \setminus X)$, then the external tensor product $\mathcal{Z} = C_0(X) [\mathcal{Y}]$, as defined in [9, §1.2] or [2, §2], is a $C_0(X) \otimes B - C_0(X) \otimes B'$-imprimitivity bimodule. A routine argument, similar to that in [2, Lemma 2.1], shows that the Rieffel correspondence [14, Theorem 3.1] between the lattices of ideals in $C_0(X) \otimes B$ and in $C_0(X) \otimes B'$ maps the balancing ideal $I = I_{C(G \setminus X)}$ in $C_0(X) \otimes B$ to the balancing ideal $J = J_{C(G \setminus X)}$ in $C_0(X) \otimes B'$. Thus [14, Corollary 3.2] implies that $\mathcal{X} = \mathcal{Z}/\mathcal{Z} \cdot J$ is a $(\nu^\star B) - \nu^\star (B')$-imprimitivity bimodule. Since $f \cdot x = x \cdot f$ for all $x \in \mathcal{X}$ and $f \in C_0(X)$, it follows from [10, Proposition 1.11] that $\mathcal{X}$ implements a Morita equivalence over $\mathcal{Z}$. More tedious but routine calculations show that the map defined on elementary tensors in $\mathcal{Z}_0 = C_0(X) \otimes \mathcal{Y}$ by $u_0^0 \cdot (f \otimes y) = \sigma_h(\tau_s(f)) \otimes v_h(y)$ extends to the completion $\mathcal{Z}$, and defines a strongly continuous map $u: G \times H \to \text{Isom}(\mathcal{X})$ such that $(\mathcal{X}, u)$ implements an equivalence between $(\nu^\star (B), \tau \sigma \otimes \beta)$ and $(\nu^\star (B'), \tau \sigma \otimes \beta')$. Thus $\Theta$ is well defined.

Observe that

$$\Theta([B, \beta][B', \beta']) = \Theta([B \otimes_{C(G \setminus X)} B', \beta \otimes \beta'])$$

(6)

= $[\nu^\star (B \otimes_{C(G \setminus X)} B'), \tau \sigma \otimes (\beta \otimes \beta')]$.

But (6) is the class of

$$\left(\text{C}_0(X) \otimes_{C(G \setminus X)} B \otimes_{C(G \setminus X)} B', \tau \sigma \otimes (\beta \otimes \beta')\right)$$

$$\sim \left(\text{C}_0(X) \otimes_{C(G \setminus X)} C_0(X) \otimes_{C(G \setminus X)} B \otimes_{C(G \setminus X)} B', \tau \sigma \otimes \beta \otimes \beta'\right)$$

$$\sim \left(\text{C}_0(X) \otimes_{C(G \setminus X)} B \otimes_{C(G \setminus X)} C_0(X) \otimes_{C(G \setminus X)} B', \tau \sigma \otimes \beta \otimes \sigma \otimes \beta'\right),$$

which represents the product of $\Theta[B, \beta]$ and $\Theta[B', \beta']$. Thus $\Theta$ is a homomorphism.

Now suppose that $(A, \alpha \gamma) \sim (A', \alpha' \gamma')$ in $\mathfrak{B}_{G \times \mathcal{H}}(X)$ via $(\mathcal{Z}, \nu)$. Then $u_s = w_{s,e,h}^0$ and $v_h = w_{h,e,0}^0$ define actions of $G$ and $H$, respectively, on $\mathcal{Z}$. In particular, $(\mathcal{Z}, \nu)$ implements an equivalence between $(A, \alpha)$ and $(A', \alpha')$ in $\mathfrak{B}_{G\times H}(X)$. It follows from [1, §6] that $\mathcal{Y}_0 = C_0(G, \mathcal{Z})$ can be completed to a $A \times_{\alpha} G - A' \times_{\alpha'} G$-imprimitivity bimodule $\mathcal{Y}$. One can verify that the induced $C_0(G \setminus X)$-actions on $\mathcal{Y}_0$ are given by $(\phi \cdot x)(t) = \phi(\cdot x(t))$ and $(x \cdot \phi)(t) = (x(t) \cdot \phi)$, and [10, Proposition 1.11] implies that $\mathcal{Y}$ is an imprimitivity bimodule over $G \setminus X$. Now define $v_{h0}^0$ on $\mathcal{Y}_0$ by $v_{h0}^0(x)(t) = v_h(x(t))$. Using the inner products defined in [1, §6],

$$A_{\alpha \gamma} G(v_{h0}^0(x), v_{h0}^0(y))(t) = \int_{G \setminus A} \langle v_{h0}^0(x)(s), \Delta(t^{-1}s) u_t(v_{h0}^0(y)(t^{-1}s)) \rangle \, ds$$

$$\int_{G \setminus A} \langle v_h(x(s)), \Delta(t^{-1}s) u_t(v_h(y(t^{-1}s))) \rangle \, ds$$

$$= \gamma_h(A_{\alpha \gamma} G(x, y)(t)), $$

where $\Delta(t^{-1}s)$ is the 2-cocycle associated with the 2-cocycle $\Delta(t)$ on $G \setminus A$.
Lemma 2.6. In the proof of [13, Theorem 2.5], it was shown that the reducible representations of $\Lambda$ can be completed to a $C_b$ by, for example, [8, Proposition 12]. By [13, Theorem 2.2], $\Lambda$ is Morita equivalent to $\Lambda' \approx (\Lambda' \approx \Lambda', \gamma' \times \text{id})$ in $\mathfrak{Br}_{G \times H}(X)$, and $\Lambda$ is well defined.

Now it suffices to show that, for $a \in \mathfrak{Br}_H(G \setminus X)$ and $b \in \mathfrak{Br}_{C_0(X)}(X)$, $\theta(a \gamma(b)) \sim b$ and $\lambda(a \alpha(b)) \sim a$. For the first of these, suppose that $(A, a \gamma) \in \mathfrak{Br}_{G \times H}(X)$. Then $\lambda(A, a \gamma) = (p^* A \times \alpha, (\tau \sigma \gamma) \times \text{id})$, which by Corollary 6 is equivalent to $(A \otimes K(L^2(G)), a \gamma \otimes \text{Ad} \rho)$, and hence to $(A, a \gamma)$. For the other direction, suppose that $(B, \beta) \in \mathfrak{Br}_H(G \setminus X)$. Then $\lambda(B, \beta) = (p^* B \times \sigma \otimes \beta, \text{id})$.

This shows that $\Lambda \circ \Theta$ is the identity, and also implies that

$$p^* B \times \sigma \otimes \beta \sim (p^* B \times \sigma \otimes \beta, H \times \beta H),$$

which proves the last assertion.

Remark 8. We showed that $\Lambda$ is a well-defined map of $\mathfrak{Br}_{G \times H}(X)$ into $\mathfrak{Br}_H(G \setminus X)$, and that it is a set-theoretic inverse for $\Theta$; since $\Theta$ is a group homomorphism, it follows that $\Lambda$ is also a homomorphism. This seems to be non-trivial: it implies that if $(A, a \alpha), (B, \beta)$ are in $\mathfrak{Br}_G(X)$, then $(A \otimes C(X, B) \times \sigma \otimes \beta) G$ is Morita equivalent to $(A \otimes C(X, B) \times \sigma \otimes \beta) G$. We do not know what general mechanism is at work here. Certainly, it is a Morita equivalence rather than an isomorphism: if $G$ is finite and the algebra commutative, one algebra is $G$-homogeneous and the other $G^2$-homogeneous. The only direct way we have found uses [8, Theorem 17], which seems an excessively heavy sledgehammer.

Proof of Theorem 1. It follows from Proposition 7 that there are isomorphisms $\Theta_H : \mathfrak{Br}_H(G \setminus X) \to \mathfrak{Br}_{G \times H}(X)$ and $\Lambda_G : \mathfrak{Br}_{G \times H}(X) \to \mathfrak{Br}_G(X/H)$. Therefore $\Lambda_G \circ \Theta_H$ is an isomorphism of $\mathfrak{Br}_H(G \setminus X)$ onto $\mathfrak{Br}_G(X/H)$. Assertion (1) also follows from Proposition 7. The isomorphism $\Lambda_G \circ \Theta_H$ maps the class of $(B, \beta)$ in $\mathfrak{Br}_H(G \setminus X)$ to the class of $(p^* B \times _w \sigma \otimes (H, \tau \otimes \beta))$, so it remains to show that the latter is equivalent to $(A, \tau, \gamma)$.

For convenience, write $I$ for the balancing ideal $I_{C_G(X, H)}$ in $C_0(X) \otimes B$. Then

$$p^*(B) \times _w \sigma \otimes H = (C_0(X) \otimes B) / \times _w \sigma \otimes H = (C_0(X, B) / \times _w \sigma \otimes H) / (I \times _w \sigma \otimes H)$$

by, for example, [8, Proposition 12]. By [13, Theorem 2.2], $X_0 \approx C_c(X, B)$ can be completed to a $C_0(X, B) \times \text{id}$. The irreducible representations of $A$ are given by $M_{(x, \pi_c, \gamma)}(x) = \pi_G(y(f(x)) \approx [13, \text{Lemma } 2.6]$. In the proof of [13, Theorem 2.5], it was shown that the representation $\Lambda^{-1}_{c(\cdot, \pi_c, \gamma)}$ of $C_0(X, B) \times _w \sigma \otimes H$ induced from $M_{(x, \pi_c, \gamma)}$ via $X$ is equivalent to $\text{Ind}_{C_c}^G N_{(x, \pi_c, \gamma)}$, where $N_{(x, \pi_c, \gamma)}$ is the analogous irreducible representation of $C_0(X, B)$. Since the orbit space for a proper action is Hausdorff, [5] implies that

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
\((C_0(X, B), H, \sigma \otimes \beta)\) is regular. Since \(R = \bigoplus_{\mathcal{E} \in X} N_{(\mathcal{E}, G, B)}\) is a faithful representation of \(p^*(B)\), it follows from [8, Theorem 24] that \(\text{Ind}_{H}^{G}(R)\) is a faithful representation of \(p^*(B) \times_{\sigma \otimes \beta} H\), and so has kernel \(I \times_{\sigma \otimes \beta} H\). On the other hand, \(\text{Ind}_{H}^{G}(R)\) is equivalent to \(\bigoplus_{\mathcal{E} \in X} \mathcal{H}^{\mathcal{E}}(G, B)\). It follows from [14, §3] that \(\mathcal{H} = \mathcal{H}/I \cdot \mathcal{H}\) is an \(p^*(B) \times_{\sigma \otimes \beta} H_{\mathcal{H}/H} A/J\)-admissible bimodule. Then the map \(u_0^0: \mathcal{H}_0 \to \mathcal{H}_0\) defined by \(u_0^0(\xi)(x) = \xi(s^{-1} \cdot x)\) induces a map \(u: G \to \text{Iso}(\mathcal{H})\) such that \((\mathcal{H}, u)\) implements the desired equivalence.

We close with two interesting special cases where the isomorphism takes a particularly elegant form. Recall that if \(B\) is a continuous-trace \(C^\ast\)-algebra with spectrum \(X\), then we may view \(B\) as the sections \(\Gamma_0(\xi)\) of a \(C^\ast\)-bundle \(\pi\) vanishing at infinity.

**Corollary 9.** Suppose that \(H\) is a closed subgroup of a second countable locally compact group \(G\), and that \(X\) is a second countable locally compact right \(H\)-space.

Then \(G \times X\) is a free and proper \(H\)-space via the diagonal action \((s, x) \cdot h = (sh, x)\). Thus \((G \times X)/H\) is a locally compact \(G\)-space via \(s \cdot [r, x] = [sr, x]\), and the map \((B, \beta) \mapsto (\text{Ind}_{H}^{G}(B, \beta), \tau)\) induces an isomorphism of \(\text{Br}(X)\) onto \(\text{Br}(G \times G)/H)\).

**Proof.** We apply Theorem 1 to \(G(G \times X)/H\), where \(G\) acts on the left of the first factor, obtaining an isomorphism of \(\text{Br}(X) \cong \text{Br}(G(G \times X))/H)\) onto \(\text{Br}(G \times G)/H)\) sending the class of \((B, \beta)\) to the class of \(\text{Ind}_{H}^{G}(B, \beta)/J\) where \(J = \{ f \in \text{Ind}_{H}^{G}(B, \beta) \mid f(x) = 0 \}\).

Given \(f \in \text{Ind}_{H}^{G}(B, \beta)\) and \(s \in G\), let \(\Phi(f)(s)\) be the function from \(X\) to \(\xi\) defined by \(\Phi(f)(s)(x) = f(s, x)(x)\). We claim \(\Phi(f)(s) \in \Gamma_0(\xi)\). If \(x_0 \in X\), then \(x_0 \mapsto f(s, x_0)(x)\) is in \(\Gamma_0(\xi)\), and \(\|\Phi(s)(x) - f(s, x_0)(x)\|\) tends to zero as \(x \to x_0\). It follows from [6, Proposition 1.6 (Corollary 1)] that \(\Phi(f)(s)\) is continuous. To see that \(\Phi(f)(s)\) vanishes at infinity, suppose that \(\{x_n\} \subset X\) satisfies 

\[\|\Phi(f)(s)(x_n)\| \geq \varepsilon > 0\]

for all \(n\). Then \(\|f(s, x_n)\| \geq \varepsilon\) for all \(n\), and passing to a subsequence and relabeling if necessary, there must be \(h_n \in H\) such that \((s \cdot h_n, x_n, h_n) \to (r, x)\). Then \(h_n \to s^{-1}r \in H\), and \(x_n \to x \cdot (r^{-1}s)\). In sum, \(\Phi(f)(s) \in \Gamma_0(\xi) = B\). Now the continuity of \(f\) easily implies that \(s \mapsto \Phi(f)(s)\) is continuous from \(G\) to \(B\). Furthermore, since \(\beta\) covers \(\sigma\) (i.e., \(\beta_h(\phi \cdot b)(x) = \phi(x \cdot h) \beta_h(b)(x)\)),

\[f(\phi r, x)(x) = \beta_h^{-1}(\Phi(f)(r))(x),\]

and \(\Phi\) is a \(\ast\)-homomorphism of \(\text{Ind}_{H}^{G}(B, \beta)\) into \(\text{Ind}_{H}^{G}(B, \beta)\), which clearly has kernel \(J\).

Finally, it is not difficult (cf., e.g., [13, Lemma 2.6]) to see that \(\Phi(\text{Ind}_{H}^{G}(B, \beta))\) is a rich subalgebra of \(\text{Ind}_{H}^{G}(B, \beta)\) as defined in [4, Definition 11.1.1]. Thus \(\Phi\) is surjective by [4, Lemma 11.1.4].

**Corollary 10.** Suppose that \(X\) is a locally compact left \(G\)-space, and that \(H\) is a closed normal subgroup of \(G\) which acts freely and properly on \(X\). Then there is an isomorphism of \(\text{Br}(G/H)(X)\) onto \(\text{Br}(X)\) taking \([B, \beta]\) to \([p^\ast(B), p^\ast(\beta)]\). 

**Proof.** View \(Y = X \times G/H\) as a left \(G\)-space via the diagonal action, and a right \(G/H\)-space via right translation on the second factor. Both actions are free, and the second action is proper. To see that the first action is proper, suppose that
\((x_n, t_n H) \to (x, t H)\) while \((s_n \cdot x_n, s_n t_n H) \to (y, r H)\). Then \(s_n H \to s H\) for some \(s \in G\). Passing to a subsequence and relabeling, we can assume that there are \(h_n \in H\) such that \(h_n s_n \to s\) in \(G\). But then \(s_n \cdot x_n \to y\) while \(h_n \cdot (s_n \cdot x_n) \to s \cdot x\). Since the \(H\)-action is proper, we can assume that \(h_n \to h\) in \(H\). Thus \(s_n \to h^{-1} s\), and this proves the claim.

The map \(G \cdot (x, t H) \to H r^{-1} \cdot x\) is a bijection \(\phi\) of \(G)/Y\) onto \(H)\). Further, \(G)\) is a right \(G\)-space and \(H)\) is a left \(G\)-space with

\[\phi(v \cdot (s^{-1}H)) = sH \cdot \phi(v)\]

(That is, \(\phi\) is equivariant when the \(G\)-action on \(G)\) is viewed as a left-action.)

Therefore,

\[\text{Br}_{G/H}(G) \cong \text{Br}_{G/H}(H)\).

Similarly, \(Y/(G)\) and \(X\) are isomorphic as left \(G\)-spaces so that

\[\text{Br}_G(Y/(G) \cong \text{Br}_G(X)\).

Finally, Theorem 1 implies that

\[\text{Br}_G(Y/(G) \cong \text{Br}_{G/H}(G)\).

Thus, Equations (7)–(9) imply that there is an isomorphism of \(\text{Br}_{G/H}(H)\) onto \(\text{Br}_G(X)\) sending \((B, \beta)\) to \((\text{Ind}_{G/H}^{X} B, \beta)/I, \tau \otimes \text{id}\) with

\[J = \{f \in \text{Ind}_{G/H}^{X} (B, \beta) : f(x, r H)(H r^{-1} \cdot x) = 0 \text{ for all } x \in X\}\].

Define \(\Phi: \text{Ind}_{G/H}^{X} (B, \beta) \to C_0(X, B)\) by \(\Phi(f)(x) = f(x, H)\). Then \(\Phi\) is onto (see, for example, the first sentence of the proof of [13, Lemma 2.6]). Since

\[
\Phi(\tau_s \otimes \text{id}(f))(x) = \tau_s \otimes \text{id}(f)(x, H) = f(s^{-1} \cdot x, s^{-1}H)
= \beta s_H(f(s^{-1} \cdot x, H)) = \tau_s \otimes \beta s_H(\Phi(f))(x),
\]

\(\Phi\) is equivariant, and it only remains to show that \(\Phi\) induces a bijection of the quotient by \(J\) with the quotient of \(C_0(X, B)\) by the balancing ideal \(I\).

However, if \(\Phi(f) \in I\), then \(f(x, H)(H \cdot x) = 0\) for all \(x \in X\). But then \(f(x, r H)(H r^{-1} \cdot x) = \beta^{-1}(f(x, H))(H r^{-1} \cdot x)\), which is zero since \(\beta\) covers the \(G\)-action on \(X\), and \(f \in J\). The argument reverses, so \(\Phi(J) = I\), and the result follows.

\[\Box\]

**References**

2. David Crocker, Alex Kumjian, Iain Raeburn, and Dana P. Williams, An equivariant Brauer group and actions of groups on \(C^*\)-algebras, preprint.

**Department of Mathematics, University of Nevada, Reno, Nevada 89557**
*E-mail address*: alex@math.unr.edu

**Department of Mathematics, University of Newcastle, Newcastle, New South Wales 2308, Australia**
*E-mail address*: iain@math.newcastle.edu.au

**Department of Mathematics, Dartmouth College, Hanover, New Hampshire 03755-3551**
*E-mail address*: dana.williams@dartmouth.edu