THE C^*-ALGEBRAS OF INFINITE GRAPHS

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(Communicated by David R. Larson)

ABSTRACT. We associate C^*-algebras to infinite directed graphs that are not necessarily locally finite. By realizing these algebras as Cuntz-Krieger algebras in the sense of Exel and Laca, we are able to give criteria for their uniqueness and simplicity, generalizing results of Kumjian, Pask, Raeburn, and Renault for locally finite directed graphs.

A directed graph $E = (E^0, E^1, r, s)$ consists of a set E^0 of vertices, a set E^1 of edges, and range and source maps $r, s : E^1 \to E^0$. The C^*-algebra of E is the universal C^*-algebra $C^*(E)$ generated by mutually orthogonal projections $\{p_v : v \in E^0\}$ and partial isometries $\{s_e : e \in E^1\}$ which satisfy $s_e^* s_e = p_{r(e)}$ and

$$(1.1) \qquad p_v = \sum_{e:s(e)=v} s_e s_e^*.$$

For finite graphs, these are precisely the Cuntz-Krieger algebras $\mathcal{O}_A$: given E, take A to be the edge matrix A_E defined by

$$(1.2) \qquad A_E(e, f) = \begin{cases} 1 & \text{if } r(e) = s(f), \\ 0 & \text{if } r(e) \neq s(f); \end{cases}$$

conversely, given an $I \times I$ matrix A, form the graph E_A with vertex set I and incidence matrix A, and then $C^*(E_A)$ is isomorphic to $\mathcal{O}_A$ in a slightly nonobvious way (see [9, Theorem 3] or [13, Proposition 4.1]). This correspondence carries over to locally finite graphs (graphs in which vertices receive and emit finitely many edges), and the classical uniqueness and simplicity theorems for Cuntz-Krieger algebras have elegant extensions to these graph algebras [11, 12].

If a vertex v emits infinitely many edges, then the Cuntz-Krieger relation (1.1) does not make sense in an abstract C^*-algebra. On the other hand, Fowler and Raeburn have recently noticed that if all vertices emit infinitely many edges, one can obtain uniqueness and simplicity theorems for graph algebras without demanding equality in (1.1) [8, Corollaries 4.2 and 4.5]. This suggests that a satisfactory theory should be possible if we merely insist that (1.1) holds when v emits finitely many edges.

Exel and Laca have studied a generalization of the Cuntz-Krieger algebras which allows arbitrary infinite matrices, and have obtained uniqueness and simplicity theorems among other interesting results [7]. We shall show that, if we define $C^*(E)$ in the way suggested by [8], then we can pass from the graph algebra $C^*(E)$

to the Cuntz-Krieger algebra $\mathcal{O}_{A_E}$, as defined in [7]. We can then use the results of [7] to obtain uniqueness and simplicity theorems for $C^*(E)$.

It is not clear whether the reverse passage from Cuntz-Krieger algebras to graph algebras is always possible for graphs which are not locally finite: the obvious analogue of the isomorphism of $\mathcal{O}_A$ onto $C^*(E_A)$ given in [9] and [13] would involve infinite sums of the sort we have to avoid in abstract C^*-algebras.

We shall begin by giving precise definitions and stating our main results.

Definition 1. Let $E = (E^0, E^1, r, s)$ be a directed graph. A *Cuntz-Krieger E-family* consists of mutually orthogonal projections $\{P_v : v \in E^0\}$ and partial isometries $\{S_e : e \in E^1\}$ with orthogonal ranges, such that $S_e^* S_e = P_{r(e)}$, $S_e S_e^* \leq P_{s(e)}$, and

$$(1.3) \quad P_v = \sum_{e:s(e)=v} S_e S_e^* \qquad \text{for every vertex } v \text{ with } 0 < \#\{e : s(e) = v\} < \infty.$$

The C^*-algebra $C^*(E)$ of the graph E is the C^*-algebra generated by a universal Cuntz-Krieger E-family $\{s_e, p_v\}$. (We find it helpful to use lowercase letters for Cuntz-Krieger families in an abstract C^*-algebra and uppercase letters for families of operators on Hilbert space.) To see that there is such a C^*-algebra, we can modify the construction of [10, Theorem 2.1] and [12, Theorem 1.2]. Alternatively, we can appeal to one of the more general constructions in [2] or [14] (our Proposition 12 identifies $C^*(E)$ with the Cuntz-Pimsner algebra of the Hilbert bimodule $X(E)$ introduced in [8, Example 1.2]).

We aim to prove the following theorems, which are our main results.

Theorem 2 (Uniqueness)**.** *Suppose that E is a directed graph in which every loop has an exit, and let $\{S_e, P_v\}$ and $\{T_e, Q_v\}$ be two Cuntz-Krieger E-families such that $P_v \neq 0 \neq Q_v$ for all v. Then there is an isomorphism of $C^*(\{S_e, P_v\})$ onto $C^*(\{T_e, Q_v\})$ taking S_e to T_e and P_v to Q_v.*

Theorem 3 (Simplicity)**.** *Suppose that E is a directed graph that is transitive, in the sense that every ordered pair of vertices is joined by a directed path. If E does not consist of a single loop, then $C^*(E)$ is simple.*

Theorem 4 (Pure infiniteness)**.** *Suppose that E is a directed graph with no sources, in which every vertex is connected to a loop and every loop has an exit. Then $C^*(E)$ is purely infinite.*

We shall prove these theorems by applying the corresponding results for $\mathcal{O}_A$ from [7] to the edge matrix A_E defined by (1.2). In order to do this we first establish, in Theorem 10, the isomorphism of $C^*(E)$ to $\mathcal{O}_{A_E}$ when E has no sinks or sources. We recall the definition of $\mathcal{O}_A$ [7, Definition 8.1].

Definition 5. Let I be any set and let $A = \{A(i, j)\}_{i,j \in I}$ be a $\{0,1\}$-valued matrix over I with no identically zero rows. The Cuntz-Krieger C^*-algebra $\mathcal{O}_A$ is the universal C^*-algebra generated by partial isometries $\{s_i : i \in I\}$ with commuting initial projections and mutually orthogonal range projections, and satisfying $s_i^* s_i s_j s_j^* = A(i, j) s_j s_j^*$ and

$$(1.4) \qquad \prod_{x \in X} s_x^* s_x \prod_{y \in Y} (1 - s_y^* s_y) \; = \; \sum_{j \in I} A(X, Y, j) s_j s_j^*$$

whenever X and Y are finite subsets of I such that the function

$$j \in I \mapsto A(X,Y,j) := \prod_{x \in X} A(x,j) \prod_{y \in Y}(1 - A(y,j))$$

is finitely supported.

Remark 6. To understand where this last relation comes from, notice that combinations of the formal infinite sums obtained from the original Cuntz-Krieger relations could give relations involving finite sums, and (1.4) says that these finite relations have to be satisfied in $\mathcal{O}_A$; see the introduction of [7] for more details. Although there is reference to a unit 1 in (1.4), this relation applies to algebras that are not necessarily unital, with the convention that if 1 still appears after expanding the product in (1.4) (i.e. if $X = \emptyset$ occurs in a certain relation), then the relation implicitly states that $\mathcal{O}_A$ is unital.

Remark 7. It is important to notice that the relation (1.4) also applies when the function $j \mapsto A(X,Y,j)$ is identically zero. This particular instance of (1.4) seems to be interesting in itself (e.g. in Proposition 14 below), so we emphasize it by stating the associated relation separately:

$$(1.5) \qquad \prod_{x \in X} s_x^* s_x \prod_{y \in Y}(1 - s_y^* s_y) \;=\; 0$$

whenever X and Y are finite subsets of I such that $A(X,Y,j) = 0$ for every $j \in I$.

For generic matrices, condition (1.4) may not yield any restriction at all, as there may not be any pair X, Y for which the support of $A(X,Y,j)$ is finite or empty. However, as the following lemma shows, the special case (1.5) always applies nontrivially to edge matrices.

Lemma 8. *Let A_E be the edge matrix of the graph E. If the partial isometries $\{S_e\}$ satisfy (1.5), then for every e and f in E^1, the projections $S_e^* S_e$ and $S_f^* S_f$ are equal when $r(e) = r(f)$, and mutually orthogonal when $r(e) \neq r(f)$.*

Proof. If $r(e) = r(f)$, then the rows of A_E labeled e and f coincide, so $A(\{e\}, \{f\}, j) = A_E(e,j)(1 - A_E(f,j)) = 0$ for every $j \in E^1$; hence (1.5) gives $S_e^* S_e(1 - S_f^* S_f) = 0$.

If $r(e) \neq r(f)$, then the rows of A_E labeled e and f are orthogonal, so $A_E(\{e,f\}, \emptyset, j) = A_E(e,j)A(f,j) = 0$ for every $j \in E^1$; hence (1.5) gives $S_e^* S_e S_f^* S_f = 0$, from which it follows that $S_e^* S_e(1 - S_f^* S_f) = S_e^* S_e$. $\qquad\square$

This lemma will allow us to remove the first obstacle to relating the two sets of Cuntz-Krieger relations, namely that the projections P_v in Cuntz-Krieger E-families do not appear explicitly in the relations defining $\mathcal{O}_{A_E}$. Given a family of partial isometries satisfying the latter, we will simply *define* P_v by choosing an edge e with $r(e) = v$, and taking $P_v = S_e^* S_e$. Lemma 8 implies that this is a consistent definition, and, provided E has no sources, it assigns a projection to each vertex.

Proposition 9. *Suppose E is a directed graph with no sources or sinks, and let A_E be the edge matrix of E.*

(i) *If $\{S_e, P_v\}$ is a Cuntz-Krieger E-family, then $\{S_e : e \in E^1\}$ is a collection of partial isometries satisfying the relations defining $\mathcal{O}_{A_E}$.*

(ii) *Conversely, if the partial isometries $\{S_e : e \in E^1\}$ satisfy the relations defining $\mathcal{O}_{A_E}$, and we define $P_v = S_e^* S_e$ for e such that $r(e) = v$ (cf. Lemma 8), then $\{S_e, P_v\}$ is a Cuntz-Krieger E-family.*

Proof. We prove (i) first. Assume $\{S_e, P_v\}$ is a Cuntz-Krieger E-family. Trivially the S_e have commuting initial projections and orthogonal range projections. If e and f are edges, we have that

$$S_e^* S_e S_f S_f^* = P_{r(e)} S_f S_f^* = \left\{ \begin{array}{ll} 0 & \text{if } r(e) \neq s(f), \\ S_f S_f^* & \text{if } r(e) = s(f), \end{array} \right.$$

because $S_f S_f^* \leq P_{s(f)}$. In view of the definition of A_E as the edge matrix of E, this says precisely that $S_e^* S_e S_f S_f^* = A_E(e, f) S_f S_f^*$.

Next we show that (1.4) also holds. Suppose that X and Y are finite sets of edges such that the function $A_E(X, Y, j)$ has finite (or empty) support in the variable $j \in E^1$. We divide the proof of (1.4) into two cases.

CASE I: $X = \emptyset$.

We claim that E^0 is finite. The support of $j \mapsto A_E(\emptyset, Y, j) := \prod_{y \in Y}(1 - A(y, j))$, which is finite (or empty) by assumption, is given by

$$F := \{j \in E^1 : A(y, j) = 0 \text{ for all } y \in Y\} = \{j \in E^1 : s(j) \notin r(Y)\}.$$

The source map is surjective because there are no sinks, so $E^0 = s(F) \sqcup r(Y)$, and since both $s(F)$ and $r(Y)$ are finite (or empty), E^0 is finite.

Since there are only finitely many vertices, $\sum_{v \in E^0} P_v$ is an identity for $C^*(E)$. Thus, splitting the sum according to $E^0 = r(Y) \sqcup s(F)$, we may write

$$1 - \sum_{v \in r(Y)} P_v = \sum_{v \in s(F)} P_v.$$

Since the P_v are mutually orthogonal, the left-hand side can be rewritten as a product:

$$(1.6) \qquad \prod_{v \in r(Y)} (1 - P_v) = \sum_{v \in s(F)} P_v.$$

- *Subcase* Ia. If F happens to be empty, the right-hand side of (1.6) vanishes, so (1.4) holds.
- *Subcase* Ib. If F is nonempty, then $s(F) \neq \emptyset$. For each $v \in s(F)$ we have that $\{j : s(j) = v\} \subset \{j : s(j) \notin r(Y)\} = F$, so $0 < \#\{j : s(j) = v\} < \infty$, and we may apply (1.3) to get $P_v = \sum_{j : s(j) = v} S_j S_j^*$. Summing over all vertices in $s(F)$, and substituting into (1.6), gives

$$\prod_{y \in Y} (1 - S_y^* S_y) = \sum_{j \in F} S_j S_j^*,$$

  which is (1.4).

CASE II: $X \neq \emptyset$.

Once again, let $F = \{j \in E^1 : s(j) = r(x) \text{ for all } x \in X, \text{ and } s(j) \neq r(y) \text{ for all } y \in Y\}$ be the support of $A(X, Y, j)$, and let $x_0 \in X$.

- *Subcase* IIa. Assume F is empty. Since there are no sinks, there exists $j \in E^1$ such that $s(j) = r(x_0)$. But $j \notin F$, so either $s(j) \neq r(f)$ for some $f \in X$, in which case $r(x_0) \neq r(f)$ and (1.5) holds because the left-hand side contains the product $S_{x_0}^* S_{x_0} S_f^* S_f = P_{r(x_0)} P_{r(f)}$, or else $s(j) = r(y)$ for some

$y \in Y$, in which case $r(x_0) = r(y)$ and (1.5) holds because the left-hand side contains the product $P_{r(x_0)}(1 - P_{r(y)})$.

- *Subcase* IIb. If F is nonempty, then $r(X)$ is the singleton $\{r(x_0)\}$ and $r(x_0) \notin r(Y)$. Thus $F = \{j \in E^1 : s(j) = r(x_0)\}$, and we have that $\prod_{x \in X} P_{r(x)} \prod_{y \in Y}(1 - P_{r(y)}) = P_{r(x_0)}$. With these simplifications, condition (1.4) now reads

$$P_{r(x_0)} = \sum_{j:s(j)=r(x_0)} S_j S_j^*,$$

and since $F = \{j : s(j) = r(x_0)\}$ is finite and nonempty, this follows from (1.3).

Next we prove (ii), so assume that $\{S_e : e \in E^1\}$ is a collection of partial isometries satisfying the relations defining $\mathcal{O}_{A_E}$. By Lemma 8 it is consistent to define $P_v = S_e^* S_e$ for e with $r(e) = v$. Since there are no sources, this gives a family of mutually orthogonal projections indexed by the vertices.

Let f be an edge; we show next that $S_f S_f^* \leq P_{s(f)}$. From the relations for $\mathcal{O}_A$ we know that $S_e^* S_e S_f S_f^* = A(e,f) S_f S_f^*$ for every pair of edges. We can choose e such that $r(e) = s(f)$, in which case we get $S_e^* S_e S_f S_f^* = S_f S_f^*$, proving that $S_f S_f^* \leq S_e^* S_e = P_{r(e)} = P_{s(f)}$.

To finish the proof it suffices to observe that if $\{j : s(j) = r(e)\}$ is a finite nonempty set, then the eth row of A is finite, so (1.3) follows from (1.4) with $X = \{e\}$ and $Y = \emptyset$. $\qquad\square$

Having established the equivalence of the two sets of relations for graphs with no sinks or sources, we may conclude that the corresponding C^*-algebras are canonically isomorphic.

Theorem 10. *Let E be a directed graph with no sinks or sources. The graph algebra $C^*(E)$ is canonically isomorphic to the Cuntz-Krieger algebra $\mathcal{O}_{A_E}$ of the edge matrix A_E of E.*

Proof. If the graph E has no sources, every vertex v is of the form $r(e)$, so $p_v = p_{r(e)} = s_e^* s_e$. Thus $C^*(E) = C^*(\{s_e : e \in E^1\})$, and the result follows from the equivalence of the presentations given in Proposition 9. $\qquad\square$

This isomorphism allows us to import the results from [7] in order to prove our main results. A bootstrap argument from [1] allows us to include graphs with sinks or sources in the uniqueness theorem.

Proof of Theorem 2. We first suppose that E has no sources or sinks, and seek to apply Corollary 13.2 of [7] to $C^*(E) \cong \mathcal{O}_{A_E}$. The hypotheses of that corollary are expressed in terms of the graph with vertex matrix A_E, which is the dual graph $\widehat{E}$ of E; however, loops in E correspond to loops in $\widehat{E}$, so every loop in $\widehat{E}$ has an exit, and the corollary applies.

If E has sources or sinks, we add a copy of the graph

$$\bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \cdots$$
$$v$$
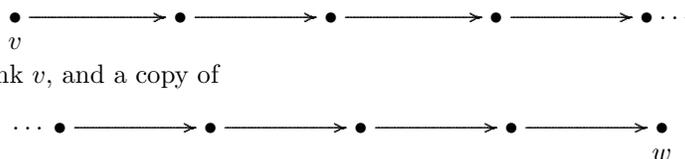
to each sink v, and a copy of

$$\cdots \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet \longrightarrow \bullet$$
$$w$$

to each source w, giving a larger graph F without sources or sinks. Now it is easy to extend the Cuntz-Krieger E-families on $\mathcal{H}$ to Cuntz-Krieger F-families by adding infinitely many copies of each $P_v\mathcal{H}$ and each $P_w\mathcal{H}$. We have not added any loops in constructing F from E, so the argument in the first paragraph applies to F; because the resulting isomorphism maps generators to generators, it restricts to an isomorphism of $C^*(S_e, P_v)$ onto $C^*(T_e, Q_v)$. $\qquad\square$

Proof of Theorem 3. Clearly E has no sources or sinks, so $C^*(E)$ is isomorphic to $\mathcal{O}_{A_E}$, by Theorem 10. The edge matrix A_E has a transitive graph, so $C^*(E)$ is simple because $\mathcal{O}_A$ is, by Theorem 14.1 of [7]. $\qquad\square$

Proof of Theorem 4. Follows from Theorem 16.2 of [7]. $\qquad\square$

Remark 11. It was pointed out in §14 of [7] that, for non-locally-finite graphs, cofinality does not imply simplicity of $\mathcal{O}_{A_E}$. The observation remains valid for graph algebras: indeed, in Example 10.8 of [7] the matrix A is the edge matrix of a graph E. Explicitly, E is the graph with $E^0 = \mathbb{Z}$, one edge from n to $n+1$ for every $n \neq 0$, and infinitely many edges from 0 to 1. This graph is cofinal, but there are representations of $C^*(E)$ such that the partial isometry S_e is zero if and only if $s(e) \geq 0$.

It is not clear at present how to relax the transitivity assumption to characterize precisely those graphs with simple C^*-algebras. Indeed, Corollary 4.5 of [8] shows that, for graphs with infinitely many edges going out of every vertex, transitivity of E is necessary as well as sufficient for simplicity of $C^*(E)$. For locally finite graphs, on the other hand, the necessary and sufficient condition is cofinality [11], which is much weaker than transitivity.

Cuntz-Pimsner algebras. Since our results were motivated by the investigations of [8] into the Toeplitz algebras of Hilbert bimodules introduced by Pimsner [14], it is interesting to note that our graph algebra $C^*(E)$ is naturally isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ of the Hilbert bimodule $X(E)$ constructed in [8, Example 1.2]. Indeed, our condition (1.3) is precisely the extra condition which distinguishes representations of $\mathcal{O}_{X(E)}$ among representations of $\mathcal{T}_{X(E)}$. This provides reassurance that we are dealing with the right notion of graph algebra, and further confirmation that Pimsner has had an important insight.

Proposition 12. *Let E be a directed graph with no sinks. The Cuntz-Pimsner algebra $\mathcal{O}_{X(E)}$ is generated by a Cuntz-Krieger E-family, and is universal for such families.*

Proof. It was shown in [8, Example 1.2 and Theorem 4.1] that the map $(\psi, \pi) \mapsto \{\psi(\delta_f), \pi(\delta_v)\}$ is a bijection between the Toeplitz representations of $X(E)$ and Toeplitz-Cuntz-Krieger E-families, and that the Toeplitz algebra $\mathcal{T}_{X(E)}$ of the bimodule $X(E)$ is universal for such families. Thus to establish the result we have to show that $\{\psi(\delta_f), \pi(\delta_v)\}$ is a Cuntz-Krieger family if and only if (ψ, π) satisfies the extra condition that characterizes representations of $\mathcal{O}_{X(E)}$, namely condition (4) of [14, Theorem 3.12].

To express this condition in the notation of [8], suppose (ψ, π) is a Toeplitz representation of a Hilbert bimodule X over A. Let $\rho^{\psi,\pi}$ be the representation of $\mathcal{L}(X)$ in $\mathcal{H}_{\psi,\pi}$ induced by (ψ, π), which is characterized by $\rho^{\psi,\pi}(S)(\psi(x)h) =$

$\psi(Sx)h$ for $S \in \mathcal{L}(X)$ and satisfies

$$\rho^{\psi,\pi}(\Theta_{x,y}) = \psi(x)\psi(y)^* \qquad x, y \in X$$

[8, Proposition 1.6]. (We note that $\rho^{\psi,\pi}$ is denoted $\psi^{(1)}$ in [14] but depends on both ψ and π.) Then (ψ, π) gives a representation of $\mathcal{O}_X$ if and only if

(1.7) $\qquad \rho^{\psi,\pi}(\phi(a)) = \pi(a)$ for every a such that $\phi(a) \in \mathcal{K}(X)$,

where $\phi : A \to \mathcal{L}(X)$ is the homomorphism describing the left action of A on X.

For the bimodule $X(E)$, we have that $\phi(a) \in \mathcal{K}(X(E))$ if and only if $a \in \overline{\mathrm{span}}\{\delta_v : v$ emits finitely many edges$\}$, and for such v

$$\phi(\delta_v) = \sum_{f:s(f)=v} \Theta_{\delta_f,\delta_f}$$

(see Proposition 4.4 of [8] and its proof). Thus (1.7) is equivalent to

$$\sum_{f:s(f)=v} \psi(\delta_f)\psi(\delta_f)^* = \pi(\delta_v)$$

whenever $0 < |s^{-1}(v)| < \infty$, which says precisely that $\{\psi(\delta_f), \pi(\delta_v)\}$ satisfies (1.3). $\qquad \square$

Remark 13. This result is not true for graphs with sinks. The point mass δ_v corresponding to a sink v acts trivially on the left of $X(E)$, and hence is in the kernel of $\phi : C_0(E^0) \to \mathcal{L}(X(E))$; thus (1.7) says that $\pi(\delta_v) = 0$. But in [11] a deliberate decision was made not to impose a relation on p_v, so that p_v is always nonzero in $C^*(E)$.

Toeplitz-Cuntz-Krieger algebras. Dropping the relation (1.4) from Definition 5 gives an extension $\mathcal{T}\mathcal{O}_A$ of $\mathcal{O}_A$, which was called the Toeplitz-Cuntz-Krieger algebra of A in [7]. Unfortunately, when A_E is the edge matrix of a graph E, the algebra $\mathcal{T}\mathcal{O}_{A_E}$ is not universal for the Toeplitz-Cuntz-Krieger E-families as defined in [8, Example 1.2], and has $\mathcal{T}_{X(E)}$ as a proper quotient: without (1.4), nothing forces the initial projections $S_i^* S_i$ to be mutually orthogonal or equal, as is required in Toeplitz-Cuntz-Krieger E-families. We shall show that the missing relations are precisely those singled out in Remark 7, which correspond to the degenerate situation in which the functions $j \mapsto A(X, Y, j)$ are identically zero.

Proposition 14. *Let E be a graph with no sources and let A_E be its edge matrix.*
   (i) *Suppose $\{S_e : e \in E^1\}$ are partial isometries with commuting initial projections and mutually orthogonal range projections such that $S_i^* S_i S_j S_j^* = A_E(i,j)S_j S_j^*$ and (1.5) holds. For $v \in E^0$, choose e with $r(e) = v$ and define $P_v := S_e^* S_e$ (cf. Lemma 8). Then $\{S_e, P_v\}$ is a Toeplitz-Cuntz-Krieger E-family.*
   (ii) *If $\{S_e, P_v\}$ is a Toeplitz-Cuntz-Krieger E-family, then $\{S_e : e \in E^1\}$ is a collection of partial isometries with commuting initial projections and orthogonal range projections such that $S_i^* S_i S_j S_j^* = A_E(i,j)S_j S_j^*$ and (1.5) holds.*
   (iii) *The C^*-algebra $\mathcal{T}_{X(E)}$ is the proper quotient of the algebra $\mathcal{T}\mathcal{O}_{A_E}$ from [7] defined by the extra relation (1.5).*

Proof. For the proof of (i), it suffices to observe that the P_v are well-defined mutually orthogonal projections, by Lemma 8; the rest has been done in the proof of Proposition 9(ii).

For the proof of (ii), it suffices to observe that, in the proof of Proposition 9(i), (1.3) is not used in subcases Ia and IIa, which proves the special case (1.5) of (1.4). (Condition (1.3) is needed to prove (1.4) only when the set F is nonempty.)

For part (iii), notice that since E has no sources, the projections P_v are automatically in the C^*-algebra generated by the S_e. By parts (i) and (ii), the relations defining Toeplitz-Cuntz-Krieger E-families are equivalent to the modification of the relations defining $\mathcal{TO}_{A_E}$ in which the requirement of commuting initial projections is replaced by the strictly stronger (1.5). This concludes the proof because, by [8, Theorem 4.1], the C^*-algebra $\mathcal{T}_{X(E)}$ is universal for Toeplitz-Cuntz-Krieger E-families. $\square$

Finally, we observe that $\mathcal{T}_{X(E)}$ inherits from $\mathcal{TO}_{A_E}$ a partial crossed product structure like that in Theorem 4.6 of [7].

Corollary 15. *Let E be a directed graph with no sources and let $\mathbb{F}$ be the free group on E^1. Then*

$$\mathcal{T}_{X(E)} \cong C(\Omega) \rtimes_\alpha \mathbb{F},$$

in which Ω is the spectrum of the relations

$$S_e S_e^* S_f S_f^* = \delta_{e,f} S_f S_f^*,$$

$$S_e^* S_e S_f S_f^* = A_E(e,f) S_f S_f^*,$$

and (1.5), and α is the restriction of the canonical partial action.

Proof. The relations defining Toeplitz-Cuntz-Krieger families are obtained by adding (1.5) to the relations defining $\mathcal{TO}_{A_E}$, and only involve the initial and final projections of the generating partial isometries, so the result follows from Theorem 4.4 of [6]. $\square$

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